Balanced MultiWavelets Theory and Design

Jérôme Lebrun and Martin Vetterli

Dept. of Electrical Engineering

Swiss Federal Institute of Technology - Lausanne

{lebrun,vetterli}@de.epfl.ch

Martin Vetterli is also with the Dept. of Electrical Engineering, University of California at Berkeley. This work was supported in part by the National Science Foundation under grant MIP-93-21302 and by the Swiss National Science Foundation under grant 2100-043136.95/1.

Abstract

This paper deals with multiwavelets which are a recent generalization of wavelets in the context of multirate filter banks and with their applications to signal processing and especially compression. By their inherent structure, multiwavelets are fit for processing multi-channel signals. First, we will recall some general results on multifilters by looking at them as time-varying filters. Then, we will link this to multiwavelets, looking closely at the convergence of the iterated matrix product leading to them and the typical properties we can expect. Then, we will define under what conditions we can apply systems based on multiwavelets to one-dimensional signals in a simple way. That means we will give some natural and simple conditions that should help in the design of new multiwavelets for signal processing. Finally, we will provide some tools in order to construct multiwavelets with the required properties, the so-called 'balanced multiwavelets'.

Keywords

Multifilter, multiwavelet, prefiltering, balancing.

I. Introduction

Wavelet constructions from iterated filter banks, as pioneered by Daubechies [4], have become a standard way to derive orthogonal and biorthogonal wavelet bases. The underlying filter banks are well studied, and thus, the design procedure is well understood. By the structure of the problem, certain issues are ruled out: the impossibility of constructing orthogonal FIR linear phase filter banks implies that there is no orthogonal wavelet with compact support and symmetry. Nevertheless, by relaxing the requirement of time-invariance, it is easy to see that new solutions are possible. As mentioned in [18], such filter banks are closely related to some matrix 2-scale equations leading to multiwavelets. The outline of the paper is as follow. First, we will review material on multifilter banks and time-varying filter banks in Section II. Then, Section III deals with multiwavelets and their link with multifilters. Here, we will mostly recall some known results but from the point of view of signal processing. Finally, in Section IV, we introduce the problems encountered when using multiwavelets in applications and give some new direction for the design of multiwavelets.

II. Multifilter banks

A. Theory

A.1 Time-varying filter banks

We define time-varying filter banks as filter banks where the filter applied to the signal varies periodically in time. Here, we are specifically interested in time-varying interpolation filters, that is, an upsampling function (typically by 2) followed by a LPTV (linear periodically time-varying) interpolation filter. In time domain, the resulting operator (we consider the case of two alternating impulse responses for simplicity) is given by

$$\mathbf{T} = \begin{pmatrix} \ddots & & & & \\ & c[0] & d[0] & \dots & \\ & c[1] & d[1] & \dots & \\ & c[2] & d[2] & 0 & \dots \\ & c[3] & d[3] & 0 & \dots \\ & c[4] & d[4] & c[0] & d[0] \\ & c[5] & d[5] & c[1] & d[1] \end{pmatrix} \quad (1)$$

where $c[k]$ and $d[k]$ are the two interpolation filter impulse responses. Clearly, when $\mathbf{T}$ is applied to a sequence $x[n]$, then $x[2n]$ and $x[2n+1]$ lead to impulses $c[k-4n]$ and $d[k-4n]$, respectively. That is, even and odd indexed samples lead to different responses, as to be expected. In z-transform domain, write sequences in terms of even and odd indexed subsequences, or polyphase components, as

$$X(z) = X_0(z^2) + z^{-1}X_1(z^2) \quad (2)$$

$$Y(z) = Y_0(z^2) + z^{-1}Y_1(z^2) \quad (3)$$

for the input and output, as well as the filters

$$C(z) = C_0(z^2) + z^{-1}C_1(z^2) \quad (4)$$

$$D(z) = D_0(z^2) + z^{-1}D_1(z^2) \quad (5)$$

Then, the polyphase components of $Y(z)$ can be written in term of the polyphase components of the input $X(z)$ as

$$\begin{pmatrix} Y_0(z) \\ Y_1(z) \end{pmatrix} = \begin{pmatrix} C_0(z) & D_0(z) \\ C_1(z) & D_1(z) \end{pmatrix} \cdot \begin{pmatrix} X_0(z^2) \\ X_1(z^2) \end{pmatrix} \tag{6}$$

Call the above matrix $\mathbf{T}(z)$. Its size is given by the number of different impulse responses, or period. In the special case when the filter is time-invariant ($d[k] = c[k-2]$), $\mathbf{T}(z)$ is

$$\mathbf{T}(z) = \begin{pmatrix} C_0(z) \\ C_1(z) \end{pmatrix} \cdot \begin{pmatrix} 1 & z^{-1} \end{pmatrix} \tag{7}$$

A.2 Properties

For convenience, we merge the two 'lowpass' filters $c[n]$ and $d[n]$ into a single matrix coefficients multifilter $\mathbf{M}[n]$ defined by

$$\mathbf{M}[k] := \begin{pmatrix} c[2k] & c[2k+1] \\ d[2k] & d[2k+1] \end{pmatrix} \tag{8}$$

We then define the z-transform of the 'lowpass' analysis multifilter

$$\mathbf{H}_0(z) := \mathbf{T}^\top(z) = \sum_k \mathbf{M}[k] z^{-k} \tag{9}$$

and in exactly the same way, we define $\mathbf{H}_1(z)$, $\mathbf{G}_0(z)$ and $\mathbf{G}_1(z)$ respectively the 'highpass' analysis, 'lowpass' synthesis and 'highpass' synthesis multifilters. Then defining $\mathbf{X}(z) := [X_0(z), X_1(z)]^\top$ the input signal, we have the familiar result for the output signal of the filterbank

$$\hat{\mathbf{X}}(z) = \frac{1}{2} \{[\mathbf{G}_0(z)\mathbf{H}_0(z) + \mathbf{G}_1(z)\mathbf{H}_1(z)]\mathbf{X}(z) \tag{10}$$

$$+[\mathbf{G}_0(z)\mathbf{H}_0(-z) + \mathbf{G}_1(z)\mathbf{H}_1(-z)]\mathbf{X}(-z)\} \tag{11}$$

Note that unlike the scalar case, the order of the product is very important, since matrix products do not commute.

- Biorthogonal multifilter banks

From (10), we have the conditions for perfect reconstruction

$$\mathbf{G}_0(z)\mathbf{H}_0(z) + \mathbf{G}_1(z)\mathbf{H}_1(z) = 2\mathbf{I}_2 \tag{12}$$

and for alias cancelation

$$\mathbf{G}_0(z)\mathbf{H}_0(-z) + \mathbf{G}_1(z)\mathbf{H}_1(-z) = \mathbf{O}_2 \tag{13}$$

Introducing the modulation matrix

$$\mathbf{H}_m(z) := \begin{pmatrix} \mathbf{H}_0(z) & \mathbf{H}_0(-z) \\ \mathbf{H}_1(z) & \mathbf{H}_1(-z) \end{pmatrix} \tag{14}$$

then we can include the two conditions into

$$[\mathbf{G}_0(z), \mathbf{G}_1(z)]\ \mathbf{H}_m(z) = 2\ [\mathbf{I}_2, \mathbf{O}_2] \tag{15}$$

Because of the non commutativity of matrix products, there is no Smith & Barnwell simple solution as in the scalar case. However, some straightforward calculation leads to

$$\mathbf{G}_0(z) \;=\; 2\mathbf{U}^{-1}(z) \tag{16}$$

$$\mathbf{G}_1(z) \;=\; -2\mathbf{U}^{-1}(z)\mathbf{H}_0(-z)\mathbf{H}_1^{-1}(-z) \tag{17}$$

where

$$\mathbf{U}(z) := \mathbf{H}_0(z) - \mathbf{H}_0(-z)\mathbf{H}_1^{-1}(-z)\mathbf{H}_1(z) \tag{18}$$

- Orthogonal multifilter banks

As usual, we are particularly interested in the case when the operator $\mathbf{T}$ in (1) is unitary, or $\mathbf{T}^T\mathbf{T} = \mathbf{I}$. Expressed in the modulation form, this gives

$$\tilde{\mathbf{H}}_m(z)\mathbf{H}_m(z) = \mathbf{H}_m(z)\tilde{\mathbf{H}}_m(z) = \mathbf{I}_2 \tag{19}$$

where $\tilde{\mathbf{H}}(z) := \mathbf{H}^\top(z^{-1})$ if we assume real coefficients for the filters. $\tilde{\mathbf{H}}(z)$ is called the paraconjuguate of $\mathbf{H}(z)$ [17]. This gives

$$\mathbf{H}_0(z)\tilde{\mathbf{H}}_0(z) + \mathbf{H}_0(-z)\tilde{\mathbf{H}}_0(-z) \;=\; \mathbf{I}_2 \tag{20}$$

$$\mathbf{H}_1(z)\tilde{\mathbf{H}}_1(z) + \mathbf{H}_1(-z)\tilde{\mathbf{H}}_1(-z) \;=\; \mathbf{I}_2 \tag{21}$$

$$\mathbf{H}_0(z)\tilde{\mathbf{H}}_1(z) + \mathbf{H}_0(-z)\tilde{\mathbf{H}}_1(-z) \;=\; \mathbf{O}_2 \tag{22}$$

$$\mathbf{H}_1(z)\tilde{\mathbf{H}}_0(z) + \mathbf{H}_1(-z)\tilde{\mathbf{H}}_0(-z) \;=\; \mathbf{O}_2 \tag{23}$$

We then obtain for perfect reconstruction and alias cancelation

$$\mathbf{G}_0(z) = \tilde{\mathbf{H}}_0(z) \tag{24}$$

$$\mathbf{G}_1(z) = \tilde{\mathbf{H}}_1(z) \tag{25}$$

- Linear phase

We say that the multifilter $\mathbf{T}(z)$ defined in (6) has linear phase if there exists c_0, c_1 real numbers such that

$$\mathbf{T}_{ij}(z) = \alpha_{ij}\, z^{2c_i - c_j}\, \mathbf{T}_{ij}(z^{-1}) \tag{26}$$

where α_{ij} is one of the following functions: $\alpha_{ij} = 1$ or $\alpha_{ij} = -1$ or $\alpha_{ij} = (-1)^{i+j}$ or $\alpha_{ij} = -(-1)^{i+j}$. We could have defined linear phase by requiring $c[n]$ and $d[n]$ to be linear phase, but this isn't enough to ensure symmetry or antisymmetry of the scaling functions obtained by iteration of the matrix product leading to multiwavelets.

B. Design of multifilter banks

Algebraically, a 2 channels time-varying filter bank with 2 phases is equivalent to a 4 channels filter bank. This is easy to see, since we have 4 distinct impulse responses that have to generate (with shifts by 4) a basis of $\ell_2(\mathbb{Z})$. More generally, a K channels multifilter bank with L phases is equivalent to a $K \times L$ channels filter bank. That is, all results known for N channels filter banks can be used immediately in the context of time-varying or multifilter banks. For example, it is clear that orthonormal, linear phase FIR solutions exist for 2 channels, 2 phases multifilter banks, since such solutions exists in the 4 channels case.

Let us now consider a specific problem: namely that of completion. In several multiwavelets constructions [5], [14], [1], scaling functions were constructed first, and multiwavelets were derived some time after. In the usual wavelet case, this is really simple, since the highpass filter which is complementary to a given lowpass filter is easily specified [19]. In the multifilter case, the problem can be stated as follows: given a 2×4 unimodular or paraunitary matrix, find a unimodular or paraunitary completion. Let us recall that a unimodular matrix $\mathbf{C}(z)$ is an $M \times N$ matrix with $M \leq N$ of Laurent polynomials such that there exists an $N \times M$ right inverse matrix $D(z)$ of Laurent polynomials. The problem of completion is that of finding an $N \times N$ matrix $\mathbf{C}'(z)$, where the first M rows equal

$\mathbf{C}(z)$, such that $\mathbf{C}'(z)$ is a square unimodular matrix. Similarly, a $M \times N$ with $M \leq N$ paraunitary matrix $\mathbf{U}(z)$ of Laurent polynomials satisfies $\mathbf{U}(z)\tilde{\mathbf{U}}(z) = \mathbf{I}_{M \times M}$. The completion problem is then to find an $N \times N$ matrix $\mathbf{U}'(z)$, with the first M rows given by $\mathbf{U}(z)$, such that $\mathbf{U}'(z)$ is a square paraunitary matrix. It turns out that the completion problem is a standard question in linear system theory and algebraic geometry. We will thus simply review the state of affairs.

First, in one dimension (single variable polynomials), the questions are well understood, and can be found in textbooks. If the original matrix contains Laurent polynomials, one can first use a change of variable to reduce it to a matrix with polynomials only. Then, standard factorization procedures for unimodular or paraunitary matrices can be used. This leads to cascades of $N \times N$ matrices followed by a left most matrix of size $M \times N$. The problem is then reduced to complete this final $M \times N$ matrix into a square matrix, either a full rank matrix (unimodular case) or an orthogonal matrix (paraunitary case). It can be shown that this covers the whole range of possible completions.

The multidimensional case (multiple variable polynomials) has been addressed only recently, in particular in H.J.Park's thesis [11], [12]. The situation is then the following:

- The transformation of Laurent polynomial into regular polynomial can be done as in the one variable case.
- Unimodular completion is solvable, in a similar way as in one dimension. It is based on a factorization procedure for multivariable unimodular matrices [11].
- Paraunitary completion is an open problem in the multidimensional case. One complication is that factorization is not always possible, and thus, the one dimensional approach to completion cannot be generalized.

C. Iterated multifilter banks

Now, it is easy to study iterated LPTV interpolators. Calling the n-times cascade transfer matrix $\mathbf{T}^{(n)}(z)$

$$\mathbf{T}^{(n)}(z) = \prod_{i=0}^{n-1} \mathbf{T}(z^{2^i}), \tag{27}$$

we get the output, after n-times upsampling and interpolation, as

$$Y^{(n)}(z) = \begin{pmatrix} 1 & z^{-1} \end{pmatrix} \mathbf{T}^{(n)}(z^2) \begin{pmatrix} X_0(z^4) \\ X_1(z^4) \end{pmatrix}. \tag{28}$$

Note that there are two impulse responses, given by

$$C^{(n)}(z) = T_{00}(z^2) + z^{-1} T_{10}(z^2), \tag{29}$$

$$D^{(n)}(z) = T_{01}(z^2) + z^{-1} T_{11}(z^2). \tag{30}$$

• Orthogonality

If $\mathbf{T}$ is unitary, so is $\mathbf{T}^{(n)}$. The important point is that $c^{(n)}[k]$ and $d^{(n)}[k]$ (the impulse responses of $C^{(n)}(z)$ and $D^{(n)}(z)$ from (29)) are of unit norm and orthogonal with respect to shifts by $2^{(n+1)}$.

• Linear phase

It is easily seen that the linear phase property of multifilter as defined in Section II.B is maintained during the iterations. For example, one easily proves that $\mathbf{T}(z^2)\mathbf{T}(z)$ has linear phase with coefficients $3c_0, c_1 - 2c_0$.

Note that in the above, we concentrated on the lowpass channel of a time-varying filter bank. For a unitary transformation, we also need a time-varying highpass channel that is orthogonal to the time-varying lowpass, as well as to its own translates. However, for all discussions concerning regularity or iteration, the lowpass channel is the key element (since that is the channel involved in the infinite iteration, while the highpass channel is only applied once).

III. Multiwavelets

Similar to the wavelet case, the multiscaling function $\phi(t) := [\phi_0(t), \ldots, \phi_{r-1}(t)]^\top$ is solution of a 2-scale equation

$$\phi(t) = \sum_{k=0}^{N} \mathbf{M}[k]\phi(2t-k) \tag{31}$$

where $\mathbf{M}[k]$ are $r \times r$ matrices of real coefficients. The properties of the scaling function are strongly dependent on the spectral behavior of the refinement mask

$$\mathbf{M}(\omega) := \frac{1}{2}\sum_{k=0}^{N} \mathbf{M}[k]e^{-j\omega k} \tag{32}$$

By defining the Fourier transform componentwise for matrix-valued functions, the 2-scale equation converts to the equivalent form

$$\hat{\phi}(2\omega) = \mathbf{M}(\omega)\hat{\phi}(\omega) \tag{33}$$

and we can then derive the behavior of the scaling function by iterating this product [2]. Once more, for simplicity and without loss of generality, we will concentrate on the case $r = 2$.

A. Convergence

In the wavelet case, $M(\omega)$ is a trigonometric polynomial satisfying the following two necessary constraints: (i) $M(0) = 1$ and (ii) $M(\pi) = 0$ for the iterated product to converge. The multiwavelet case is more tedious. As in [18], we define $D(\omega)$ the determinant of $\mathbf{M}(\omega)$, and $\{\lambda_0(\omega), \lambda_1(\omega)\}$ the eigenvalues of $\mathbf{M}(\omega)$. We also define

$$\mathbf{\Phi}^{(n)}(\omega) := \mathbf{M}(\omega/2) \cdot \mathbf{M}(\omega/4) \ldots \mathbf{M}(\omega/2^n) \cdot \Theta(\omega) \tag{34}$$

where $\Theta(\omega)$ is the normalized interpolation function

$$\Theta(\omega) := e^{-j\omega/2^{n+1}} \cdot \frac{\sin(\omega/2^{n+1})}{\omega/2^{n+1}} \tag{35}$$

Note that $\mathbf{\Phi}^{(n)}(\omega)$ satisfies

$$\|\Phi_{i0}^{(n)}(\omega)\|_2^2 + \|\Phi_{i1}^{(n)}(\omega)\|_2^2 = 1 \tag{36}$$

given that we have orthogonality

$$\sum_k \mathbf{M}[k]\mathbf{M}[2l+k]^\top = 2\delta_{0l}\mathbf{I} \quad \forall l \tag{37}$$

Also, $\Theta(\omega) \to 1$ for any finite ω and large n, and can thus be ignored. In the following, we will be interested in the limit

$$\mathbf{\Phi}(\omega) := \lim_{n\to\infty} \mathbf{\Phi}^{(n)}(\omega) = \prod_{i=1}^{\infty} \mathbf{M}(\omega/2^i) \tag{38}$$

Note that

$$\mathbf{\Phi}(\omega) = \mathbf{M}(\omega/2) \cdot \mathbf{\Phi}(\omega/2) \tag{39}$$

or

$$\begin{pmatrix} \phi_{00}(\omega) & \phi_{01}(\omega) \\ \phi_{10}(\omega) & \phi_{11}(\omega) \end{pmatrix} = \begin{pmatrix} M_{00}(\frac{\omega}{2}) & M_{01}(\frac{\omega}{2}) \\ M_{10}(\frac{\omega}{2}) & M_{11}(\frac{\omega}{2}) \end{pmatrix} \cdot \begin{pmatrix} \phi_{00}(\frac{\omega}{2}) & \phi_{01}(\frac{\omega}{2}) \\ \phi_{10}(\frac{\omega}{2}) & \phi_{11}(\frac{\omega}{2}) \end{pmatrix} \tag{40}$$

A.1 Unconstrained convergence

Following the terminology developed in [6], we say that 'unconstrained' solutions occur when the infinite matrix product

$$\mathbf{\Phi}(\omega) := \lim_{n\to\infty} \mathbf{\Phi}^{(n)}(\omega) = \prod_{i=1}^{\infty} \mathbf{M}(\omega/2^i) \tag{41}$$

converges for every ω. Following closely what was already developed by one of this paper's authors in [18], we get then that first, $\mathbf{\Phi}(0)$ has to be finite, and thus, neither eigenvalue of $\mathbf{M}(0)$ can be larger than 1 in absolute value. If both are smaller than 1 in absolute value, $\mathbf{\Phi}(0)$ will be the zero matrix, which contradicts the requirement that it represents scaling functions, or lowpass filters. Thus, either $|\lambda_0(0)| = |\lambda_1(0)| = 1$ or $|\lambda_0(0)| = 1$ and $|\lambda_1(0)| < 1$. For convergence of the infinite product at $\omega = 0$, it is further necessary that eigenvalues of absolute value 1 are actually equal to 1, since otherwise, at least one of the entries will not be a Cauchy sequence. Thus, for pointwise convergence at $\omega = 0$, $\mathbf{M}(0)$ has either (i) $\lambda_0(0) = \lambda_1(0) = 1$, that is $\mathbf{M}(0) = \mathbf{I}$ or (ii) $\lambda_0(0) = 1$ and $|\lambda_1(0)| < 1$.

Let us now investigate conditions on $\mathbf{M}(\pi)$. We assume that the infinite product converges pointwise, and want to see what condition it imposes on $\mathbf{M}(\omega)$. Write

$$\mathbf{M}(\omega) = \mathbf{M}_e(2\omega) + e^{-j\omega}\mathbf{M}_o(2\omega) \tag{42}$$

where $\mathbf{M}_e(2\omega)$ and $\mathbf{M}_o(2\omega)$ correspond to even and odd polyphase components of $\mathbf{M}(\omega)$. Also, call $\mathbf{M}^{(n)}(\omega)$ the n-times iteration. Then

$$\begin{aligned} \mathbf{M}^{(n)}(\omega) &= \mathbf{M}(2^{n-1}\omega) \cdot \mathbf{M}(2^{n-2}\omega) \ldots \mathbf{M}(\omega) \\ &= \mathbf{M}^{(n-1)}(2\omega)[\mathbf{M}_e(2\omega) + e^{-j\omega}\mathbf{M}_o(2\omega)] \end{aligned} \tag{43}$$

Consider the even and odd polyphase components of $\mathbf{M}^{(n)}(\omega)$,

$$\mathbf{M}_e^{(n)}(\omega) = \mathbf{M}^{(n-1)}(\omega) \cdot \mathbf{M}_e(\omega) \tag{44}$$

$$\mathbf{M}_o^{(n)}(\omega) \;=\; \mathbf{M}^{(n-1)}(\omega) \cdot \mathbf{M}_o(\omega) \tag{45}$$

Associate piecewise constant approximations with unit elements of length $1/2^n$ in the usual manner, and take the limit as $n \to \infty$. That is, ω is divided by 2^n. Then, $\mathbf{M}_e^{(n)}(\omega/2^n)$ goes towards $\mathbf{\Phi}(\omega/2)$, as do $\mathbf{M}_o^{(n)}(\omega/2^n)$ and $\mathbf{M}^{(n-1)}(\omega)$. On the other hand, $\mathbf{M}_e(\omega/2^n)$ goes towards $\mathbf{M}_e(0)$, and $\mathbf{M}_o(\omega/2^n)$ towards $\mathbf{M}_o(0)$ for any finite ω. Therefore, (44-45) become

$$\mathbf{\Phi}(\omega/2) \;=\; \mathbf{\Phi}(\omega/2) \cdot \mathbf{M}_e(0) \tag{46}$$

$$\mathbf{\Phi}(\omega/2) \;=\; \mathbf{\Phi}(\omega/2) \cdot \mathbf{M}_o(0) \tag{47}$$

and we get

$$\mathbf{\Phi}(\omega/2) \cdot \mathbf{M}_e(0) = \mathbf{\Phi}(\omega/2) \cdot \mathbf{M}_o(0). \tag{48}$$

There are two cases:

(i) $\mathbf{\Phi}(\omega)$ has full rank for some ω

$$\mathbf{M}_e(0) = \mathbf{M}_o(0) \Leftrightarrow \mathbf{M}(\pi) = \mathbf{0}. \tag{49}$$

(ii) $\mathbf{\Phi}(\omega)$ has rank 1 for some ω

$$\mathbf{\Phi}(\omega) \cdot [\mathbf{M}_e(0) - \mathbf{M}_o(0)] = \mathbf{\Phi}(\omega) \cdot \mathbf{M}(\pi) = \mathbf{0}. \tag{50}$$

Consider case (i). $\mathbf{M}(\omega)$ satisfies the matrix Smith-Barnwell condition:

$$\mathbf{M}(\omega)\mathbf{M}^T(-\omega) + \mathbf{M}(\omega+\pi)\mathbf{M}^T(-\omega+\pi) = \mathbf{I}. \tag{51}$$

At $\omega = 0$, since $\mathbf{M}(\pi) = \mathbf{0}$, we get

$$\mathbf{M}(0)\mathbf{M}^T(0) = \mathbf{I}. \tag{52}$$

that is, $\mathbf{M}(0)$ is unitary, or orthonormal since we assume real filters. That is, it is a rotation matrix, and in order for $\mathbf{M}^{(n)}(0)$ to converge, $\mathbf{M}(0)$ has to be the identity.

Consider case (ii) and (50) at $\omega = 0$,

$$\mathbf{\Phi}(0) \cdot \mathbf{M}(\pi) = \mathbf{0}. \tag{53}$$

Thus, $\mathbf{\Phi}(0)$ is of rank 1, and its rows are colinear with the left eigenvector $\mathbf{r}_0$ attached to the eigenvalue $\lambda_0(0) = 1$ (since $\mathbf{\Phi}(0) = \lim_{n\to\infty} \mathbf{M}^{(n)}(0)$). Therefore, a necessary condition is

$$\mathbf{r}_0 \cdot \mathbf{M}(\pi) = \mathbf{0}. \tag{54}$$

We can summarize our findings so far.

Proposition III.1

Given an infinite matrix product of size 2 by 2

$$\mathbf{\Phi}(\omega) = \prod_{i=1}^{\infty} \mathbf{M}(\omega/2^i) \tag{55}$$

where $\mathbf{M}(\omega)$ *satisfies a matrix Smith-Barnwell condition (51), a necessary condition for convergence to a scaling matrix* $\mathbf{\Phi}(\omega)$ *such that* $\mathbf{\Phi}(0)$ *is non-zero and bounded is*

(i) $\mathbf{M}(0) = \mathbf{I}$, $\mathbf{M}(\pi) = \mathbf{0}$ *(note:* $\mathbf{\Phi}(\omega)$ *has rank 2)*

(ii) $\mathbf{M}(0)$ *has eigenvalue* $\lambda_0(0) = 1$ *and* $|\lambda_1(0)| < 1$, $\mathbf{M}(\pi)$ *has rank 1 and satisfies* $\mathbf{r}_0 \cdot \mathbf{M}(\pi) = \mathbf{0}$ *(note:* $\mathbf{\Phi}(\omega)$ *has rank 1)*

A.2 Constrained convergence

Following [6], we define 'constrained' convergence when the matrix product

$$\mathbf{\Phi}(\omega) := \lim_{n\to\infty} \mathbf{\Phi}^{(n)}(\omega) = \prod_{i=1}^{\infty} \mathbf{M}(\omega/2^i) \tag{56}$$

does not converge (for example, if 1 is not the unique largest eigenvalue of $\mathbf{M}(0)$). However, we have the convergence of the matrix product

$$g(\omega) := \lim_{n\to\infty} (\mathbf{\Phi}^{(n)}(\omega) u) = \lim_{n\to\infty} \left[\prod_{i=1}^{n} \mathbf{M}(\omega/2^i) \right] u \tag{57}$$

converges nevertheless for u some 1-right eigenvector of $\mathbf{M}(0)$. We have then the result given in [10] with $\|.\|$ standing for the spectral norm (norm-2) of vector or matrix.

Theorem III.1 (Massopust)

If there exists $C > 0$ *and* $0 < \alpha \leq 1$ *such that for large* k

$$\|u - \mathbf{M}(2^{-k}\omega)u\| \leq C|2^{-k}\omega|^{\alpha} \tag{58}$$

and if $\mathbf{M}(0)$ *has spectral radius* 1*, then*

$$g(\omega) := \lim_{n\to\infty} \left[\prod_{i=1}^{n} \mathbf{M}(\omega/2^i) \right] u \tag{59}$$

converges pointwise and $g(\omega)$ *satisfies*

(i) $g(\omega) = \mathbf{M}(\omega/2) g(\omega/2)$

(ii) $g(0) = u$

Furthermore , if $g_0, g_1 \in L^2(\mathbb{R})$ *then there exists* ϕ *with* $\phi_0, \phi_1 \in L^2(\mathbb{R})$ *such that* $\hat{\phi}(0) = u$ *and* $\hat{\phi} = g$

This provides a way to construct iteratively ϕ as shown by the following theorem from [10]

Theorem III.2 (Massopust)

Assume $\mathbf{M}(\omega)$ *verifies the hypotheses of theorem III.1 and suppose* $\exists f \in L^2(\mathbb{R})$ *such that* $\hat{f}(0) = 1$ *and* f *is continuous at* 0*. Then defining* $\phi_{[0]}(\omega) := \hat{f}(\omega)\, u$ *and assuming* $\exists\, C > 0, \eta > 0$ *such that* $\forall n, \omega$

$$\left| \prod_{i=1}^{n} \mathbf{M}(2^{-i}\omega) \hat{\phi}_{[0]}(2^{-n}\omega) \right| \leq \frac{C}{(1 + |\omega|)^{\eta}} \tag{60}$$

Then defining

$$\phi_{[n]}(x) := \sum_{k=0}^{N} \mathbf{M}[k] \phi_{[n-1]}(2x - k) \tag{61}$$

we get $\forall x$

$$\lim_{n\to\infty} \phi_{[n]}(x) = \phi(x) \tag{62}$$

with uniform convergence.

B. Properties

B.1 Support

Defining $\operatorname{supp}\phi := \operatorname{supp}\phi_0 \cup \operatorname{supp}\phi_1$, we have as a direct consequence of theorem III.2, that if $\phi_{[0]}$ has compact support, then ϕ has compact support. A more general result from [1] gives also

Proposition III.2

Let ϕ be a solution of

$$\phi(t) = \sum_{k=0}^{N} \mathbf{M}[k]\phi(2t-k)$$

then ϕ is compactly supported with $\mathrm{supp}\,\phi \subseteq [0,\mathrm{N}]$ *Moreover*

1. *if* $\mathbf{M}$*[0] is nilpotent, then* $\mathrm{supp}\,\phi \subseteq [\frac{1}{3},\mathrm{N}]$
2. *if* $\mathbf{M}$*[N] is nilpotent, then* $\mathrm{supp}\,\phi \subseteq [0,\mathrm{N}-\frac{1}{3}]$
3. *if both* $\mathbf{M}$*[0] and* $\mathbf{M}$*[N] are not nilpotent, then* $\mathrm{supp}\,\phi = [0,\mathrm{N}]$

B.2 Symmetry

A proposition given in [1] links symmetry of the scaling functions with the property of linear phase of the refinement mask.

Proposition III.3 (Chui)

Let ϕ be a scaling functions satisfying (31) with the refinement mask $\mathbf{M}$ *such that* $\mathrm{supp}\,\phi_\mathrm{i} = [\mathrm{a_i},\mathrm{b_i}] \subseteq [0,\mathrm{N}]$. *Then ϕ_0 is symmetric and ϕ_1 antisymmetric i.e. for $i=0,1$*

$$\phi_i(x) = (-1)^i \phi_i(a_i+b_i-x) \tag{63}$$

if and only if the refinement mask $\mathbf{M}$ *verifies*

$$\mathbf{M}_{ij}(z) = (-1)^{i+j}\, z^{2(a_i+b_i)-(a_j+b_j)}\mathbf{M}_{ij}(z^{-1}) \tag{64}$$

B.3 Approximation power

One says that ϕ has approximation power m if one can exactly decompose polynomials up to degree $m-1$ using only ϕ_0,ϕ_1 and their translates. Calling $\mathbf{M}^{(k)}(\omega) := \frac{d^k}{d\omega^k}\mathbf{M}(\omega)$, we have the following theorem

Theorem III.3 (Plonka)

Let ϕ be a integrable scaling functions satisfying (31) with the refinement mask $\mathbf{M}$ *such that the integer translates of ϕ_0,ϕ_1 are independent. Then ϕ has approximation power m if and only if there exist vectors* $\mathbf{y}_0,\ldots,\mathbf{y}_{(m-1)} \in \mathbb{C}^2$ *with* $\mathbf{y}_0 \neq \mathbf{0}$ *such that for $l=0,\ldots,m-1$*

$$\sum_{k=0}^{l} \left(\begin{array}{c} l \\ k \end{array}\right) (2j)^{k-l}\mathbf{y}_k\mathbf{M}^{(l-k)}(0) \quad = \quad 2^{-n}\mathbf{y}_l \tag{65}$$

$$\sum_{k=0}^{l} \begin{pmatrix} l \\ k \end{pmatrix} (2j)^{k-l} \mathbf{y}_k \mathbf{M}^{(l-k)}(\pi) \quad = \quad \mathbf{0} \tag{66}$$

B.4 Smoothness

In [2], an interesting result shows some link between the approximation power and the smoothness (number of continuous derivatives) of the scaling functions.

Theorem III.4

Assume that $\mathbf{M}$ *can be decomposed in the form*

$$\mathbf{M}(\omega) = \frac{1}{2^m} \mathbf{T}_{r_{m-1}}(2\omega) \ldots \mathbf{T}_{r_0}(2\omega) \mathbf{M}_0(\omega) \mathbf{T}_{r_0}(\omega)^{-1} \ldots \mathbf{T}_{r_{m-1}}(\omega)^{-1} \tag{67}$$

with $\mathbf{M}_0(\omega)$, $\mathbf{M}(\omega)$ *and* $\mathbf{T}_{r_k}(\omega)$ $k = 0, \ldots, m-1$ *satisfying the Plonka conditions [13]. Furthermore, suppose that the spectral radius verifies* $\sigma(\mathbf{M}_0(0) < 2$ *and that* $\inf_{k \geq 1} \gamma_k < m - d$ *where*

$$\gamma_k := \frac{1}{k} \log_2 \sup_{\omega} \|\mathbf{M}_0(2^{-1}\omega) \ldots \mathbf{M}_0(2^{-k}\omega)\| \tag{68}$$

Let $g(t)$ *be defined by*

$$\hat{g}(\omega) := \lim_{n \to \infty} \left[\prod_{i=1}^{n} \mathbf{M}(\omega / 2^i) \right] u \tag{69}$$

where u *is a right eigenvector of* $\mathbf{M}(0)$ *for the eigenvalue* 1. *Then* $g(t)$ *has compact support and is a* $d-1$ *times continuously differentiable solution of the 2-scale equation with the refinement mask* $\mathbf{M}(\omega)$. *Moreover,* g *has an approximation power of at least* m.

However, this result is not really practical to evaluate the smoothness of the scaling functions given the refinement mask $\mathbf{M}$. In the scalar case, a simple way was to use the method of invariant cycles of the mapping $\omega \to 2\omega \pmod{2\pi}$ to find upper bounds on smoothness [4]. This method is now based on the eigenvalues of the matrix products in the cycle. For example, with the matrix $\mathbf{M}$ and $\omega = 2\pi/3$, we have the invariant cycle $\{2\pi/3, 4\pi/3\}$. Computing the eigenvalues of $\mathbf{M}(4\pi/3) \cdot \mathbf{M}(2\pi/3)$ can be used to show upper bounds on the smoothness of the scaling functions by low bounding the decay of the Fourier transforms.

IV. Balanced multiwavelets

A. Introduction

An important point to remember is that a multiwavelet filter bank (often abbreviated multifilter bank) is fundamentally a MIMO (multi-input multi-output) system that requires vectorization of the input signal which is usually one-dimensional to produce an input signal which is 2-dimensional. However, due to some differences in the spectral behavior of the components of the scaling function vector, the 'lowpass' multifilter may have 'unbalanced' channels that complicate this vectorization. In that case, simple methods for the vectorization like splitting the input signal into blocks of size 2 lead to a mixing of coarse resolution and details creating strong oscillations in the reconstructed signal after compression as seen in Fig. 2. Namely, one of the important issues with wavelets in signal compression is the behavior of truncated series, i.e. robustness to truncation of the 'details' subbands. One would then expect some class of smooth signals to be well reproduced, i.e. one expect some kind of 'eigensignals' for the coarse approximation. For example, defining

$$\mathbf{L} := \begin{pmatrix} \dots & & & & & \\ \dots & M[1] & M[0] & & & \\ \dots & M[3] & M[2] & M[1] & M[0] & \\ \dots & M[5] & M[4] & M[3] & M[2] & \dots \\ & & & & & \dots \end{pmatrix} \tag{70}$$

it would be reasonable to require $[1,1,\dots,1,\dots]^\top$ to be preserved by the operator L i.e.

$$L\,[\dots,1,1,\dots,1,\dots]^\top = [\dots,1,1,\dots,1,\dots]^\top \tag{71}$$

However, most of the multiwavelets constructed so far don't even verify this simple requirement as illustrated in Fig. 1.

B. Prefiltering

A solution proposed in [16] and generalized in [20] is to add some pre/post filtering of the input/output signal to adapt it to the spectral imbalance of the filter bank.

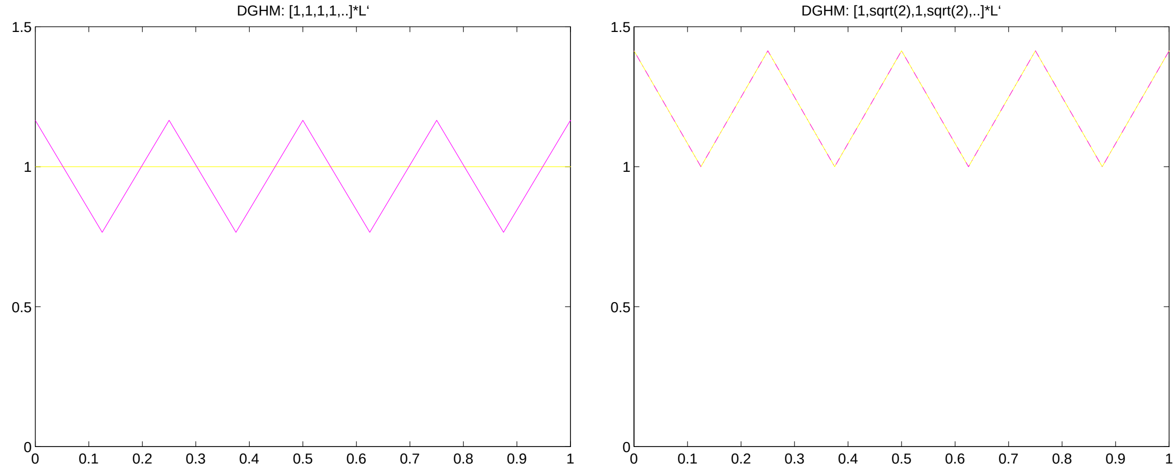

Fig. 1. Left: Reproduction of the input signal $[1, 1, \ldots, 1]$ by a GHM (Geronimo, Hardin, Massopust) multiwavelet [5] based filter bank without prefiltering, Right: reproduction of the eigensignal $[1, \sqrt{2}, 1, \sqrt{2}, \ldots, 1]$

B.1 Critical sampling

A natural way of prefiltering is to partition the input signal into size 2 vectors chunks and apply on the sequence of vectors the refinement mask $\mathbf{A}(\omega) := \sum_k \mathbf{A}[k]e^{-j\omega k}$ where $\mathbf{A}[k]$ are 2×2 matrices in order to get some adapted input sequence of vectors. In that case, we maintain critical sampling, with the only restriction that the input signal must be of size 2^K for some K. The reconstruction is easily processed applying the refinement mask $\mathbf{B}$ inverse of $\mathbf{A}$ onto the output signal. A simple way of understanding prefiltering is then to see it as a transform such that the eigensignal $[1, 1, \ldots, 1]$ is mapped into some genuine vector eigensignal associated to the eigenvalue 1 of $\mathbf{M}$, for example, in the GHM case we have

$$L\,[\ldots, 1, \sqrt{2}, 1, \sqrt{2}, \ldots, 1, \ldots]^\top = [\ldots, 1, \sqrt{2}, 1, \sqrt{2}, \ldots, 1, \ldots]^\top \tag{72}$$

The results obtained (Fig. 2) using this 'trick' are of the same order as the ones obtained using a plain Daubechies filter bank with 4 taps. However, the new system constructed that way is no more orthogonal.

B.2 Non-critical sampling

Another way of doing pre/post filtering is to allow non critical sampling and to construct some projection of the input signal on V_0. A simple way of doing so in the case of the

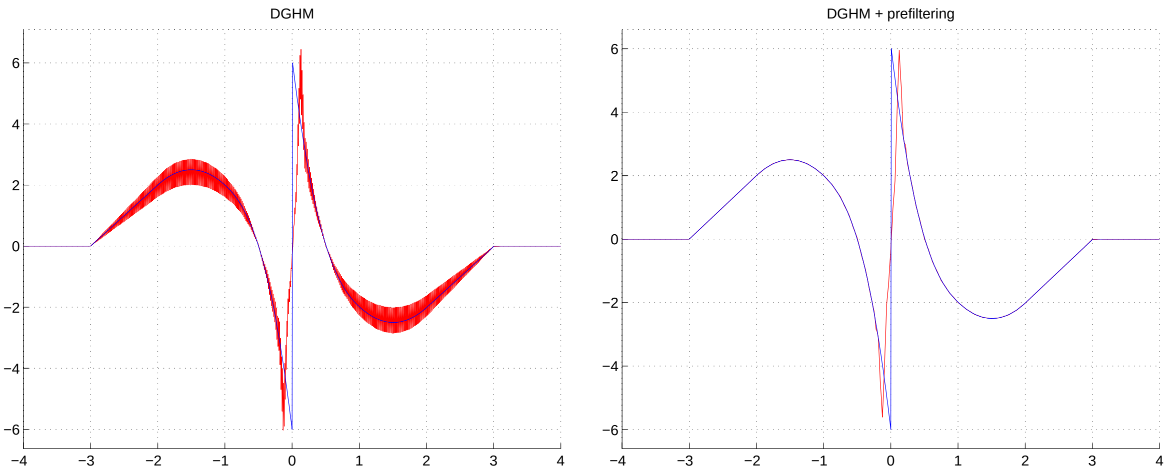

Fig. 2. Truncation of the expansion associated to the details in a level 1 filter bank based on GHM multiwavelet. It shows on the left the poor robustness of systems based on GHM without pre/post filtering. The results are greatly enhanced with some pre/post filtering as seen on the graph on the right.

GHM multiwavelet is starting from an input signal $[x[0], \ldots, x[2^K]]$ to transform it into some vector-valued input sequence

$$[\begin{pmatrix} x[0] \\ \sqrt{2}x[0] \end{pmatrix}, \begin{pmatrix} x[1] \\ \sqrt{2}x[1] \end{pmatrix}, \ldots, \begin{pmatrix} x[2^K] \\ \sqrt{2}x[2^K] \end{pmatrix}] \tag{73}$$

This preprocessing is often called the 'repeated signal' approach. It doubles the size of the input signal, but allows to maintain the orthogonality of the system. However, by the redundancy it creates, one can't use this approach in the case of signal compression.

As mentioned in [20], [21], an issue is then to maintain orthogonality and critical sampling at the same time in the case of prefiltering. Thus, one may rather directly design orthogonal multiwavelets with good balance between the two scaling functions.

C. Balancing

In [18], [2], a necessary condition for the balancing of the scaling functions has been given: in the case $r = 2$, we need $[1, 1]^\top$ to be a right eigenvector associated to the eigenvalue 1 of $\mathbf{M}(0)$. This is easily understood by looking closely at (71). Furthermore, this implies that $\hat{\phi}(0) = [1, 1]^\top$ i.e. ϕ_0, ϕ_1 are bona-fide lowpass scaling functions, and so the approximation rule on which the Mallat algorithm [4] is based apply:

$$\int x(t)\phi_i(t-n)\,dt \approx x(n) \tag{74}$$

C.1 Direct construction

A simple way to construct balanced multiwavelets of arbitrary order is to derive them from the complex Daubechies filters. Daubechies filters are constructed using the halfband filter:

$$P(z) := c(1+z^{-1})^N(1+z)^N R(z) \tag{75}$$

such that $P(z)+P(-z)=1$ with $R(e^{j\omega}) \geq 0$ and $R(e^{j\omega}) = R(e^{-j\omega})$. One gets the usual Daubechies lowpass filters: $D_N(z) := (1+z^{-1})^N B(z)$ where $B(z)$ is a spectral factor of $R(z)$ with real coefficients. We can't achieve orthogonality and symmetry with real coefficients, however by allowing complex coefficients in the spectral factorization, one can construct symmetric, orthogonal FIR filters [9]. Writing $[a[0],\ldots,a[N],a[N],\ldots,a[0]]$ for the lowpass filter, we construct the matrix coefficients:

$$A[i] := \begin{pmatrix} -Im(a[i]) & Re(a[i]) \\ Re(a[i]) & Im(a[i]) \end{pmatrix} \tag{76}$$

and the refinement mask is then with $z = e^{j\omega}$

$$\mathbf{M}(\omega) := \frac{1}{2}\left(\sum_{i=0}^{N} A[i]z^{-i} + z^{-(N+1)}\sum_{i=0}^{N} A[N-i]z^{-i}\right) \tag{77}$$

The multifilter bank is clearly orthogonal and it is easily seen that the smoothness and approximation power of the Daubechies complex scaling functions and wavelets transfer to the multiscaling functions and multiwavelets. Namely, by defining

$$\varphi(x) := \phi_1(x) + j\phi_0(x) \tag{78}$$

where $[\phi_0,\phi_1]$ is the multiscaling function associated to $\mathbf{M}(\omega)$, we get that φ verifies the 2-scale equation

$$\varphi(x) = \sum_{k=0}^{N} a[k]\varphi(2x-k) + \sum_{k=N+1}^{2N+1} a[2N+1-k]\varphi(2x-k) \tag{79}$$

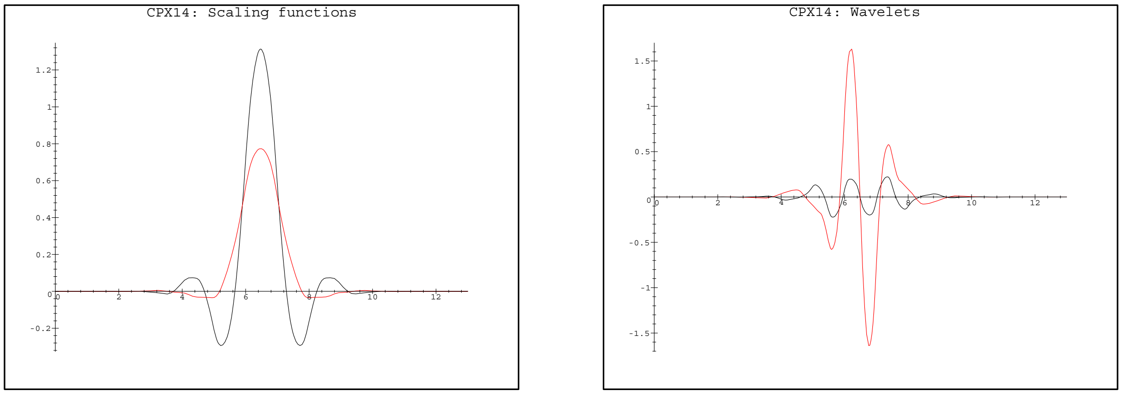

Fig. 3. Highly regular Daubechies based Multiwavelets (same approximation power and smoothness as D14). Left: scaling functions, Right: multiwavelets.

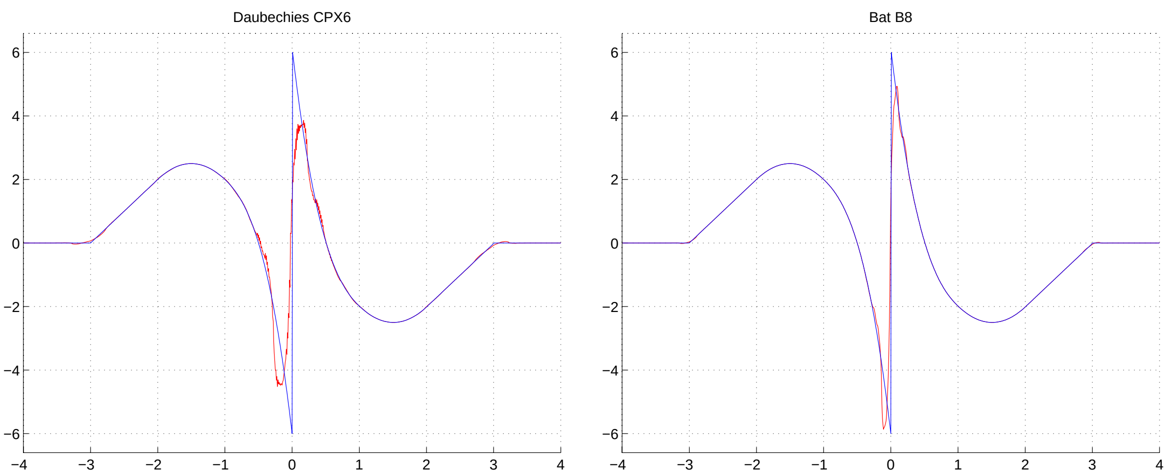

Fig. 4. Robustness to truncation of the first order details subband of a 6 2x2 taps complex Daubechies based multiwavelet filter bank and the Chui based balanced multiwavelet with 8 2x2 taps filter bank.

so φ is the scaling function associated to the complex Daubechies filters, hence we get the same smoothness and approximation power for the multiscaling functions and the multiwavelets. Using proposition III.3, we also easily derive that the multiscaling functions and multiwavelets are symmetric/antisymmetric as seen in Fig.3. However, this refinement mask when iterated doesn't converge properly because $\mathbf{M}(0)$ has eigenvalues $1, -1$ with eigenvectors $[1,1]^\top, [1,-1]^\top$. Then, we get only constrained convergence as defined in theorem III.1, hence the poor behavior of this multiwavelet in applications as seen in Fig 4.

C.2 Balancing the non-balanced

Another interesting way of constructing balanced multiwavelets is to balance already existing multiwavelets like the ones constructed in [1] or [5]. The point is that we want $[1,1]^\top$ to be a right eigenvector associated with eigenvalue 1 of $\mathbf{M}(0)$. The way to achieve this is to use the unitary matrix R such that

$$R^\top \mathbf{M}(0)\, R \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{80}$$

Defining the new refinement mask

$$\mathbf{P}(\omega) := R^\top \mathbf{M}(\omega)\, R \tag{81}$$

and the new 2-scale equation

$$\hat{\varphi}(2\omega) = \mathbf{P}(\omega)\hat{\varphi}(\omega) \tag{82}$$

we then verify that

$$\hat{\varphi}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{83}$$

We notice that in the iteration, $R^\top$ and R cancel, except for the first and last term. The convergence of the matrix product for $\mathbf{M}$ imply the convergence for $\mathbf{P}$ and the smoothness and approximation power are therefore unchanged. However the symmetry of the scaling functions is usually lost. Nevertheless, the symmetry/antisymmetry of the multiwavelets can be maintained, by taking for the highpass refinement mask

$$\mathbf{Q}(\omega) := \mathbf{N}(\omega)\, R \tag{84}$$

where $\mathbf{N}(\omega)$ is the highpass refinement mask associated to $\mathbf{M}(\omega)$. Namely

$$(\mathbf{N}(\omega)\, R \prod_{i=1}^{\infty} (R^\top \mathbf{M}(\omega/2^i) R)) \begin{pmatrix} 1 \\ 1 \end{pmatrix}) = (\mathbf{N}(\omega) \prod_{i=1}^{\infty} \mathbf{M}(\omega/2^i))u = \begin{pmatrix} \hat{\varphi}_0(\omega) \\ \hat{\varphi}_1(\omega) \end{pmatrix} \tag{85}$$

Using Chui multiwavelets [1], we obtained orthogonal, compactly supported multiscaling functions / multiwavelets with symmetry and good approximation for the multiwavelets and also verifying the $[1,1]^\top$ right eigenvector condition (Fig. 5). These balanced multiwavelets (Bat) have shown very good robustness in compression algorithm without any pre/post filtering (Fig. 4).

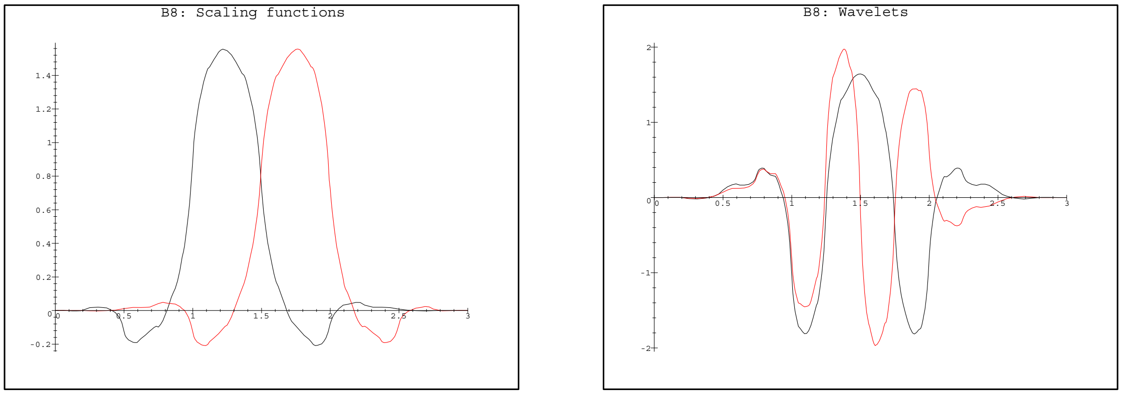

Fig. 5. Balanced multiwavelet with 2^{nd} order of approximation. Left: scaling functions, Right: multiwavelets.

C.3 Higher order balancing

One can generalize what was previously done for balancing non-balanced multiwavelets to higher order polynomial input signals. Namely, in the case of GHM, we have approximation power of order 2

$$1 = \sum_k (\sqrt{2}\phi_0(t-k) + \phi_1(t-k) \tag{86}$$

$$t = \sum_k (\sqrt{2}(k+\frac{1}{2})\phi_0(t-k) + (k+1)\phi_1(t-k)) \tag{87}$$

So if we want to preserve the sampled version of 1 and t as input signals, we should transform them into some eigensignals of the GHM based filter bank. So we get the equations

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \quad \text{and} \quad \mathrm{A}\begin{pmatrix} n \\ n+\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \sqrt{2}(n+\frac{1}{2}) \\ n+1 \end{pmatrix} \tag{88}$$

We then get

$$A = \begin{pmatrix} 0 & \sqrt{2} \\ -1 & 2 \end{pmatrix} \tag{89}$$

Defining the new refinement mask

$$\mathbf{P}(\omega) := A^{-1}\,\mathbf{M}(\omega)\,A \tag{90}$$

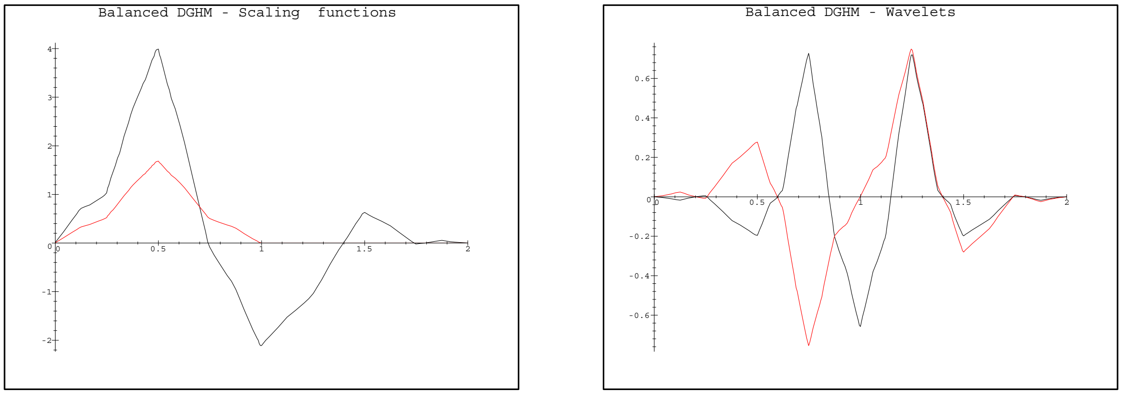

Fig. 6. Balanced GHM multiwavelets (order 2), Left: scaling functions, Right: multiwavelets.

and the new 2-scale equation

$$\hat{\varphi}(2\omega) = \mathbf{P}(\omega)\hat{\varphi}(\omega) \tag{91}$$

the multifilter bank based on this refinement mask keeps unchanged constant and linear input signal. Once more, the convergence of the matrix product for $\mathbf{M}$ imply the convergence for $\mathbf{P}$ and the smoothness and approximation power are therefore unchanged. However, this time not only the symmetry but also the orthogonality of the scaling functions are lost. Nevertheless, the symmetry/antisymmetry of the multiwavelets can again be maintained (Fig. 6), by taking for the highpass refinement mask

$$\mathbf{Q}(\omega) := \mathbf{N}(\omega)\, A \tag{92}$$

where $\mathbf{N}(\omega)$ is the highpass refinement mask associated to $\mathbf{M}(\omega)$.

A more general issue is then to describe some general design method for constructing bona-fide multiwavelets with all the desired properties. Recently Plonka and Strela proposed in [13], [15] a method to increase the approximation order of a given scaling function by what they called the 2-scale similarity transform. This transform applied to the refinement mask $\mathbf{M}(\omega)$ determines a new scaling function with higher approximation order. This last one is derived from the new refinement mask $\mathbf{M}_T(\omega)$ given by

$$\mathbf{M}_T(\omega) := \mathbf{T}(2\omega)\mathbf{M}(\omega)\mathbf{T}^{-1}(\omega) \tag{93}$$

where $\mathbf{T}(\omega)$ is the transform matrix. Although this method showed some good results, as mentioned in [15], it is not clear how to maintain orthogonality and compact support at

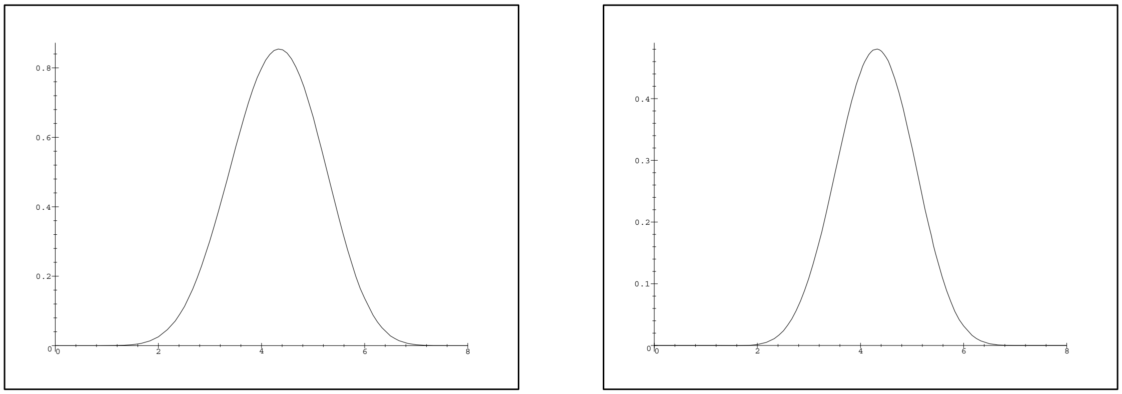

Fig. 7. Highly regular scaling functions obtained from the Bat multiwavelet (support is $[0, 8]$).

the same time. Moreover, this transform is not preserving the eigenvectors or even the eigenvalues of $\mathbf{M}(0)$. So, we made some modification of this method by defining the new refinement mask $\mathbf{P}(\omega)$ as

$$\mathbf{P}(\omega) := \mathbf{T}(\omega)\mathbf{M}(\omega)\mathbf{T}(\omega) \tag{94}$$

where now the transform matrices

$$\mathbf{T}(\omega) := \mathbf{T}_1^{-1}(\omega)\mathbf{T}_2(2\omega) \tag{95}$$

verifies some weaker conditions than the ones required in [13]. This enables greater freedom in the design of the new refinement mask and allows especially to maintain the $[1, 1]^\top$ eigenvector associated to the 1 eigenvalue condition on $\mathbf{P}(0)$. As seen in Fig. 7, we constructed this way some highly regular biorthogonal balanced multiwavelets with compact support and symmetry starting from Chui's 1^{st} order balanced multiwavelet and using for example

$$\mathbf{T}_1(\omega) = \mathbf{T}_2(\omega) = \begin{pmatrix} (1-z)^2 & -z(1-z) \\ (1-z)^2 & (1-z)^2 \end{pmatrix} \tag{96}$$

where $z = e^{-j\omega}$. Nevertheless, the issue of maintaining the orthogonality during this process remains open.

V. Conclusion

After recalling some basic facts about multiwavelets, we introduced some of the problems we face applying multiwavelets in signal processing. We gave a new way to solve these problems: the balanced multiwavelets. However, some questions remain open. We still have to develop some systematic and simple way to construct orthogonal balanced multiwavelets with the desired approximation power. An important issue is also the preservation of higher order polynomial signals by orthogonal multiwavelet based system. This will certainly bring some further developments and applications in the fields of one dimensional signal processing.

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