

CONVEXITY OF RESISTIVE CIRCUIT CHARACTERISTICS

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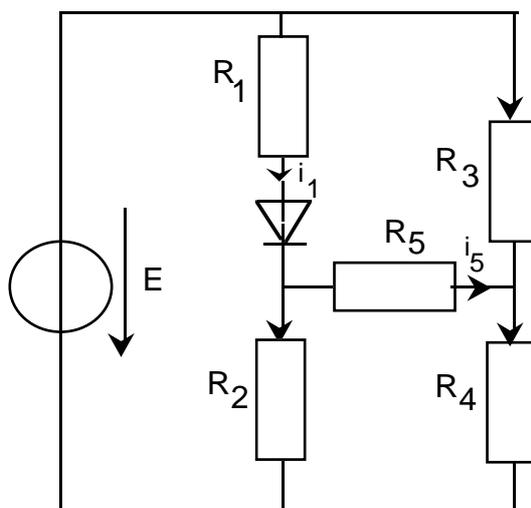
Abstract

We give topological criteria for the convexity and the concavity of a current or a voltage in a resistive circuit, as a function of a source voltage or current, when the nonlinear resistor characteristics are all either convex or concave. When the criteria are satisfied, all circuits with the same structure will have a convex, or all will have a concave transfer characteristic. The application of the criterion to ladder circuits leads to explicit and easily verifiable conditions.

1. INTRODUCTION

When resistive circuit elements such as sources, linear resistors, diodes, transistors, operational amplifiers, etc. are connected, various types of transfer characteristics may be obtained. By transfer characteristic, we mean the following: A voltage or current source of value α is connected to the circuit and the voltage or current of branch k in the circuit is observed. The functions $v_k(\alpha)$ and $i_k(\alpha)$ are transfer characteristics.

Clearly, when the circuit consists of only linear resistive elements (sources, linear resistors, operational amplifiers, linear controlled sources), all transfer characteristics are linear, or, to be precise, affine functions. Properties other than linearity do not carry over so easily from the single circuit elements to the transfer characteristics. E.g., a circuit composed of elements with monotonic characteristics does not necessarily have monotonic transfer characteristics. A simple example is given in Fig.1.



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Fig.1. Circuit with monotonic elements (from [1])

With standard diode model parameters and $R_1=R_2=R_3=2R_4=R_5= 1\Omega$, the transfer characteristic $i_5(E)$ obtained by simulation is represented in Fig.2 [1]. It is clearly nonmonotonic.

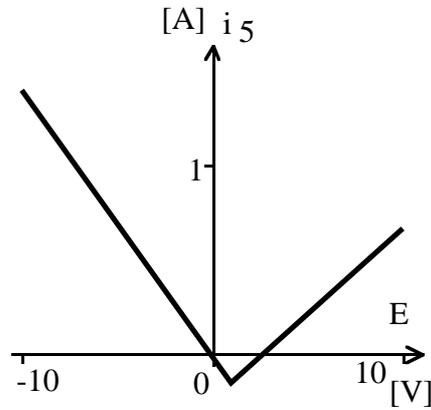


Fig.2. Nonmonotonic transfer characteristic (from [1])

Similarly, if the elements of a resistive circuit have convex or concave characteristics, the transfer characteristics of the circuit in general are neither convex nor concave. A simple example is given in Fig.3. The transfer characteristic $i(E)$ is neither convex nor concave (Fig.4).

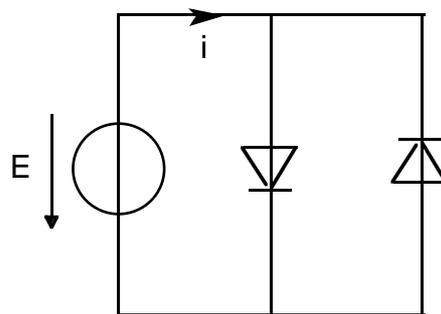


Fig.3. Circuit with convex/concave elements

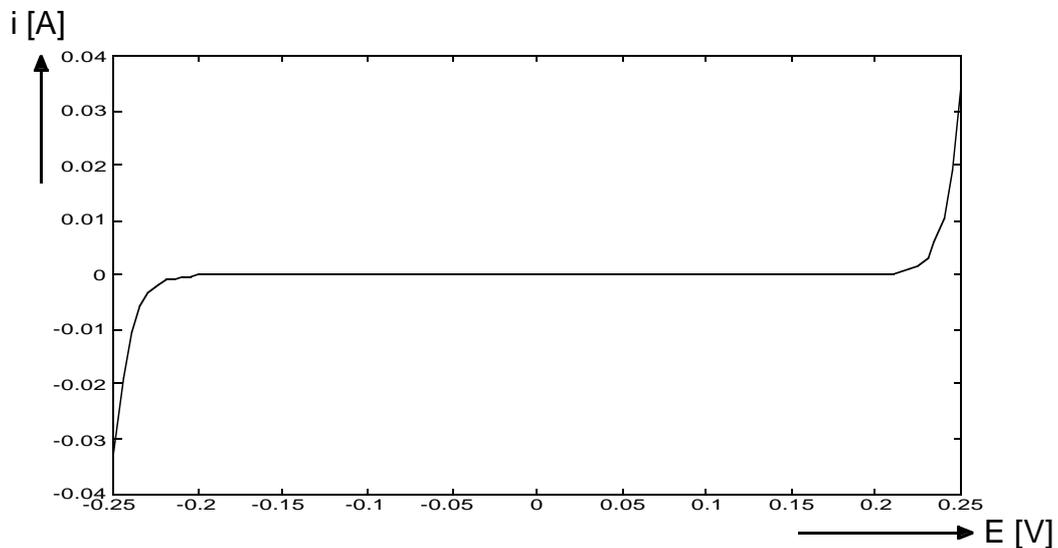


Fig.4.Non concave/convex transfer characteristic $i(E)$

The purpose of this paper is to give conditions which assure convexity/concavity of transfer characteristics.

To cite again the example of monotonicity, when R_4 in the circuit of Figure 1 is changed from 1Ω to 2Ω , the transfer characteristic $i_5(E)$ becomes monotonic (Fig.5).

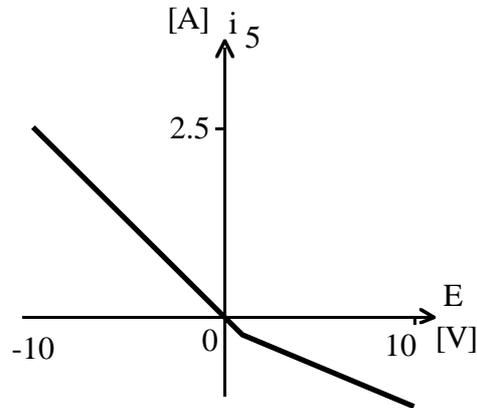


Fig.5. Monotonic transfer characteristic

Thus, the monotonicity of a transfer characteristic may depend on the values of the circuit elements. This is the case of $i_5(E)$ in the circuit of Fig.1. On the other hand, it is possible to show that $i_1(E)$ is always a strictly increasing function of E , irrespective of the particular values of R_1, \dots, R_5 , as long as they are positive, and irrespective of the particular form of the diode characteristic, as long as it is strictly increasing [1].

Similarly, the convexity/concavity of a transfer characteristic may depend on the values of the circuit elements. E.g. if in the circuit of Fig.3 the diodes are replaced by resistors with characteristic $i = Av^4 + Bv^2$ and $i = -Cv^2$, respectively, the transfer characteristic $i(E)$ is convex for $B > C$, and neither convex nor concave for $B < C$, even though both resistor characteristics have a definite sign of the second derivative.

The purpose of this paper is to give the conditions which allow to assure that a given transfer characteristic is always convex, or always concave, **irrespective of the particular values and characteristics of the circuit elements**, as long as the signs of the first and the second derivative of their characteristics are not changed.

2. SECOND DERIVATIVES

We suppose that a resistive circuit, composed of nonlinear and positive linear resistors, voltage and current sources, and an equal number of nullators (constitutive relations $u = 0, i = 0$) and norators (no constitutive relations) is given. The nonlinear resistors are supposed to be either *V-resistors*, i.e. given by a constitutive relation $i = g(v)$, where g is a strictly increasing function defined for all real values of v , or *I-resistors*, i.e. given by a constitutive relation $v = h(i)$, where h is a strictly increasing function defined for all real values of i . One among the sources is varied and the current or the voltage of a resistor is considered as a function of the

source value. E.g. we vary the value α of a voltage source on branch s and we consider the current of branch r (Fig.6).

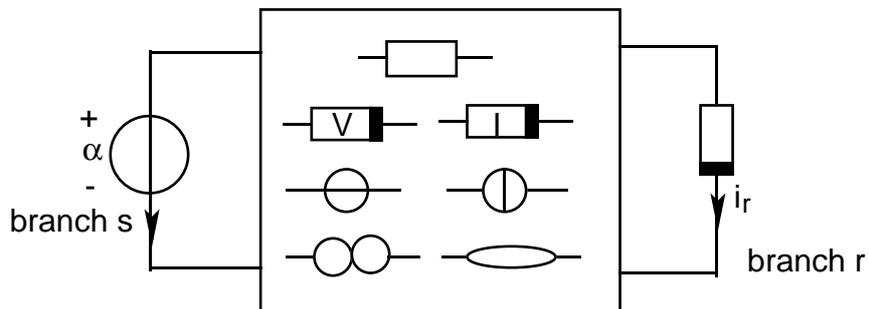


Fig.6. Class of circuits considered

Suppose that all circuits with the same *structure*, i.e. all circuits that have the same graph and the same type of element (linear resistor, I-resistor, V-resistor, voltage source, current source, nullator, norator) on the same branch of the graph have exactly one solution. By solution, we mean a vector of b branch currents and b branch voltages

$$\mathbf{i} = \begin{pmatrix} i_1 \\ M \\ i_b \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} v_1 \\ M \\ v_b \end{pmatrix} \quad (1)$$

that satisfy both Kirchoff's laws and the constitutive relations of the elements. The unique solution property implies in particular that the solution of the circuit as a function of α is well-defined. We shall give in the next section topological criteria which assure the validity of these hypotheses.

Suppose, furthermore, that the constitutive relations of the nonlinear resistors are twice differentiable. Then the solution is also twice differentiable in α [2]. If we differentiate the circuit equations

$$\left. \begin{aligned} \mathbf{A}\mathbf{i}(\alpha) &\equiv 0 \\ \mathbf{B}\mathbf{v}(\alpha) &\equiv 0 \end{aligned} \right\} \text{Kirchoff's equations}$$

$$v_s(\alpha) = \alpha$$

$$i_k(\alpha) = g(v_k(\alpha)) \text{ for branches } k \text{ carrying V-resistors}$$

$$v_k(\alpha) = h(i_k(\alpha)) \text{ for branches } k \text{ carrying I-resistors} \quad (2)$$

$$i_k(\alpha) \equiv I_k \text{ (const.) for branches } k \text{ carrying current sources}$$

$$v_k(\alpha) \equiv E_k \text{ (const.) for branches } k \neq s \text{ carrying voltage sources}$$

$$v_k(\alpha) \equiv i_k(\alpha) \equiv 0 \text{ for branches } k \text{ carrying nullators}$$

twice, we obtain

$$\begin{aligned}
 \mathbf{A} \frac{d^2 \mathbf{i}}{d\alpha^2} &\equiv 0 \\
 \mathbf{B} \frac{d^2 \mathbf{v}}{d\alpha^2} &\equiv 0 \\
 \frac{d^2 v_s}{d\alpha^2} &\equiv 0 \\
 \frac{d^2 i_k}{d\alpha^2} &= g''(v_k) \left(\frac{dv_k}{d\alpha} \right)^2 + g'(v_k) \frac{d^2 v_k}{d\alpha^2} \quad \text{for V-resistors} \\
 \frac{d^2 v_k}{d\alpha^2} &= h''(i_k) \left(\frac{di_k}{d\alpha} \right)^2 + h'(i_k) \frac{d^2 i_k}{d\alpha^2} \quad \text{for I-resistors} \\
 \frac{d^2 v_k}{d\alpha^2} &\equiv 0 \quad \text{for voltage sources (} k \neq s \text{) and nullators} \\
 \frac{d^2 i_k}{d\alpha^2} &\equiv 0 \quad \text{for current sources and nullators}
 \end{aligned} \tag{3}$$

If we consider $d^2 i_k / d\alpha^2$ and $d^2 v_k / d\alpha^2$ in (3) as unknown currents and voltages, respectively, but $i_k(\alpha)$, $v_k(\alpha)$, $di_k/d\alpha$ and $dv_k/d\alpha$, as given, the system (3) describes a linear circuit. The branches that originally carried V-resistors and I-resistors become the composite branches of Figs. 7 and 8, respectively.

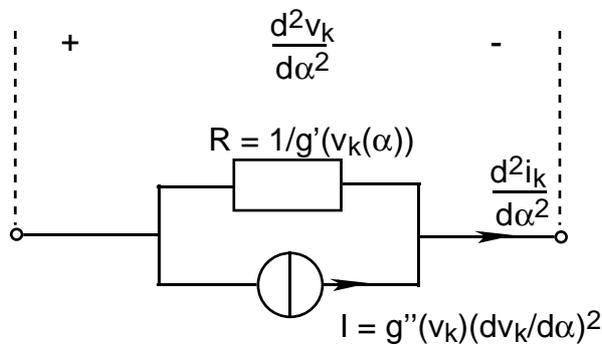


Fig.7. Composite branch of the circuit for the second derivative, corresponding to a V-resistor in the original circuit.

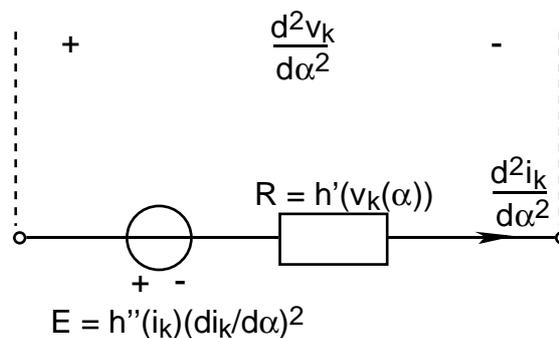


Fig.8. Composite branch of the circuit for the second derivative, corresponding to an I-resistor in the original circuit.

Remark 1:

- a) Note that linear resistors can be considered as V- and I-resistors and they remain unchanged when passing to the circuit for the second derivative, since g'' and h'' are both 0.
- b) The linear resistors in Figs. 7 and 8 are positive, since the functions g and h are supposed to be strictly increasing. Thus, the two composite branches of Figs. 7 and 8 are equivalent. Nevertheless, we shall use the composite branch of Fig.7 for V-resistors and the composite branch of Fig.8 for I-resistors.
- c) The second derivative of the current through the voltage controlled nonlinear resistor is not the "current" through the linear resistor in Fig.7, but the total "current" of the composite branch. Similarly, the second derivative of the voltage across the current controlled nonlinear resistor is not the voltage across the linear resistor in Fig.8, but the total voltage across the composite branch.

Definition 1

Suppose a resistive circuit is composed of linear resistors, V-resistors, I-resistors, voltage- and current sources, nullators and norators. Suppose that one of the voltage or current sources is varied. Its value is denoted by α . Suppose that the solution of the circuit is a well-defined, twice differentiable function of α . Then the *associated D_2 -circuit* is the linear resistive circuit with the same graph, obtained by the following substitution of elements:

- linear resistor --> linear resistor of the same value.
- V-resistor defined by $i = g(v)$ --> composite branch with element values depending on α as indicated in Fig.7
- I-resistor defined by $v = h(i)$ --> composite branch with element values depending on α as indicated in Fig.8
- voltage source --> short circuit
- current source --> open circuit
- nullator --> nullator
- norator --> norator

The following theorem is a consequence of the derivation of (3) from (2).

Theorem 1

Let a resistive circuit satisfy the same hypotheses as in definition 1. Then the second derivative of its solution with respect to α is a solution of the associated D_2 -circuit.

Examples:

The D_2 -circuit associated with the circuit of Fig.1 is represented in Fig.9.

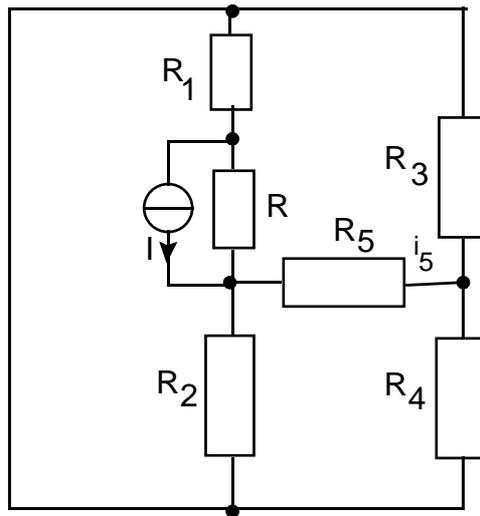


Fig.9. D₂-circuit associated with the circuit of Fig.1

If the constitutive relation for the diode is

$$i = I_s \left(e^{\frac{v}{nV_T}} - 1 \right) \quad (4)$$

where I_s , n and V_T are constants and the voltage of the diode in Fig.1 as a function of the source voltage E is $v(E)$, then the values of R and I are given by

$$R = \frac{nV_T}{I_s} e^{-\frac{v(E)}{nV_T}} \quad (5)$$

$$I = \frac{I_s}{n^2 V_T^2} e^{\frac{v(E)}{nV_T}} \left(\frac{dv}{dE} \right)^2$$

The D₂-circuit associated with the circuit of Fig.3 is represented in Fig.10

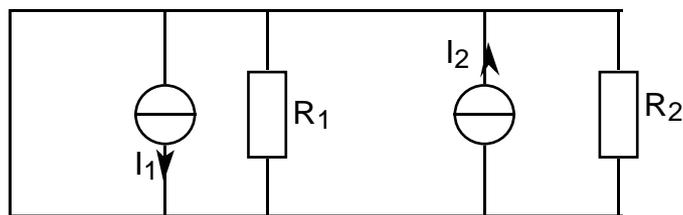


Fig.10. D₂-circuit associated with the circuit of Fig.3

From the constitutive relation (4) for the diodes, one obtains, since the voltage on the diodes is directly E :

$$\begin{aligned}
 R_1 &= \frac{nV_T}{I_s} e^{-\frac{E}{nV_T}} \\
 I_1 &= \frac{I_s}{n^2 V_T^2} e^{\frac{E}{nV_T}} \\
 R_2 &= \frac{nV_T}{I_s} e^{\frac{E}{nV_T}} \\
 I_2 &= \frac{I_s}{n^2 V_T^2} e^{-\frac{E}{nV_T}}
 \end{aligned} \tag{6}$$

3. TOPOLOGICAL CONDITIONS FOR CONVEX/CONCAVE TRANSFER CHARACTERISTICS

We start by giving the topological conditions for the basic assumptions about the nonlinear resistive circuit. A few definitions are needed. They are slightly extended versions of definitions in [1,2].

Definition 2

A *resistive circuit structure* S is a graph whose branches are labeled with one of the seven element types V-resistor, I-resistor, linear resistor, voltage source, current source, nullator, norator. If there are no V- and I-resistor branches, S is a *linear resistive circuit structure*.

A circuit C has structure S , if it has the same graph as S and the same element type on each branch.

Definition 3

Two trees t_1 and t_2 of the circuit graph are *conjugate* if the following conditions hold:

- a) t_1 contains all norators, all voltage sources and some resistors, but no current sources and no nullators.
- b) t_2 contains all nullators, all voltage sources and the same resistors as t_1 , but no current sources and no norators.

Definition 4

A *partial orientation* of the resistor branches in a resistive circuit structure is *uniform*, if the following conditions hold:

- a) Every oriented resistor is part of a uniform loop composed exclusively of oriented resistors, norators and voltage sources.
- b) Every oriented resistor is part of a uniform cutset composed exclusively of oriented resistors, norators and current sources.

A loop (cutset) is *uniform* if all oriented branches are evenly directed with respect to it. A *partial orientation* of a graph is *trivial*, if no branch is oriented.

Theorem 2 [1, 3]

Let S be a resistive circuit structure. All circuits with structure S have exactly one solution, if and only if

- S has a pair of conjugate trees such that all I-resistors are on the tree and all V-resistors are on the cotree, and
- S has no nontrivial partial uniform orientation of the resistors

Furthermore, if the nonlinear resistors are defined by constitutive relations $i = g(v)$ and $v = h(i)$ with v times continuously differentiable functions g and h , the solution, as a function of any source value, is v times continuously differentiable.

This theorem gives the criteria which assure that the second derivative of the solution of a nonlinear resistive circuit with respect to a source value is well defined. The circuit has to possess a pair of conjugate trees, no nontrivial uniform partial orientation of the resistors and nonlinear resistors whose constitutive relations are at least twice differentiable. From theorem 1 we know that the second derivatives constitute a solution of the associated D_2 -circuit. Is this solution unique?

Definition 5

Let S be a resistive circuit structure. The *associated D_2 -structure S_2* is the linear resistive circuit structure obtained by carrying the following substitutions in S out:

- linear resistor branch --> linear resistor branch.
- V-resistor branch --> parallel connection of a current source branch and a linear resistor branch
- I-resistor branch --> series connection of a voltage source branch and a linear resistor branch
- voltage source branch --> short circuit
- current source branch --> open circuit
- nullator branch --> nullator branch
- norator branch --> norator branch

If C is a circuit with structure S , we say that C has the D_2 -structure S_2 .

Corollary 1

Let S be a resistive circuit structure and S_2 the associated D_2 -structure. If all resistive circuits with structure S have exactly one solution, then all linear resistive circuits with structure S_2 have exactly one solution.

Proof:

It is easy to construct from a pair of conjugate trees in S , as required by part b) of theorem 2 a pair of conjugate trees in S_2 . Similarly, a uniform partial orientation of the resistor branches in S_2 leads to a uniform partial orientation of the resistor branches in S .

Consider now a resistive circuit with nonlinear resistor characteristics that are either convex or concave. Since the inverse of a convex function is concave and vice versa, the distinction between convex and concave is understood relative to a constitutive relation $i = g(v)$ or $v = h(i)$. We say that a V-resistor has a convex (concave) characteristic if g is a convex (concave) function, i.e. if $g''(v) \geq 0$ ($g''(v) \leq 0$). Similarly, an I-resistor has a convex (concave) characteristic if h is a convex (concave) function, i.e. if $h''(v) \geq 0$ ($h''(v) \leq 0$).

Note that the distinction between convexity and concavity of the characteristic of a nonlinear resistor not only depends on whether we consider it to be a V-resistor or an I-resistor, but also on the reference direction of its branch. Thus, the reference direction can always be chosen such that the resistor has a convex characteristic when considered as a V-resistor, or when considered as an I-resistor. If this reference direction is carried over to the corresponding source in the associated D_2 -circuit, the source value is always positive.

Consider a source branch s and a resistor branch r . When is the transfer characteristic from the source value on s to the current or voltage of r convex or concave? We shall be able to answer this question using again the associated D_2 -circuit. Again, the distinction between convex and concave depends on the reference direction of branch r , but not on the reference direction of branch s .

Definition 6

An *oriented D_2 -structure* S_2 is a D_2 -structure with oriented source branches. In order to distinguish these orientations from other orientations we are using, we call them the *reference orientations* of S_2 .

A resistive circuit C has the oriented D_2 -structure S_2 if it has the (non-oriented) D_2 -structure S_2 (without orientations) and

- its nonlinear resistor characteristics are twice continuously differentiable and either convex or concave, and
- its nonlinear resistor characteristics are convex, if the reference orientations of S_2 are used as reference directions on the corresponding nonlinear resistors of C .

Note that the associated D_2 -structure S_2 of a resistive circuit structure S with N nonlinear V- and I-resistor branches can be provided with in 2^N different reference orientations. A given resistive circuit C with convex/concave element characteristics has exactly one among them as oriented D_2 -structure. In Figs.11 and 12 the resistive circuit structures and the oriented D_2 -structures of the circuits of Fig. 1 and 3 are shown. The orientations of the current sources result from the fact that the diode has a convex characteristic when its reference orientation is taken in the direction of conduction.

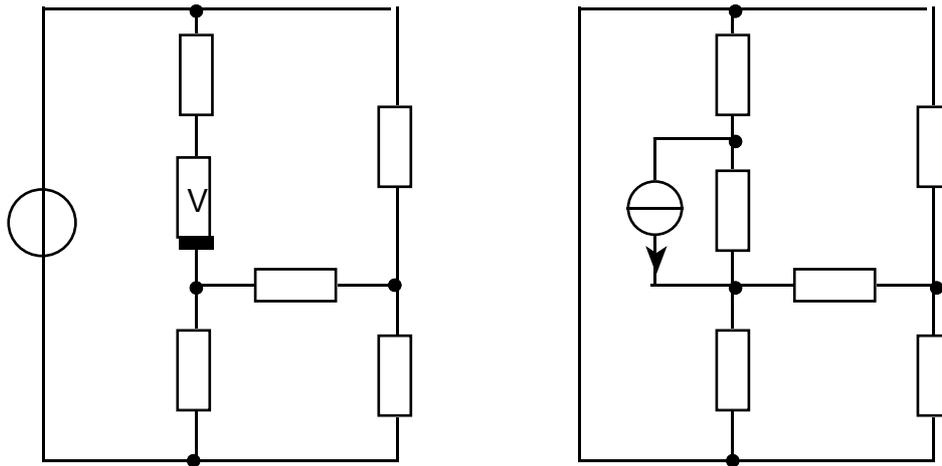


Fig.11. Resistive circuit structure (left) and oriented D_2 -structure (right) of the circuit of Fig.1.

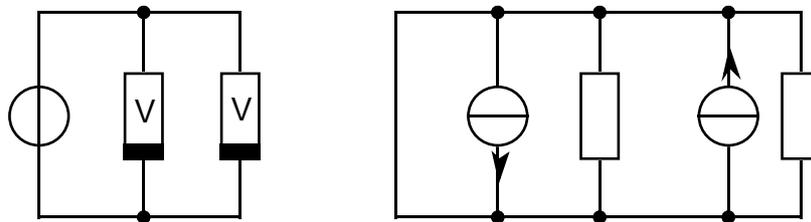


Fig.12. Resistive circuit structure (left) and oriented D_2 -structure (right) of the circuit of Fig.3.

The question of whether or not the second derivative of the transfer characteristic from a source on branch s to the voltage or current of a resistor on branch r has always the same sign, be it positive or negative, depends on whether or not the solution of the D_2 -circuit has always the same sign on branch r . Since the D_2 -circuit is linear, its solution is obtained by superposition of the solutions generated separately by the different sources that are associated with the V- and I-resistors. It follows that all the partial solutions generated by a single source also have to have always the same sign; separately and with respect to each other. When a circuit contains many nonlinear resistors, this is quite a restrictive condition.

When the resistor on branch r is nonlinear, it is replaced by a composite branch according to Fig.7 or 8 in the associated D_2 -circuit. According to remark 1, the sign of the current and voltage of the resistor in the composite branch expresses the convexity or concavity of the transfer characteristic to either the current or the voltage of branch r in the original circuit, but not both. The source on branch s on which the solution of the nonlinear circuit depends, is short-circuited in the D_2 -circuit, like any other source. Thus, convexity/concavity of a transfer characteristic from branch s to branch r does depend on the choice of r , but not of s .

The question whether the solution of a linear circuit with a single source always has the same sign in a given branch r is a question of monotonic dependence of r on the source. Indeed, if the source is set to zero, the solution is zero. If, say, the current i_r is a strictly increasing function of the source value, then for a positive source value, i_r is always positive, irrespective of the other element values. Thus, the criteria for convexity/concavity can be deduced from the monotonicity criteria of [1] which we recall here, adapted to our needs.

First, we have to extend definition 4 to the case where also sources can be oriented. In the application an oriented source has a variable value.

Definition 7

A *partial orientation* of the resistor **and source branches** in a resistive circuit structure is *uniform*, if

- Every oriented resistor and every oriented current source is part of a loop composed exclusively of oriented resistors, oriented current sources, norators and voltage sources such that all oriented resistors and current sources (but not necessarily the oriented voltage sources) within the loop are evenly directed, and
- Every oriented resistor and every oriented voltage source is part of a cutset composed exclusively of oriented resistors, oriented voltage sources, norators and current sources such that all oriented resistors and voltage sources (but not necessarily the oriented current sources) within the cutset are evenly directed.

Proposition 1 [1]:

A partial orientation of the resistor and source branches in a resistive circuit structure is uniform if and only if

- There are no evenly directed loops composed exclusively of oriented resistors, voltage sources, oriented or not, and nullators, containing at least one oriented branch.
- There are no evenly directed cutsets composed exclusively of oriented resistors, current sources, oriented or not, and nullators, containing at least one oriented branch.

Theorem 3 [1]

Let S_L be a linear resistive circuit structure. Suppose s is a source and r a resistor branch. Introduce arbitrary reference directions on s and r . Consider uniform partial orientations of the resistor and source branches such that

- the source on s is oriented as its reference direction, and
- no other source is oriented

Call these orientations *admissible*.

Then all circuits with structure S_L and a source with positive value on branch s (with respect to the reference direction on s) have

- a) a nonnegative voltage and current on r if and only if on all admissible orientations on S_L branch r is oriented as its reference direction or not oriented .
- b) a nonpositive voltage and current on r if and only if on all admissible orientations on S_L branch r is oriented against its reference direction or not oriented.

Definition 8

Let S be a nonlinear resistive circuit structure and S_2 an associated oriented D_2 -structure. A uniform partial orientation of the resistors and the sources in S_2 is *admissible* if

- Exactly one source is oriented, and
- this source is oriented according to its reference orientation in S_2 .

The different definitions and theorems can now be combined to formulate the main theorem of this paper, the conditions for convex/concave transfer characteristics.

Theorem 4

Let S be a resistive circuit structure and S_2 an associated oriented D_2 -structure. Suppose that

- S has a pair of conjugate trees
- S has no nontrivial uniform partial orientation of the resistors

Let r be a resistor branch of S . Suppose that

- In all admissible uniform partial orientations of the resistors and the sources in S_2 the resistor branch corresponding to r has the same orientation, or is not oriented.

Use this orientation as the reference direction of r .

Then all resistive circuits C with structure S and oriented D_2 -structure S_2 have convex transfer characteristics from all sources of C to

- a) the voltage and the current of r , if r is a linear resistor branch in S ,
- b) the voltage of r , if r is a nonlinear resistor branch in S that is replaced by a current source in parallel with a linear resistor branch in S_2 ,
- c) the current of r , if r is a nonlinear resistor branch in S that is replaced by a voltage source in series with a linear resistor branch in S_2 .

Remark 2:

If the convexity/concavity of the transfer characteristic from a source value to the current of a V-resistor has to be established, theorem 4 is not directly applicable. Two tricks can be used:

- a) If a linear resistor or an I-resistor is connected in series with the V-resistor, the transfer characteristic to the former can be considered, since both resistors carry the same current
- b) In the associated D_2 -structure the V-resistor is not replaced by a parallel connection of the a current source branch and a linear resistor branch, but by a series connection of a voltage source branch and a linear resistor branch. The voltage source has to be oriented opposite to the direction with respect to which the V-resistor is convex.

An analogous procedure can be applied when the convexity/concavity of the transfer characteristic to the voltage of an I-resistor is sought.

4. EXAMPLES

4.1. Circuit with diode

Consider the circuit of Fig.1. It has been shown in [1] that the circuit has a pair of conjugate trees and no nontrivial uniform partial orientation of the resistors. The diode has a convex characteristic when the direction of conductance is taken as its reference direction. Now consider the associated oriented D_2 -structure S_2 (Fig.11). Using proposition 1 (cf.[1] for the

details of a similar reasoning), it is easy to see that the uniform partial orientation of the resistors and the source represented in Fig. 13 is the only admissible one.

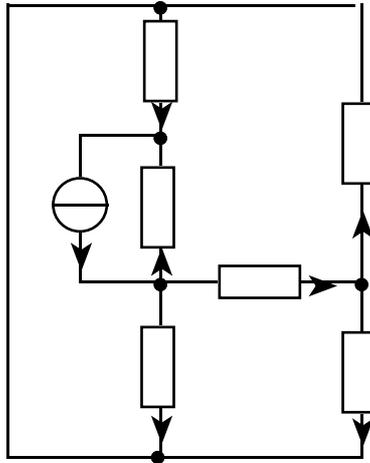


Fig.13. Only admissible uniform partial orientation of the resistors and the source in S_2 .

It follows from theorem 4 that, if we take the orientations of Fig.13 as reference directions for the linear resistor branches, all transfer characteristics from the voltage source in Fig.1 to the voltages and currents of the resistors are convex. For the diode, take the direction of conduction as the reference orientation. Then, the transfer characteristic from the voltage source to the voltage of the diode is concave, whereas the transfer characteristic from the voltage source to the current of the diode is convex. This last statement follows from remark 2a).

As far as the the current i_5 is concerned, this result is illustrated in Figs. 2 and 5. Indeed, for both choices of circuit parameters, $i_5(E)$ is convex. It is interesting to note that in this case we have structural convexity, but not structural monotonicity.

4.2. Circuit with two diodes

Consider the circuit of Fig.3. It easy to see that there is a pair of conjugate trees (actually a single tree consisting of only the voltage source) and no nontrivial partial uniform orientation of the resistors. We are interested in the convexity/concavity of the current i in Fig.3. Since this is not a current through a resistor branch, theorem 4 cannot be applied directly. However, using the equivalence of Fig.14 for the associated D_2 -circuit of Fig.10, i becomes a resistor current, namely the current through R_3 .

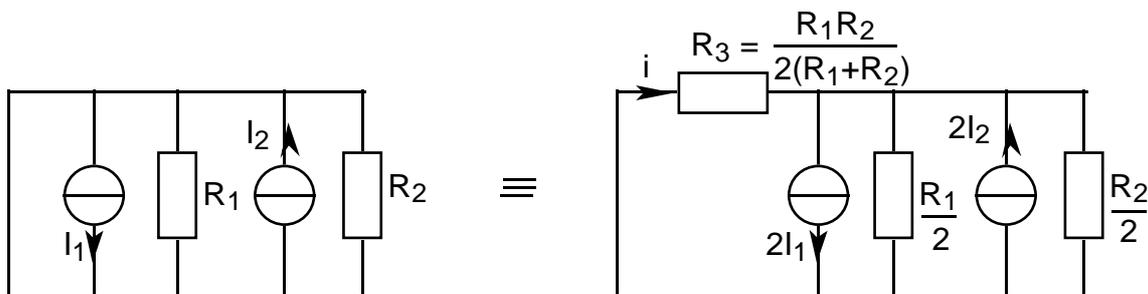


Fig.14. Equivalent circuit to the associated D_2 -circuit of Fig.10

The corresponding linear resistive circuit structure has two admissible uniform partial orientations of the resistors and the sources, represented in Fig.15. They have opposite orientation of the resistor branch, and thus, no structural convexity or concavity of i as a function of E can be deduced from theorem 4. Of course, this must be the case, since Fig.4 shows an example of a neither convex nor concave function $i(E)$.

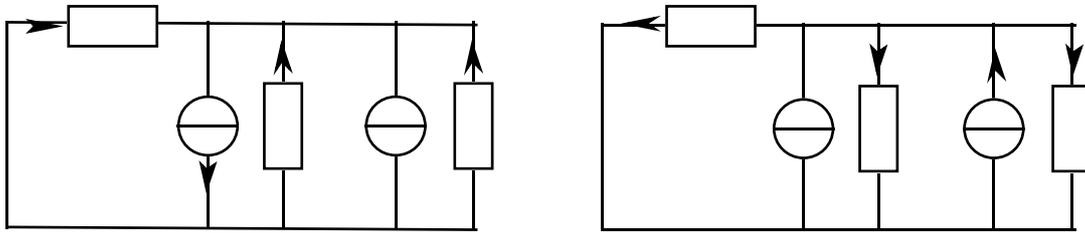


Fig.15. Admissible uniform partial orientations of the resistors and the sources

4.3. Ladder circuits

Consider a general ladder circuit composed of nonlinear resistors with convex/concave characteristics (Fig.16). We use the reference orientation indicated in Fig.16 in order to distinguish between convex and concave element characteristics. For the sake of simplicity, we suppose that all series resistors are I-resistors and all shunt resistors V-resistors. The analysis for other combinations of V- and I-resistors is analogous.

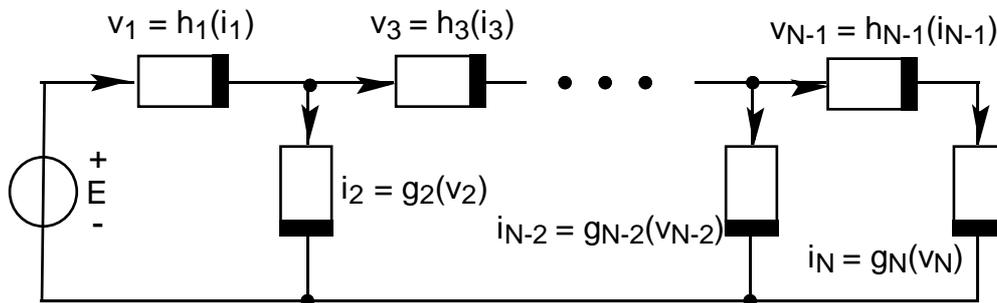


Fig.16. Ladder circuit

Let us determine under which conditions the transfer characteristics from the source voltage to a given resistor voltage is convex. The condition for concavity can be deduced subsequently by changing all convex resistors to concave resistors and vice versa.

The I-resistors together with the voltage source form a tree which, taken twice, fulfils the requirements for a pair of conjugate trees. It is not difficult to show that there is no nontrivial uniform partial orientation of the resistors.

Suppose, temporarily, that the resistors have convex characteristics with respect to the reference orientations of Fig.16. Consider the associated oriented D_2 -structure S_2 (Fig.17).

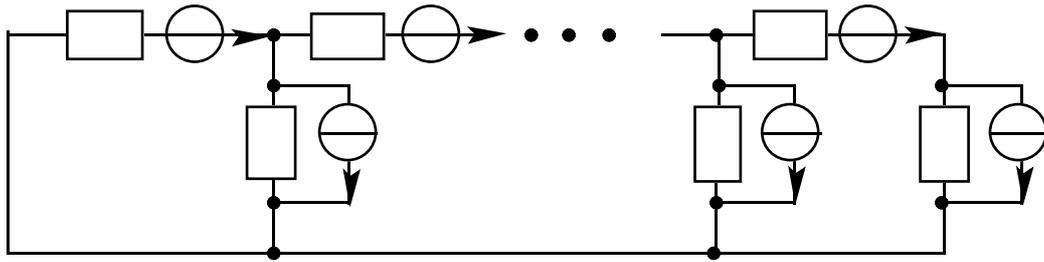


Fig.17. Associated D_2 -structure of the ladder circuit

In Fig.18, an admissible uniform partial orientation of the resistor and source branches of S_2 is represented, where one current source is oriented. In fact, this is the only such orientation with

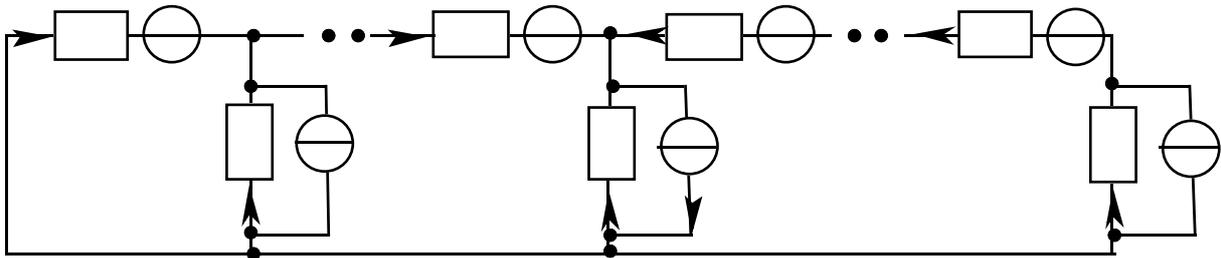


Fig.18. Admissible uniform partial orientation of S_2 with one oriented current source

the given orientation of the current source. This can be seen as follows. Start orienting the series resistor to the utmost left. Then the orientation of the first shunt resistor from the left is determined in order to avoid a "bad" loop according to proposition 1. For the next series resistor there is also only one choice of orientation, because a "bad" cutset according to proposition 1 has to be avoided (Fig.19). In this way the resistors are successively oriented from left to right, until the current source is reached. In the same way the resistors are oriented successively from the right to the left, until the oriented current source is reached. In this procedure, the leftmost and the rightmost resistor is oriented arbitrarily. By considering the 9 possible orientation of the extremal resistors, one can exclude 8 by proposition 1 and there remains only the orientation shown in Fig.18.

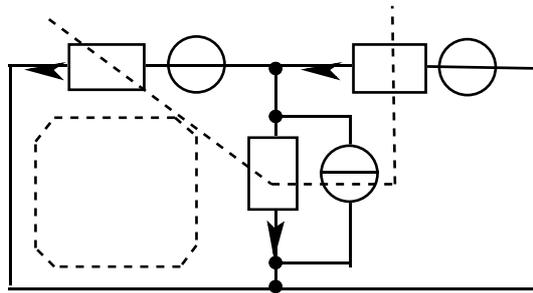


Fig.19. Orientation of the resistors starting from an arbitrary orientation of the leftmost resistor.

In Fig.20, an admissible uniform partial orientation of the resistor and source branches of S_2 is represented, where one voltage source is oriented. Again, this is the only admissible orientation for the given orientation of the voltage source, and the reasoning leading to this conclusion is similar as in the case of an oriented current source.

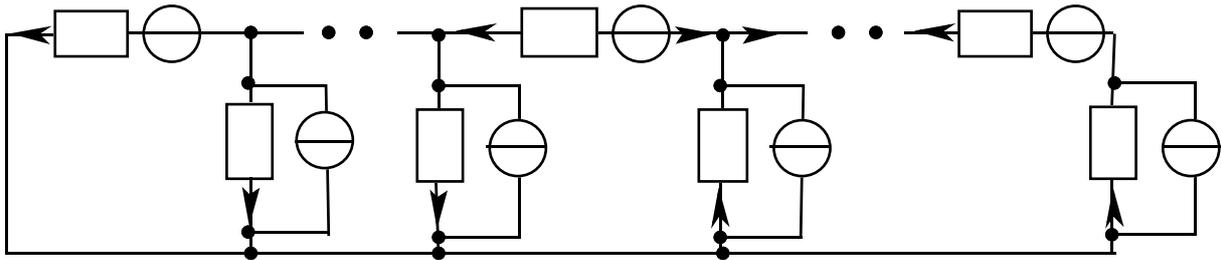


Fig.20. Admissible uniform partial orientation of S_2 with one oriented voltage source

If all admissible orientations, as given by Fig.18 and Fig.20 are considered simultaneously, it appears that only the last shunt resistor has always the same orientation. If, however, not all of the nonlinear resistors in the original circuit are convex but some are concave, a different resistor may always have the same orientation in all admissible orientations of S_2 . The results of this analysis are summarized in the following theorem.

Theorem 5.

Consider the ladder circuit of Fig.16, where the functions g and h are continuous, strictly increasing and defined on the whole real line.

- a) The transfer characteristic of from the voltage source (or from any other voltage source in series with a resistor, or from any other current source in parallel with a resistor) to the voltage of a shunt resistor is concave, if all functions g are convex, all functions h of the series resistors to the left of the shunt resistor are convex and all those to the right concave (Fig.21).

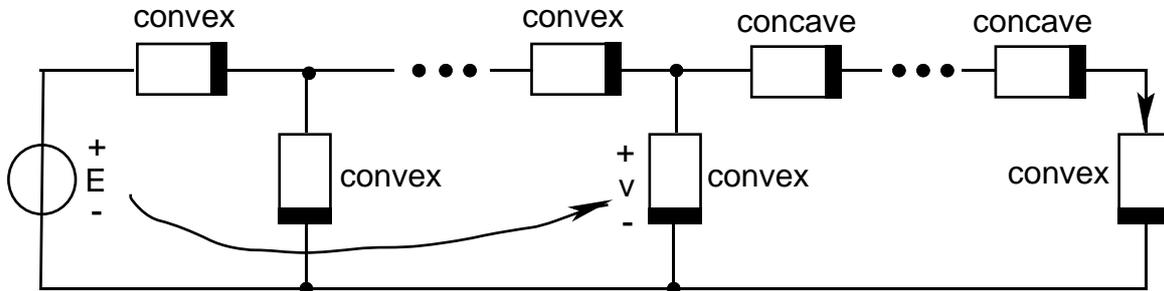


Fig.21. Convexity/concavity conditions which guarantee that $v(E)$ is concave

- b) The transfer characteristic from the voltage source (or from any other voltage source in series with a resistor, or from any other current source in parallel with a resistor) to the current of a series resistor is concave, if all functions h are convex, all functions g of the shunt resistors to the left of the series resistor are convex and all those to the right concave (Fig.22).

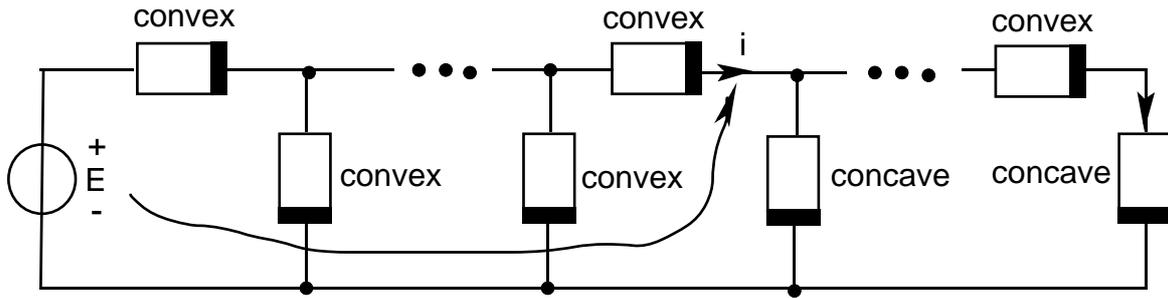


Fig.22. Convexity/concavity conditions which guarantee that $i(E)$ is concave

To check theorem 5 with a numerical example, we take a ladder of 2 series and two shunt resistors with characteristics g and h of the form (Fig.23). Note that both the first and second

$$f(x) = a(x - x_0) + \begin{cases} 0 & \text{if } x \leq x_0 \\ (b - a) \left[(x - x_0)^3 - 0.5(x - x_0)^4 \right] & \text{if } x_0 < x < x_0 + 1 \\ (b - a)(x - x_0 - 0.5) & \text{if } x_0 + 1 \leq x \end{cases} \quad (7)$$

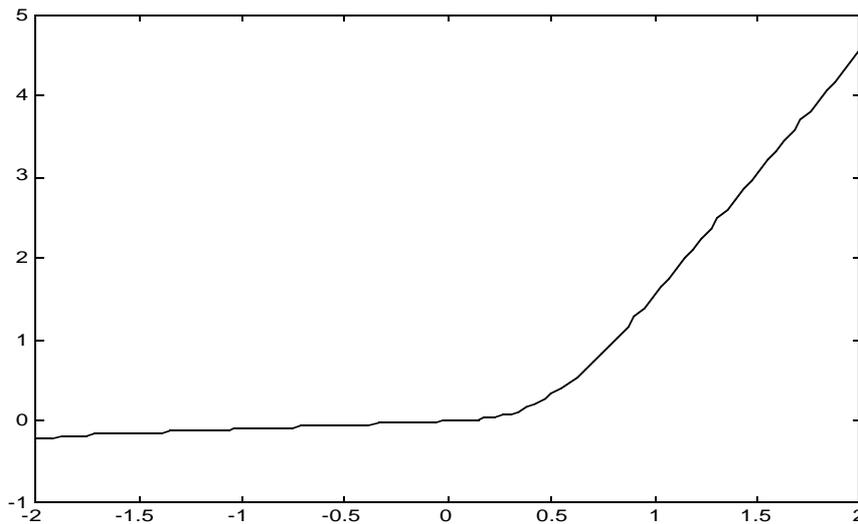


Fig.23. Resistor characteristics with parameters $a = 0.1$, $b = 3$ and $x_0 = 0$

derivatives are continuous. Furthermore, the function f is convex if $a \leq b$ and concave if $a \geq b$. We have chosen for the following parameters for the 4 resistors:

Resistor	series/shunt	convex/concave	a	b	x_0
1	series	convex	0.2	1	6
2	shunt	convex	1	2	3
3	series	convex	0.1	3	0
4	shunt	concave	2	0.2	-5

Table 1. Parameters of the nonlinear resistors (cf. equation 7)

Theorem 5 implies that the transfer characteristic $i_3(E)$ is concave (Fig.27), but it does not give any conclusion for the other transfer characteristics. Indeed, in our example none of the latter is convex nor concave (Figs.25,26,28).

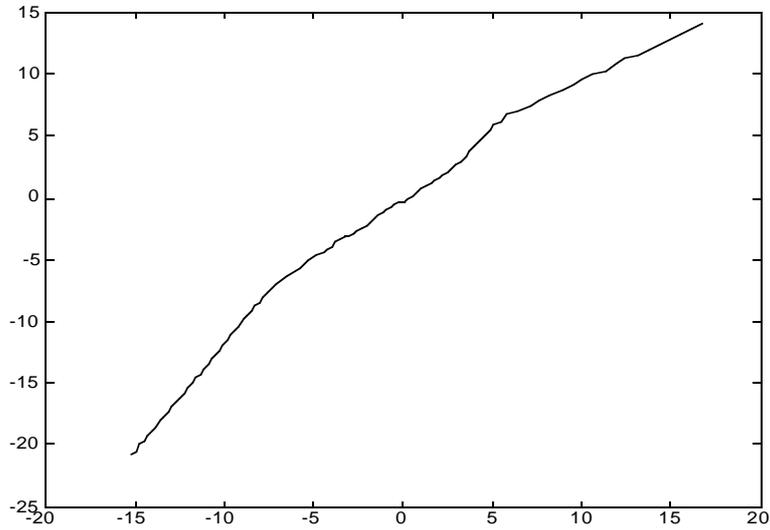


Fig. 25. Transfer characteristic $i_1(E)$. Neither convex nor concave.

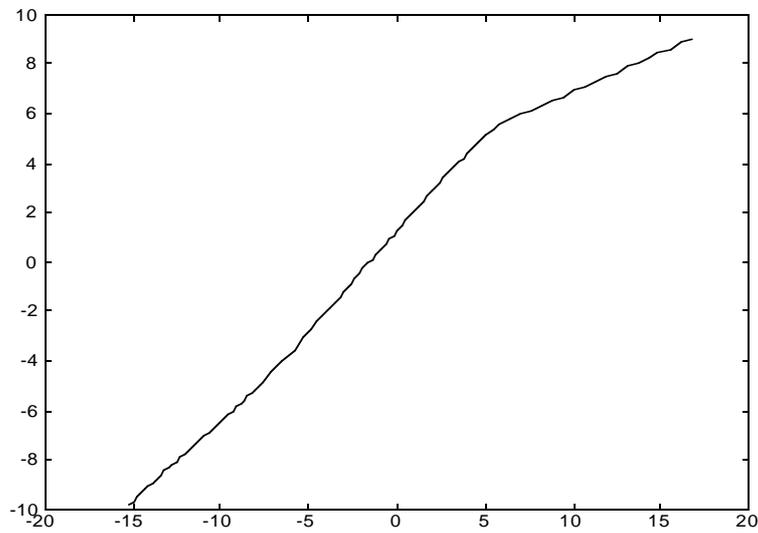


Fig. 26. Transfer characteristic $v_2(E)$. Neither convex nor concave.

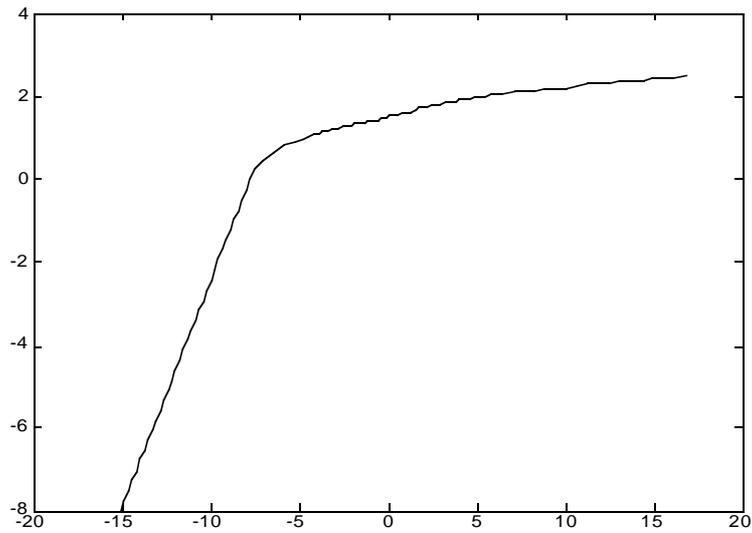


Fig. 27. Transfer characteristic $i_3(E)$. Concave

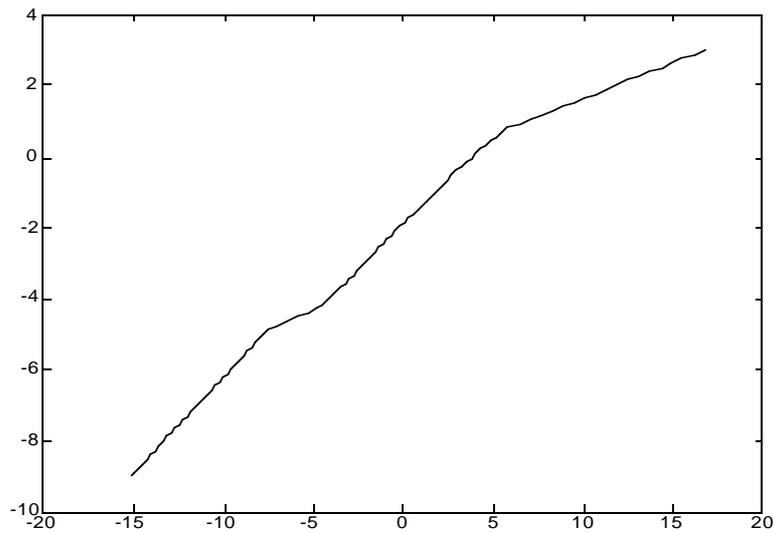


Fig. 28. Transfer characteristic $v_4(E)$. Neither convex nor concave.

5. CONCLUSION

We have given topological criteria for convexity or concavity of the transfer characteristics in a nonlinear resistive circuit where each resistor has either a convex or a concave element characteristic. These criteria do not need any special form of the nonlinearity, other than a positive slope and a second derivative of a given sign. In small circuits, the analysis can be performed by pencil and paper, whereas in medium size circuits a combinatorial algorithm has to be run. We have applied the criteria to a few examples. One of them concerns the whole class of

ladder circuits, where the criteria lead to a direct condition for the concavity or the convexity of transfer characteristics. A numerical example confirms the pertinence of the criteria.

We think that similar results could be obtained by the equation based approaches of Sandberg-Willson [4] and Nishi [5]. It would also be possible to extend the results to dependencies on circuit parameters other than source values, thereby generalizing [6].

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