QUASILINEAR SECOND ORDER ELLIPTIC SYSTEMS AND TOPOLOGICAL DEGREE

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Version abrégée

Dans ce travail, nous considérons une grande classe de systèmes elliptiques quasilinéaires du 2ème ordre de la forme

$$-\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x, u(x), \nabla u(x))\partial^2_{\alpha\beta}u(x) + b(x, u(x), \nabla u(x)) = 0,$$

où x varie dans un domaine Ω non borné de l'espace Euclidien $\mathbb{R}^N$, et $u = (u^1, \ldots, u^m)$ est un vecteur de fonctions inconnues. Ces systèmes engendrent des opérateurs agissant entre les espaces de Sobolev $W^{2,p}(\Omega, \mathbb{R}^m)$ et $L^p(\Omega, \mathbb{R}^m)$ pour $p > N$. Nous examinons alors les propriétés de Fredholm et des applications propres, de ces opérateurs, et l'interaction entres elles.

Ces propriétés fonctionnelles jouent des rôles importants dans le domaine des équations différentielles nonlinéaires, et sont aussi liées à deux degrés topologiques récents.

Une première partie de ce travail constitue une généralisation de résultats récemment obtenus par Rabier et Stuart qui ont traité le cas scalaire (une seule équation) sur $\mathbb{R}^N$. Notre étude couvre donc le cas de plusieurs équations (équations couplées) définies sur des domaines plus généraux. Nous étudions aussi la question de décroissance exponentielle des solutions.

Les résultats obtenus dans notre cadre général nous permettent ensuite d'explorer de nouvelles situations plus spécifiques: systèmes stationnaires de réaction-diffusion et élasticité non linéaire, où grâce au degré topologique, nous démontrons de nouveaux résultats d'existence et de continuation globale.

Abstract

We consider a large class of quasilinear second order elliptic systems of the form

$$-\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x,u(x),\nabla u(x))\partial^2_{\alpha\beta}u(x)+b(x,u(x),\nabla u(x))=0,$$

where x varies in an unbounded domain Ω of the Euclidean space $\mathbb{R}^N$ and $u=(u^1,\ldots,u^m)$ is a vector of functions. These systems generate operators acting between the Sobolev spaces $W^{2,p}(\Omega,\mathbb{R}^m)$ and $L^p(\Omega,\mathbb{R}^m)$ for $p>N$. We investigate then the Fredholm and properness properties of these operators and the connections between them.

These functional properties play important roles in the existence theory of nonlinear differential equations, and they are related to two recent topological degrees.

A first part of this work is an extension of recent results obtained by Rabier and Stuart who studied the scalar case (a single equation) on $\mathbb{R}^N$. Our results cover the case of several equations (coupled equations) defined on more general domains. We also study the question of exponential decay of solutions.

The general results obtained in our framework are then applied to more specific and new situations: steady reaction-diffusion systems and nonlinear elasticity, where by means of the topological degree, we prove new existence and global continuation results.

A la mémoire de mon père, à ma mère, à mon professeur Charles A. Stuart, et à tous ceux qui voient dans les mathématiques une discipline qui permet à l'homme de développer ses capacités humaines.

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Contents

Chapter 1

Introduction

This work lies in a vast domain of analysis which can be called "Topological methods in nonlinear analysis". We assume that the reader is familiar with the Leray-Schauder degree and some of its applications to differential equations. Otherwise, there are many textbooks which introduce to the subject or treat advanced topics of it. A good place to start is the recent book by Brown [**7**]. The books of Lloyd [**24**] and Deimling [**11**] are now classical in this field.

Before describing the contents of the present thesis, we think it is important to present its nearest context through a brief survey of some recent works. In doing so, we also try to bring out progressively the problematic of this research.

1.1 Some recent works

One of the modern approaches for studying differential equations, consists first in formulating the differential equation together with its boundary conditions, as an operator equation between function spaces. Once the problem is defined in a functional framework, there are many methods to analyze it. One of the powerful tools being the topological degree, since it can give answers to the questions of existence, multiplicity, and bifurcation of solutions. Furthermore the fundamental work of Rabinowitz [**35**] showed how a clever use of the degree can lead to global information about branches of solutions.

For regular boundary-value problems on bounded domains, the compactness of some imbeddings (for instance in Sobolev or Hölder spaces) can reduce the problem to a compact perturbation of the identity, and so the Leray-Schauder degree can be used. However the situation is different when dealing with differential equations on unbounded domains, since the imbeddings -when they still hold- are no longer compact in general. To overcome this difficulty several approaches were developed including construction of more general degrees (there are also 'ad hoc' constructions for particular problems, see [**20**] and [**42**] for a recent example). In this direction, the research of Fitzpatrick and Pejsachowicz in the 1980's prepared the way for the construction of a degree for proper Fredholm maps of index zero. It was presented in a complete and concise form by Fitzpatrick, Pejsachowicz and Rabier first for mappings of class C^2 [**13**], and later extended to the C^1 setting [**29**].

A first application of this degree was made by Jeanjean, Lucia and Stuart [**21**]. In that paper, they deal with a semilinear problem of the form

$$\begin{cases} -\Delta u + f(x,u) = \lambda u, & x \in \mathbb{R}^N \\ \lim\limits_{|x|\to\infty} u(x) = 0 \end{cases}$$

where $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies $f(x,0) = 0$, and λ is a real parameter.

They use the Sobolev spaces $X = W^{2,p}(\mathbb{R}^N)$ and $Y = L^p(\mathbb{R}^N)$ with $p \in (\frac{N}{2}, \infty)$, and study the functional properties of the operator $F : \mathbb{R} \times X \to Y$ defined by

$$F(\lambda, u) = -\Delta u + f(x, u) - \lambda u.$$

They formulate conditions on the nonlinearity f and the location of the parameter λ which ensure that F is well defined and has all the properties required for the use of the degree (see §1.3 and below), and subsequently establish global bifurcation results in the spirit of **[35]**. In that work the maximum principle plays a crucial role in proving the properness of the operator, and this forces λ to lie in an interval below the essential spectrum of the linearization $-\Delta u + \partial_2 f(x, 0)$.

The more general case of a quasilinear second order elliptic equation was studied by Rabier and Stuart. In a series of papers **[31]**, **[32]**, **[33]**, they discuss the problem

$$\begin{cases} -\displaystyle\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x, u, \nabla u)\partial^2_{\alpha\beta} u + b(x, u, \nabla u, \lambda) = 0, & x \in \mathbb{R}^N \\ \displaystyle\lim_{|x|\to\infty} u(x) = 0. \end{cases}$$

Here again, they work in the Sobolev spaces $X = W^{2,p}(\mathbb{R}^N)$ and $Y = L^p(\mathbb{R}^N)$ but with $p \in (N, \infty)$, and this incorporates the decay to zero of the solutions and their derivatives at infinity. The problem is formulated as the search of zeros of the operator $F : \mathbb{R} \times X \to Y$ defined by

$$F(\lambda, u) = -\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(., u, \nabla u)\partial^2_{\alpha\beta} u + b(., u, \nabla u, \lambda).$$

Then, the main goal is to find ways for verifying the conditions required for the use of the degree, and which are precisely

(a) $F \in C^1(\mathbb{R} \times X, Y)$.

(b) $D_u F(\lambda, u) \in \Phi_0(X, Y)$ for all $u \in X$.

(c) $F : \mathbb{R} \times X \to Y$ is proper on the closed bounded subsets of $\mathbb{R} \times X$.

They concentrate on the situation where the problem is asymptotically periodic as $|x| \to \infty$, in the sense that there are functions $a^\infty_{\alpha\beta}$ and b^∞ such that $a^\infty_{\alpha\beta}(., \xi)$ and $b^\infty(., \xi, \lambda)$ are periodic on $\mathbb{R}^N$ with the same period for all ξ and λ and

$$\lim_{|x|\to\infty} \left|a_{\alpha\beta}(x, \xi) - a^\infty_{\alpha\beta}(x, \xi)\right| = 0 \quad \text{and} \quad \lim_{|x|\to\infty} |\partial_{\xi_i} b(x, \xi, \lambda) - \partial_{\xi_i} b^\infty(x, \xi, \lambda)| = 0.$$

Setting

$$F^\infty(\lambda, u) = -\sum_{\alpha,\beta=1}^{N} a^\infty_{\alpha\beta}(., u, \nabla u)\partial^2_{\alpha\beta} u + b^\infty(., u, \nabla u, \lambda),$$

the main results of **[31]**, show that under mild smoothness assumptions which imply (a), the important properties (b) and (c) are equivalent to

(i) $D_u F(\lambda, 0) \in \Phi_0(X, Y)$.

(ii) $\{u \in X, F^\infty(\lambda, u) = 0\} = \{0\}$.

Then, they explore some important situations where condition (i) can be explicitly verified, essentially by using the spectral theory of Schrödinger operators.

To verify condition (ii) they consider three situations and accordingly give three approaches. The first is based on the maximum principle. The second is based on integral identities of Pohozaev-type, and this was carried out in **[33]** after a study of the exponential decay of solutions. The third approach deals with the case when $F^{\infty}(\lambda, u)$ is linear in u, so spectral theory is used. Each situation then leads to a global bifurcation theorem.

A more detailed survey of the works of Rabier and Stuart can be found in **[34]**.

Independently of Fitzpatrick, Pejsachowicz and Rabier, Benevieri and Furi constructed another degree for C^1 Fredholm maps of index zero. We introduce these tools in §1.3.2.

1.2 Description of the present report

Many problems in science can be described mathematically as a system of several differential equations of several unknown functions (consider for instance Navier-Stokes equations, the equations of 3 dimensional elasticity, reaction-diffusion equations in mathematical ecology and so on). These equations are sometimes called coupled equations. Suppose that we are given m quasilinear partial differential equations of second order containing m unknown functions $u^1, u^2, \dots, u^m$ defined on a domain $\Omega \subset \mathbb{R}^N$, then we can write the system in the compact form

$$-\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x, u, \nabla u)\partial^2_{\alpha\beta} u + b(x, u, \nabla u) = 0, \qquad x = (x_1, \dots, x_N) \in \Omega, \tag{1.1}$$

where

- u is the vector $\left(u^1, u^2, \dots, u^m\right)^T$, ∇u is the matrix of partial derivatives $\dfrac{\partial}{\partial x_j} u^i$, $j = 1, \dots, N$, $i = 1, \dots, m$.
- $a_{\alpha\beta}(x, u, \nabla u)$ is an $m \times m$ matrix.
- $b(x, u, \nabla u)$ is a vector of height m.

In the physical examples mentioned above, these equations correspond to the stationary case, and x represents the independent space variables.

The first objective of this research is to extend the approach of Rabier and Stuart **[31]** to cover such systems on unbounded domains of $\mathbb{R}^N$, and which are also elliptic in some sense.

The first difficulty is the choice of an ellipticity condition. For a single linear second order equation, there is essentially one such condition which states that the matrix of the leading coefficients is positive definite. Indeed, different refinements such as strong or uniform ellipticity capture different behavior of the coefficients when they are not constant. However, in the case of a system, even with constant coefficients, there are at least three different conditions of ellipticity: that of Agmon-Douglis-Nirenberg, the strong Legendre-Hadamard, and the strong Legendre conditions. After an exploratory investigation of these different conditions and some of their functional implications, we decided to concentrate on an intermediate condition between Agmon-Douglis-Nirenberg and the strong Legendre-Hadamard. This condition is known as ellipticity in the sense of Petrovskii, and for which L^p-*a priori* estimates due to Koshelev are available under weak assumptions about the smoothness of the coefficients. Furthermore, Petrovskii ellipticity enables us to use the same function space for all the components of the vector u introduced above. The interest in ellipticity stems from its strong connections with the Fredholm property of linear partial differential operators. For problems on bounded domains, an important connection is given by the theorems on complete collections of isomorphisms, which establish an equivalence between (i) ellipticity (in the sense of Agmon-Douglis-Nirenberg) together with a boundary condition (known as Lopatinskii condition), (ii) *a priori* estimates, and (iii) the Fredholm property, in a scale of function spaces. See Roïtberg **[36]**, Agranovich **[2]** or Wloka and al. **[43]** for precise statements. We mention, however, that these theorems are stated and proved in the literature under the strong assumptions that the domain has a C^∞ boundary and the coefficients of the operator are C^∞ on the domain (it is mentioned sometimes that these assumptions can be weakened, but they are still strong to us in order to exploit them in the quasilinear problem).

Thus, serious difficulties arise in the transition from the scalar problem to the vectorial one. Also, the fact that we consider more general domains than $\mathbb{R}^N$, introduces further technical complications which have to be resolved in order to obtain a complete generalization of the work of Rabier and Stuart.

Let us now proceed to the presentation of this work. In section 1.3 we introduce the needed tools from functional analysis and degree theory.

In chapter 2, we consider a large class of quasilinear second order operators

$$F(u) = -\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(.,u,\nabla u)\partial^2_{\alpha\beta}u + b(.,u,\nabla u) \tag{1.2}$$

where $b : \Omega \times \left(\mathbb{R}^m \times \mathbb{R}^{m\times N}\right) \to \mathbb{R}^m$, $a_{\alpha\beta} : \Omega \times \left(\mathbb{R}^m \times \mathbb{R}^{m\times N}\right) \to \mathbb{R}^{m\times m}$ $(\alpha, \beta = 1, \dots, N)$ is a family of matrix-valued maps, and Ω is an open subset of $\mathbb{R}^N$, whose boundary is Lipschitz continuous and bounded.

By requiring that Ω has a bounded boundary, we cover three situations: Ω is bounded (in this case, we claim to no novelty), Ω is the exterior of a bounded set, and $\Omega = \mathbb{R}^N$.

The goal is then the investigation of the Fredholm and properness properties of F in (1.2) and the interplay between them, in the context of Sobolev spaces. More precisely, we study these functional properties of the operator F acting between $W^{2,p}(\Omega,\mathbb{R}^m) \cap W^{1,p}_0(\Omega,\mathbb{R}^m)$ and $L^p(\Omega,\mathbb{R}^m)$, where p is a real number $> N$.

Indeed, we formulate conditions on the coefficients of F, and the domain Ω which ensure that F is well defined i.e. maps $W^{2,p}(\Omega,\mathbb{R}^m)$ into $L^p(\Omega,\mathbb{R}^m)$, and is of class C^1. This is done in §2.1, after a preparatory study of some Nemytskii operators. The needed notation are introduced and fixed on pp. 25-26. The conditions on the coefficients are formulated on p. 33 and they are implicitly assumed in all the relevant theorems of chapter 2.

In §2.2 we introduce two ellipticity conditions for linear systems. The study of the Fredholm property of F begins in §2.3. The main result there is Theorem 2.3 which states that, if the coefficients of F satisfy an ellipticity condition (in the sense of Petrovskii), and the Fréchet derivative $\mathrm{D}F(u)$ is semi-Fredholm (with index $\neq \infty$) for some $u \in W^{2,p}(\Omega,\mathbb{R}^m) \cap W^{1,p}_0(\Omega,\mathbb{R}^m)$, then $\mathrm{D}F(u)$ is semi-Fredholm of the same index for every u. Therefore, to prove the Fredholm property, one should only consider the linearization at a particular point. We end §2.3 by a result on the factorization of F.

The study of properness begins in §2.4, where we relate - through some technical results - properness on closed bounded subsets, to the Fredholm property, and to a notion of uniform decay of sequences of functions. We end that section by proving that if F is proper on the closed bounded subsets of $W^{2,p}(\Omega,\mathbb{R}^m) \cap W^{1,p}_0(\Omega,\mathbb{R}^m)$, then F is semi-Fredholm of index $\neq \infty$ (Corollary 2.2).

In §2.5, we consider the situation where the coefficients of F are asymptotically periodic as $|x| \to \infty$, in the sense that there are functions $a^\infty_{\alpha\beta}$ and c^∞_i $1 \le \alpha,\beta \le N$, $0 \le i \le N$ such that

$a^{\infty}_{\alpha\beta}(.,\xi)$ and $c^{\infty}_{i}(.,\xi)$ are periodic on $\mathbb{R}^N$ with the same period for all $\xi \in \mathbb{R}^m \times \mathbb{R}^{m\times N}$ and

$$\lim_{|x|\to\infty} \left| a_{\alpha\beta}(x,\xi) - a^{\infty}_{\alpha\beta}(x,\xi) \right| = 0 \quad \text{and} \quad \lim_{|x|\to\infty} \left| \int_0^1 \nabla_{\xi_i} b(x,t\xi)\,\mathrm{d}t - c^{\infty}_i(x,\xi) \right| = 0.$$

The periodicity condition forces the limit coefficients $a^{\infty}_{\alpha\beta}(.,\xi)$ and $c^{\infty}_{i}(.,\xi)$ to be defined on the whole space $\mathbb{R}^N$ even if the coefficients $a^{\infty}_{\alpha\beta}$ and c^{∞}_{i} are only defined on $\Omega \neq \mathbb{R}^N$ (which is indeed required to be unbounded).

Setting $b^{\infty}(x,\xi) = \sum_{i=0}^{N} c^{\infty}_i(x,\xi)\xi_i$, we define a limit operator

$$F^{\infty}(u) = -\sum_{\alpha,\beta=1}^{N} a^{\infty}_{\alpha\beta}(.,u,\nabla u)\partial^2_{\alpha\beta} u + b^{\infty}(.,u,\nabla u).$$

Then, after a series of technical lemmas, we succeed in finding sufficient conditions for the properness (on closed bounded subsets) of the operator F in (1.2). Namely, we prove that if (i) F is semi-Fredholm of index $\neq \infty$, and (ii) the limit problem $F^{\infty}(u) = 0$ has only the trivial solution $u = 0$, then F is proper on the closed bounded subsets of $W^{2,p}(\Omega,\mathbb{R}^m) \cap W^{1,p}_0(\Omega,\mathbb{R}^m)$. These conditions are also necessary if $\Omega = \mathbb{R}^N$ (Theorem 2.7 and Theorem 2.8). We point out that Lemma 2.18 plays a crucial role in this issue.

The aim of §2.6 is to find verifiable conditions which ensure that a linear elliptic operator L is an isomorphism or Fredholm of index zero (between our function spaces). We begin with the case when $\Omega = \mathbb{R}^N$ and the linear operator is elliptic in the sense of Petrovskii and has constant coefficients. Using Fourier transforms, we find a simple algebraic condition on the coefficients of L which implies that L is an isomorphism (Corollary 2.5). When the coefficients are variable, we assume that they have a limit as $|x| \to \infty$. The resulting limit operator L^{∞} then has constant coefficients. We prove that if the limit operator satisfies the mentioned algebraic condition (and so it is an isomorphism), and furthermore all the operators $L_t = tL + (1-t)L^{\infty}$ $(t \in [0,1])$ are elliptic, then L is Fredholm of index zero (Theorem 2.10). When L is elliptic in the stronger sense of Legendre-Hadamard, the condition that L_t are elliptic is automatically satisfied. When $\Omega \neq \mathbb{R}^N$, we obtained only a partial result (Theorem 2.11).

§2.7 concerns the question of exponential decay of solutions of $F(u) = f$, when f has exponential decay as $|x| \to \infty$. First, we present and complement some results obtained recently by Rabier in an abstract setting. Then, we consider linear and quasilinear systems. The main result states that if F is a Fredholm map (of any index) and $f \in L^p(\Omega,\mathbb{R}^m)$ has exponential decay, then any possible solution in $W^{2,p}(\Omega,\mathbb{R}^m) \cap W^{1,p}_0(\Omega,\mathbb{R}^m)$ of $F(u) = f$ has also an exponential decay. We give some refinements in Proposition 2.2 and Theorem 2.14.

Chapter 3 and 4 illustrate the use of the previous results and the topological degree in more specific and new situations.

In chapter 3, we begin by discussing a particular homotopy, and relate the problem of existence to finding *a priori* estimates. Then we consider a special case of steady reaction-diffusion systems, where we are able to find *a priori* bounds by means of a maximum principle and obtain a first existence result. A second homotopy is considered in §3.3 leading to a second existence result. We note that the maximum principle also plays an important role in the properness issue by proving that the limit problem has only a trivial solution.

Chapter 4 is devoted to a study of a model in three dimensional elastostatics, where the elastic body fills the whole space $\mathbb{R}^3$. There, we are confronted with a new difficulty concerning the injectivity and orientation preserving of admissible deformations of the elastic body. We discuss this issue in §4.1, where we define the mathematical problem and show how it fits in the general framework of chapter 2. In §4.2, we prove that the elasticity operator acting between

$W^{2,p}(\mathbb{R}^3,\mathbb{R}^3)$ and $L^p(\mathbb{R}^3,\mathbb{R}^3)$, is Fredholm of index zero and proper on the closed bounded subsets of $W^{2,p}(\mathbb{R}^3,\mathbb{R}^3)$. The Fredholm issue is treated as follows. Using the results of §2.6, we first show that the linearization at 0 is a compact perturbation of an isomorphism (hence Fredholm of index zero), then using the results on exponential decay of solutions, we prove that it has a trivial kernel, and therefore it is an isomorphism. Theorem 2.3 ensures then that the operator is Fredholm of index zero. For the properness issue, we first establish identities of Pohozaev type for the limit problem by using the exponential decay of solutions. Then we prove that the limit problem has only a trivial solution. Together with the Fredholm property, this implies that the operator is proper on the closed bounded subsets of $W^{2,p}(\mathbb{R}^3,\mathbb{R}^3)$.

In §4.3, we introduce a parameter in the problem and obtain global continuation results using the topological degree.

The appendix contains some more or less known results about sequences and function spaces, which we used throughout this work.

1.3 Fundamental concepts and notation

1.3.1 Compact, Fredholm and proper maps

In this subsection, we collect some results about compact, Fredholm and proper maps. Although these concepts are well known, there is no universally accepted terminology for them. For instance, some authors call Noetherian what we call a Fredholm operator, others call completely continuous what we call a compact map. Therefore, we think it is important to have a precise language.

In what follows, X, Y and Z are real Banach spaces. $\mathcal{L}(X,Y)$ is the Banach space of all linear and bounded operators from X to Y, and $GL(X,Y)$ is the open subset of isomorphisms. $X' = \mathcal{L}(X,\mathbb{R})$ is the dual of X. $X \hookrightarrow Y$ means that X is continuously imbedded in Y, and $X \underset{\text{comp}}{\hookrightarrow} Y$ means that the imbedding is compact.

A. Let $G : X \to Y$ be an operator (not necessary linear or continuous). G is compact if it maps bounded subsets onto relatively compact ones i.e. with compact closure. G is completely continuous if it transforms weakly convergent sequences into strongly convergent ones. Then one can prove the following.

(1) If X is reflexive and G is completely continuous, then G is compact.

(2) If G is weakly continuous and compact, then it is completely continuous (hence continuous). One can argue by contradiction.

Therefore, when X is reflexive and G is linear and bounded (and therefore weakly continuous), complete continuity and compactness for G are equivalent.

Remark 1.1 The reflexivity of X is equivalent to the following condition (Eberlein-Smulyan theorem): every bounded sequence from X contains a weakly convergent subsequence. It is in this form that reflexivity is used throughout this work.

Remark 1.2 Weak continuity, as continuity relative to weak (non metrizable) topologies, is not considered in our context, nor Moore-Smith sequences. So to simplify some statements, we use 'weak continuity' in the sense of **weak sequential continuity**.

B. $L \in \mathcal{L}(X,Y)$ is called semi-Fredholm, if $\operatorname{rge} L$ is closed and at least one among $\dim\ker L$ and $\operatorname{codim}\operatorname{rge} L$ is finite. The index of L is $\mu = \dim\ker L - \operatorname{codim}\operatorname{rge} L \in \mathbf{Z} \cup \{\pm\infty\}$. We denote by $\Phi_\mu(X,Y)$ the set of semi-Fredholm operators of index μ. In this work, the case $\mu=+\infty$ is not of particular interest, and we set

$$\Phi_+(X,Y)=\bigcup_{\mu\in\mathbf{Z}\cup\{-\infty\}}\Phi_\mu(X,Y).$$

When $G\in C^1(X,Y)$ is not necessarily linear, it is semi-Fredholm if for every $u\in X$, the Fréchet derivative $\mathrm{D}G(u)\in\mathcal{L}(X,Y)$ is semi-Fredholm. A semi-Fredholm operator with finite index is called Fredholm. We recall two fundamental properties of semi-Fredholm operators. See for instance **[23]** pp. 78-79.

(a) The index of semi-Fredholm operators is a locally constant function, or equivalently $\Phi_\mu(X,Y)$ is open in $\mathcal{L}(X,Y)$.

(b) If $L\in\Phi_\mu(X,Y)$ and $K\in\mathcal{L}(X,Y)$ is compact, then $L+K\in\Phi_\mu(X,Y)$ (stability under a compact perturbation).

C. Let $\mathcal{O}$ be a subset of X and $G:\mathcal{O}\to Y$ be a continuous operator. G is called proper if for every compact subset $\mathcal{C}$ from Y, we have that $\overset{-1}{G}(\mathcal{C})$ is compact. We are mainly interested in operators defined on X and which are proper on the closed bounded subsets, that is, their restriction to any closed bounded subset of X is proper. This condition is clearly equivalent to the following one.

(i) Every bounded sequence (u_n) from X, such that $(G(u_n))$ converges, contains a convergent subsequence.

When X is reflexive, properness on the closed bounded subsets is equivalent to the following.

(ii) For every sequence $(u_n)\subset X$ such that $u_n\rightharpoonup u$ and $(G(u_n))$ converges, we have $u_n\to u$, where $\rightharpoonup$ denotes weak convergence.

Proof. (ii) $\Rightarrow$ (i). Let (u_n) be bounded and $(G(u_n))$ convergent. Since X is reflexive, there is a subsequence $(u_{\varphi(n)})$ [1] converging weakly to some u. But $\big(G(u_{\varphi(n)})\big)$ also converges, therefore $u_{\varphi(n)}\to u$.

(i) $\Rightarrow$ (ii). Let $u_n\rightharpoonup u$ and $(G(u_n))$ converges. If (u_n) does not converge to u, there exist $\varepsilon_0>0$ and a subsequence $(u_{\varphi(n)})$ such that $\|u_{\varphi(n)}-u\|\geq\varepsilon_0$. But $(u_{\varphi(n)})$ has the same properties of (u_n), therefore it contains a convergent subsequence (to u by uniqueness of the weak limit). But this contradicts the above inequality. $\square$

When the operator depends on one or several parameters, the following result is useful.

Lemma 1.1 *Let J be a subset of $\mathbb{R}^d$ where d is a positive integer, and consider a continuous operator $G:J\times X\to Y$, with the following properties.*

(a) *For each $t\in J$, the partial operator $G(t,.):X\to Y$ is proper on the closed bounded subsets of X.*

(b) *For each bounded subset $B\subset X$, the collection $(G(.,u))_{u\in B}$ is equicontinuous.*

Then, for any compact subset A of J and any closed bounded subset $B\subset X$, the restriction $G|_{A\times B}$ is proper.

[1] φ is a strictly increasing function from $\mathbb{N}$ to $\mathbb{N}$.

Proof. Let K be a compact set of Y and consider a sequence $((t_n, u_n))$ from $G^{-1}(K) \cap (A \times B)$. We need to show that $((t_n, u_n))$ contains a convergent subsequence (the limit belongs then to $G^{-1}(K) \cap (A \times B)$ because this a closed set in $\mathbb{R} \times X$). First, since (t_n) belongs to the compact subset A by assumption, it contains a subsequence (still denoted by (t_n)) which converges to an element $t \in A$. Next, since the sequence $(G(t_n, u_n))$ belongs to K, it contains a subsequence $G(t_{\varphi(n)}, u_{\varphi(n)})$ which converges to an element v of Y. Now,

$$G(t, u_{\varphi(n)}) = G(t, u_{\varphi(n)}) - G(t_{\varphi(n)}, u_{\varphi(n)}) + G(t_{\varphi(n)}, u_{\varphi(n)}),$$

and it follows from the equicontinuity assumption that $G(t, u_n) - G(t_n, u_n) \to 0$ as $n \to \infty$. Therefore, $G(t, u_{\varphi(n)}) \to v$. Since $(u_{\varphi(n)})$ is bounded, it follows from the properness of $G(t, .)$ that $(u_{\varphi(n)})$ has a convergent subsequence. $\square$

We end this subsection with the following result. See Proposition 9.3 in **[11]** for a proof.

Yood's criterion. For a linear operator $L \in \mathcal{L}(X, Y)$, the following statements are equivalent.

(i) L is proper on the closed bounded subsets of X.

(ii) $L \in \Phi_+(X, Y)$.

1.3.2 Degrees for C^1 Fredholm maps of index zero

Here, we outline the construction of two recent degrees for C^1 Fredholm maps of index zero, and mention their most important properties. These degrees are defined for operators between Banach manifolds, but we restrict our brief description to the Banach space setting.

The degree of Fitzpatrick-Pejsachowicz-Rabier

This degree is constructed upon the concept of parity which we first introduce. Let $I = [a, b]$ be an interval of the real line, and consider a path[2] $\alpha : I \to \Phi_0(X, Y)$. There exists a path $k : I \to \mathcal{L}(X, Y)$ such that $k(t)$ is compact for every $t \in I$ and $\alpha(t) + k(t) \in GL(X, Y)$ (see Proposition 2.3 of **[12]** for a more general statement). Let $\beta(t) = (\alpha(t) + k(t))^{-1}$, then $\beta : I \to GL(Y, X)$ is a path such that for all $t \in I$, $\beta(t)\alpha(t)$ is a compact perturbation of the identity in X, and it is called a parametrix of α.

Now suppose that the ends of the path $\alpha(a)$ and $\alpha(b)$ are invertible. For any open neighborhood U of zero in X, the Leray-Schauder degree of $\beta(a)\alpha(a)$ at zero relative to U is well defined and takes values in $\{-1, 1\}$, independently of U. Accordingly, let $\deg_{\mathrm{L.S}}(\beta(a)\alpha(a))$ denote this degree. Indeed, the same things can be said about $\beta(b)\alpha(b)$. Given a parametrix β of the path α, the number

$$\sigma(\alpha) = \deg_{\mathrm{L.S}}(\beta(a)\alpha(a)) \deg_{\mathrm{L.S}}(\beta(b)\alpha(b))$$

which is either 1 or -1, does not depend on the choice of the parametrix β as shown in **[12]**. This number is called the parity of α. It satisfies some important properties such as homotopy invariance, multiplicativity, and invariance under reparametrizations. Furthermore the parity is 1 if and only if the path is homotopic to a path of isomorphisms.

Let $\mathcal{O}$ be an open connected and simply connected subset of X and $F : \mathcal{O} \to Y$ be a C^1 Fredholm map of index zero. A base point of F is any point $p \in \mathcal{O}$ at which $\mathrm{D}F(p)$ is an isomorphism. Assume that there exists a base point p for F, and consider an open subset $\mathcal{B} \subset \mathcal{O}$ such that F can be extended as a proper map to the closure of $\mathcal{B}$. Then if $y \notin F(\partial\mathcal{B})$ is a

[2] Let us agree that a path or a curve are continuous by definition.

regular value of F, the set $F^{-1}(y) \cap \mathcal{B} = F^{-1}(y) \cap \overline{\mathcal{B}}$ is compact (possibly empty) by properness, and discrete as it follows from the inverse function theorem. Therefore it is finite. If it is not empty, let $u \in F^{-1}(y) \cap \mathcal{B}$ and γ_u be a curve joining p to u (this is possible because $\mathcal{O}$ is open and connected and therefore path connected), then indeed $\mathrm{D}F \circ \gamma_u$, has invertible endpoints and so the parity $\sigma(\mathrm{D}F \circ \gamma_u)$ is well defined. Furthermore it is independent of γ_u as a consequence of the homotopy invariance of the parity and the simple connectivity of $\mathcal{O}$. Accordingly, the degree of F at y relative to $\mathcal{B}$ and p is defined by

$$\deg_p(F, \mathcal{B}, y) = \sum_{u \in F^{-1}(y) \cap \mathcal{B}} \sigma(\mathrm{D}F \circ \gamma_u).$$

When $F^{-1}(y) \cap \mathcal{B}$ is empty, the degree is naturally defined to be zero.

Now, if $y \notin F(\partial\mathcal{B})$ is not necessarily a regular value of F, there exists a ball U centered at y and contained in $Y \backslash F(\partial\mathcal{B})$, and as a consequence of the Quinn-Sard theorem, this ball contains regular values of F. But then, it is shown that $\deg_p(F, \mathcal{B}, y_1) = \deg_p(F, \mathcal{B}, y_2)$ for any regular values $y_1, y_2 \in U$. Accordingly, $\deg_p(F, \mathcal{B}, y)$ is defined to be this common number.

A change of the base point from p to q, changes the degree by the factor $\sigma(\mathrm{D}F \circ \theta) \in \{-1, 1\}$, where θ is any curve joining p to q.

This degree satisfies the usual properties of existence (called there normalization property), excision, additivity on domains, while its behavior under homotopy is given by the following reformulation of Theorem 5.1 and Corollary 5.5 of **[29]**.

Theorem 1.1 *Let $h \in C^1([0,1] \times \mathcal{O}, Y)$ be Fredholm of index 1 (this is equivalent to saying that $\mathrm{D}_u h(t,.) \in \Phi_0(X,Y)$ for all $t \in [0,1]$), and proper on $[0,1] \times \overline{\mathcal{B}}$. Such a homotopy is called $\mathcal{B}-$admissible. Suppose that $y \notin h([0,1] \times \partial\mathcal{B})$ and $p_0 \in \mathcal{O}$ is a base point of $h(0,.)$. Then we have the following.*

(i) *If $p_1 \in \mathcal{O}$ is a base point for $h(1,.)$, then*

$$\deg_{p_0}(h(0,.), \mathcal{B}, y) = \sigma(\mathrm{D}_u h \circ \gamma) \deg_{p_1}(h(1,.), \mathcal{B}, y),$$

where γ is any curve joining $(0, p_0)$ to $(1, p_1)$ in $[0,1] \times \mathcal{O}$.

(ii) *If for some $t \in (0,1]$, $h(t,.)$ has no base point or the equation $h(t,u) = y$ has no solution in $\mathcal{B}$, then $\deg_{p_0}(h(0,.), \mathcal{B}, y) = 0$.*

It is sometimes useful to consider the absolute degree of $(F, \mathcal{B}, y)$, which is defined to be zero if F has no base point, and to be the absolute value of $\deg_p(F, \mathcal{B}, y)$ whenever p is a base point. Indeed, this is not a degree because it is not additive, however, it satisfies the excision property and it is homotopy invariant. And as noticed on p. 24 in **[13]**, these two properties are sufficient to prove the generalized homotopy invariance as in the Leray-Schauder degree. For future reference, we state

Lemma 1.2 *Let J be an open interval of the real line, $\Upsilon \subset J \times \mathcal{O}$, $G : \Upsilon \to Y$ be C^1 Fredholm of index 1, and extendable by continuity to a proper map on the closure of Υ. Set for $t \in J$, $\Upsilon_t = \{u \in \mathcal{O}, (t,u) \in \Upsilon\}$. Suppose that $y \neq G(t,u)$ for all $u \in \partial(\Upsilon_t)$. Then, for any a and b from J,*

$$|\deg|(G(a,.), \Upsilon_a, y) = |\deg|(G(b,.), \Upsilon_b, y).$$

Finally, we mention the connection with the Leray-Schauder's degree. When $X = Y$, $\mathcal{B}$ is bounded and F is a compact perturbation of the identity, we have

$$\deg_p(F, \mathcal{B}, y) = (-1)^n \deg_{\mathrm{L.S}}(F, \mathcal{B}, y),$$

where n is the number of negative eigenvalues of $\mathrm{D}F(p)$.

The degree of Benevieri-Furi

This degree is based upon an algebraic concept of orientation for linear Fredholm maps of index zero. Let $L : X \to Y$ be a Fredholm operator of index zero between two vector spaces (no additional structure is needed at this stage). Then, there exists a linear operator $A : X \to Y$ with finite rank, such that $L + A$ is an algebraic isomorphism[3]. An operator A having this property is called a corrector of L and the set of correctors of L is denoted by $C(L)$. On $C(L)$ is defined an equivalence relation in the following way. Given $A, B \in C(L)$, the automorphism of X:

$$T = (L + B)^{-1}(L + A) = I - (L + B)^{-1}(B - A)$$

is such that $I - T$ has finite rank. Hence, given any nontrivial finite dimensional subspace X_0 containing $\operatorname{rge}(I - T)$, the restriction of T to X_0 is an automorphism of X_0. Therefore, its determinant is well defined and nonzero, and furthermore it can be shown that it is independent of the choice of X_0. Accordingly, A is said to be $L-$equivalent to B if $\det\big((L+B)^{-1}(L+A)\big) > 0$. This is indeed an equivalence relation on $C(L)$ with just two equivalence classes. An orientation of L is then one among these two classes, and the elements of the chosen class are called the positive correctors of L.

When L is an isomorphism, the trivial operator 0 is indeed a corrector of L and therefore belongs to a class in $C(L)$, this class is called the natural orientation of L . However, if L is already oriented, the sign of L is defined to be 1 if the trivial operator belongs to the orientation of L and -1 otherwise.

Now let X and Y be Banach spaces. In this context, it is natural to consider only bounded correctors of bounded Fredholm operators of index zero. Because $GL(X, Y)$ is open in $\mathcal{L}(X, Y)$, it follows that if A is a corrector of $L \in \Phi_0(X, Y)$, then it is still a corrector of all operators L' sufficiently close to L. Furthermore, if A is a positive corrector of L, then L' can be oriented with the L-class to which A belongs. This stability property permits to define a concept of orientation for a continuous map $h : \Lambda \to \Phi_0(X, Y)$ where Λ is a topological space. An orientation of h is a continuous choice of an orientation $\alpha(\lambda)$ of $h(\lambda)$, where continuity means that for every $\lambda \in \Lambda$, there exists $A_\lambda \in \alpha(\lambda)$ which is a positive corrector of $h(\lambda')$ for all λ' in a neighborhood of λ. A map is then called orientable if it admits an orientation and oriented when an orientation is chosen.

In particular, let $\mathcal{O}$ be an open subset of X, and $F : \mathcal{O} \to Y$ be a C^1 Fredholm operator of index zero. An orientation of F is an orientation (if it exists) of the continuous map $\mathrm{D}F : \mathcal{O} \to \Phi_0(X, Y)$.

With this in mind, we may proceed to the definition of the degree for regular triples. Given an element $y \in Y$, an open subset $\mathcal{B} \subset \mathcal{O}$, and an oriented map $F : \mathcal{O} \to Y$, the triple $(F, \mathcal{B}, y)$ is called admissible if $F^{-1}(y) \cap \mathcal{B}$ is compact[4]. If y is a regular value of F then $F^{-1}(y) \cap \mathcal{B}$ is finite and the degree of $(F, \mathcal{B}, y)$ is

$$\deg_{\mathrm{B.F}}(F, \mathcal{B}, y) = \sum_{u \in F^{-1}(y) \cap \mathcal{B}} \operatorname{sign} \mathrm{D}F(u),$$

where, as in the general case, $\operatorname{sign} \mathrm{D}F(u) = 1$ if the trivial operator is a positive corrector of the oriented isomorphism $\mathrm{D}F(u)$, and $\operatorname{sign} \mathrm{D}F(u) = -1$ otherwise.

The assumption that y is a regular value is then removed by means of the classical Sard's theorem, after showing that given two neighborhoods U_1 and U_2 of $F^{-1}(y)$, one has $\deg_{\mathrm{B.F}}(F, U_1, y_2) = \deg_{\mathrm{B.F}}(F, U_2, y_2)$, for any regular values y_1, y_2 sufficiently close to y.

[3] Let A_1 be an isomorphism between $\ker L$ and some complement of $\operatorname{rge} L$, and let P be a projection from X onto $\ker L$. Then, one check that $L + A_1 P$ is an isomorphism. If in addition X and Y are Banach space and L is bounded, then by taking P continuous, we see that $L + A_1 P$ is in fact a topological isomorphism.

[4] It is called strongly admissible if F has a proper extension to the closure of $\mathcal{B}$ and $y \notin F(\partial\mathcal{B})$.

As expected, this degree satisfies the usual properties of existence, additivity and invariance under oriented homotopies. More precisely,

Definition 1.1 A homotopy of Fredholm maps of index zero from $\mathcal{O} \subset X$ to Y is a continuous map $h : [0,1] \times \mathcal{O} \to Y$, which is differentiable with respect to the second variable and such that, for any $(t,u) \in [0,1] \times \mathcal{O}$, the partial derivative $\mathrm{D}_u h(t,u) \in \Phi_0(X,Y)$, and furthermore the map $\mathrm{D}_u h : [0,1] \times \mathcal{O} \to \Phi_0(X,Y)$ is continuous. Then, an orientation of h is an orientation of the continuous map $\mathrm{D}_u h : [0,1] \times \mathcal{O} \to \Phi_0(X,Y)$.

The invariance of the degree under oriented homotopies is stated in the following form.

Let $h : [0,1] \times \mathcal{O} \to Y$ be an oriented homotopy and $y : [0,1] \to Y$ be a path. If the set $\{(t,u) \in [0,1] \times \mathcal{O} \mid h(t,u) = y(t)\}$ is compact, then $\deg_{\mathrm{B.F}}(h(t,.), \mathcal{O}, y(t))$ is well defined and does not depend on $t \in [0,1]$.

Note that the requirements on the homotopy are weaker than those in the Fitzpatrick-Pejsachowicz-Rabier theory, where the homotopy is required to be C^1 in both variables, and this is needed there in order to prove the homotopy variance of the degree at regular values via an approximation theorem -Theorem 2.1- (see the proof of Theorem 4.1 in [**29**]).

Now the following question arises: how can we check that a Fredholm map or a homotopy of Fredholm maps is orientable? Benevieri and Furi give the following answer (which is a consequence of Theorem 3.11 in [**4**]). Any continuous map $h : \Lambda \to \Phi_0(X,Y)$ is orientable provided that Λ is simply connected and locally path connected. In particular the map $\mathrm{D}F : \mathcal{O} \to \Phi_0(X,Y)$ is orientable if $\mathcal{O}$ is simply connected (since $\mathcal{O}$, being an open subset of a Banach space is locally path connected).

For the orientation of a homotopy, another answer is given by Theorem 4.3 in [**4**] which ensures that a homotopy $h : [0,1] \times \mathcal{O} \to Y$, is orientable if and only if for some $t_0 \in [0,1]$, $h(t_0,.) : \mathcal{O} \to Y$ is orientable. In this case, an orientation of $h(t_0,.)$ is the restriction of a unique orientation of h.

The two degree theories that we have just sketched are strongly connected to each other, and an instructive comparison can already be found in [**4**], §5. We mention, in particular, the following (Proposition 5.6 in [**4**]). For an oriented path $\gamma : [0,1] \to \Phi_0(X,Y)$ with invertible endpoints $\gamma(0)$ and $\gamma(1)$ we have

$$\sigma(\gamma) = \operatorname{sign}\gamma(0)\operatorname{sign}\gamma(1).$$

As a consequence, when $(F, \mathcal{B}, y)$ is admissible and regular in both theories, and p is a base point for F,

$$\begin{aligned}
\deg_p(F,\mathcal{B},y) &= \sum_{u \in F^{-1}(y)\cap\mathcal{B}} \sigma(\mathrm{D}F \circ \gamma_u) \quad \text{where } \gamma_u \text{ is a path joining } p \text{ to } u \\
&= \sum_{u \in F^{-1}(y)\cap\mathcal{B}} \operatorname{sign}\mathrm{D}F(p)\operatorname{sign}\mathrm{D}F(u) \\
&= \operatorname{sign}\mathrm{D}F(p) \sum_{u \in F^{-1}(y)\cap\mathcal{B}} \operatorname{sign}\mathrm{D}F(u) \\
&= \operatorname{sign}\mathrm{D}F(p)\deg_{\mathrm{B.F}}(F,\mathcal{B},y),
\end{aligned}$$

and from the general definition of each degree, this relation continues to hold for any triple which is admissible in both theories.

Thus, we can draw the following picture. Each of the above degrees is defined on a class of admissible triples. The two classes are distinct but not disjoint, and on the intersection of these classes, the degrees coincide up to a sign. The intersection -in which we are interested- contains the triples $(F, \mathcal{B}, y)$ satisfying

(a) F is a continuously differentiable Fredholm map of index zero from an open connected and simply connected subset $\mathcal{O} \subset X$, with values in Y.

(b) $\mathcal{B} \subset \mathcal{O}$ is open.

(c) F has a proper extension to $\overline{\mathcal{B}}$.

(d) $y \notin F(\partial\mathcal{B})$.

(e) F has a base point $p \in \mathcal{O}$.

In the light of the above, Theorem 1.1 (i) remains true under a weaker assumption on the homotopy. We see this as a theorem on the base point degree (but from outside the theory).

Theorem 1.2 *Let $h : [0,1] \times \mathcal{O} \to Y$ be a homotopy of Fredholm maps of index 0, which is proper on $[0,1] \times \overline{\mathcal{B}}$. Suppose that $y \notin h([0,1] \times \partial\mathcal{B})$ and $p_0, p_1 \in \mathcal{O}$ are respectively base points for $h(0,.)$ and $h(1,.)$, then*

$$\deg_{p_0}(h(0,.),\mathcal{B},y) = \sigma(\mathrm{D}_u h \circ \gamma) \cdot \deg_{p_1}(h(1,.),\mathcal{B},y),$$

where γ is any curve joining $(0,p_0)$ to $(1,p_1)$ in $[0,1] \times \mathcal{O}$.

Proof. Note first that the map $h(0,.) : \mathcal{O} \to Y$ is orientable in the sense of Benevieri-Furi because $\mathcal{O}$ is simply connected. Therefore h is orientable (Theorem 4.3 in **[4]**)[5]. Choose accordingly an orientation of h. Then

$$\deg_{p_0}(h(0,.),\mathcal{B},y) = \mathrm{sign}\,(\mathrm{D}_u h(0,p_0))\deg_{\mathrm{B.F}}(h(0,.),\mathcal{B},y).$$

Now the set

$$\{(t,u) \in [0,1] \times \mathcal{B} \mid h(t,u) = y\} = h^{-1}(y) \cap \big([0,1] \times \mathcal{B}\big) = h^{-1}(y) \cap \big([0,1] \times \overline{\mathcal{B}}\big)$$

is compact by the properness assumption on h. Therefore, by the invariance of the Benevieri-Furi degree under oriented homotopies

$$\deg_{\mathrm{B.F}}(h(0,.),\mathcal{B},y) = \deg_{\mathrm{B.F}}(h(1,.),\mathcal{B},y).$$

But $\deg_{\mathrm{B.F}}(h(1,.),\mathcal{B},y) = \mathrm{sign}\,\mathrm{D}_u h(1,p_1)\ \deg_{p_1}(h(1,.),\mathcal{B},y)$, and so

$$\begin{aligned}\deg_{p_0}(h(0,.),\mathcal{B},y) &= \mathrm{sign}\,\mathrm{D}_u h(0,p_0)\,\mathrm{sign}\,\mathrm{D}_u h(1,p_1)\deg_{p_1}(h(1,.),\mathcal{B},y)\\ &= \sigma(\mathrm{D}_u h \circ \gamma)\deg_{p_1}(h(1,.),\mathcal{B},y),\end{aligned}$$

where γ is any curve joining $(0,p_0)$ to $(1,p_1)$ in $[0,1] \times \mathcal{O}$. $\square$

We end by some remarks. First, we mention that the degree of Fitzpatrick-Pejsachowicz-Rabier is available for a class of admissible triples larger than we presented. As in the Benevieri-Furi theory, for the admissibility of a triple $(F,\mathcal{B},y)$, the properness assumption, and the condition $y \notin F(\partial\mathcal{B})$ can be replaced by the requirement that $F^{-1}(y) \cap \mathcal{B}$ is compact. This is discussed in section 7 of **[29]**.

Another extension is discussed in section 8, where it is shown that the assumption on the simple connectedness of $\mathcal{O}$ can be removed provided that one considers oriented maps (in the sense of Fitzpatrick-Pejsachowicz-Rabier). See **[4]** §5 for a comparison.

[5] This also follows from the fact that $[0,1] \times \mathcal{O}$ is simply connected and path connected.

Chapter 2

Fredholm and properness properties of quasilinear second order elliptic systems

Before beginning the study of Fredholm and properness properties, we need first to introduce some notation which will be used in the rest of this work.

Notation

Let N and m be two integers ≥ 1. N will always denote the dimension of the space of the independent variable i.e. $\mathbb{R}^N$, and m the dimension of the system (m equations with m unknown functions). The real number p (which appears in the Sobolev spaces L^p and $W^{2,p}$) will always satisfy $N < p < \infty$. The elements of $\mathbb{R}^m$ are viewed as columns.

Let Ω be an open subset of $\mathbb{R}^N$, with bounded and Lipschitz continuous boundary $\partial\Omega$. For a vector-valued function $u = \left(u^1, u^2, \ldots, u^m\right)^T : \Omega \to \mathbb{R}^m$, we denote by $\partial_i u$ the vector $(\partial_i u^1, \ldots, \partial_i u^m)$ and by ∇u the $m \times N$ matrix with columns $\partial_i u$, $i = 1 \ldots, N$. For convenience, we set $\partial_0 u = u$.

If $z_1, z_2 \in \mathbb{R}^m$, and A is an $m \times m$ matrix, $z_1 \cdot z_2$ denotes the scalar product of z_1 and z_2, and Az_1 denotes the usual matrix-vector multiplication. For any integer $d \geq 1$, the Euclidean norm on $\mathbb{R}^d$ is denoted by $|\cdot|$.

In the sequel we deal with functions $f : \Omega \times \left(\mathbb{R}^m \times \mathbb{R}^{m\times N}\right) \to \mathbb{R}$. The arguments of f will often be denoted by $x \in \Omega$ and $\xi \in \mathbb{R}^m \times \mathbb{R}^{m\times N}$. Whenever we need to display the components of $\xi \in \mathbb{R}^m \times \mathbb{R}^{m\times N}$, we shall write $\xi = (\xi_0, \xi')$, where

$$\xi' = [\xi_1 \; \xi_2 \; \cdots \; \xi_N] \,, \text{ with } \; \xi_k = \begin{pmatrix} \xi_k^1 \\ \vdots \\ \xi_k^m \end{pmatrix} \text{ for } k = 0, \ldots, N.$$

When we deal with Nemytskii operators, we form the expression $f(x, u(x), \nabla u(x))$, in which $u(x)$ takes the place of ξ_0, and $\partial_1 u(x), \ldots \partial_N u(x)$ take the place of $\xi_1, \ldots, \xi_N$ respectively.

Remark 2.1 The space $\mathbb{R}^{m\times N}$ of $m \times N$ matrices is indeed isomorphic to the space $\mathbb{R}^{mN}$ and therefore it is possible to consider ∇u as an element of $\mathbb{R}^{mN}$. This notation was used in **[17]**. What is important in both cases is the block decomposition of ∇u which simplifies the subsequent calculations and statements, by avoiding having to handle a lot of indices.

The Nemytskii operator generated by f: $\;u \mapsto f(., u, \nabla u)$ will be denoted by $\boldsymbol{f}$ i.e.

$$\boldsymbol{f}(u)(x) = f(x, u(x), \nabla u(x)).$$

We denote by

$$\nabla_{\xi_k} f(x,\xi) = \left(\partial_{\xi_k^1}, \dots, \partial_{\xi_k^m}\right) f(x,\xi),$$

the partial gradient with respect to the ξ_k block variable. $\nabla_\xi f$ is the gradient of f with respect to ξ.

We use the standard notation for the Lebesgue and Sobolev spaces. Let $O \subset \mathbb{R}^N$ be an open set, $l \in \mathbb{N}$, $q \in [1,\infty]$, the norm in $\left(W^{l,q}(O)\right)^m = W^{l,q}(O, \mathbb{R}^m)$ is the norm in a Cartesian product of Banach spaces and will be denoted by $\|u\|_{l,q,O}$, (i.e. if $u = (u^1, \dots, u^m) \in W^{l,q}(O; \mathbb{R}^m)$, then $\|u\|_{l,q,O} = \|u^1\|_{W^{l,q}(O)} + \dots + \|u^m\|_{W^{l,q}(O)}$). To simplify the writing we often use: $Y_p(\Omega) = (L^p(\Omega))^m$, and $X_p(\Omega) = \left(W^{2,p}(\Omega)\right)^m$, when $\Omega = \mathbb{R}^N$, we write X_p and Y_p.

To deal with the Dirichlet problem, we introduce the space $D_p(\Omega) = \left(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\right)^m$. Note that $D_p(\Omega)$ is a closed subspace of $X_p(\Omega)$ and so it is also reflexive. Finally $D_p(\mathbb{R}^N) = X_p(\mathbb{R}^N) = X_p$.

$C^1(\overline{\Omega})$ is the subspace of $C^0(\overline{\Omega}) \cap C^1(\Omega)$ of the functions v for which ∇v has a continuous extension to $\overline{\Omega}$. We also use the space $C_d^1(\overline{\Omega})$ introduced in **[31]**.

$$C_d^1(\overline{\Omega}) = \left\{ v \in C^1(\overline{\Omega}) : \lim_{x\in\Omega, |x|\to\infty} |v(x)| = \lim_{x\in\Omega, |x|\to\infty} |\nabla v(x)| = 0 \right\}.$$

This is a Banach space for the norm: $\max\left(\max_{x\in\overline{\Omega}} |v(x)|, \max_{x\in\overline{\Omega}} |\nabla v(x)|\right)$. Note also that $C_d^1(\overline{\Omega}) \subset W^{1,\infty}(\Omega)$ and $C_d^1(\overline{\Omega}) = C^1(\overline{\Omega})$ when Ω is bounded. Some important properties of the spaces used here are recalled in the appendix.

Remarks on the domain Ω

Ω will always have a bounded and Lipschitz boundary $\partial\Omega$ (possibly empty). So that Ω can be a bounded domain, an exterior domain, or $\mathbb{R}^N$ itself. In the main results of this work, it is furthermore assumed that Ω has a C^2 boundary, and it is explicitly mentioned. This implies some remarks that will be useful later.

Remark 2.2 We have two cases: either Ω is bounded or not. If Ω is unbounded, then necessarily $\complement^\Omega$ is bounded. This is due to the boundedness of the boundary. Indeed, let B_r be a ball containing $\partial\Omega$, we claim that B_r contains $\complement^\Omega$. If not, there is a point $x \in \complement^\Omega \cap \complement^{B_r}$. Since Ω is unbounded, there is $y \in \Omega \cap \complement^{B_r}$. Now recall that $\complement^{B_r}$ is path connected, so we can join x to y by a path in $\complement^{B_r}$. This path joining an exterior point to an interior point of Ω, should meet the boundary, but it does not since the boundary lies inside the ball B_r. Therefore $K = \complement^\Omega$ is bounded (compact).

Remark 2.3 Let Ω be unbounded. For every ball B_r containing $\partial\Omega$ we have

$$\partial\left(\Omega \cap B_r\right) = \partial\Omega \cup \partial B_r.$$

Proof. We clearly have $\overline{\Omega \cap B_r} \subset \overline{\Omega} \cap \overline{B}_r$. Let us prove the reverse inclusion. Let $x \in \overline{\Omega} \cap \overline{B}_r$. Then either (i) $x \in B_r$ or (ii) $x \in \partial B_r$. Let first $x \in B_r$ and V be an open neighborhood of x. If $V \cap (\Omega \cap B_r) = \varnothing$, then $x \in V \cap B_r \subset K := \complement^\Omega$ which means that x is an interior point of K. But this is impossible since

$$x \in \overline{\Omega} = \overline{\complement^K} = \complement^{\overset{\circ}{K}}.$$

Therefore $x \in \overline{\Omega \cap B_r}$. Next if $x \in \partial B_r \subset \complement^{B_r} \subset \Omega$, then for all $\varepsilon > 0$ sufficiently small, $B(x,\varepsilon) \subset \Omega$. Now clearly $B(x,\varepsilon) \cap B_r \neq \varnothing$ and so $B(x,\varepsilon) \cap (\Omega \cap B_r) \neq \varnothing$, and once again $x \in \overline{\Omega \cap B_r}$. Finally $\overline{\Omega \cap B_r} = \overline{\Omega} \cap \overline{B}_r$.

On the other hand,

$$\overline{\complement^{\Omega\cap B_r}} = \overline{\complement^{\Omega}} \cup \overline{\complement^{B_r}}.$$

So

$$\partial(\Omega\cap B_r) = \overline{\Omega}\cap\overline{\complement^{\Omega}}\cap\overline{B}_r \bigcup \overline{B}_r\cap\overline{\complement^{B_r}}\cap\overline{\Omega} = \partial\Omega\cap\overline{B}_r\bigcup\partial B_r\cap\overline{\Omega} = \partial\Omega\bigcup\partial B_r,$$

since $\partial\Omega\subset B_r\subset\overline{B}_r$ and $\partial B_r\subset\complement^{B_r}\subset\Omega\subset\overline{\Omega}$. □

Remark 2.4 The connection between the Dirichlet problem and the space

$$D_p(\Omega) = \left(W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\right)^m$$

introduced above is given by the following theorem (see Brézis **[6]** Théorème IX. 17 for example). Let Ω have a C^1 boundary and $u\in W^{1,q}(\Omega)\cap C(\overline{\Omega})$ with $1\le q<\infty$. Then the following conditions are equivalent.

(i) $u=0$ on $\partial\Omega$.

(ii) $u\in W_0^{1,q}(\Omega)$.

2.1 Smoothness of some Nemytskii operators

Our first task is to make sure that the operator in (1.2) maps $W^{2,p}(\Omega,\mathbb{R}^m)$ to $L^p(\Omega,\mathbb{R}^m)$ and has enough smoothness for the subsequent discussion. Therefore it is necessary to study the smoothness of the Nemytskii operators $u\mapsto b(.,u,\nabla u)$ and $u\mapsto a_{\alpha\beta}(.,u,\nabla u)$, entering in F. This leads us to consider maps of the type $f:\Omega\times\left(\mathbb{R}^m\times\mathbb{R}^{m\times N}\right)\to\mathbb{R}^d$. Note that if $f=(f^1,\ldots,f^d)$ and each component f^j gives rise to a Nemytskii operator $\boldsymbol{f}^j$, then the Nemytskii operator associated with f is $\boldsymbol{f}=(\boldsymbol{f}^1,\ldots,\boldsymbol{f}^d)$, and any smoothness property of $\boldsymbol{f}$ is equivalent to the same property of each component. So it is sufficient to study scalar-valued maps. It is clear that the smoothness of a Nemytskii operator generated by $f:\Omega\times\left(\mathbb{R}^m\times\mathbb{R}^{m\times N}\right)\to\mathbb{R}$ should be derived from smoothness assumptions on f. In this work as in **[31]**, the following property of equicontinuity plays an important role (M and d are two integers ≥ 1).

Definition 2.1 We say that $f:\Omega\times\mathbb{R}^M\to\mathbb{R}^d$ is an equicontinuous C^0-bundle map if f is continuous and the collection $(f(x,.))_{x\in\Omega}$ is equicontinuous at every point of $\mathbb{R}^M$. If $k\ge 0$ is an integer, we say that f is an equicontinuous $C^k_\xi-$bundle map if the partial derivatives $\mathrm{D}^\gamma_\xi f$, $|\gamma|\le k$, exist and are equicontinuous C^0-bundle maps.

We recall that equicontinuity of $(f(x,.))_{x\in\Omega}$ at a point $\eta_0\in\mathbb{R}^M$ means that for all $\varepsilon>0$ there is $\delta=\delta(\eta_0,\varepsilon)>0$ such that $|\eta-\eta_0|\le\delta\Rightarrow|f(x,\eta)-f(x,\eta_0)|<\varepsilon$ for all $x\in\Omega$. When δ can be chosen independently of η_0 for η_0 in some set B, we have uniform equicontinuity on B.

Note that $f=(f^1,\ldots,f^d)$ is an equicontinuous $C^k_\xi-$bundle map if and only if each component f^j is an equicontinuous $C^k_\xi-$bundle map. Note also that a sum of equicontinuous $C^k_\xi-$bundle maps, is an equicontinuous $C^k_\xi-$bundle map.

Now we give some important properties and examples of equicontinuous $C^k_\xi-$bundle maps.

Lemma 2.1 *Let $f:\Omega\times\mathbb{R}^M\to\mathbb{R}^d$ be an equicontinuous C^0-bundle map. Then we have the following.*

(i) *The collection $(f(x,.))_{x\in\Omega}$ is uniformly equicontinuous on the compact subsets of $\mathbb{R}^M$.*

(ii) *If A is a measurable subset of Ω and $f(.,0) \in L^\infty(A)$, the collection $(f(x,.))_{x\in A}$ is equibounded on the bounded subsets of* $\mathbb{R}^M$.

Proof. (i) If not, there exist a compact set $K \subset \mathbb{R}^M$, $\varepsilon_0 > 0$, and 3 sequences $(x_n) \subset \Omega$, $(\xi_n), (\eta_n) \subset K$ such that for all $n \in \mathbb{N}$,

$$|\xi_n - \eta_n| \leq \frac{1}{n} \text{ and } |f(x_n, \xi_n) - f(x_n, \eta_n)| \geq \varepsilon_0.$$

But (ξ_n) belongs to a compact set, so it contains a subsequence $(\xi_{\varphi(n)})$ converging to some ξ, which also implies that $\eta_{\varphi(n)} \to \xi$. By the equicontinuity of $(f(x,.))_x$ at ξ we have, for all n large enough,

$$|f(x_{\varphi(n)}, \xi_{\varphi(n)}) - f(x_{\varphi(n)}, \xi)| < \frac{\varepsilon_0}{4} \text{ and } |f(x_{\varphi(n)}, \eta_{\varphi(n)}) - f(x_{\varphi(n)}, \xi)| < \frac{\varepsilon_0}{4},$$

and therefore $|f(x_{\varphi(n)}, \xi_{\varphi(n)}) - f(x_{\varphi(n)}, \eta_{\varphi(n)})| < \frac{\varepsilon_0}{2}$, a contradiction.

(ii) Let K be a bounded subset of $\mathbb{R}^M$, and B be a closed ball in $\mathbb{R}^M$ containing 0 and K. By part (i), there is $\delta > 0$ such that, for all $x \in \Omega$, $|f(x,\xi) - f(x,\eta)| < 1$ whenever $|\xi - \eta| \leq \delta$ and $\xi, \eta \in B$. For any $\xi \in K$, one can divide the segment joining 0 to ξ into $[|\xi|/\delta] + 1$ segments of length not greater than δ. Thus for $x \in A$,

$$|f(x,\xi)| \leq |f(x,0)| + |f(x,\xi) - f(x,0)| < \|f(.,0)\|_{L^\infty(A)} + \left[\frac{|\xi|}{\delta}\right] + 1.$$

But $|\xi|$ is bounded by the diameter of B, so the proof is complete. □

Remark 2.5 Let $g : \Omega \times \left(\mathbb{R}^m \times \mathbb{R}^{m\times N}\right) \to \mathbb{R}^m$ be an equicontinuous C^0-bundle map, such that $g(.,0) \in L^\infty(\Omega, \mathbb{R}^m)$. Then, for $i = 0, \ldots, N$, $(x,\xi) \mapsto g(x,\xi) \cdot \xi_i$ is a scalar-valued equicontinuous C^0-bundle map.

Proof. Fix $\eta \in \mathbb{R}^m \times \mathbb{R}^{m\times N}$. Then

$$g(x,\xi) \cdot \xi_i - g(x,\eta) \cdot \eta_i = g(x,\xi) \cdot (\xi_i - \eta_i) + (g(x,\xi) - g(x,\eta)) \cdot \eta_i.$$

The result follows from the equicontinuity of $(g(x,.))_{x\in\Omega}$ at η, and its equiboundedness on bounded subsets of $\mathbb{R}^m \times \mathbb{R}^{m\times N}$ (Lemma 2.1 (ii)). □

Remark 2.6 Let $f : \Omega \times \left(\mathbb{R}^m \times \mathbb{R}^{m\times N}\right) \to \mathbb{R}$ be an equicontinuous $C^1_\xi-$bundle map. Define $g : \Omega \times \left(\mathbb{R}^m \times \mathbb{R}^{m\times N}\right) \to \mathbb{R}^m$ by

$$g(x,\xi) = \int_0^1 \nabla_{\xi_i} f(x, t\xi)\, \mathrm{d}t.$$

Then g is an equicontinuous C^0-bundle map.

Proof. Fix $\eta \in \mathbb{R}^m \times \mathbb{R}^{m\times N}$. Then

$$g(x,\xi) - g(x,\eta) = \int_0^1 \left(\nabla_{\xi_i} f(x,t\xi) - \nabla_{\xi_i} f(x,t\eta)\right) \mathrm{d}t.$$

If $|\xi - \eta| \leq 1$ then $t\xi$ and $t\eta$ belong to the closed ball with center 0 and radius $|\eta| + 1$. Thus the conclusion follows from Lemma 2.1 (i) applied to $\nabla_{\xi_i} f$. □

Remark 2.7 If f is of class C^k and $f(.,\xi)$ is periodic in x with period $T=(T_1,\dots,T_N)$ for every $\xi\in \mathbb{R}^m\times\mathbb{R}^{m\times N}$, then f is an equicontinuous $C^k_\xi-$bundle map. This follows from the uniform continuity of $\mathrm{D}^\gamma_\xi f$ on $[0,T_1]\times\ldots\times[0,T_N]\times K$ for every compact $K\subset\mathbb{R}^m\times\mathbb{R}^{m\times N}$ and $|\gamma|\le k$ (see §2.5).

Lemma 2.2 *Let $f:\Omega\times\mathbb{R}^m\times\mathbb{R}^{m\times N}\to\mathbb{R}$ be an equicontinuous C^0-bundle map. Suppose that $f(.,0)\in L^\infty(\Omega)$. Then the Nemytskii operator $\boldsymbol{f}$ has the following properties.*

(i) *It is well defined and continuous from $\left(C^1_d(\overline{\Omega})\right)^m$ to $L^\infty(\Omega)$.*

(ii) *It is well defined and continuous from $\left(W^{2,p}(\Omega)\right)^m$ to $L^\infty(\Omega)$ and maps bounded subsets onto bounded subsets.*

(iii) *If Ω is bounded, it is completely continuous from $\left(W^{2,p}(\Omega)\right)^m$ to $L^\infty(\Omega)$ (hence also to $L^q(\Omega), 1\le q\le\infty$).*

(iv) *The multiplication $(u,v)\in\left(W^{2,p}(\Omega)\right)^m\times L^p(\Omega)\mapsto f(.,u,\nabla u)v\in L^p(\Omega)$ is weakly sequentially continuous.*

Proof. (i) If $u\in C^1_d(\overline{\Omega},\mathbb{R}^m)$, the function $x\in\overline{\Omega}\to f(x,u(x),\nabla u(x))$ is continuous and hence measurable. From the boundedness of u and ∇u on $\overline{\Omega}$, there is a bounded subset $K\subset\mathbb{R}^m\times\mathbb{R}^{m\times N}$ containing $(u(x),\nabla u(x))$ for all $x\in\overline{\Omega}$. Therefore by Lemma 2.1 (ii) there is a constant $M_K>0$ such that $|f(x,u(x),\nabla u(x))|\le M_K\ \forall x\in\overline{\Omega}$. This means that $f(.,u,\nabla u)\in L^\infty(\Omega)$.

To prove the continuity, let $u_n,u\in C^1_d(\overline{\Omega},\mathbb{R}^m)$ and $u_n\to u$ in $C^1_d(\overline{\Omega},\mathbb{R}^m)$. Then since $\{u\}\cup\{u_n,n\in\mathbb{N}\}$ is compact and hence bounded in $C^1_d(\overline{\Omega},\mathbb{R}^m)$, there is a compact $K\subset\mathbb{R}^m\times\mathbb{R}^{m\times N}$ containing $(u(x),\nabla u(x))$ and $(u_n(x),\nabla u_n(x))$ for all $x\in\overline{\Omega}$ and $n\in\mathbb{N}$. But $|(u_n(x),\nabla u_n(x))-(u(x),\nabla u(x))|$ can be made arbitrary small uniformly in $x\in\overline{\Omega}$, for n large enough, so by Lemma 2.1 (i), given $\varepsilon>0$, we have

$$|f(x,u_n(x),\nabla u_n(x))-f(x,u(x),\nabla u(x))|\le\varepsilon\quad\forall x\in\overline{\Omega}.$$

Lastly if $\mathcal{B}\subset C^1_d(\overline{\Omega},\mathbb{R}^m)$ is bounded, there is a bounded subset $K\subset\mathbb{R}^m\times\mathbb{R}^{m\times N}$ containing $(u(x),\nabla u(x))$ for all $x\in\overline{\Omega}$ and $u\in\mathcal{B}$. The boundedness of $\boldsymbol{f}(\mathcal{B})$ follows from Lemma 2.1 (ii).

(ii) Follows from the imbedding $W^{2,p}(\Omega,\mathbb{R}^m)\hookrightarrow C^1_d(\overline{\Omega},\mathbb{R}^m)$.

(iii) The above imbedding is compact when Ω is bounded.

(iv) Let $u_n\rightharpoonup u$ in $W^{2,p}(\Omega,\mathbb{R}^m)$ and $v_n\rightharpoonup v$ in $L^p(\Omega)$. From part (ii), the sequence $(\boldsymbol{f}(u_n))$ is bounded in $L^\infty(\Omega)$, and hence $(\boldsymbol{f}(u_n)v_n)$ is bounded in $L^p(\Omega)$. Let $\Omega'\subset\Omega$ be any open ball. By part (iii), $\boldsymbol{f}(u_n)|_{\Omega'}\to\boldsymbol{f}(u)|_{\Omega'}$ in $L^\infty(\Omega')$, which implies $\boldsymbol{f}(u_n)v_n|_{\Omega'}\rightharpoonup\boldsymbol{f}(u)v|_{\Omega'}$ in $L^p(\Omega')$. Now if a subsequence of $(\boldsymbol{f}(u_n)v_n)$ converges weakly to w in $L^p(\Omega)$ and hence in $L^p(\Omega')$, we have $w|_{\Omega'}=\boldsymbol{f}(u)v|_{\Omega'}$, and therefore $w=\boldsymbol{f}(u)v$ since the ball is arbitrary. This means that $(\boldsymbol{f}(u_n)v_n)$ has a unique weak cluster point, which yields $\boldsymbol{f}(u_n)v_n\rightharpoonup\boldsymbol{f}(u)v$ in $L^p(\Omega)$, by Note A1 of the appendix. □

Lemma 2.3 *Let $f:\Omega\times\mathbb{R}^m\times\mathbb{R}^{m\times N}\to\mathbb{R}$ have the form*

$$f(x,\xi)=f_0(x)+\sum_{i=0}^N g_i(x,\xi)\cdot\xi_i \tag{2.1}$$

where g_i is a C^0-bundle map, with $g_i(.,0)\in L^\infty(\Omega,\mathbb{R}^m)$, $0\le i\le N$. Suppose that $f_0\in L^p(\Omega)$. In particular, the above conditions hold if f is a $C^1_\xi-$bundle map with $f(.,0)\in L^p(\Omega)$ and $\nabla_\xi f(.,0)$ bounded in Ω. Then the Nemytskii operator has the following properties.

(i) *It is well defined and continuous from* $W^{2,p}(\Omega, \mathbb{R}^m)$ *to* $L^p(\Omega)$ *and maps bounded subsets onto bounded subsets.*

(ii) *It is weakly sequentially continuous from* $W^{2,p}(\Omega, \mathbb{R}^m)$ *to* $L^p(\Omega)$.

Proof. To see the "in particular", note that for an equicontinuous C^1_ξ−bundle map f, one can write

$$\begin{aligned} f(x,\xi) - f(x,0) &= \int_0^1 \frac{\partial}{\partial t} f(x,t\xi)\,\mathrm{d}t \\ &= \int_0^1 \sum_{i=0}^N \nabla_{\xi_i} f(x,t\xi)\cdot \xi_i\,\mathrm{d}t \\ &= \sum_{i=0}^N \left(\int_0^1 \nabla_{\xi_i} f(x,t\xi)\,\mathrm{d}t \right)\cdot \xi_i. \end{aligned} \tag{2.2}$$

Take

$$g_i(x,\xi) = \int_0^1 \nabla_{\xi_i} f(x,t\xi)\,\mathrm{d}t,$$

then by Remark 2.6, g_i is an equicontinuous C^0−bundle map. Furthermore $g_i(.,0) = \nabla_{\xi_i} f(.,0) \in L^\infty(\Omega, \mathbb{R}^m)$.

(i) Applying Lemma 2.2 (ii) to each component of g_i, we have that $\boldsymbol{g}_i : W^{2,p}(\Omega, \mathbb{R}^m) \to L^\infty(\Omega, \mathbb{R}^m)$ is continuous and maps bounded subsets onto bounded ones. As a result, the operator (recall that $\partial_0 u = u$)

$$u \mapsto \sum_{i=0}^N \boldsymbol{g}_i(u)\cdot \partial_i u \ \in L^p(\Omega) \tag{2.3}$$

is continuous and maps bounded subsets onto bounded ones. By (2.1), this is $\boldsymbol{f} - f_0$, and the conclusion follows from the assumption $f_0 \in L^p(\Omega)$.

(ii) Let $u_n \rightharpoonup u$ in $W^{2,p}(\Omega, \mathbb{R}^m)$. By part (i), $(\boldsymbol{f}(u_n))$ is bounded in $L^p(\Omega)$. Let $\Omega' \subset \Omega$ be an open ball. Since $f - f_0$ is an equicontinuous C^0−bundle map (see Remark 2.5) and vanishes when $\xi = 0$, Lemma 2.2 (iii) applies and yields $\boldsymbol{f}(u_n)|_{\Omega'} \to \boldsymbol{f}(u)|_{\Omega'}$ in $L^p(\Omega')$. Now if a subsequence of $(\boldsymbol{f}(u_n))$ converges weakly to some w in $L^p(\Omega)$ and hence in $L^p(\Omega')$, we have $w|_{\Omega'} = \boldsymbol{f}(u)|_{\Omega'}$, and therefore $w = \boldsymbol{f}(u)$ since the ball is arbitrary. This means that $(\boldsymbol{f}(u_n))$ has a unique weak cluster point, and thus $\boldsymbol{f}(u_n) \rightharpoonup \boldsymbol{f}(u)$ in $L^p(\Omega)$. □

Theorem 2.1 *Let* $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times N} \to \mathbb{R}$ *be an equicontinuous* C^1_ξ*−bundle map. Suppose that* $f(.,0) \in L^\infty(\Omega)$ *(resp.* $f(.,0) \in L^p(\Omega)$*) and that* $\nabla_\xi f(.,0)$ *is bounded on* Ω*. Then the Nemytskii operator* $\boldsymbol{f}$ *is of class* C^1 *from* $W^{2,p}(\Omega, \mathbb{R}^m)$ *to* $L^\infty(\Omega)$*, (resp* $L^p(\Omega)$*) with derivative*

$$\mathrm{D}\boldsymbol{f}(u)v = \sum_{i=0}^N \nabla_{\xi_i} f(.,u,\nabla u)\cdot \partial_i v. \tag{2.4}$$

Furthermore $\mathrm{D}\boldsymbol{f}$ *is bounded on the bounded subsets of* $W^{2,p}(\Omega, \mathbb{R}^m)$*, and hence* $\boldsymbol{f}$ *is uniformly continuous on these subsets.*

Proof. Define for $u \in W^{2,p}(\Omega, \mathrm{IR}^m)$,

$$T_u v = \sum_{i=0}^{N} \nabla_{\xi_i} f(., u, \nabla u) \cdot \partial_i v.$$

By Lemma 2.2 (ii) applied to each component of $\nabla_{\xi_i} f$, we have $\nabla_{\xi_i} f(., u, \nabla u)$ is bounded on Ω. Thus

$$\|T_u v\|_{0,p,\Omega} \leq \sum_{i=0}^{N} \|\nabla_{\xi_i} f(., u, \nabla u)\|_{0,\infty,\Omega} \|\partial_i v\|_{0,p,\Omega} \leq \text{const.} \times \|v\|_{2,p,\Omega}$$

and

$$\|T_u v\|_{0,\infty,\Omega} \leq \text{const.} \times \|v\|_{1,\infty,\Omega} \leq \text{const.} \times \|v\|_{2,p,\Omega}.$$

Therefore T_u is linear and bounded from $W^{2,p}(\Omega, \mathrm{IR}^m)$ to $L^p(\Omega)$ and to $L^\infty(\Omega)$.

Note that

$$\begin{aligned} f(., u+v, \nabla(u+v)) - f(., u, \nabla u) &= \int_0^1 \frac{\partial}{\partial t} f(., u+tv, \nabla u + t\nabla v)\, \mathrm{d}t \\ &= \sum_{i=0}^{N} \left(\int_0^1 \nabla_{\xi_i} f(., u+tv, \nabla u + t\nabla v)\, \mathrm{d}t \right) \cdot \partial_i v. \end{aligned}$$

So

$$\begin{aligned} &f(., u+v, \nabla(u+v)) - f(., u, \nabla u) - T_u v \\ &\qquad = \sum_{i=0}^{N} \left(\int_0^1 \nabla_{\xi_i} f(., u+tv, \nabla u + t\nabla v) - \nabla_{\xi_i} f(., u, \nabla u)\, \mathrm{d}t \right) \cdot \partial_i v. \end{aligned}$$

Thus if we define

$$k_{u,i}(x, \xi) := \int_0^1 (\nabla_{\xi_i} f(\, x, u(x) + t\xi_0, \nabla u(x) + t\xi'\,) - \nabla_{\xi_i} f(x, u(x), \nabla u(x)))\, \mathrm{d}t$$

where $\xi' = [\xi_1 \ \cdots \ \xi_N]$, we get

$$f(., u+v, \nabla(u+v)) - f(., u, \nabla u) - T_u v = \sum_{i=0}^{N} k_{u,i}(., v, \nabla v) \cdot \partial_i v. \tag{2.5}$$

Now, one can check as in Remark 2.6, that $k_{u,i}$ is an equicontinuous C^0-bundle map satisfying $k_{u,i}(., 0) = 0 \in L^\infty(\Omega, \mathrm{IR}^m)$. Therefore by Lemma 2.2 (ii) applied to each component of $k_{u,i}$ we have that $\boldsymbol{k}_{u,i}$ is continuous from $W^{2,p}(\Omega, \mathrm{IR}^m)$ to $L^\infty(\Omega, \mathrm{IR}^m)$. So, given $\varepsilon > 0$, we have that $\|\boldsymbol{k}_{u,i}(v)\|_{0,\infty,\Omega} \leq \varepsilon$ provided $\|v\|_{2,p,\Omega}$ is small enough.

Now, if $f(., 0) \in L^\infty(\Omega)$, $\boldsymbol{f}$ maps $W^{2,p}(\Omega, \mathrm{IR}^m)$ to $L^\infty(\Omega)$ (Lemma 2.2 (ii)). By (2.5), we obtain

$$\|\boldsymbol{f}(u+v) - \boldsymbol{f}(u) - T_u v\|_{0,\infty,\Omega} \leq \text{const.} \times \varepsilon \|v\|_{1,\infty,\Omega} \leq \text{const.} \times \varepsilon \|v\|_{2,p,\Omega},$$

which means that $\boldsymbol{f}$ is differentiable and $\mathrm{D}\boldsymbol{f}(u) = T_u$.

If $f(., 0) \in L^p(\Omega)$, $\boldsymbol{f}$ maps $W^{2,p}(\Omega, \mathrm{IR}^m)$ to $L^p(\Omega)$ (Lemma 2.3 (i)). By (2.5), we obtain

$$\|\boldsymbol{f}(u+v) - \boldsymbol{f}(u) - T_u v\|_{0,p,\Omega} \leq \text{const.} \times \varepsilon \|v\|_{1,p,\Omega} \leq \text{const.} \times \varepsilon \|v\|_{2,p,\Omega},$$

which means that $\boldsymbol{f}$ is differentiable and $\mathrm{D}\boldsymbol{f}(u) = T_u$.

To prove the continuity of $\mathrm{D}\boldsymbol{f}$, note that from (2.4), we get

$$\|\mathrm{D}\boldsymbol{f}(u) - \mathrm{D}\boldsymbol{f}(u^0)\| \leq \|\nabla_\xi f(., u, \nabla u) - \nabla_\xi f(., u^0, \nabla u^0)\|_{0,\infty,\Omega}$$

where $\|\mathrm{D}\boldsymbol{f}(u) - \mathrm{D}\boldsymbol{f}(u^0)\|$ denotes either the norm in $\mathcal{L}(X_p(\Omega), L^\infty(\Omega))$ if $f(.,0) \in L^\infty(\Omega)$, or the norm in $\mathcal{L}(X_p(\Omega), L^p(\Omega))$ if $f(.,0) \in L^p(\Omega)$. The result then follows from Lemma 2.2 (ii) applied to each component of $\nabla_\xi f$, and which also ensures that $\mathrm{D}\boldsymbol{f}$ is bounded on the bounded subsets of $W^{2,p}(\Omega, \mathbb{R}^m)$. □

Remark 2.8 If f takes values in $\mathbb{R}^m$, the derivative of the Nemytskii operator generated by f is just:

$$\left.\begin{aligned} \mathrm{D}\boldsymbol{f}(u)v = \big(\mathrm{D}\boldsymbol{f}^1(u)v, \dots, \mathrm{D}\boldsymbol{f}^m(u)v\big) \quad \text{with for } \ k = 1, \dots, m \\ \mathrm{D}\boldsymbol{f}^k(u)v = \sum_{i=0}^N \nabla_{\xi_i} f^k(., u, \nabla u) \cdot \partial_i v \end{aligned}\right\} \tag{2.6}$$

Similarly if $f = \big(f^{k,j}\big)_{k,j=1,\dots,m}$ is an $m \times m$ matrix, the derivative of the Nemytskii operator $\boldsymbol{f}$ is the matrix

$$\left.\begin{aligned} \mathrm{D}\boldsymbol{f}(u)v = \big(\mathrm{D}\boldsymbol{f}^{k,j}(u)v\big)_{k,j=1,\dots,m} \quad \text{with} \\ \mathrm{D}\boldsymbol{f}^{k,j}(u)v = \sum_{i=0}^N \nabla_{\xi_i} f^{k,j}(., u, \nabla u) \cdot \partial_i v \end{aligned}\right\} \tag{2.7}$$

Lemma 2.4 (Lemma 2.9 of [31]) *Let X, Y and Z be normed spaces with $X \hookrightarrow Y$ and let $\boldsymbol{f} : X \to Z$ be uniformly continuous on the bounded subsets of X. Suppose that there is a dense subset $D \subset X$ such that whenever $u \in D$ and $(u_n) \subset X$ is a bounded sequence with $u_n \to u$ in Y, we have $\boldsymbol{f}(u_n) \to \boldsymbol{f}(u)$ in Z. Then the restriction of $\boldsymbol{f}$ to the bounded subsets of X remains continuous for the topology induced by Y.*

Lemma 2.5 *Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times N} \to \mathbb{R}$ be an equicontinuous $C^1_\xi-$bundle map. Suppose that $f(.,0) \in L^p(\Omega)$ and that $\nabla_{\xi_i} f(.,0) \in (L^p(\Omega) \cap L^\infty(\Omega))^m$, $0 \leq i \leq N$. Then the restriction of the Nemytskii operator to any bounded subset of $W^{2,p}(\Omega, \mathbb{R}^m)$ is continuous into $L^p(\Omega)$ for the topology of $C^1_d(\overline{\Omega}, \mathbb{R}^m)$.*

Proof. Recall that $\boldsymbol{f}$ is uniformly continuous on the bounded subsets of $W^{2,p}(\Omega, \mathbb{R}^m)$ by Theorem 2.1. Note also that if

$$D = \big\{u \in C^\infty(\Omega, \mathbb{R}^m) \mid \exists\, v \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^m) \text{ such that } v|_\Omega = u\big\}\,,$$

then D is dense in $W^{2,p}(\Omega, \mathbb{R}^m)$ (Adams [**1**] Theorem 3.18). We show that if $u \in D$ and (u_n) is a bounded sequence from $X_p(\Omega)$ converging to u in $C^1_d(\overline{\Omega}, \mathbb{R}^m)$, then $\boldsymbol{f}(u_n) \to \boldsymbol{f}(u)$ in $L^p(\Omega)$. The result will follow from Lemma 2.4 with $X = W^{2,p}(\Omega, \mathbb{R}^m)$, $Y = C^1_d(\overline{\Omega}, \mathbb{R}^m)$, $Z = L^p(\Omega)$ and D defined above.

In Lemma 2.3 we have already established that

$$\boldsymbol{f}(u) = f(.,0) + \sum_{i=0}^N \boldsymbol{g}_i(u) \cdot \partial_i u,$$

where

$$g_i(x,\xi) = \int_0^1 \nabla_{\xi_i} f(x, t\xi)\, \mathrm{d}t,$$

and by Remark 2.6, the g_i are equicontinuous C^0-bundle maps. Hence by Lemma 2.2 (i), applied to each component of g_i, $\boldsymbol{g}_i : C^1_d(\overline{\Omega}, \mathbb{R}^m) \to L^\infty(\Omega, \mathbb{R}^m)$ is continuous.

Clearly the problem reduces to showing that $\boldsymbol{g}_i(u_n)\cdot\partial_i u_n \to \boldsymbol{g}_i(u)\cdot\partial_i u$ in $L^p(\Omega)$, $0 \le i \le N$. To see this we write

$$\boldsymbol{g}_i(u_n)\cdot\partial_i u_n - \boldsymbol{g}_i(u)\cdot\partial_i u = (\boldsymbol{g}_i(u_n) - \boldsymbol{g}_i(u))\cdot\partial_i u_n + \boldsymbol{g}_i(u)\cdot\partial_i(u_n - u). \tag{2.8}$$

The first term tends to zero in $L^p(\Omega)$ because $\boldsymbol{g}_i(u_n) \to \boldsymbol{g}_i(u)$ in $L^\infty(\Omega, \mathbb{R}^m)$, and $(\partial_i u_n)$ is bounded in $L^p(\Omega, \mathbb{R}^m)$. On the other hand $(u_n - u) \to 0$ in $C^1_d(\overline{\Omega}, \mathbb{R}^m)$ which is continuously imbedded in $W^{1,\infty}(\Omega, \mathbb{R}^m)$. The last term of (2.8) tends to zero if we show that $\boldsymbol{g}_i(u) \in L^p(\Omega, \mathbb{R}^m)$. And this is true for the following reason: let $\Omega' \subset \Omega$ be the support of $u \in D$. Then first $\boldsymbol{g}_i(u) \in L^\infty(\Omega, \mathbb{R}^m) \subset L^\infty(\Omega', \mathbb{R}^m) \subset L^p(\Omega', \mathbb{R}^m)$, and secondly, when $x \in \Omega\backslash\Omega'$, we have $\boldsymbol{g}_i(u)(x) = g_i(x, u(x), \nabla u(x)) = g_i(x, 0) = \nabla_{\xi_i} f(x, 0)$. But $\nabla_{\xi_i} f(., 0) \in L^p(\Omega, \mathbb{R}^m)$, therefore $\boldsymbol{g}_i(u) \in L^p(\Omega\backslash\Omega', \mathbb{R}^m)$. And thus $\boldsymbol{g}_i(u) \in L^p(\Omega, \mathbb{R}^m)$ as claimed. □

Lemma 2.6 *Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times N} \to \mathbb{R}$ be an equicontinuous $C^1_\xi-$bundle map. Suppose that $f(., 0) \in L^p(\Omega)$ and that $\nabla_\xi f(., 0)$ is bounded on Ω (so that the Nemytskii operator $\boldsymbol{f}$ is of class C^1 from $W^{2,p}(\Omega, \mathbb{R}^m)$ to $L^p(\Omega)$ by Theorem 2.1). If $(u_n) \subset W^{2,p}(\Omega, \mathbb{R}^m)$ is a bounded sequence and $u \in W^{2,p}(\Omega, \mathbb{R}^m)$ is such that $u_n \to u$ in $C^1_d(\overline{\Omega}, \mathbb{R}^m)$ (hence $u_n \rightharpoonup u$ in $W^{2,p}(\Omega, \mathbb{R}^m)$, by Note A2 in the appendix), we have*

$$\boldsymbol{f}(u_n) - \boldsymbol{f}(u) - \mathrm{D}\boldsymbol{f}(u)(u_n - u) \to 0 \quad \textit{in} \;\; L^p(\Omega). \tag{2.9}$$

Proof. Let $v_n = u_n - u$ so that $v_n \rightharpoonup 0$ in $W^{2,p}(\Omega, \mathbb{R}^m)$, and $v_n \to 0$ in $C^1_d(\overline{\Omega}, \mathbb{R}^m)$. Then the left hand side of (2.9) is $\boldsymbol{f}(u + v_n) - \boldsymbol{f}(u) - \mathrm{D}\boldsymbol{f}(u)v_n = \boldsymbol{g}(v_n) - \boldsymbol{g}(0)$ where $\boldsymbol{g}(v) := \boldsymbol{f}(u + v) - \mathrm{D}\boldsymbol{f}(u)v$ for $v \in W^{2,p}(\Omega, \mathbb{R}^m)$. Note that $\boldsymbol{g}$ is the Nemytskii operator associated with (see (2.4))

$$g(x, \xi) := f(x, u(x) + \xi_0, \nabla u(x) + \xi') - \sum_{i=0}^{N} \nabla_{\xi_i} f(x, u(x), \nabla u(x)) \cdot \xi_i$$

where $\nabla_{\xi_i} f(., u, \nabla u)$ is continuous and bounded (Lemma 2.2 (ii)). So one can check using Lemma 2.1 (i), that g is an equicontinuous $C^1_\xi-$bundle map with

$$\nabla_{\xi_i} g(x, \xi) = \nabla_{\xi_i} f(x, u(x) + \xi_0, \nabla u(x) + \xi') - \nabla_{\xi_i} f(x, u(x), \nabla u(x)).$$

Furthermore $\boldsymbol{g}(0) = \boldsymbol{f}(u) \in L^p(\Omega)$ (Lemma 2.3 (i)), and $\nabla_{\xi_i} g(., 0) = 0 \in L^p(\Omega, \mathbb{R}^m) \cap L^\infty(\Omega, \mathbb{R}^m)$. Thus g verifies the conditions of Lemma 2.5, and therefore $\boldsymbol{g}(v_n) \to \boldsymbol{g}(0)$ in $L^p(\Omega)$, which completes the proof. □

Smoothness of F

Let the coefficients of F in (1.2) satisfy the following assumptions:

$$a_{\alpha\beta} \text{ are equicontinuous } C^1_\xi-\text{bundle maps, } 1 \le \alpha, \beta \le N \tag{2.10}$$

$$a_{\alpha\beta}(., 0) \text{ and } \nabla_\xi a_{\alpha\beta}(., 0) \text{ are bounded on } \Omega,\ 1 \le \alpha, \beta \le N \tag{2.11}$$

$$b \text{ is an equicontinuous } C^1_\xi-\text{bundle map} \tag{2.12}$$

$$b(., 0) \in L^p(\Omega, \mathbb{R}^m),\ \nabla_\xi b(., 0) \text{ is bounded on } \Omega. \tag{2.13}$$

Lemma 2.7 *The operator F in (1.2) is both continuous and weakly sequentially continuous from $W^{2,p}(\Omega, \mathbb{R}^m)$ to $L^p(\Omega, \mathbb{R}^m)$ and it maps bounded subsets onto bounded subsets.*

Proof. By Lemma 2.2 (ii) applied to each component of $a_{\alpha\beta}$, the Nemytskii operators $\boldsymbol{a}_{\alpha\beta}$ are continuous from $W^{2,p}(\Omega, \mathbb{R}^m)$ to $L^\infty(\Omega, \mathbb{R}^{m\times m})$ and they map bounded subsets onto bounded subsets. By Lemma 2.3 (i) (applied to each component of b), $\boldsymbol{b}$ is continuous from $W^{2,p}(\Omega, \mathbb{R}^m)$ to $L^p(\Omega, \mathbb{R}^m)$ and maps bounded subsets onto bounded ones. This proves the continuity and the boundedness properties.

Now if $(u_n) \subset W^{2,p}(\Omega, \mathbb{R}^m)$ converges weakly to u, we have $\partial^2_{\alpha\beta} u_n \rightharpoonup \partial^2_{\alpha\beta} u$ in $L^p(\Omega, \mathbb{R}^m)$[1]. By Lemma 2.2 (iv), $\boldsymbol{a}_{\alpha\beta}(u_n)\partial^2_{\alpha\beta} u_n \rightharpoonup \boldsymbol{a}_{\alpha\beta}(u)\partial^2_{\alpha\beta} u$ in $L^p(\Omega, \mathbb{R}^m)$. Next by Lemma 2.3 (ii), $\boldsymbol{b}(u_n) \rightharpoonup \boldsymbol{b}(u)$ in $L^p(\Omega, \mathbb{R}^m)$. This proves the weak continuity of F. □

Remark 2.9 Note that the proof of the above lemma requires only the following weaker assumptions: $a_{\alpha\beta}$ are equicontinuous C^0-bundle maps, with $a_{\alpha\beta}(.,0)$ bounded, and

$$b(x,\xi) = b_0(x) + \sum_{i=0}^{N} c_i(x,\xi)\xi_i$$

where $b_0 \in L^p(\Omega, \mathbb{R}^m)$, and c_i are equicontinuous C^0-bundle maps with $c_i(.,0)$ bounded. This will be used in §2.5.

Theorem 2.2 *The operator F in (1.2) is of class C^1 from $X_p(\Omega) = W^{2,p}(\Omega, \mathbb{R}^m)$ to $Y_p(\Omega) = L^p(\Omega, \mathbb{R}^m)$, with derivative*

$$\mathrm{D}F(u)v = -\sum_{\alpha,\beta=1}^{N} \boldsymbol{a}_{\alpha\beta}(u)\partial^2_{\alpha\beta} v + \mathrm{D}\boldsymbol{b}(u)v - \sum_{\alpha,\beta=1}^{N} \left(\mathrm{D}\boldsymbol{a}_{\alpha\beta}(u)v\right)\partial^2_{\alpha\beta} u \tag{2.14}$$

where $\mathrm{D}\boldsymbol{b}(u)$ and $\mathrm{D}\boldsymbol{a}_{\alpha\beta}(u)$ are given by (2.6) and (2.7) respectively.

In particular, the restriction of F to the subspace $D_p(\Omega) = \left(W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)\right)^m$ is C^1 from $D_p(\Omega)$ to $L^p(\Omega, \mathbb{R}^m)$.

Proof. Recall that

$$F(u) = -\sum_{\alpha,\beta=1}^{N} \boldsymbol{a}_{\alpha\beta}(u)\partial^2_{\alpha\beta} u + \boldsymbol{b}(u).$$

By Theorem 2.1, $\boldsymbol{b} \in C^1(X_p(\Omega), Y_p(\Omega))$, and $\boldsymbol{a}_{\alpha\beta} \in C^1\left(X_p(\Omega), L^\infty(\Omega, \mathbb{R}^{m\times m})\right)$. Now let $G(u) := \boldsymbol{a}_{\alpha\beta}(u)\partial^2_{\alpha\beta} u$, and $B : L^\infty(\Omega, \mathbb{R}^{m\times m}) \times Y_p(\Omega) \to Y_p(\Omega)$ be the bounded bilinear operator defined by $B(\mathcal{M}, \mathcal{X}) = \mathcal{M}\mathcal{X}$. Then $G = B \circ (\boldsymbol{a}_{\alpha\beta}, \partial^2_{\alpha\beta})$, and the result follows from the chain rule. □

2.2 Ellipticity and examples

Ellipticity is defined in general for linear systems. In the nonlinear case, one formulates a condition which implies that the linearization is elliptic in some sense. Here we introduce two ellipticity conditions for linear systems of second order.

[1] $\partial^2_{\alpha\beta} : X_p(\Omega) \to Y_p(\Omega)$ is linear and bounded and therefore weakly continuous.

Let $A_{\alpha\beta},\ B_\alpha,\ C\ (\alpha,\beta = 1,\dots,N)$ be (matrix-valued) functions from Ω to $\mathbb{R}^{m\times m}$. Define a second order linear differential operator L by

$$Lv := -\sum_{\alpha,\beta=1}^{N} A_{\alpha\beta}(x)\partial^2_{\alpha\beta}v + \sum_{\alpha=1}^{N} B_\alpha(x)\partial_\alpha v + C(x)v, \tag{2.15}$$

where $v:\Omega \to \mathbb{R}^m$.

For the majority of our results in this chapter, the following condition of ellipticity is sufficient.

Definition 2.2 (Petrovskii) An operator L of the form (2.15) is said to be elliptic at x in the sense of Petrovskii, if there exists a positive constant $\gamma(=\gamma(x))$ such that

$$\det\left(\sum_{\alpha,\beta=1}^{N}\eta_\alpha\eta_\beta A_{\alpha\beta}(x)\right) \geq \gamma|\eta|^{2m} \quad \forall\, \eta \in \mathbb{R}^N. \tag{2.16}$$

We say that L is strictly elliptic on a subset $K \subset \Omega$, if in the above definition one can choose the same γ for all $x \in K$.

We mention right away that the (linear) Stokes system is not elliptic in the sense of Petrovskii. But a study of the Fredholm and properness properties of the Navier-Stokes operator on unbounded domains, was already carried out by Galdi and Rabier **[15]**, **[16]**. The two physical examples we have in mind are steady reaction diffusion systems and elasticity, and these satisfy a stronger condition of ellipticity known as the strong Legendre-Hadamard (see **[8]**, **[25]**).

Definition 2.3 (The Strong Legendre-Hadamard ellipticity) An operator of the form (2.15) is called strongly elliptic at x in the sense of Legendre-Hadamard, if there is $\gamma = \gamma(x) > 0$, such that

$$\zeta^T\left(\sum_{\alpha,\beta=1}^{N}\eta_\alpha\eta_\beta A_{\alpha\beta}(x)\right)\zeta \geq \gamma|\eta|^2|\zeta|^2 \quad \text{for all } \ \eta\in\mathbb{R}^N,\ \zeta\in\mathbb{R}^m. \tag{2.17}$$

We say that L is strongly elliptic on a subset $K \subset \mathbb{R}^N$ in the sense of Legendre-Hadamard, if (2.17) holds with the same $\gamma > 0$ for all $x \in K$.

Indeed this is stronger than Petrovskii ellipticity, since (2.17) means that the matrix

$$\sum_{\alpha,\beta=1}^{N}\eta_\alpha\eta_\beta A_{\alpha\beta}(x)$$

is positive definite for $\eta \neq 0$ (and so all its real eigenvalues are positive), whereas Petrovskii condition means that it has a positive determinant. Note that the strong Legendre-Hadamard condition is "convex" in the sense that all operators on the segment joining two elliptic operators (in the sense of Legendre-Hadamard) are elliptic. However, this is not true for Petrovskii-ellipticity. On the other hand, both conditions concern only the higher order coefficients of the system, and furthermore they are stable under small enough perturbation of the leading coefficients. We mention finally that the Petrovskii condition is available for higher order systems (see **[22]**).

Steady reaction-diffusion of particles in a fluid flow

For $N \leq 3$, let $v : \mathbb{R}^N \to \mathbb{R}^N$ denote the velocity field of a stationary flow containing m types of particle in suspension. Let $u^k(x,t)$ denote the density of particles of type k at time t and position $x \in \mathbb{R}^N$. The particles move with the fluid, diffuse with diffusion coefficients $D_k > 0$ and take part in a chemical reaction. According to **[14]**, the evolution of the particle densities in the fluid is governed by the system

$$\frac{\partial}{\partial t}u^k = \frac{D_k}{2}\Delta u^k + v(x).\nabla u^k + f_k(x, u^1, ..., u^m) \quad \text{for } k = 1, 2..., m.$$

In a matrix-vector form, steady states of this system satisfy

$$-\sum_{\alpha,\beta=1}^{N} \begin{pmatrix} \frac{D_1}{2}\delta_{\alpha\beta} & 0 & \cdots & 0 \\ 0 & \frac{D_2}{2}\delta_{\alpha\beta} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{D_m}{2}\delta_{\alpha\beta} \end{pmatrix} \begin{pmatrix} \partial^2_{\alpha\beta}u^1 \\ \vdots \\ \vdots \\ \partial^2_{\alpha\beta}u^m \end{pmatrix} - \begin{pmatrix} v \cdot \nabla u^1 + f_1(x,u) \\ \vdots \\ \vdots \\ v \cdot \nabla u^m + f_m(x,u) \end{pmatrix} = 0,$$

which indeed is a system of the form (1.1) with

$$a^{ij}_{\alpha\beta}(x, \xi_0, [\xi_1 \cdots \xi_N]) = \frac{D_i}{2}\delta_{ij}\delta_{\alpha\beta} \quad \text{for } \alpha, \beta = 1, \ldots, N \text{ and } i, j = 1, \ldots, m,$$
$$b_j(x, \xi_0, [\xi_1 \cdots \xi_N]) = -\left\{ \sum_{\alpha=1}^{N} v_\alpha(x)\xi^j_\alpha + f_j(x, \xi_0) \right\}.$$

Now the linearization of the above system is of the form (2.15) with $A_{\alpha\beta} = a_{\alpha\beta}$. Indeed

$$\sum_{\alpha,\beta=1}^{N} \eta_\alpha\eta_\beta a_{\alpha\beta} = \sum_{\alpha=1}^{N} \eta^2_\alpha a_{\alpha\alpha} = \frac{|\eta|^2}{2}\mathrm{diag}\,(D_1, \ldots, D_m),$$

and therefore

$$\zeta^T \left(\sum_{\alpha,\beta=1}^{N} \eta_\alpha\eta_\beta a_{\alpha\beta} \right) \zeta \geq \frac{\min D_k}{2}|\eta|^2|\zeta|^2.$$

Thus the system is strongly elliptic in the sense of Legendre-Hadamard.

Our conditions (2.10)$-$(2.13) are satisfied provided that

(i) $v_\alpha \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and

(ii) $f_j : \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}$ is an equicontinuous $C^1_\xi-$ bundle map with $f_j(x, 0) = 0$ and $\nabla_\xi f_j(\cdot, 0)$ is bounded on $\mathbb{R}^N$.

2.3 Fredholmness

We now begin the investigation of the Fredholmness of the second order differential operator (1.2). To the hypotheses (2.10)$-$(2.13), we add an ellipticity condition, which implies that the linearization $\mathrm{D}F(u)$ is Petrovskii-elliptic.

$$\left.\begin{array}{c} \text{For all } \eta \in \mathbb{R}^N,\ (x,\xi) \in \Omega \times \left(\mathbb{R}^m \times \mathbb{R}^{m\times N}\right) \\ \det\left(\sum_{\alpha,\beta=1}^{N} \eta_\alpha\eta_\beta a_{\alpha\beta}(x,\xi) \right) \geq \gamma(x,\xi)|\eta|^{2m} \end{array}\right\} \tag{2.18}$$

where $\gamma : \Omega \times \left(\mathbb{R}^m \times \mathbb{R}^{m\times N}\right) \to (0,\infty)$ is bounded from below by a positive constant on every compact subset of $\Omega \times \left(\mathbb{R}^m \times \mathbb{R}^{mN}\right)$.

Note that in the case of a single equation ($m = 1$), this condition reduces to

$$\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(x,\xi)\eta_\alpha\eta_\beta \geq \gamma(x,\xi)|\eta|^2 \quad \forall\, \eta \in \mathbb{R}^N,\ (x,\xi) \in \Omega \times \mathbb{R}^{N+1},$$

and this is the ellipticity condition used in **[31]** with $\Omega = \mathbb{R}^N$. In the remainder of this chapter, the coefficients of the operator F in (1.2) will satisfy the hypotheses (2.10)−(2.13) and (2.18).

Note that

$$\mathrm{D}F(u)v = L(u)v - \sum_{\alpha,\beta=1}^{N} (\mathrm{D}\boldsymbol{a}_{\alpha\beta}(u)v)\,\partial^2_{\alpha\beta}u,$$

where

$$L(u)v := -\sum_{\alpha,\beta=1}^{N} \boldsymbol{a}_{\alpha\beta}(u)\partial^2_{\alpha\beta}v + \mathrm{D}\boldsymbol{b}(u)v \tag{2.19}$$

and clearly $L(u) \in \mathcal{L}\left(X_p(\Omega), Y_p(\Omega)\right)$.

Lemma 2.8 *Let $u \in X_p(\Omega)$. Then the difference $\mathrm{D}F(u) - L(u)$ is compact between $X_p(\Omega)$ and $Y_p(\Omega)$. Therefore, given $\mu \in \mathbf{Z} \cup \{\pm\infty\}$, and any closed subspace E of $X_p(\Omega)$, we have*

$$\mathrm{D}F(u) \in \Phi_\mu(E, Y_p(\Omega)) \Longleftrightarrow L(u) \in \Phi_\mu(E, Y_p(\Omega)).$$

Proof. If we show that the difference is compact, the second statement will follow from the stability of $\Phi_\mu(E, Y_p(\Omega))$ under compact perturbations.

By (2.19) we have

$$\mathrm{D}F(u)v - L(u)v = -\sum_{\alpha,\beta=1}^{N} (\mathrm{D}\boldsymbol{a}_{\alpha\beta}(u)v)\,\partial^2_{\alpha\beta}u.$$

Clearly it suffices to show that each $(\mathrm{D}\boldsymbol{a}_{\alpha\beta}(u)v)\,\partial^2_{\alpha\beta}u$, is compact, and again for that it suffices to show that each component $\sum\limits_{j=1}^{m} (\mathrm{D}\boldsymbol{a}_{\alpha\beta}(u)v)_{k,j}\,\partial^2_{\alpha\beta}u^j$ ($k = 1,\ldots,m$), is compact. Now each term of this sum is by (2.7)

$$\left(\sum_{i=0}^{N} \nabla_{\xi_i} a^{k,j}_{\alpha\beta}(.,u,\nabla u)\cdot\partial_i v\right)\partial^2_{\alpha\beta}u^j.$$

Once again writing the components, we deal with the terms $\left(\partial_{\xi^l_i} a^{k,j}_{\alpha\beta}(.,u,\nabla u)\partial^2_{\alpha\beta}u^j\right)\partial_i v^l$, $k,j,l = 1,\ldots,m$. Now indeed $\partial_{\xi^l_i} a^{k,j}_{\alpha\beta}(.,u,\nabla u)\partial^2_{\alpha\beta}u^j \in L^p(\Omega)$. But since $N < p < \infty$, the multiplication by a fixed function of $L^p(\Omega)$ is a compact operator from $W^{1,p}(\Omega)$ to $L^p(\Omega)$ (see the appendix). Thus $w \mapsto Tw := \left(\partial_{\xi^l_i} a^{k,j}_{\alpha\beta}(.,u,\nabla u)\partial^2_{\alpha\beta}u^j\right)\partial_i w$ is compact from $W^{2,p}(\Omega)$ to[2] $L^p(\Omega)$. Now going back through the steps, we see that $\mathrm{D}F(u) - L(u)$ is a compact operator from $X_p(\Omega)$ to $Y_p(\Omega)$, and therefore also from $D_p(\Omega)$ to $Y_p(\Omega)$. □

[2] $w_n \rightharpoonup w$ in $W^{2,p} \Rightarrow \partial_i w_n \rightharpoonup \partial_i w$ in $W^{1,p} \Rightarrow Tw_n \to Tw$ in L^p.

Fix u and let $A_{\alpha\beta}(x) = \boldsymbol{a}_{\alpha\beta}(u)(x)$, $B_\alpha(x)$ be the matrix with lines $\nabla_{\xi_\alpha} b^k(x, u(x), \nabla u(x))$, $k = 1, \ldots, m$, and $C(x)$ be the matrix with lines $\nabla_{\xi_0} b^k(x, u(x), \nabla u(x))$, $k = 1, \ldots, m$. Then

$$L(u)v = -\sum_{\alpha,\beta=1}^{N} A_{\alpha\beta}(x)\partial^2_{\alpha\beta} v + \sum_{\alpha=1}^{N} B_\alpha(x)\partial_\alpha v + C(x)v$$

is a linear second order differential operator, with continuous and bounded coefficients. Now condition (2.18) implies

$$\det\left(\sum_{\alpha,\beta=1}^{N} \eta_\alpha\eta_\beta A_{\alpha\beta}(x)\right) \geq \gamma(x, u(x), \nabla u(x))|\eta|^{2m}.$$

As x varies over a compact set, the continuity of u and ∇u ensures that $(x, u(x), \nabla u(x))$ remain in a compact set K. Therefore by (2.18) there exists $\gamma_K > 0$ such that

$$\det\left(\sum_{\alpha,\beta=1}^{N} \eta_\alpha\eta_\beta A_{\alpha\beta}(x)\right) \geq \gamma_K|\eta|^{2m}.$$

Thus, for each fixed $u \in X_p(\Omega)$ the differential operator $L(u)$ is strictly Petrovskii-elliptic on the compact subsets of Ω.

Lemma 2.9 (Koshelev [22], Theorem 17 pp. 150-151) *Let $\Omega' \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary, and $1 < q < \infty$. Let the linear operator L in (2.15) be strictly Petrovskii-elliptic in Ω', with continuous coefficients on $\overline{\Omega'}$. If $v \in W^{2,q}(\Omega', \mathbb{R}^m) \cap W_0^{1,q}(\Omega', \mathbb{R}^m)$, then v satisfies the a priori estimate*

$$\|v\|_{2,q,\Omega'} \leq c\left(\|Lv\|_{0,q,\Omega'} + \|v\|_{0,1,\Omega'}\right), \tag{2.20}$$

where c is a positive constant.

Lemma 2.10 *Assume that $\partial\Omega$ is of class C^2. Let L be a second order linear differential operator, strictly Petrovskii-elliptic on the compact subsets of Ω, with continuous bounded coefficients*[3]*. If $(u_n) \subset D_p(\Omega)$, is a sequence converging weakly to zero in $D_p(\Omega)$, and $Lu_n \to 0$ in $Y_p(\Omega)$, then $u_n \to 0$ in $X_p(\Omega')$ for all open and bounded subsets $\Omega' \subset \Omega$.*

Proof. We distinguish between two cases.

Case 1 (Ω is bounded). Since $u_n \rightharpoonup 0$ in $W^{2,p}(\Omega, \mathbb{R}^m) \underset{\text{comp}}{\hookrightarrow} L^1(\Omega; \mathbb{R}^m)$, we have $u_n \to 0$ in $L^1(\Omega, \mathbb{R}^m)$. On the other hand $Lu_n \to 0$ in $L^p(\Omega, \mathbb{R}^m)$. Note that $(u_n), L$ and Ω satisfy the conditions of Lemma 2.9. So by letting $q = p$ and $v = u_n$ in (2.20), we get: $u_n \to 0$ in $W^{2,p}(\Omega, \mathbb{R}^m)$.

Case 2 (Ω is unbounded). For every $r > 0$, set $B_r = \{x \in \mathbb{R}^N; |x| < r\}$, and $\Omega_r = \Omega \cap B_r$. Clearly it is equivalent to show that the result hold when $\Omega' = \Omega_r$ for $r > 0$ large enough[4].

So let B_r be a ball containing $\partial\Omega$, and $R > r$. It follows from the remarks made about Ω at the beginning of this chapter, that $\partial\Omega_R = \partial\Omega \cup \partial B_R$ so that $\partial\Omega_R$ is C^2 since $\partial\Omega \cap \partial B_R = \varnothing$.

[3]The boundedness of the coefficients ensures that L maps continuously $W^{2,p}(\Omega, \mathbb{R}^m)$ into $L^p(\Omega, \mathbb{R}^m)$.

[4]Because Ω_r is open and bounded, and every open and bounded subset of Ω is contained in a subset of the form Ω_r.

Now define $\varphi : \mathbb{R}^N \to \mathbb{R}$ to be a C^∞ function with compact support such that $\varphi = 1$ on B_r, $\varphi = 0$ outside B_R, and $\|\varphi\|_{0,\infty} \le 1$. Define a new sequence (v_n) by $v_n = \varphi u_n$, so that $u_n = v_n$ on $\Omega \cap B_r$, and $v_n \in D_p(\Omega_R)$.

Now recall that $u_n \rightharpoonup 0$ in $D_p(\Omega) \underset{\text{comp}}{\hookrightarrow} W^{1,p}(\Omega_R, \mathbb{R}^m) \hookrightarrow L^1(\Omega_R, \mathbb{R}^m)$, so that $u_n \to 0$ in $W^{1,p}(\Omega_R, \mathbb{R}^m)$ as well as in $L^1(\Omega_R, \mathbb{R}^m)$. And therefore also $v_n \to 0$ in $L^1(\Omega_R, \mathbb{R}^m)$ since φ is bounded.

On the other hand a direct calculation leads to

$$Lv_n = \varphi L u_n + \sum_{\alpha,\beta} \left(\partial^2_{\alpha\beta}\varphi A_{\alpha\beta}\right) u_n + \sum_{\alpha,\beta} \left(\partial_\alpha \varphi A_{\alpha\beta}\right) \partial_\beta u_n + \sum_{\alpha,\beta} \left(\partial_\beta \varphi A_{\alpha\beta}\right) \partial_\alpha u_n + \sum_{\alpha} \left(\partial_\alpha \varphi B_\alpha\right) u_n.$$

Thus due to the boundedness of φ, of its derivatives and of the coefficients of L, we get $Lv_n \to 0$ in $L^p(\Omega_R, \mathbb{R}^m)$.

Estimate (2.20) now gives that $v_n \to 0$ in $X_p(\Omega_R)$, and therefore also in $X_p(\Omega_r)$. This finally implies that $u_n \to X_p(\Omega_r)$. □

For the next results, we need the following concept introduced in **[31]**.

Definition 2.4 Let X and Y be real Banach spaces with X reflexive and let $T, L \in \mathcal{L}(X, Y)$ be given. We say that T is compact modulo L if, for every sequence $(u_n) \subset X$, we have $\{u_n \rightharpoonup 0 \text{ in } X,\ Lu_n \to 0 \text{ in } Y\} \Rightarrow Tu_n \to 0$ in Y.

Lemma 2.11 (Lemma 3.7 of [31]) *Let X and Y be real Banach spaces with X reflexive and let L_0, $L_1 \in \mathcal{L}(X, Y)$ be given. Suppose $L_0 - L_1$ is compact modulo both L_0 and L_1. Then we have the following.*

(i) *If $(u_n) \subset X$ is a sequence converging weakly to zero, we have $L_0 u_n \to 0$ in Y if and only if $L_1 u_n \to 0$.*

(ii) *$L_0 \in \Phi_+(X, Y)$ if and only if $L_1 \in \Phi_+(X, Y)$.*

For $t \in [0, 1]$ define $L_t := tL_1 + (1-t)L_0$. If $L_0 - L_1$ is compact modulo L_t $\forall\, t \in [0, 1]$, then

(iii) *$L_t \in \Phi_+(X, Y)$ for all $t \in [0, 1]$ if and only if this holds for some $t_0 \in [0, 1]$, and in this case, the index of L_t is independent of t.*

Lemma 2.12 *Assume that $\partial\Omega$ is C^2. For $L(u)$ defined by (2.19), the relation*

$$L(u) \in \Phi_+ \left(D_p(\Omega), Y_p(\Omega)\right)$$

holds for every $u \in D_p(\Omega)$, if and only if it holds for some $u^0 \in D_p(\Omega)$.

Proof. We shall prove that $L(u) - L(u^0)$ is compact modulo $L(u)$. By exchanging the roles of u and u^0, this shows that $L(u) - L(u^0)$ is compact modulo both $L(u)$ and $L(u^0)$, the conclusion follows from Lemma 2.11 (ii).

Let then $(v_n) \subset D_p(\Omega)$ be such that $v_n \rightharpoonup 0$ in $D_p(\Omega)$ and $L(u)v_n \to 0$ in $Y_p(\Omega)$. From the equicontinuity of $a_{\alpha\beta}$ and $D_\xi b$ at $\xi = 0$, we have that, given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|a_{\alpha\beta}(x, \xi) - a_{\alpha\beta}(x, 0)| < \frac{\varepsilon}{2} \quad \text{and} \quad |\nabla_\xi b(x, \xi) - \nabla_\xi b(x, 0)| < \frac{\varepsilon}{2}$$

for $|\xi| < \delta$ and all $x \in \Omega$. Now due to the embedding $W^{2,p}(\Omega, \mathbb{R}^m) \hookrightarrow C^1_d(\overline{\Omega}, \mathbb{R}^m)$ and the definition of $C^1_d(\overline{\Omega})$, there is $r > 0$ such that $|(u(x), \nabla u(x))| < \delta$, and $|(u^0(x), \nabla u^0(x))| < \delta$ for $|x| \geq r$ (we can choose r such that $\partial\Omega \subset B_r$). Therefore

$$|a_{\alpha\beta}(x, u(x), \nabla u(x)) - a_{\alpha\beta}(x, u^0(x), \nabla u^0(x))| < \varepsilon$$

and

$$|\nabla_\xi b(x, u(x), \nabla u(x)) - \nabla_\xi b(x, u^0(x), \nabla u^0(x))| < \varepsilon$$

whenever $|x| \geq r$. Now let $\Omega_r = \{x \in \Omega \mid |x| < r\}$, $\tilde{\Omega}_r = \{x \in \Omega \mid |x| > r\}$, and recall that

$$\left(L(u) - L(u^0)\right) v = -\sum_{\alpha,\beta=1}^{N} \left(\boldsymbol{a}_{\alpha\beta}(u) - \boldsymbol{a}_{\alpha\beta}(u^0)\right) \partial^2_{\alpha\beta} v + \sum_{i=0}^{N} \left(\nabla_{\xi_i} b(., u, \nabla u) - \nabla_{\xi_i} b(., u^0, \nabla u^0)\right) \partial_i v.$$

Therefore

$$\| \left(L(u) - L(u^0)\right) v\|_{0,p,\tilde{\Omega}_r} \leq m^2(N^2 + N + 1)\varepsilon \|v\|_{2,p,\tilde{\Omega}_r} \quad \text{for } v \in D_p(\Omega).$$

Hence

$$\| \left(L(u) - L(u^0)\right) v_n\|_{0,p,\tilde{\Omega}_r} \leq m^2 M(N^2 + N + 1)\varepsilon \tag{2.21}$$

where M is a bound for $\|v_n\|_{2,p,\Omega}$.

As already observed $L(u)$ verifies the conditions required in Lemma 2.10, thus $v_n \to 0$ in $X_p(\Omega_r)$, so $L(u^0)v_n$ and $L(u)v_n$ converge to zero in $Y_p(\Omega_r)$[5], which means that for any $\varepsilon > 0$ and n large enough,

$$\| \left(L(u) - L(u^0)\right) v_n\|_{0,p,\Omega_r} \leq \varepsilon. \tag{2.22}$$

Together (2.21) and (2.22), yield that $\| \left(L(u) - L(u^0)\right) v_n\|_{0,p,\Omega}$ can be made arbitrary small for n large enough. This completes the proof. $\square$

Theorem 2.3 *Let $\partial\Omega$ be of class C^2. The operator F in (1.2) is semi-Fredholm of index $\mu \in \mathbf{Z} \cup \{-\infty\}$ (i.e. $\mathrm{D}F(u) \in \Phi_\mu(D_p(\Omega), Y_p(\Omega))$ for every $u \in D_p(\Omega)$), if and only if there is some $u^0 \in D_p(\Omega)$ such that $\mathrm{D}F(u^0) \in \Phi_\mu(D_p(\Omega), Y_p(\Omega))$.*

Proof. By Lemma 2.8 and 2.12,

$$\begin{aligned} DF(u^0) \in \Phi_+(D_p(\Omega), Y_p(\Omega)) &\iff L(u^0) \in \Phi_+(D_p(\Omega), Y_p(\Omega)) \\ &\iff L(u) \in \Phi_+(D_p(\Omega), Y_p(\Omega)) \quad \forall\, u \in D_p(\Omega) \\ &\iff \mathrm{D}F(u) \in \Phi_+(D_p(\Omega), Y_p(\Omega)) \quad \forall\, u \in D_p(\Omega). \end{aligned}$$

Now by Theorem 2.2, $\mathrm{D}F$ is continuous as a map from $D_p(\Omega)$ into $\mathcal{L}(D_p(\Omega), Y_p(\Omega))$. Recall also that the index of a semi-Fredholm operator is locally constant, whence $u \mapsto \operatorname{index} \mathrm{D}F(u)$ is locally constant and therefore constant since $D_p(\Omega)$ is connected. $\square$

[5] $L(u) \in \mathcal{L}(X_p(\Omega_r), Y_p(\Omega_r))$ by Theorem 2.2 with Ω replaced by Ω_r.

A useful factorization of F

Theorem 2.4 *There exists an operator $G : X_p(\Omega) \to \mathcal{L}(X_p(\Omega), Y_p(\Omega))$, having the following properties.*

(a) *$F(u) - F(0) = G(u)u$, for every $u \in X_p(\Omega)$.*

(b) *For every $\mu \in \mathbf{Z} \cup \{-\infty\}$ and $u \in D_p(\Omega)$, we have $\mathrm{D}F(u) \in \Phi_\mu(D_p(\Omega), Y_p(\Omega))$ if and only if $G(u) \in \Phi_\mu(D_p(\Omega), Y_p(\Omega))$.*

(c) *G is continuous and maps bounded subsets of $X_p(\Omega)$ into bounded subsets of $\mathcal{L}(X_p(\Omega), Y_p(\Omega))$.*

Proof. (a) Recall that

$$F(u) = -\sum_{\alpha,\beta=1}^{N} \boldsymbol{a}_{\alpha\beta}(u)\partial^2_{\alpha\beta}u + \boldsymbol{b}(u), \tag{2.23}$$

and $\boldsymbol{b}(u)$ can be written as

$$\boldsymbol{b}(u) = \boldsymbol{b}(0) + \sum_{i=0}^{N} \boldsymbol{c}_i(u)\partial_i u \tag{2.24}$$

where c_i is the Nemytskii operator generated by the (matrix-valued) equicontinuous C^0-bundle map

$$c_i(x,\xi) = \int_0^1 \nabla_{\xi_i} b(x, t\xi)\, \mathrm{d}t.$$

By Theorem 2.2,

$$\begin{aligned}
\mathrm{D}F(u)v &= -\sum_{\alpha,\beta=1}^{N} \boldsymbol{a}_{\alpha\beta}(u)\partial^2_{\alpha\beta}v + \mathrm{D}\boldsymbol{b}(u)v - \sum_{\alpha,\beta=1}^{N} \left(\mathrm{D}\boldsymbol{a}_{\alpha\beta}(u)v\right)\partial^2_{\alpha\beta}u \\
&= -\sum_{\alpha,\beta=1}^{N} \boldsymbol{a}_{\alpha\beta}(u)\partial^2_{\alpha\beta}v + \sum_{i=0}^{N} \boldsymbol{b}_i(u)\partial_i v + K(u)v
\end{aligned} \tag{2.25}$$

where indeed $K(u)$ is compact as already observed in Lemma 2.8, and $\boldsymbol{b}_i$ is the Nemytskii operator associated with $\nabla_{\xi_i} b(x,\xi)$. In particular when $v = u$ we obtain

$$\mathrm{D}F(u)u = -\sum_{\alpha,\beta=1}^{N} \boldsymbol{a}_{\alpha\beta}(u)\partial^2_{\alpha\beta}u + \sum_{i=0}^{N} \boldsymbol{b}_i(u)\partial_i u + K(u)u. \tag{2.26}$$

Combining (2.23), (2.24) and (2.26) we get since $F(0) = \boldsymbol{b}(0)$,

$$F(u) - F(0) = \mathrm{D}F(u)u - K(u)u - \sum_{i=0}^{N} \left(\boldsymbol{b}_i(u) - \boldsymbol{c}_i(u)\right)\partial_i u. \tag{2.27}$$

Letting

$$T(u)v = \sum_{i=0}^{N} \left(\boldsymbol{b}_i(u) - \boldsymbol{c}_i(u)\right)\partial_i v, \tag{2.28}$$

and

$$G(u) = \mathrm{D}F(u) - K(u) - T(u) = L(u) - T(u), \tag{2.29}$$

where $L(u)$ is defined in (2.19), we get $F(u) - F(0) = G(u)u$.

Note that since $T(0) = K(0) = 0$, we have $G(0) = L(0) = \mathrm{D}F(0)$.

(b) We show that $T(u)$ is compact. From the equicontinuity of $(\nabla_{\xi_i} b(x,.))_x$ at $\xi = 0$, given any $\varepsilon > 0$, there is $\delta > 0$ such that $\forall\, x \in \Omega$

$$|\nabla_{\xi_i} b(x,\xi) - \nabla_{\xi_i} b(x,0)| < \frac{\varepsilon}{2}$$

whenever $|\xi| < \delta$. Therefore $\forall\, x \in \Omega$, $\forall\, t \in [0,1]$, we have $|\nabla_{\xi_i} b(x,\xi) - \nabla_{\xi_i} b(x,t\xi)| < \varepsilon$ if $|\xi| < \delta$. But since $u \in C^1_d(\overline{\Omega}, \mathbb{R}^m)$ there is an $r > 0$ such that $|\,(u(x), \nabla u(x))\,| < \delta$ whenever $|x| > r$. All this means that $\forall\, t \in [0,1]$, $|x| > r$, we have

$$|\nabla_{\xi_i} b(x, u(x), \nabla u(x)) - \nabla_{\xi_i} b(x, tu(x), t\nabla u(x))| < \varepsilon,$$

and so by integrating with respect to t we get

$$\lim_{|x|\to\infty} \left(\nabla_{\xi_i} b(x, u(x), \nabla u(x)) - \int_0^1 \nabla_{\xi_i} b(x, tu(x), t\nabla u(x))\, \mathrm{d}t \right) = 0.$$

But multiplication by a bounded function vanishing at infinity is a compact operator from $W^{1,q}$ to L^q for all $1 < q < \infty$ (see Note E2 in the appendix). Therefore $T(u)$ is compact from $D_p(\Omega)$ to $Y_p(\Omega)$ for all $u \in D_p(\Omega)$.

(c)

$$G(u)v = L(u)v - T(u)v = -\sum_{\alpha,\beta=1}^{N} \boldsymbol{a}_{\alpha\beta}(u)\partial^2_{\alpha\beta} v + \sum_{\alpha=0}^{N} \boldsymbol{c}_\alpha(u)\partial_\alpha v.$$

Therefore,

$$\begin{aligned} \|G(u)v - G(u_0)v\|_{0,p,\Omega} \le & \sum_{\alpha,\beta=1}^{N} \|\boldsymbol{a}_{\alpha\beta}(u) - \boldsymbol{a}_{\alpha\beta}(u_0)\|_{0,\infty,\Omega} \|\partial^2_{\alpha\beta} v\|_{0,p,\Omega} \\ & + \sum_{\alpha=0}^{N} \|\boldsymbol{c}_\alpha(u) - \boldsymbol{c}_\alpha(u_0)\|_{0,\infty,\Omega} \|\partial_\alpha v\|_{0,p,\Omega}. \end{aligned}$$

As already observed, the Nemytskii operators $\boldsymbol{a}_{\alpha\beta}$ and $\boldsymbol{c}_\alpha$ are continuous from $X_p(\Omega)$ to $L^\infty(\Omega, \mathbb{R}^{m\times m})$. Accordingly, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|\boldsymbol{a}_{\alpha\beta}(u) - \boldsymbol{a}_{\alpha\beta}(u_0)\|_{0,\infty,\Omega} \le \varepsilon \quad \text{and} \quad \|\boldsymbol{c}_\alpha(u) - \boldsymbol{c}_\alpha(u_0)\|_{0,\infty,\Omega} \le \varepsilon,$$

whenever $\|u - u_0\| \le \delta$. Consequently,

$$\|G(u)v - G(u_0)v\|_{0,p,\Omega} \le \text{const.} \times \varepsilon \|v\|_{2,p,\Omega},$$

and therefore,

$$\|G(u) - G(u_0)\|_{\mathcal{L}(X_p(\Omega), Y_p(\Omega))} \le \text{const.} \times \varepsilon.$$

The boundedness property follows from the boundedness of $\boldsymbol{a}_{\alpha\beta}$ and $\boldsymbol{c}_\alpha$ (Lemma 2.2). □

2.4 Properness

Lemma 2.13 *Let $u \in W^{2,p}(\Omega, \mathbb{R}^m)$ and $(u_n) \subset W^{2,p}(\Omega, \mathbb{R}^m)$ be a bounded sequence converging to u in $C^1_d(\overline{\Omega}, \mathbb{R}^m)$. Then $F(u_n) - F(u) - \mathrm{D}F(u)(u_n - u) \to 0$ in $L^p(\Omega, \mathbb{R}^m)$.*

Proof. Note first that $u_n \rightharpoonup u$ in $W^{2,p}(\Omega, \mathbb{R}^m)$, by Note A2 of the appendix.

$$\begin{aligned} &F(u_n) - F(u) - \mathrm{D}F(u)(u_n - u) \\ = &- \sum_{\alpha,\beta=1}^{N} \Big(\boldsymbol{a}_{\alpha\beta}(u_n) - \boldsymbol{a}_{\alpha\beta}(u)\Big)\partial^2_{\alpha\beta} u_n + \sum_{\alpha,\beta=1}^{N} \Big(\mathrm{D}\boldsymbol{a}_{\alpha\beta}(u)(u_n - u)\Big)\partial^2_{\alpha\beta} u \\ &+ \boldsymbol{b}(u_n) - \boldsymbol{b}(u) - \mathrm{D}\boldsymbol{b}(u)(u_n - u). \end{aligned} \tag{2.30}$$

As already observed in the proof of Lemma 2.8, $v \mapsto (\mathrm{D}\boldsymbol{a}_{\alpha\beta}(u)v) \cdot \partial^2_{\alpha\beta} u$ is a compact linear operator from $X_p(\Omega)$ to $Y_p(\Omega)$, therefore

$$\sum_{\alpha,\beta=1}^{N} \Big(\mathrm{D}\boldsymbol{a}_{\alpha\beta}(u)(u_n - u)\Big)\partial^2_{\alpha\beta} u \to 0 \quad \text{in } L^p(\Omega, \mathbb{R}^m). \tag{2.31}$$

By Lemma 2.6

$$\boldsymbol{b}(u_n) - \boldsymbol{b}(u) - \mathrm{D}\boldsymbol{b}(u)(u_n - u) \to 0 \quad \text{in } L^p(\Omega, \mathbb{R}^m). \tag{2.32}$$

By Lemma 2.2 (i), $\boldsymbol{a}_{\alpha\beta}(u_n) \to \boldsymbol{a}_{\alpha\beta}(u)$ in $L^\infty(\Omega, \mathbb{R}^{m\times m})$, and since $\partial^2_{\alpha\beta} u_n$ is bounded in $L^p(\Omega, \mathbb{R}^m)$

$$\sum_{\alpha,\beta=1}^{N} \big(\boldsymbol{a}_{\alpha\beta}(u_n) - \boldsymbol{a}_{\alpha\beta}(u)\big)\partial^2_{\alpha\beta} u_n \to 0 \quad \text{in } L^p(\Omega, \mathbb{R}^m). \tag{2.33}$$

□

Theorem 2.5 *Let Ω have a C^2 boundary. Suppose that there exists $u^0 \in D_p(\Omega)$ for which $\mathrm{D}F(u^0) \in \Phi_+(D_p(\Omega), Y_p(\Omega))$. The following properties are equivalent.*

(i) *$F : D_p(\Omega) \to Y_p(\Omega)$ is proper on the closed bounded subsets of $D_p(\Omega)$.*

(ii) *Every bounded sequence $(u_n) \subset D_p(\Omega)$ such that $(F(u_n))$ converges in $Y_p(\Omega)$, contains a subsequence converging in $C^1_d(\overline{\Omega}, \mathbb{R}^m)$.*

Proof. (i) $\Rightarrow$ (ii) is evident, since $D_p(\Omega) \hookrightarrow C^1_d(\overline{\Omega}, \mathbb{R}^m)$.

(ii) $\Rightarrow$ (i). Let $(u_n) \subset D_p(\Omega)$ be bounded and such that $(F(u_n))$ converges in $Y_p(\Omega)$. By assumption, there is a subsequence $(u_{\phi(n)})$ converging to some u in $C^1_d(\overline{\Omega}, \mathbb{R}^m)$, and hence, by Lemma 2.13, $F(u_{\phi(n)}) - F(u) - \mathrm{D}F(u)(u_{\phi(n)} - u) \to 0$ in $Y_p(\Omega)$. By Note A2 in the appendix, $u_{\phi(n)} \rightharpoonup u$ in $X_p(\Omega)$, and F is weakly continuous (Lemma 2.7), so $F(u_{\phi(n)}) \rightharpoonup F(u)$ hence $F(u_{\phi(n)}) \to F(u)$ in $Y_p(\Omega)$. Thus $\mathrm{D}F(u)(u_{\phi(n)} - u) \to 0$ in $Y_p(\Omega)$. But we know that $\mathrm{D}F(u) \in \Phi_+(D_p(\Omega), Y_p(\Omega))$ (Theorem 2.3), and hence it is proper by Yood's criterion. Therefore $u_{\phi(n)} \to u$ in $D_p(\Omega)$, by Note C (ii) in §1.3. □

As in **[31]**, we can give an equivalent formulation of Theorem 2.5 in terms of sequences vanishing uniformly at infinity.

Definition 2.5 We say that the sequence $(u_n) \subset C^1_d(\overline{\Omega}, \mathbb{R}^m)$ vanishes uniformly at infinity in the sense of $C^1_d(\overline{\Omega})$, if the following condition holds: $\forall\, \varepsilon > 0$, $\exists\, R > 0$, $\exists\, n_0 \in \mathbb{N}$ such that

$$|u_n(x)| + |\nabla u_n(x)| \le \varepsilon \quad \text{for all } |x| \ge R \text{ and } n \ge n_0.$$

Lemma 2.14 *Let $(u_n) \subset X_p(\Omega)$ be a bounded sequence. For $u \in X_p(\Omega)$, the following conditions are equivalent.*

(i) $u_n \to u$ *in* $C^1_d(\overline{\Omega}, \mathbb{R}^m)$.

(ii) $u_n \rightharpoonup u$ *in* $X_p(\Omega)$, *and* (u_n) *vanishes uniformly at infinity in the sense of* $C^1_d(\overline{\Omega})$.

Proof. Let $\Omega_r = \{x \in \Omega : |x| < r\}$ and $\tilde{\Omega}_r = \{x \in \Omega : |x| > r\}$ for every $r > 0$.

(i) $\Rightarrow$ (ii). It follows from note A2 in the appendix that $u_n \rightharpoonup u$ in $X_p(\Omega)$. Next let $\varepsilon > 0$ be given. There is $r > 0$ for which $|u(x)| + |\nabla u(x)| \leq \varepsilon/2$ whenever $x \in \tilde{\Omega}_r$. Let n_0 be such that $\|u_n - u\|_{1,\infty,\Omega} \leq \varepsilon/2$ for $n \geq n_0$. Then for every $x \in \tilde{\Omega}_r$ and $n \geq n_0$, we have $|u_n(x)| + |\nabla u_n(x)| \leq \varepsilon$.

(ii) $\Rightarrow$ (i). Let $\varepsilon > 0$ be given and let $r > 0$ and $n_0 \in \mathbb{N}$ be such that $|u_n(x)| + |\nabla u_n(x)| \leq \varepsilon/2$ whenever $x \in \tilde{\Omega}_r$ and $n \geq n_0$. After increasing r if necessary we may assume that $|u(x)| + |\nabla u(x)| \leq \varepsilon/2$. Hence

$$|u(x) - u_n(x)| + |\nabla u(x) - \nabla u_n(x)| \leq \varepsilon, \quad \forall x \in \tilde{\Omega}_r \quad \forall\, n \geq n_0.$$

Next, since $X_p(\Omega_r) \underset{\text{comp}}{\hookrightarrow} C^1_d(\overline{\Omega}_r, \mathbb{R}^m)$ there is $n_1 \in \mathbb{N}$ such that

$$|u(x) - u_n(x)| + |\nabla u(x) - \nabla u_n(x)| \leq \varepsilon, \quad \forall x \in \overline{\Omega}_r \quad \forall\, n \geq n_1.$$

Thus finally, we have $\|u_n - u\|_{1,\infty,\Omega} \leq \varepsilon$ for $n \geq \max(n_0, n_1)$, which shows that $u_n \to u$ in $C^1_d(\overline{\Omega}, \mathbb{R}^m)$ as claimed. □

Corollary 2.1 *Let Ω have a C^2 boundary. Suppose there exists $u^0 \in D_p(\Omega)$ for which $DF(u^0) \in \Phi_+(D_p(\Omega), Y_p(\Omega))$. The following conditions are equivalent.*

(i) $F : D_p(\Omega) \to Y_p(\Omega)$ *is proper on the closed bounded subsets of* $D_p(\Omega)$.

(ii) *Every bounded sequence* $(u_n) \subset D_p(\Omega)$ *such that* $(F(u_n))$ *converges in* $Y_p(\Omega)$, *vanishes uniformly at infinity in the sense of* $C^1_d(\overline{\Omega})$.

(iii) *Every bounded sequence* $(u_n) \subset D_p(\Omega)$ *such that* $(F(u_n))$ *converges in* $Y_p(\Omega)$, *contains a subsequence vanishing uniformly at infinity in the sense of* $C^1_d(\overline{\Omega})$.

Proof. (i) $\Rightarrow$ (ii). Let (u_n) be a bounded sequence from $D_p(\Omega)$ such that $(F(u_n))$ converges in $Y_p(\Omega)$, and suppose that (u_n) does not vanish uniformly at infinity. Set $\theta_n(x) = |u_n(x)| + |\nabla u_n(x)|$. Then there are $\varepsilon_0 > 0$, a subsequence $u_{\phi(n)}$, and a sequence $(x_n) \subset \Omega$ such that $|x_n| \geq n$ and $\theta_{\phi(n)}(x_n) = |u_{\phi(n)}(x_n)| + |\nabla u_{\phi(n)}(x_n)| \geq \varepsilon_0$. Now $(u_{\phi(n)})$ is also bounded and its image by F convergent, so by Theorem 2.5 it contains a subsequence $u_{\phi(\psi(n))}$ converging in C^1_d and therefore vanishing uniformly at infinity. Accordingly, there is $n_0 \in \mathbb{N}$, and $r > 0$ such that $\theta_{\phi(\psi(n))}(x) < \frac{\varepsilon_0}{2}$ whenever $|x| \geq r$ an $n \geq n_0$. So for $n \geq \max(r, n_0)$ (since $\psi(n) \geq n$), we have $\varepsilon_0 \leq \theta_{\phi(\psi(n))}(x_{\psi(n)}) < \frac{\varepsilon_0}{2}$. Contradiction.

(ii) $\Rightarrow$ (iii) is evident.

(iii) $\Rightarrow$ (i). Let (u_n) be a bounded sequence from $D_p(\Omega)$ such that $(F(u_n))$ converges, by assumption it contains a subsequence $(u_{\phi(n)})$ vanishing uniformly at infinity in the sense of C^1_d. But this subsequence is also bounded in $D_p(\Omega)$ and therefore it contains a subsequence $(u_{\phi(\psi(n))})$ converging weakly to some u in $D_p(\Omega)$. So, by Lemma 2.14, $u_{\phi(\psi(n))} \to u$ in $C^1_d(\overline{\Omega}, \mathbb{R}^m)$. Hence F is proper by Theorem 2.5 (ii). □

Lemma 2.15 *Let $(u_n) \subset W^{2,p}(\Omega)$, be a sequence converging to zero in $W^{2,p}(\Omega')$, for every bounded and open subset $\Omega' \subset \Omega$. Then, given $v \in W^{2,p}(\Omega)$, $\varepsilon \in (0,1)$ and $n_0 \in \mathbb{N}$, there is $n_1 \in \mathbb{N}, n_1 \geq n_0$, such that for every $n \geq n_1$:*

(i) $\|v\|^p_{2,p,\Omega} + \|u_n\|^p_{2,p,\Omega} - \varepsilon \leq \|v+u_n\|^p_{2,p,\Omega} \leq \|v\|^p_{2,p,\Omega} + \|u_n\|^p_{2,p,\Omega} + \varepsilon$, *and*

(ii) $\|v+u_n\|_{1,\infty,\Omega} \leq \max\left(\|v\|_{1,\infty,\Omega}, \|u_n\|_{1,\infty,\Omega}\right) + \varepsilon$.

Proof. Let $\Omega_r = \{x \in \Omega : |x| < r\}$, and $\tilde{\Omega}_r = \{x \in \Omega : |x| > r\}$. Since $v \in X_p(\Omega) \hookrightarrow C^1_d(\overline{\Omega}, \mathbb{R}^m)$, there is $r > 0$ such that

$$\|v\|_{2,p,\tilde{\Omega}_r} \leq \varepsilon \quad \text{and} \quad \|v\|_{1,\infty,\tilde{\Omega}_r} \leq \varepsilon \tag{2.34}$$

By assumption, we have $u_n \to 0$ in $W^{2,p}(\Omega_r)$ and hence also in $C^1(\overline{\Omega}_r)$. Thus for n large enough

$$\|u_n\|_{2,p,\Omega_r} \leq \varepsilon \quad \text{and} \quad \|u_n\|_{1,\infty,\Omega_r} \leq \varepsilon. \tag{2.35}$$

Using the preceding inequalities and $|a-b|^p \geq a^p - p(a+b)^{p-1}b$, we get

$$\begin{aligned}\|v+u_n\|^p_{2,p,\Omega} &\geq \left|\|v\|_{2,p,\Omega_r} - \|u_n\|_{2,p,\Omega_r}\right|^p + \left|\|v\|_{2,p,\tilde{\Omega}_r} - \|u_n\|_{2,p,\tilde{\Omega}_r}\right|^p \\ &\geq \|v\|^p_{2,p,\Omega_r} + \|u_n\|^p_{2,p,\tilde{\Omega}_r} - 2p(M+\varepsilon)^{p-1}\varepsilon,\end{aligned}$$

where M is a bound for $\|v\|_{2,p,\Omega}$ and $\|u_n\|_{2,p,\Omega}$. We also deduce from (2.34) and (2.35) that

$$\|v\|^p_{2,p,\Omega} \leq \|v\|^p_{2,p,\Omega_r} + \varepsilon^p \quad \text{and} \quad \|u_n\|^p_{2,p,\Omega} \leq \|u_n\|^p_{2,p,\tilde{\Omega}_r} + \varepsilon^p.$$

Thus

$$\|v\|^p_{2,p,\Omega} + \|u_n\|^p_{2,p,\Omega} - 2(p(M+1)^{p-1}+1)\varepsilon \leq \|v+u_n\|^p_{2,p,\Omega}. \tag{2.36}$$

Analogously, using (2.34) and (2.35) and $(a+b)^p \leq a^p + p(a+b)^{p-1}b$, we prove that

$$\|v+u_n\|^p_{2,p,\Omega} \leq \|v\|^p_{2,p,\Omega} + \|u_n\|^p_{2,p,\Omega} + 2p(M+1)^{p-1}\varepsilon.$$

This proves (i) since ε is arbitrary.

For (ii), we have

$$\begin{aligned}\|v+u_n\|_{1,\infty,\Omega} &= \max\left(\|v+u_n\|_{1,\infty,\Omega_r}, \|v+u_n\|_{1,\infty,\tilde{\Omega}_r}\right) \\ &\leq \max\left(\|v\|_{1,\infty,\Omega_r} + \varepsilon, \|u_n\|_{1,\infty,\tilde{\Omega}_r} + \varepsilon\right) \\ &= \max\left(\|v\|_{1,\infty,\Omega_r}, \|u_n\|_{1,\infty,\tilde{\Omega}_r}\right) + \varepsilon \\ &\leq \max\left(\|v\|_{1,\infty,\Omega}, \|u_n\|_{1,\infty,\Omega}\right) + \varepsilon.\end{aligned}$$

□

Lemma 2.16 *Assume that Ω has a C^2 boundary. Let*

$$L = -\sum_{\alpha,\beta=1}^{N} A_{\alpha\beta}(x)\partial^2_{\alpha\beta} + \sum_{\alpha=1}^{N} B_\alpha(x)\partial_\alpha + C(x)$$

be a differential operator which is strictly Petrovskii-elliptic on the compact subsets of Ω, with continuous and bounded coefficients. Suppose that there is a sequence (u_n) in $D_p(\Omega)$ such that $u_n \rightharpoonup 0$ in $D_p(\Omega)$, $Lu_n \to 0$ in $Y_p(\Omega)$, and (u_n) contains no subsequence converging to 0 in $D_p(\Omega)$. Then, there is a sequence $(w_n) \subset D_p(\Omega)$ such that $w_n \rightharpoonup 0$ in $D_p(\Omega)$ and $Lw_n \to 0$ in $Y_p(\Omega)$, (w_n) contains no subsequence converging to 0 in $D_p(\Omega)$, but furthermore, $w_n \to 0$ in $C^1_d(\overline{\Omega}, \mathbb{R}^m)$.

Proof. For simplicity we denote by $\|u\|_{k,p}$ the norm of u in $W^{k,p}(\Omega,\mathbb{R}^m)$.

Since (u_n) contains no subsequence converging to 0, there exist $\gamma>0$ and $n_0\in\mathbb{N}$ such that $\|u_n\|_{2,p}\geq\gamma$ for $n\geq n_0$. Therefore, at least one component $(u_n^{l_n})$ of (u_n) verifies $\|u_n^{l_n}\|_{2,p}\geq\frac{\gamma}{m}=\delta$. Since $(l_n)\subset\{1,\ldots,m\}$ is finite, it contains a constant subsequence $l_{\psi(n)}=l$ so that $\|u^l_{\psi(n)}\|_{2,p}\geq\delta$. In the remainder of the proof, l is fixed, and for more simplicity we denote by u_n the subsequence $u_{\psi(n)}$.

Note that the hypotheses made about (u_n) imply by Lemma 2.10 that $u_n\to 0$ in $W^{2,p}(\Omega',\mathbb{R}^m)$ for every open and bounded subset $\Omega'\subset\Omega$. Therefore $u_n^j\to 0$ in $W^{2,p}(\Omega')$ $\forall\, j=1,\ldots,m$.

Let ε_n be a sequence from (0,1) such that $\sum\limits_{n=0}^{\infty}\varepsilon_n=\delta^p$.

We construct a sequence (v_n) in $D_p(\Omega)$, and a subsequence $(u_{\varphi(n)})$ verifying $v_{n+1}=v_n+u_{\varphi(n+1)}$. Recall that $Lu_n\to 0$ so there is an integer $\varphi(0)$ for which $\|Lu_{\varphi(0)}\|_{0,p}\leq\varepsilon_0$. Set $v_0=u_{\varphi(0)}$. In Lemma 2.15 let $v=u^j_{\varphi(0)},\varepsilon=\varepsilon_1$ and $n_0=\varphi(0)$. This produces a integer $n_1(j)$. Also there is a $n_2\in\mathbb{N}$ such that $k\geq n_2\Rightarrow\|Lu_k\|_{0,p}\leq\varepsilon_1$. Set then $\varphi(1)=\max\{n_2,n_1(j),1\leq j\leq m\}+1$, and $v_1=v_0+u_{\varphi(1)}$. By induction suppose (v_n) and $\varphi(n)$ already constructed. Let then $v=v_n^j$ and $\varepsilon=\varepsilon_{n+1}$ and $n_0=\varphi(n)$ in Lemma 2.15. This produces an integer $n_1(j)$ from which the estimates of this Lemma hold. Also there is a $n_2\in\mathbb{N}$ such that $k\geq n_2\Rightarrow\|Lu_k\|_{0,p}\leq\varepsilon_{n+1}$. Set then $\varphi(n+1)=\max\{n_2,n_1(j):1\leq j\leq m\}+1$, and $v_{n+1}=v_n+u_{\varphi(n+1)}$. Note that by construction $\|Lu_{\varphi(n)}\|_{0,p}\leq\varepsilon_n$. Note also that the relation defining v_n, shows by induction that $v_n\in D_p(\Omega)$.

By Lemma 2.15 (i), we have

$$\|v_k^j\|^p_{2,p}+\|u^j_{\varphi(k+1)}\|^p_{2,p}-\varepsilon_{k+1}\leq\|v^j_{k+1}\|^p_{2,p}\leq\|v^j_k\|^p_{2,p}+\|u^j_{\varphi(k+1)}\|^p_{2,p}+\varepsilon_{k+1}\quad\forall\,k\in\mathbb{N}.$$

Thus by summation for $n\geq 1$,

$$\sum_{k=0}^{n}\|u^j_{\varphi(k)}\|^p_{2,p}-\sum_{k=1}^{n}\varepsilon_k\leq\|v_n^j\|^p_{2,p}\leq\sum_{k=0}^{n}\|u^j_{\varphi(k)}\|^p_{2,p}+\sum_{k=1}^{n}\varepsilon_k. \tag{2.37}$$

Now taking $j=l$ in the above, we get for $n\geq n_0$ (l and n_0 are defined in the beginning of the proof) $(n-n_0)\delta^p\leq\|v_n^l\|^p_{2,p}$ and therefore

$$(n-n_0)^{1/p}\,\delta\leq\|v_n^l\|_{2,p}\leq\|v_n\|_{2,p}\quad\text{for }\ n\geq n_0. \tag{2.38}$$

Let $M_j\geq 1$ be a bound for $\|u_n^j\|_{2,p}$, so that $M=\sum_{j=1}^m M_j$ is a bound of $\|u_n\|_{2,p}$. The second inequality of (2.37) yields: $\|v_n^j\|_{2,p}\leq M_j\,(n+1+\delta^p)^{1/p}$. Therefore

$$\|v_n\|_{2,p}\leq M\,(n+1+\delta^p)^{1/p}\,. \tag{2.39}$$

From Lemma 2.15 (ii),

$$\|v^j_{n+1}\|_{1,\infty}\leq\max\{\|v_n^j\|_{1,\infty},\|u^j_{\varphi(n+1)}\|_{1,\infty}\}+\varepsilon_{n+1}.$$

Hence, by induction

$$\|v_n^j\|_{1,\infty}\leq\max\{\|u^j_{\varphi(k)}\|_{1,\infty}\ :0\leq k\leq n\}+\sum_{k=1}^{n}\varepsilon_k,$$

and thus

$$\|v_n\|_{1,\infty}\leq m\,(C+\delta^p)\,, \tag{2.40}$$

where C is a bound for $\|u_n\|_{1,\infty}$.

Next,

$$\|Lv_n\|_{0,p} \le \sum_{k=0}^{n} \|Lu_{\varphi(k)}\|_{0,p} \le \sum_{k=0}^{n} \varepsilon_k.$$

Therefore,

$$\|Lv_n\|_{0,p} \le \delta^p. \tag{2.41}$$

Now set $w_n = n^{-1/p} v_n$. Then first, by (2.38) $\|w_n\|_{2,p} \ge \delta\,(1-(n_0/n))^{1/p}$ so that (w_n) contains no subsequence converging to 0 in $D_p(\Omega)$. Secondly, by (2.39) (w_n) is bounded in $D_p(\Omega)$. Next by (2.40) $\|w_n\|_{1,\infty} \le$ const. $\times\, n^{-1/p}$, whence $w_n \to 0$ in $C^1_d(\overline{\Omega}, \mathbb{R}^m)$. Lastly by (2.41), $\|Lw_n\|_{0,p} \le \delta^p n^{-1/p}$ which implies that $Lw_n \to 0$ in $Y_p(\Omega)$. That $w_n \rightharpoonup 0$ in $D_p(\Omega)$ follows from its boundedness in $D_p(\Omega)$ and its convergence to 0 in $C^1_d(\overline{\Omega}, \mathbb{R}^m)$. □

Theorem 2.6 *Let Ω have a C^2 boundary, and L be an elliptic operator as in the preceding lemma. Then the following statements are equivalent.*

(i) $L \in \Phi_+(D_p(\Omega), Y_p(\Omega))$.

(ii) *Every bounded sequence $(u_n) \subset D_p(\Omega)$ converging to zero in $C^1_d(\overline{\Omega}, \mathbb{R}^m)$ and such that $Lu_n \to 0$ in $Y_p(\Omega)$, contains a subsequence converging to zero in $D_p(\Omega)$*[6].

Proof. (i) $\Rightarrow$ (ii). Recall that a bounded sequence in $D_p(\Omega)$ converging to 0 in $C^1_d(\overline{\Omega}, \mathbb{R}^m)$, is weakly convergent to zero in $D_p(\Omega)$, and by Yood's criterion L is proper on the closed bounded subsets of $D_p(\Omega)$. Then, the result follows from Note C in §1.3.

(ii) $\Rightarrow$ (i). It suffices to show (by the same note) that if (u_n) is sequence in $D_p(\Omega)$, such that $u_n \rightharpoonup 0$ in $D_p(\Omega)$ and $Lu_n \to 0$ in $Y_p(\Omega)$, then $u_n \to 0$ in $D_p(\Omega)$. If this is false, then there is a subsequence $(u_{\phi(n)})$ bounded away from zero in $D_p(\Omega)$ (which implies that it contains no subsequence converging to zero). Hence $(u_{\phi(n)})$ satisfies the conditions of Lemma 2.16, and accordingly, there is a sequence (w_n) having the same properties as $(u_{\phi(n)})$ and furthermore converging to zero in $C^1_d(\overline{\Omega}, \mathbb{R}^m)$. Then by assumption (w_n) contains a subsequence converging to zero in $D_p(\Omega)$. But this is impossible since (w_n) contains no subsequence converging to 0 in $D_p(\Omega)$. □

Corollary 2.2 *Let Ω have a C^2 boundary. Suppose that every bounded sequence $(u_n) \subset D_p(\Omega)$ converging to zero in $C^1_d(\overline{\Omega}, \mathbb{R}^m)$ and such that $F(u_n) \to F(0)$ in $Y_p(\Omega)$, contains a subsequence converging to zero in $D_p(\Omega)$. (It is so if F is proper on the closed bounded subsets of $D_p(\Omega)$). Then $\mathrm{D}F(u) \in \Phi_+(D_p(\Omega), Y_p(\Omega))$ for all $u \in D_p(\Omega)$.*

Proof. By Theorem 2.3 it suffices to show that $\mathrm{D}F(0) \in \Phi_+(D_p(\Omega), Y_p(\Omega))$. According to Theorem 2.6 with $L = \mathrm{D}F(0)$, it is sufficient to show that if (u_n) is a bounded sequence from $D_p(\Omega)$ converging to zero in $C^1_d(\overline{\Omega}, \mathbb{R}^m)$ and $(\mathrm{D}F(0)u_n)$ converges to 0 in $Y_p(\Omega)$, then (u_n) contains a subsequence converging to 0 in $D_p(\Omega)$. By Lemma 2.13 we have

$$F(u_n) - F(0) - \mathrm{D}F(0)u_n \to 0 \ \text{ in } Y_p(\Omega).$$

But $\mathrm{D}F(0)u_n \to 0$, and therefore $F(u_n) \to F(0)$. Hence by assumption (u_n) contains a subsequence converging to 0 in $D_p(\Omega)$. □

[6] which in turn implies that $u_n \to 0$ in $D_p(\Omega)$, by Note A3 of the appendix.

2.5 Operators with asymptotically periodic coefficients

In this section, we consider the case where F has a limit operator with periodic coefficients in a sense precised below. Here Ω is unbounded and so $K = \complement\Omega$ is bounded according to our assumptions.

When we deal with periodic functions, it is necessary to assume them defined on the whole space $\mathbb{R}^N$. So let $T = (T_1, \ldots, T_N) \in \mathbb{R}^N$ with $T_i > 0$. A mapping f defined on $\mathbb{R}^N$ is said to be periodic with period T if $f(x_1, \ldots, x_i + T_i, \ldots, x_N) = f(x_1, \ldots, x_N)\ \forall x \in \mathbb{R}^N$. We use the following notation for $n \in \mathbf{Z}$ and T as above, $nT = (nT_1, \ldots, nT_N)$, and for $l \in \mathbf{Z}^N$, $lT = (l_1T_1, \ldots, l_NT_N)$.

We maintain the previous notation for $r > 0$: B_r is the ball of center 0 and radius r, $\tilde{B}_r = \{x \in \mathbb{R}^N : |x| > r\}$, $\Omega_r = \Omega \cap B_r$ and $\tilde{\Omega}_r = \Omega \cap \tilde{B}_r$.

Assume that there are two families of matrix valued functions,

$$a^\infty_{\alpha\beta} : \mathbb{R}^N \times \left(\mathbb{R}^m \times \mathbb{R}^{m\times N}\right) \to \mathbb{R}^{m\times m}, \quad 1 \le \alpha, \beta \le N,$$

and

$$c^\infty_i : \mathbb{R}^N \times \left(\mathbb{R}^m \times \mathbb{R}^{m\times N}\right) \to \mathbb{R}^{m\times m}, \quad 0 \le i \le N,$$

both continuous and periodic in x with the same period T, and satisfying

$$\lim_{|x|\to\infty} \left|a_{\alpha\beta}(x,\xi) - a^\infty_{\alpha\beta}(x,\xi)\right| = 0, \tag{2.42}$$

$$\lim_{|x|\to\infty} \left|\int_0^1 \nabla_{\xi_i} b(x, t\xi)\,\mathrm{d}t - c^\infty_i(x,\xi)\right| = 0, \tag{2.43}$$

the convergence being uniform on the compact subsets of $\mathbb{R}^m \times \mathbb{R}^{m\times N}$. We set

$$b^\infty(x,\xi) = \sum_{i=0}^N c^\infty_i(x,\xi)\xi_i, \tag{2.44}$$

so that $b^\infty(x,0) = 0$.

Note that by Remark 2.7, $a^\infty_{\alpha\beta}$ and b^∞ are equicontinuous C^0-bundle maps. Now we define the limit operator F^∞ by

$$F^\infty(u) = -\sum_{\alpha,\beta=1}^N a^\infty_{\alpha\beta}(.,u,\nabla u)\partial^2_{\alpha\beta}u + b^\infty(.,u,\nabla u). \tag{2.45}$$

Observe that by Lemma 2.7 and Remark 2.9, F^∞ is continuous and weakly continuous from X_p to Y_p, as well as from $X_p(\Omega)$ to $Y_p(\Omega)$ and maps bounded subsets onto bounded ones. Note also that

$$\begin{aligned} F^\infty(v) - F^\infty(0) - \left(-\sum_{\alpha,\beta=1}^N a^\infty_{\alpha\beta}(.,0)\partial^2_{\alpha\beta}v + \sum_{\alpha=0}^N c^\infty_\alpha(.,0)\partial_\alpha v\right) \\ = -\sum_{\alpha,\beta=1}^N \left(a^\infty_{\alpha\beta}(.,v,\nabla v) - a^\infty_{\alpha\beta}(.,0)\right)\partial^2_{\alpha\beta}v + \sum_{\alpha=0}^N \left(c^\infty_\alpha(.,v,\nabla v) - c^\infty_\alpha(.,0)\right)\partial_\alpha v \end{aligned}$$

So it follows from the equicontinuity of $a^\infty_{\alpha\beta}$ and c^∞_α at $\xi = 0$, that F^∞ is differentiable at 0 with derivative

$$\mathrm{D}F^\infty(0)v = -\sum_{\alpha,\beta=1}^N a^\infty_{\alpha\beta}(.,0)\partial^2_{\alpha\beta}v + \sum_{\alpha=0}^N c^\infty_\alpha(.,0)\partial_\alpha v \tag{2.46}$$

Lemma 2.17 *Let $\tilde{\Omega}_r = \{x \in \Omega : |x| > r\}$ and $\mathcal{B} \subset X_p(\Omega)$ be a bounded subset. Then for every $\varepsilon > 0$ there is an $r > 0$ such that for every $u \in \mathcal{B}$, the following hold.*

(i) $\|F(u) - F(0) - F^\infty(u)\|_{0,p,\tilde{\Omega}_r} \le \varepsilon$, *and*

(ii) $\|\mathrm{D}F(0)u - \mathrm{D}F^\infty(0)u\|_{0,p,\tilde{\Omega}_r} \le \varepsilon$.

Proof. (i) Since $\mathcal{B}$ is bounded in $X_p(\Omega)$ and therefore in $C^1_d(\overline{\Omega}, \mathbb{R}^m)$, there is a compact set $K \subset \mathbb{R}^m \times \mathbb{R}^{m\times N}$ such that $(u(x), \nabla u(x)) \in K$ for every $u \in \mathcal{B}$ and $x \in \Omega$. Since the limit in (2.42) is uniform in $\xi \in K$, $|a_{\alpha\beta}(x,\xi) - a^\infty_{\alpha\beta}(x,\xi)| \le \varepsilon \ \forall x \in \tilde{\Omega}_r \ \forall \xi \in K$ if r is large enough. Thus

$$\|\boldsymbol{a}_{\alpha\beta}(u) - \boldsymbol{a}^\infty_{\alpha\beta}(u)\|_{0,\infty,\tilde{\Omega}_r} \le \varepsilon \quad \forall\, u \in \mathcal{B}. \tag{2.47}$$

A similar argument based on (2.43) yields

$$\left| \int_0^1 \nabla_{\xi_i} b(x, t\xi)\,\mathrm{d}t - c_i^\infty(x,\xi) \right| \le \varepsilon \quad \forall\, x \in \tilde{\Omega}_r \quad \forall\, \xi \in K$$

if r is large enough, and thus

$$|b(x,\xi) - b(x,0) - b^\infty(x,\xi)| \le \varepsilon \sum_{i=0}^{N} |\xi_i|.$$

Therefore for $k = 1, \dots, m$,

$$\begin{aligned}
\|\boldsymbol{b}^k(u) - \boldsymbol{b}^k(0) - \boldsymbol{b}^{\infty,k}(u)\|_{0,p,\tilde{\Omega}_r} &\le \varepsilon \sum_{i=0}^{N} \|\partial_i u\|_{0,p,\tilde{\Omega}_r} \\
&\le \varepsilon \sum_{i=0}^{N} \sum_{j=1}^{m} \left\|\partial_i u^j\right\|_{0,p,\tilde{\Omega}_r} \\
&\le \varepsilon m(N+1) \|u\|_{2,p,\Omega} \quad \forall\, u \in \mathcal{B}.
\end{aligned}$$

Thus

$$\|\boldsymbol{b}(u) - \boldsymbol{b}(0) - \boldsymbol{b}^\infty(u)\|_{0,p,\tilde{\Omega}_r} \le \varepsilon m^2 (N+1) \|u\|_{2,p,\Omega}. \tag{2.48}$$

With (2.47) we get

$$\|F(u) - F(0) - F^\infty(u)\|_{0,p,\tilde{\Omega}_r} \le \varepsilon m^2 (N^2 + N + 1) \|u\|_{2,p,\Omega} \quad \forall\, u \in \mathcal{B}.$$

Lastly note that ε is arbitrary and $\|u\|_{2,p,\Omega}$ is bounded. Hence the desired result follows.

(ii) The proof is similar. Recall that

$$\mathrm{D}F(0)u - \mathrm{D}F^\infty(0)u = -\sum_{\alpha,\beta=1}^{N} \left(a_{\alpha\beta}(.,0) - a^\infty_{\alpha\beta}(.,0)\right) \partial^2_{\alpha\beta} u + \sum_{\alpha=0}^{N} \left(\nabla_{\xi_\alpha} b(.,0) - c^\infty_\alpha(.,0)\right) \partial_\alpha u.$$

Thus $\|\mathrm{D}F(0)u - \mathrm{D}F^\infty(0)u\|_{0,p,\tilde{\Omega}_r} \le \varepsilon m^2 (N^2 + N + 1) \|u\|_{2,p,\Omega}$. □

Corollary 2.3 *Let (u_n) be a bounded sequence from $X_p(\Omega)$ such that $u_n \to 0$ in $X_p(\Omega')$ for every bounded open subset $\Omega' \subset \Omega$. Then we have the following.*

(i) $F(u_n) - F(0) - F^\infty(u_n) \to 0$ *in* $Y_p(\Omega)$.

(ii) $(\mathrm{D}F(0)-\mathrm{D}F^\infty(0))u_n \to 0$ *in* $Y_p(\Omega)$.

Proof. (i) Let $\varepsilon >$ be given. Since (u_n) is bounded in $X_p(\Omega)$, it follows from Lemma 2.17 (i) that for $r>0$ large enough we have $\|F(u_n)-F(0)-F^\infty(u_n)\|_{0,p,\tilde{\Omega}_r} \le \varepsilon \; \forall n \in \mathbb{N}$. Next recall that $u_n \to 0$ in $X_p(\Omega_r)$ by hypotheses and that F and F^∞ are continuous from $X_p(\Omega_r)$ to $Y_p(\Omega_r)$ by Lemma 2.7 and Remark 2.9. Therefore $F(u_n)\to F(0)$ and $F^\infty(u_n)\to 0$ in $Y_p(\Omega_r)$, which means that $\|F(u_n)-F(0)-F^\infty(u_n)\|_{0,p,\Omega_r} \le \varepsilon$ for n large enough. Thus $\|F(u_n)-F(0)-F^\infty(u_n)\|_{0,p,\Omega}$ can be made arbitrary small for n large enough.

(ii) The proof is similar. First by Lemma 2.17 (ii) we have $\|\mathrm{D}F(0)u_n - \mathrm{D}F^\infty(0)u_n\|_{0,p,\tilde{\Omega}_r} \le \varepsilon$. Next $\mathrm{D}F(0)\in\mathcal{L}(X_p(\Omega_r),Y_p(\Omega_r))$ by Theorem 2.2 with $\Omega=\Omega_r$ and it is clearly seen from (2.46) that $\mathrm{D}F^\infty(0)\in\mathcal{L}(X_p(\Omega_r),Y_p(\Omega_r))$. Therefore $(\mathrm{D}F(0)-\mathrm{D}F^\infty(0))\,u_n \to 0$ in $Y_p(\Omega_r)$. And thus $\|(\mathrm{D}F(0)-\mathrm{D}F^\infty(0))\,u_n\|_{0,p,\Omega}$ can be made arbitrary small for n large enough. □

Given $h\in\mathbb{R}^N$ and a function $f:\mathbb{R}^N\to\mathbb{R}$, we denote by $\tau_h(f):\mathbb{R}^N\to\mathbb{R}$ the function

$$\tau_h(f)(x) = f(x+h)$$

Corollary 2.4 *Let $\mathcal{B}\subset X_p(\Omega)$ be a bounded subset, and $\Omega'\subset\mathbb{R}^N$ be a bounded open subset. Then for every $\varepsilon>0$, we have $\|\tau_h(F(u)-F(0))-\tau_h F^\infty(u)\|_{0,p,\Omega'}\le\varepsilon$ for every $u\in\mathcal{B}$ provided $|h|$ is large enough.*

Proof. Choose $r>0$ as in Lemma 2.17 and increase it if necessary to have $\tilde{B}_r\subset\Omega$. Then for $|h|$ large enough we have $\Omega'+h\subset\tilde{B}_r\subset\Omega$. By the translation invariance of the Lebesgue measure,

$$\|\tau_h(F(u)-F(0))-\tau_h F^\infty(u)\|_{0,p,\Omega'} = \|F(u)-F(0)-F^\infty(u)\|_{0,p,\Omega'+h}.$$

Now, by Lemma 2.17

$$\|F(u)-F(0)-F^\infty(u)\|_{0,p,\Omega'+h} \le \|F(u)-F(0)-F^\infty(u)\|_{0,p,\tilde{B}_r} \le \varepsilon$$

for every $u\in\mathcal{B}$. □

Lemma 2.18 (Shifted subsequence Lemma) *Let $T=(T_1,\dots,T_N)$ with $T_i>0$. If (u_n) is a bounded sequence from $X_p(\Omega)$, then either*

(i) *(u_n) vanishes uniformly at infinity in the sense of $C^1_d(\overline{\Omega})$, or*

(ii) *there exist a sequence $(l_n)\subset\mathbf{Z}^N$ with $\lim_{n\to\infty}|l_n|=\infty$, a subsequence $(u_{\phi(n)})$, and a nonzero element $\bar{u}\in X_p = W^{2,p}\left(\mathbb{R}^N,\mathbb{R}^m\right)$ such that the sequence $\tilde{u}_n$ defined by $\tilde{u}_n(x)=u_{\phi(n)}(x+l_nT)$ is weakly convergent to $\bar{u}$ in $X_p(B_k)$ for all $k\in\mathbb{N}^*$.*

Proof. First of all the definition of $\tilde{u}_n$ makes sense. Indeed, the domain of such shifted subsequence is $\Omega - l_nT$. Let B_R be a ball containing $K=\complement\Omega$, and $\hat{T}=\min T_i$. Given $k\in\mathbb{N}^*$ there is $n_k\in\mathbb{N}^*$ such that $|l_n|>\frac{k+R}{\hat{T}}$ for $n\ge n_k$. Then for $x\in B_k$, $|x+l_nT|\ge|l_n|\hat{T}-|x|\ge|l_n|\hat{T}-k>R$ for all $n\ge n_k$, and so $x+l_nT\in\Omega$ for all $x\in B_k$ and $n\ge n_k$. Hence B_k is in the domain of definition of $\tilde{u}_n$ for $n\ge n_k$. Furthermore $\tilde{u}_n\in X_p(B_k)$ with

$$\|\tilde{u}_n\|_{2,p,B_k}\le\|u_{\phi(n)}\|_{2,p,\Omega}\le M \tag{2.49}$$

for some constant M and for all $n \geq n_k$. Therefore it makes sense to consider $\lim_{n\to\infty} \tilde{u}_n|_{B_k}$ for any $k \in \mathbb{N}^*$.

Now we go to the proof of the alternative. Let $Q_0 = (0, T_1) \times \cdots \times (0, T_N)$. Suppose that (i) does not hold, so that there is an $\varepsilon_0 > 0$, a sequence (x_n) such that $|x_n| \geq n$, and a subsequence $(u_{\psi(n)})$ such that $|u_{\psi(n)}(x_n)| + |\nabla u_{\psi(n)}(x_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$. Since $\mathbb{R}^N = \bigcup_{l \in \mathbf{Z}^N} \left(\overline{Q_0} + lT\right)$, there is $z_n \in \mathbf{Z}^N$ for which $y_n = x_n - z_n T \in \overline{Q_0}$. Clearly $\lim_{n\to\infty} |z_n| = \infty$. Define $v_n(x) = u_{\psi(n)}(x + z_n T)$.

According to what has been said at the beginning of the proof, for every $k \in \mathbb{N}^*$ there is $n_k \in \mathbb{N}^*$ from which $v_n \in X_p(B_k)$, and furthermore $(v_n)_{n \geq n_k}$ is bounded in $X_p(B_k)$. In particular $(v_n)_{n \geq n_1}$ is bounded in $X_p(B_1)$, and so there is a subsequence $(v_{\theta_1(n)})$ converging weakly to some $\bar{u}_1 \in X_p(B_1)$. But for $n \geq n_2$, $(v_{\theta_1(n)}) \subset (v_n)$ is bounded in $X_p(B_2)$, and again there is a subsequence $(v_{\theta_2(n)}) \subset (v_{\theta_1(n)})$ converging weakly to some $\bar{u}_2 \in X_p(B_2)$. Clearly $\bar{u}_2|_{B_1} = \bar{u}_1$. Continuing the process, we construct a sequence of subsequences $(v_{\theta_k(n)})$, each of which converging weakly to $\bar{u}_k \in X_p(B_k)$, and furthermore $\bar{u}_{k+1}$ is an extension of $\bar{u}_k$.

Now we define $(\tilde{u}_n)$ as the diagonal subsequence $(v_{\theta_n(n)})$, i.e.

$$\tilde{u}_n(x) = u_{\psi(\theta_n(n))}(x + z_{\theta_n(n)} T) = u_{\phi(n)}(x + l_n T)$$

if we set $l_n = z_{\theta_n(n)}$ and $\phi(n) = \psi(\theta_n(n))$. On the other hand, we see that there is a function $\bar{u} : \mathbb{R}^N \to \mathbb{R}^m$ naturally defined by: $\bar{u}(x) = \bar{u}_k(x)$ if $x \in B_k$.

Now since $(\tilde{u}_n)_{n \geq n_k}$ is a subsequence of $(v_{\theta_k(n)})$, we have that $(\tilde{u}_n)$ converges weakly to $\bar{u}$ in $X_p(B_k)$ for all $k \in \mathbb{N}^*$. Therefore according to (2.49)

$$\|\bar{u}\|_{2,p,B_k} \leq \liminf_{n\to\infty} \|\tilde{u}_n\|_{2,p,B_k} \leq M$$

for all $k \in \mathbb{N}^*$, and so $\bar{u} \in X_p$.

It remains to show that $\bar{u} \neq 0$. Choose $k \in \mathbb{N}^*$ such that $\overline{Q_0} \subset B_k$. By the compactness of the embedding $W^{2,p}(Q_0, \mathbb{R}^m) \hookrightarrow C^1(\overline{Q_0}, \mathbb{R}^m)$ we have that $\tilde{u}_n \to \bar{u}$ in $C^1(\overline{Q_0}, \mathbb{R}^m)$, hence $\|\tilde{u}_n\|_{1,\infty,Q_0} \to \|\bar{u}\|_{1,\infty,Q_0}$. But

$$\begin{aligned}
\|\tilde{u}_n\|_{1,\infty,Q_0} &\geq \frac{1}{2}\left(|\tilde{u}_n(y_{\theta_n(n)})| + |\nabla \tilde{u}_n(y_{\theta_n(n)})|\right) \\
&= \frac{1}{2}\left(|u_{\psi(\theta_n(n))}(x_{\theta_n(n)})| + |\nabla u_{\psi(\theta_n(n))}(x_{\theta_n(n)})|\right) \\
&\geq \frac{1}{2}\varepsilon_0.
\end{aligned}$$

Therefore, $\|\bar{u}\|_{1,\infty,Q_0} \geq \frac{1}{2}\varepsilon_0$, whence $\bar{u} \neq 0$. □

Theorem 2.7 *Let Ω have a C^2 boundary. Suppose that*

(i) *there is $u^0 \in D_p(\Omega)$ for which $\mathrm{D}F(u^0) \in \Phi_+\left(D_p(\Omega), Y_p(\Omega)\right)$, and*

(ii) $\{u \in X_p \mid F^\infty(u) = 0\} = \{0\}$.

Then F is proper on the closed bounded subsets of $D_p(\Omega)$.

Proof. By Corollary 2.1 it suffices to show that if (u_n) is a bounded sequence from $D_p(\Omega)$ and $(F(u_n))$ converges to some y in $Y_p(\Omega)$, then (u_n) vanishes uniformly at infinity in the sense of $C^1_d(\overline{\Omega})$. After replacing F by $F - F(0)$ and y by $y - F(0)$, we can assume that $F(0) = 0$. Let us show that the case (ii) of Lemma 2.18 cannot occur. By contradiction suppose there is a sequence $(l_n) \subset \mathbf{Z}^N$ with $\lim_{n\to\infty} |l_n| = \infty$ and a subsequence $(u_{\phi(n)})$ such that the sequence $(\tilde{u}_n)$ defined by $\tilde{u}_n(x) = u_{\phi(n)}(x + l_n T)$ has a nonzero weak limit $\bar{u}$ in $X_p(B_k)$.

It is enough to show that $F^\infty(\bar{u}) = 0$. Let $\tilde{y}_n$ be defined by $\tilde{y}_n(x) = y(x + l_n T) = \tau_{l_n T}(y)(x)$. According to the proof of Lemma 2.18, for all $k \in \mathbb{N}^*$, $\tilde{y}_n \in X_p(B_k)$ from a certain rank n_k, and it is bounded by a constant independent of k.

Let $\psi \in C_0^\infty(\mathbb{R}^N)$, and choose $k \in \mathbb{N}^*$ such that B_k contains the support of ψ. Recall that $\tilde{u}_n \rightharpoonup \bar{u}$ in $X_p(B_k)$ and $F^\infty : X_p(B_k) \to Y_p(B_k)$ is weakly continuous, and so $F(\tilde{u}_n) \rightharpoonup F(\bar{u})$ in $Y_p(B_k)$. Thus

$$\begin{aligned}\int_{\mathbb{R}^N} \psi F^\infty(\bar{u})\,\mathrm{d}x &= \int_{B_k} \psi F^\infty(\bar{u})\,\mathrm{d}x = \lim_{n\to\infty} \int_{B_k} \psi F^\infty(\tilde{u}_n)\,\mathrm{d}x \\ &= \lim_{n\to\infty} \int_{B_k} \psi \tau_{l_n T} F^\infty(u_{\phi(n)})\,\mathrm{d}x. \end{aligned} \tag{2.50}$$

On the other hand according to Corollary 2.4 (we assumed that $F(0) = 0$),

$$\tau_{l_n T} F(u_{\phi(n)}) - \tau_{l_n T} F^\infty(u_{\phi(n)}) \underset{n\to\infty}{\longrightarrow} 0 \text{ in } Y_p(B_k). \tag{2.51}$$

Next for $n \geq n_k$, and $j = 1, \ldots, m$, we have

$$\begin{aligned}\left\| \left(\tau_{l_n T} F(u_{\phi(n)}) - \tilde{y}_n\right)_j \right\|_{0,p,B_k}^p &= \int_{B_k} \left|\tau_{l_n T} F_j(u_{\phi(n)})(x) - \tau_{l_n T} y_j(x)\right|^p \mathrm{d}x \\ &= \int_{|z - l_n T| < k} \left|F_j(u_{\phi(n)})(z) - y_j(z)\right|^p \mathrm{d}z \\ &\leq \left\|F_j(u_{\phi(n)}) - y_j\right\|_{0,p,\Omega}^p \underset{n\to\infty}{\longrightarrow} 0 \quad \text{by assumption.} \end{aligned} \tag{2.52}$$

Now together equations (2.50), (2.51) and (2.52), give

$$\int_{\mathbb{R}^N} \psi F^\infty(\bar{u})\,\mathrm{d}x = \lim_{n\to\infty} \int_{B_k} \psi \tilde{y}_n\,\mathrm{d}x.$$

But for each component $(\tilde{y}_n)_j$, $j = 1, \ldots, m$, we have

$$\begin{aligned}\int_{B_k} \psi (\tilde{y}_n)_j\,\mathrm{d}x &= \int_{B_k} \psi(x) y_j(x + l_n T)\,\mathrm{d}x = \int_{B(l_n T, k)} \psi(z - l_n T) y_j(z)\,\mathrm{d}z \\ &\leq \Bigg(\int_{B(l_n T,k)} |y_j|^p\Bigg)^{\frac{1}{p}} \Bigg(\int_{B(l_n T,k)} |\psi(z - l_n T)|^q\,\mathrm{d}z\Bigg)^{\frac{1}{q}} \qquad \Big(\frac{1}{p} + \frac{1}{q} = 1\Big) \\ &\leq \Bigg(\int_{B(l_n T,k)} |y_j|^p\Bigg)^{\frac{1}{p}} \Bigg(\int_{\mathbb{R}^N} |\psi|^q\Bigg)^{\frac{1}{q}} \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

because $B(l_n T, k) \subset \tilde{B}_{|l_n T| - k}$ so that the result follows from Note B3 in the appendix.

Thus, finally

$$\int_{\mathbb{R}^N} \psi F^\infty(\bar{u})\,\mathrm{d}x = 0 \quad \text{ for all } \psi \in C_0^\infty(\mathbb{R}^N),$$

and consequently $F^{\infty}(\bar{u}) = 0$. □

The last theorem shows that together the semi-Fredholmness of F and the nonexistence of nontrivial solutions of the limit problem $F^{\infty}(u) = 0$, are sufficient conditions for the properness of F on $D_p(\Omega)$. Are they also necessary conditions? We already know by Corollary 2.2 that the semi-Fredholmness is necessary. It turns out that in the case of $\Omega = \mathbb{R}^N$, the second condition is also necessary.

Accordingly assume in the sequel that conditions (2.10)–(2.13) and (2.18) are satisfied on the hole space $\mathbb{R}^N$. Consequently the results obtained so far, are true and will be applied with $\Omega = \mathbb{R}^N$.

Lemma 2.19 *Let $1 < q < \infty$ and $k \in \mathbb{N}$. Given $u \in W^{k,q}(\mathbb{R}^N, \mathbb{R}^m)$ and a sequence $(h_n) \subset \mathbb{R}^N$ such that $\lim_{n\to\infty} |h_n| = \infty$, set $\tilde{u}_n(x) = u(x + h_n)$. Then $\tilde{u}_n \to 0$ in $W^{k,q}(\Omega', \mathbb{R}^m)$ for every bounded open subset $\Omega' \subset \mathbb{R}^N$. In particular $\tilde{u}_n \rightharpoonup 0$ in $W^{k,q}(\mathbb{R}^N, \mathbb{R}^m)$.*

Proof. For $m = 1$, this is Lemma 4.8 of **[31]**. The conclusion is then clear, since convergence, respectively weak convergence in $W^{k,q}(\mathbb{R}^N, \mathbb{R}^m)$, is equivalent to this type of convergence of each component in $W^{k,q}(\mathbb{R}^N)$. □

Theorem 2.8 *The following statements are equivalent.*

(i) *F is proper on the closed bounded subsets of X_p.*

(ii) *Every sequence $(u_n) \subset X_p$ such that $u_n \rightharpoonup 0$ in X_p and $F(u_n) \to F(0)$ in Y_p, contains a subsequence converging in X_p.*

(iii) *There is $u^0 \in X_p$ for which $\mathrm{D}F(u^0) \in \Phi_+(X_p, Y_p)$ and the equation $F^{\infty}(u) = 0$ has no nonzero solution in X_p.*

Proof. (i) ⇒ (ii) is evident since a weakly convergent sequence is bounded. (iii) ⇒ (i) is Theorem 2.7 (with $\Omega = \mathbb{R}^N$). It remains to prove (ii) ⇒ (iii). That $\mathrm{D}F(0) : X_p \to Y_p$ is semi-Fredholm follows from Corollary 2.2. Consider now an element $u \in X_p$ such that $F^{\infty}(u) = 0$. Set $u_n(x) = u(x + nT)$ so that $F^{\infty}(u_n) = 0$ by the periodicity of the the coefficients of F^{∞}. By Lemma 2.19, $u_n \to 0$ in $X_p(\Omega')$ for every open bounded subset $\Omega' \subset \mathbb{R}^N$, and hence by Corollary 2.3 (i) $F(u_n) - F(0) \to 0$ in Y_p. Also $u_n \rightharpoonup 0$ in X_p, and therefore by hypotheses (u_n) contains a convergent subsequence $(u_{\phi(n)})$. Its limit is necessarily 0 (by the uniqueness of the weak limit in X_p). But recall that $\|u_n\|_{2,p,\mathbb{R}^N} = \|u\|_{2,p,\mathbb{R}^N}$, hence $u = 0$. □

Remark 2.10 Note that strict Petrovskii-ellipticity on the compact subsets of $\mathbb{R}^N$ and strict ellipticity on $\mathbb{R}^N$ are equivalent for F^{∞}. This is due to the periodicity of $a^{\infty}_{\alpha\beta}$.

On the other hand note that $\det\left(\sum_{\alpha,\beta=1}^{N} \eta_\alpha \eta_\beta a_{\alpha\beta}(x, 0)\right)$ is a homogeneous polynomial of order $2m$ in η, so it could be written as[7] $P(x, \eta) = \sum_{|\gamma|=2m}^{N} p_\gamma(x)\eta^\gamma$, and the coefficients are algebraic combination of the components of the matrices $a_{\alpha\beta}$. Similarly ,

$$\det\left(\sum_{\alpha,\beta=1}^{N} \eta_\alpha \eta_\beta a^{\infty}_{\alpha\beta}(x, 0)\right) = P^{\infty}(x, \eta) = \sum_{|\gamma|=2m}^{N} p^{\infty}_\gamma(x)\eta^\gamma.$$

[7] $\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathbb{N}^N$ is a multi-index

Thus from (2.42) it follows that given $\varepsilon > 0$, $|p_\gamma(x) - p_\gamma^\infty(x)| \leq \varepsilon$ for $|x|$ large enough. Therefore $|P(x,\eta) - P^\infty(x,\eta)| \leq N^{2m}\varepsilon|\eta|^{2m}$. And thus if $P(x,\eta) \geq \lambda|\eta|^{2m}$ we have

$$P^\infty(x,\eta) \geq (\lambda - N^{2m}\varepsilon)|\eta|^{2m}.$$

Therefore the strict ellipticity condition (in $\mathbb{R}^N$) for $a_{\alpha\beta}(.,0)$ is equivalent to the strict ellipticity of $a_{\alpha\beta}^\infty(.,0)$.

Note that this reasoning also proves the stability of the ellipticity condition i.e. an elliptic system remains elliptic after a small enough perturbation of its leading coefficients.

2.6 Linear systems

2.6.1 Linear systems on $\mathbb{R}^N$

Let $S = S(\mathbb{R}^N)$ denote the Schwartz space of rapidly decreasing functions and S' its dual, the space of tempered distributions. Recall that $L^p(\mathbb{R}^N) \subset S'$ for $1 \leq p \leq \infty$, in the sense that $T_f \in S'$ for all $f \in L^p(\mathbb{R}^N)$ where T_f is defined by

$$\langle T_f, \varphi\rangle = \int_{\mathbb{R}^N} f\varphi \,\mathrm{d}x \quad \text{for all } \varphi \in S.$$

Multiplication by any function $\Phi \in C^\infty(\mathbb{R}^N)$ which is slowly increasing (that is, Φ and all its partial derivatives have at most polynomial growth at infinity) defines a continuous mapping of S into itself and consequently a multiplication in S', denoted by $\Phi(\eta) : S' \to S'$, can be defined through

$$\langle \Phi(\eta)u, \varphi\rangle = \langle u, \Phi\varphi\rangle \quad \text{for all } u \in S' \text{ and } \varphi \in S,$$

and is a continuous mapping of S' into itself. In the same way, the Fourier transform $\mathcal{F}$ defined by

$$(\mathcal{F}\varphi)(\eta) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-ix\cdot\eta}\varphi(x)\,\mathrm{d}x$$

which is a continuous bijection of S onto itself, induces a continuous bijection $\mathcal{F} : S' \to S'$ by

$$\langle \mathcal{F}u, \varphi\rangle = \langle u, \mathcal{F}\varphi\rangle \quad \text{for all } u \in S' \text{ and } \varphi \in S.$$

Recall also that (see for instance **[38]** Theorem 2.3.3. p. 177), for $1 < p < \infty$,

$$W^{2,p}(\mathbb{R}^N) = \left\{ g \in L^p(\mathbb{R}^N) \,\middle|\, \mathcal{F}^{-1}(1+|\eta|^2)\mathcal{F}T_g = T_f \text{ for some } f \in L^p(\mathbb{R}^N) \right\}$$

and that there is a constant $K > 0$ such that

$$\|g\|_{2,p} \leq K \|f\|_{0,p}.$$

A function $\Phi : \mathbb{R}^N \to \mathbb{C}$ is called a multiplier in L^p if $u \mapsto \mathcal{F}^{-1}\Phi(\eta)\mathcal{F}u$ defines a continuous linear map from $L^p(\mathbb{R}^N)$ into itself. As a special case of a result due to Miklin (Theorem 2, Appendix **[27]**) we can formulate the following sufficient condition for a function to be a multiplier.

(M) $\Phi \in C^\infty(\mathbb{R}^N)$ is a slowly increasing function and there exists a constant $M > 0$ such that, for all $k \in \{0, 1, ..., N\}$,

$$|\eta|^k \left| \frac{\partial^k \Phi(\eta)}{\partial\eta_{j_1}...\partial\eta_{j_k}} \right| \leq M \quad \text{for all } \eta \in \mathbb{R}^N,$$

where $1 \le j_1 < j_2 < \cdots < j_k \le N$.

If Φ satisfies **(M)**, then, for $1 < p < \infty$, there is a constant $C_p > 0$ such that

$$\left|\left\langle \mathcal{F}^{-1}\Phi(\eta)\mathcal{F}T_f, \varphi\right\rangle\right| \le C_p \|f\|_{0,p} \|\varphi\|_{0,p'} \quad \text{for all } \ f \in L^p(\mathbb{R}^N) \text{ and } \varphi \in S,$$

where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Hence there is a unique element $\theta_\Phi(f) \in L^p(\mathbb{R}^N)$ such that

$$\left\langle \mathcal{F}^{-1}\Phi(\eta)\mathcal{F}T_f, \varphi\right\rangle = \int_{\mathbb{R}^N} \varphi\theta_\Phi(f)\,\mathrm{d}x \qquad \text{for all } \ \varphi \in S.$$

Thus $\mathcal{F}^{-1}\Phi(\eta)\mathcal{F}T_f = T_{\theta_\Phi(f)}$ and $\theta_\Phi \in \mathcal{L}(L^p(\mathbb{R}^N), L^p(\mathbb{R}^N))$.

We can now formulate a condition ensuring that $\theta_\Phi \in \mathcal{L}(L^p(\mathbb{R}^N), W^{2,p}(\mathbb{R}^N))$.

(W) Let Φ and Ψ satisfy the condition **(M)** where $\Psi(\eta) = (1 + |\eta|^2)\Phi(\eta)$.

If **(W)** is satisfied by Φ and $f \in L^p(\mathbb{R}^N)$, we have that $T_{\theta_\Phi(f)} = \mathcal{F}^{-1}\Phi(\eta)\mathcal{F}T_f \in S'$ and

$$\mathcal{F}^{-1}(1 + |\eta|^2)\mathcal{F}T_{\theta_\Phi(f)} = \mathcal{F}^{-1}(1 + |\eta|^2)\Phi(\eta)\mathcal{F}T_f = T_{\theta_\Psi(f)}$$

where $\theta_\Psi(f) \in L^p(\mathbb{R}^N)$ since Ψ also satisfies **(M)**. Thus $\theta_\Phi(f) \in W^{2,p}(\mathbb{R}^N)$ and there exists a constant $K > 0$ such that

$$\|\theta_\Phi(f)\|_{2,p} \le K \|\theta_\Psi(f)\|_{0,p} \le K_1 \|f\|_{0,p} \qquad \text{for all } f \in L^p(\mathbb{R}^N),$$

showing that $\theta_\Phi \in \mathcal{L}(L^p(\mathbb{R}^N), W^{2,p}(\mathbb{R}^N))$ for $1 < p < \infty$.

All the above discussion extends to vector valued functions and distributions in an obvious way.

For a second order linear $m \times m-$ system with constant coefficients

$$Lu = -\sum_{\alpha,\beta=1}^{N} A_{\alpha\beta}\partial^2_{\alpha\beta}u + \sum_{\alpha=1}^{N} B_\alpha\partial_\alpha u + Cu \tag{2.53}$$

where $A_{\alpha\beta}, B_\alpha$ and C are a real $m \times m$ matrices, its characteristic polynomial (or symbol) is the $m \times m$ matrix polynomial $S(\eta)$ defined by

$$S(\eta) = \sum_{\alpha,\beta=1}^{N} \eta_\alpha\eta_\beta A_{\alpha\beta} + i\sum_{\alpha=1}^{N} \eta_\alpha B_\alpha + C.$$

Note the the characteristic polynomial of the formal adjoint operator

$$L^t u = -\sum_{\alpha,\beta=1}^{N} A^T_{\alpha\beta}\partial^2_{\alpha\beta}u - \sum_{\alpha=1}^{N} B^T_\alpha\partial_\alpha u + C^T u$$

is the matrix $S(\eta)^* = \overline{S(\eta)^T}$ where T denotes the transpose of a matrix.

(S) There exist constants $\mu > 0$ and $\gamma > 0$ such that

$$|\det S(\eta)| \ge \mu\,|\eta|^{2m} + \gamma \qquad \text{for all } \eta \in \mathbb{R}^N.$$

Then, since

$$S(\eta)^{-1} = \frac{1}{\det S(\eta)}[\mathrm{Cof}\, S(\eta)]^T,$$

and the elements of the cofactor matrix $\mathrm{Cof}\, S(\eta)$ are polynomials in η of degree at most $2m-2$, it follows that all the elements of the matrix $S(\eta)^{-1}$ satisfy the condition **(W)** when **(S)** holds.

Recall that L acts on S', through the relation

$$\langle Lu, \varphi \rangle = \langle u, L^t\varphi \rangle \quad \text{for all } u \in S' \text{ and } \varphi \in S.$$

Furthermore, $\mathcal{F}Lu = S(\eta)\mathcal{F}u$ for all $u \in S'$ and so

$$\mathcal{F}L^t u = S(\eta)^*\mathcal{F}u.$$

Theorem 2.9 *Let L be a linear differential operator of the form (2.53) whose characteristic polynomial satisfies* **(S)**. *Then $L : W^{2,p}(\mathbb{R}^N, \mathbb{R}^m) \to L^p(\mathbb{R}^N, \mathbb{R}^m)$ is an isomorphism.*

Proof. Let $f \in L^p(\mathbb{R}^N, \mathbb{R}^m)$. The equation $Lu = T_f$ in S' is equivalent to

$$\begin{aligned} &\langle \mathcal{F}Lu, \varphi\rangle = \langle \mathcal{F}T_f, \varphi\rangle \quad \text{for all } \varphi \in S \\ \iff &\langle S(\eta)\mathcal{F}u, \varphi\rangle = \langle \mathcal{F}T_f, \varphi\rangle \quad \text{for all } \varphi \in S \\ \iff &\langle u, \varphi\rangle = \left\langle \mathcal{F}^{-1} S(\eta)^{-1} \mathcal{F}T_f, \varphi \right\rangle \quad \text{for all } \varphi \in S. \end{aligned}$$

Hence $w := \mathcal{F}^{-1}S(\eta)^{-1}\mathcal{F}T_f \in S'$ and it is the unique solution in S' of the equation $Lu = T_f$. Since the elements of the matrix $S(\eta)^{-1}$ satisfy the condition **(W)** it follows that there exists $g \in W^{2,p}(\mathbb{R}^N, \mathbb{R}^m)$ such that $T_g = w$. Thus, for all $\varphi \in S$,

$$\int_{\mathbb{R}^N} (Lg)\cdot\varphi \,\mathrm{d}x = \int_{\mathbb{R}^N} (L^t\varphi)\cdot g\,\mathrm{d}x = \langle T_g, L^t\varphi\rangle = \langle w, L^t\varphi\rangle = \langle Lw, \varphi\rangle = \langle T_f, \varphi\rangle = \int_{\mathbb{R}^N} f\cdot\varphi\,\mathrm{d}x,$$

showing that $Lg = f$. $\square$

Lemma 2.20 *Let L be a linear differential operator of the form (2.53) that is elliptic in the sense of Petrovskii. Then its characteristic polynomial satisfies* **(S)** *if and only if $\det S(\eta) \neq 0$ for all $\eta \in \mathbb{R}^N$.*

Proof. If there exists a point $\eta \in \mathbb{R}^N$ such that $\det S(\eta) = 0$ then $S(\eta)$ clearly cannot satisfy **(S)**.

If L is elliptic, it follows from (2.16) that there exists $\gamma > 0$ such that

$$\det\left(\sum_{\alpha,\beta=1}^{N} \eta_\alpha\eta_\beta A_{\alpha\beta}\right) \geq \gamma\,|\eta|^{2m} \quad \text{for all } \eta \in \mathbb{R}^N.$$

We claim that there exists a constant $R > 0$ such that

$$\left|\det\left(|\eta|^{-2}\, S(\eta)\right)\right| \geq \frac{\gamma}{2} \quad \text{for all } \eta \in \mathbb{R}^N \text{ with } |\eta| \geq R.$$

If not, there exists a sequence $(\eta^k) \subset \mathbb{R}^N$ such that

$$|\eta^k| \to \infty \quad \text{and} \quad \left|\det\left(|\eta^k|^{-2} S(\eta^k)\right)\right| < \frac{\gamma}{2}.$$

Setting $\zeta^k = \dfrac{\eta^k}{|\eta^k|}$ and passing to a subsequence, we have that $\zeta^k \to \zeta$ where $|\zeta| = 1$, and

$$\frac{1}{|\eta^k|^2} S(\eta^k) = \sum_{\alpha,\beta=1}^{N} \zeta_\alpha^k \zeta_\beta^k A_{\alpha\beta} + i \sum_{\alpha=1}^{N} \frac{\zeta_\alpha^k}{|\eta^k|} B_\alpha + \frac{1}{|\eta^k|^2} C \longrightarrow \sum_{\alpha,\beta=1}^{N} \zeta_\alpha \zeta_\beta A_{\alpha\beta}.$$

By the continuity of the determinant this implies that

$$\det \left(\sum_{\alpha,\beta=1}^{N} \zeta_\alpha \zeta_\beta A_{\alpha\beta} \right) \le \frac{\gamma}{2} = \frac{\gamma}{2} |\zeta|^{2m},$$

contradicting the choice of γ. It follows that, for all $\eta \in \mathbb{R}^N$ with $|\eta| \ge R$,

$$|\det S(\eta)| \ge \frac{\gamma}{2} |\eta|^{2m} \ge \frac{\gamma}{4} |\eta|^{2m} + \frac{\gamma R^{2m}}{4}.$$

If $\det S(\eta) \neq 0$ for all $\eta \in \mathbb{R}^N$, there exists $\delta > 0$ such that $|\det S(\eta)| \ge \delta$ for all $\eta \in \mathbb{R}^N$ with $|\eta| \le R$. Thus, setting $\nu = \delta/(2R^{2m})$, we have that

$$|\det S(\eta)| \ge \frac{\delta}{2} + \nu R^{2m} \ge \nu |\eta|^{2m} + \frac{\delta}{2}$$

for all $\eta \in \mathbb{R}^N$ with $|\eta| \le R$, showing that $S(\eta)$ satisfies **(S)**. □

Corollary 2.5 *Let L be a linear differential operator of the form (2.53) that is elliptic in the sense of Petrovskii and such that the determinant of its characteristic polynomial has no zeros in $\mathbb{R}^N$. Then $L : W^{2,p}(\mathbb{R}^N, \mathbb{R}^m) \to L^p(\mathbb{R}^N, \mathbb{R}^m)$ is an isomorphism.*

Finally we consider the case of linear systems with variable coefficients.

$$Lu = -\sum_{\alpha,\beta=1}^{N} A_{\alpha\beta}(x) \partial^2_{\alpha\beta} u + \sum_{\alpha=1}^{N} B_\alpha(x) \partial_\alpha u + C(x) u \tag{2.54}$$

where
(C) $A_{\alpha\beta}, B_\alpha$ and C are continuous functions from $\mathbb{R}^N$ into the space of real $m \times m$ matrices, and there exist matrices $A^\infty_{\alpha\beta}, B^\infty_\alpha$ and C^∞ such that

$$A_{\alpha\beta}(x) \to A^\infty_{\alpha\beta}, \quad B_\alpha(x) \to B^\infty_\alpha \text{ and } C(x) \to C^\infty \text{ as } |x| \to \infty.$$

Let

$$L^\infty u = -\sum_{\alpha,\beta=1}^{N} A^\infty_{\alpha\beta} \partial^2_{\alpha\beta} u + \sum_{\alpha=1}^{N} B^\infty_\alpha \partial_\alpha u + C^\infty u$$

and let $S^\infty(\eta)$ be its characteristic polynomial.

Theorem 2.10 *Under the hypothesis* **(C)***, suppose that the operators $L_t = tL + (1-t)L^\infty$ are strictly elliptic on $\mathbb{R}^N$ in the sense of Petrovskii for all $t \in [0,1]$ and that $\det S^\infty(\eta) \neq 0$ for all $\eta \in \mathbb{R}^N$. Then $L \in \Phi_0(X_p, Y_p)$.*

Proof. Our assumptions ensure that L^∞ is elliptic and so by Corollary 2.5 we have that $L^\infty : X_p \to Y_p$ is an isomorphism. Furthermore $L_t : X_p \to Y_p$ is a bounded linear operator for all $t \in [0,1]$.

We now consider $T = L - L^\infty$ and claim that it is compact modulo L_t for all $t \in [0,1]$. To see this, we consider a sequence $(u_n) \subset X_p$ such that $u_n \rightharpoonup 0$ in X_p and $L_t u_n \to 0$ in Y_p for some $t \in [0,1]$. Thus (u_n) is bounded in X_p and it follows as in Lemma 2.17, that for any $\varepsilon > 0$, there exists $r > 0$ such that $\|Tu_n\|_{0,p,\tilde{B}_r} \leq \varepsilon$ for all n. By Lemma 2.10, we have that $u_n \to 0$ in $X_p(B_r)$ and so $Tu_n \to 0$ in $Y_p(B_r)$. Thus we may conclude that $Tu_n \to 0$ in Y_p, showing that T is compact modulo L_t. Since $L^\infty : X_p \to Y_p$ is an isomorphism, $L_0 = L^\infty \in \Phi_0(X_p, Y_p)$. The conclusion now follows from Lemma 2.11 (iii) since $L = L_1$. $\square$

As already noticed in §2.2, all operators on the segment joining two operators which are strongly elliptic in the sense of Legendre-Hadamard, are also elliptic. Therefore if this new condition is used, then the assumption "$L_t = tL + (1-t)L^\infty$ are elliptic " in the preceding theorem becomes redundant, and we have

Corollary 2.6 *Let condition* **(C)** *holds, and L be strongly elliptic on $\mathbb{R}^N$ in the sense of Legendre-Hadamard. Suppose that $\det S^\infty(\eta) \neq 0$ for all $\eta \in \mathbb{R}^N$. Then $L \in \Phi_0(X_p, Y_p)$.*

Proof. Letting $|x| \to \infty$ in (2.17), we see that L^∞ is also elliptic in the sense of Legendre-Hadamard. Then L_t is elliptic in the sense of Legendre-Hadamard and therefore in the sense of Petrovskii. Thus the conclusion follows from Theorem 2.10. $\square$

2.6.2 Linear systems on Ω

The operator L in (2.54) acts as an operator from X_p to Y_p, but also as an operator from $D_p(\Omega)$ to $Y_p(\Omega)$. To distinguish them we denote the second operator by L_Ω, i.e. $L_\Omega \in \mathcal{L}(D_p(\Omega), Y_p(\Omega))$, and indeed $L \in \mathcal{L}(X_p, Y_p)$.

Then what is the connection between the Fredholmness of L and L_Ω? It turns out that their semi-Fredholmness are equivalent.

Theorem 2.11 *Let Ω be unbounded and have a C^2 boundary. Then*

$$L_\Omega \in \Phi_+(D_p(\Omega), Y_p(\Omega)) \iff L \in \Phi_+(X_p, Y_p).$$

Proof. (i) First let $L_\Omega \in \Phi_+(D_p(\Omega), Y_p(\Omega))$. To prove the semi-Fredholmness of L, consider a sequence (v_n) from X_p weakly convergent to 0, and such that $Lv_n \to 0$ in Y_p. We show that $v_n \to 0$ in X_p.

Let B_r be a ball containing $\partial\Omega$ (and consequently also $K = \complement\Omega$), $R > r$, and $\phi \in C_0^\infty(\mathbb{R}^N)$ be such that $\phi = 1$ outside B_R, $\phi = 0$ on B_r. Then consider the restriction to Ω of (ϕv_n) that we denote by (u_n). We have clearly $(u_n) \subset D_p(\Omega)$, and furthermore:

• $u_n \rightharpoonup 0$ in $D_p(\Omega)$. Indeed $v_n \rightharpoonup 0$ in Y_p so $\phi v_n \rightharpoonup 0$ in Y_p (because if $f \in L^{p'}$, then also $f\phi \in L^{p'}$, so that $\int f\phi v_n \to 0\ \forall f \in L^{p'}$). For $1 \leq \alpha, \beta \leq N$, $\partial_\alpha(\phi v_n) = (\partial_\alpha\phi)v_n + \phi\partial_\alpha v_n$, and so for the same reasons as above, we have $\partial_\alpha(\phi v_n) \rightharpoonup 0$ in Y_p. Next since

$$\partial^2_{\alpha\beta}(\phi v_n) = (\partial^2_{\alpha\beta}\phi)v_n + (\partial_\beta\phi)\partial_\alpha v_n + (\partial_\alpha\phi)\partial_\beta v_n + \phi\partial^2_{\alpha\beta}v_n,$$

we have $\partial^2_{\alpha\beta}(\phi v_n) \rightharpoonup 0$ in Y_p.
Now weak convergence to 0 in $W^{2,p}$ of a sequence is equivalent to the weak convergence to 0 in L^p of all its derivatives up to order 2 (see Note C in the appendix). Thus $\phi v_n \rightharpoonup 0$ in X_p and so also in $D_p(\Omega)$.

• Since $u_n = v_n$ outside B_R, we have that $L_\Omega u_n \to 0$ in $Y_p(\tilde{B}_R)$. But a direct calculation already used in the proof of Lemma 2.10 shows that

$$L_\Omega u_n =$$

$$\left. \phi L v_n + \sum_{\alpha,\beta} \left(\partial^2_{\alpha\beta}\phi A_{\alpha\beta}\right) v_n + \sum_{\alpha,\beta} \left(\partial_\alpha\phi A_{\alpha\beta}\right) \partial_\beta v_n + \sum_{\alpha,\beta} \left(\partial_\beta\phi A_{\alpha\beta}\right) \partial_\alpha v_n + \sum_\alpha \left(\partial_\alpha\phi B_\alpha\right) v_n \right|_\Omega .$$

Recalling that the imbedding $W^{2,p}(\mathbb{R}^N) \hookrightarrow W^{1,p}(\Omega_R)$ is compact, we see that $v_n \to 0$ in $W^{1,p}(\Omega_R, \mathbb{R}^m)$. This fact together with the boundedness of ϕ, its derivatives and of the coefficients of L, implies that $L_\Omega u_n \to 0$ in $Y_p(\Omega_R)$. So in fact $L_\Omega u_n \to 0$ in $Y_p(\Omega)$.

Now by hypothesis $L_\Omega \in \Phi_+ (D_p(\Omega), Y_p(\Omega))$, therefore $u_n \to 0$ in $D_p(\Omega)$. Next as already observed $v_n = u_n$ outside B_R, so $v_n \to 0$ in $X_p(\tilde{B}_R)$. And by Lemma 2.10 (with $\Omega = \mathbb{R}^N$) $v_n \to 0$ in $X_p(B_R)$. Thus finally $v_n \to 0$ in X_p, and the proof of the first part is complete.

(ii) Let now $L \in \Phi_+ (X_p, Y_p)$, and (u_n) be a sequence from $D_p(\Omega)$ converging weakly to 0, and $L_\Omega u_n$ converging to 0 in $Y_p(\Omega)$ (by note A4 of the appendix, weak convergence in $D_p(\Omega)$ is the same as weak convergence in $X_p(\Omega)$). Take the same r, R and ϕ as in the proof of (i), and define the sequence (v_n) by $v_n(x) = \phi(x)u_n(x)$ if $x \in \Omega$ and 0 elsewhere.

Now one can check that $v_n \in X_p$, and furthermore as in the proof of the first part, that $v_n \rightharpoonup 0$ in X_p.
On the other hand, by construction $u_n = v_n$ outside B_R, therefore $L_\Omega u_n \to 0$ in $Y_p(\tilde{B}_R)$ and since

$$Lv_n|_\Omega = \phi L_\Omega u_n + \sum_{\alpha,\beta} \left(\partial^2_{\alpha\beta}\phi A_{\alpha\beta}\right) u_n + \sum_{\alpha,\beta} \left(\partial_\alpha \phi A_{\alpha\beta}\right) \partial_\beta u_n + \sum_{\alpha,\beta} \left(\partial_\beta \phi A_{\alpha\beta}\right) \partial_\alpha u_n + \sum_{\alpha} \left(\partial_\alpha \phi B_\alpha\right) u_n,$$

we deduce as in the proof of the first part that $Lv_n \to 0$ in $Y_p(r < |x| < R)$. Next, $v_n = 0$ on B_r, so $Lv_n \to 0$ in $Y_p(B_r)$. Therefore $Lv_n \to 0$ in X_p.

The semi-Fredholmness of L implies that $v_n \to 0$ in X_p, and so $u_n \to 0$ in $X_p(\tilde{B}_R)$. By Lemma 2.10, $u_n \to 0$ in $X_p(\Omega_R)$. Thus finally $u_n \to 0$ in $X_p(\Omega)$ and the proof is complete. □

2.7 Exponential decay

Exponential decay of solutions is an important question in the field of partial differential equations. Consider the operator F in (1.2), and let $f \in Y_p(\Omega)$ decays exponentially at infinity (in particular this happens when $f = 0$), then, does all possible solutions of $F(u) = f$ have exponential decay? In our context, this question is related to the properness issue. In **[34]**, by establishing exponential decay of possible solutions of quasilinear elliptic equations on $\mathbb{R}^N$, Rabier and Stuart prove identities of Pohozaev type for the limit problem, and in some situations, this implies that the limit problem has only the trivial solution. If in addition the operator is Fredholm, then Theorem 2.7 ensures that F is proper on the closed bounded subset of $W^{2,p}(\mathbb{R}^N)$. But as we shall see, this question has also another strong connection with the Fredholm property.

In a recent paper **[30]**, Rabier noticed the following. For $u \in L^p(\mathbb{R}^N)$, the intuitive idea that $u(x)$ decays exponentially as $|x| \to \infty$ is usually captured by the condition that $\|u\|_{L^p(|x|>r)} = O(e^{-sr})$ for some $s > 0$ as $r \to \infty$, and this happens in particular if $u = e^{-s|x|}v$ for some $v \in L^p(\mathbb{R}^N)$ and $s > 0$. The remark that multiplication by $e^{-s|x|}$ generates a semigroup on $\mathcal{L}(L^p(\mathbb{R}^N))$ leads Rabier to study the problem in an abstract setting. We briefly describe his results.

Recall first that given a Banach space X, a C_0 or strongly continuous semigroup of bounded linear operators on X is a family $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ satisfying

(i) $T(0)$ is the identity operator on X.

(ii) $T(s+t) = T(t)T(s)$ for every $t, s \geq 0$.

(iii) $\lim_{t\to 0} T(t)x = x$ for every $x \in X$.

The uniform boundedness principle then implies that the function $t \mapsto \|T(t)\|_{\mathcal{L}(X)}$ is bounded on the bounded subsets of $[0, \infty)$. In turn, this fact implies that, for every $x \in X$, the map $t \mapsto T(t)x$ is continuous from $[0, \infty)$ into X (see **[28]**).

Now let X and Y be two reflexive Banach spaces. Assume that there are two injective C_0 semigroups $(T(s))_{s\geq 0} \subset \mathcal{L}(X)$ and $(S(s))_{s\geq 0} \subset \mathcal{L}(Y)$. Let $L \in \mathcal{L}(X, Y)$ be a Fredholm operator such that for some $\sigma > 0$, we have

$$\operatorname{rge} LT(s) \subset \operatorname{rge} S(s), \qquad \forall\, s \in [0, \sigma] \tag{2.55}$$

$$S(s)^{-1} LT(s) \in \mathcal{L}(X, Y) \quad \forall\, s \in [0, \sigma] \tag{2.56}$$

$$\lim_{s\to 0} \|S(s)^{-1} LT(s) - L\|_{\mathcal{L}(X,Y)} = 0. \tag{2.57}$$

Under the above assumptions Rabier proves the following theorem.

Theorem 2.12 *There is $s_0 \in (0, \sigma]$, such that the following property holds. If $f = S(s)g$ for some $g \in Y$, and $s > 0$, and if $u \in X$ satisfies $Lu = f$, then there is $v \in X$ such that $u = T(\min(s, s_0))v$.*

The proof is given in three steps. First the result is proved when L is surjective, and in this case the index of L is nonnegative. The second step treats the case when L is injective (and then the index is nonpositive). Finally, the general case is reduced to one of the previews situations (according to the sign of the index) by adding a finite dimensional operator, and using the following lemma.

Lemma 2.21 *Let $E \in \mathcal{L}(X, Y)$ have finite rank. Then the mappings $S(.)E : [0, \infty) \to \mathcal{L}(X, Y)$ and $ET(.) : [0, \infty) \to \mathcal{L}(X, Y)$ are continuous.*

Indeed the proof becomes simpler if the index is zero, since by adding a finite dimensional operator, the problem is reduced to the case when L is an isomorphism.

Now Theorem 2.12 leaves us with the following question. Suppose that $s > 0$ is fixed, but g, and the possible solutions u of $Lu = S(s)g$ are allowed to vary. Then how does the element v behave with respect to u and g? Our next proposition answers the question in the zero index case, and reproves by the way the existence of v. Note that since $T(\min(s, s_0))$ is injective by hypothesis, v is uniquely determined by u (for each g).

Proposition 2.1 *Let L be Fredholm of index zero in Theorem 2.12. Suppose that $s > 0$ is fixed, and $S(s)g \in \operatorname{rge} L$. Then the following hold.*

(i) *For each g, the correspondence $u \mapsto v$ is an affine continuous map from $L^{-1}(S(s)g)$ to X.*

(ii) *In particular, when $g = 0$, the map $u \mapsto v$ is linear and continuous from $\ker L$ to X.*

(iii) *Let g vary in a bounded subset B of Y, and u vary in a bounded subset of $\bigcup_{g\in B} L^{-1}(S(s)g)$. Then, v also varies in a bounded subset of X.*

Proof. Since L is Fredholm of index zero, there is a finite rank operator $E \in \mathcal{L}(X, Y)$ such that $L + E \in GL(X, Y)$. It follows from (2.55) that $LT(t) + S(t)ET(t) \subset \operatorname{rge} S(t)$ for all $t \in [0, \sigma]$, therefore the operator

$$L_t = S(t)^{-1} \left(L + S(t)E\right) T(t) \tag{2.58}$$

is well defined. Indeed, $L_t = S(t)^{-1} LT(t) + ET(t)$, and it follows from (2.57) and lemma 2.21 that $L_t \in \mathcal{L}(X, Y)$ and

$$\lim_{t\to 0} \|L_t - (L + E)\|_{\mathcal{L}(X,Y)} = 0. \tag{2.59}$$

Since $GL(X,Y)$ is open in $\mathcal{L}(X,Y)$, there is $s_0 \in (0,\sigma]$ such that both L_t and $L+S(t)E$ are in $GL(X,Y)$ for all $t \in [0,s_0]$. Take in particular $t = \min(s_0,s)$ and let

$$v = L_t^{-1}\left(S(s-t)g + Eu\right), \tag{2.60}$$

where $Lu = S(s)g$. Then

$$(L+S(t)E)T(t)v = S(t)S(s-t)g + S(t)Eu = S(s)g + S(t)Eu = Lu + S(t)Eu,$$

which yields

$$(L+S(t)E)(T(t)v-u) = 0.$$

Since $L+S(t)E \in GL(X,Y)$, we have $u = T(t)v$.

Therefore (i) and (ii) follow from (2.60), which also yields

$$\|v\|_X \le \|L_t^{-1}\|_{\mathcal{L}(Y,X)}\left\{\|S(s-t)\|_{\mathcal{L}(Y)}\|g\|_Y + \|E\|_{\mathcal{L}(X,Y)}\|u\|_X\right\}. \tag{2.61}$$

Thus, (iii) follows from the boundedness of $\|g\|$ and $\|u\|$. □

Before we apply the preceding results to elliptic systems of second order, we extend Lemma 2.21 to the case when E is compact.

Lemma 2.22 *Let $E \in \mathcal{L}(X,Y)$ be compact. Then the mappings $S(.)E : [0,\infty) \to \mathcal{L}(X,Y)$ and $ET(.) : [0,\infty) \to \mathcal{L}(X,Y)$ are continuous.*

Proof. The result can be deduced from Lemma 2.21 by approximating E with finite-rank operators. But we give a direct proof.

(i) We prove the continuity of $t \mapsto S(t)E$. Let $t \ge 0$ and $\varepsilon > 0$ be given. Consider a bounded neighborhood I of t in $[0,\infty)$, and let $M > 0$ be a bound of $\|T(s)\|$ for $s \in I$. Let B denote the closed unit ball in X. The set $E(B)$ is relatively compact in Y and therefore it is totally bounded. Accordingly, there exist $u_1,\dots,u_k \in B$ such that $E(B)$ is covered by the open balls centered at Eu_j with radius $\frac{\varepsilon}{3M}$. It follows from the strong continuity of $S(s)$ at Eu_j that

$$\|S(s)Eu_j - S(t)Eu_j\|_Y \le \frac{\varepsilon}{3},$$

for every $s \in I$ close enough to t, and every $j = 1,\dots,k$. Now, for every $u \in B$ there is some j such that

$$\|Eu - Eu_j\|_Y < \frac{\varepsilon}{3M}.$$

Then,

$$\begin{aligned}\|S(s)Eu - S(t)Eu\| &\le \|S(s)Eu - S(s)Eu_j\| + \|S(s)Eu_j - S(t)Eu_j\| + \|S(t)Eu_j - S(t)Eu\| \\ &\le \|S(s)\|\|Eu - Eu_j\| + \|S(s)Eu_j - S(t)Eu_j\| + \|S(t)\|\|Eu_j - Eu\| \\ &\le M\frac{\varepsilon}{3M} + \frac{\varepsilon}{3} + M\frac{\varepsilon}{3M} = \varepsilon.\end{aligned}$$

Since this holds for every $u \in B$, we have $\|S(s)E - S(t)E\|_{\mathcal{L}(X,Y)} \le \varepsilon$, and the proof is complete.

(ii) Let E^* and $T^*(t)$ denote the adjoints of E and $T(t)$ respectively. Since X is reflexive, Corollary 10.6 in Pazy **[28]** ensures that T^* is a C_0 semigroup on X'. Now,

$$\|ET(s) - ET(t)\|_{\mathcal{L}(X,Y)} = \|T^*(s)E^* - T^*(t)E^*\|_{\mathcal{L}(Y',X')}.$$

But E^* is compact, therefore, the result follows from part (i). □

2.7.1 Linear systems

Let $\Omega \subset \mathbb{R}^N$ be unbounded and have a Lipschitz bounded boundary. Consider the second order differential operator of the form (2.15)

$$Lv := -\sum_{\alpha,\beta=1}^{N} A_{\alpha\beta}(x)\partial^2_{\alpha\beta}v + \sum_{\alpha=1}^{N} B_\alpha(x)\partial_\alpha v + C(x)v$$

with its coefficients being continuous and bounded on Ω.

We already know that $L \in \mathcal{L}\left(X_p(\Omega), Y_p(\Omega)\right)$.

Let B_r be a ball containing $\partial\Omega$ (and consequently also $K = \complement\Omega$), $R > r$, and $\phi \in C_0^\infty(\mathbb{R}^N)$ be such that $\phi = 1$ outside B_R, $\phi = 0$ on B_r. Set $\theta(x) = \phi(x)|x|$. Then $\theta \in C^\infty(\mathbb{R}^N)$, $\theta(x) = |x|$ when $|x| \geq R$, and $\theta(x) = 0$ when $|x| \leq r$.

Furthermore for $x \neq 0$

$$\frac{\partial\theta}{\partial x_i} = \phi\frac{x_i}{|x|} + \partial_i\phi|x| \tag{2.62}$$

$$\frac{\partial^2\theta}{\partial x_j\partial x_i} = \phi\cdot\left(\frac{\delta_{ij}}{|x|} - \frac{x_ix_j}{|x|^3}\right) + \partial_j\phi\frac{x_i}{|x|} + \partial_i\phi\frac{x_j}{|x|} + \partial^2_{ij}\phi|x| \tag{2.63}$$

When $|x| \leq r$, $\theta = 0$ and when $|x| > R$, $\phi = 1$ so its derivatives are zero. Therefore from (2.62) and (2.63), we deduce that all the derivatives of θ are bounded on $\mathbb{R}^N$.

Define for every $s \geq 0$ and $u \in Y_p(\Omega)$, $S(s)u := e^{-s\theta}u$. Let $T(s)$ be the restriction of $S(s)$ to $X_p(\Omega)$. Now we check that T and S satisfy all the assumptions of Theorem 2.12.

• We clearly have that $e^{-s\theta} \in C^\infty(\mathbb{R}^N)$ and is bounded by 1 (because $\theta \geq 0$), therefore $S(s)$ maps $Y_p(\Omega)$ into itself, $T(s)$ maps $X_p(\Omega)$ into itself, and $D_p(\Omega)$ into itself. T and S clearly satisfy conditions (i) and (ii) in the definition of a semigroup. They are also injective since $e^{-s\theta} > 0$.

We check condition (iii). Let $u \in Y_p(\Omega)$ be fixed, then it follows from the Lebesgue dominated convergence that

$$\lim_{s\to 0} e^{-s\theta}u = u \ \text{ in } \ Y_p(\Omega) \tag{2.64}$$

Therefore S is a C_0 semigroup on $\mathcal{L}(Y_p(\Omega))$.

But for the same reason when $u \in X_p(\Omega)$

$$\lim_{s\to 0} e^{-s\theta}\partial_\alpha u = \partial_\alpha u \quad \text{in} \ \ Y_p(\Omega) \tag{2.65}$$

$$\lim_{s\to 0} e^{-s\theta}\partial^2_{\alpha\beta} u = \partial^2_{\alpha\beta} u \quad \text{in} \ \ Y_p(\Omega). \tag{2.66}$$

for all $1 \leq \alpha, \beta \leq N$. Now a direct calculation (in $Y_p(\Omega)$) shows that

$$\partial_\alpha(e^{-s\theta}u) - e^{-s\theta}\partial_\alpha u = -s\partial_\alpha\theta e^{-s\theta}u, \tag{2.67}$$

$$\partial^2_{\alpha\beta}(e^{-s\theta}u) - e^{-s\theta}\partial^2_{\alpha\beta}u = -se^{-s\theta}\left(\partial_\beta\theta\partial_\alpha u + \partial_\alpha\theta\partial_\beta u + \partial^2_{\alpha\beta}\theta u - s\partial_\alpha\theta\partial_\beta\theta u\right). \tag{2.68}$$

Therefore

$$\lim_{s\to 0}\left(\partial_\alpha(e^{-s\theta}u) - e^{-s\theta}\partial_\alpha u\right) = 0 \tag{2.69}$$

$$\lim_{s\to 0}\left(\partial^2_{\alpha\beta}(e^{-s\theta}u) - e^{-s\theta}\partial^2_{\alpha\beta}u\right) = 0. \tag{2.70}$$

Now, recalling (2.65) and (2.66), we see that $\partial_\alpha(e^{-s\theta}u) \to \partial_\alpha u$, and $\partial^2_{\alpha\beta}(e^{-s\theta}u) \to \partial^2_{\alpha\beta}u$ in $Y_p(\Omega)$ when $s \to 0$. All this means that $e^{-s\theta}u \to u$ in $X_p(\Omega)$, that is $\lim_{s\to 0} T(s)u = u$ in $X_p(\Omega)$. This proves the strong continuity the semigroups T and S.

• Now, we prove that condition (2.55) is satisfied. Let $u \in X_p(\Omega)$, from (2.67) we see that $B_\alpha \partial_\alpha (e^{-s\theta} u) = e^{-s\theta} \eta_\alpha$ where η_α is some function in $Y_p(\Omega)$ (because B_α is bounded). From (2.68) we see that $A_{\alpha\beta} \partial^2_{\alpha\beta}(e^{-s\theta} u) = e^{-s\theta} \zeta_{\alpha\beta}$ where $\zeta_{\alpha\beta} \in Y_p(\Omega)$. Hence

$$L(e^{-s\theta} u) = e^{-s\theta} \left(\sum_{\alpha,\beta=1}^{N} \zeta_{\alpha\beta} + \sum_{\alpha=1}^{N} \eta_\alpha + Cu \right).$$

That is $LT(s)u = L(e^{-s\theta} u) = e^{-s\theta} g = S(s)g$ for some g in $Y_p(\Omega)$, and every $s \geq 0$.

• We prove (2.56) and (2.57). Again using (2.67) an (2.68) we see that for all $u \in X_p(\Omega)$

$$\|S(s)^{-1} LT(s)u - Lu\|_{0,p,\Omega} \leq sM \|u\|_{2,p,\Omega} \tag{2.71}$$

where M is a suitable bound of the coefficients $A_{\alpha\beta}$, B_α, C and the derivatives of θ. This proves at the same time (2.56) and (2.57), when $X = X_p(\Omega)$ and indeed also when $X = D_p(\Omega)$.

As a result of the preceding and Theorem 2.12, we have

Theorem 2.13 *, Suppose that the differential operator L in (2.15) is Fredholm between $D_p(\Omega)$ and $Y_p(\Omega)$. Then, whenever $u \in D_p(\Omega)$ satisfies $e^{s\theta} Lu \in Y_p(\Omega)$ for some $s > 0$, we have $u = e^{-t\theta} v$ for some $v \in D_p(\Omega)$, where $t = \min(s, s_0)$. Consequently, since $p > N$, we have*

$$|u(x)| \leq e^{-t\theta(x)} |v(x)| \leq \|v\|_{0,\infty} e^{-t\theta(x)} \leq \text{const.} \times \|v\|_{2,p} e^{-t\theta(x)}, \tag{2.72}$$

and

$$|\nabla u(x)| \leq \text{const.} \times \|v\|_{2,p} e^{-t\theta(x)}, \tag{2.73}$$

for all $x \in \Omega$.

From Proposition 2.1, we have a uniform exponential decay when L is Fredholm of index zero.

Proposition 2.2 *Let the differential operator L in (2.15) be Fredholm of index zero between $D_p(\Omega)$ and $Y_p(\Omega)$. Suppose that for some fixed $s > 0$, $e^{s\theta} Lu$ belongs to a bounded subset of $Y_p(\Omega)$ for all u in a bounded subset $B \subset D_p(\Omega)$. Then, there exists a positive constant $C(s, B)$ such that*

$$|u(x)| \leq C(s, B) e^{-t\theta(x)}, \tag{2.74}$$

for all $u \in B$ and all $x \in \Omega$.

2.7.2 Quasilinear systems

We consider now the operator F in (1.2) under the assumptions (2.10)−(2.13) and (2.18). Furthermore, we assume that $F(0) = 0$.

It follows from Theorem 2.4, that $F(u) = G(u)u$, where for a fixed $u \in D_p(\Omega)$, $G(u)$ is a differential operator of the form (2.15), and so it satisfies conditions (2.55), (2.56) and (2.57). As a result, we get

Proposition 2.3 *Let the operator $F : D_p(\Omega) \to Y_p(\Omega)$ in (1.2) be Fredholm. Let $f \in Y_p(\Omega)$ satisfy $e^{s\theta} f \in Y_p(\Omega)$, for some $s > 0$. Then for all possible solutions $u \in D_p(\Omega)$ of $F(u) = f$, there is $t = t(u, s) > 0$ and $v \in D_p(\Omega)$, such that $u = e^{-t\theta} v$.*

Theorem 2.14 *With the additional assumption that b is a C^2_ξ−bundle map, with $\nabla^2_\xi b(.,0)$ bounded, the number t in Proposition 2.3 can be chosen independently of u.*

Proof. We begin with some notation. For a scalar equicontinuous C^1_ξ−bundle map $h : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times N} \to \mathbb{R}$, we know from the proof of Lemma 2.3 that h can be written in the form (see (2.2))

$$h(x,\xi) = h(x,0) + \sum_{\alpha=0}^{N} \left(\int_0^1 \nabla_{\xi_\alpha} h(x,t\xi)\,\mathrm{d}t \right) \cdot \xi_\alpha. \tag{2.75}$$

If now h takes values in $\mathbb{R}^m$, applying the above equality to each component h^k and letting $\nabla_{\xi_\alpha} h$ denote the matrix of lines $\nabla_{\xi_\alpha} h^k$, then (2.75) also holds for h in this case. We make one more step in the matrix-vector notation. If h takes values in $\mathbb{R}^{m\times m}$, we apply what has been said to each column h_j. Let then $\nabla_{\xi_\alpha} h(x,\xi)$ denote the array of length m of matrices, with each component being the matrix $\nabla_{\xi_\alpha} h_j(x,\xi)$. Then (2.75) still holds for h in this case as well.

Now we go to the proof of the Theorem. Since b is an equicontinuous C^1_ξ−bundle map with $b(x,0)=0$, we can write

$$b(x,\xi) = \sum_{\alpha=0}^{N} c_\alpha(x,\xi)\xi_\alpha.$$

But b is supposed to be C^2_ξ and so one can check that c_α is an equicontinuous C^1_ξ−bundle map. According to what have been said, there is a family of equicontinuous C^0−bundle map c^ℓ_α such that

$$c_\alpha(x,\xi) = c_\alpha(x,0) + \sum_{\ell=0}^{N} c^\ell_\alpha(x,\xi)\xi_\ell.$$

Note that $b_\alpha(.,0) = c_\alpha(.,0)$, and condition (2.13) implies that they are bounded functions.

Now, again, we can write

$$a_{\alpha\beta}(x,\xi) = a_{\alpha\beta}(x,0) + \sum_{\ell=0}^{N} a^\ell_{\alpha\beta}(x,\xi)\xi_\ell,$$

where $a^\ell_{\alpha\beta}$ are equicontinuous C^0−bundle maps.
Then

$$F(u) = -\sum_{\alpha,\beta=1}^{N} \left(\boldsymbol{a}_{\alpha\beta}(0) + \sum_{\ell=0}^{N} \boldsymbol{a}^\ell_{\alpha\beta}(u)\partial_\ell u \right) \partial^2_{\alpha\beta} u + \sum_{\alpha=0}^{N} \left(\boldsymbol{c}_\alpha(0) + \sum_{\ell=0}^{N} \boldsymbol{c}^\ell_\alpha(u)\partial_\ell u \right) \partial_\alpha u,$$

and

$$G(0)u = \mathrm{D}F(0)u = -\sum_{\alpha,\beta=1}^{N} \boldsymbol{a}_{\alpha\beta}(0)\partial^2_{\alpha\beta} u + \sum_{\alpha=0}^{N} \boldsymbol{b}_\alpha(0)\partial_\alpha u.$$

Therefore, since $\boldsymbol{b}_\alpha(0) = \boldsymbol{c}_\alpha(0)$, we get

$$\begin{aligned} G(0)u - f &= G(0)u - F(u) \\ &= \sum_{\alpha,\beta=1}^{N} \sum_{\ell=0}^{N} \left(\boldsymbol{a}^\ell_{\alpha\beta}(u)\partial_\ell u \right) \partial^2_{\alpha\beta} u - \sum_{\alpha=0}^{N} \sum_{\ell=0}^{N} \left(\boldsymbol{c}^\ell_\alpha(u)\partial_\ell u \right) \partial_\alpha u. \end{aligned} \tag{2.76}$$

Now, let s_0 be the number given by Theorem 2.12, for the linear operator $G(0)$. Indeed it suffices to prove the result, when $s \le s_0$, for if $s > s_0$, then the condition $e^{s\theta} f \in Y_p(\Omega)$, implies that $e^{s_0\theta} f \in Y_p(\Omega)$.

Proposition 2.3, already ensures that $u = e^{-t\theta}v$, for some t (depending on u) and some $v \in D_p(\Omega)$. Recalling (2.67) and (2.68), we see that the the derivatives of u have also an exponential decay i.e. they have the form $e^{-t\theta}y$ where $y \in Y_p(\Omega)$. Therefore, replacing u by $e^{-t\theta}v$ in the right hand side of (2.76), we see that we get the factor $e^{-t\theta}e^{-t\theta} = e^{-2t\theta}$, and what is left is a function we call g_0, i.e.

$$G(0)u - f = e^{-2t\theta}g_0.$$

We claim that $g_0 \in Y_p(\Omega)$. First, conditions (2.11) and (2.13) imply that $a^\ell_{\alpha\beta}(.,0)$, $c^\ell_\alpha(.,0)$ are bounded. The assumption that $\nabla^2_\xi b(.,0)$ is bounded implies that $b^\ell_\alpha(.,0)$ are bounded as well. Therefore, according to Lemma 2.2 (ii), $\boldsymbol{a}^\ell_{\alpha\beta}(u)$, and $\boldsymbol{c}^\ell_\alpha(u)$ are bounded functions. Second, v and its first derivatives are in L^∞ and in L^p, and the second derivatives are in L^p. Therefore, g_0 is a sum of functions in $Y_p(\Omega)$, and the claim is proved.

Consequently, we can write

$$G(0)u = f + e^{-2t\theta}g_0 = e^{-\min(s,2t)\theta}\left(e^{\min(s,2t)\theta}f + e^{(-2t+\min(s,2t))\theta}g_0\right).$$

Now, $|e^{\min(s,2t)\theta(x)}f(x)| \le |e^{s\theta(x)}f(x)|$ for every $x \in \Omega$ and so $e^{\min(s,2t)\theta}f \in Y_p(\Omega)$. And similarly, $e^{(-2t+\min(s,2t))\theta}g_0 \in Y_p(\Omega)$.

As a result, Theorem 2.12 for $G(0)$ yields an element $v_1 \in D_p(\Omega)$, such that

$$u = e^{-\min(s_0,\min(s,2t))\theta}v_1 = e^{-\min(s,2t)\theta}v_1,$$

since $\min(s,2t) \le s \le s_0$ by assumption.

If $s \le 2t$, then $u = e^{-s\theta}v_1$ and we are done. If not, replace u by $e^{-2t\theta}v_1$ in (2.76), to get

$$G(0)u = f + e^{-4t\theta}g_1 = e^{-\min(s,4t)\theta}\left(e^{\min(s,4t)\theta}f + e^{(-4t+\min(s,4t))\theta}g_0\right).$$

Then, again, Theorem 2.12 yields an element $v_2 \in D_p(\Omega)$ such that

$$u = e^{-\min(s,4t)\theta}v_2.$$

If $s \le 4t$, then $u = e^{-s\theta}v_2$ and this is the claimed result. If not, then $u = e^{-4t\theta}v_2$, and replacing this value of u in (2.76), leads again to an alternative. But this procedure ends in a finite number k of steps (depending on u), for which $s \le 2^k u$, and we have

$$u = e^{-s\theta}v_k,$$

for some $v_k \in D_p(\Omega)$. □

Proposition 2.4 *$S(s)^{-1}G(u)T(s)$ converges to $G(u)$ as $s \to 0$, uniformly with respect to u in bounded subsets.*

Proof.

$$\begin{aligned}
G(u)T(s)v = G(u)(e^{-s\theta}v) &= -\sum_{\alpha,\beta=1}^{N}\boldsymbol{a}_{\alpha\beta}(u)\partial^2_{\alpha\beta}(e^{-s\theta}v) + \sum_{\alpha=0}^{N}\boldsymbol{c}_\alpha(u)\partial_\alpha(e^{-s\theta}v)\\
&= -\sum_{\alpha,\beta=1}^{N}\boldsymbol{a}_{\alpha\beta}(u)\Big(-se^{-s\theta}\left(\partial_\beta\theta\partial_\alpha v + \partial_\alpha\theta\partial_\beta v + \partial^2_{\alpha\beta}\theta v - s\partial_\alpha\theta\partial_\beta v\right) + e^{-s\theta}\partial^2_{\alpha\beta}v\Big)\\
&\quad + \sum_{\alpha=1}^{N}\boldsymbol{c}_\alpha(u)\left(-s\partial_\alpha\theta e^{-s\theta}v + e^{-s\theta}\partial_\alpha v\right) + c_0(u)e^{-s\theta}v
\end{aligned}$$

Therefore

$$S(s)^{-1}G(u)T(s)v - G(u)v = s\sum_{\alpha,\beta=1}^{N} \boldsymbol{a}_{\alpha\beta}(u)\Big(\partial_\alpha\theta\partial_\beta v + \partial_\beta\theta\partial_\alpha v + \partial^2_{\alpha\beta}\theta v - s\partial_\alpha\theta\partial_\beta\theta v\Big)$$
$$- s\sum_{\alpha=1}^{N} c_\alpha(u)(\partial_\alpha\theta v).$$

Let u belong to a bounded subset of $X_p(\Omega)$. It follows that there is a positive constant M such that

$$\|S(s)^{-1}G(u)T(s)v - G(u)v\|_{0,p,\Omega} \leq sM\|v\|_{2,p,\Omega}.$$ □

Chapter 3

Existence from *a priori* bounds

3.1 A general result

Let Ω be an open subset of $\mathbb{R}^N$ with a C^2 bounded boundary. Consider the quasilinear operator F defined by (1.2), under the assumptions (2.10)−(2.13) and (2.18). The purpose of this section is to give a condition ensuring that for every $h \in Y_p(\Omega)$, the equation $F(u) = h$ has at least one solution $u \in D_p(\Omega)$. To simplify the writing and since no other domain than Ω is involved in this part, we write D_p and Y_p instead of $D_p(\Omega)$ and $Y_p(\Omega)$.

In order to use the degree for Fredholm maps defined in **[29]**, $\mathrm{D}F(u)$ must be invertible at some point $u_0 \in D_p$. Otherwise F has no base point and so the absolute degree $|\deg|(F, B, h)$ (when defined) is always zero, and therefore no conclusion about existence can be derived using the degree in the usual way. But in our setting this ensures that F is Fredholm of index zero (Theorem 2.3). It is not restrictive then to assume that $u_0 = 0$, and $F(u_0) = F(0) = 0$. Indeed set $\tilde{F}(u) = F(u + u_0) - F(u_0)$. Then the properness of F is equivalent to the properness of $\tilde{F}$. Also $\mathrm{D}\tilde{F}(u) = \mathrm{D}F(u + u_0)$ so the Fredholmness of F is equivalent to the Fredholmness (of the same index) of $\tilde{F}$. And of course the solvability of the equation $F(u) = h$ for every h is equivalent to the solvability of $\tilde{F}(u) = h$ for every h.

We assume that

• $\mathrm{D}F(0) \in GL(D_p, Y_p)$.

• F is proper on the closed bounded subsets of D_p. This happens for instance if Ω is unbounded and F has a periodic limit operator F^∞ such that $\{u \in X_p(\mathbb{R}^N);\ F^\infty(u) = 0\} = \{0\}$. But F may be proper for different reasons[1].

We consider the homotopy H defined by

$$H(t,u) = \begin{cases} \dfrac{1}{t}F(tu) - th & \text{if } t \neq 0 \\ \mathrm{D}F(0)u & \text{if } t = 0 \end{cases} \tag{3.1}$$

Lemma 3.1 *H has the following properties.*

1. *For every $t \in \mathbb{R}$, $H(t,.) \in C^1(D_p, Y_p)$, and $H \in C^1(\mathbb{R}^* \times D_p, Y_p)$.*

2. $\mathrm{D}_u H(t,u) = \mathrm{D}F(tu) \in \Phi_0(D_p, Y_p),\ \ \forall\,(t,u) \in \mathbb{R} \times D_p$.

3. *For every bounded $\mathcal{B} \subset D_p$, the collection $(H(.,u))_{u\in\mathcal{B}}$ is equicontinuous.*

4. *The restriction of H to any closed bounded subset of $\mathbb{R} \times D_p$ is proper.*

[1] Theorem 2.8 states that when $\Omega = \mathbb{R}^N$ and F has a limit operator, then the properness of F implies the nonexistence of nontrivial solutions of $F^\infty(u) = 0$. But it may happen that no periodic limit problem exists as in **[21]** for example.

Proof. Indeed $(t,u) \mapsto -th$ is C^1, constant with respect to u, and compact since one dimensional, so it is enough to verify the above properties for G defined by $G(t,u) = H(t,u)+th$.

1. and **2.** Since F is C^1,we see by the chain rule that for fixed t, the partial map $G(t,.)$ is C^1 and that $\mathrm{D}_u G(t,u)v = \mathrm{D}F(tu)v$ for all $v \in D_p$. So this proves point **2.** On the other hand $G(.,u)$ is differentiable with respect to t at every point $t \neq 0$, and $\mathrm{D}_t G(t,u) = \frac{-1}{t^2}F(tu) + \frac{1}{t}\mathrm{D}F(tu)u$, so $D_t G$ is continuous at every point (t,u) with $t \neq 0$ (see the remark after the proof). Therefore G has continuous partial derivatives on $\mathbb{R}^* \times D_p$ and this is equivalent to the second statement of point **1.**

3. The equicontinuity at $t = 0$ is a consequence of the differentiability of F at 0. Indeed $G(t,u) - G(0,u) = \frac{1}{t}F(tu) - \mathrm{D}F(0)u$. Now for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\|F(v) - \mathrm{D}F(0)v\|_p \le \varepsilon \|v\|_{2,p} \quad \text{if } \|v\|_{2,p} \le \delta.$$

Therefore if $M > 0$ is a bound for $u \in \mathcal{B}$, and $|t| \le \delta/M$, we have $\|G(t,u) - G(0,u)\|_p \le \varepsilon M$. Let now $t_0 \neq 0$, then

$$\begin{aligned} G(t,u) - G(t_0,u) &= \frac{1}{t}F(tu) - \frac{1}{t_0}F(t_0 u) \\ &= -\sum_{\alpha,\beta=1}^{N} \big(\boldsymbol{a}_{\alpha\beta}(tu) - \boldsymbol{a}_{\alpha\beta}(t_0 u)\big)\partial^2_{\alpha\beta}u + \sum_{\alpha=0}^{N} \big(\boldsymbol{c}_\alpha(tu) - \boldsymbol{c}_\alpha(t_0 u)\big)\partial_\alpha u \end{aligned}$$

where $\boldsymbol{c}_\alpha$ is the Nemytskii operator generated by the (matrix-valued) C^0 bundle map

$$c_\alpha(x,\xi) = \int_0^1 \nabla_{\xi_\alpha} b(x,\tau\xi)\,\mathrm{d}\tau.$$

Let $M > 0$ be a bound for $u \in \mathcal{B}$, so in particular $|u(x)|$ and $|\nabla u(x)|$ are bounded by M for every $u \in \mathcal{B}$ and every $x \in \Omega$. Now by Lemma 2.1 (i), $(a_{\alpha\beta}(x,.))_{x\in\Omega}$ is uniformly equicontinuous on the bounded subsets of $\mathbb{R}^m \times \mathbb{R}^{m\times N}$. Accordingly for every $\varepsilon > 0$, there is $\delta > 0$ such that $|a_{\alpha\beta}(x,tu(x),t\nabla u(x)) - a_{\alpha\beta}(x,t_0u(x),t_0\nabla u(x))| \le \varepsilon$ if $|t - t_0| \le \delta/M$, that is

$$\|\boldsymbol{a}_{\alpha\beta}(tu) - \boldsymbol{a}_{\alpha\beta}(t_0 u)\|_\infty \le \varepsilon.$$

On the other hand using the equicontinuity for c_α, we see that $\|\boldsymbol{c}_\alpha(tu) - \boldsymbol{c}_\alpha(t_0u)\|_\infty \le \varepsilon$ if $|t-t_0|$ is sufficiently small. Therefore for $|t - t_0|$ sufficiently small and all $u \in \mathcal{B}$, we have

$$\|G(t,u) - G(t_0,u)\|_p \le m^2 M(N^2 + N + 1)\varepsilon.$$

Since ε is arbitrary, this inequality means that $(G(.,u))_{u\in\mathcal{B}}$ is equicontinuous at $t_0 \neq 0$. Together with point **1.**, this also shows that $G \in C(\mathbb{R} \times D_p, Y_p)$.

4. We begin by proving the properness of $G(t,.)$. If $t = 0$ then $G(0,.)$ is an isomorphisms and so it is proper. If $t \neq 0$, let $(u_n) \subset D_p$ be a bounded sequence such that $(G(t,u_n))$ converges. Then $(F(tu_n))$ converges. But (tu_n) is bounded, so the properness of F implies that it has a convergent subsequence. Since $t \neq 0$, (u_n) has a convergent subsequence.

Since for every bounded subset $\mathcal{B} \subset D_p$, $(G(.,u))_{u\in\mathcal{B}}$ is equicontinuous, the conclusion follows from Lemma 1.1. □

Remark 3.1 H need not be differentiable at zero. Take for example $F(u) = -\Delta u + |u|^{1/2}u$, then $\frac{1}{t}F(tu) = -\Delta u + |t|^{1/2}|u|^{1/2}u$. Indeed this situation does not occur if more regularity on F is assumed. For instance if F is C^2, it follows from the Taylor expansion of F about zero that $\frac{1}{t^2}F(tu) - \frac{1}{t}\mathrm{D}F(0)u \to \frac{1}{2}\mathrm{D}^2F(0)(u,u)$ as $t \to 0$ (uniformly with respect to u on bounded subsets), and so $D_tG(0,u) = \frac{1}{2}\mathrm{D}^2F(0)(u,u)$. On the other hand as $t \to 0$,

$$\begin{aligned}\mathrm{D}_tG(t,u) &= -\frac{1}{t^2}F(tu) + \frac{1}{t}\mathrm{D}F(0)u - \frac{1}{t}\mathrm{D}F(0)u + \frac{1}{t}\mathrm{D}F(tu)u \\ &\to -\frac{1}{2}\mathrm{D}^2F(0)(u,u) + \mathrm{D}^2F(0)(u,u) = \frac{1}{2}\mathrm{D}^2F(0)(u,u).\end{aligned}$$

Therefore G is C^1 in this case.

Now since F is only C^1, $H|_{[0,1]}$ need not be an admissible homotopy in the sense of Definition 4.2 of **[29]**. However it is admissible in the sense of Benevieri-Furi.

Proposition 3.1 *Suppose that there is $R > 0$ such that*

$$\|u\|_{2,p} \le R \quad \text{whenever } H(t,u) = 0 \text{ for some } t \in [0,1]. \tag{E}$$

Then $F(u) = h$ is solvable in D_p.

Proof. Let $B \subset D_p$ be the ball of center 0 and radius $R+1$. The condition (E) ensures that $0 \notin H([0,1], \partial B)$. Next by Lemma 3.1 (**2.**), $D_uH(t,0) = DF(0) \in GL(D_p, Y_p)$, i.e. 0 is a base point of $H(t,.)$ for all $t \in \mathbb{R}$. Therefore, using Theorem 1.2, we get

$$\begin{aligned}\deg_0(F-h, B, 0) &= \deg_0(H(1,.)), B, 0) \\ &= \deg_0(H(0,.), B, 0) \\ &= \deg_0(\mathrm{D}F(0), B, 0) \\ &= 1.\end{aligned}$$

Hence the existence of an element $u \in D_p$ such that $F(u) - h = 0$. □

Remark 3.2 We can still prove Proposition 3.1 without appealing to Theorem 1.2 (which uses the Benevieri-Furi degree). Indeed, note first that, as a consequence of Lemma 3.1, $H|_{[\epsilon,1]\times\bar{B}}$ is B-admissible for every $\epsilon \in (0,1)$ and every bounded subset $B \subset D_p$. In particular, taking B as in the first proof above, we get

$$\deg_0(H(1,.), B, 0) = \deg_0(H(\epsilon,.), B, 0).$$

So let us calculate $\deg_0(H(\epsilon,.), B, 0)$ for ϵ small enough.

From the inverse function theorem, there is a neighborhood U_0 of zero in D_p and a neighborhood V_0 of zero in Y_p such that $F : U_0 \to V_0$ is a diffeomorphism. Now choose $\epsilon_0, \epsilon_1 \in (0,1)$ small enough so that $\epsilon_0^2 h \in V_0$, and $\epsilon_1 B \subset U_0$, and let $\epsilon = \min(\epsilon_0, \epsilon_1)$. Then there exists $u_0 \in U_0$ for which $F(u_0) = \epsilon^2 h$. Letting then $u_1 = \frac{1}{\epsilon}u_0$, this is equivalent to $H(\epsilon, u_1) = 0$. Furthermore u_1 is the unique solution of $H(\epsilon, u) = 0$ (for this choice of ϵ indeed). For if u_2 is another solution, then firstly, $F(\epsilon u_2) = \epsilon^2 h$, and secondly, $u_2 \in B$ by (E) and so $\epsilon u_2 \in U_0$. But $F|_{U_0}$ is injective, so $\epsilon u_1 = \epsilon u_2$ and therefore $u_1 = u_2$.

Now since $F : U_0 \to V_0$ is a diffeomorphism, $\mathrm{D}F(u) \in GL(D_p, Y_p)$ for all $u \in U_0$, thus $D_uH(\epsilon, u) = \mathrm{D}F(\epsilon u) \in GL(D_p, Y_p)$ for all $u \in B$. This means that 0 is regular value of $H(\epsilon,.)$. Therefore from the very definition of the degree at regular values

$$\deg_0(H(\epsilon,.), B, 0) = \sigma(\mathrm{D}_uH(\epsilon, \mathcal{C}), [0,1]),$$

where $\mathcal{C}(\lambda) = \lambda u_1$ is the segment joining 0 to u_1. But from the above, $D_uH(\epsilon, \lambda u_1)$ is an isomorphism, and since the parity of a path of isomorphisms is 1, we conclude that

$$\deg_0(H(\epsilon,.), B, 0) = 1.$$

□

Indeed the most difficult thing to verify in Proposition 3.1 , is the *a priori* estimate for the homotopy. And there may be situations where no bounds exist, in this case Proposition 3.1 is useless. So we shall present a special case where a priori bounds can be derived, showing that all this business is not empty.

3.2 A special case (reaction - diffusion systems)

Let now $\Omega = \mathbb{R}^N$. We consider F of the special form

$$F(u) = Lu + f(x, u) \tag{3.2}$$

which we begin to define. Let for $k = 1, \dots, m,$ and $\alpha, \beta = 1, \dots, N,$ $A^k_{\alpha\beta}$ be real numbers satisfying

$$\exists\, \gamma_0 > 0 \qquad \forall\, \eta \in \mathbb{R}^N \quad \sum_{\alpha,\beta=1}^{N} A^k_{\alpha\beta}\eta_\alpha\eta_\beta \geq \gamma_0|\eta|^2. \tag{3.3}$$

Let also $d^k \in \mathbb{R}^N$ be given vectors. Define for each k the (scalar) elliptic differential operator L^k by

$$L^k u^k = -\sum_{\alpha,\beta=1}^{N} A^k_{\alpha\beta}\partial^2_{\alpha\beta}u^k + d^k \cdot \nabla u^k \qquad \text{for } u^k \in W^{2,p}, \tag{3.4}$$

and set for $u = (u^1, \dots, u^m)^T \in X_p$

$$Lu = L\begin{pmatrix} u^1 \\ \vdots \\ u^m \end{pmatrix} = \begin{pmatrix} L^1u^1 \\ \vdots \\ L^m u^m \end{pmatrix}. \tag{3.5}$$

Let $f : \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^m$ be an equicontinuous $C^1_\zeta-$bundle map with $f(x, 0) = 0$ and $\nabla_\zeta f(\cdot, 0)$ bounded[2]. We assume that there is a matrix-valued function $g^\infty \in C(\mathbb{R}^m, \mathbb{R}^{m\times m})$ such that

$$\lim_{|x|\to\infty} \left| \int_0^1 \nabla_\zeta f(x, t\zeta)\,\mathrm{d}t - g^\infty(\zeta) \right| = 0, \tag{3.6}$$

uniformly for ζ in compact subsets of $\mathbb{R}^m$, and we set $f^\infty(\zeta) = g^\infty(\zeta)\zeta$.

Consider then the operator F defined by (3.2). It has the form (1.2)

$$F(u) = -\sum_{\alpha,\beta=1}^{N} a_{\alpha\beta}(., u, \nabla u)\partial^2_{\alpha\beta}u + b(., u, \nabla u), \tag{3.7}$$

where

$$a_{\alpha\beta}(x, \xi_0, .., \xi_N) = \begin{pmatrix} A^1_{\alpha\beta} & 0 & \cdots & 0 \\ 0 & A^2_{\alpha\beta} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A^m_{\alpha\beta} \end{pmatrix}, \tag{3.8}$$

and

$$b(x, \xi_0, .., \xi_N) = \begin{pmatrix} \sum_{\alpha=1}^{N} d^1_\alpha \xi^1_\alpha + f_1(x, \xi_0) \\ \vdots \\ \sum_{\alpha=1}^{N} d^m_\alpha \xi^m_\alpha + f_m(x, \xi_0) \end{pmatrix}. \tag{3.9}$$

[2]here ζ replace the original ξ_0.

One can check that L is strongly elliptic in the sense of Legendre-Hadamard. This, together with the assumptions made on the nonlinearity f, ensure that conditions (2.10)−(2.13) and (2.18), are satisfied. Furthermore F has a limit operator F^∞

$$F^\infty(u) = Lu + \boldsymbol{f}^\infty(u) \tag{3.10}$$

which is differentiable at 0 with (according to (2.46))

$$\mathrm{D}F^\infty(0)u = Lu + g^\infty(0)u. \tag{3.11}$$

Therefore, $\mathrm{D}F^\infty(0)$ is an elliptic operator with constant coefficients. Our last assumption on the nonlinearity is

$$\exists\,\delta > 0 \quad f_k(x,\zeta)\zeta^k \geq \delta|\zeta^k|^2 \quad \forall\,(x,\zeta) \in \mathbb{R}^N \times \mathbb{R}^m. \tag{3.12}$$

Example. One of the simplest examples that satisfy all the previous assumptions is

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\Delta u + u(u^2+v^2+1) \\ -\Delta v + v(u^2+v^2+1) \end{pmatrix}.$$

Its linearization at zero

$$\mathrm{D}F(0)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\Delta u + u \\ -\Delta v + v \end{pmatrix}$$

is an isomorphism between X_p and Y_p.

For properness and a priori bounds, we need a maximum principle.

Lemma 3.2 (Maximum principle) *Let z be a continuous map from $\mathbb{R}^N$ to $\mathbb{R}$ vanishing at infinity, $\mathscr{L}$ be a second order scalar elliptic operator of the form*

$$\mathscr{L}v = -\sum_{\alpha,\beta=1}^{N} p_{\alpha\beta}(x)\partial^2_{\alpha\beta}v + B(x)\cdot\nabla v,$$

where the matrix $(p_{\alpha\beta}(x))$ is positive definite. Then the following hold.

(i) $\mathscr{L}z \geq 0$ *on* $\mathbb{R}^N \Longrightarrow z \geq 0$ *on* $\mathbb{R}^N$.

(ii) *If Θ is an open non trivial*[3] *subset of $\mathbb{R}^N$ such that $z(x) = 0$ on $\partial\Theta$, then*

$$\mathscr{L}z \leq 0 \;\; on \;\; \Theta \Longrightarrow z \leq 0 \;\; on \;\; \Theta.$$

(iii) $(\mathscr{L}z)z \leq 0$ *on* $\mathbb{R}^N \Longrightarrow z = 0$ *on* $\mathbb{R}^N$.

Proof. (i) Let $x \in \mathbb{R}^N$. Consider the ball B_R of center 0 and radius $R > |x|$. By Theorem 9.6 of **[18]**,

$$z(x) \geq \sup_{x\in\partial B_R} |z(x)| \to 0 \text{ as } R \to \infty$$

And therefore $z(x) \geq 0$ for all $x \in \mathbb{R}^N$.

(ii) If Θ is bounded, this is once again Theorem 9.6 of **[18]**. So let Θ be unbounded, and choose R sufficiently large so that $\Theta \cap B_R$ is not empty. Then applying the same theorem to $\Theta \cap B_R$, and noting that $\partial(\Theta \cap B_R) \subset \partial\Theta \cup B_R$, we have $\forall x \in \Theta$,

$$z(x) \leq \sup_{x\in\partial\Theta\cap B_R} z(x) \leq \sup_{x\in\partial B_R} |z(x)| \to 0 \text{ as } R \to \infty$$

[3]this means that $\Theta \neq \varnothing$ and $\Theta \neq \mathbb{R}^N$, which is equivalent to saying $\partial\Theta \neq \varnothing$.

Therefore $z(x) \leq 0$ for all $x \in \Theta$.

(iii) Suppose that $\Theta = \{x \in \mathbb{R}^N, z(x) > 0\} \neq \varnothing$. If $\Theta = \mathbb{R}^N$, then $\mathcal{L}z \leq 0$ on $\mathbb{R}^N$, but then $z \leq 0$ by (i), contradiction. Next if $\Theta \neq \mathbb{R}^N$ then $\partial\Theta \neq \varnothing$, and one can show that $z(x) = 0$ on $\partial\Theta$, and once again we get a contradiction from (ii). Therefore $\Theta = \varnothing$. Replacing z by $-z$ in this reasoning, we get $\{x \in \mathbb{R}^N, z(x) < 0\} = \varnothing$. Thus finally $z = 0$ on $\mathbb{R}^N$. □

Lemma 3.3 *Under the above conditions, if* $\mathrm{D}F(0) \in GL(X_p, Y_p)$*, then* $F : X_p \to Y_p$ *is Fredholm of index zero, and proper on the closed bounded subsets of* X_p*.*

Proof. We need to show that $F^\infty(u) = 0 \Rightarrow u = 0$. Note first that condition (3.6) implies that as $|x| \to \infty$

$$f(x, \zeta) = \left(\int_0^1 \nabla_\zeta f(x, t\zeta)\, \mathrm{d}t \right) \zeta \longrightarrow g^\infty(\zeta)\zeta = f^\infty(\zeta).$$

So letting $|x| \to \infty$ in (3.12), we get

$$f_k^\infty(\zeta)\zeta^k \geq \delta|\zeta^k|^2 \quad \forall\, \zeta \in \mathbb{R}^m. \tag{3.13}$$

Now if $u = (u^1, \ldots, u^m) \in X_p$ satisfies $F^\infty(u) = 0$, then for all $k = 1, \ldots, m$,

$$(L^k u^k)u^k = -\boldsymbol{f}_k^\infty(u)u^k \leq -\delta(u^k)^2 \leq 0.$$

Therefore $u^k = 0$ from Lemma 3.2 (iii). From Theorem 2.7 (with $\Omega = \mathbb{R}^N$), we conclude that F is proper on the closed bounded subsets of X_p. □

Proposition 3.2 *If* $\mathrm{D}F(0) \in GL(X_p, Y_p)$*, then for every* $h \in Y_p$*,* $F(u) = h$ *is solvable in* X_p*.*

Proof. F satisfies all the properties required in §3.1. So to prove solvability it is enough to obtain *a priori* estimates. Set for $t \in [0,1]$, $E_t = \{u \in X_p, H(t,u) = 0\}$. Since $E_0 = \{\mathrm{D}F(0)^{-1}(h)\}$, we need to show that $E' = \bigcup_{0<t\leq 1} E_t$ is bounded.

We know that $L^k + \delta$ is an isomorphism from $W^{2,p}(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$ (apply for example Corollary 2.5 with $m = 1$). So let for $k = 1, \ldots, m$, $\psi^k = (L^k + \delta)^{-1}(|h^k|)$. Then by Lemma 3.2 (i), $\psi_k \geq 0$. Now consider an element $u \in E'$. We claim that $|u^k(x)| \leq \psi^k(x)$ for every $x \in \mathbb{R}^N$ and every $k = 1, \ldots, m$. Suppose first that for some $k \in \{1, \ldots, m\}$ and some $x \in \mathbb{R}^N$, we have $u^k(x) > \psi^k(x)$. Let then $z = u^k - \psi^k$, and $\Theta = \left\{x \in \mathbb{R}^N,\, z(x) > 0\right\}$. Then, for all $x \in \Theta$

$$\begin{aligned} L^k z &= L^k u^k - L^k \psi^k \\ &= -\frac{1}{t}\boldsymbol{f}_k(tu) + th^k - (L^k + \delta)\psi^k + \delta\psi^k \\ &= -\frac{1}{t}\boldsymbol{f}_k(tu) + th^k - |h^k| + \delta\psi^k. \end{aligned}$$

But for $x \in \Theta$, $tu^k(x) > 0$, and so by condition (3.12)

$$f_k(x, tu(x)) \geq \delta t u^k(x) \quad \text{hence} \quad -\frac{1}{t} f_k(x, tu(x)) \leq -\delta u^k(x)$$

Therefore

$$L^k z \leq -\delta u^k + \delta\psi^k < 0.$$

If $\Theta = \mathbb{R}^N$, then by Lemma 3.2 (i) $z \le 0$ on $\mathbb{R}^N$, contradiction. If $\Theta \neq \mathbb{R}^N$, then $z(x) = 0$ for $x \in \partial\Theta$, and $z \le 0$ on Θ from Lemma 3.2 (ii). But this contradicts the definition of Θ. Thus $u^k(x) \le \psi^k(x)$ for every $x \in \mathbb{R}^N$ and every $k = 1, \dots, m$.

Next, suppose that $u^k(x) < -\psi^k(x)$ for some $k \in \{1, \dots, m\}$ and some $x \in \mathbb{R}^N$, and let $w = u^k + \psi^k$, and $\Sigma = \{x \in \mathbb{R}^N,\ w(x) < 0\}$. Then

$$\begin{aligned} L^k w &= L^k u^k + L^k \psi^k \\ &= -\frac{1}{t}\boldsymbol{f}_k(tu) + th^k + (L^k + \delta)\psi^k - \delta\psi^k \\ &= -\frac{1}{t}\boldsymbol{f}_k(tu) + th^k + |h^k| - \delta\psi^k \\ &\ge -\delta u^k - \delta\psi^k > 0, \end{aligned}$$

because $f_k(x, tu(x)) \le \delta t u^k(x)$ since $u^k(x) < 0$ when $x \in \Sigma$. Once again by Lemma 3.2 (i) and (ii), $w \ge 0$ on Σ, and we have a contradiction. Consequently $|u(x)| \le |\psi(x)|$ and so $\|u\|_\infty \le \|\psi\|_\infty$ and $\|u\|_{0,p} \le \|\psi\|_{0,p}$.

Now as already observed several times f can be written as $f(x,\zeta) = g(x,\zeta)\cdot\zeta$, where g is the equicontinuous C^0-bundle map defined by $g(x,\zeta) = \int_0^1 \nabla_\zeta f(x, t\zeta)\, \mathrm{d}t$. And then by Lemma 2.1 (ii) $(g(x,.))_x$ is equibounded on the bounded subset of $\mathbb{R}^m$ because $g(.,0) = \nabla_\zeta f(.,0)$ is bounded. Accordingly, for every bounded $K \subset \mathbb{R}^M$ there is a constant M_K such that $|f(x,\zeta)| \le M_K|\zeta|$ for all $x \in \mathbb{R}^N$ and $\zeta \in K$. Consequently, since $\|u\|_\infty \le \|\psi\|_\infty$, there is $M = M_{\|\psi\|_\infty}$ such that in particular

$$\|\boldsymbol{f}(tu)\|_{0,p} \le M\|tu\|_{0,p}.$$

But for $u \in E'$

$$(L+1)u = -\frac{1}{t}\boldsymbol{f}(tu) + th + u,$$

and since $(L+1) \in GL(X_p, Y_p)$, there is $C > 0$ such that $\|u\|_{2,p} \le C\|(L+1)u\|_{0,p}$, and consequently

$$\|u\|_{2,p} \le C\|(L+1)u\|_{0,p} \le C\left((M+1)\|\psi\|_{0,p} + \|h\|_{0,p}\right).$$

□

3.3 A second homotopy

We continue to deal with the problem of the previous section, but we consider another homotopy

$$H(t,u) = Lu + t\boldsymbol{f}(u) + (1-t)\delta u - th, \quad \text{where } \delta > 0.$$

The advantage of this homotopy is that we do not need to assume explicitly that $\mathrm{D}F(u)$ is invertible at some point (the existence of a point u_0 at which $\mathrm{D}F(u_0)$ is invertible will follow from degree arguments: see the remark after Proposition 3.3). But to ensure the Fredholmness of this homotopy we make a further assumption. Consider the family of elliptic operators with constant coefficients ($t \in [0,1]$)

$$Lu + tg^\infty(0)u + (1-t)\delta u. \tag{3.14}$$

We assume that its characteristic polynomial has no real zeros

$$\det S_t^\infty(\xi) \neq 0 \quad \text{for all } \xi \in \mathbb{R}^N \text{ and } t \in [0,1]. \tag{3.15}$$

Lemma 3.4 *H has the following properties.*

(i) $H \in C^1(\mathbb{R} \times X_p, Y_p)$.

(ii) $\mathrm{D}_u H(t,u) \in \Phi_0(X_p, Y_p)\ \forall (t,u) \in [0,1] \times X_p$.

(iii) *For every bounded* $\mathcal{B} \subset X_p$*, the collection* $(H(.,u))_{u\in\mathcal{B}}$ *is equicontinuous.*

(iv) *The restriction of* H *to any closed bounded subset of* $[0,1] \times X_p$ *is proper.*

Proof. Indeed it is sufficient to prove the above properties for $G(t,u) = H(t,u) + th$.

(i) is straightforward.

(ii) For each $t \in [0,1]$, $G(t,.)$ has the same form of F with $f(x,\zeta)$ replaced by $tf(x,\zeta)+(1-t)\delta\zeta$ which satisfies exactly the same assumptions on f. Thus, once again, all theorems of chapter 2 apply to $G(t,.)$. In particular by Theorem 2.3, it is enough to prove that $\mathrm{D}_u G(t,0) \in \Phi_0(X_p, Y_p)$. But $\mathrm{D}_u G(t,0)u = Lu + t\mathrm{D}\boldsymbol{f}(0)u + (1-t)\delta u$, and it has (3.14) as a limit operator, which is assumed to be an isomorphism for all $t \in [0,1]$. Therefore the result follows from Corollary 2.6.

(iii) Let $t, t_0 \in [0,1]$, then $G(t,u) - G(t_0,u) = (t-t_0)\boldsymbol{f}(u) + (t_0-t)\delta u$. And the result follows from the fact that $\boldsymbol{f}(u)$ is bounded in Y_p when u is bounded in X_p (see for example Lemma 2.3 (i)).

(iv) We prove that each partial $G(t,.)$ map is proper. Then recalling point (iii), we proceed exactly as in the proof of Lemma 3.1 (**4.**). Now for each $t \in [0,1]$, $G(t,.)$ has a limit problem

$$G^\infty(t,u) = Lu + t\boldsymbol{f}^\infty(u) + (1-t)\delta u$$

Let $u \in X_p$ satisfies $G^\infty(t,u) = 0$. Then

$$(L^k u^k)u^k = -t\boldsymbol{f}_k^\infty(u)u^k - (1-t)\delta(u^k)^2 \leq 0$$

as it follows from (3.12). Therefore the maximum principle implies $u^k = 0$. By Theorem 2.7 (recalling (ii)), $G(t.,)$ is proper on the closed bounded subsets of X_p. □

Corollary 3.1 *For every bounded subset* B *in* X_p*,* $H|_{[0,1]\times\bar{B}}$ *is* B*-admissible.*

Lemma 3.5 *Let* $E = \{u \in X_p, H(t,u) = 0$ *for some* $t \in [0,1]\}$*. Then* E *is bounded.*

Proof. Let for $k = 1,\dots,m$, $\psi^k = (L^k+\delta)^{-1}(|h^k|)$. Then by Lemma 3.2 (i), $\psi_k \geq 0$. We claim that $|u^k(x)| \leq \psi^k(x)$ for every $x \in \mathbb{R}^N$ and every $k = 1,\dots,m$. Suppose first that for some $k \in \{1,\dots,m\}$ and some $x \in \mathbb{R}^N$, we have $u^k(x) > \psi^k(x)$. Let then $z = u^k - \psi^k$, and $\Theta = \{x \in \mathbb{R}^N,\ z(x) > 0\}$. Then for all $x \in \Theta$

$$\begin{aligned}
L^k z &= L^k u^k - L^k \psi^k \\
&= -t\boldsymbol{f}_k(u) - (1-t)\delta u^k + th^k - (L^k+\delta)\psi^k + \delta\psi^k \\
&= -t\boldsymbol{f}_k(tu) - (1-t)\delta u^k + th^k - |h^k| + \delta\psi^k \\
&\leq -\delta u^k + \delta\psi^k < 0,
\end{aligned}$$

which leads to a contradiction by the maximum principle. Therefore $u^k(x) \leq \psi^k(x)$ for every $x \in \mathbb{R}^N$ and every $k = 1,\dots,m$.

A similar discussion as in the proof of Proposition 3.2 shows that $u^k(x) \geq -\psi^k(x)$ for every $x \in \mathbb{R}^N$ and every $k = 1,\dots,m$.

Consequently, $|u(x)| \leq |\psi(x)|$ and so $\|u\|_\infty \leq \|\psi\|_\infty$ and $\|u\|_{0,p} \leq \|\psi\|_{0,p}$. The conclusion follows exactly as in the end of Proposition's 3.2 proof. □

Proposition 3.3 *For every $h \in Y_p$, $F(u) = h$ is solvable in X_p.*

Proof. Let B be a ball of center 0 and containing E. Then indeed $0 \notin H([0,1], \partial B)$. Now $H(0,.) = L + \delta \in GL(X_p, Y_p)$, therefore 0 is a base point of $H(0,.)$ and $\deg_0(H(0,.), B, 0) = 1$. The contraposition of Theorem 1.1 (ii) ensures the existence of an element $u \in B$ for which $H(1,u) = F(u) - h = 0$. □

Remark 3.3 The contraposition of Theorem 1.1 (ii) ensures also the existence of a base point u_0 of $H(1,.)$, and consequently of F. Then, $\deg_{u_0}(F - h, B, 0) = \pm 1$.

Chapter 4

Global continuation in nonlinear elasticity

4.1 Setting of the problem

In three dimensional elasticity the following notation is often used when formulating the basic field equations.

$$\begin{aligned}
&M \text{ is the set of all real } 3\times 3 \text{ matrices,}\\
&M_+ = \{E \in M \mid \det E > 0\},\\
&O_+ = \left\{E \in M \mid E^T E = EE^T = I \text{ and } \det E = 1\right\},
\end{aligned}$$

where I is the identity matrix, and E^T is the transpose of E. A scalar product on M is defined by

$$\langle E, G\rangle = \operatorname{trace} EG^T = \operatorname{trace} E^T G.$$

and the associated norm is denoted by $\|\cdot\|$. This norm satisfies $\|E\| = \|E^T\|$, and $\|QE\| = \|EQ\| = \|E\|$ for all $E \in M$, and $Q \in O_+$.

We consider an elastic body in an unstressed reference configuration which fills $\mathbb{R}^3$. In this case a **deformation** of the body is described by a map φ such that

(i) $\varphi \in C^1(\mathbb{R}^3, \mathbb{R}^3)$.

(ii) $\nabla\varphi(x) \in M_+$ for all $x \in \mathbb{R}^3$.

(iii) $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ is injective.

We suppose that the **mass density** in the reference configuration is given by a function $\rho : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\rho \in C(\mathbb{R}^3) \text{ and } \rho(x) > 0 \text{ for all } x \in \mathbb{R}^3.$$

The **external forces** acting on the body are given by a density $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \text{ is an equicontinuous } C^1_\xi - \text{bundle map.}$$

A particular example reminiscent of the Hook's law is

$$f(x,\zeta) = -k(\zeta - x) + f_0(x),$$

where $k > 0$, and $f_0 : \mathbb{R}^3 \to \mathbb{R}^3$ is a continuous function with compact support. Under these assumptions, the equations of equilibrium for a deformation φ are

$$\operatorname{div} T(x) + \rho(x) f(x, \varphi(x)) = 0 \quad \text{for } x \in \mathbb{R}^3, \tag{4.1}$$

$$T(x)[\nabla\varphi(x)]^T = [\nabla\varphi(x)][T(x)]^T \quad \text{for } x \in \mathbb{R}^3, \tag{4.2}$$

where $T(x)$ denotes the **first Piola-Kirchhoff stress** tensor at x.

We suppose that the body is asymptotically undeformed in the sense that

$$\varphi(x) - x \to 0 \quad \text{as } |x| \to \infty. \tag{4.3}$$

To complete the specification of the problem (4.1) – (4.3), we must say how the stress in the material is determined by its deformation. This is done through a constitutive relation of the form

$$T(x) = \widehat{T}(\nabla\varphi(x)),$$

where $\widehat{T} : M_+ \to M$ is the **elastic response function**, which depends on the material. The frame indifference requires that $\widehat{T}(QE) = Q\widehat{T}(E)$, for all $E \in M_+$ and $Q \in O_+$. We also assume that the reference configuration is unstressed, and so $\widehat{T}(I) = 0$.

In this chapter we concentrate on a class of materials known as **Mooney-Rivlin materials**, for which the response function derives from a potential energy or **stored energy function** $W : M_+ \to \mathbb{R}$ of the form

$$W(E) = a\,\|E\|^2 + b\,\|\operatorname{Cof} E\|^2 + h(\det E), \tag{4.4}$$

where $a > 0$, $b \geq 0$ are constants, and $h : (0, \infty) \to \mathbb{R}$ is a C^3-convex function. Usually $h(t) \to \infty$ as $t \to 0$, and a typical example is $h(t) = ct^2 - d\ln t$ with $c, d > 0$. Note that by adding a constant, we can normalize the energy by $W(I) = 0$, and this is equivalent to

$$3a + 3b + h(1) = 0,$$

since $\|I\|^2 = \operatorname{trace} I = 3$, and $\operatorname{Cof} I = I$. This will be assumed in the sequel.

The fact that $\widehat{T}$ derives from the energy W means that $\widehat{T}(E) = \operatorname{grad} W(E)$, which is also expressed by the condition

$$\left\langle \widehat{T}(E), G \right\rangle = \mathrm{D}W(E)G \quad \text{for all } E \in M_+ \text{ and } G \in M.$$

Remark 4.1 Here, $\mathrm{D}W(E)$ is the Fréchet derivative of W at E, and so it is a linear function from M to $\mathbb{R}$, whereas $\widehat{T}(E)$ is a matrix.

Remark 4.2 Materials for which the response function derives from a potential energy are called **hyperelastic**.

If we write φ in the form $\varphi(x) = x + u(x)$, (the function $u : \mathbb{R}^3 \to \mathbb{R}^3$ is called the **displacement** corresponding to the deformation φ), the equations (4.1), (4.2) and (4.3) are then equivalent to

$$\operatorname{div} \widehat{T}\,(I + \nabla u(x)) + \rho(x) f\,(x, x + u(x)) = 0 \quad \text{for } x \in \mathbb{R}^3, \tag{4.5}$$

$$\lim_{|x|\to\infty} u(x) = 0. \tag{4.6}$$

We note that, for $i = 1, 2, 3$,

$$\begin{aligned}\Big(\operatorname{div}\widehat{T}\,(I+\nabla u(x))\Big)_i &= \sum_{\alpha=1}^{3}\frac{\partial}{\partial x_\alpha}\Big(\widehat{T}_{i\alpha}\,(I+\nabla u(x))\Big)\\ &= \sum_{\alpha=1}^{3}\sum_{\beta,j=1}^{3}\frac{\partial\widehat{T}_{i\alpha}\,(I+\nabla u(x))}{\partial E_{j\beta}}\partial_\alpha[\nabla u(x)]_{j\beta}\\ &= \sum_{\alpha=1}^{3}\sum_{\beta,j=1}^{3}\frac{\partial\widehat{T}_{i\alpha}(I+\nabla u(x))}{\partial E_{j\beta}}\partial^2_{\alpha\beta}u_j(x).\end{aligned}$$

Before going further, let us write down explicitly the expression of $\widehat{T}$, for a Mooney-Rivlin material, In doing so, we are led naturally to establish some useful identities that will be needed in the sequel.

Lemma 4.1 *For $E \in M_+$,*

$$\widehat{T}(E) = 2aE + 2b(\det E)^2\left(\left\|E^{-1}\right\|^2 E^{-T} - E^{-T}E^{-1}E^{-T}\right) + h'(\det E)(\det E)E^{-T}, \tag{4.7}$$

where $\;E^{-T} = (E^{-1})^T = (E^T)^{-1}$.

Proof. Define for $E \in M_+$

$$A(E) = a\,\|E\|^2\,, \qquad B(E) = b\,\|\operatorname{Cof}E\|^2\,, \qquad C(E) = h(\det E).$$

We compute the derivatives of A, B, and C. Indeed,

$$\mathrm{D}A(E)G = 2a\langle E, G\rangle$$
$$\mathrm{D}B(E)G = 2b\langle \operatorname{Cof}E, \mathrm{D}(\operatorname{Cof}E)G\rangle \tag{4.8}$$
$$\mathrm{D}C(E)G = h'(\det E)\mathrm{D}(\det E)G \tag{4.9}$$

Therefore, we are led to compute D(det) and D(Cof).

Now, $(\operatorname{Cof}E)_{ij} = (-1)^{i+j}\det\widehat{E}_{ij}$, where $\widehat{E}_{ij}$ is the matrix obtained from E by deleting the i^{th} row and j^{th} column, and then

$$E^{-1} = \frac{(\operatorname{Cof}E)^T}{\det E}, \quad \text{or} \quad \operatorname{Cof}E = (\det E)E^{-T}.$$

Furthermore, for every $i = 1, 2, 3$,

$$\det E = \sum_{j=1}^{3} E_{ij}(\operatorname{Cof}E)_{ij} \qquad (\text{expanding } \det E \text{ along the } i^{\text{th}} \text{ row}).$$

Therefore,

$$\begin{aligned}\frac{\partial\det E}{\partial E_{ik}} &= \sum_{j=1}^{3}\frac{\partial E_{ij}}{\partial E_{ik}}(\operatorname{Cof}E)_{ij} + E_{ij}\frac{\partial(\operatorname{Cof}E)_{ij}}{\partial E_{ik}}\\ &= \sum_{j=1}^{3}\delta_{jk}(\operatorname{Cof}E)_{ij} + E_{ij}\times 0, \qquad \text{since } (\operatorname{Cof}E)_{ij} \text{ contains no terms from row } i\\ &= (\operatorname{Cof}E)_{ik}.\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{D}(\det E)G &= \sum_{i,k=1}^{3} \frac{\partial \det E}{\partial E_{ik}} G_{ik} = \sum_{i,k=1}^{3} (\operatorname{Cof} E)_{ik} G_{ik} \\
&= \operatorname{trace}(\operatorname{Cof} E)^T G = \langle \operatorname{Cof} E, G\rangle. \\
&= (\det E)\langle E^{-T}, G\rangle.
\end{aligned}$$

Now, by differentiating the identity $EE^{-1} = I$, we get $\mathrm{D}(E^{-1})G = -E^{-1}GE^{-1}$, and by the chain rule, $\mathrm{D}(E^{-T})G = \left(-E^{-1}GE^{-1}\right)^T = -E^{-T}G^TE^{-T}$. Since $\operatorname{Cof} E = (\det E)E^{-T}$, we get

$$\mathrm{D}(\operatorname{Cof} E)G = (\det E)\langle E^{-T}, G\rangle E^{-T} + (\det E)(-E^{-T}G^TE^{-T}).$$

Replacing $\operatorname{Cof} E$ and $\mathrm{D}(\operatorname{Cof} E)G$ by their values in (4.8), using the properties of the scalar product $\langle\cdot,\cdot\rangle$, and the identity $\langle E^{-T}, E^{-T}G^TE^{-T}\rangle = \langle E^{-T}E^{-1}E^{-T}, G\rangle$, we finally obtain

$$\mathrm{D}B(E)G = 2b(\det E)^2 \left(\left\|E^{-1}\right\|^2 \langle E^{-T}, G\rangle - \left\langle E^{-T}E^{-1}E^{-T}, G\right\rangle\right).$$

Equation (4.9) becomes

$$\mathrm{D}C(E)G = h'(\det E)(\det E)\langle E^{-T}, G\rangle.$$

Therefore,

$$\begin{aligned}
\mathrm{D}W(E)G = {} & 2a\langle E, G\rangle + 2b(\det E)^2 \left(\left\|E^{-1}\right\|^2 \langle E^{-T}, G\rangle - \left\langle E^{-T}E^{-1}E^{-T}, G\right\rangle\right) \\
& + h'(\det E)(\det E)\langle E^{-T}, G\rangle,
\end{aligned}$$

and so,

$$\widehat{T}(E) = 2aE + 2b(\det E)^2 \left(\left\|E^{-1}\right\|^2 E^{-T} - E^{-T}E^{-1}E^{-T}\right) + h'(\det E)(\det E)E^{-T}.$$

□

Remark 4.3 In fact, the identities of the previous lemma, hold for arbitrary $n \times n-$ matrices with nonzero determinant, and not just for $E \in M_+$.

It follows that $\widehat{T}(I) = (2a+4b+h'(1))I$, and so the condition that the reference configuration is unstressed (i.e. $\widehat{T}(I) = 0$) becomes

$$2a + 4b + h'(1) = 0. \tag{4.10}$$

Corollary 4.1 *$\widehat{T}$ has the following properties.*

R(i) $\widehat{T} \in C^2(M_+, M)$, *and* $\widehat{T}(I) = 0$.

R(ii) $\widehat{T}(QE) = Q\widehat{T}(E)$ *for all* $Q \in O_+$ *and* $E \in M_+$.

R(iii) $\widehat{T}(E)E^T = E[\widehat{T}(E)]^T$ *for all* $E \in M_+$.

Remark 4.4 The property R(ii) means that the response function is frame-indifferent. By R(iii), the equation (4.2) for the balance of the moments of the forces is always satisfied.

Differentiating the terms of $\widehat{T}$ and using the identities of the previous lemma, we get

Lemma 4.2 *Let*

$$\widehat{B}(E) = (\det E)^2 \left(\left\|E^{-1}\right\|^2 E^{-T} - E^{-T}E^{-1}E^{-T}\right)$$
$$\widehat{C}(E) = h'(\det E)(\det E)E^{-T}.$$

Then

$$\begin{aligned}\mathrm{D}\widehat{B}(E)G &= 2(\det E)^2\langle E^{-T}, G\rangle \left(\|E^{-1}\|^2 E^{-T} - E^{-T}E^{-1}E^{-T}\right)\\ &\quad - (\det E)^2 \left(2\langle E^{-1}, E^{-1}GE^{-1}\rangle E^{-T} + \|E^{-1}\|^2 E^{-T}G^T E^{-T}\right)\\ &\quad + (\det E)^2 E^{-T}\left(G^T E^{-T}E^{-1} + E^{-1}GE^{-1} + E^{-1}E^{-T}G^T\right)E^{-T}\end{aligned}$$

and

$$\begin{aligned}\mathrm{D}\widehat{C}(E)G &= h''(\det E)(\det E)^2\langle E^{-T}, G\rangle E^{-T} + h'(\det E)(\det E)\left(\langle E^{-T}, G\rangle E^{-T} - E^{-T}G^T E^{-T}\right)\\ &= h''(\det E)\langle \mathrm{Cof}\, E, G\rangle \mathrm{Cof}\, E + h'(\det E)(\det E)\left(\langle E^{-T}, G\rangle E^{-T} - E^{-T}G^T E^{-T}\right)\end{aligned}$$

□

Since $\widehat{T}(E) = 2aE + 2b\widehat{B}(E) + \widehat{C}(E)$, we have $\mathrm{D}\widehat{T}(E)G = 2aG + 2b\mathrm{D}\widehat{B}(E)G + \mathrm{D}\widehat{C}(E)G$ for all $G \in M$. In particular, when $E = I$, we get

$$\mathrm{D}\widehat{T}(I)G = [4b + h''(1) + h'(1)]\mathrm{trace}\,(G)I + 2(a+b)G - (2b + h'(1))G^T.$$

But from (4.10), it follows that $2(a+b) = -2(b + h'(1))$, and thus, we can state

Corollary 4.2

$$\mathrm{D}\widehat{T}(I)G = \lambda \mathrm{trace}\,(G)I + \mu(G^T + G)$$

where

$$\lambda = 4b + h'(1) + h''(1) = -2a + h''(1),$$
$$\mu = 2(a+b) > 0,$$

are the Lamé constants, and satisfy

$$\lambda + 2\mu = 2a + 4b + h''(1) > 0.$$

□

At this point, two issues remain to be resolved before we can use the results of chapter 2 in this context. First of all, the equation (4.5) is not defined for all smooth functions $u : \mathbb{R}^3 \to \mathbb{R}^3$ but only for those such that $I + \nabla u(x) \in M_+$ for all $x \in \mathbb{R}^3$. Secondly, amongst all possible solutions of (4.5), only those such that $x \mapsto x + u(x)$ is injective on $\mathbb{R}^3$ correspond to deformations of the elastic body.
To deal with the first point, we clearly have to extend the definition of $\widehat{T}$ from M_+ to all of M. For most response functions, this cannot be done in a smooth way because

$$\left\|\widehat{T}(E)\right\| \to \infty \text{ when } \det E \to 0.$$

Hence we must restrict $\widehat{T}$ to a slightly smaller set and then find a smooth extension of this restriction.

Lemma 4.3 *Let $h \in C^3(0,\infty)$ satisfy $h''(t) \geq 0$ for all $t > 0$ (this is equivalent to the convexity of h). Then, for any $\varepsilon > 0$, there exists a function $h_\varepsilon \in C^3(\mathbb{R})$ such that*

$$h''_\varepsilon(t) \geq 0 \;\; \text{for all } t \in \mathbb{R} \quad \text{and} \quad h_\varepsilon(t) = h(t) \;\; \text{for all } t \geq \varepsilon.$$

Proof. Define h_ε on $(-\infty,\varepsilon)$ as follows

$$h_\varepsilon(t) = \begin{cases} h(\varepsilon) + h'(\varepsilon)(t-\varepsilon) + \dfrac{1}{2}h''(\varepsilon)(t-\varepsilon)^2 + \dfrac{1}{6}h'''(\varepsilon)(t-\varepsilon)^3 & \text{if } h'''(\varepsilon) \leq 0, \\[2ex] h(\varepsilon) + \left(h'(\varepsilon) - \dfrac{h''(\varepsilon)^2}{h'''(\varepsilon)}\right)(t-\varepsilon) + \dfrac{h''(\varepsilon)^3}{h'''(\varepsilon)^2}\left(\exp\left(\dfrac{h'''(\varepsilon)}{h''(\varepsilon)}(t-\varepsilon)\right) - 1\right) & \text{if } h'''(\varepsilon) > 0. \end{cases}$$

Note that in the second case $h''(\varepsilon) > 0$, because h'' is strictly increasing in a neighborhood of ε. Now, one can check that, in both cases, $h_\varepsilon^{(i)}(\varepsilon) = h^{(i)}(\varepsilon)$ for $i = 0,1,2,3$, and that $h''_\varepsilon(t) \geq 0$ for all $t \leq \varepsilon$. □

Setting

$$W^\varepsilon(E) = a\,\|E\|^2 + b\,\|\mathrm{Cof}\,E\|^2 + h_\varepsilon(\det E) \quad \text{for all } E \in M, \tag{4.11}$$

we clearly have that $W^\varepsilon \in C^3(M,\mathbb{R})$ and $W^\varepsilon(E) = W(E)$ for all $E \in M_\varepsilon$, where

$$M_\varepsilon = \{E \in M \mid \det E > \varepsilon\}\,. \tag{4.12}$$

Thus we can define $\widehat{T}^\varepsilon \in C^2(M,M)$ by

$$\left\langle \widehat{T}^\varepsilon(E), G \right\rangle = \mathrm{D}W^\varepsilon(E)G \quad \text{for all } E, G \in M,$$

and it follows that $\widehat{T}^\varepsilon(E) = \widehat{T}(E)$ for all $E \in M_\varepsilon$.

We fix $\varepsilon > 0$ and define a matrix valued function $a^\varepsilon_{\alpha\beta} : \mathbb{R}^3 \times \left(\mathbb{R}^3 \times M\right) \to M$ by

$$\left(a^\varepsilon_{\alpha\beta}(x,\xi_0,\xi')\right)_{i,j} = \frac{\partial \widehat{T^\varepsilon}_{i\alpha}\,(I+\xi')}{\partial E_{j\beta}} \tag{4.13}$$

where $\alpha,\beta,i,j = 1,2,3$. We also define a function $b : \mathbb{R}^3 \times \mathbb{R}^3 \times M \to \mathbb{R}^3$ by

$$b(x,\xi) = -\rho(x) f\,(x, x+\xi_0) \tag{4.14}$$

We replace (4.5) by the equation

$$\mathrm{div}\,\widehat{T}^\varepsilon\,(I + \nabla u(x)) + \rho(x) f\,(x, x+u(x)) = 0 \quad \text{for } x \in \mathbb{R}^3, \tag{4.15}$$

which is then in the from (1.1) since,

$$\begin{aligned} \left(\mathrm{div}\,\widehat{T}^\varepsilon\,(I+\nabla u(x))\right)_i &= \sum_{\alpha=1}^{3}\sum_{\beta,j=1}^{3} \frac{\partial \widehat{T}^\varepsilon_{i\alpha}\,(I+\nabla u(x))}{\partial E_{j\beta}} \partial^2_{\alpha\beta} u_j(x) \\ &= \left[\sum_{\alpha,\beta=1}^{3} a^\varepsilon_{\alpha\beta}(x,u(x),\nabla u(x))\partial^2_{\alpha\beta} u\right]_i . \end{aligned}$$

We prove that this system satisfies the ellipticity condition (2.18), and we first claim that

$$\left\langle D\widehat{T}^\varepsilon(E)(\zeta\otimes\eta), \zeta\otimes\eta \right\rangle \geq 2a|\zeta|^2|\eta|^2, \tag{4.16}$$

for all $\zeta, \eta \in \mathbb{R}^3$ and all $E \in M$, and where $(\zeta \otimes \eta)_{ij} = \zeta_i \eta_j$.

One way to show this, is to make a direct calculation based on Lemma 4.2, after noting that some of the identities hold in fact for every $E \in M$, because the set $\{E \in M \mid \det E \neq 0\}$ is dense in M[1].

We give however, a more conceptual proof. In doing so, we introduce two fundamental concepts in elasticity that we need later.

Note that the function W^ε is not convex in general (unless $b = h_\varepsilon = 0$), however, if we define $\widetilde{W} : M \times M \times \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{W}(E_1, E_2, \delta) = a \|E_1\|^2 + b \|E_2\|^2 + h_\varepsilon(\delta),$$

then $\widetilde{W}$ is convex in its arguments, and we have $W^\varepsilon(E) = \widetilde{W}(E, \operatorname{Cof} E, \det E)$. A function having this property is called **polyconvex**, a concept introduced by J. Ball in the context of the calculus of variations.

Now, it can be shown (using the Jensen inequality), that a polyconvex function $P : M \to \mathbb{R}$ satisfies

$$P(E) \leq \frac{1}{\operatorname{meas} D} \int_D P(E + \nabla v)\, \mathrm{d}x,$$

for every bounded measurable subset $D \subset \mathbb{R}^3$, every $E \in M$ and every $v \in W_0^{1,\infty}(D, \mathbb{R}^3)$. A function satisfying the preceding inequality is called **quasiconvex**.

Finally a C^2-quasiconvex function P satisfies the (not strong) Legendre-Hadamard condition

$$\mathrm{D}^2 P(E)(\zeta \otimes \eta, \zeta \otimes \eta) \geq 0 \quad \text{for all } E \in M, \text{ and all } \zeta, \eta \in \mathbb{R}^3.$$

For more details and proofs, see for instance **[10]**.

Now we prove inequality (4.16). Note that $W^\varepsilon(E) = A(E) + B(E) + C_\varepsilon(E)$ where

$$A(E) = a \|E\|^2, \qquad B(E) = b \|\operatorname{Cof} E\|^2, \qquad C_\varepsilon(E) = h_\varepsilon(\det E).$$

For any $E, G \in M$,

$$\mathrm{D}^2 A(E)(G, G) = 2a \|G\|^2,$$

and so,

$$\mathrm{D}^2 A(E)(\zeta \otimes \eta, \zeta \otimes \eta) = 2a \|\zeta \otimes \eta\|^2 = 2a|\zeta|^2|\eta|^2 \quad \text{for all } \zeta, \eta \in \mathbb{R}^3.$$

Note that the parts B and C_ε are themselves polyconvex functions on M, and consequently

$$\begin{aligned} &\mathrm{D}^2 B(E)(\zeta \otimes \eta, \zeta \otimes \eta) \geq 0, \\ &\mathrm{D}^2 C_\varepsilon(E)(\zeta \otimes \eta, \zeta \otimes \eta) \geq 0, \end{aligned}$$

for all $\zeta, \eta \in \mathbb{R}^3$. But

$$\left\langle \mathrm{D}\widehat{T^\varepsilon}(E) G, G \right\rangle = \mathrm{D}^2 W^\varepsilon(E)(G, G) \quad \text{for all } E, G \in M,$$

and for $G = \zeta \otimes \eta$, we have just shown that,

$$\mathrm{D}^2 W^\varepsilon(E)(G, G) \geq 2a \|G\|^2 = 2a |\zeta|^2 |\eta|^2 . \qquad \square$$

We now show that the ellipticity condition (2.18) is satisfied. Since

$$\left[\sum_{\alpha,\beta=1}^{3} \eta_\alpha \eta_\beta a^\varepsilon_{\alpha\beta}(x, \xi) \right]_{ij} = \sum_{\alpha,\beta=1}^{3} \frac{\partial \widehat{T^\varepsilon}_{i\alpha}(I + \xi')}{\partial E_{j\beta}} \eta_\alpha \eta_\beta$$

[1] Since the equation $\det(E - \lambda I) = 0$ has finitely many solutions in $\lambda \in \mathbb{R}$, then $\det(E - \frac{1}{n} I) \neq 0$ for $n \in \mathbb{N}$ large enough, and indeed $E - \frac{1}{n} I \to E$ as $n \to \infty$.

we have that

$$\begin{aligned}\sum_{i,j=1}^{3}\left[\sum_{\alpha,\beta=1}^{3}\eta_\alpha\eta_\beta a^\varepsilon_{\alpha\beta}(x,\xi)\right]_{ij}\zeta_i\zeta_j &= \sum_{i,j=1}^{3}\sum_{\alpha,\beta=1}^{3}\frac{\partial\widehat{T}^\varepsilon_{i\alpha}\left(I+\xi'\right))}{\partial E_{j\beta}}\eta_\alpha\eta_\beta\zeta_i\zeta_j\\ &= \left\langle \mathrm{D}\widehat{T}^\varepsilon\left(I+\xi'\right)(\zeta\otimes\eta),\zeta\otimes\eta\right\rangle\\ &= \mathrm{D}^2W^\varepsilon(I+\xi')(\zeta\otimes\eta,\zeta\otimes\eta)\geq 2a|\zeta|^2|\eta|^2,\end{aligned}$$

for all $\zeta,\eta\in\mathbb{R}^3\backslash\{0\}$ and $(x,\xi)\in\mathbb{R}^3\times\left(\mathbb{R}^3\times M\right)$.

Thus, (since $a>0$), the Strong Legendre-Hadamard condition (2.17) is satisfied and so also (2.18).

Henceforth, we assume that the mass density $\rho(x)$ and the body forces $f(x,\zeta)$ have the following property.

(B) There exist a real number $p>3$ such that the function

$$x\mapsto\rho(x)f(x,x)\ \text{ belongs to } L^p(\mathbb{R}^3,\mathbb{R}^3),$$

and the function

$$x\mapsto\rho(x)\nabla_\zeta f(x,x)\ \text{ belongs to } L^\infty(\mathbb{R}^3,M).$$

Then the hypotheses (2.10) to (2.13) are satisfied by (4.13) and (4.14). Solutions of the system

$$u\in X_p=W^{2,p}(\mathbb{R}^3,\mathbb{R}^3)\text{ and}\tag{4.17}$$

$$\operatorname{div}\widehat{T^\varepsilon}(I+\nabla u(x))+\rho(x)f\left(x,x+u(x)\right)=0\text{ for }x\in\mathbb{R}^3\tag{4.18}$$

satisfy (4.6), because $3<p<\infty$.

To conclude we need to discuss the extent to which solutions of (4.18) correspond to deformations of the elastic body satisfying (4.3). For this we set

$$\begin{aligned}\mathcal{O}&=\left\{u\in X_p\,|\,I+\nabla u(x)\in M_+\ \text{ for all } x\in\mathbb{R}^3\right\}\quad\text{ and}\\ \mathcal{O}^\varepsilon&=\left\{u\in X_p\,|\,I+\nabla u(x)\in M_\varepsilon\ \text{ for all }\ x\in\mathbb{R}^3\right\}\quad\text{ for } 0<\varepsilon<1.\end{aligned}$$

Lemma 4.4 *For any real number $p>3$, the sets $\mathcal{O}$ and $\mathcal{O}^\varepsilon$ have the following properties.*

(i) *$\mathcal{O}$ and $\mathcal{O}^\varepsilon$ are open subsets of X_p. For any $u\in\mathcal{O}$ there exists $\varepsilon>0$ such that $I+\nabla u(x)\in M_\varepsilon$ for all $x\in\mathbb{R}^3$. Thus $\mathcal{O}=\bigcup\limits_{0<\varepsilon<1}\mathcal{O}^\varepsilon$.*

(ii) *For any $u\in\mathcal{O}$, the function φ defined by $\varphi(x)=x+u(x)$ is a deformation.*

(iii) *For any $\mu\in(0,1)$ there exists $\varepsilon>0$ such that $\{u\in X_p:\|\nabla u\|_\infty<\mu\}\subset\mathcal{O}^\varepsilon$.*

Proof. Since $\det I=1$, there exists $\delta>0$ such that

$$\det E\geq\frac{1}{2}\quad\text{ for all }\ E\in Z=\{E\in M:\|E-I\|<\delta\}\,.$$

(i) Fix $u\in\mathcal{O}$. Since $u\in X_p\subset C^1_d(\mathbb{R}^3,\mathbb{R}^3)$, there exists $R>0$ such that

$$|u(x)|+\|\nabla u(x)\|<\frac{\delta}{2}\quad\text{ whenever }\ |x|\geq R.$$

In particular, $I + \nabla u(x) \in Z$ for all $|x| \geq R$ and consequently $\det(I + \nabla u(x)) \geq \frac{1}{2}$ whenever $|x| \geq R$.
But $\det(I + \nabla u(x)) > 0$ for all x and hence $\gamma \equiv \inf\{\det(I + \nabla u(x)) : |x| \leq R\} > 0$ by the continuity of ∇u. Thus we see that $I + \nabla u(x) \in M_\varepsilon$ for all $x \in \mathbb{R}^3$ for any $\varepsilon < \min\{\frac{1}{2}, \gamma\}$, and hence $u \in \mathcal{O}^\varepsilon$.

Furthermore, there exists $\nu > 0$ such that $\det(I + E) \geq \frac{\gamma}{2}$ provided that $\|E - \nabla u(x)\| < \nu$ for some x with $|x| \leq R$. By the continuity of the embedding of X_p in $C^1_d(\mathbb{R}^3, \mathbb{R}^3)$, there exists $\sigma > 0$ such that $\|\nabla v(x) - \nabla u(x)\| < \min\{\frac{\delta}{2}, \nu\}$ for all $x \in \mathbb{R}^3$ and all $v \in X_p$ such that $\|v - u\|_{2,p} < \sigma$. Then for $v \in X_p$ with $\|v - u\|_{2,p} < \sigma$, we have that

$$\|\nabla v(x)\| \leq \|\nabla u(x)\| + \|\nabla v(x) - \nabla u(x)\| < \frac{\delta}{2} + \frac{\delta}{2} \quad \text{for } |x| \geq R,$$

and so $I + \nabla v(x) \in Z$ for all $|x| \geq R$, whereas $\|\nabla v(x) - \nabla u(x)\| < \nu$ for $|x| \leq R$, and so $\det(I + \nabla v(x)) \geq \frac{\gamma}{2}$ for all $|x| \leq R$.
It follows that $\det(I + \nabla v(x)) \geq \min\{\frac{1}{2}, \frac{\gamma}{2}\}$ for all $x \in \mathbb{R}^3$ and all $v \in X_p$ with $\|v - u\|_{2,p} < \sigma$. This proves that $\mathcal{O}$ is an open subset of X_p. The proof that $\mathcal{O}^\varepsilon$ is open is similar.

(ii) Fix $u \in \mathcal{O}$ and set $\varphi(x) = x + u(x)$. There exists $r > 0$ such that

$$|u(x)| + \|\nabla u(x)\| < \frac{1}{2} \quad \text{whenever } |x| \geq r.$$

We show first that φ is injective on $\{x \in \mathbb{R}^3 : |x| \geq r + 1\}$.

Suppose that $|x|, |y| \geq r + 1$ and that $\varphi(x) = \varphi(y)$. Then

$$|x - y| = |u(x) - u(y)| \leq |u(x)| + |u(y)| \leq 1.$$

But

$$u(y) - u(x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} u(ty + (1-t)x)\,\mathrm{d}t = \int_0^1 \nabla u(ty + (1-t)x)\,\mathrm{d}t[y - x]$$

and so

$$|x - y| = |u(y) - u(x)| \leq |y - x| \max_{0 \leq t \leq 1} \|\nabla u(ty + (1-t)x)\|.$$

However

$$|ty + (1-t)x| = |x - t(y - x)| \geq |x| - t|y - x| \geq r$$

since $|x| \geq r + 1$ and $|y - x| \leq 1$. Therefore, $\|\nabla u(ty + (1-t)x)\| < \frac{1}{2}$ for all $t \in [0, 1]$ and consequently,

$$|x - y| = |u(y) - u(x)| \leq \frac{1}{2}|y - x|,$$

so we must have $x = y$, proving the injectivity of φ on the region $\{x \in \mathbb{R}^3 : |x| \geq r + 1\}$. From this we can now deduce that φ is injective on all of $\mathbb{R}^3$ by appealing to a well-known result due to Meisters and Olech **[26]**,**[9]**. Indeed, given two elements $x, y \in \mathbb{R}^3$ such that $\varphi(x) = \varphi(y)$, there exists $R > r + 1$ such that $x, y \in B$, where $B = \{z \in \mathbb{R}^3 : |z| \leq R\}$. Now as we have just shown, φ restricted to ∂B is injective and $\det \nabla\varphi(x) > 0$ for all x since $u \in \mathcal{O}$. Hence φ restricted to B is injective by the result of Meisters and Olech, and therefore, $x = y$.

(iii) Let $U = \{E \in M : \|E - I\| < 1\}$ and recall that for all $E \in U$, E is invertible with $\det E > 0$. For any $\mu < 1$,

$$\varepsilon(\mu) \equiv \inf\{\det E : \|E - I\| \leq \mu\} > 0.$$

Now consider an element $u \in X_p$ such that $\|\nabla u\|_\infty < \mu$. Then $\det(I + \nabla u(x)) \geq \varepsilon(\mu)$ for all $x \in \mathbb{R}^3$, so $u \in \mathcal{O}^\varepsilon$ for all $\varepsilon < \varepsilon(\mu)$. $\square$

Let us end by summarizing the situation. Our aim is to treat the equilibrium, under the applications of body forces and no deformation at infinity, of an elastic body whose reference configuration occupies $\mathbb{R}^3$. This amounts to finding those deformations φ which satisfy the equations (4.1), (4.2), (4.3). We can begin by studying the solutions u of the problem (4.17),(4.18) which, as we have shown, satisfies the hypotheses of the main results of chapter 2, provided that f has the property (B). If, in addition, $u \in \mathcal{O}^\varepsilon$, then by Lemma 4.4, $\varphi(x) = x + u(x)$ is a deformation and it satisfies the equations (4.1), (4.2), (4.3). Conversely, for any equilibrium deformation φ, there exists an $\varepsilon > 0$ such that $u(x) = \varphi(x) - x$ lies in $\mathcal{O}^\varepsilon$ and satisfies (4.17),(4.18). Since $\mathcal{O}^\varepsilon$ is an open neighborhood of 0 in $W^{2,p}(\mathbb{R}^3, \mathbb{R}^3)$, solutions of (4.18) that are obtained by continuation from $u \equiv 0$ will, at least initially lie in the subset $\mathcal{O}^\varepsilon$.

Now, we are ready to study the Fredholm and properness properties of the map

$$F_\varepsilon(u)(x) = -\operatorname{div} \widehat{T^\varepsilon}(I + \nabla u(x)) - \rho(x) f(x, x + u(x)). \tag{4.19}$$

We recall, for future references, the assumptions on $\widehat{T}^\varepsilon$.

(T) $\widehat{T}^\varepsilon$ is the gradient of the function $W^\varepsilon : M \to \mathbb{R}$ defined by

$$W^\varepsilon(E) = a\,\|E\|^2 + b\,\|\operatorname{Cof} E\|^2 + h_\varepsilon(\det E),$$

where $a > 0$, $b \geq 0$ are constants, and $h_\varepsilon : \mathbb{R} \to \mathbb{R}$ is a C^3-convex function. Furthermore, $W^\varepsilon(I) = 0$ and $\widehat{T}^\varepsilon(I) = 0$.

4.2 Fredholm and properness properties

So far, we assumed the following about the density and the forces.

$$\rho \in C(\mathbb{R}^3) \text{ and } \rho(x) > 0 \text{ for all } x \in \mathbb{R}^3, \tag{4.20}$$

$$f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \text{ is an equicontinuous } C^1_\zeta - \text{bundle map}, \tag{4.21}$$

$$\text{the functions } x \mapsto \rho(x)\nabla_\zeta f(x,x) \text{ and } x \mapsto \rho(x) f(x,x) \text{ belong respectively to } L^\infty(\mathbb{R}^3, M) \text{ and } L^p(\mathbb{R}^3, \mathbb{R}^3) \text{ for some } p \in (3, \infty). \tag{4.22}$$

Now, we add the following assumptions.

$$\lim_{|x| \to \infty} \rho(x) = \rho^\infty > 0, \tag{4.23}$$

there exists a function $f^\infty \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ with $f^\infty(0) = 0$ such that

$$\lim_{|x| \to \infty} \|\nabla_\zeta f(x, x+\zeta) - \nabla f^\infty(\zeta)\| = 0, \text{ uniformly for } \zeta \text{ in compact subsets of } \mathbb{R}^3, \tag{4.24}$$

$$\text{there is } k > 0 \text{ such that } (\nabla_\zeta f(x,x)\eta) \cdot \eta \leq -k|\eta|^2, \ \forall\, x, \eta \in \mathbb{R}^3. \tag{4.25}$$

Let

$$g(\zeta) = \rho^\infty f^\infty(\zeta). \tag{4.26}$$

(G) g (or equivalently f^∞) is conservative i.e. there exist a function $\mathcal{G} \in C^1(\mathbb{R}^3)$ such that $\nabla \mathcal{G}(\zeta) = g(\zeta)$ for all $\zeta \in \mathbb{R}^3$. Then, by adding a constant, we consider that $\mathcal{G}(0) = 0$.

Our last hypothesis is

$$f^\infty(\zeta) \cdot \zeta \leq -k|\zeta|^2. \tag{4.27}$$

Under the previous assumptions we have a limit operator in the sense of §2.5.

$$F^\infty_\varepsilon(u) = -\operatorname{div} \widehat{T}(I + \nabla u) - \rho^\infty f^\infty(u) \tag{4.28}$$

Remark 4.5 It follows that $\underline{\rho} := \inf_{x\in\mathbb{R}^3} \rho(x) > 0$. Indeed, from (4.23), there is $R > 0$ such that $\rho(x) \geq \frac{1}{2}\rho^\infty$, whenever $|x| > R$. From (4.20), ρ attains its (positive) minimum on the compact set $\overline{B_R}$ at a point x_0. But then $\underline{\rho} \geq \min(\frac{1}{2}\rho^\infty, \rho(x_0)) > 0$.

Remark 4.6 Under assumptions (4.20) and (4.23), condition (4.22) is equivalent to

$$x \mapsto \nabla_\zeta f(x,x) \in L^\infty(\mathbb{R}^3, M), \ \text{ and } \ x \mapsto f(x,x) \in L^p(\mathbb{R}^3, \mathbb{R}^3). \tag{4.29}$$

Indeed, if (4.22) holds, then (4.29) holds as well because $\|\nabla_\zeta f(x,x)\| \leq \frac{1}{\underline{\rho}}\|\rho(x)\nabla_\zeta f(x,x)\|$, and $|f(x,x)| \leq \frac{1}{\underline{\rho}}|\rho(x)f(x,x)|$, for every $x \in \mathbb{R}^3$. Conversely, if (4.29) holds then (4.22) holds, because $\|\rho(x)\nabla_\zeta f(x,x)\| \leq \|\rho\|_{0,\infty}\|\nabla_\zeta f(x,x)\|$, and $|\rho(x)f(x,x)| \leq \|\rho\|_{0,\infty}|f(x,x)|$.

Remark 4.7 It follows from assumptions (4.23)−(4.24) that

$$\zeta^T \nabla f^\infty(0)\zeta \leq -k|\zeta|^2.$$

Remark 4.8 Assumption ***(G)*** is weaker than the fact that f is conservative with respect to ζ. Indeed, the condition that there is $\Phi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ such that $\nabla_\zeta \Phi(x,\zeta) = f(x,\zeta)$, is equivalent to the condition that $\mathrm{rot}_\zeta f(x,\zeta) = 0$ for all x and $\zeta \in \mathbb{R}^3$, since $\mathbb{R}^3$ is simply connected. Then letting $|x| \to \infty$, we get $\mathrm{rot}\, f^\infty(\zeta) = 0$ and so f^∞ is conservative.

Remark 4.9 The field $f(x,\zeta) = -k(\zeta - x) + f_0(x)$, given in §4.1 satisfies all our assumptions, with $f^\infty(\zeta) = -k\zeta$.

Proposition 4.1 *Under assumptions **(T)** and (4.20)−(4.22), the operator F_ε in (4.19) is of class C^1 from X_p to Y_p. Its derivative at zero is*

$$\mathrm{D}F_\varepsilon(0)u = -(\lambda+\mu)\nabla(\mathrm{div}\, u) - \mu\Delta u - \rho(x)\nabla_\zeta f(x,x)u, \tag{4.30}$$

where λ and μ are given in Corollary 4.2.

Proof. That F_ε is C^1, follows from Theorem 2.2, since the coefficients of F_ε satisfy the required assumptions (2.10)−(2.13). In particular, from the chain rule and the linearity of the operators div and ∇, it follows that

$$\mathrm{D}F_\varepsilon(0)u = -\mathrm{div}\, \mathrm{D}\widehat{T}^\varepsilon(I)\nabla u - \rho(x)\nabla_\zeta f(x,x)u,$$

From Corollary 4.2, we have, since $\widehat{T}^\varepsilon(I) = \widehat{T}(I)$,

$$\mathrm{D}\widehat{T}^\varepsilon(I)G = \lambda \mathrm{trace}\,(G)I + \mu(G^T + G),$$

and therefore,

$$\begin{aligned}\mathrm{div}\, \mathrm{D}\widehat{T}^\varepsilon(I)\nabla u &= \mathrm{div}\,(\lambda \mathrm{trace}\,(\nabla u)I + \mu(\nabla u^T + \nabla u)) \\ &= \mathrm{div}\,(\lambda(\mathrm{div}\, u)I + \mu(\nabla u^T + \nabla u)).\end{aligned}$$

Or componentwise, for $i = 1,2,3$,

$$\begin{aligned}\left(\mathrm{div}\, \mathrm{D}\widehat{T}^\varepsilon(I)\nabla u\right)_i &= \lambda\sum_{j=1}^3 \partial_j(\mathrm{div}\, u)\delta_{ij} + \mu\sum_{j=1}^3 \partial_j(\partial_i u_j + \partial_j u_i) \\ &= \lambda\partial_i(\mathrm{div}\, u) + \mu\partial_i(\mathrm{div}\, u) + \mu\Delta u_i \\ &= (\lambda+\mu)\left(\nabla(\mathrm{div}\, u)\right)_i + \mu\Delta u_i.\end{aligned}$$

□

Lemma 4.5 *Let $\theta \in C^\infty(\mathbb{R}^3)$ satisfy $\theta(x) = |x|$ for $|x| > 2$, and $\theta(x) = 0$ when $|x| < 1$. Let $u = e^{-s\theta} v$ for some $v \in X_p$ and some $s > 0$. Then the following hold.*

(i) $u \in W^{1,q}$ *for all* $q \in [1, \infty]$.

(ii) $\partial_\alpha u_i \partial_\beta u_j \in W^{1,1}(\mathbb{R}^3)$, *for every* $i, j = 1,2,3$ *and* $\alpha, \beta = 0,1,2,3$. *In particular,*

$$\|\nabla u\|^2,\ (\operatorname{div} u)^2,\ \|\operatorname{rot} u\|^2,\ \Delta u \cdot u,\ \nabla(\operatorname{div} u)\cdot u \in L^1(\mathbb{R}^3),$$

and

$$(\operatorname{div} u) u,\ (\nabla u)^T u \in W^{1,1}(\mathbb{R}^3, \mathbb{R}^3).$$

(iii) $\displaystyle\int_{\mathbb{R}^3} \partial_\alpha w = 0$ *for every* $w \in W^{1,1}(\mathbb{R}^3)$, *and every* $\alpha = 1,2,3$.

Proof. (i) Note first, that since $v \in L^\infty$ and $e^{-s\theta} \in L^q$ for any $q \in [1,\infty]$, we have $u \in L^q$. Second, by a direct computation already made in §2.7, we have for $\alpha = 1,2,3$,

$$\partial_\alpha(e^{-s\theta} v) = e^{-s\theta}\partial_\alpha v - s\partial_\alpha\theta e^{-s\theta} v.$$

Using the fact that the derivatives of θ are bounded functions, we see that $\partial_\alpha u$ belongs to L^q as well.

(ii) In particular $\partial_\alpha u_i \in L^{p'}$, where $\frac{1}{p'} + \frac{1}{p} = 1$. And since $\partial_\beta u_j \in L^p$, we have from Hölder inequality $\partial_\alpha u_i \partial_\beta u_j \in L^1$. Next, $\partial_k(\partial_\alpha u_i \partial_\beta u_j) = \partial_{k\alpha} u_i \partial_\beta u_j + \partial_\alpha u_i \partial_{k\beta} u_j$. And for the same reason as before, we have $\partial_k(\partial_\alpha u_i \partial_\beta u_j) \in L^1$.

(iii) follows from the density of $C_0^\infty(\mathbb{R}^3)$ in $W^{1,1}(\mathbb{R}^3)$.

Proposition 4.2 *Retain assumptions **(T)** and (4.20)–(4.25). Then, $\mathrm{D}F_\varepsilon(0) \in GL(X_p, Y_p)$.*

Proof. The proof is made in two steps. First, we show that $\mathrm{D}F_\varepsilon(0) \in \Phi_0(X_p, Y_p)$, because it is a compact perturbation of an isomorphism, and then we show that $\ker \mathrm{D}F_\varepsilon(0) = \{0\}$.

First step.

$$\mathrm{D}F_\varepsilon(0)u = -(\lambda+\mu)\nabla(\operatorname{div} u) - \mu\Delta u - \rho^\infty \nabla f^\infty(0) u + \left(\rho^\infty \nabla f^\infty(0) - \rho(x)\nabla_\zeta f(x,x)\right) u.$$

Now, let

$$\begin{aligned} Lu &= -(\lambda+\mu)\nabla(\operatorname{div} u) - \mu\Delta u - \rho^\infty\nabla f^\infty(0)u \\ &= -(\lambda+\mu)\begin{pmatrix} \partial_1^2 & \partial_1\partial_2 & \partial_1\partial_3 \\ \partial_2\partial_1 & \partial_2^2 & \partial_2\partial_3 \\ \partial_3\partial_1 & \partial_3\partial_2 & \partial_3^2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} - \mu \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} - \rho^\infty \nabla f^\infty(0)\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}. \end{aligned}$$

Then, the characteristic polynomial of L is

$$S(\eta) = (\lambda+\mu)\begin{pmatrix} \eta_1^2 & \eta_1\eta_2 & \eta_1\eta_3 \\ \eta_2\eta_1 & \eta_2^2 & \eta_2\eta_3 \\ \eta_3\eta_1 & \eta_3\eta_2 & \eta_3^2 \end{pmatrix} + \mu\begin{pmatrix} |\eta|^2 & 0 & 0 \\ 0 & |\eta|^2 & 0 \\ 0 & 0 & |\eta|^2 \end{pmatrix} - \rho^\infty\nabla f^\infty(0)$$

We claim that the matrix $S(\eta)$ is positive definite. Indeed,

$$\begin{pmatrix} \zeta_1 & \zeta_2 & \zeta_3 \end{pmatrix}\begin{pmatrix} \eta_1^2 & \eta_1\eta_2 & \eta_1\eta_3 \\ \eta_2\eta_1 & \eta_2^2 & \eta_2\eta_3 \\ \eta_3\eta_1 & \eta_3\eta_2 & \eta_3^2 \end{pmatrix}\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \zeta^T(\eta\otimes\eta)\zeta = (\zeta\cdot\eta)^2,$$

and

$$\begin{pmatrix}\zeta_1 & \zeta_2 & \zeta_3\end{pmatrix}\begin{pmatrix}|\eta|^2 & 0 & 0\\ 0 & |\eta|^2 & 0\\ 0 & 0 & |\eta|^2\end{pmatrix}\begin{pmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{pmatrix} = |\eta|^2|\zeta|^2.$$

Therefore,

$$\zeta^T S(\eta)\zeta = (\lambda+\mu)(\zeta\cdot\eta)^2 + \mu|\eta|^2|\zeta|^2 - \rho^\infty\zeta^T\nabla f^\infty(0)\zeta$$

By Cauchy inequality, $|\eta|^2|\zeta|^2 \geq (\zeta\cdot\eta)^2$, and from Remark 4.7

$$-\zeta^T\nabla f^\infty(0)\zeta \geq k|\zeta|^2.$$

So

$$\begin{aligned}\zeta^T S(\eta)\zeta &\geq (\lambda+2\mu)(\zeta\cdot\eta)^2 + k|\zeta|^2\\ &\geq k|\zeta|^2,\end{aligned}$$

which proves the claim. But then $S(\eta)$ has a positive determinant for all $\eta \in \mathbb{R}^3$, and it follows from Corollary 2.5 that L is an isomorphism from X_p and Y_p.

On the other hand, the operator $u \mapsto (\rho^\infty\nabla f^\infty(0) - \rho(x)\nabla_\zeta f(x,x))\,u$ is compact (see Note E2 in the appendix). Therefore, $\mathrm{D}F_\varepsilon(0) \in \Phi_0(X_p, Y_p)$.

Second step. Let $u = (u_1,u_2,u_3) \in \ker \mathrm{D}F_\varepsilon(0)$. Since $\mathrm{D}F_\varepsilon(0)$ is Fredholm, it follows from Theorem 2.13 that u has exponential decay, that is, there is $s > 0$ and $v \in X_p$ such that $u = e^{-s\theta}v$. Therefore from Lemma 4.5

$$\int_{\mathbb{R}^3} \operatorname{div}\,((\operatorname{div} u)u)\;\mathrm{d}x = 0,$$

and since

$$\operatorname{div}\,((\operatorname{div} u)u) = \nabla(\operatorname{div} u)\cdot u + (\operatorname{div} u)^2,$$

with all terms in L^1 by the same lemma, we get

$$-\int_{\mathbb{R}^3}\nabla(\operatorname{div} u)\cdot u\,\mathrm{d}x = \int_{\mathbb{R}^3}(\operatorname{div} u)^2\,\mathrm{d}x.$$

For the same reasons, we deduce from the identity

$$\operatorname{div}\,\big((\nabla u)^T u\big) = \Delta u\cdot u + \langle\nabla u, \nabla u\rangle\,,$$

that

$$-\int_{\mathbb{R}^3}\Delta u\cdot u\,\mathrm{d}x = \int_{\mathbb{R}^3}\langle\nabla u,\nabla u\rangle\;\mathrm{d}x = \int_{\mathbb{R}^3}\|\nabla u\|^2\,\mathrm{d}x.$$

By a direct calculation,

$$\begin{aligned}\|\operatorname{rot} u\|^2 + (\operatorname{div} u)^2 = \|\nabla u\|^2 + 2\big\{&\partial_2(u_2\partial_1 u_1) - \partial_1(u_2\partial_2 u_1) + \partial_3(u_3\partial_1 u_1) - \partial_1(u_3\partial_3 u_1)\\ &+ \partial_3(u_3\partial_2 u_2) - \partial_2(u_3\partial_3 u_2)\big\}.\end{aligned}$$

It follows from Lemma 4.5, that all terms in the preceding identity belong to $L^1(\mathbb{R}^3)$ and furthermore $\int_{\mathbb{R}^3}\partial_i(u_j\partial_l u_k)\,\mathrm{d}x = 0$ for all $i,j,k,l \in \{1,2,3\}$. Therefore

$$\int_{\mathbb{R}^3}\|\operatorname{rot} u\|^2\,\mathrm{d}x + \int_{\mathbb{R}^3}(\operatorname{div} u)^2\,\mathrm{d}x = \int_{\mathbb{R}^3}\|\nabla u\|^2\,\mathrm{d}x$$

and so,

$$\begin{aligned}
-\int_{\mathbb{R}^3} \operatorname{div} \mathrm{D}\widehat{T}(I)\nabla u \cdot u \,\mathrm{d}x &= (\lambda+\mu)\int_{\mathbb{R}^3} (\operatorname{div} u)^2 \,\mathrm{d}x + \mu \int_{\mathbb{R}^3} \|\nabla u\|^2 \,\mathrm{d}x \\
&= (\lambda+\mu)\int_{\mathbb{R}^3} (\operatorname{div} u)^2 \,\mathrm{d}x + \mu \int_{\mathbb{R}^3} \|\operatorname{rot} u\|^2 + (\operatorname{div} u)^2 \,\mathrm{d}x \\
&= (\lambda+2\mu)\int_{\mathbb{R}^3} (\operatorname{div} u)^2 \,\mathrm{d}x + \mu \int_{\mathbb{R}^3} \|\operatorname{rot} u\|^2 \,\mathrm{d}x \\
&\geq \min(\mu, \lambda+2\mu) \int_{\mathbb{R}^3} \|\nabla u\|^2 \,\mathrm{d}x.
\end{aligned}$$

Now, since $\nabla_\zeta f(x,x) \in L^\infty$ and $u \in L^2$, we get from assumption (4.25)

$$-\int_{\mathbb{R}^3} \rho(x)\,(\nabla_\zeta f(x,x)u)\cdot u \,\mathrm{d}x \geq k\underline{\rho}\int_{\mathbb{R}^3} |u|^2 \,\mathrm{d}x.$$

Hence,

$$0 = \int_{\mathbb{R}^3} \mathrm{D}F_\varepsilon(0)u \cdot u \,\mathrm{d}x \geq \min(\mu,\lambda+2\mu)\int_{\mathbb{R}^3} \|\nabla u\|^2 \,\mathrm{d}x + k\underline{\rho}\int_{\mathbb{R}^3} |u|^2 \,\mathrm{d}x \geq 0.$$

Therefore, $u = 0$, and so $\ker \mathrm{D}F_\varepsilon(0) = \{0\}$. Thus, $\mathrm{D}F_\varepsilon(0) \in GL(X_p, Y_p)$. □

Proposition 4.3 (Pohozaev identities) *Retain assumptions **(T)**, (4.20)–(4.25) and **(G)**. Let $u \in X_p$ be a solution of $F_\varepsilon^\infty(u) = 0$. Then,*

(i) *the functions $\left\langle \widehat{T}^\varepsilon(I+\nabla u), \nabla u \right\rangle$, $W^\varepsilon(I+\nabla u)$, $g(u)\cdot u$, and $\mathcal{G}(u)$ belong to $L^1(\mathbb{R}^3)$.*

(ii)

$$\int_{\mathbb{R}^3} g(u)\cdot u \,\mathrm{d}x = \int_{\mathbb{R}^3} \left\langle \widehat{T}^\varepsilon(I+\nabla u), \nabla u \right\rangle \mathrm{d}x \tag{4.31}$$

$$= 3\int_{\mathbb{R}^3} W^\varepsilon(I+\nabla u)\,\mathrm{d}x - 3\int_{\mathbb{R}^3} \mathcal{G}(u)\,\mathrm{d}x. \tag{4.32}$$

Proof. (i) Observe first, that, under our assumptions, the operator F_ε^∞ is of class C^1 between X_p and Y_p, and its derivative at zero is the operator

$$\mathrm{D}F_\varepsilon^\infty(0) = Lu = -(\lambda+\mu)\nabla(\operatorname{div} u) - \mu\Delta u - \rho^\infty \nabla f^\infty(0)u$$

But we showed in Proposition 4.2, that L an isomorphism from X_p to Y_p. Since the coefficients of F_ε^∞ have all the properties required in chapter 2, we deduce from Theorem 2.3 that F_ε^∞ is Fredholm of index zero. Because F_ε^∞ is Fredholm, it follows from Proposition 2.3, that u has exponential decay.

Consider now the term $W^\varepsilon(I+\nabla u)$. Let ξ vary in a ball B centered at 0 in M. From the mean value theorem we have

$$|W^\varepsilon(I+\xi)| = |W^\varepsilon(I+\xi) - W^\varepsilon(I)| \leq \max_{\xi\in B} \|\mathrm{D}W^\varepsilon(I+\xi)\|\|\xi\|$$

for all $\xi \in B$. In particular, since $\nabla u(x)$ belong to the ball $B = B(0, \|u\|_{1,\infty})$, for all $x \in \mathbb{R}^3$, we have

$$|W^\varepsilon(I+\nabla u(x))| \leq \max_{\xi\in B} \|\mathrm{D}W^\varepsilon(I+\xi)\|\|\nabla u(x)\|.$$

Since ∇u decays exponentially we get

$$|W^\varepsilon(I + \nabla u(x))| \le \text{const.} \times e^{-s\theta(x)}, \tag{4.33}$$

for all $x \in \mathbb{R}$. In particular the function $x \mapsto W^\varepsilon(I + \nabla u(x))$ belongs to $L^1(\mathbb{R}^3)$.

Remark. The number s and the constant appearing in the last inequality depend on u, but note that for the purpose of this proposition, we do not need any kind of uniform decay as in Theorem 2.14 or Proposition 2.2.

For the same reasons, i.e. the mean value theorem, the exponential decay of ∇u, and the condition $\mathcal{G}(0) = 0$, the function $\mathcal{G}(u)$ has exponential decay, and so belongs also to $L^1(\mathbb{R}^3)$.

For the function $g(u) \cdot u$, note that since $|u(x)| \le \|u\|_{1,\infty}$ for all $x \in \mathbb{R}^3$, it follows from the continuity of g, that $x \mapsto g(u(x))$ belongs to $L^\infty(\mathbb{R}^3, \mathbb{R}^3)$. Since u has exponential decay, $g(u) \cdot u$ has also an exponential decay, and therefore belongs to L^1. Similarly, the function $\left\langle \widehat{T}^\varepsilon(I + \nabla u), \nabla u \right\rangle$ has exponential decay.

(ii) The idea to prove these identities, is to multiply the equation $F_\varepsilon^\infty(u) = 0$ i.e.

$$\operatorname{div} \widehat{T}^\varepsilon(I + \nabla u) + g(u) = 0,$$

successively by u and $(x \cdot \nabla)u = (\nabla u)x$, and integrate by parts.

As before, it follows from the mean value theorem, and the exponential decay of u and ∇u that $\left(\widehat{T}^\varepsilon(I + \nabla u)\right)^T u \in W^{1,1}(\mathbb{R}^3, \mathbb{R}^3)$. Therefore,

$$\int_{\mathbb{R}^3} \operatorname{div} \left[\left(\widehat{T}^\varepsilon(I + \nabla u)\right)^T u \right] \mathrm{d}x = 0,$$

by Lemma 4.5 (iii). But

$$\operatorname{div} \left[\left(\widehat{T}^\varepsilon(I + \nabla u)\right)^T u \right] = \left[\operatorname{div} \widehat{T}^\varepsilon(I + \nabla u)\right] \cdot u + \left\langle \widehat{T}^\varepsilon(I + \nabla u), \nabla u \right\rangle,$$

with all terms in $L^1(\mathbb{R}^3)$. Hence,

$$-\int_{\mathbb{R}^3} \left[\operatorname{div} \widehat{T}^\varepsilon(I + \nabla u)\right] \cdot u \, \mathrm{d}x = \int_{\mathbb{R}^3} \left\langle \widehat{T}^\varepsilon(I + \nabla u), \nabla u \right\rangle \mathrm{d}x,$$

and equation (4.31) follows. Now,

$$\begin{aligned} g(u) \cdot (\nabla u)x &= \sum_{i,j=1}^{3} g_i(u)(\partial_j u_i)x_j \\ &= \sum_{j=1}^{3} \sum_{i=1}^{3} \partial_i \mathcal{G}(u)(\partial_j u_i)x_j \\ &= \sum_{j=1}^{3} x_j \frac{\partial}{\partial x_j} \mathcal{G}(u). \end{aligned}$$

Note that for any polynomial map $P : \mathbb{R}^3 \to \mathbb{R}$, the function $P(x)e^{-s\theta(x)}$, decays exponentially at infinity. Consequently, all the terms above are in L^1, and the function $x_j \mathcal{G}(u)$ belongs to $W^{1,1}(\mathbb{R}^3)$. Therefore,

$$0 = \int_{\mathbb{R}^3} \partial_j \left[x_j \mathcal{G}(u)\right] \mathrm{d}x = \int_{\mathbb{R}^3} \mathcal{G}(u) \, \mathrm{d}x + \int_{\mathbb{R}^3} x_j \frac{\partial}{\partial x_j} \mathcal{G}(u) \, \mathrm{d}x,$$

and so

$$\sum_{j=1}^{3}\int_{\mathbb{R}^3} x_j \frac{\partial}{\partial x_j}\mathcal{G}(u)\,\mathrm{d}x = -3\int_{\mathbb{R}^3}\mathcal{G}(u)\,\mathrm{d}x.$$

Similarly,

$$\begin{aligned}
-\int_{\mathbb{R}^3} \operatorname{div}\widehat{T}^{\varepsilon}(I+\nabla u)\cdot(\nabla u)x\,\mathrm{d}x &= -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\frac{\partial}{\partial x_j}\widehat{T}^{\varepsilon}_{ij}(I+\nabla u)x_k\partial_k u_i\,\mathrm{d}x\\
&= \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\widehat{T}^{\varepsilon}_{ij}(I+\nabla u)\partial_j(x_k\partial_k u_i)\,\mathrm{d}x\\
&= \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\widehat{T}^{\varepsilon}_{ij}(I+\nabla u)(\delta_{kj}\partial_k u_i + x_k\partial_{jk}u_i)\,\mathrm{d}x\\
&= \int_{\mathbb{R}^3}\sum_{i,j=1}^{3}\widehat{T}^{\varepsilon}_{ij}(I+\nabla u)\partial_j u_i\,\mathrm{d}x + \sum_{k=1}^{3}\int_{\mathbb{R}^3} x_k\partial_k W^{\varepsilon}(I+\nabla u)\,\mathrm{d}x\\
&= \int_{\mathbb{R}^3}\langle\widehat{T}^{\varepsilon}(I+\nabla u),\nabla u\rangle\,\mathrm{d}x - 3\int_{\mathbb{R}^3}W^{\varepsilon}(I+\nabla u)\,\mathrm{d}x.
\end{aligned}$$

Therefore,

$$-3\int_{\mathbb{R}^3}\mathcal{G}(u)\,\mathrm{d}x = \int_{\mathbb{R}^3}\langle\widehat{T}^{\varepsilon}(I+\nabla u),\nabla u\rangle\,\mathrm{d}x - 3\int_{\mathbb{R}^3}W^{\varepsilon}(I+\nabla u)\,\mathrm{d}x,$$

and equation (4.32) follows. □

Theorem 4.1 *Retain assumptions **(T)**, (4.20)–(4.25), **(G)**, and (4.27). Then $F_\varepsilon : X_p \to Y_p$ is Fredholm of index zero, and proper on the closed bounded of X_p.*

Proof. The only point that remains to prove is the nonexistence of nontrivial solutions of $F_\varepsilon^\infty(u) = 0$ (Theorem 2.8). So let $u \in X_p$ satisfy $F_\varepsilon^\infty(u) = 0$. Then, from Proposition 4.3

$$3\int_{\mathbb{R}^3}W^{\varepsilon}(I+\nabla u)\,\mathrm{d}x = 3\int_{\mathbb{R}^3}\mathcal{G}(u)\,\mathrm{d}x + \int_{\mathbb{R}^3}g(u)\cdot u\,\mathrm{d}x.$$

From assumption (4.27), we have $g(\zeta)\cdot\zeta \le -k\rho^\infty|\zeta|^2$. As a consequence,

$$\begin{aligned}
\mathcal{G}(\zeta) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(t\zeta)\,\mathrm{d}t = \int_0^1 \nabla\mathcal{G}(t\zeta)\cdot\zeta\,\mathrm{d}t\\
&= \int_0^1 g(t\zeta)\cdot\zeta\,\mathrm{d}t\\
&\le \int_0^1 -k\rho^\infty t|\zeta|^2\,\mathrm{d}t\\
&= -\frac{1}{2}k\rho^\infty|\zeta|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
3\int_{\mathbb{R}^3}W^{\varepsilon}(I+\nabla u)\,\mathrm{d}x &\le -\frac{3}{2}k\rho^\infty\int_{\mathbb{R}^3}|u|^2\,\mathrm{d}x - k\rho^\infty\int_{\mathbb{R}^3}|u|^2\,\mathrm{d}x\\
&= -\frac{5}{2}k\rho^\infty\int_{\mathbb{R}^3}|u|^2\,\mathrm{d}x \le 0 \qquad (4.34)
\end{aligned}$$

so $\int_{\mathbb{R}^3} W^\varepsilon(I+\nabla u)\,\mathrm{d}x \le 0$.

We show that the reverse inequality also holds. It follows from the quasiconvexity of W^ε that for any $v \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$ such that $\operatorname{supp} v \subset B_R$

$$\int_{\mathbb{R}^3} W^\varepsilon(I+\nabla v)\,\mathrm{d}x = \int_{B_R} W^\varepsilon(I+\nabla v)\,\mathrm{d}x \ge (\operatorname{meas} B_R) W^\varepsilon(I) = 0.$$

Let $\psi : \mathbb{R} \to \mathbb{R}$ be C^∞ function satisfying $\psi(t) = 1$ for $t < 0$, and $\psi(t) = 0$ for $t > 1$. Define for $r > 0$, $\phi_r : \mathbb{R}^3 \to \mathbb{R}$ by

$$\phi_r(x) = \begin{cases} 1 & \text{if } |x| < r \\ \psi(|x|-r) & \text{if } |x| \ge r. \end{cases}$$

Then, indeed, ϕ_r is a C^∞ function with compact support, and furthermore, its derivatives are bounded by a constant independent of r. As for (4.33), there is a constant $C > 0$ independent of r, such that

$$\left|W^\varepsilon\big(I+\nabla(\phi_r u(x))\big)\right| \le Ce^{-s\theta(x)},$$

for all $x \in \mathbb{R}^3$ and $r > 0$. Consequently, $\displaystyle\int_{|x|>r} |W^\varepsilon(I+\nabla(\phi_r u))|\,\mathrm{d}x$ tends to zero as $r \to \infty$. Given $\delta > 0$, choose r large enough such that both

$$\int_{|x|>r} |W^\varepsilon(I+\nabla u)|\,\mathrm{d}x \quad \text{and} \quad \int_{|x|>r} |W^\varepsilon(I+\nabla(\phi_r u))|\,\mathrm{d}x$$

are less than δ. Then

$$\begin{aligned} \int_{\mathbb{R}^3} W^\varepsilon(I+\nabla u)\,\mathrm{d}x &= \int_{\mathbb{R}^3} W^\varepsilon(I+\nabla u) - W^\varepsilon(I+\nabla(\phi_r u))\,\mathrm{d}x + \int_{\mathbb{R}^3} W^\varepsilon(I+\nabla(\phi_r u))\,\mathrm{d}x \\ &= \int\limits_{|x|>r} W^\varepsilon(I+\nabla u)\,\mathrm{d}x - \int\limits_{|x|>r} W^\varepsilon(I+\nabla(\phi_r u))\,\mathrm{d}x + \int\limits_{\mathbb{R}^3} W^\varepsilon(I+\nabla(\phi_r u))\,\mathrm{d}x \\ &\ge -2\delta. \end{aligned}$$

But this is true for any $\delta > 0$, and therefore, $\displaystyle\int_{\mathbb{R}^3} W^\varepsilon(I+\nabla u)\,\mathrm{d}x \ge 0$. Hence,

$$\int_{\mathbb{R}^3} W^\varepsilon(I+\nabla u)\,\mathrm{d}x = 0.$$

But then, it follows from (4.34) that

$$-\frac{5}{2}k\rho^\infty \int_{\mathbb{R}^3} |u|^2\,\mathrm{d}x = 0,$$

which implies $u = 0$. □

4.3 Global continuation

Theorem 4.2 (Global continuation) *Let X, Y be two Banach spaces, and J be an open interval of the real line containing 0. Consider a continuous operator $F : J \times X \to Y$ having the following properties.*

- *F is differentiable with respect to the second variable, $\mathrm{D}_u F(t,u) \in \Phi_0(X,Y)$, for all $(t,u) \in J \times X$, and the map $(t,u) \mapsto D_u F(t,u)$ is continuous.*

- *For any compact subset A of J and any closed bounded subset $B \subset X$, the restriction $F|_{A\times B}$ is proper.*
- $\mathrm{D}_u F(0,0) \in GL(X,Y)$ *and* $F(0,0) = 0$.

Let $\mathcal{O}$ be an open subset of X containing 0, and $\mathcal{C}$ be the maximal connected subset of $(J \times \mathcal{O}) \cap F^{-1}(0)$ which contains $(0,0)$. Let $P_{\mathbb{R}}$ and P_X denote respectively the projections of $\mathbb{R} \times X$ on $\mathbb{R}$ and X. Then, at least one of the following holds.

(i) *$\mathcal{C}$ is unbounded.*

(ii) *There exists $\bar{u} \in X, \bar{u} \neq 0$ such that $(0,\bar{u}) \in \mathcal{C}$.*

(iii) *The closure in $\mathbb{R}$ of $P_{\mathbb{R}}(\mathcal{C})$ intersects ∂J.*

(iv) *$P_X(\overline{\mathcal{C}})$ intersects $\partial\mathcal{O}$, where $\overline{\mathcal{C}}$ is the closure of $\mathcal{C}$ in $\mathbb{R} \times X$.*

Remark 4.10 If alternative (iii) holds, then at a point $t \in \partial J$, the map $F(t,.)$ may no longer be defined, or it may loose its properness or Fredholm properties. The meaning of alternative (iv) will be clear in Theorem 4.3 (see Remark 4.13).

Remark 4.11 If $\partial(J \times \mathcal{O}) = \varnothing$ (this happens if and only if $J = \mathbb{R}$ and $\mathcal{O} = X$), then neither of alternative (iii) or (iv) can hold. Therefore, we prove the theorem under the assumption that $\partial(J \times \mathcal{O}) \neq \varnothing$. Note that $\partial(J \times \mathcal{O}) = \big(\partial J \times \overline{\mathcal{O}}\big) \cup \big(\overline{J} \times \partial\mathcal{O}\big)$.

Proof. Suppose that none of the alternatives hold. Then,

1. Claim. $\mathcal{C}$ is compact. Indeed, $\overline{P_{\mathbb{R}}(\mathcal{C})}$ is contained in $\overline{J}$ but does not meet ∂J since (iii) does not hold. Hence it is contained in J. Similarly, $P_X(\overline{\mathcal{C}}) \subset \mathcal{O}$, because (iv) does not hold. Now observe that $\overline{\mathcal{C}} \subset P_{\mathbb{R}}(\overline{\mathcal{C}}) \times P_X(\overline{\mathcal{C}}) \subset \overline{P_{\mathbb{R}}(\mathcal{C})} \times P_X(\overline{\mathcal{C}}) \subset J \times \mathcal{O}$. But $\mathcal{C}$, being a connected component of $J \times \mathcal{O}$, is closed in $J \times \mathcal{O}$. Therefore, $\mathcal{C} = \overline{\mathcal{C}}$, that is, $\mathcal{C}$ is closed in $\mathbb{R} \times X$.

Because alternative (i) does not hold, $\mathcal{C}$ is bounded, and accordingly, $\overline{P_{\mathbb{R}}(\mathcal{C})}$ is compact. It follows then from the properness assumption on F, that $F^{-1}(0) \cap \Big(\overline{P_{\mathbb{R}}(\mathcal{C})} \times \overline{P_{\mathbb{R}}(\mathcal{C})}\Big)$ is compact. But this set contains the closed set $\mathcal{C}$, which is, therefore, compact. Therefore, in fact, $P_{\mathbb{R}}(\mathcal{C})$ is a compact interval $[a,b] \subset J$, and we can choose two points $a', b' \in J$, such that $[a,b] \subset (a',b')$.

2. Since $\mathcal{C} \cap \partial(J \times \mathcal{O}) = \varnothing$, we have

$$\begin{aligned} d_1 &:= \mathrm{dist}\ (\mathcal{C}, \partial(J \times \mathcal{O})) \\ &= \inf_{(t,u)\in\mathcal{C}} \mathrm{dist}\ ((t,u), \partial(J \times \mathcal{O})) > 0, \end{aligned}$$

because the infinimum is attained at a point (t_0,u_0) in the compact set $\mathcal{C}$, and $(t_0,u_0) \notin \partial(J\times\mathcal{O})$.

3. The implicit function theorem ensures that there exist $\delta > 0$, a C^1- map $\widetilde{u} : (-\delta,\delta) \to X$ and a neighborhood V_δ of (0,0) such that

$$(t,u) \in V_\delta \text{ and } F(t,u) = 0 \Leftrightarrow t \in (-\delta,\delta) \text{ and } u = \widetilde{u}(t).$$

Let $r_\delta > 0$ be such that $B((0,0), r_\delta) \subset V_\delta$ and let $\mathcal{C}_1 = B((0,0), \frac{r_\delta}{2}) \cap \mathcal{C}$, so that $\mathcal{C}_1$ is open in $\mathcal{C}$ and hence $\mathcal{C}\backslash\mathcal{C}_1$ is compact.

Therefore, since (ii) does not hold,

$$d_2 := \mathrm{dist}\ (\mathcal{C}\backslash\mathcal{C}_1, \{0\} \times X) > 0,$$

because $\mathcal{C}\backslash\mathcal{C}_1$ is compact and $\{0\} \times X$ is closed.

Define

$$\begin{aligned}\Sigma &= \{(t,u) \in \mathbb{R} \times X \mid \text{dist}\,((t,u),\mathcal{C}) < \alpha\} \cap (a',b') \times \mathcal{O} \\ &= \bigcup_{(t_0,u_0)\in\mathcal{C}} B\left((t_0,u_0),\alpha\right) \cap (a',b') \times \mathcal{O},\end{aligned}$$

where $\alpha = \dfrac{1}{2}\min\left(d_1, d_2, \dfrac{r_\delta}{2}\right)$.[2]

Then, indeed, $\overline{\Sigma} \subset J \times \mathcal{O}$, and Σ is open and bounded.

Claim. $F(0,u) = 0$ and $(0,u) \in \overline{\Sigma} \Rightarrow u = 0$.

If not, there exist $u \neq 0$ such that $F(0,u) = 0$ and $(0,u) \in \overline{\Sigma}$. Since the unique solution of $F(0,u) = 0$ in V_δ is $u = 0$, then $(0,u)$ cannot lie in V_δ an so neither in $B((0,0),r_\delta)$. Therefore $\|u\| = \|(0,u)\| \geq r_\delta$. Then, (from the triangle inequality)

$$\text{dist}\,\left((0,u), B_{r_\delta/2}\right) \geq \frac{r_\delta}{2},$$

and since $\mathcal{C}_1 \subset B_{\frac{r_\delta}{2}}$, we have

$$\text{dist}\,((0,u),\mathcal{C}_1) \geq \frac{r_\delta}{2}.$$

$(0,u) \in \{0\} \times X \Rightarrow \text{dist}\,((0,u),\mathcal{C}\backslash\mathcal{C}_1) \geq d_2$. Therefore,

$$\begin{aligned}\text{dist}\,((0,u),\mathcal{C}) &= \min\{\text{dist}\,((0,u),\mathcal{C}_1)\,, \text{dist}\,((0,u),\mathcal{C}\backslash\mathcal{C}_1)\} \\ &\geq \min\left\{\frac{r_\delta}{2}, d_2\right\} \geq 2\alpha.\end{aligned}$$

But $(0,u) \in \overline{\Sigma} \subset \{(t,u) \in \mathbb{R} \times X \,|\, \text{dist}\,((t,u),\mathcal{C}) \leq \alpha\}$, so $\text{dist}\,((0,u),\mathcal{C}) \leq \alpha$, contradiction.

4. Let

$$K = \overline{\Sigma} \cap F^{-1}(0),$$

which is compact by the properness of F. Since $\mathcal{C} \subset \Sigma$, then $\mathcal{C} \cap \partial\Sigma = \varnothing$.

If there were a closed connected subset $\widetilde{\mathcal{C}}$ of K intersecting both $\mathcal{C}$ and $\partial\Sigma$, then, $\mathcal{C} \bigcup \widetilde{\mathcal{C}}$ would be connected (as the union of two connected sets with non empty intersection), contradicting the maximality of $\mathcal{C}$.

Therefore, the celebrated separation theorem[3] implies that there exist two disjoint closed subset A and B of K such that

$$\begin{aligned}&K = A \cup B, \\ &\mathcal{C} \subset A, \\ &\partial\Sigma \cap F^{-1}(0) \subset B.\end{aligned}$$

Let

$$\Upsilon = \left\{(t,u) \in \Sigma \mid \text{dist}\,((t,u),A) < \frac{1}{2}\text{dist}\,(A,B)\right\},$$

which is open and bounded and contains A. So $\partial\Upsilon \cap A = \varnothing$ and indeed $\partial\Upsilon \cap B = \varnothing$, hence $\partial\Upsilon \cap K = \varnothing$, i.e. $\partial\Upsilon \cap \overline{\Sigma} \cap F^{-1}(0) = \varnothing$. Since $\partial\Upsilon \subset \overline{\Sigma}$, we get

$$\partial\Upsilon \cap F^{-1}(0) = \varnothing.$$

---

[2]By taking α smaller, we can ensure that $\bigcup_{(t_0,u_0)\in\mathcal{C}} B\left((t_0,u_0),\alpha\right) \subset (a',b') \times \mathcal{O}$. But this is not essential.

[3]See Brown **[7]**- chapter 14, for a topological proof (which uses no metric) of this theorem.

5. Define, for $t \in J$,

$$\Upsilon_t = \{u \in \mathcal{O} \mid (t,u) \in \Upsilon\}.$$

Claim. $0 \notin F(t, \partial\Upsilon_t)$, for all $t \in J$.
One can show that $u \in \partial\Upsilon_t \Rightarrow (t,u) \in \partial\Upsilon$.
If $0 \in F(t,\partial\Upsilon_t)$ for some $t \in J$, then there exists $u \in \partial\Upsilon_t$ such that $F(t,u) = 0$, so $(t,u) \in \partial\Upsilon$ and $(t,u) \in F^{-1}(0)$. But $F^{-1}(0) \cap \partial\Upsilon = \varnothing$, a contradiction which proves the claim.

6. It follows from the claim in point **3.**, that $u = 0$ is the unique solution of $F(0,u) = 0$ in Υ_0. Therefore, by the definition of the base point degree at regular values

$$\deg_0(F(0,.), \Upsilon_0, 0) = 1.$$

By the generalized homotopy invariance of the absolute degree (since $0 \notin F(t,\partial\Upsilon_t)$ for all $t \in J$), we should have

$$|\deg|(F(t,.), \Upsilon_t, 0) = 1, \quad \text{for all } t \in J.$$

However, since $\Upsilon \subset \Sigma \subset (a', b') \times \mathcal{O}$, we have $\Upsilon_t = \varnothing$, for $t \in (b', \sup J)$, and so

$$|\deg|(F(t,.), \Upsilon_t, 0) = 0,$$

contradiction. □

Remark 4.12 Consider the homotopy (3.1) defined in chapter 3. It satisfies all the assumptions of the previous theorem with $J = \mathbb{R}$ and $\mathcal{O} = D_p(\Omega)$. Therefore, alternatives (iii) and (iv) cannot hold. Alternative (ii) cannot hold either because the only solution of $H(0,u) = \mathrm{D}F(0)u = 0$ is $u = 0$. Then, we deduce that there exists an unbounded branch of solutions (t,u) of $F(tu) = t^2 h$ emanating from $(0,0)$.

Consider now the operator F_ε in (4.19) under assumption ***(T)***, with the body forces

$$f(t,x,\zeta) = -k(\zeta - x) + t f_0(x),$$

where t is a real parameter, $k > 0$, and $f_0 : \mathbb{R}^3 \to \mathbb{R}^3$ is a continuous function with compact support. Then, we can define F_ε as an operator acting between $\mathbb{R} \times X_p$ and Y_p, by

$$F_\varepsilon(t,u)(x) = -\mathrm{div}\,\widehat{T^\varepsilon}(I + \nabla u(x)) + k\rho(x)u(x) - t\rho(x)f_0(x) \tag{4.35}$$

Theorem 4.3 *Let F_ε be given by (4.35) and*

$$\mathcal{O} = \left\{u \in X_p \mid \det(I + \nabla u(x)) > 0 \textit{ for all } x \in \mathbb{R}^3\right\}.$$

Let $\mathcal{C}$ be the maximal connected subset of $(\mathbb{R} \times \mathcal{O}) \cap F_\varepsilon^{-1}(0)$ which contains $(0,0)$. Then, either

(i) *$\mathcal{C}$ is unbounded, or*

(ii) *$P_{X_p}(\overline{\mathcal{C}})$ intersects $\partial\mathcal{O}$.*

Remark 4.13 In alternative (ii), the projection of $\overline{\mathcal{C}}$ on X_p meets the boundary of $\mathcal{O}$ at a point u which consequently satisfies $\det(I + \nabla u(x)) = 0$ for some $x \in \mathbb{R}^3$. Then, the deformation $\varphi(x) = x + u(x)$ is not orientation preserving. In this case, the global branch attains the limits of elasticity.

Proof. Note first, that, for each $t \in \mathbb{R}$ the partial map $F_\varepsilon(t,.)$ satisfies all the assumptions of the previous section, and accordingly it is proper on the closed bounded subsets of X_p, and Fredholm of index zero. Furthermore, it is clear that $(F_\varepsilon(.,u))_{u\in X_p}$ is equicontinuous. Thus, by Lemma 1.1, F_ε is proper on the closed bounded subsets of $\mathbb{R} \times X_p$. Since indeed $F_\varepsilon \in C^1(\mathbb{R} \times X_p, Y_p)$, and $D_u F_\varepsilon(t,0)$ is an isomorphism (independent of t), we see that F_ε satisfies the required conditions of Theorem 4.2. Accordingly, we have 4 alternatives, but alternative (iii) of that theorem cannot happen since $J = \mathbb{R}$. Alternative (ii) cannot occur either because $F_\varepsilon(0,u) = F_\varepsilon^\infty(0,u)$, and we showed in Theorem 4.1 that the only solution of $F_\varepsilon^\infty(0,u) = 0$ is $u = 0$. □

Now we go back to the original problem (without restriction and extension). It turns out that if we define the operator $F(t,u)$, by

$$F(t,u)(x) = -\mathrm{div}\,\widehat{T}(I + \nabla u(x)) + k\rho(x)u(x) - t\rho(x)f_0(x),$$

then F maps $\mathbb{R} \times \mathcal{O}$ into $L^p(\mathbb{R}^3, \mathbb{R}^3)$.

Indeed, for every $u \in \mathcal{O}$, there exists $\varepsilon \in (0,1)$ such that $u \in \mathcal{O}^\varepsilon$ (see Lemma 4.4 (i)). But then, it follows from the definition of F_ε, that $F(t,u) = F_\varepsilon(t,u) \in Y_p$.

We formulate a global continuation result for this original operator.

Theorem 4.4 *Let $\mathcal{C}$ be the connected component of $F^{-1}(0) \cap (\mathbb{R} \times \mathcal{O})$ which contains $(0,0)$. Then either $\mathcal{C}$ is unbounded, or the projection onto X_p of $\mathcal{C}$ meets $\partial\mathcal{O}^\varepsilon$ for every $\varepsilon \in (0,1)$.*

The second alternative is equivalent to[4] $\mathcal{C} \not\subseteq \mathbb{R} \times \mathcal{O}^\varepsilon\ \forall\, \varepsilon \in (0,1)$ indicating that the branch gets closer and closer to a break down of orientation preserving.

Proof. Clearly it is equivalent to show that there is a connected set satisfying one of the two alternatives.

Now Theorem 4.3 can be restated as follows. Let $\mathcal{C}_\varepsilon$ be the connected component of $F_\varepsilon^{-1}(0) \cap (\mathbb{R} \times \mathcal{O}_\varepsilon)$ which contains $(0,0)$, then for every $\varepsilon \in (0,1)$, either $\mathcal{C}_\varepsilon$ is unbounded or $P_{X_p}(\overline{\mathcal{C}_\varepsilon}) \cap \partial\mathcal{O}_\varepsilon \neq \varnothing$.

Let $\mathcal{C} = \cup_{0<\varepsilon<1}\overline{\mathcal{C}_\varepsilon}$. Then $\mathcal{C}$ is connected as the union of connected sets having a common point. Note that $\mathcal{C} \subset \mathbb{R} \times \cup_{0<\varepsilon<1}\overline{\mathcal{O}^\varepsilon} = \mathbb{R} \times \mathcal{O}$, and that $\mathcal{C} \subset F^{-1}(0)$ because if $(t,u) \in \mathcal{C}$, then there is $\varepsilon \in (0,1)$ s.t. $(t,u) \in \mathcal{C}_\varepsilon \subset \mathbb{R} \times \mathcal{O}^\varepsilon$. But then $F(t,u) = F_\varepsilon(t,u) = 0$.

Now we have two cases. Either (i) there is $\varepsilon \in (0,1)$ s.t. $\mathcal{C}_\varepsilon$ is unbounded, and then $\mathcal{C}$ is unbounded. Or (ii) for every $\varepsilon \in (0,1)$, $\mathcal{C}_\varepsilon$ is bounded, and then by Theorem 4.3 $P_{X_p}(\overline{\mathcal{C}_\varepsilon}) \cap \partial\mathcal{O}^\varepsilon \neq \varnothing$. But since $P_{X_p}(\overline{\mathcal{C}_\varepsilon}) \subset P_{X_p}(\mathcal{C})$, we get $P_{X_p}(\mathcal{C}) \cap \partial\mathcal{O}^\varepsilon \neq \varnothing$ for every $\varepsilon \in (0,1)$. □

Second proof. The key relation between the operator $F : \mathbb{R} \times \mathcal{O} \to Y_p$ and the family of operators $F_\varepsilon : \mathbb{R} \times X_p \to Y_p$ is that

$$F(t,u) = F_\varepsilon(t,u) \quad \text{whenever } u \in \mathcal{O}^\varepsilon,$$

and so F coincides with F_ε on a open subset of X_p and this has the following implications.

(a) $F \in C^1(\mathbb{R} \times \mathcal{O}, Y_p)$.

(b) For every $u \in \mathcal{O}$, there is $\varepsilon \in (0,1)$ such that $u \in \mathcal{O}^\varepsilon$, and so $\mathrm{D}_u F(t,u) = \mathrm{D}F_\varepsilon(t,u)$ is Fredholm of index zero.

(c) For every $\varepsilon \in (0,1)$ (and every $t \in \mathbb{R}$), the restriction of $F(t,.)$ to any closed bounded subset $A \subset \mathcal{O}^\varepsilon$ is proper.

[4] $\mathcal{C} \not\subseteq \mathbb{R} \times \mathcal{O}^\varepsilon$ means that $\mathcal{C}$ intersects the complement of $\mathbb{R} \times \mathcal{O}^\varepsilon$. Since the connected set $\mathcal{C}$ intersects $\mathbb{R} \times \mathcal{O}^\varepsilon$ at (0,0), it meets the boundary of $\mathbb{R} \times \mathcal{O}^\varepsilon$.

We do not know if the set $\mathcal{O}$ is connected or simply connected, but in the proof of the global continuation theorem, all we used is the absolute degree, and this does not requires any condition of orientability. Accordingly, the theorem applies to the operator F and so we recover the alternatives. $\square$

Generalization. We can treat more general body forces f that depend on a real parameter t which lies in an interval J. But the list of technical assumptions increases.

Consider a continuous map $f : J \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ with the following properties. For each $t \in J$, $f(t,.) : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is an equicontinuous C^1_ζ−bundle map, and $\partial_t \nabla_\zeta f$ is an equicontinuous C^0−bundle maps. Furthermore,

$$f(0,x,x) = 0, \text{ for all } x \in \mathbb{R}^3.$$

$$\text{For each } t \in J\text{, the function } x \mapsto \nabla_\zeta f(t,x,x) \text{ belongs to } L^\infty(\mathbb{R}^3, M), \text{ and}$$
$$\text{the function } x \mapsto f(t,x,x) \text{ belongs to } L^p(\mathbb{R}^3,\mathbb{R}^3). \tag{4.36}$$
$$\text{For each } t \in J\text{, the function } x \mapsto \nabla_\zeta \partial_t f(t,x,x) \text{ belongs to } L^\infty(\mathbb{R}^3, M). \tag{4.37}$$
$$\text{For every } t \in J,\ \int_{\mathbb{R}^3} |f(t+h,x,x) - f(t,x,x)|^p \, \mathrm{d}x \to 0 \text{ as } h \to 0. \tag{4.38}$$
$$\text{For every } t \in J\text{, there exists a function } f_t^\infty \in C^1(\mathbb{R}^3,\mathbb{R}^3) \text{ with } f_t^\infty(0) = 0 \text{ for all } t \in J$$
$$\text{and such that } \lim_{|x|\to\infty} \|\nabla_\zeta f(t,x,x+\zeta) - \nabla_\zeta f_t^\infty(\zeta)\| = 0, \tag{4.39}$$
$$\text{uniformly for } \zeta \text{ in compact subsets of } \mathbb{R}^3.$$
$$\text{For each } t \in J\text{, there is } k_t > 0 \text{ such that } (\nabla_\zeta f(t,x,x)\eta)\cdot\eta \le -k_t|\eta|^2,\ \forall\, x,\eta \in \mathbb{R}^3. \tag{4.40}$$

Let

$$g_t(\zeta) = \rho^\infty f_t^\infty(\zeta). \tag{4.41}$$

We assume that for each $t \in J$, there exists a function $\mathcal{G}_t \in C^1(\mathbb{R}^3)$ such that $\nabla \mathcal{G}_t(\zeta) = g_t(\zeta)$ for all $\zeta \in \mathbb{R}^3$. And the last hypothesis is

$$f_t^\infty(\zeta)\cdot\zeta \le -k_t|\zeta|^2. \tag{4.42}$$

Under the above conditions, we consider the elasticity operator which depends now on a parameter $t \in J$,

$$F_\varepsilon(t,u)(x) = -\mathrm{div}\, \widehat{T^\varepsilon}(I + \nabla u(x)) - \rho(x) f(t,x,x+u(x)), \tag{4.43}$$

and satisfies $F_\varepsilon(0,0) = 0$.

Then our assumptions ensure that for each $t \in J$, the map $F_\varepsilon(t,.)$ is proper on the closed bounded subset of X_p, and Fredholm of index zero, and $\mathrm{D}_u F(t,0)$ is an isomorphism for all $t \in J$. Also, one check that $(t,u) \mapsto D_u F(t,u)$ is continuous. To prove that $F_\varepsilon|_{A\times B}$ is proper whenever A is compact in J and $B \subset X_p$ is closed and bounded, it is enough to show by Lemma 1.1 that the collection $(F_\varepsilon(.,u))_{u\in B}$ is equicontinuous. This will follow from

Lemma 4.6 *Let $\boldsymbol{f}(t,u)(x) = f(t,x,x+u(x))$, be the Nemytskii operator associated with f. Then, for every bounded subset $B \subset X_p$, the collection $(\boldsymbol{f}(.,u))_{u\in B}$ is equicontinuous.*

Proof.

$$
\begin{aligned}
\boldsymbol{f}(t+h,u)(x)-\boldsymbol{f}(t,u)(x) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau} f(t+\tau h, x, x+u(x))\,\mathrm{d}\tau \\
&= h\int_0^1 \partial_t f(t+\tau h, x, x+u(x))\,\mathrm{d}\tau \\
&= h\int_0^1 \left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\partial_t f\,(t+\tau h, x, x+su(x))\ \mathrm{d}s + \partial_t f(t+\tau h, x, x)\right)\,\mathrm{d}\tau \\
&= h\left(\int_0^1\int_0^1 \nabla_\varsigma \partial_t f\,(t+\tau h, x, x+su(x))\ \mathrm{d}s\,\mathrm{d}\tau\right) u(x) \\
&\quad + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau} f(t+\tau h, x, x)\,\mathrm{d}\tau. \qquad (4.44)
\end{aligned}
$$

The assumptions made on f ensure (as in Lemma 2.1) that there is $M>0$ such that

$$\|\nabla_\varsigma \partial_t f\,(t+\tau h, x, x+su(x))\,\| \le M$$

for all $s,\tau\in[0,1],\ x\in\mathbb{R}^3$, and $u\in B$. The last term in (4.44) is $f(t+h,x,x)-f(t,x,x)$, and it norm in L^p tends to zero as $h\to 0$ by assumption (4.38). Therefore, given any $\delta>0$, we have

$$\|\boldsymbol{f}(t+h,u)-\boldsymbol{f}(t,u)\|_{0,p} \le |h|M\|u\|_{0,p}+\delta,$$

for $|h|$ small enough. $\square$

Appendix A

Some results about sequences and function spaces

In what follows X, Y and Z are real Banach spaces.

Note A1. Let X be reflexive, $u \in X$ and (u_n) be a bounded sequence from X. Suppose that every weakly convergent subsequence of (u_n) converges weakly to u (i.e. the limit is independent of the subsequence, or (u_n) has a unique weak cluster point). Then $u_n \rightharpoonup u$.

Proof. If not, there exist $f \in X'$, $\varepsilon_0 > 0$, and a subsequence $(u_{\varphi(n)})$ such that

$$|\langle f, u_{\varphi(n)} \rangle - \langle f, u \rangle| \geq \varepsilon_0 \quad \text{for all } n \in \mathbb{N}.$$

But $(u_{\varphi(n)})$ is bounded, therefore it contains a subsequence $(u_{\varphi(\psi(n))})$ converging weakly to some l. By assumption, $l = u$, so that $u_{\varphi(\psi(n))} \rightharpoonup u$. But this contradicts the above inequality. $\square$

Note A2. Let X be reflexive, $X \hookrightarrow Y$, $u \in X$ and (u_n) be a bounded sequence in X such that $u_n \to u$ in Y. Then $u_n \rightharpoonup u$ in X.

Proof. If not, there exist $f \in X'$, $\varepsilon_0 > 0$ and a subsequence $(u_{\varphi(n)})$ such that

$$\left|\langle f, u_{\varphi(n)} \rangle - \langle f, u \rangle\right| \geq \varepsilon_0 \quad \text{for all } n \in \mathbb{N}.$$

But $(u_{\varphi(n)})$ is bounded so it contains a subsequence $(u_{\varphi(\psi(n))})$ converging weakly to some v in X (and hence in Y). By the uniqueness of the weak limit in Y, we have $v = u$. Therefore $u_{\varphi(\psi(n))} \rightharpoonup u$ in X. But this contradicts the definition of $(u_{\varphi(n)})$. $\square$

Application. $X = W^{2,p}(\Omega, \mathbb{R}^m)$, $Y = C_d^1(\overline{\Omega}, \mathbb{R}^m)$, with $N < p < \infty$, and $\partial\Omega$ bounded and Lipschitz.

Note A3. Let $L : X \to Z$ have the following property: if (u_n) is bounded in X, $u_n \to 0$ in Y and $L(u_n)$ converges in Z then (u_n) contains a subsequence converging to zero in X. Then for (u_n) as above, we have in fact $u_n \to 0$ in X.

Proof. If not, there is a subsequence whose norm is bounded away from zero. But this subsequence has all the properties of (u_n), so by hypotheses, it contains a subsequence converging to zero in X, which contradicts its definition. $\square$

Note A4. Let E be a subspace of X. Consider a sequence (u_n) and an element u from E. Then $u_n \rightharpoonup u$ in $E \Leftrightarrow u_n \rightharpoonup u$ in X.

Proof. Since $X' \subset E'$, if $u_n \rightharpoonup u$ in E, then $u_n \rightharpoonup u$ in X. Conversely let $u_n \rightharpoonup u$ in X, and $f \in E'$. By Hahn-Banach theorem, we can extend f to an element $\tilde{f}$ of X'. Then $\langle f, u_n - u\rangle = \langle \tilde{f}, u_n - u\rangle \to 0$. Thus $u_n \rightharpoonup u$ in E. $\square$

Note B1. Let $A \subset \mathbb{R}^N$ be a measurable set, and $1 \le q \le \infty$. Let $u_n \to u$ in $L^\infty(A)$ and $v_n \to v$ in $L^q(A)$. Then $u_n v_n \to uv$ in $L^q(A)$.

Proof. This is because $\|u_n v_n - uv\|_q \le \|u_n - u\|_\infty \|v_n\|_q + \|u\|_\infty \|v_n - v\|_q$. $\square$

Note B2. Let $u_n \to u$ in $L^\infty(A)$ and $v_n \rightharpoonup v$ in $L^q(A)$, for $1 \le q < \infty$. Then $u_n v_n \rightharpoonup uv$ in $L^q(A)$.

Proof. Let $f \in L^{q'}(A)$, where $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. Then

$$\int_A f(u_n v_n - uv) = \int_A f(u_n v_n - uv_n) + \int_A fu(v_n - v).$$

The result follows from the facts that $fu \in L^{q'}(A)$ and

$$\left| \int_A f(u_n v_n - uv_n) \right| \le \|f\|_{q'} \|v_n\|_q \|u_n - u\|_\infty. \qquad \square$$

Note B3. Let $f \in L^1(A)$. Then the functional μ defined on the measurable subsets of A by $\mu(G) = \int_G |f|$, is a measure on A, and as any measure it satisfies $\lim\limits_{n\to\infty} \mu(G_n) = \mu(\bigcap G_n)$ for every decreasing family of subsets (G_n) from A. This is why

$$\lim_{n\to\infty} \int_{|x|>n} |f| = \int_{\bigcap \tilde{B}_n} |f| = 0,$$

since $\bigcap \tilde{B}_n = \varnothing$.

C. Let $G \subset \mathbb{R}^N$ be an open set, $k \in \mathbb{N}$, $1 \le q < \infty$, and $(u_n) \subset W^{k,q}(G)$ such that $u_n \rightharpoonup u$. Then for every multi-index γ with $|\gamma| \le k$, $D^\gamma : W^{k,q} \to W^{k-|\gamma|,q}$ is linear and bounded and hence weakly continuous, therefore $\mathrm{D}^\gamma u_n \rightharpoonup \mathrm{D}^\gamma u$ in $W^{k-|\gamma|,q}(G) \subset L^q(G)$.

The converse results from a theorem on the representation of elements of $\left(W^{k,q}\right)'$, see for instance Adams [**1**] Theorem 3.8, which states that if $f \in (W^{k,q}(G))'$ then there is a family $(v_\gamma) \subset L^{q'}(G)$ $(|\gamma| \le k)$ such that

$$\langle f, u\rangle = \sum_{|\gamma|\le k} \int_G (D^\gamma u)\, v_\gamma.$$

Note D. Let $\Omega \subset \mathbb{R}^N$ be an open set, with $\partial\Omega$ bounded and Lipschitz and $N < p < \infty$. Then the functions of $W^{1,p}(\Omega)$ are bounded and Hölder continuous on $\overline{\Omega}$ (Adams [**1**], Theorem 5.4). Now let $u \in W^{1,p}(\Omega)$. Since $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\Omega)$ ([**1**], Theorem 3.18), there is a sequence (u_n) in $C_0^\infty(\mathbb{R}^N)$ converging to u in $W^{1,p}(\Omega)$ and hence in $L^\infty(\Omega)$. Accordingly, for any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\|u - u_{n_0}\|_{0,\infty,\Omega} \le \varepsilon$, and therefore $\forall\, x \in \Omega$, $|u(x)| \le |u_{n_0}(x)| + \varepsilon$. But for $|x|$ large enough $u_{n_0}(x) = 0$ so $|u(x)| \le \varepsilon$ which means that $\lim\limits_{x\in\Omega, |x|\to\infty} |u(x)| = 0$.

From the preceding we deduce that $W^{2,p}(\Omega) \hookrightarrow C^1_d(\overline{\Omega})$. The imbeddings are compact if in addition Ω is bounded.

Note E1. Let $N < p < \infty$, and $u \in L^p(\Omega)$. Define the operator T by $Tv := uv$. Then $T : W^{1,p}(\Omega) \to L^p(\Omega)$ is compact.

Proof. $Tv \in L^p(\Omega)$ because $v \in W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. Define ϕ_n for $n \in \mathbb{N}$ by $\phi_n(x) = 1$ if $|x| \leq n$ and zero elsewhere. Let $T_n v = \phi_n uv$, and $\Omega_n = \{x \in \Omega : |x| < n\}$. Since $W^{1,p}(\Omega_n) \underset{\text{comp}}{\hookrightarrow} L^\infty(\Omega_n)$, $T_n : W^{1,p}(\Omega) \to L^p(\Omega)$ is compact. Now

$$\|Tv - T_n v\|_p^p = \int\limits_{|x|>n} |u|^p |v|^p \leq \|v\|_\infty^p \int\limits_{|x|>n} |u|^p \leq \text{const.} \times \|v\|_{1,p}^p \int\limits_{|x|>n} |u|^p.$$

This means that T is the uniform limit of a sequence of compact operators, and so it is itself compact. $\square$

Note E2. Let A be an open unbounded subset of $\mathbb{R}^N$, and $\varphi \in L^\infty(A)$ tending to zero at infinity. Define for $u \in W^{1,q}(A)$, $Mu := \varphi u$. Then $M : W^{1,q}(A) \to L^q(A)$ is compact.

Proof. Indeed using the compactness of Sobolev imbeddings on bounded domains, one can see that M is a uniform limit of compact operators (as in Note E1). Equivalently, one can also argue like this. Let (u_n) be a sequence of $W^{1,q}(A)$ converging weakly to zero, $K > 0$ be a bound of $\left(\int_A |u_n|^q\right)$ and $\varepsilon > 0$ be given. Then there is $r > 0$ such that $|\varphi(x)|^q \leq \dfrac{\varepsilon}{2K}$ for almost every $x \in A \cap \tilde{B}_r$. By the compactness of the imbedding $W^{1,q}(A) \hookrightarrow L^q(A \cap B_r)$, we have $u_n \to 0$ in $L^q(A \cap B_r)$, and so for n large enough we have $\|\varphi\|_\infty^q \int\limits_{|x|<r} |u_n|^q \leq \dfrac{\varepsilon}{2}$. Therefore

$$\int\limits_A |\varphi u_n|^q = \int\limits_{|x|<r} |\varphi u_n|^q + \int\limits_{|x|>r} |\varphi u_n|^q \leq \varepsilon$$

for n large enough. This means that $Mu_n \to 0$ in $L^q(A)$ and so M is completely continuous and hence compact because $W^{1,q}$ is reflexive. $\square$

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Curriculum Vitae

I was born the 27th of March 1976 in Beirut, Lebanon. I studied at the Carmel Saint Joseph school until I passed the French Baccalauréat (série C), and the Lebanese Baccalauréat (Mathématiques élémentaires) in 1994. Then, I attended the Lebanese University, Faculty of Sciences, where I obtained a Master in Mathematics in 1998.

In the same year, I attended a joint program between several French-speaking universities (including the EPFL), at the end of which I obtained a 'Diplôme d'études approfondies' in mathematical modelling and scientific computations (1999). In the context of this program, I made a training course with Prof. Jean Descloux at EPFL. The following months, I worked as an assistant at the American University of Beirut, where I did some numerical simulations with Prof. M. El Ghoul.

I work in the group of Prof. C. A. Stuart since October 2000, where I have some teaching activities, in parallel to the preparation of the present dissertation.