Connectivity in ad-hoc and hybrid networks

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***Abstract*—We consider a large-scale wireless network, but with a low density of nodes per unit area. Interferences are then less critical, contrary to connectivity. This paper studies the latter property for both a purely ad-hoc network and a hybrid network, where fixed base stations can be reached in multiple hops. We assume here that power constraints are modeled by a maximal distance above which two nodes are not (directly) connected.**

We find that the introduction of a sparse network of base stations does significantly help in increasing the connectivity, but only when the node density is much larger in one dimension than in the other. We explain the results by percolation theory. We obtain analytical expressions of the probability of connectivity in the 1-dim. case. We also show that at a low spatial density of nodes, bottlenecks are unavoidable. Results obtained on actual population data confirm our findings.

I. Introduction

Early enthusiasm for large-scale ad-hoc networks, which would eliminate the need for any fixed infrastructure, has been recently dampened by their reduced capacity when the density of nodes per area unit [5], which we denote here by λ, becomes large. Interferences and contention for medium access are the cause of this fundamental limitation. The introduction of base stations can overcome this limitation, and fortunately, most highly populated areas, such as cities, are covered by cellular wireless networks. Another approach is to rely on multi-user diversity, such as mobility, which can improve substantially the capacity of the network [7]. Connectivity on the other hand is not a problem: when $\lambda \to \infty$, and when the distance $r(\lambda)$ below which nodes can connect, decreases at a rate slower than $\sqrt{\log \lambda / \lambda}$, Gupta and Kumar have proven that all nodes are almost surely connected [6].

The situation is reversed when the nodes density λ becomes small. Power constraints become then a serious impediment [11]. If the maximal radius r ensuring direct communication between two nodes is too small, some nodes will be disconnected. Unfortunately, those geographical areas, such as rural areas which are not much populated, tend to be poorly covered by cellular technology, because of cost limitations. One is therefore lead to investigate whether the much more economical solution where nodes connect to a base station in more than one hop, using other relaying nodes, is feasible when the number of these nodes per unit area, λ, is small, and when power limits the maximal radius r ensuring connectivity in one hop.

This paper does address this question, for the Poisson Boolean model, which is described in more detail in Section II.

We begin with a purely ad-hoc network. We first examine the probability that a pair of arbitrary nodes are connected. This problem is strongly related to the field of percolation theory, which was already applied in the proof in [6], in a different context. The percolation probability is the probability that an arbitrary node belongs to a cluster of infinite size. The main result of percolation theory is that there exists a finite, positive value λ_c of λ, under which the percolation probability is zero (sub-critical phase) and above which it is non zero (super-critical phase). We compute (analytically in 1-dim and numerically in 2-dim) the connection probability in Section III for three geometries of the domain on which nodes are scattered: a line, a plane and an infinite strip of finite width.

Although in the super-critical phase, there are almost always a large number alternate paths between two distant nodes belonging to the infinite cluster, we find in Section IV that the size of the minimal cut-set encircling an arbitrary node may be reached far away from the node, unless λ is quite large. We formalize the resulting structure of the network, which appears as a set of islands connected by a few links. Nodes within each island are connected via multiple routes, but links between islands form "hot spots" limiting the number of alternate paths between two nodes located in different islands.

We then move to a hybrid network, and investigate the benefit of having a sparse network of base stations regularly placed in the network in Section V. Base stations are supposed to be all connected to each other by a wired network, so that two nodes connected to base stations can always communicate, no matter the distance that separates them. Because of percolation, we will see that there is no benefit in terms of connectivity in the super-critical case. We also show that the benefit in the sub-critical case remains marginal, unless the nodes spatial distribution is

close to 1-dim. In the latter case, we obtain a bound on the distance between base stations guaranteeing a prescribed probability of connectivity for any node.

Finally, Section VI shows that the above results hold not only for a Poisson point process, but for more general point processes as well. We show on real data from the Swiss population density, that base stations are needed to guarantee connectivity in elongated geographic areas where population is concentrated in a narrow space, such as Alpine valleys, and not in wide areas where population is well scattered in 2-dim, such as the city and suburbs of Zurich.

II. Models

A. Mathematical model

To describe the random network, we use a Poisson Boolean model $\mathcal{B}(\lambda, r)$, where the positions of the nodes are distributed according to a Poisson point process of constant, finite intensity λ in $\mathbb{R}^d$, with $d = 1$ or 2. We associate to each node a closed ball of fixed radius $r/2$, as shown in Figure 1. The plane is thus partitioned into two regions: the *occupied* region covered by the balls, and the *vacant* region $\mathcal{V}$, which is the complement of the occupied region.

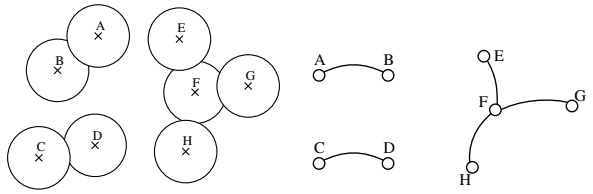

Fig. 1. The Boolean model (left) and the associated graph (right).

Two nodes are *directly connected* or *immediate neighbors* if the intersection of their associated balls is non-empty. In other words, we assume in this model that two nodes are able to communicate together through a wireless channel if the distance between them is less than a characteristic *range* r. A *cluster* is a connected component of the occupied region. Finally, two nodes are said to be *connected* together if they belong to the same cluster.

Furthermore, one can associate with the random model $\mathcal{B}(\lambda, r)$ the graph $\mathcal{G}(\lambda, r)$ by associating a vertex to each node of $\mathcal{B}(\lambda, r)$ and an edge with each direct connection in $\mathcal{B}(\lambda, r)$. $\mathcal{G}(\lambda, r)$ is called the *associated graph* of $\mathcal{B}(\lambda, r)$.

In this paper, we consider the simple case where r is fixed (it is the maximal radius allowed by power constraints). The case where r is randomly chosen for each node would be interesting as it leads to small-world type of associated graphs [13]. Most of the present results extend to this case, but we leave them for further work.

B. Simulation model

As an infinite network cannot be handled by a computer, simulations were performed on a finite domain of 65536×65536 pixels. The maximal range r is measured in pixels, whereas the density λ of the Poisson process is measured in pixels^{-2}. For practical reasons, we vary sometimes the range r instead of varying λ. Indeed, each realization of the model $\mathcal{B}(\lambda, r)$ can be scaled by a ratio r/r' in the d-dim space, to coincide with a realization of model $\mathcal{B}(\lambda', r')$ with $\lambda' = (r/r')^d \lambda$, so that their associated graphs are identical: $\mathcal{G}(\lambda, r) = \mathcal{G}(\lambda', r')$. All connectivity properties can be reformulated by a suitable scaling ([9] pp 30-31).

In the various plots in the paper, the experimental average value of the quantity of interest is plotted together with vertical bars delimiting the 99% confidence interval.

III. Connectivity of a pure ad-hoc network

We begin with the connectivity of the Poisson Boolean model. More specifically, our goal is to evaluate the probability $P_c(x)$ that two arbitrary nodes A and B, whose Euclidean distance in $\mathbb{R}^d$ is denoted by $d(A, B) = x$, are connected to each other, as a function of λ, r and x.

A. One-dimensional case : line

We begin with the 1-dim. case, where two nodes at a distance x of each other are connected if the entire interval between them is occupied. If there is a vacant region, the information cannot be relayed from a node to the other. A hole appears if the interval between two consecutive nodes is longer than r. As $\lambda < \infty$, the probability for a vacant interval to occur between two consecutive nodes is strictly positive, whatever the value of λ and r. Since there is an infinite number of intervals, and since their lengths are independent, the probability of having no hole is zero. Hence $\lim_{x\to\infty} P_c(x) = 0$.

Related results include the computation that a given number of nodes on a finite interval are all connected to each other [12]. A recursive formula giving the average number of hops between two connected nodes is given in [3].

Here we compute the probability distribution $P_c(x)$ for an infinite domain. Clearly, $P_c(x) = 1$ if $0 \le x < r$. When $x \ge r$, $P_c(x)$ is the probability that there is another node between A and B, located at a distance ξ from B, with $0 \le \xi < r$, and that this node is connected to A. We use this argument to condition $P_c(x)$ recursively on $P_c(x-\xi)$, with $0 \le \xi < r$, and we establish the following result.

Theorem 1: The probability $P_c(x)$ that two nodes dis-

tant of x space units are connected is

$$P_c(x) = \begin{cases} 1 & \text{if } 0 \le x < r \\ \sum_{i=0}^{\lfloor x/r \rfloor} \frac{\left(-\lambda e^{-\lambda r}(x-ir)\right)^i}{i!} & \\ \quad -e^{-\lambda r} \sum_{i=0}^{\lfloor x/r \rfloor - 1} \frac{\left(-\lambda e^{-\lambda r}(x-(i+1)r)\right)^i}{i!} & \\ & \text{if } x \ge r \end{cases} \tag{1}$$

with $\lfloor x/r \rfloor$ denoting the largest integer smaller than or equal to x/r.

Proof: If $x \ge r$, $P_c(x)$ is the probability that (i) the next node towards Node A is located at a distance ξ from Node B, with $0 \le \xi < r$, and that (ii) this new node is connected to A. This second event occurs with probability $P_c(x-\xi)$. The interval separating two consecutive nodes has an exponentially distributed length, whose pdf is thus $\lambda e^{-\lambda \xi}$ for $0 \le \xi < x$. Therefore

$$\begin{aligned} P_c(x) &= \int_0^r P_c(x-\xi)\lambda e^{-\lambda\xi} d\xi \qquad (2) \\ &= \lambda e^{-\lambda x} \int_{x-r}^{x} P_c(y) e^{\lambda y} dy. \end{aligned}$$

Taking the derivative of this expression with respect to x, we obtain

$$\frac{dP_c}{dx}(x) = -\lambda e^{-\lambda r} P_c(x-r) \tag{3}$$

for $x \ge r$. This is a first order linear delay differential equation, with initial condition defined on interval $[0, r]$. We know that $P_c(x) = 1$ for $0 \le x < r$. At $x = r$ however, $P_c(x)$ has a discontinuity: indeed, solving (2) for $x = r$, we find $P_c(r) = 1 - e^{-\lambda r}$. Integrating (3) with this initial condition on $[0, r]$, we obtain, after some manipulations, (1). ∎

The following bounds on $P_c(x)$ will be needed in Section V.

Theorem 2: If $x \ge 2r$,

$$\begin{aligned} \left(1 - e^{-\lambda r}\right) e^{-\lambda(x-2r)e^{-\lambda r}} - \lambda e^{-\lambda r} \le \\ P_c(x) \le \left(1 - e^{-\lambda r}\right) e^{-\lambda(x-r)e^{-\lambda r}}. \end{aligned} \tag{4}$$

Proof: Since $P_c(x)$ is a decreasing function of x, $P_c(x) \le P_c(x-r)$ for any $x \ge r$. Inserting this inequality in (3), we get

$$\frac{dP_c}{dx}(x) \le -\lambda e^{-\lambda r} P_c(x) \tag{5}$$

for $x \ge r$. Since $P_c(r) = 1 - e^{-\lambda r}$, we have from the theory of Gronwall-type differential inequalities (see e.g [8], Chap. 1) that the solution of (3) is upper bounded by the solution of the linear ode (5), (with the inequality sign replaced by an equality sign), which is

$$P_c(x) \le \left(1 - e^{-\lambda r}\right) e^{-\lambda(x-r)e^{-\lambda r}} \tag{6}$$

for all $x \ge r$. This is the upper bound in (4).

To obtain a lower bound, let us note that (6) provides an upper bound on $P_c(x-r)$, by replacing x by $(x-r)$ in this expression. Inserting it in (3), we get

$$\frac{dP_c}{dx}(x) \ge -\lambda e^{-\lambda r}\left(1 - e^{-\lambda r}\right) e^{-\lambda(x-2r)e^{-\lambda r}}$$

which we integrate for $x \ge 2r$ to obtain

$$P_c(x) \ge -\lambda r e^{-\lambda r} + \left(1 - e^{-\lambda r}\right) e^{-\lambda(x-2r)e^{-\lambda r}}$$

using the fact that $P_c(2r) = 1 - e^{-\lambda r} - \lambda r e^{-\lambda r}$ because of (1). This is the lower bound in (4). ∎

We can thus conclude that in one dimension, the network is almost surely divided into an infinite number of bounded clusters, between which no communication is possible. A large-scale ad hoc network will therefore not work in this scenario.

B. Two-dimensional case : plane

The picture is very different in 2 dim. For example, the existence of an unbounded cluster does not imply full connectivity. The following result from percolation theory is our starting point.

Theorem 3—[9] pp 45-68: Consider a Poisson Boolean model $\mathcal{B}(\lambda, r)$ in $\mathbb{R}^2$. There exists a critical density $\lambda_c > 0$ such that

- in the *sub-critical case*, defined by $\lambda < \lambda_c$, all clusters are bounded almost surely (a.s.)
- in the *supercritical case*, defined by $\lambda > \lambda_c$, there exists a unique unbounded cluster $\mathcal{U}$ a.s.

In terms of networking, this means that the sub-critical case is similar to the 1-dim. case, where the network is partitioned in an infinite number of bounded clusters. In the supercritical case however, the result is much more encouraging, because of the existence of an unbounded cluster.

In the supercritical phase, one can divide the nodes into two categories: those belonging to the unbounded cluster $\mathcal{U}$, and the others. The first ones can communicate with nodes located arbitrarily far away, whereas the others are restricted to a finite area. Thus, the quality of the connectivity is related to the fraction θ of nodes belonging to the first category. Note that θ is most frequently defined as the probability of an arbitrary node to belong to the unbounded cluster, and is called *percolation probability*. Because of the spatial invariance of the Poisson process, the position of this node can be taken, without loss of generality, at the origin. One easily shows that $P_c(x) \ge \theta^2$. Moreover, $\inf_{x>0} P_c(x) = \theta^2$.

To date, there is unfortunately no explicit expression of θ, nor of λ_c. Bounds on λ_c have been obtained in [4], [10], [9], whereas θ can be evaluated by simulation. Figure 2 shows θ as a function of λ.

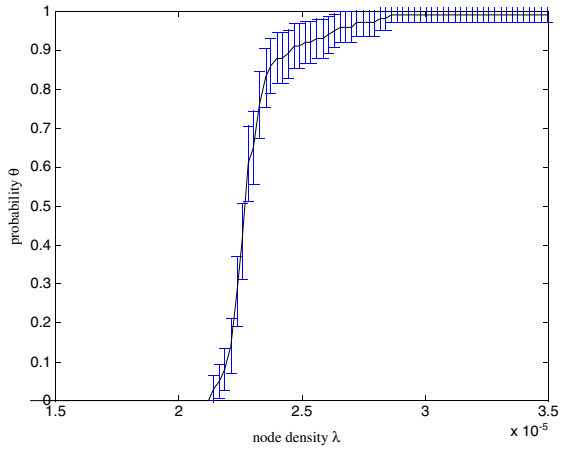

Fig. 2. Percolation probability θ in 2 dim. for a fixed ball radius $r/2 = 125$ pixels. The critical density λ_c is roughly $2.26 \cdot 10^{-5}$ node per square pixel. The presence of a tail below λ_c is due to the finiteness of the domain, a simulation artifact.

C. Two-dimensional case : strip

The last geometry we consider is an strip of infinite length and of fixed, finite width d. We set the density of the Poisson point process to $\lambda = \frac{\lambda_1}{d} + \lambda_2$. For $d = 0$, we have $\lambda = \infty$, but the surface of the strip is zero; we obtain therefore a 1-dim. network with density λ_1 on $\mathbb{R}$. For $d \to \infty$ we obtain a 2-dim. network with density $\lambda = \lambda_2$ on $\mathbb{R}^2$.

Percolation never occurs for a finite width d. This can be shown by projecting the network on the horizontal axis. One obtains a 1-dim. Poisson point process of density $\lambda = \lambda_1 + d\lambda_2$. As the distances are smaller in the projected space, for a constant ball radius, the existing connections are preserved, while others are created. However, percolation never occurs in 1 dim, as we have seen above. This implies that percolation never occurs on the strip. The existence of an unbounded cluster requires the domain to be infinite in both dimensions.

IV. Number of paths and bottlenecks

In the previous section, we discussed the existence of at least one path between two nodes. In order to study the reliability of such a network, and to identify possible bottlenecks, we now examine the number of alternate paths, which do not share any link with each other, in the super-critical phase. This amounts to compute the size of the minimal cutset in the associated graph, between both nodes.

A. Number of alternate paths between two nodes

Clearly, the more distant two nodes, the lower the size of the minimal cutset in the associated graph $G(\lambda, r)$ between them. In fact, we will first consider one of these two nodes, call it A. Suppose that $A \in \mathcal{U}$. We denote by $N_A(p)$ the size of the minimal cutset within a circle of radius p centered on A, that separates A from the nodes located outside of this circle.

More formally, denote by $\mathcal{C}_A(p)$ the set of the closed curves surrounding a node A, and which are included in the disk of radius p centered on A. For each curve $\boldsymbol{C} \in \mathcal{C}_A(p)$, denote by $\mathcal{E}(\boldsymbol{C})$ the set of the edges of $\mathcal{G}(\lambda, r)$ intersected by $\boldsymbol{C}$. Function $N_A(p)$ associated with each node $A \in \mathcal{U}$ is therefore given by

$$N_A(p) = \min_{\boldsymbol{C} \in \mathcal{C}_A(p)} \text{card}\left(\mathcal{E}(\boldsymbol{C})\right) \tag{7}$$

This function is clearly decreasing in p because $\mathcal{C}_A(p_1) \subseteq \mathcal{C}_A(p_2)$ if $p_1 < p_2$. Furthermore, $N_A(p) > 0$, because $A \in \mathcal{U}$. Therefore, $N_A(p)$ admits a limit when $p \to \infty$, which we denote N_A^∞:

$$\lim_{p \to \infty} N_A(p) = \inf_{p > 0} N_A(p) = N_A^\infty. \tag{8}$$

This number can be seen as the size of the minimal cutset separating A from what we call the "rest of the network", i.e. nodes located far away from A.

Figure 3 shows typical values of $N_A(p)$ (scaled to the number of immediate neighbors of A, which is clearly equal to $\lim_{p \to 0} N_A(p)$).

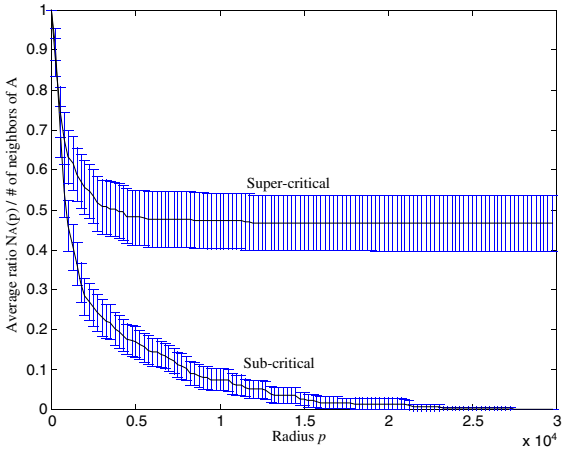

Fig. 3. Average ratio between $N_A(p)$ and the number of neighbors of A for sub- and super-critical densities. For sub-critical densities, $N_A(p)$ goes to zero, because A belongs a.s. to a bounded cluster. For supercritical densities, $N_A(p)$ tends to a limit $N_A(\infty)$ noticeably smaller than $\lim_{p \to 0} N_A(p)$.

Thanks to this number, we can now compute the size of the minimal cutset between two nodes $A, B \in \mathcal{U}$. But we need first the following result.

Lemma 1: In the supercritical case, any unbounded[1] curve $\boldsymbol{C}$ crosses an infinite number of edges of the associated graph.

[1] A curve $\boldsymbol{C} \subset \mathbb{R}^2$ is *unbounded* if for any two points $\boldsymbol{a}$,$\boldsymbol{b}$ on the curve, $\sup_{\boldsymbol{a},\boldsymbol{b} \in \boldsymbol{C}} d(\boldsymbol{a}, \boldsymbol{b}) = \infty$ where $d(\boldsymbol{a}, \boldsymbol{b})$ is the Euclidean distance in $\mathbb{R}^2$ between $\boldsymbol{a}$ and $\boldsymbol{b}$.

Proof: Consider a portion $\boldsymbol{D}$ of the curve $\boldsymbol{C}$ that does not cross any edge. Denote by $\mathcal{W}$ the occupied region, and by $\mathcal{V}$ the vacant region of $\mathbb{R}^2$. Take a point $\boldsymbol{a}$ on $\boldsymbol{D}$. Either $\boldsymbol{a} \in \mathcal{V}$, or $\boldsymbol{a} \in \mathcal{W}$. If $\boldsymbol{a} \in \mathcal{W}$, it can be included in the ball of a node placed on either side of the curve, but not on both, as otherwise, there would be an edge that crosses the curve. Denote by d be the distance between this node and $\boldsymbol{a}$. We are then sure that there is a vacant region on the other side of the curve at a distance $r/2 - d$ from $\boldsymbol{a}$. Therefore, there is always a vacant region at a distance smaller than $r/2$ from any point of the curve.

One can thus draw a new curve $\tilde{\boldsymbol{D}}$ such that for any point $\boldsymbol{b}$ of $\tilde{\boldsymbol{D}}$, $d(\boldsymbol{b}, \boldsymbol{D}) \leq r/2$ and $\boldsymbol{b}$ is in a vacant region. $\tilde{\boldsymbol{D}}$ is thus included in a single vacant region. However, there is no unbounded vacant region in the supercritical case (see [9] pp 108-116). Hence, $\tilde{\boldsymbol{D}}$ must be bounded. As every point of $\boldsymbol{D}$ is at a finite distance of $\tilde{\boldsymbol{D}}$, $\boldsymbol{D}$ must also be bounded.

One can then conclude that the portion of curve between two crossings must be bounded. As the curve $\boldsymbol{C}$ is unbounded, there are infinitely many crossings. ■

We can now compute the following lower bound on the size of the minimal cutset, and thus the number of alternate paths, between two nodes A and B belonging to the unbounded cluster.

Theorem 4: The number of alternate paths between two nodes A and B is lower-bounded by $\min\{N_A^\infty, N_B^\infty\}$.

Proof: We can assume without loss of generality that $N_A^\infty \leq N_B^\infty$. Suppose that there are less than N_A^∞ paths from A to B. Then there exists a curve dividing $\mathbb{R}^2$ into two parts, one including A and the other including B, and that intersects less than N_A^∞ edges. This curve is either closed and surrounds A, either closed and surrounds B, or unbounded. The two first cases are excluded by assumption. The third possibility is excluded by Lemma 1. ■

The lower bound in Theorem 4 is tight. Indeed, since $N_A(p)$ is an integer-valued function, the infimum in (8) is reached for a finite p, which we denote by L_A. In other words,

$$L_A = \inf\{p > 0 \text{ such that } N_A(p) = N_A^\infty\}. \qquad (9)$$

If the distance between A and B is larger than $\max\{L_A, L_B\}$, then the number of alternate paths is exactly equal to $\min\{N_A^\infty, N_B^\infty\}$. It follows that the number of paths between two nodes is limited by two local values (N_A^∞ and N_B^∞), which depend exclusively on their respective neighborhoods.

B. Bottlenecks

In this section, we now analyze the structure of graph $\mathcal{G}(\lambda, r)$. In particular, we would like to identify the possible bottlenecks that can form in the network.

Parameter L_A is important for locating hot spots and bottlenecks. Indeed, it is the smallest distance from Node A at which the minimal cutset (which has size N_A^∞) with "the rest of the network" is reached. A small value of L_A means that the number of alternate paths between A and a node located far away is approximately equal to the number of paths between A and its close neighbors. As a result, the number of paths is simply limited by the number of immediate neighbors of end-nodes. On the contrary, if L_A is large, then long-distance communications are constrained by bottlenecks located far way from the end nodes.

As we can see on Figure 4, L_A is in average non zero. In other words, N_A^∞ is in average less than the number of immediate neighbors of A (what we had already observed on Figure 3). Nevertheless, L_A tends to zero when the density λ increases, for a fixed r. This means that for high densities, the number of paths is simply limited by the number of immediate neighbors of end-nodes, and that "hot spots" do not occur, as already pointed out in [5].

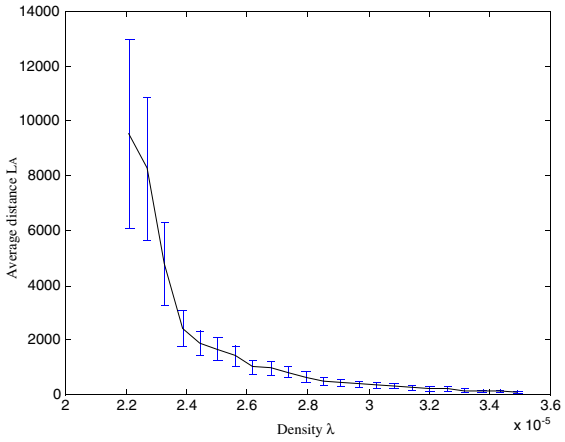

Fig. 4. $E[L_A]$ as a function of λ. We see that L_A goes to zero when the density increases, meaning that N_A^∞ tends to the number of neighbors of A

The case where λ is just above the critical density is the situation where bottlenecks will appear. The network appears then as a set of "islands", inside which nodes are well connected. Islands are however connected by a few links only, those forming the minimal cutsets. Nodes belonging to different islands are thus connected by a number of alternate paths given by Theorem 4. These links are the bottlenecks of the network.

The rest of this section is devoted to a formal definition of the concept of island. We begin first by that, more general, of a domain of a node.

Definition 1: For each node A, consider the closed curve $\boldsymbol{C}_A \in \mathcal{C}_A(L_A)$ with card $(\mathcal{E}(\boldsymbol{C}_A)) = N_A^\infty$ that surrounds the minimum number of nodes. We call *domain* of A the set $\mathcal{S}_A$ of nodes surrounded by $\boldsymbol{C}_A$.

For all $B \in \mathcal{S}_A$ we have $N_B^\infty \leq N_A^\infty$, because $\boldsymbol{C}_A$

surrounds also B. In other words, all the elements of the domain $\mathcal{S}_A$ of Node A share the same N_A^∞ paths to the "rest of the network".

Domain $\mathcal{S}_A$ of Node A is called an island if no other domain contains $\mathcal{S}_A$. In other words:

Definition 2: Let $A \in \mathcal{U}$. Then $\mathcal{S}_A$ is an *island* if there exists no node $B \in \mathcal{U}$ such that $\mathcal{S}_B \supset \mathcal{S}_A$.

Islands are disjoint, and form a partition of the unbounded cluster $\mathcal{U}$. It is a consequence of the following lemma.

Lemma 2: Consider two nodes A and $B \in \mathcal{U}$. Then either $\mathcal{S}_A \cap \mathcal{S}_B = \emptyset$, either $\mathcal{S}_A \subseteq \mathcal{S}_B$, or $\mathcal{S}_B \subseteq \mathcal{S}_A$.

Proof: We discuss first the case where $B \in \mathcal{S}_A$. If $\mathcal{S}_B \nsubseteq \mathcal{S}_A$, as illustrated in Figure 5, $\boldsymbol{C}_A$ and $\boldsymbol{C}_B$ have two intersection points X and Y.

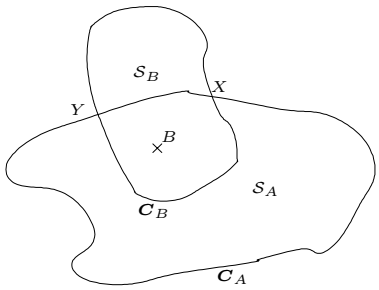

Fig. 5. Curves $\boldsymbol{C}_A$ and $\boldsymbol{C}_B$ when $B \in \boldsymbol{C}_A$

If the number of edges cut by $\boldsymbol{C}_A$ between X and Y is less or equal to the number of edges cut by the portion of $\boldsymbol{C}_B$ located between X and Y and inside $\mathcal{S}_A$, then we can construct a new curve $\boldsymbol{C'}_B$ surrounding B by replacing in $\boldsymbol{C}_B$ the latter portion between X and Y by the former one. This new curve cuts fewer edges or surrounds fewer nodes, which is impossible by definition of $\boldsymbol{C}_B$.

If the number of edges cut by $\boldsymbol{C}_A$ between X and Y is strictly more than the number of edges cut by the corresponding portion of $\boldsymbol{C}_B$, then one could construct a new curve $\boldsymbol{C'}_A$ surrounding A in the same way as above. This new curve cuts fewer edges than $\boldsymbol{C}_A$, what is impossible by definition.

Therefore, the situation represented in Figure 5 is not possible, and Curve $\boldsymbol{C}_B$ is always surrounded by $\boldsymbol{C}_A$. This implies $\mathcal{S}_B \subseteq \mathcal{S}_A$. Similarly, $A \in \mathcal{S}_B$ implies $\mathcal{S}_A \subseteq \mathcal{S}_B$.

Let us discuss now the case where $A \notin \mathcal{S}_B$ and $B \notin \mathcal{S}_A$. Suppose that $\mathcal{S}_A \cap \mathcal{S}_B \neq \emptyset$. $\mathcal{U}$ is therefore partitioned into four disjoint sets, $\mathcal{N} = \mathcal{S}_A \cap \mathcal{S}_B$, $\mathcal{A} = \mathcal{S}_A \setminus \mathcal{S}_B$, $\mathcal{B} = \mathcal{S}_B \setminus \mathcal{S}_A$ and $\tilde{\mathcal{U}} = \mathcal{U} \setminus (\mathcal{S}_A \cup \mathcal{S}_B)$. Denote by $n_{\mathcal{AB}}$ the number of edges between elements of $\mathcal{A}$ and elements of $\mathcal{B}$, by $n_{\mathcal{AN}}$ the number of edges between element of $\mathcal{A}$ and elements of $\mathcal{N}$, and so on (see Figure 6).

Remember that $\boldsymbol{C}_A$ surrounds the elements of $\mathcal{S}_A = \mathcal{A} \cup \mathcal{N}$. Consider now a curve that surrounds the elements of $\mathcal{A}$ only. By definition of $\boldsymbol{C}_A$, this curve must intersect

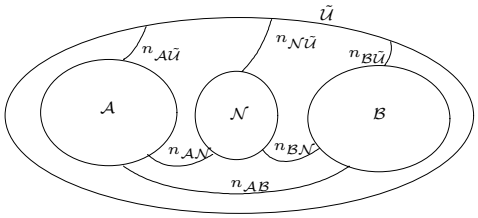

Fig. 6. Number of edges between sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{N}$ and $\tilde{\mathcal{U}}$.

strictly more edges than $\boldsymbol{C}_A$, because $\mathcal{N} \neq \emptyset$. In other words, if we count these edges, we must have:

$$n_{\mathcal{A}\tilde{\mathcal{U}}} + n_{\mathcal{N}\tilde{\mathcal{U}}} + n_{\mathcal{BN}} + n_{\mathcal{AB}} < n_{\mathcal{A}\tilde{\mathcal{U}}} + n_{\mathcal{AB}} + n_{\mathcal{AN}}$$

and thus

$$n_{\mathcal{N}\tilde{\mathcal{U}}} + n_{\mathcal{BN}} < n_{\mathcal{AN}}. \tag{10}$$

Similarly, by considering $\mathcal{S}_B = \mathcal{B} \cup \mathcal{N}$, we obtain:

$$n_{\mathcal{N}\tilde{\mathcal{U}}} + n_{\mathcal{AN}} < n_{\mathcal{BN}} \tag{11}$$

Combining (10) and (11) together, we obtain a contradiction, which proves that $N = \mathcal{S}_A \cap \mathcal{S}_B = \emptyset$. ■

A consequence of Lemma 2 is that for each node $B \in \mathcal{S}_A$, $\mathcal{S}_B \subseteq \mathcal{S}_A$. It follows then from Definition 2 and Lemma 2 that islands are disjoint. Moreover, every node of $\mathcal{U}$ belongs to one (and only one) island:

Theorem 5: For each node $A \in \mathcal{U}$, there exists a unique island $\mathcal{I}_A$ containing A, and which is given by

$$\mathcal{I}_A = \bigcup_{B \in \mathcal{U}, \mathcal{S}_B \ni A} \mathcal{S}_B. \tag{12}$$

Proof: (i) Existence: We have to prove that the set defined in (12) is actually an island. Consider B_1 and $B_2 \in \mathcal{U}$ with $\mathcal{S}_{B_i} \ni A$, $i = 1, 2$. As $\mathcal{S}_{B_1}$ and $\mathcal{S}_{B_2}$ are not disjoint, we know from Lemma 2 that either $\mathcal{S}_{B_1} \subseteq \mathcal{S}_{B_2}$ or $\mathcal{S}_{B_2} \subseteq \mathcal{S}_{B_1}$. Thus, the union in (12) is equal to $\mathcal{S}_C$ for some node C.

On the other hand, the condition in Definition 2 is verified by construction. $\mathcal{I}_A$ is therefore an island.

(ii) Uniqueness. As islands are disjoint, two islands cannot contain A simultaneously. ■

Since every node of $\mathcal{U}$ belongs to an island, the unbounded cluster $\mathcal{U}$ can be partitioned into islands. We can thus define a new simplified graph with islands as nodes, and connections between islands as edges. This new graph, with a coarser granularity than $\mathcal{G}(\lambda, r)$, describes the "backbone" structure of the unbounded cluster.

The existence of this backbone structure can be intuitively explained as follows : below the critical density, the graph is made of an infinite number of bounded clusters. As the density increases, new connections appear, making clusters merge together. At the critical density, an

infinite number of clusters merge to create the unbounded cluster $\mathcal{U}$. Above the critical density, the islands are more or less the remnants of the bounded clusters, whereas the backbone structure is made of brand-new connections. Figure 7 illustrates this phenomenon.

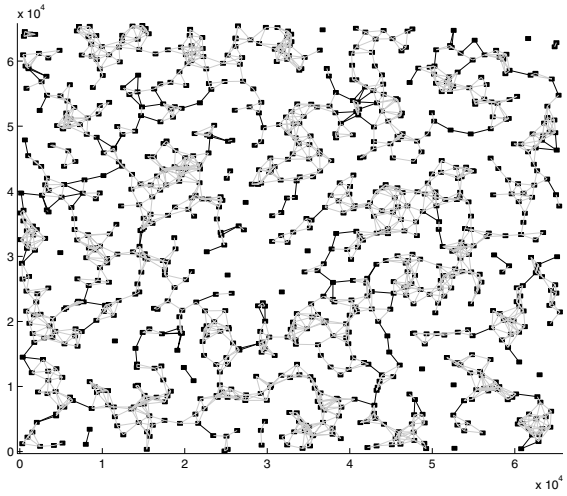

Fig. 7. Phase transition : islands and bottlenecks. At sub-critical density, the graph is divided into many bounded clusters (gray). When the density becomes supercritical, new edges (in black) appear and connect the clusters together (they become the *islands* of the unbounded cluster). The number of paths between two nodes inside an island remains in average higher than the number of paths between two islands.

V. Placement of fixed base stations

The introduction of base stations will allow distant nodes to communicate through a fixed, wired infrastructure. It represents a trade-off between today's cellular networks and large-scale ad-hoc networks. In the first case, every node connects to the nearest base station. To prevent nodes from being isolated from the network, base stations must therefore cover the whole space. In the second case, as we have seen in the previous sections, the density and radius must be large enough to keep the proportion of nodes which do not belong to $\mathcal{U}$ small enough.

In this section we want to evaluate whether the introduction of base stations is helpful in decreasing the probability that an arbitrary node remains unconnected to the network, for the three geometries of a line, a plane and a strip. We assume here that the radius of connectivity of the base stations is the same as the one of the wireless nodes.

A. Nodes and base stations on a line

Base stations are placed every L distance units, say at every nL for $n \in \mathbb{Z}$, and therefore define intervals $[nL, (n+1)L]$ of length L on the line $\mathbb{R}$.

With no loss of generality, we only consider here nodes that are located in the first interval $[0, L]$. Such a node is connected to the left (respectively, right) base station if and only if all intervals separating any pair of consecutive nodes located in $[0, x]$ (resp., $[x, L]$) have a length less than r. Because of the Poisson assumption, these intervals are independent, so that the probability that the node located at position x be connected to both base stations is the product of $P_c(x)$ (the probability that the node is connected to the left base station) and $P_c(L-x)$ (the probability that the node is connected to the right base station). Therefore the probability that an arbitrary node located at distance $0 \leq x \leq L$ from the origin is connected to a base station is

$$P_{cb}(x) = P_c(x) + P_c(L-x) - P_c(x)P_c(L-x)$$

where $P_c(x)$ is given by (1).

Of course, the closer we pick a node from a base station, the more likely it will be connected. The average of the connection probability of a node to a base station, over all positions that this node can have (which are uniformly distributed over a given interval, because of the Poisson assumption) is $\int_0^L P_{cb}(x)dx/L$. A more interesting value is a lower bound on P_{cb}, which is valid for the worst possible location of a node, which is at mid distance between the base stations:

$$\begin{aligned} P_{cb}^+ &= \inf_{0\leq x\leq L}\{P_{cb}(x)\} = P_{cb}(L/2) \\ &= 2P_c(L/2) - P_c^2(L/2). \end{aligned} \tag{13}$$

Using the bounds (4), we obtain the following lower bound on P_{cb}^+.

Theorem 6: P_{cb}^+ is lower bounded by:

$$P_{cb}^+ \geq 1 - (1+\lambda L/2)^2\, e^{-2\lambda r}. \tag{14}$$

Proof: If $0 \leq L \leq 2r$, then $P_{cb}^+ = 1$, obviously: in this case, the base stations cover the entire line. If $2r \leq L \leq 4r$, then $P_c(L/2) = 1 - e^{-\lambda r} - \lambda L e^{-\lambda r}/2$ because of (1). Inserting this value in (13) yields $P_{cb}^+ = 1 - (1 + \lambda L/2)^2\, e^{-2\lambda r}$.

Finally, if $L \geq 4r$, we insert the bounds (4) in (13), which becomes

$$\begin{aligned} P_{cb}^+ \;\geq\; & 2\left(1-e^{-\lambda r}\right) e^{-\lambda(L-4r)e^{-\lambda r}/2} \\ & -2\lambda r e^{-\lambda r} - \left(1-e^{-\lambda r}\right)^2 e^{-\lambda(L-2r)e^{-\lambda r}} \end{aligned}$$

The expansion of the right hand side of this inequality in Taylor series yields (14) after lengthy manipulations omitted here for lack of space. ■

One deduces from (14) the maximal spacing between consecutive base stations to keep the probability of finding a node not connected to a base station, in an arbitrary interval, below a given value $P_{uc}^+ = 1 - P_{cb}^+$. This spacing is at least equal to $2r$, in which case the base stations do cover the entire line ($P_{uc}^+ = 0$). Combining this with (14),

we see that the maximal distance between base stations is lower bounded by the following expression

$$\left(\frac{L}{r}\right)_{\max} \geq \max\left\{2, \frac{2}{\lambda r}\left(e^{\lambda r}\sqrt{P_{uc}^{+}} - 1\right)\right\}. \quad (15)$$

The distance between base stations can grow exponentially with λr, provided it is large enough to bring the right hand side of this inequality above 2.

B. Nodes and base stations on a plane

We now place the base stations on the nodes of a square lattice, with edges of length equal to L. In the 2-dim. case, P_{cb} cannot be computed analytically. We compute it therefore numerically. Figure 8 shows P_{cb} as a function of L in both the super- and sub-critical cases.

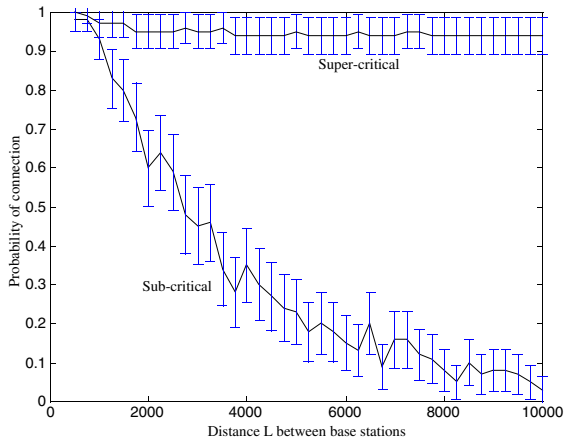

Fig. 8. Probability of connection P_{cb} in 2 dim. with base stations for a sub-critical density $\lambda = 2.1 \cdot 10^{-5}$ and a super-critical density $\lambda = 2.56 \cdot 10^{-5}$.

For sub-critical densities, the probability of connection is now non-zero, but still far away from one, except in the limit case where base stations cover almost the whole space. Connectivity is thus not significantly better than with a standard cellular network, where only one hop is allowed between nodes and base stations.

For supercritical densities, we can see that the probability of connection is slightly better than without base stations. Furthermore, the probabilities take almost identical values in both cases for high densities. Let us explain why.

If $\lambda > \lambda_c$, there exists a unique unbounded cluster $\mathcal{U}$. This cluster is in contact with a base station almost surely (the probability for each base station to be connected to the unbounded cluster is positive, and there are an infinite number of them). All nodes in $\mathcal{U}$ are thus connected to the base stations. The situation for this subset of nodes is actually the same as that of a purely ad hoc network. The difference comes from nodes that are connected to a base station through a bounded cluster.

The quantity in which we are interested is then the fraction of bounded clusters connected to a base station. We know from [9] that the average number of nodes contained in a bounded cluster tends to one when $\lambda \to \infty$. Therefore the probability that a bounded cluster is connected to a base station tends to the fraction of space covered by the base stations, which is $\pi r^2/L^2$. The total probability that an arbitrary node is connected is thus

$$P_{cb} \overset{\lambda \to \infty}{\simeq} \theta(\lambda) + (1 - \theta(\lambda))\frac{\pi r^2}{L^2}.$$

This shows that the gain of connectivity for high densities grows linearly with the fraction of space covered by the base stations.

We saw in Section IV that one can divide the unbounded cluster into *islands*, and that the bottlenecks appear right between them. Inside an island, the connectivity is better, meaning that if a base station is present inside an island, the number of paths between the nodes of this island and the rest of the network will be increased. As the size of the islands is of order of L_A, we can see on the graph of Fig 4 that this size decreases quickly with an increasing density, meaning the probability of an island to contain a base station becomes small. In other words, for high densities, the benefit (in terms of connectivity and number of paths) of inserting a fixed infrastructure of base stations is small. For intermediate densities, the benefit needs further study. One needs to investigate how the lattice of base stations covers the islands, following probably the approach of [1], [2].

C. Nodes and base stations on a strip

The behavior of the network is very different in one and in two dimensions. It is therefore interesting to explore the transition between these two behaviors, thanks to the infinite strip of width d of Section III-C. Remember that percolation never occurs for d finite: one expects therefore to benefit from base stations for connectivity. This will certainly be the case for very small values of d. On the other hand, as $d \to \infty$, one gets closer a network on $\mathbb{R}^2$, and the need for base stations to ensure connectivity should vanish.

Figure 9 confirms our expectation. It shows the maximum distance L between base stations, as a function of the strip width d, in order to achieve a probability of connection larger than 90% (computed by simulation). Since $\lambda = \lambda_1/d + \lambda_2$, with λ_2 chosen such that $\lambda_2 \gg \lambda_c$, the 90% connectivity is achieved without base station for $d = \infty$. One can see that $L(d)$ grows regularly, meaning that there is continuous transition between the 1-dim. and 2-dim. cases.

VI. Results on Swiss demographic data

In this section, instead of generating the positions of the nodes with a Poisson point process, we use node dis-

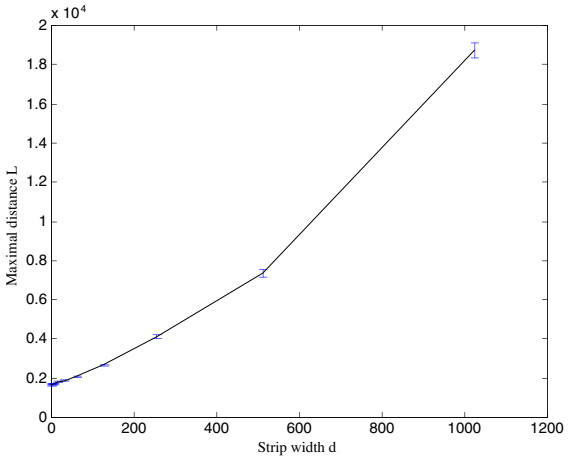

Fig. 9. Maximal distance L between the base stations as a function of the width d of the strip. $r = 300$ pixels, $\lambda_1 = 8 \cdot 10^{-3}$ and $\lambda_2 = 2.3 \cdot 10^{-5}$

tributions that are based on real population data[2]. We consider two regions of Switzerland: the Zurich area, and the Alpine Valley of Surselva (Fig 10). The first region present a quite uniform population 2-dim. density, whereas the second one has an almost 1-dim. shape. We assume that each resident has a probability 0.01 to own a (turned-on) device, and we simulate the network for different values of r.

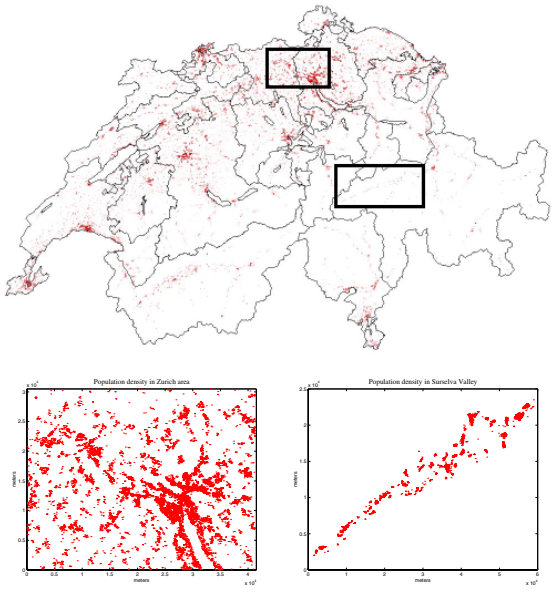

Fig. 10. Swiss population density in 1990. In the frames, the two considered regions of Zurich (zoom at bottom left) and Surselva Valley (zoom at bottom right).

Let us discuss the region of Zurich first. In the pure ad-hoc case, when r increases, we do observe the phase transition predicted by the model (see Figure 11). Furthermore, when r is sufficiently large, nodes are connected with high probability (more than 99.9% for $r = 2500$m). From a connectivity point of view, a pure ad-hoc network would therefore be feasible in this region.

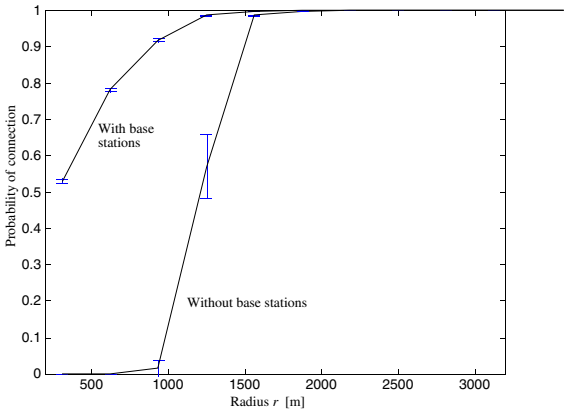

Fig. 11. Probability of connection in the region of Zurich without base stations and with base stations every 5km as a function of r.

If we insert base stations every 5km, the gain of connectivity is not significant above $r = 1500$ m. We actually obtain full connectivity only for $r = 2500$ m, as with a pure ad-hoc network.

In the Surselva Valley, the picture is very different. In fact, without base stations, we never observe clusters that spread form one end of the valley to the other. It means that long range communication is impossible. If we add base stations every 5km (i.e. 11 units), the probability of connection then becomes much larger, already for transmission ranges of a few hundreds of meters. However, a 99 % connection probability is only reached when base stations cover almost entirely the area. To allow more distance between base stations, one needs a larger fraction of active nodes than 0.01 in this little populated region.

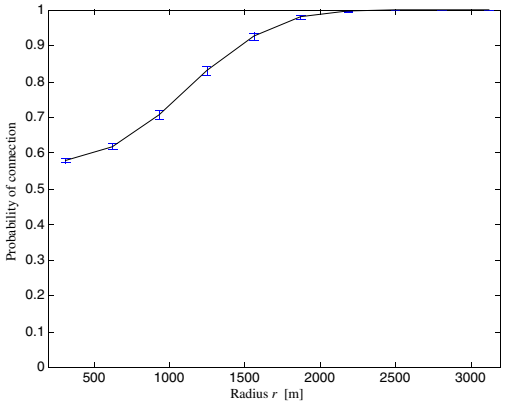

Fig. 12. Probability of connection in Surselva Valley with base stations every 5km as a function of r.

[2]Source : Population census 1990, OFS GEOSTAT

We can conclude from these simulations that the qualitative behavior of the network remains the same with a realistic node distribution, as with a Poisson distribution.

VII. Conclusion

Connectivity in 1 dim. is limited to short range communications. In 2 dim., a phase transition occurs at a critical node density. Above this value, the fraction of nodes being connected to the network grows rapidly. However, the number of paths between two arbitrary nodes remains limited by some bottlenecks. The node density has to be far above criticality in order to have a well connected network with numerous paths form one node to the other.

If we introduce a fixed infrastructure of base stations in the network, the 1-dim. connectivity improves drastically. The maximal distance between base stations to secure a given probability of connection grows exponentially with the density. On the other hand, in 2-dim, the introduction of base stations does not significantly modify the behavior of the network; the probability of connection and the number of paths remains of the same order of magnitude. If nodes are distributed on a strip of finite width and infinite length, fixed infrastructure improves connectivity, but this improvement becomes less and less significant as the width of the strip becomes large. In this case, we observe thus a smooth transition from 1-dim to 2-dim.

We should insist that these results are focused on the sole aspect of connectivity. It is clear that the presence of base stations can be a good answer to the problem of low capacity at large node density. We conclude that in a wireless network, a relatively sparse fixed infrastructure is well suited for very populated regions and valleys (almost 1-dim. networks), whereas pure ad-hoc networking can be used for areas with a relatively low density of nodes, but that are well scattered in 2-dim. However, if in some region the density is sub-critical, only cellular network can offer an acceptable connectivity.

The dimensionality of the spatial node distribution appears thus as a key element to require or not a fixed infrastructure for connectivity, independently from other considerations of capacity. This is valid for a Poisson distribution, as well as real data, as exemplified by Section VI.

Acknowledgments

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References

[1] F. Baccelli and B. Blaszczyszyn. 'On a Coverage Process Ranging from the Boolean Model to the Poisson Voronoi Tessellation, with Applications to Wireless Communications, Report RR-4019, INRIA, Sophia-Antipolis, France, Oct. 2000.

[2] F. Baccelli, B. Blaszczyszyn and F. Tournois. 'Spatial Averages of Coverage Characteristics in Large CDMA Networks,' Report RR-3435, INRIA, Sophia-Antipolis, France, June 2001.

[3] Y.-C. Cheng and T. G. Robertazzi, 'Critical Connectivity Phenomena in Multihop Radio Models,' *IEEE Trans. on Communications*, pp. 770-777, vol. 37, July 1989.

[4] E. N. Gilbert, 'Random plane networks,' *SIAM J.*, vol 9, pp. 533-543, 1961.

[5] P. Gupta and P. R. Kumar, 'The Capacity of Wireless Networks,' *IEEE Transactions on Information Theory*, pp. 388-404, vol. IT-46(2) March 2000.

[6] P. Gupta and P. R. Kumar, 'Critical Power for Asymptotic Connectivity in Wireless Networks,' pp. 547-566, in *Stochastic Analysis, Control, Optimization and Applications : A Volume in Honor of W.H. Fleming*. Edited by W.M. McEneany, G. Yin, and Q. Zhang, Birkhauser, Boston, 1998.

[7] M. Grossglauser and D. Tse, 'Mobility increases the capacity of ad-hoc wireless networks, in *Proc. INFOCOM*, Anchorage, April 2001.

[8] V. Lakshmikantham, S. Leela, A. A. Martynyuk, *Stability analysis of nonlinear systems*, M. Dekker, NY, 1989.

[9] R. Meester and R. Roy, *Continuum percolation*, Cambridge University Press, Cambridge, 1996.

[10] T.K. Philips, S.S. Panwar and A.N. Tantawi, 'Connectivity Properties of a Packet Radio Network Model,' *IEEE Transactions on Information Theory*, Vol. 35, pp. 1044-1047, September 1989.

[11] M Sanchez, P. Manzoni, and Z.J. Haas, 'Determination of Critical Transmission Range in Ad-Hoc Networks', *Proc. MMT'99*, Venice, Italy, Oct. 1999.

[12] P. Santi, D. M. Blough and F. Vainstein, 'A Probabilistic Analysis for the Range Assignment Problem in Ad Hoc Networks,' *Proc. Mobihoc'2001*, Long Beach (CA), pp. 212-220, October 2001.

[13] D. Watts and S. Strogatz, 'Collective Dynamics of Small-World Networks,' *Nature* pp. 202-204, vol. 363, June 1998