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POWER SPECTRA OF RANDOM SPIKE FIELDS AND RELATED PROCESSES

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Abstract

This article reviews known results and contains new ones concerning the power spectra of large classes of signals and random fields driven by an underlying point process, such as spatial shot noises (with random impulse response and arbitrary basic stationary point processes described by their Bartlett spectrum), and signals or fields sampled at random times or points (where again the sampling point process is quite general). We also obtain the Bartlett spectrum for the general linear Hawkes spatial branching point process (with random fertility rate and general immigrant process described by its Bartlett spectrum). We then obtain the Bochner spectrum of general spatial linear birth and death processes. Finally we adress the issue of random sampling, and of the linear reconstruction of a signal from its random samples, reviewing and extending former results.

Keywords: Shot noise; random sampling; point process; Bochner power spectral measure; Bartlett power spectral measure; Hawkes process; random fields; branching process; linear birth and death process.

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1. Introduction

Classes of signals related to point processes

This article is concerned with the second-order properties of "signals" (stochastic processes) related to random spike fields, that is, spatial point processes. More specifically, we shall consider three types of signals:

(a) the random *spike fields* themselves;

(b) the *filtered* random spike fields;

(c) the *modulated* random spike fields.

These types of signals are depicted in Figure 1 in the unidimensional case (spike fields are then called *spike trains* or *Dirac combs*). The second category of signals is also known under the name of *shot noises*, and the third category arises in particular in *random sampling*. As for the first category, that is point processes, they form the basic element on which are constructed the other signals of interest in this research/review article.

In this article, we give the power spectrum (sometimes in a generalized sense) for the above three categories of signals. This is done in quite general cases, in particular, concerning shot noises, cluster point processes, jittered point processes and Hawkes branching point processes, we do not require the basic point process to be either a homogeneous Poisson process, a renewal process, or a Cox process.

Shot noises have received much attention in the applied literature, whether in physics or in electrical engineering. They model: thermoionic noise in conductors (Schottky [28]) (see the references in Bondesson [6]); queuing systems, for instance under the form of $M/GI/\infty$ pure delay system, traffic flows in communications systems; delayed claims in insurance risk analysis (Klüppelberg and Mikosch [16], Samorodnitsky [27]). The signals arising in neurophysiology are typically non-Poisson shot noises and the interference field in a mobile communication system is aptly modeled as a spatial shot noise (see for instance Baccelli and Blaszczyszyn [3]). Shot noises also arise naturally in wavelet signal analysis when the analyzed signal is a point process, since the wavelet coefficients are in this case samples of shot noises. Wavelet statistical analysis has been proposed to detect and compute the Hurst parameter in classical signals and the method applies equally well to random Dirac combs with long-range dependence properties (Abry and Flandrin [1], Abry and Veitch [2]). For the references on *clustered point processes* and *jittered point processes* the reader is referred to [13]. In the present article, they are together with shot noises, considered as special cases of shot noises (the impulse function being replaced by a point process measure).

A modulated Dirac comb is a Dirac comb with pulses of varying height. In random sampling, the height of a pulse is equal to the value of the signal sampled at this time. Random sampling has been extensively studied in view of spectral analysis, the object being to recover the power spectrum of the signal from the modulated sample comb, or even from the sample sequence (without timing information); a specific domain of application is laser velocimetry, where the samples are collected only at the passage of a reflecting particle through the laser beam. Early investigation on *random sampling* (Shapiro and Silverman [29]) was mostly motivated by the search for alias-free sampling schemes, that is, sampling schemes leading to a one-to-one relation between the spectrum of the sample comb to that of the sampled signal. The first detailed analyses of randomly sampled signals were based on the modeling of the sample comb using the Dirac (pseudo) process δ. Beutler and Leneman [5, 17, 4] obtained formulas for the moments of the sample comb that lead to the expression of the correlation of the sample comb as a function of the correlation of the sampled signal. Leneman and Lewis [18] investigated the reconstruction error for several interpolators of the random samples. Such results depend on the sampling scheme through statistics related to the intervals between successive points of the sampler. Modulated random spike fields are studied in Section 5.

As for the spike fields themselves, we recall the basic theory of Bartlett spectra in Section 2. The *Hawkes branching point processes* are studied in Section 4. Hawkes processes were introduced, under the name of self-exciting point processes, by Hawkes [14], and further studied in Hawkes and Oakes [15]; see also Daley and Vere-Jones [13]. Such branching point processes are of interest in epidemics, and also in seismology (see Vere-Jones and Davies [30]), where they are known as ETAS models (Ogata [23]).

The generalized linear birth and death process (not necessarily Markovian), are shot noises where the basic point process is a Hawkes process. Note that such process can be viewed as a shot noise on a Hawkes process, the results of Section 3 do not apply since the "shots" are in this case not independent of the basic point process.

Review results and novel results

The present paper can be considered partly as a review, sometimes with extensions (most of the times trivial) to the spatial case. For instance, the results on clustered point processes and jittered point processes appear in [13] (respectively Example 8.2 (d) and Exercise 8.2.6), and those on modulated point processes appear in Example 8.4 (c), and previously, in the seminal papers of Masry [20, 19]. However this article is not an exhaustive review of the subject, and concentrates on the actual computation of the spectra of *complex signals*, the motivation for this work being applications to ultrawide band communications and multipath fading channels (see the dissertation [24], as well as the articles [25, 26]. For more exhaustive reviews, the reader is referred to Brillinger ([10], [9]), and Daley ([11]), and of course to Daley and Vere-Jones ([13]).

The article presents the results on shot noises, cluster point processes and jittered point processes, in a unified manner, showing that they can be derived from a single formula (the fundamental isometry formula) which appears for the first time in the present article. The conditions of validity of this formula are most important in that they allow a precise description of the test functions used in the definition of the Bochner or Bartlett spectra of the processes considered.

Another new result concerns the Bartlett spectrum of Hawkes branching point processes with random fertility rate and general ancestor point processes. Earlier results in this direction are in Brémaud and Massoulié [8]. (Note however, that the method of [8] does not work.) See also Brémaud and Massoulié [7], where the critical case, leading to long-range dependence, was considered. Note that, although Hawkes processes are a particular case of cluster point processes, their spectrum cannot be obtained easily from the general formula of cluster point processes recalled in Section 3. Our analysis is more direct than that of [8], and it allows to obtain more insight as to the test functions intervening in the definition of the Bartlett spectrum, as we already mentioned.

Among the new results, we obtain the Cramer spectrum of the generalized spatial birth and death processes under the same general conditions (general birth process and general lifetimes) as for the Hawkes processes.

Concerning modulated point processes, our contribution does not go far beyond the original results of Masry [20, 19] (and the presentation given in [13]). There, the spectrum of the sample sequence was expressed as a function of the spectrum of the sampled signal and of the second order quantities of the point process, and then, by reformulating the alias-free concept, alias-free sampling schemes were proved to lead to a consistent spectral estimator. This work is closest to ours, and our method of proof is the same as in [13], our contribution being to give more details for the proof thereof (given in Example 8.4 (c)), details which, again, turn out to be useful in determining the class of test funtions for which the defining formula of the spectrum is true. The novel results in this article concerning modulated point processeses are: the power spectra of modulated spike fields when the sampler is possibly dependent from the signal, and, in the independent case, the expression of the error when the signal is approximated by a filtered version of the samples, that is, the reconstruction error.

2. Covariance and spectral measure

2.1. The covariance measure

Let N be a *simple and locally bounded* point process on $\mathbb{R}^m$. It is called a *second-order* point process if for all bounded Borel sets $C \subset \mathbb{R}^m$

$$E\left[N(C)^2\right] < \infty\,. \tag{1}$$

The formula

$$\nu(C) = E[N(C)] \tag{2}$$

defines a Radon (that is, locally finite) measure ν on $\mathbb{R}^m$, called the *mean measure*, or *intensity measure* of N. By Campbell's theorem, for all measurable functions $\varphi : \mathbb{R}^m \to \mathbb{R}$, that are nonnegative or such that belong to $L^1_{\mathbb{C}}(\nu)$, the sum $\sum_{t\in N} \varphi(t) = N(\varphi)$ is well-defined and

$$E[N(\varphi)] = E[\nu(\varphi)]\ . \tag{3}$$

Moreover, the measure M_2 on $\mathbb{R}^m \times \mathbb{R}^m$ defined by

$$M_2(A \times B) = E[N(A)\,N(B)]\,,$$

is a Radon measure.

Definition 1. By definition, $L^2_N(M_2)$ is the collection of measurable funtions $\varphi : \mathbb{R}^m \to \mathbb{C}$ such that

$$E\left[N(|\varphi|)^2\right] < \infty\,,$$

the latter implying $\varphi \in L^1_{\mathbb{C}}(\nu)$.

Clearly $L^2_N(M_2)$ is a vector space that contains all bounded functions with compact support, and if $\varphi, \psi \in L^2_N(M_2)$,

$$E\left[\left(\int_{\mathbb{R}^m} \varphi(t)\,N(dt)\right)\left(\int_{\mathbb{R}^m} \psi(t)\,N(dt)\right)^*\right] = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \varphi(t)\,\psi^*(s)\,M_2(dt \times ds)\ . \tag{4}$$

We now assume that N is of second order and stationary, in which case

$$\nu(C) = \lambda\,\ell^m(C)$$

(where ℓ^m is the Lebesgue measure on $\mathbb{R}^m$) for some $\lambda \in \mathbb{R}_+$, called the *intensity.* From a previous remark,

$$L^2_N(M_2) \subseteq L^1_{\mathbb{C}}(\mathbb{R}^m)\,. \tag{5}$$

By stationarity again, for all Borel sets $A,\ B \subseteq \mathbb{R}^m$, all $t \in \mathbb{R}^m$

$$M_2\left((A+t)\times(B+t)\right) = M_2\left(A\times B\right)$$

It will follow from Lemma A2.7.II, p. 409 of Daley and Vere-Jones ([13]), that for all $\varphi, \psi \in L^2_N(M_2)$,

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \varphi(t)\,\psi^*(s)\,M_2(dt\times ds) = \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m} \varphi(t)\,\psi^*(s+t)\,dt\right)\sigma(ds) \tag{6}$$

for some Radon measure σ, and we have from (4) and (6) that for $\varphi, \psi \in L^2_N(M_2)$,

$$\mathrm{cov}\left(\int_{\mathbb{R}^m}\varphi(t)\,N(dt)\,,\int_{\mathbb{R}^m}\psi(s)\,N(ds)\right) = \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\varphi(t)\,\psi^*(t+s)\,dt\right)\Gamma_N(ds)$$

where the Radon measure

$$\Gamma_N := \sigma - \lambda^2\ell^m \tag{7}$$

is called the *covariance measure* of the stationary second-order point process N.

2.2. Spectral measure

Definition 2. Let N be a simple second-order stationary point process on $\mathbb{R}^m$ with intensity λ. Let B_N be a vector space of functions, such that $B_N \subseteq L^2_N(M_2)$. A measure μ_N on $\mathbb{R}^m$ is called the Bartlett spectral measure of N on the domain B_N if for all $\varphi \in B_N$, the identity

$$\mathrm{Var}\left(\int_{\mathbb{R}^m}\varphi(t)\,N(dt)\right) = \int_{\mathbb{R}^m}\left|\widehat{\varphi}(\nu)\right|^2\mu_N(d\nu) \tag{8}$$

holds, the two terms of the equality being finite. The space B_N is also called a test function space for the point process N.

By polarization of (8), we have that for all $\varphi,\ \psi \in B_N$,

$$\mathrm{cov}\left(N(\varphi)\,,\,N(\psi)\right) = \int_{\mathbb{R}^m}\widehat{\varphi}(\nu)\widehat{\psi}^*(\nu)\mu_N(d\nu). \tag{9}$$

A suitable space B_N of test functions will be determined in each situation. Ideally we want to determine the largest possible domain B_N. We are looking for conditions like, for instance, $\varphi \in L^1_{\mathbb{C}}(\mathbb{R}^m)\cap L^2_{\mathbb{C}}(\mathbb{R}^m)$. Note that it is necesarily contained in $L^1_{\mathbb{C}}(\mathbb{R}^m)$ since, as we observed earlier, $L^2_N(M^2) \subseteq L^1_{\mathbb{C}}(\mathbb{R}^m)$. In particular the Fourier transform of any $\varphi \in B_N$ is well-defined.

The existence and uniqueness of the Bartlett spectrum is the content of Theorem 1 below that can be found in [22]. The theorem also shows that it is always possible to take for B_N the space of functions that are $O\left(1/\left|t\right|^2\right)$ as $|t| \to \infty$, together with their Fourier transform.

Theorem 1. *([22]) Let N be a stationary, second order point process, and let σ be the corresponding Radon measure as in (6). There exists a unique non-negative Radon measure $\widehat{\sigma}$ on $(\mathbb{R}^m, \mathcal{B}^m)$ such that, if f and its Fourier transform are $O\left(1/\left|t\right|^2\right)$ as $|t| \to \infty$, then*

$$\int_{\mathbb{R}^m} f(\nu)\,\widehat{\sigma}(d\nu) = \int_{\mathbb{R}^m} \widehat{f}(t)\,\sigma(dt)\ , \tag{10}$$

and, if g satisfies the same conditions as f,

$$E\left[N(f)\,N(g)\right] = \lambda \int_{\mathbb{R}^m} \widehat{f}(\nu)\,\check{\widehat{g}}(\nu)\,\widehat{\sigma}(d\nu)\ . \tag{11}$$

Theorem 8.6.III of [13] contains a general condition on the test functions, which however is not explicit in that it is stated in terms of the spectral measure (f is required to be integrable with respect to σ with a Fourier transform that is integrable with the Bartlett spectral measure). One of the purposes of the present article is to find explicit spaces of test functions, and to show how these are modified through the various transformations that we study.

Example 1. *Regular grid.* Consider the point process on $\mathbb{R}^2$ whose points form a regular (T_1, T_2)-grid on $\mathbb{R}^2$ with random origin, that is

$$N = \left\{(n_1T_1 + U_1, n_2T_1 + U_2), (n_1, n_2) \in \mathbb{Z}^2\right\}$$

where $T_1 > 0$, $T_2 > 0$, and U_1, U_2 are independent uniform random variables on $[0, T_1]$, $[0, T_2]$ respectively. The point process is obviously a second-order stationary with average intensity $\lambda = 1/(T_1T_2)$. Its Bartlett spectral measure is

$$\mu_N = \frac{1}{T_1^2T_2^2} \sum_{(n_1,n_2)\neq(0,0)} \varepsilon_{\left(\frac{n_1}{T_1},\frac{n_2}{T_2}\right)}\ , \tag{12}$$

and we can take

$$B_N = \left\{\varphi \in L^1_{\mathbb{C}}(\mathbb{R}^2) \text{ and } \sum_{n_1,n_2\in\mathbb{Z}} \left|\hat{\varphi}\left(\frac{n_1}{T_1}, \frac{n_2}{T_2}\right)\right| < \infty\right\}\ . \tag{13}$$

Proof. Conditions (13) guarantee that the weak Poisson formula holds true. More precisely[1], the left-hand side of the following equality

$$\sum_{n_1,n_2\in\mathbb{Z}} \varphi(u_1 + n_1T_1, u_2 + n_2T_2) = \frac{1}{T_1T_2} \sum_{n_1,n_2\in\mathbb{Z}} \widehat{\varphi}\left(\frac{n_1}{T_1}, \frac{n_2}{T_2}\right) e^{2i\pi\left(\frac{n_1}{T_1}u_1 + \frac{n_2}{T_2}u_2\right)} \tag{14}$$

[1] See for instance P. Brémaud, *Mathematical Principles of Signal Processing*, Theorem A2.3

is well-defined, and the equality holds for almost-all $(u_1, u_2) \in \mathbb{R}^2$ (with respect to the Lebesgue measure). By (14)

$$\int_{\mathbb{R}^2} \varphi(t)\, N(dt) = \sum_{n_1,n_2\in\mathbb{Z}} \varphi(U_1 + n_1T_1, U_2 + n_2T_2) = \frac{1}{T_1T_2} \sum_{n_1,n_2\in\mathbb{Z}} \widehat{\varphi}\left(\frac{n_1}{T_1}, \frac{n_2}{T_2}\right) e^{2i\pi\left(\frac{n_1}{T_1}U_1 + \frac{n_2}{T_2}U_2\right)}.$$

The rest of the proof is straightforward, noting that the finiteness of the sum in (14) implies

$$\sum_{n_1,n_2\in\mathbb{Z}} \left|\hat{\varphi}\left(\frac{n_1}{T_1}u_1, \frac{n_2}{T_2}u_2\right)\right|^2 < \infty.$$

Example 2. *Cox process.* Let N be a Cox point process on $\mathbb{R}^m$ with stochastic intensity $\{\lambda(t)\}_{t\in\mathbb{R}^m}$. By this, the following is meant. Firstly, $\{\lambda(t)\}_{t\in\mathbb{R}^m}$ is a non-negative a.s. locally integrable process; and secondly, *conditionally on this process*, N is a Poisson process with intensity $\lambda(t)$. We suppose that $\{\lambda(t)\}_{t\in\mathbb{R}^m}$ is a WSS process with mean λ and Bochner spectral measure μ_λ. Then the Bartlett spectrum of N is

$$\mu_N(d\nu) = \mu_\lambda(d\nu) + \lambda d\nu\,, \tag{15}$$

on the domain $B_N = L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m)$. Moreover, in this case this is the maximal domain, since $B_N = L^2_N(M_2)$.

Proof. We omit the proof, the details of which can be found in [24]

3. Basic isometry formula

3.1. Filtered point process fields

Consider a marked point process on $\mathbb{R}^m$ with marks in the measurable space $(K, \mathcal{K})$. Let N be its basic point process on $\mathbb{R}^m$, assumed locally finite and simple, and let $\{Z(t)\}_{t\in\mathbb{R}^m}$ be its mark process. Assume that the family of random variables $\{Z(t)\}_{t\in\mathbb{R}^m}$ is i.i.d. with common probability distribution Q, and independent of N. Also assume that N is a second order stationary point process with Bartlett spectral measure μ_N on the domain B_N.

Let Z be a random element with distribution Q. We introduce (or recall) a notation: $L^p_{\mathbb{C}}(\ell\times Q)$ is the set of functions $\varphi : \mathbb{R}^m \times K \to \mathbb{C}$ such that

$$\int_{\mathbb{R}^m} E\left[|\varphi(t,Z)|^p\right]\, dt < \infty\,.$$

In particular, $\varphi(t,Z) \in L^p_{\mathbb{C}}(P)$ for almost all $t \in \mathbb{R}$ (with respect to the Lebesgue measure).

Let $\varphi : \mathbb{R}^m \times K \to \mathbb{R}$ be a measurable function such that

$$\varphi \in L^1_{\mathbb{C}}(\ell \times Q). \tag{16}$$

In particular, $\varphi(t,Z)\in L^p_{\mathbb{C}}(P)$ for almost all $t\in\mathbb{R}$ (with respect to the Lebesgue measure) and we can define for almost all t

$$\overline{\varphi}(t) := E\left[\varphi(t,Z)\right] .$$

It also follows from assumption (16) that $\overline{\varphi}\in L^1_{\mathbb{C}}(\mathbb{R}^m)$ and for Q-almost all $z\in K$, $\varphi(\cdot,z)\in L^1_{\mathbb{C}}(\mathbb{R}^m)$. Let the Fourier transforms of these two functions be denoted by $\widehat{\overline{\varphi}}$ and $\widehat{\varphi}(\cdot,z)$ respectively. Suppose moreover that

$$\varphi\in L^2_{\mathbb{C}}(\ell\times Q). \tag{17}$$

Note that condition (17) implies that $\int_{\mathbb{R}^m}\left|E\left[\varphi(t,Z)\right]\right|^2\,dt\;<\infty$, that is $\overline{\varphi}\in L^2_{\mathbb{C}}(\mathbb{R}^m)$, and for Q-almost all $z\in K$, $\varphi(\cdot,z)\in L^2_{\mathbb{C}}(\mathbb{R}^m)$. Observe that

$$\widehat{\overline{\varphi}}(\nu) = E\left[\widehat{\varphi}(\nu,Z)\right] := \overline{\widehat{\varphi}}(\nu).$$

Finally, suppose that

$$\overline{\varphi}\in B_N\,. \tag{18}$$

We can now state a fundamental *isometry formula.*

Theorem 2. *Let N and $\{Z(t)\}_{t\in\mathbb{R}^m}$ be as above, and $\varphi,\psi:\mathbb{R}^m\times K\to\mathbb{R}$ satisfy conditions (16), (17) and (18). Then*

$$\operatorname{cov}\left(\sum_{t\in N}\varphi(t,Z(t))\,,\,\sum_{t\in N}\psi(t,Z(t))\right) = \int_{\mathbb{R}^m}\widehat{\overline{\varphi}}(\nu)\,\widehat{\overline{\psi}}^*(\nu)\mu_N(d\nu) + \lambda\int_{\mathbb{R}^m}\operatorname{cov}\big(\widehat{\varphi}(\nu,Z),\widehat{\psi}^*(\nu,Z)\big)d\nu, \tag{19}$$

where Z is a K-valued random variable with distribution Q.

Proof. Formally:

$$\begin{aligned}
&E\left[\left(\sum_{t\in N}\varphi(t,Z(t))\right)\left(\sum_{t\in N}\psi(t,Z(t))\right)\right]\\
&\quad= E\left[\sum_{t,t'\in N,t\neq t'}\varphi(t,Z(t))\psi(t',Z(t'))\right] + E\left[\sum_{t\in N}\varphi(t,Z(t))\psi^*(t,Z(t))\right]\\
&\quad= E\left[\sum_{t,t'\in N,t\neq t'}\overline{\varphi}(t)\overline{\psi}^*(t')\right] + E\left[\sum_{t\in N}\varphi(t,Z)\psi^*(t,Z)\right]\\
&\quad= E\left[\left(\sum_{t\in N}\overline{\varphi}(t)\right)\left(\sum_{t'\in N}\overline{\psi}^*(t')\right)\right] - E\left[\sum_{t\in N}\overline{\varphi}(t)\overline{\psi}^*(t)\right] + E\left[\sum_{t\in N}\varphi(t,Z)\psi^*(t,Z)\right].
\end{aligned}$$

Denote $a-b+c$ the last line. The above formal computations are justified because all the three terms are, when φ and ψ are replaced by their absolute values, finite. This follows from Schwarz's inequality, and the facts that (for a and b) $\overline{\varphi}$ and $\overline{\psi}$ are in $L^2_N(M_2)$ and in $L^2_{\mathbb{C}}(\mathbb{R}^m)$; and for c because of condition (17). Since $E\left[\sum_{t\in N}\varphi(t,Z(t))\right] = E\left[\sum_{t\in N}\overline{\varphi}(t)\right]$,

$$\operatorname{cov}\left(\sum_{t\in N}\varphi(t,Z(t)),\sum_{t\in N}\psi(t,Z(t))\right) =$$
$$\operatorname{cov}\left(\sum_{t\in N}\overline{\varphi}(t),\sum_{t\in N}\overline{\psi}(t)\right) - E\left[\sum_{t\in N}\overline{\varphi}(t)\overline{\psi}^*(t)\right] + E\left[\sum_{t\in N}\varphi(t,Z)\psi^*(t,Z)\right].$$

Denote by A, B and C the three terms in the right-hand side of the above equation, which then reads $A-B+C$. By definition of the Bartlett spectrum, and the hypothesis (18),

$$A = \int_{\mathbb{R}^m}\widehat{\overline{\varphi}}(\nu)\widehat{\overline{\psi}}^*(\nu)\mu_N(d\nu).$$

By definition of the intensity λ,

$$B = \lambda\int_{\mathbb{R}^m}\overline{\varphi}(t)\overline{\psi}(t)^*dt,\ \ C = \lambda\int_{\mathbb{R}^m}E\left[\varphi(t,Z)\psi(t,Z)^*\right]dt.$$

By the Plancherel-Parseval identity,

$$B = \lambda\int_{\mathbb{R}^m}\widehat{\overline{\varphi}}(\nu)\widehat{\overline{\psi}}(\nu)^*d\nu = \lambda\int_{\mathbb{R}^m}\overline{\widehat{\varphi}}(\nu)\overline{\widehat{\psi}}(\nu)^*d\nu = \lambda\int_{\mathbb{R}^m}E\left[\widehat{\varphi}(\nu,Z)\right]E\left[\widehat{\psi}(\nu,Z)^*\right]d\nu,$$

and

$$C = \lambda E\left[\int_{\mathbb{R}^m}\widehat{\varphi}(\nu,Z)\widehat{\psi}(\nu,Z)^*d\nu\right],$$

and the result (19) follows.

We can now compute the power spectrum of a shot noise.

Corollary 1. *Consider the above marked point process (with independent i.i.d. marks) and let $h:\mathbb{R}^m\times K\to\mathbb{R}$ satisfy conditions (18), and (17). Define the shot noise $\{X(t)\}_{t\in\mathbb{R}^m}$ by*

$$X(t) = \sum_{s\in N}h(t-s,Z(t)).$$

Its Bochner spectral measure is given by the formula

$$\mu_X(d\nu) = \left|E\left[\hat{h}(\nu,Z)\right]\right|^2\mu_N(d\nu) + \lambda\operatorname{Var}\left(\hat{h}(\nu,Z)\right)d\nu\,. \tag{20}$$

Proof. It suffices to apply the fundamental isometry formula to $\varphi(t,z) = h(u-t,z)$, $\psi(t,z) = h(v-t,z)$ to obtain

$$\operatorname{cov}\left(X(u),X(v)\right) = \int_{\mathbb{R}^m}\left|\overline{\hat{h}}(\nu)\right|^2 e^{-2i\pi<\nu,u-v>}\mu_N(d\nu) + \lambda\int_{\mathbb{R}^m}\operatorname{Var}\left(\hat{h}(\nu,Z)\right)e^{-2i\pi<\nu,u-v>}d\nu.$$

3.2. Jittered point processes

Knowing the Bartlett spectrum μ_N of a WSS point process N, what is the Bartlett spectrum $\mu_{\widetilde{N}}$ of the point process obtained by independent and identically distributed displacements of the points of N? We have the following result.

Corollary 2. *Consider the marked point process of Theorem 2, with $K = \mathbb{R}^m$. A point process $\widetilde{N}$ is defined by*

$$\widetilde{N} = \{t + Z(t), t \in N\}.$$

Then, calling λ the intensity of N, and $\mu_{\widetilde{N}}$ the Bartlett spectrum of $\widetilde{N}$,

$$\mu_{\widetilde{N}}(d\nu) = |\psi_Z(\nu)|^2 \mu_N(d\nu) + \lambda \left(1 - |\psi_Z(\nu)|^2\right) d\nu, \tag{21}$$

where

$$\psi_Z(\nu) = E\left[e^{2i\pi<\nu,Z>}\right] \tag{22}$$

is the characteristic function of the random displacements distributed as Q. We can take

$$B_{\widetilde{N}} = \left\{\widetilde{\varphi};\, E\left[\widetilde{\varphi}(t+Z)\right] \in B_N \text{ and } \widetilde{\varphi} \in L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m)\right\} \tag{23}$$

Proof. Define $\varphi(t,z) = \widetilde{\varphi}(t+z)$. Conditions (16) and (17) for the function φ are equivalent to conditions $\tilde{\varphi} \in L^1_{\mathbb{C}}(\mathbb{R}^m)$ and $\tilde{\varphi} \in L^2_{\mathbb{C}}(\mathbb{R}^m)$ respectively, since for any $p \geq 0$,

$$E\left[\int_{\mathbb{R}^m} |\varphi(t,Z)|^p \, dt\right] = \int_{\mathbb{R}^m} |\tilde{\varphi}(t)|^p \, dt\,.$$

Condition (18) for the function φ is satisfied by the *ad hoc* definition of $B_{\widetilde{N}}$. We may therefore apply Theorem 2. We have

$$\begin{aligned}
&\widehat{\varphi}(\nu, z) = e^{2i\pi<\nu,z>}\widehat{\widetilde{\varphi}}(\nu),\\
&\widehat{\overline{\varphi}}(\nu) = \overline{\widehat{\varphi}}(\nu) = \psi_Z(\nu)\widehat{\widetilde{\varphi}}(\nu),\\
&\text{cov}\left(\widehat{\varphi}(\nu, Z), \widehat{\varphi}(\nu, Z)^*\right) = \left(1 - |\psi_Z(\nu)|^2\right)\left|\widehat{\widetilde{\varphi}}(\nu)\right|^2.
\end{aligned}$$

Also,

$$\text{Var}\left(\int_{\mathbb{R}^m} \widetilde{\varphi}(t)\widetilde{N}(dt)\right) = \text{Var}\left(\sum_{t\in N} \tilde{\varphi}(t + Z(t))\right),$$

and therefore, applying formula (19),

$$\text{Var}\left(\int_{\mathbb{R}^m} \widetilde{\varphi}(t)\widetilde{N}(dt)\right) = \int_{\mathbb{R}^m} \left|\widehat{\widetilde{\varphi}}(\nu)\right|^2 \mu_{\widetilde{N}}(d\nu),$$

where $\mu_{\widetilde{N}}$ is as in (21).

Example 3. *Jittered regular grid.* We consider the case where N is the grid process of Example 1. We can take

$$B_{\tilde{N}} = \left\{\tilde{\varphi};\, \sum_{n_1,n_2\in\mathbb{Z}} \left|\hat{\tilde{\varphi}}\left(\frac{n_1}{T_1}, \frac{n_2}{T_2}\right)\right| < \infty \text{ and } \tilde{\varphi} \in L^1_{\mathbb{C}}(\mathbb{R}^2) \cap L^2_{\mathbb{C}}(\mathbb{R}^2)\right\}.$$

Proof. Indeed, observing that

$$\left| E\,[\widehat{\tilde{\varphi}(\cdot + Z)}](\nu) \right| = \left| \hat{\tilde{\varphi}}(\nu) \right| ,$$

we see that condition $E\,[\tilde{\varphi}(t+Z)] \in B_N$ is equivalent to $\sum_{n_1,n_2\in\mathbb{Z}} \left| \hat{\tilde{\varphi}}(\frac{n_1}{T_1}, \frac{n_2}{T_2}) \right| < \infty$.

Example 4. *Jittered Cox process.* We consider the case where N is the Cox process of Example 2. We can take

$$B_{\tilde{N}} = \left\{ \tilde{\varphi}\,;\, \tilde{\varphi} \in L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m) \right\}.$$

Indeed condition $E\,[\tilde{\varphi}(t+Z)] \in B_N$, that is, in this particular case, $E\,[\tilde{\varphi}(t+Z)] \in L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m)$, is exactly $\tilde{\varphi} \in L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m)$.

3.3. Cluster point processes

Let N be a stationary point process on $\mathbb{R}^m$ with intensity $\lambda > 0$, Bartlett spectral measure μ_N, and set of test functions $\mathcal{B}_N$. Let $\{Z\,(t)\}_{t\in\mathbb{R}^m}$ be an i.i.d. collection of *point processes* on $\mathbb{R}^m$, independent of N. Denote $Z\,(t)\,(C) = Z\,(t,C)$, and let Z be a point process on $\mathbb{R}^m$ with the same distribution as the common distribution of the $Z\,(t)$'s. Let

$$\psi_Z\,(\nu) = \mathrm{E}\left[\int_{\mathbb{R}^m} e^{2i\pi\langle\nu,t\rangle} Z\,(dt)\right]$$

Existence and finiteness of $\psi_Z\,(\nu)$ is guaranteed under the condition

$$\mathrm{E}\,[Z\,(\mathbb{R}^m)] < \infty.$$

In particular, Z is almost surely a finite point process.

We now define two point process on $\mathbb{R}^m$, $\widetilde{N}$ and $\widehat{N}$, by

$$\begin{aligned}
\widetilde{N}(C) =& N\,(C) + \sum_{t\in N} Z\,(t, C-t)\,, \\
\widehat{N}(C) =& \sum_{t\in N} Z\,(t, C-t)\,.
\end{aligned}$$

We would like to compute the Bartlett spectrum of $\widetilde{N}$ and $\widehat{N}$. We start with $\widetilde{N}$. Formally

$$\begin{aligned}
\mathrm{Var}\left(\sum_{t\in\widetilde{N}} \varphi\,(t)\right) =& \mathrm{Var}\left(\sum_{t\in N} \left\{\varphi\,(t) + \int_{\mathbb{R}^m} \varphi\,(t+s)\,Z\,(t,ds)\right\}\right) \\
=& \mathrm{Var}\left(\sum_{t\in N} \varphi\,(t, Z\,(t))\right),
\end{aligned}$$

where

$$\varphi\,(t,z) = \varphi\,(t) + \int_{\mathbb{R}^m} \varphi\,(t+s)\,z\,(ds)\,.$$

Formally

$$\mathrm{E}\left[\varphi\left(t,z\right)\right]=\varphi\left(t\right)+\mathrm{E}\left[\int_{\mathbb{R}^{m}}\varphi\left(t+s\right)z\left(ds\right)\right]$$

$$\begin{aligned}\widehat{\varphi}\left(\nu,z\right)&=\widehat{\varphi}\left(\nu\right)+\int_{\mathbb{R}^{m}}\left(\int_{\mathbb{R}^{m}}\varphi\left(t+s\right)z\left(ds\right)\right)e^{-2i\pi\langle\nu,t\rangle}dt\\&=\widehat{\varphi}\left(\nu\right)+\int_{\mathbb{R}^{m}}\left(\int_{\mathbb{R}^{m}}\varphi\left(t+s\right)e^{-2i\pi\langle\nu,t\rangle}dt\right)z\left(ds\right)\\&=\widehat{\varphi}\left(\nu\right)+\int_{\mathbb{R}^{m}}\widehat{\varphi}\left(\nu\right)e^{2i\pi\langle\nu,s\rangle}z\left(ds\right)\\&=\widehat{\varphi}\left(\nu\right)\left(1+\int_{\mathbb{R}^{m}}e^{2i\pi\langle\nu,s\rangle}z\left(ds\right)\right)\end{aligned}$$

Note that the exchange of order of integration is not a problem if z is a finite point process, in particular if z is replaced by its random version Z. Note also that $\varphi\in\mathcal{B}_{\widetilde{N}}$ (not yet identified) is necessarily in $\mathrm{L}^{1}_{\mathbb{C}}\left(\mathbb{R}^{m}\right)$, since $\mathcal{B}_{\widetilde{N}}\subseteq\mathrm{L}^{1}_{\mathbb{C}}\left(\mathbb{R}^{m}\right)$.

Also

$$\widehat{\widehat{\varphi}}\left(\nu\right)=\widehat{\varphi}\left(\nu\right)\left(1+\psi_{Z}\left(\nu\right)\right)$$

Applying formally Theorem 2, we obtain

$$\begin{aligned}\mathrm{Var}\left(\sum_{t\in N}\varphi\left(t,Z\left(t\right)\right)\right)&=\int_{\mathbb{R}^{m}}\left|\widehat{\varphi}\left(\nu\right)\right|^{2}\left|1+\psi_{Z}\left(\nu\right)\right|^{2}\mu_{N}\left(d\nu\right)\\&\quad+\lambda\int_{\mathbb{R}^{m}}\left|\widehat{\varphi}\left(\nu\right)\right|^{2}\mathrm{Var}\left(1+\int_{\mathbb{R}^{m}}e^{2i\pi\langle\nu,s\rangle}Z\left(ds\right)\right)d\nu.\end{aligned}$$

Observe that

$$\mathrm{Var}\left(1+\int_{\mathbb{R}^{m}}e^{2i\pi\langle\nu,s\rangle}Z\left(ds\right)\right)=\mathrm{Var}\left(\int_{\mathbb{R}^{m}}e^{2i\pi\langle\nu,s\rangle}Z\left(ds\right)\right)$$

to obtain

$$\mathrm{Var}\left(\sum_{t\in\widetilde{N}}\varphi\left(t\right)\right)=\int_{\mathbb{R}^{m}}\left|\widehat{\varphi}\left(\nu\right)\right|^{2}\mu_{\widetilde{N}}\left(d\nu\right)$$

where

$$\mu_{\widetilde{N}}\left(d\nu\right)=\left|1+\psi_{Z}\left(\nu\right)\right|^{2}\mu_{N}\left(d\nu\right)+\lambda\mathrm{Var}\left(\int_{\mathbb{R}^{m}}e^{2i\pi\langle\nu,s\rangle}Z\left(ds\right)\right)d\nu.$$

is the Bartlett spectrum of $\widetilde{N}$. Similar computations lead to

$$\mu_{\widehat{N}}\left(d\nu\right)=\left|\psi_{Z}\left(\nu\right)\right|^{2}\mu_{N}\left(d\nu\right)+\lambda\mathrm{Var}\left(\int_{\mathbb{R}^{m}}e^{2i\pi\langle\nu,s\rangle}Z\left(ds\right)\right)d\nu.$$

To obtain the corresponding domains $\mathcal{B}_{\widetilde{N}}$ and $\mathcal{B}_{\widehat{N}}$, it suffices to ask that the condition for $\varphi\left(t,z\right)$ in Theorem 2 are satisfied.

4. Spatial Hawkes processes

4.1. Spatial branching point process

The Hawkes point process N on $\mathbb{R}^m$ is a spatial *branching point process*. It is constructed as follows:

Let N_0 be a simple second order stationary point process with Bartlett spectrum μ_0 and set of admissible functions B_{N_0}. This point process is called the "ancestors process". Let $\{Z_n(t)\}_{n\geq 0, t\in\mathbb{R}^m}$, be a family of i.i.d. random variables with values in the measurable space $(K, \mathcal{K})$ and common distribution Q, and independent of N_0. Let

$$N = \sum_{n\geq 0} N_n,$$

where each N_n is the basic point process on $\mathbb{R}^m$ of a marked point process $\overline{N}_n$ on $\mathbb{R}^m \times K$ with the i.i.d. marks $\{Z_n(t)\}_{t\in\mathbb{R}^m}$, that is,

$$\overline{N}_n(C \times L) = \sum_{t\in N_n} 1_C(t) 1_L(Z_n(t)),$$

where $C \subseteq \mathbb{R}^m$ is a Borel set and $L \in \mathcal{K}$. The sequence of point processes $\{N_n\}_{n\geq 1}$ is constructed recursively as follows. First we are given a nonnegative *rate function* $h : \mathbb{R}^m \times K \to \mathbb{R}$ such that the quantity

$$\rho := \int_{\mathbb{R}^m} E[h(t, Z)]\, dt$$

is finite, where Z is a K-valued random variable with distribution Q (the general mark distribution). We denote by $\mathcal{F}_n$ the sigma-field recording all the events relative to $\overline{N}_0, \ldots, \overline{N}_n$. Then N_n is, conditionally on $\mathcal{F}_{n-1}$, a Poisson process on $\mathbb{R}^m$ with the intensity

$$\lambda_n(t) = \int_{\mathbb{R}^m} \int_K h(t-s, z) \overline{N}_{n-1}(ds \times dz) \left(= \sum_{s\in N_{n-1}} h(t-s, Z_{n-1}(s)) \right). \tag{24}$$

N_n is called the n-th generation point process. The interpretation is the following: each point $a \in N_{n-1}$ of generation $n-1$ creates descendants in the next generation according to a Poisson process of intensity $h(t-a, Z_{n-1}(a))$. We therefore have for each ancestor (a point $a \in N_0$) ρ direct descendants on the average. Denote

$$N' = \sum_{n\geq 1} N_n, \quad \overline{N}' = \sum_{n\geq 1} \overline{N}_n.$$

From (24) and the Campbell formula we see that, denoting $\lambda_n = E[\lambda_n(t)]$,

$$\lambda_n = \lambda_{n-1} \int_{\mathbb{R}^m} E[h(t, Z)]\, dt$$

and therefore the average intensity λ' of N' verifies

$$\lambda' = \rho\lambda_0 + \rho\lambda'.$$

Therefore, if $\lambda_0 > 0$, in order for N' to have a finite intensity, it is necessary that

$$\rho < 1\,. \tag{25}$$

In this case, each ancestor (point of N_0) is the root of an eventually extinguishing branching process, because its average progeny is strictly less than 1. Condition (25) will be assumed throughout.

Lemma 1. *For* $\varphi \in L^1_{\mathbb{C}}(\ell \times Q) \cap L^2_{\mathbb{C}}(\ell \times Q)$,

$$Var\left(\int_{\mathbb{R}^m}\int_K \varphi(t,z)\,\overline{M}'(dt \times dz)\right) = \lambda' \int_{\mathbb{R}^m} E\left[|\varphi(t,Z)|^2\right] dt, \tag{26}$$

where

$$\overline{M}'(dt \times dz) = \overline{N}'(dt \times dz) - \lambda'(t)\,dt\,Q\,(dz)$$
$$\lambda'(t) = \int_{\mathbb{R}^m}\int_K h\,(t-s,z')\,\overline{N}_0\,(dt \times dz') + \int_{\mathbb{R}^m}\int_K h\,(t-s,z')\,\overline{N}'\,(dt \times dz')$$

Proof. We shall use simplified notation of the kind $\int\int \varphi(t,z)\,\overline{M}'(dt \times dz) = \int \varphi\, d\overline{M}'$. We have

$$\int \varphi\, d\overline{M}' = \sum_{n\geq 1} \int \varphi\, d\overline{M}_n,$$

where $\overline{M}_n(dt \times dz) = \overline{N}_n(dt \times dz) - \lambda_n(t)dt\,Q(dz)$. Given $\mathcal{F}_{n-1}$, $\overline{N}_n$ is a Poisson process with mean measure $\lambda_n(t)Q(dz)dt$, and therefore, by standard properties of Poisson processes,

$$\text{Var}\left(\int \varphi\, d\overline{M}_n \middle| \mathcal{F}_{n-1}\right) = \int_{\mathbb{R}^m} E\left[\varphi^2(t,Z)\right]\lambda_n(t)dt,$$

and

$$E\left[\int \varphi\, d\overline{M}_n \middle| \mathcal{F}_{n-1}\right] = 0.$$

Therefore, by the conditional variance formula

$$\begin{aligned}\text{Var}\left(\int \varphi\, d\overline{M}_n\right) &= E\left[\text{Var}\left(\int \varphi\, d\overline{M}_n \middle| \mathcal{F}_{n-1}\right)\right] + \text{Var}\left(E\left[\int \varphi\, d\overline{M}_n \middle| \mathcal{F}_{n-1}\right]\right)\\ &= \lambda_n \int_{\mathbb{R}^m} E\left[\varphi^2(t,Z)\right] dt.\end{aligned}$$

Also for $j, k \geq 1$,

$$E\left[\left(\int \varphi\, d\overline{M}_j\right)\left(\int \varphi\, d\overline{M}_{j+k}\right)\right] = E\left[\left(\int \varphi\, d\overline{M}_j\right) E\left[\int \varphi\, d\overline{M}_{j+k} \middle| \mathcal{F}_{j+k-1}\right]\right] = 0.$$

Therefore

$$\begin{aligned}\operatorname{Var}\left(\int \varphi\, d\overline{M}'\right) &= \sum_{n\geq 1} \operatorname{Var}\left(\int \varphi\, d\overline{M}_n\right)\\ &= \left(\sum_{n\geq 1}\lambda_n\right)\int_{\mathbb{R}^m} E\left[\varphi^2(t,Z)\right] dt\\ &= \lambda' \int_{\mathbb{R}^m} E\left[\varphi^2(t,Z)\right] dt.\end{aligned}$$

4.2. SPECTRUM OF THE HAWKES PROCESS

Lemma 2. *A. Suppose that*

$$E\left[\left(\int_{\mathbb{R}^m} h\,(t,Z)\,dt\right)^2\right] < \infty. \tag{27}$$

There exists, for any given $F \in L^1_{\mathbb{C}}(\ell\times Q)\cap L^2_{\mathbb{C}}(\ell\times Q)$, *a unique function* $\varphi \in L^1_{\mathbb{C}}(\ell\times Q)\cap L^2_{\mathbb{C}}(\ell\times Q)$ *such that*

$$\varphi(t,z) - \int_{\mathbb{R}^m} h(s-t,z)E\left[\varphi(s,Z)\right] ds = F(t,z). \tag{28}$$

B. For given $f\in L^1_{\mathbb{C}}(\mathbb{R}^m)\cap L^2_{\mathbb{C}}(\mathbb{R}^m)$, *there exists a unique* $\varphi \in L^1_{\mathbb{C}}(\ell\times Q)\cap L^2_{\mathbb{C}}(\ell\times Q)$ *such that*

$$\varphi(t,z) - \int_{\mathbb{R}^m} h(s-t,z)E\left[\varphi(s,Z)\right] ds = f(t). \tag{29}$$

Proof. A. For a function $v\,(t,z)$, denote $E\left[v\,(t,Z)\right]$ by $\overline{v}\,(t)$ and $v\,(-t,z)$ by $\check{v}\,(t,z)$. Observe that $F\in L^1_{\mathbb{C}}(\ell\times Q)\cap L^2_{\mathbb{C}}(\ell\times Q)$ implies $\overline{F}\in L^1_{\mathbb{C}}(\mathbb{R}^m)\cap L^2_{\mathbb{C}}(\mathbb{R}^m)$. Let h and F be as in the statement of the above lemma, and consider the renewal equation

$$g = \overline{F} + \check{\overline{h}} * g. \tag{30}$$

Since $\overline{F}\in L^1_{\mathbb{C}}(\mathbb{R}^m)\cap L^2_{\mathbb{C}}(\mathbb{R}^m)$, and since condition (25) holds, there exists a unique solution $g\in L^1_{\mathbb{C}}(\mathbb{R}^m)\cap L^2_{\mathbb{C}}(\mathbb{R}^m)$ given by

$$g = \sum_{n\geq 0} \overline{F} * \check{\overline{h}}^{*n} \tag{31}$$

(the convergence of the series in $L^1_{\mathbb{C}}(\mathbb{R}^m)$ as well as in $L^2_{\mathbb{C}}(\mathbb{R}^m)$ is guaranteed by the inequalities $\|a*b\|_{L^1}\leq \|a\|_{L^1}\|b\|_{L^1}$ and $\|a*b\|_{L^2}\leq \|a\|_{L^1}\|b\|_{L^2}$; uniqueness follows from the equality $g-g' = \check{\overline{h}}*(g-g')$, where g' is another candidate solution, which implies $\|g-g'\|_{L^1}\leq \left\|\overline{h}\right\|_{L^1}\|g-g'\|_{L^1}$, and hence under condition (25), necessarily $\|g-g'\|_{L^1}=0$). The Fourier transform of g is

$$\widehat{g}\,(\nu) = \frac{E\left[\widehat{F}\,(\nu,Z)\right]}{1-E\left[\hat{h}\,(\nu,Z)^*\right]}. \tag{32}$$

Define now $\varphi(t,z)$ by

$$\varphi(t,z) = \int_{\mathbb{R}^m} h(s-t,z)\, g(s)\, ds + F(t,z)\,. \tag{33}$$

We have

$$E\left[\int_{\mathbb{R}^m} |\varphi(t,Z)|\, dt\right] \le E\left[\int_{\mathbb{R}^m} |F(t,Z)|\, dt\right] + E\left[\int_{\mathbb{R}^m} |h(t,Z)|\, dt\right]\int_{\mathbb{R}^m} |g(t)|\, dt < \infty$$

because $g \in L^1_{\mathbb{C}}(\mathbb{R}^m)$ and $F, h \in L^1_{\mathbb{C}}(\ell \times Q)$. Therefore $\varphi \in L^1_{\mathbb{C}}(\ell \times Q)$. We now show that $\varphi \in L^2_{\mathbb{C}}(\ell \times Q)$. It suffices to show that $\widehat{\varphi}(\nu,z) \in L^2_{\mathbb{C}}(\ell \times Q)$ because then, using the Plancherel-Parseval identity,

$$E\left[\int_{\mathbb{R}^m} |\varphi(t,Z)|^2\, dt\right] = E\left[\int_{\mathbb{R}^m} |\widehat{\varphi}(\nu,Z)|^2\, d\nu\right] < \infty.$$

For this purpose, we take for fixed z the Fourier transform of both sides of (33)

$$\widehat{\varphi}(\nu,z) = \hat{h}(\nu,z)^*\, \widehat{g}(\nu) + \widehat{F}(\nu,z)\,,$$

or (for future reference) in view of (32)

$$\widehat{\varphi}(\nu,z) = \widehat{F}(\nu,z) + \frac{\hat{h}(\nu,z)^*\, E\left[\widehat{F}(\nu,Z)\right]}{1 - E\left[\hat{h}(\nu,Z)^*\right]}\,. \tag{34}$$

We show that $\widehat{\varphi}(\nu,z) \in L^2_{\mathbb{C}}(\ell \times Q)$. Since $F(t,z) \in L^2_{\mathbb{C}}(\ell \times Q)$, it follows by the Plancherel–Parseval identity that $\widehat{F}(\nu,z) \in L^2_{\mathbb{C}}(\ell \times Q)$. It remains to show that

$$\hat{h}(\nu,z)\, \hat{g}(\nu) \in L^2_{\mathbb{C}}(\ell \times Q)\,. \tag{35}$$

This follows from the fact that $\widehat{g} \in L^2_{\mathbb{C}}(\mathbb{R}^m)$ and that

$$E\left[\left|\hat{h}(\nu,Z)\right|^2\right] = E\left[\left|\int_{\mathbb{R}^m} h(t,Z)\, e^{2i\pi\nu t} dt\right|^2\right] \le E\left[\left|\int_{\mathbb{R}^m} h(t,Z)\, dt\right|^2\right]$$

a finite constant (by hypothesis (27)), independent of ν.

B. This is clearly a particular case of A. We note for future reference that in this case the following holds:

$$\widehat{\varphi}(\nu,z) = \widehat{f}(\nu)\left(1 + \frac{\widehat{h}^*(\nu,z)}{1 - E\left[\widehat{h}^*(\nu,Z)\right]}\right). \tag{36}$$

Theorem 3. *Let $h(t,z)$ verify (25) and (27). The Bartlett spectrum of N defined above is*

$$\mu_N(d\nu) = \frac{1}{\left|1 - E\left[\hat{h}(\nu,Z)\right]\right|^2}\left[\mu_0(d\nu) + \lambda' d\nu + \lambda\, Var\left(\hat{h}(\nu,Z) d\nu\right)\right], \tag{37}$$

where $\lambda = \lambda_0/(1-\rho)$. We can take for B_N the set of functions $f \in L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m)$ such that the solution of (29) satisfies $E[\varphi(\cdot,Z)] \in B_{N_0}$.

Proof. Let φ be the solution of (29).

$$\begin{aligned}\int_{\mathbb{R}^m}\int_K \varphi(t,z)\overline{M}'(dt\times dz) &= \int_{\mathbb{R}^m}\int_K \varphi(t,z)\left(\overline{N}'(dt\times dz)-\lambda'(t)Q(dz)dt\right)\\ &= \int_{\mathbb{R}^m}\int_K \varphi(t,z)\overline{N}'(dt\times dz)-\int_{\mathbb{R}^m}\int_K \varphi(t,z)\lambda'(t)Q(dz)dt.\end{aligned}$$

Also,

$$\begin{aligned}\int_{\mathbb{R}^m}\int_K \varphi(t,z)\lambda'(t)Q(dz)dt &= \int_{\mathbb{R}^m}\int_K \varphi(t,z)\left(\int_{\mathbb{R}^m}\int_K h(t-s,z')\overline{N}(ds\times dz')\right)Q(dz)dt\\ &= \int_{\mathbb{R}^m\times\mathbb{R}^m\times K} h(t-s,z')E\left[\varphi(t,Z)\right]dt\overline{N}(ds\times dz')\\ &= \int_{\mathbb{R}^m}\int_K \left(\check{h}(s-\cdot,z')*E\left[\varphi(\cdot,Z)\right]\right)(s)\,\overline{N}(ds\times dz').\end{aligned}$$

Therefore, since $\overline{N}=\overline{N}'+\overline{N}_0$,

$$\begin{aligned}\int_{\mathbb{R}^m}\int_K \varphi(t,z)\overline{M}'(dt\times dz) = &\int_{\mathbb{R}^m}\int_K \left(\varphi(t,z)-\left(\check{h}(t-\cdot,z)*E\left[\varphi(\cdot,Z)\right]\right)(t)\right)\overline{N}(dt\times dz)\\ &-\int_{\mathbb{R}^m}\int_K \varphi(t,z)\overline{N}_0(dt\times dz).\end{aligned}$$

Take $f\in B_N$ (in particular $f\in L^1_{\mathbb{C}}(\mathbb{R}^m)\cap L^2_{\mathbb{C}}(\mathbb{R}^m)$), and let $\varphi(t,z)$ be the solution of (29). We have

$$\int_{\mathbb{R}^m}\int_K \varphi(t,z)\overline{M}'(dt\times dz)+\int_{\mathbb{R}^m}\int_K \varphi(t,z)\overline{N}_0(dt\times dz)=\int_{\mathbb{R}^m} f(t)N(dt).$$

Also, by the isometry lemma,

$$\mathrm{Var}\left(\int_{\mathbb{R}^m}\int_K \varphi(t,z)\overline{M}'(dt\times dz)\right)=\lambda'\int_{\mathbb{R}^m} E\left[|\varphi(t,Z)|^2\right]dt=\lambda'\int_{\mathbb{R}^m} E\left[|\widehat{\varphi}(\nu,Z)|^2\right]d\nu.$$

Now,

$$\begin{aligned}&E\left[\int_{\mathbb{R}^m}\int_K \varphi(t,z)\overline{M}'(dt\times dz)\int_{\mathbb{R}^m}\int_K \varphi(t,z)\overline{N}_0(dt\times dz)\right]\\ &\qquad= E\left[E\left[\int_{\mathbb{R}^m}\int_K \varphi(t,z)\overline{M}'(dt\times dz)\middle|\mathcal{F}_0\right]\int_{\mathbb{R}^m}\int_K \varphi(t,z)\overline{N}_0(dt\times dz)\right]=0.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathrm{Var}\left(\int_{\mathbb{R}^m}\int_K \varphi(t,z)\overline{M}'(dt\times dz)+\int_{\mathbb{R}^m}\int_K \varphi(t,z)\overline{N}_0(dt\times dz)\right)=\\ &\quad\mathrm{Var}\left(\int_{\mathbb{R}^m}\int_K \varphi(t,z)\overline{M}'(dt\times dz)\right)+\mathrm{Var}\left(\int_{\mathbb{R}^m}\int_K \varphi(t,z)\overline{N}_0(dt\times dz)\right)=\\ &\qquad\lambda'\int_{\mathbb{R}^m} E\left[|\varphi(t,Z)|^2\right]dt+\mathrm{Var}\left(\int_{\mathbb{R}^m}\int_K \varphi(t,z)\overline{N}_0(dt\times dz)\right).\end{aligned}$$

On the other hand,

$$\operatorname{Var}\left(\int_{\mathbb{R}^m}\int_K \varphi(t,z)\overline{N}_0(dt\times dz)\right) = \int_{\mathbb{R}^m} |E[\widehat{\varphi}(\nu,Z)]|^2 \mu_0(d\nu) + \lambda_0 \int_{\mathbb{R}^m} \operatorname{Var}(\widehat{\varphi}(\nu,Z))\, d\nu.$$

Combining the above, we have

$$\begin{aligned}\operatorname{Var}\left(\int_{\mathbb{R}^m} f(t)N(dt)\right) &= \lambda' \int_{\mathbb{R}^m} E\left[|\widehat{\varphi}(\nu,Z)|^2\right] d\nu + \int_{\mathbb{R}^m} |E[\widehat{\varphi}(\nu,Z)]|^2 \mu_0(d\nu) \\ &\quad + \lambda_0 \int_{\mathbb{R}^m} \operatorname{Var}(\widehat{\varphi}(\nu,Z))\, d\nu \\ &= A + B + C.\end{aligned}$$

By Formula (36),

$$\begin{aligned} A &= \lambda' \int_{\mathbb{R}^m} |\widehat{f}(\nu)|^2 \frac{1+\operatorname{Var}\left(\hat{h}(\nu,Z)\right)}{|1-E\left[\hat{h}(\nu,Z)\right]|^2} d\nu, \\ B &= \int_{\mathbb{R}^m} |\widehat{f}(\nu)|^2 \frac{1}{|1-E\left[\hat{h}(\nu,Z)\right]|^2} \mu_0(d\nu), \\ C &= \int_{\mathbb{R}^m} |\widehat{f}(\nu)|^2 \lambda_0 \frac{\operatorname{Var}\left(\hat{h}(\nu,Z)\right)}{|1-E\left[\hat{h}(\nu,Z)\right]|^2} d\nu. \end{aligned}$$

Recalling that $\lambda' = \rho\lambda_0/(1-\rho)$, we obtain finally that for all $f \in L^1_{\mathbb{C}}(\mathbb{R}^m)\cap L^2_{\mathbb{C}}(\mathbb{R}^m)$,

$$\operatorname{Var}\left(\int_{\mathbb{R}^m} f(t)N(dt)\right) = \int_{\mathbb{R}^m} |\widehat{f}(\nu)|^2 \left(\frac{1}{|1-E\left[\hat{h}(\nu,Z)\right]|^2}\right)\left(\mu_0(d\nu) + \lambda' d\nu + \lambda \operatorname{Var}\left(\hat{h}(\nu,Z)\right) d\nu\right),$$

and this allows us to identify μ_N as (37).

Example 5. *Coxian ancestor process.* In the particular case where N_0 is a Cox process as in Example 2, we may take $B_N = L^1_{\mathbb{C}}(\mathbb{R}^m)\cap L^2_{\mathbb{C}}(\mathbb{R}^m)$.

Example 6. *The original Hawkes process.* In the particular case

$$h(t,Z) = h(t),$$

$\hat{h}(\nu,Z) = \hat{h}(\nu)$, and we have

$$\mu_N(d\nu) = \frac{1}{|1-\hat{h}(\nu)|^2}\left[\mu_0(d\nu) + \lambda' d\nu\right].$$

If in addition N_0 is a Poisson process with average intensity α, since $\alpha + \lambda' = \lambda$, we have the original formula of Hawkes

$$\mu_N(d\nu) = \frac{\lambda d\nu}{|1-\hat{h}(\nu)|^2}.$$

4.3. Spatial linear birth and death process

We consider a shot noise based on the Hawkes branching point process N of the previous section, defined by

$$X(t) = \sum_{s\in N} \alpha(t-s, Z(s)). \tag{38}$$

Note that its spectral characteristics cannot be derived from Theorem 2, since now the marks $Z(s)$ and the process N are not independent.

Example 7. In the univariate case $\mathbb{R}^m = \mathbb{R}$,

$$X(t) = \sum_{n\in\mathbb{Z}} \alpha(t-T_n, Z_n).$$

To further specialize this example, take

$$h(t,z) = \beta 1_{[0,z]}(t),$$

and

$$\alpha(t,z) = 1_{[0,z]}(t).$$

Therefore interpreting T_n as the birth time of individual n in colony, and Z_n as its lifetime,

$$X(t) = \sum_{n\in\mathbb{Z}} 1_{(-\infty,t]}(T_n) 1_{(t,+\infty)}(T_n + Z_n)$$

is the number of individuals in the colony. If moreover we assume that Z_n is exponentially distributed with parameter γ, and that the process N_0 of ancestors is Poisson with intensity λ_0, the process $\{X(t)\}$ is a Markov birth and death process with infinitesimal generator Q given by its non-null terms $q_{i,i+1} = \lambda_0 + \beta i$, $q_{i,i-1} = \gamma i$.

Theorem 4. *Consider the process $\{X(t)\}$ defined by (38), where $\alpha \in L^1_{\mathbb{C}}(\ell \times Q) \cap L^2_{\mathbb{C}}(\ell \times Q)$, and where the conditions stated in Theorem 3 are satisfied for N. Suppose moreover that the solution φ of (28) of Lemma 2 for $F(s,z) = \int_{\mathbb{R}^m} \alpha(t-s,z) f(t)\, dt$ is such that $E[\varphi(\cdot, Z)] \in B_{N_0}$ for any $f \in L^1_{\mathbb{C}}(\mathbb{R}^m)$. Then the Bochner spectral measure μ_X is given by the expression*

$$\left|1 - E\left[\hat{h}(\nu, Z)\right]\right|^2 \mu_X(d\nu) = \left|E\left[\widehat{\alpha}(\nu, Z)\right]\right|^2 \left(\mu_0(d\nu) + \frac{\rho\lambda_0}{1-\rho} d\nu\right)$$
$$\frac{\lambda_0}{1-\rho} \operatorname{Var}\left\{\widehat{\alpha}(\nu, Z)\left(1 - E\left[\hat{h}(\nu, Z)\right]\right) + \hat{h}(\nu, Z) E\left[\widehat{\alpha}(\nu, Z)\right]\right\} d\nu. \tag{39}$$

Proof. We seek a measure μ_X such that for all $f \in L^1_{\mathbb{C}}(\mathbb{R}^m)$

$$\operatorname{Var}\left(\int_{\mathbb{R}^m} f(t) X(t)\, dt\right) = \int_{\mathbb{R}^m} \left|\widehat{f}(\nu)\right|^2 \mu_X(d\nu). \tag{40}$$

But

$$\int_{\mathbb{R}^m} f(t) X(t)\, dt = \int_{\mathbb{R}^m} f(t) \left(\int_{\mathbb{R}^m} \int_K \alpha(t-s,z) \overline{N}(ds \times dz) \right) dt$$
$$= \int_{\mathbb{R}^m} \int_K F(s,z) \overline{N}(ds \times dz),$$

where

$$F(s,z) = \int_{\mathbb{R}^m} \alpha(t-s,z) f(t)\, dt$$
$$= \left(\breve{\alpha}(\cdot, z) * f \right)(s)$$

is a function in $L^1_{\mathbb{C}}(\ell \times Q) \cap L^2_{\mathbb{C}}(\ell \times Q)$ (because $\alpha \in L^1_{\mathbb{C}}(\ell \times Q) \cap L^2_{\mathbb{C}}(\ell \times Q)$, and $f \in L^1_{\mathbb{C}}(\mathbb{R}^m)$). Therefore we seek μ_X such that

$$\mathrm{Var} \left(\int_{\mathbb{R}^m} \int_K F(s,z) \overline{N}(ds \times dz) \right) = \int_{\mathbb{R}^m} \left| \widehat{f}(\nu) \right|^2 \mu_X(d\nu).$$

Following the same calculations as in the proof of Theorem 3 up to the 3rd displayed equation thereof, and letting φ be the unique solution in $L^1_{\mathbb{C}}(\ell \times Q) \cap L^2_{\mathbb{C}}(\ell \times Q)$ of equation (28) of Lemma 2, we have

$$\int_{\mathbb{R}^m} \int_K F(s,z) \overline{N}(ds \times dz) = \int_{\mathbb{R}^m} \int_K \varphi(t,z) \overline{M}'(dt \times dz) + \int_{\mathbb{R}^m} \int_K \varphi(t,z) \overline{N}_0(dt \times dz).$$

Resuming the proof of Theorem 3 after the 4th displayed equation thereof, we obtain

$$\mathrm{Var} \left(\int_{\mathbb{R}^m} f(t) X(t)\, dt \right) = \lambda' \int_{\mathbb{R}^m} E\left[\left| \widehat{\varphi}(\nu, Z) \right|^2 \right] d\nu + \int_{\mathbb{R}^m} \left| E\left[\widehat{\varphi}(\nu, Z) \right] \right|^2 \mu_0(d\nu)$$
$$+ \lambda_0 \int_{\mathbb{R}^m} \mathrm{Var}\left(\widehat{\varphi}(\nu, Z) \right) d\nu$$
$$= A + B + C$$

where, using the expression for $\widehat{\varphi}(\nu, z)$

$$\widehat{\varphi}(\nu, z) = \widehat{F}(\nu, z) + \frac{\hat{h}(\nu, z)^* E\left[\widehat{F}(\nu, Z) \right]}{1 - E\left[\hat{h}(\nu, Z)^* \right]}$$
$$= \widehat{f}(\nu) \left[\widehat{\alpha}(\nu, z)^* + \frac{\hat{h}(\nu, z)^* E\left[\widehat{\alpha}(\nu, Z)^* \right]}{1 - E\left[\hat{h}(\nu, Z)^* \right]} \right]$$

we find that

$$A = \lambda' \int_{\mathbb{R}^m} \left|\widehat{f}(\nu)\right|^2 \frac{E\left[\left|\widehat{\alpha}(\nu,Z)\left(1-E\left[\hat{h}(\nu,Z)\right]\right)+\hat{h}(\nu,Z)E\left[\widehat{\alpha}(\nu,Z)\right]\right|^2\right]}{\left|1-E\left[\hat{h}(\nu,Z)\right]\right|^2} d\nu,$$

$$B = \int_{\mathbb{R}^m} \left|\widehat{f}(\nu)\right|^2 \frac{\left|E\left[\widehat{\alpha}(\nu,Z)\right]\right|^2}{\left|1-E\left[\hat{h}(\nu,Z)\right]\right|^2} \mu_0(d\nu),$$

$$C = \lambda_0 \int_{\mathbb{R}^m} \left|\widehat{f}(\nu)\right|^2 \frac{\mathrm{Var}\left(\widehat{\alpha}(\nu,Z)\left(1-E\left[\hat{h}(\nu,Z)\right]\right)+\hat{h}(\nu,Z)E\left[\widehat{\alpha}(\nu,Z)\right]\right)}{\left|1-E\left[\hat{h}(\nu,Z)\right]\right|^2} d\nu.$$

Therefore, using the expression $\lambda' = \rho\lambda_0/(1-\rho)$, we find after rearrangement (40) with $\mu_X(d\nu)$ given by (39).

Example 8. Consider the set-up of the previous example, i.e. that $\{X(t)\}$ is a Markov birth and death process with non-null transition rates $q_{i,i+1} = \lambda_0 + \beta i$, $q_{i,i-1} = \gamma i$. We then have the following identifications: $\rho = \beta/\gamma$, $\mu_o(d\nu) = \lambda_0 d\nu$,

$$\widehat{\alpha}(\nu,z) = \frac{1-e^{-2i\pi\nu z}}{2i\pi\nu}, \quad \widehat{h}(\nu,z) = \beta\frac{1-e^{-2i\pi\nu z}}{2i\pi\nu}.$$

From this we obtain

$$E\left[\widehat{\alpha}(\nu,Z)\right] = \frac{1}{\gamma+2i\pi\nu},$$

$$\widehat{\alpha}(\nu,Z)\left(1-E\left[\hat{h}(\nu,Z)\right]\right)+\hat{h}(\nu,Z)E\left[\widehat{\alpha}(\nu,Z)\right] = \widehat{\alpha}(\nu,Z),$$

$$\mathrm{Var}\left(\left(1-E\left[\hat{h}(\nu,Z)\right]\right)+\hat{h}(\nu,Z)E\left[\widehat{\alpha}(\nu,Z)\right]\right) = \frac{1}{\gamma^2+4\pi^2\nu^2}.$$

Combined with Formula (39), this then yields the following formula:

$$\mu_X d\nu) = \frac{2\lambda_0}{(\gamma^2+4\pi^2\nu^2)(1-\beta/\gamma)}. \tag{41}$$

5. Modulated spike fields

5.1. Random sampling

Random sampling of a continuous time random signal $\{X(t)\}_{t\in\mathbb{R}}$, yields a sequence of samples

$$\{X(T_n)\}_{n\in\mathbb{Z}} \tag{42}$$

where $\{T_n\}_{n\in\mathbb{Z}}$, is the sequence of points (times of events) of a point process. At the extremities of the spectrum of randomness, we find the completely random sampling, or Poisson sampling, where $\{T_n\}_{n\in\mathbb{Z}}$, is a homogeneous Poisson process, and the regular sampling, with $T_n = nT$ where $T > 0$.

The signal $\{X(t)\}_{t\in\mathbb{R}}$, is called the *sampled signal*, the point process $\{T_n\}_{n\in\mathbb{Z}}$, the *sampler*, the sequence (42) is the *sample sequence*, and the process

$$Y(t) = \sum_{n\in\mathbb{Z}} X(T_n)\,\delta(t - T_n) \tag{43}$$

where $\delta(t)$ is the Dirac pseudo-function, is called the *sample comb.*

The sampled signal and the sampler are assumed independent, and stationary (or at least wide-sense stationary for the sampled process). However, we shall also consider dependent sampling.

The average intensity λ of the sampler is, by definition, the average number of samples per unit time, and the sampling frequency is then $\nu_s = \lambda$. Two well know results concern regular sampling and Poisson sampling, the two extremal cases.

Example 9. In regular sampling, the Fourier spectrum of the sample comb is an aliased version of that of the sampled signal. For instance, in the case of a power spectral density

$$f_Y(\nu) = \sum_{n\in\mathbb{Z}} f_X\left(\nu - \frac{n}{T}\right)$$

and the sampled signal can be entirely recovered from the sample comb provided the former is band-limited, with band width $2B < \nu_s = \frac{1}{T}$. It suffices to filter the sample comb with a low-pass of cutoff frequency B.

Example 10. In Poisson sampling, the Bartlett spectrum of the sample comb is (density case)

$$f_Y(\nu) = \lambda^2 f_X(\nu) + \lambda\sigma_X^2$$

where $\sigma_X^2 = \mathrm{Var}(X(t))$ is the power of the sampled signal. Therefore, whatever the sampling frequency $\nu_s = \lambda$, there is no aliasing, and the spectrum of the sampled signal can be recovered from that of the sample comb. However, if we apply the sample comb to a low-pass of cutoff frequency $\nu_s = \lambda$, the output signal, $\{Z(t)\}_{t\in\mathbb{R}}$, is the *worse* reconstruction of the sampled signal, assumed band-limited with bandwidth 2B, in the sense that

$$E\left[|Z(t) - X(t)|^2\right] = \sigma_X^2 .$$

We formulate random sampling in the general spatial case. Here the sampled signal is a wide-sense stationary process $\{X(t)\}_{t\in\mathbb{R}^m}$ with mean m_X, autocovariance function $C_X(\tau)$, power spectral measure μ_X, and Cramér-Khinchin's decomposition $\{Z_X(A)\}_{A\subseteq\mathcal{B}(\mathbb{R}^m)}$. Recall that the latter is a complex-valued stochastic process, with centered and orthogonal increments, and that $E\left[|Z_X(C)|^2\right] = \mu_X(C)$. Also, we have the Cramér-Khinchin decomposition

$$X(t) = \int_{\mathbb{R}^m} e^{2i\pi<\nu,t>} Z_X(d\nu) + m_X \tag{44}$$

where the integral thereof is a Wiener integral. Note that for all functions $g \in L^2_{\mathbb{C}}(\mu_X)$, the Wiener integral $\int_{\mathbb{R}^m} g(\nu) Z_X(d\nu)$ is well defined, and it is in $L^2_{\mathbb{C}}(\mathrm{P})$; moreover

$$E\left[\left|\int_{\mathbb{R}^m} g(\nu) Z_X(d\nu)\right|^2\right] = \int_{\mathbb{R}^m} |g(\nu)|^2 \mu_X(d\nu). \tag{45}$$

The sample "brush"

$$Y(t) = \sum_{s\in N} X(s)\,\delta(t-s)$$

is identified with the measure

$$\sum_{s\in N} X(s)\,\varepsilon_s(\cdot)\ , \tag{46}$$

where ε_s is the Dirac measure at s.We define the generalized Bochner spectrum of the sample brush to be a Radon measure $\mu_Y(d\nu)$ such that, for any $\varphi(t) \in B_Y$,

$$\mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(t)\,X(t)\,N(dt)\right) = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Y(d\nu)\ , \tag{47}$$

where B_Y is a large enough vector space of functions, here also called the "test functions". By "large enough", we mean that there cannot be two different Radon measures μ_Y verifying (47) for all $\varphi \in B_Y$. Observe that, since, formally,

$$\begin{aligned}\int_{\mathbb{R}^m} \varphi(t)\,Y(t)\,dt &= \int_{\mathbb{R}^m} \varphi(t)\left(\sum_{s\in N} X(s)\,\delta(t-s)\right) dt\\ &= \sum_{s\in N} \varphi(s)\,X(s)\\ &= \int_{\mathbb{R}^m} \varphi(t)\,X(t)\,N(dt)\ ,\end{aligned}$$

equality (47) becomes, formally,

$$\mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(t)\,Y(t)\,dt\right) = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Y(d\nu). \tag{48}$$

A first result concerns the situation when the sampler is independent from the signal. Let N be a wide-sense stationary simple point process on $\mathbb{R}^m$ with intensity $\lambda < \infty$, Bartlett spectrum μ_N with a vector space of test functions B_N.

Theorem 5. *Suppose that the real signal* $\{X(t)\}$ *and the point process* N *are independent. Then, the generalized process*

$$Y(t) = \sum_{s \in N} X(s)\, \delta(t-s)$$

admits the extended Bochner power spectral measure

$$\mu_Y = \mu_N * \mu_X + \lambda^2 \mu_X + |m_X|^2 \mu_N. \tag{49}$$

If B_N *is stable with respect to multiplications by complex exponential functions, we can take* $B_Y = B_N$.

Proof. Using (44), we have

$$\begin{aligned}\int_{\mathbb{R}^m} \varphi(t)\, X(t)\, N(dt) &= \int_{\mathbb{R}^m} \varphi(t) \left(\int_{\mathbb{R}^m} e^{2i\pi<\nu,t>} Z_X(d\nu) + m_X \right) N(dt) \\ &= \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} \varphi(t)\, e^{2i\pi<\nu,t>} N(dt) \right) Z_X(d\nu) \\ &\quad + m_X \int_{\mathbb{R}^m} \varphi(t)\, N(dt),\end{aligned}$$

where we have formally exchanged the order of integration. Since the integrals with respect to $N(dt)$ and with respect to $Z_X(d\nu)$ are of a different nature (one is a usual infinite sum, the other is a Wiener integral), this exchange must be formally justified, which we do after the proof. Using the conditional variance formula, we have

$$\begin{aligned}&\mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(t)\, X(t)\, N(dt) \right) \\ &\quad = E\left[\mathrm{Var}\left(\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} \varphi(t)\, e^{2i\pi<\nu,t>} N(dt) \right) Z_X(d\nu) + m_X \int_{\mathbb{R}^m} \varphi(t)\, N(dt) \middle| \mathcal{F}_\infty^N \right) \right] \\ &\quad + \mathrm{Var}\left(E\left[\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} \varphi(t)\, e^{2i\pi<\nu,t>} N(dt) \right) Z_X(d\nu) + m_X \int_{\mathbb{R}^m} \varphi(t)\, N(dt) \middle| \mathcal{F}_\infty^N \right] \right) = \alpha + \beta.\end{aligned}$$

Observe that, since $\varphi \in L^2(M_2)$,

$$\left| \int_{\mathbb{R}^m} \varphi(t)\, e^{2i\pi<\nu,t>} N(dt) \right|^2 \le \left| \int_{\mathbb{R}^m} |\varphi(t)|\, N(dt) \right|^2 < \infty\,,\ P - \text{a-s} \tag{50}$$

Using the fact that, when N is fixed, $m_X \int_{\mathbb{R}^m} \varphi(t) N(dt)$ is deterministic,

$$\begin{aligned}
\alpha &= E\left[\operatorname{Var}\left(\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\varphi(t)\,e^{2i\pi<\nu,t>}N(dt)\right)Z_X(d\nu)\middle|\mathcal{F}_\infty^N\right)\right] \\
&= E\left[\int_{\mathbb{R}^m}\left|\int_{\mathbb{R}^m}\varphi(t)\,e^{2i\pi<\nu,t>}N(dt)\right|^2\mu_X(d\nu)\right] \text{(by eq. (45) and (50))} \\
&= \int_{\mathbb{R}^m} E\left[\left|\int_{\mathbb{R}^m}\varphi(t)\,e^{2i\pi<\nu,t>}N(dt)\right|^2\right]\mu_X(d\nu) \\
&= \int_{\mathbb{R}^m}\left(\operatorname{Var}\int_{\mathbb{R}^m}\varphi(t)\,e^{2i\pi<\nu,t>}N(dt)+\left|\mathrm{E}\int_{\mathbb{R}^m}\varphi(t)\,e^{2i\pi<\nu,t>}N(dt)\right|^2\right)\mu_X(d\nu) \\
&= \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}|\widehat{\varphi}(x-\nu)|^2\mu_N(dx)+\left|\int_{\mathbb{R}^m}\varphi(t)\,e^{2i\pi<\nu,t>}\lambda dt\right|^2\right)\mu_X(d\nu) \quad (\text{ hypothesis on } B_N) \\
&= \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}|\widehat{\varphi}(x-\nu)|^2\mu_N(dx)\right)\mu_X(d\nu)+\lambda^2\int_{\mathbb{R}^m}|\widehat{\varphi}(-\nu)|^2\mu_X(d\nu) \\
&= \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}|\widehat{\varphi}(x+\nu)|^2\mu_N(dx)\right)\mu_X(d\nu)+\lambda^2\int_{\mathbb{R}^m}|\widehat{\varphi}(+\nu)|^2\mu_X(d\nu) \\
&= \int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2(\mu_N*\mu_X)(d\nu)+\lambda^2\int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2\mu_X(d\nu)
\end{aligned}$$

and, since $E\left[\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\varphi(t)\,e^{2i\pi<\nu,t>}N(dt)\right)Z_X(d\nu)\middle|\mathcal{F}_\infty^N\right]=0$,

$$\beta = \operatorname{Var}\left(m_X\int_{\mathbb{R}^m}\varphi(t)\,N(dt)\right) = |m_X|^2\int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2\mu_N(d\nu) \ \ (\text{since } \varphi\in B_N).$$

Finally,

$$\operatorname{Var}\left(\int_{\mathbb{R}^m}\varphi(t)\,Y(t)\,dt\right) = \int_{\mathbb{R}^m}|\widehat{\varphi}(\nu)|^2\left(\mu_N*\mu_X+\lambda^2\mu_X+|m_X|^2\mu_N\right)(d\nu),$$

that is, $\{Y(t)\}_{t\in\mathbb{R}^m}$ admits an extended Bochner spectral measure given by equation (49).

It now remains to validate the exchange of integrals perpetrated at the begining of the proof.

Lemma 3. *Let N be a simple locally bounded stationary point process defined on $\mathbb{R}^m$ and admitting a Bartlett spectrum μ_N. Let M_2 be its second moment measure. Let $\{X(t)\}_{t\in\mathbb{R}^m}$ be a w.s.s. random field with Cramér decomposition Z_X and power spectral measure μ_X. Then, for all $\varphi\in L^2(M_2)$*

$$\int_{\mathbb{R}^m}\varphi(t)\,X(t)\,N(dt) = \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m}\varphi(t)\,e^{2i\pi<\nu,t>}N(dt)\right)Z_X(d\nu). \tag{51}$$

Proof. We do the proof in the univariate case. The multivariate case follows the same lines, with more notation. The left-hand side of (51) is

$$A = \sum_{n\in\mathbb{Z}}\varphi(T_n)\,X(T_n) = \lim_{c\uparrow\infty}\sum_{n\in\mathbb{Z}}\varphi(T_n)\,X(T_n)\,1_{[-c,+c]}(T_n) = \lim_{c\uparrow\infty}A(c)$$

where the limit is in $L^1(P)$. Indeed

$$E[|A-A(c)|] \le E\left[\int_{[-c,+c]} |\varphi(t)X(t)|\,N(dt)\right]$$
$$= \int_{[-c,+c]} |\varphi(t)|\,E[|X(t)|]\,\lambda dt \le \lambda K \int_{[-c,+c]} |\varphi(t)|\,dt$$

where $K = \sup_t E[|X(t)|] < \infty$ (by Schwarz's inequality, $E[|X(t)|] \le E\left[|X(t)|^2\right]^{\frac{1}{2}} = E\left[|X(0)|^2\right]^{\frac{1}{2}}$). Therefore, since $\varphi \in L^1$, $\lim_{c\uparrow\infty} \mathrm{E}[|A-A(c)|] = 0$.

The right-hand side is

$$B = \lim_{c\uparrow\infty} \int_{\mathbb{R}} \left(\int_{[-c,+c]} \varphi(t)\,e^{2i\pi<\nu,t>} N(dt)\right) Z_X(d\nu) = \lim_{c\uparrow\infty} B(c)$$

where the limit is in $L^2(P)$. Indeed

$$E\left[|B-B(c)|^2\right] = E\left[\left|\int_{\mathbb{R}} \left(\int_{[-c,+c]} \varphi(t)\,e^{2i\pi\nu t} N(dt)\right) Z_X(d\nu)\right|^2\right]$$
$$= E\left[E\left[\left|\int_{\mathbb{R}} \left(\int_{[-c,+c]} \varphi(t)\,e^{2i\pi\nu t} N(dt)\right) Z_X(d\nu)\right|^2 \middle| \mathcal{F}_\infty^N\right]\right]$$
$$= E\left[\int_{\mathbb{R}} \left|\int_{[-c,+c]} \varphi(t)\,e^{2i\pi\nu t} N(dt)\right|^2 \mu_X(d\nu)\right].$$

Denote $\varphi_c(t) = \varphi(t)\,1_{[-c,c]}(t)$. Then

$$\mathrm{E}\left[\int_{\mathbb{R}} \left|\int_{\mathbb{R}} \varphi_c(t)\,e^{2i\pi\nu t} N(dt)\right|^2 \mu_X(d\nu)\right] = \int_{\mathbb{R}} \mathrm{E}\left[\left|\int_{\mathbb{R}} \varphi_c(t)\,e^{2i\pi\nu t} N(dt)\right|^2\right] \mu_X(d\nu).$$

But

$$\mathrm{E}\left[\left|\int_{\mathbb{R}} \varphi_c(t)\,e^{2i\pi\nu t} N(dt)\right|^2\right] \le \mathrm{E}\left[\left(\int_{\mathbb{R}} |\varphi_c(t)|\,N(dt)\right)^2\right]$$
$$= \int_{\mathbb{R}\times\mathbb{R}} |\varphi_c(t)|\,|\varphi_c(s)|\,M_2(dt\times ds),$$

a quantity that tends to 0 as $c \uparrow \infty$, by dominated convergence. Dominated convergence applied to the *finite* measure μ_X then yields the desired L^2 convergence.

But

$$\begin{aligned} A(c) &= \sum_{n\in\mathbb{Z}} \varphi(T_n) X(T_n) 1_{[-c,+c]} \\ &= \sum_{n\in\mathbb{Z}} \varphi(T_n) \left(\int_{\mathbb{R}} e^{2i\pi\nu T_n} Z_X(d\nu) \right) 1_{[-c,+c]}(T_n) \\ &= \int_{\mathbb{R}} \left(\sum_{n\in\mathbb{Z}} \varphi(T_n) e^{2i\pi\nu T_n} 1_{[-c,+c]}(T_n) \right) Z_X(d\nu) \\ &= B(c), \end{aligned}$$

where we have used the fact that the sums involved are finite. Thus

$$\lim_{c\uparrow\infty} A(c) = \begin{cases} A & \text{in } L^1 \\ B & \text{in } L^2 \end{cases}$$

from which it follows that $A = B$, a.s. (use the fact that if a sequence of random variables converges in L^1 or L^2 to some r.v., one can extract a subsequence that converges a.s. to the same r.v.).

Example 11. Let N be a Cox process with a wide-sense stationary intensity $\{\lambda(t)\}_{t\in\mathbb{R}^m}$ with Bochner spectrum μ_λ. Then

$$\mu_Y = \mu_\lambda * \mu_X + \lambda^2 \mu_X + |m_X|^2 \mu_\lambda + \lambda R_X(0) \ell^m \tag{52}$$

where ℓ^m is the Lebesgue measure. In particular, when the spike field is a homogeneous Poisson process,

$$\mu_Y = \lambda^2 \mu_X + \lambda R_X(0) \ell^m. \tag{53}$$

5.2. Reconstruction of the signal from its sampled version

We now consider the problem of approximating $X(t)$ by a filtered version of $\{Y(t)\}$

$$\int_{\mathbb{R}^m} \varphi(t-s) Y(s) \, ds$$

where $\varphi \in \mathrm{L}^1 \cap \mathrm{L}^2$.

The article [21] gives motivation to this kind of problem in Geophysics.

The difference between $X(t)$ and its approximation, that is, the reconstruction error, is measured by

$$\epsilon = \mathrm{E}\left[\left| \int_{\mathbb{R}^m} \varphi(t-u) Y(u) \, du - X(t) \right|^2 \right].$$

Then, we have the following expression for the reconstruction error.

Theorem 6. *Reconstructing the signal $\{X(t)\}_{t\in\mathbb{R}}$ by filtering the sample comb $\{Y(t)\}_{t\in\mathbb{R}}$ with a filter $\varphi \in \mathrm{L}^1 \cap \mathrm{L}^2$ gives the following error*

$$\epsilon = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Y(d\nu) + \int_{\mathbb{R}^m} \mu_X(d\nu) - \lambda \int_{\mathbb{R}^m} (\widehat{\varphi}(\nu) + \widehat{\varphi}^*(\nu)) \mu_X(d\nu) + |m_X|^2 |1 - \lambda\widehat{\varphi}(0)|^2 . \tag{54}$$

Proof. We have

$$\begin{aligned}
\epsilon &= \mathrm{E}\left[\left|\int_{\mathbb{R}^m} \varphi(t-u) X(u) N(du) - X(t)\right|^2\right] \\
&= \mathrm{E}\left[\left|\int_{\mathbb{R}^m} \varphi(t-u) X(u) N(du)\right|^2\right] - 2\Re\left\{\mathrm{E}\left[\int_{\mathbb{R}} \varphi(t-u) X(t) X(u) N(du)\right]\right\} + \mathrm{E}\left[|X(t)|^2\right] \\
&= A - 2\Re\{B\} + C.
\end{aligned}$$

In this expression,

$$A = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Y(d\nu) + \lambda^2 |m_X|^2 \left|\int_{\mathbb{R}^m} \varphi(t)\, dt\right|^2 = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Y(d\nu) + \lambda^2 |m_X|^2 |\widehat{\varphi}(0)|^2 ,$$

$$\begin{aligned}
B = \mathrm{E}\left[\int_{\mathbb{R}^m} \varphi(t-u) X(t) X(u) N(du)\right] &= \lambda \int_{\mathbb{R}^m} \varphi(t-u) R_X(t-u)\, du \\
&= \lambda \int_{\mathbb{R}^m} \varphi(t) R_X(t)\, dt \\
&= \lambda \int_{\mathbb{R}^m} \varphi(t) C_X(t)\, dt + \lambda |m_X|^2 \int_{\mathbb{R}^m} \varphi(t)\, dt \\
&= \lambda \int_{\mathbb{R}^m} \widehat{\varphi}(\nu) \mu_X(d\nu) + \lambda |m_X|^2 \widehat{\varphi}(0),
\end{aligned}$$

and

$$C = \int_{\mathbb{R}^m} \mu_X(d\nu) + |m_X|^2 .$$

Therefore

$$\begin{aligned}
\mathrm{E}\left[\left|\int_{\mathbb{R}^m} \varphi(t-u) X(u) N(du) - X(t)\right|^2\right] &= \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Y(d\nu) \\
&- \lambda \int_{\mathbb{R}^m} \left(\widehat{\varphi}(\nu) + \widehat{\varphi}(\nu)^*\right) \mu_X(d\nu) \\
&+ |m_X|^2 \left(1 - \lambda\left(\widehat{\varphi}(0) + \widehat{\varphi}(0)^*\right) + \lambda^2 |\widehat{\varphi}(0)|^2\right) + \int_{\mathbb{R}^m} \mu_X(d\nu)
\end{aligned}$$

In particular, in the case $m_X = 0$ the error is

$$\epsilon = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Y(d\nu) - 2\lambda\Re\left\{\int_{\mathbb{R}^m} \widehat{\varphi}(\nu) \mu_X(d\nu)\right\} + \mu_X(\mathbb{R}^m). \tag{55}$$

We now give some examples of reconstruction error for different sampling schemes. For notation ease, we consider that the signal is centered, that is, $m_X = 0$. Moreover, some parts of the examples are developed in the univariate case. We develop the computations in the "classical" situation of a band-limited signal $X(t)$, filtered with a band-limited (low-pass) filter $\varphi(\nu)$. More precisely, let S be the support of μ_X, with length $2B = \ell(S)$, then, we consider

$$\widehat{\varphi}(\nu) = \begin{cases} \frac{1}{\lambda} & \text{on } S \\ 0 & \text{otherwise} \end{cases}$$

where λ is the intensity of the spike comb.

Example 12. When N is a homogeneous Poisson process with intensity λ, μ_Y is given by (53) and then the error is

$$\epsilon = \int_{\mathbb{R}^m} |\lambda\widehat{\varphi}(\nu) - 1|^2 \mu_X(d\nu) + \lambda C_X(0) \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 (d\nu). \tag{56}$$

In the "classical" band-limited case described above, we have

$$\begin{aligned} \epsilon &= \lambda C_X(0) \int_{\mathbb{R}} |\widehat{\varphi}(\nu)|^2 (d\nu) \\ &= \lambda C_X(0) \int_{\mathbb{R}} \frac{1}{\lambda^2} 1_S(\nu)\, d\nu, \end{aligned}$$

that is

$$\epsilon = C_X(0) \frac{2B}{\lambda}\cdot$$

Therefore, sampling at the Nyquist rate $\lambda = 2B$ gives very poor performances, not better than the estimate based on no observation at all.

This does not mean, however, that below the rate $\lambda = 2B$, there is no information (or in a sense as the result suggests "negative information") concerning the process itself contained in its samples. A better choice of a filter would indeed give a linear estimate with error less than $\sigma^2 = C_X(0)$. For instance, if we let $\widehat{\varphi}$ be real, we find for the error

$$\epsilon = \int_{\mathbb{R}} \left[(\lambda\widehat{\varphi}(\nu) - 1)^2 f_X(\nu) + \lambda\sigma^2 \widehat{\varphi}(\nu)^2 \right] d\nu,$$

where it is assumed that $\{X(t)\}_{t\in\mathbb{R}}$ has the power spectral density $f_X(\nu)$. The minimum occurs for

$$\widehat{\varphi}(\nu) = \frac{\lambda f_X(\nu)}{\lambda^2 f_X(\nu) + \lambda\sigma^2}$$

and then

$$\epsilon = \sigma^2 \left(1 - \int_{\mathbb{R}} \frac{\lambda \widetilde{f}_X(\nu)}{1 + \lambda \widetilde{f}_X(\nu)} \widetilde{f}_X(\nu)\, d\nu \right)\cdot$$

where $\widetilde{f}_X(\nu)$ is the normalized power spectral density

$$\widetilde{f}_X(\nu) = \frac{f_X(\nu)}{\int_{\mathbb{R}} f_X(\nu')\,d\nu'} = \frac{f_X(\nu)}{\sigma^2}.$$

Therefore $\epsilon = \sigma^2(1-\rho)$ where $\rho = \int_{\mathbb{R}} \frac{\lambda \widetilde{f}_X(\nu)}{1+\lambda \widetilde{f}_X(\nu)} \widetilde{f}_X(\nu)\,d\nu$ can be interpreted as the correlation coefficient between $X(t)$ and N for fixed t.

Example 13. When the sampled comb is derived by T-uniform sampling, the reconstruction error (55) reads

$$\begin{aligned}\epsilon &= \frac{1}{T^2}\int_{\mathbb{R}} |\widehat{\varphi}(\nu)|^2 \mu_X(d\nu) - \frac{2}{T}\Re\left(\int_{\mathbb{R}} \widehat{\varphi}(\nu)\,\mu_X(d\nu)\right) + \int_{\mathbb{R}} \mu_X(d\nu) \\ &= \int_{\mathbb{R}} \left|\frac{1}{T}\widehat{\varphi}(\nu) - 1\right|^2 \mu_X(d\nu).\end{aligned} \tag{57}$$

In the band-limited case, if we consider $T = 1/2B$, that is, $\lambda = 2B$, equation (57) gives an error equal to zero. Therefore, the signal is perfectly reconstructed by

$$\begin{aligned}X(t) &= \int_{\mathbb{R}} \varphi(t-s)\,X(s)\,N(ds) \\ &= \sum_{n\in\mathbb{Z}} X(T_n)\,\mathrm{sinc}(t-T_n),\end{aligned}$$

where $\mathrm{sinc}(t) = \sin(2\pi Bt)/(2\pi Bt)$, which is the usual reconstruction formula (Shannon–Nyquist theorem).

Example 14. The reconstruction error from uniform samples in the presence of jitter is obtained plugging the extended Bochner spectrum μ_Y with μ_N corresponding to a jittered uniform grid (see Example 1 and Corollary 2) into the error formula (55). The previous example showed that within the "classical" sampling framework the signal may be perfectly reconstructed. Now, in the presence of jitter this is not possible and the reconstruction error is given by

$$\epsilon = \frac{1}{2B}\left(\int_{-B}^{B} \sigma^2\left(1 - \left(|\psi_Z|^2 * \widetilde{f}_X\right)(\nu)\right) d\nu\right) \tag{58}$$

where $\widetilde{f}_X$ is the normalized power spectral density of the signal $X(t)$.

5.3. Signal dependent rate of sampling

We consider the case where the sampling rate depends on the process. The model for the sampler is now a Cox process [12] on $\mathbb{R}^m$ with the conditional (w.r.t. X) intensity of the form

$$\lambda(t) = \lambda(t, X).$$

For instance, in the univariate case, $\lambda(t) = |X(t)|^2$, $\lambda(t) = \left|\dot{X}(t)\right|^2$ where $\dot{X}$ is the derivative at t of $t \to X(t)$. More complicated functionals can be considered.

Theorem 7. *Assume that* $\mathrm{E}\left[X(t)^2 \lambda(t,X)^2\right] < \infty$, $\forall t \in \mathbb{R}^m$, *and that* $\{\lambda(t)\}_{t\in\mathbb{R}^m}$ *is a locally integrable process. Let* μ_Z *be the power spectrum of the stationary process*

$$Z(t) = X(t)\lambda(t).$$

Then,

$$\mu_Y(d\nu) = \mu_Z(d\nu) + \overline{X^2\lambda}d\nu \tag{59}$$

where we have denoted $\overline{X^2\lambda} = \mathrm{E}\left[X(t)^2\lambda(t)\right]$ *(independent of* t*).*

Proof. In order to compute the Bartlett spectrum of $Y(t)$, we have, as in the independent case, to evaluate the variance of

$$\int_{\mathbb{R}^m} \varphi(t) Y(t)\,dt = \int_{\mathbb{R}^m} \varphi(t) X(t) N(dt)$$

for all $\varphi \in \mathrm{L}^1 \cap \mathrm{L}^2$. It holds that

$$\begin{aligned}
\mathrm{Var}&\left(\int_{\mathbb{R}^m} \varphi(t) X(t) N(dt)\right) = \\
&\mathrm{E}\left[\mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(t) X(t) N(dt)\,|X\right)\right] + \mathrm{Var}\left(\mathrm{E}\left[\int_{\mathbb{R}^m} \varphi(t) X(t) N(dt)\,|X\right]\right) \\
&\qquad = \mathrm{E}\left[\int_{\mathbb{R}^m} |\varphi(t)|^2 |X(t)|^2 \lambda(t,X)\,dt\right] + \mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(t) X(t) \lambda(t,X)\,dt\right).
\end{aligned}$$

By definition of μ_Z, we have

$$\mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(u) X(u) \lambda(u)\,du\right) = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Z(d\nu).$$

Therefore, recalling the notation $\mathrm{E}\left[X(t)^2\lambda(t)\right] = \overline{X^2\lambda}$ (independent of t), we have

$$\begin{aligned}
\mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(t) X(t) N(dt)\right) &= \mathrm{Var}\left(\int_{\mathbb{R}^m} \varphi(u) X(u) \lambda(u)\,du\right) + \overline{X^2\lambda}\int_{\mathbb{R}^m} |\varphi(t)|^2\,dt \\
&= \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Z(d\nu) + \overline{X^2\lambda}\int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2\,d\nu \\
&= \int_{\mathbb{R}^m} \widehat{\varphi}(\nu)\left(\mu_Z(d\nu) + \overline{X^2\lambda}d\nu\right)
\end{aligned}$$

and result (59) follows.

As particular cases of the above result, for $X(t) \equiv 1$, we recover the formula

$$\mu_N(d\nu) = \mu_\lambda(d\nu) + \lambda d\nu$$

of the Bartlett spectrum of the Cox process, and for $X(t) = \lambda(t)$, we have

$$\mu_Y(d\nu) = \mu_{\lambda^2}(d\nu) + \mathrm{E}\left[\lambda^3(0)\right] d\nu.$$

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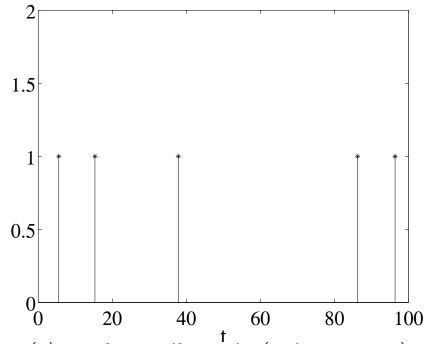
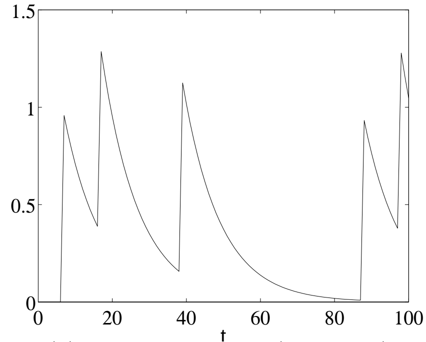
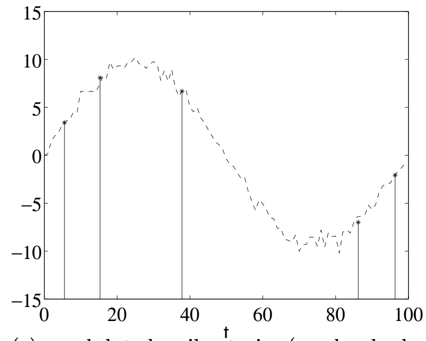


(a) random spike train (point process)



(b) filtered spike train (shot noise)



(c) modulated spike train (randomly located samples)

FIGURE 1: Random spike train and related processes