Design of Multidimensional Filter Banks for Non-Separable Sampling

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Abstract

This paper presents some results in design of multidimensional filter banks with arbitrary sampling patterns. It concentrates on the particular cases of quincunx and hexagonal sampling. We point out applications in obtaining directional subband decomposition with hexagonal filters and decomposition of interlaced and progressively scanned television with quincunx ones.

1 Introduction

In the last decade subband coding has become one of the most commonly used tools in compressing still images and video. However, most of the proposed schemes use separable processing in two or three dimensions. Only recently have some results emerged setting up the theory of general multidimensional filter banks using arbitrary non-separable sampling lattices as well as general non-separable filters [1], [2].

First, in section 2 we review some results on perfect reconstruction filter banks in multiple dimensions. Basically, the known one-dimensional results were extended to the general non-separable two dimensional case. This leads to conditions for alias cancellation and perfect FIR reconstruction. The role of polyphase decomposition is a central component as is the notion of paraunitary matrices [2].

Then we concentrate on some particular cases of interest. In section 3 we explore filter banks with quincunx sampling, which applies naturally to television systems allowing elegant solutions to the problems of compatibility and compression [3]. We state conditions on what is and what is not achievable given specific design constraints, such as perfectly diamond shaped filters, linear phase and paraunitariness. Useful cascade structures are given and the issue of completeness is addressed. A similar analysis is carried out in section 4 for the hexagonal case and a structure allowing perfect reconstruction while yielding linear phase and paraunitary filters is proposed. Finally, in section 5 we point out some applications.

2 Two Dimensional Perfect Reconstruction Filter Banks

We will keep our discussion to the two dimensional FIR case, but most results hold for an arbitrary number of dimensions. Given an input lattice indexed by (n_1, n_2), a location on the output lattice (u_1, u_2) can be written as [4], [5]:

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} d_{00} & d_{01} \\ d_{10} & d_{11} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathbf{D} \cdot \mathbf{u}. \qquad (1)$$

Note that $\mathbf{D}$ is not unique for a given sampling pattern and that two matrices representing the same sampling process are related by a matrix with determinant equal to one [4]. The subsampling factor when going to the lattice (u_1, u_2) is given by $N = det[D] = d_{00}d_{11} - d_{01}d_{10}$. The subsampling operation is clearly space variant, since samples of the input at locations $\mathbf{D} \cdot \mathbf{u}$ are kept while all the others are dropped. It is this space variance that leads to aliased versions of the input appearing in the output. In a subband coding system, dropped samples will be replaced by zeroes (corresponding to an upsampling) before entering the synthesis bank, and thus, the down-and-upsampling process is equivalent to a modulation by a function $f(n_1, n_2)$ which equals 1 at locations $\mathbf{D} \cdot \mathbf{u}$ and zero elsewhere [2], producing $N-1$ aliased versions. A convenient way to take care of the space-variance of such a multidimensional multirate system is to decompose signals and filters into so-called polyphase components, which correspond to samples on the subsampling lattice and all its cosets with respect to the input lattice (that is N polyphase components). In this polyphase domain, the system becomes space invariant. Thus, signals at the output of the analysis bank can be represented in terms of the input signal and the analysis polyphase matrix $\mathbf{H}_p(z_1, z_2)$ (that is matrix containing polyphase components of the analysis filters), while the output signal can be represented in terms of the input channel signals and the synthesis polyphase matrix $\mathbf{G}_p(z_1, z_2)$ (that is matrix containing polyphase components of the synthesis filters). For the definitions of polyphase matrices see [6].

Conditions for aliasing cancellation are given in [2] and are given in terms of the transfer function matrix $\mathbf{T}_p(z_1, z_2)$ obtained as the product of the polyphase matrices of the synthesis and analysis bank:

$$\mathbf{T}_p(z_1, z_2) = \mathbf{G}_p(z_1, z_2) \cdot \mathbf{H}_p(z_1, z_2). \qquad (2)$$

In order to make the alias-free condition clear we will state it for a one dimensional system first. The output of a system with an up/downsampling factor of N is given by:

$$\hat{X}(z) \;=\; z^{-2(N-1)} \bar{Z}(z)^T \; \mathbf{G}_p(z^N) \mathbf{H}_p(z^N) \begin{pmatrix} X_{p_0}(z^N) \\ X_{p_1}(z^N) \\ \vdots \\ X_{p_{N-1}}(z^N) \end{pmatrix} \qquad (3)$$

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where $\mathbf{T}_p(z^N) = \mathbf{G}_p(z^N)\mathbf{H}_p(z^N)$ while the vector

$$\bar{Z}(z)^T = (1 \quad z \quad \ldots \quad z^{(N-1)})^T \tag{4}$$

is the non-causal version of the inverse polyphase transform. Now it can be shown that aliasing is canceled *if and only if*:

$$\bar{Z}(z)^T \, \mathbf{T}_p(z^N) \;=\; T(z) \, \bar{Z}(z)^T, \tag{5}$$

where $T(z)$ is a scalar polynomial. The above condition means that $\bar{Z}(z)^T$ is the left eigenvector of $\mathbf{T}_p(z^N)$ with an eigenvalue $T(z)$. The interpretation is that impulses at various polyphase locations are thus put in the right place and convolved with the same eigenvalue $T(z)$. This result is equivalent to $\mathbf{T}_p(z)$ being a pseudo-circulant matrix [7]. In two dimensions the result is similar except that the vector corresponding to the inverse polyphase transform is given by $\bar{Z}_2(z_2)^T \otimes \bar{Z}_1(z_1)^T$ where $\otimes$ stands for Kronecker product. This vector has to be the left eigenvector of $\mathbf{T}_p(z_1^{d_{00}} z_2^{d_{10}}, z_1^{d_{01}} z_2^{d_{11}})$ with an eigenvalue $T(z_1, z_2)$ which is a scalar polynomial [2]. Unlike the one-dimensional case this condition does not stipulate a simple structure of the matrix $\mathbf{T}_p(z_1, z_2)$ except when the sampling is separable in which case it yields a block pseudo-circulant matrix [8].
Now it is obvious that perfect reconstruction is achieved when $\mathbf{T}_p(z_2, z_2) = \mathbf{I}$ or a shifted version thereof. Thus, the filter design problem we are faced with is to find useful set of filters so that $\mathbf{T}_p(z_1, z_2)$ corresponds to perfect reconstruction. Several approaches are possible depending on which constraints have to be met. For example, one can require the filter bank to be paraunitary [9], that is (assuming real filters):

$$\mathbf{H}_p(z_1^{-1}, z_2^{-1})^T \cdot \mathbf{H}_p(z_1, z_2) = \mathbf{I}. \tag{6}$$

In addition one can impose linear phase on all filters.
In the previous discussion the filters involved were general two-dimensional, including both separable and non-separable filters. Let us point out that the non-separable solutions have some advantages over their separable counterparts. Consider for example a two-channel one-dimensional system. It is known that there are no linear phase paraunitary filter banks achieving perfect reconstruction [10], and thus no separable solutions exist in a four channel two-dimensional case. However, there does exist a non-separable linear phase paraunitary solution in two dimensions [2], demonstrating the greater freedom offered by non-separable systems.
In what follows, we will need the linear phase testing condition in order to analyze linear phase solutions [2]:

$$\mathbf{H}_{p+} \;=\; z_1^{-w_1} z_2^{-w_2} \, \mathbf{S}\mathbf{H}_{p-}\mathbf{J}, \tag{7}$$

where $(w_1 + 1)$ and $(w_2 + 1)$ are the numbers of unit cells along the axes in the polyphase domain and $\mathbf{J}$ is an exchange matrix. $\mathbf{S}$ is a diagonal matrix with elements 1 or -1 depending on the symmetry of the corresponding filter. The subscripts $+$ and $-$ stand for (z_1, z_2) and (z_1^{-1}, z_2^{-1}) respectively. Note that $\mathbf{S}^2 = \mathbf{I}$. A matrix $\mathbf{U}$ satisfying $\mathbf{U} = \mathbf{JUJ}$ is called "persymmetric".

3 Analysis of the Quincunx Case

In the quincunx case the sampling process will be represented by $\mathbf{D}_q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ with the corresponding sampling factor $N = 2$.

3.1 The Linear Phase Case

Let us first restrict our attention to the FIR filters of the desired diamond shape. Note that a two-dimensional "persymmetric" polynomial $p(x, y)$ is the one for which $p(x, y) = x^n y^m p(x^{-1}, y^{-1})$. The product of two persymmetric polynomials is persymmetric and the sum of two persymmetric polynomials of the same size is again persymmetric.
Bearing in mind the above properties of the persymmetric polynomials we can investigate possibilities yielding perfect reconstruction solutions, or alternatively a polyphase matrix with a monomial determinant. It can be shown that the only possible solution of having diamond shaped, linear phase filters is when both filters are made causal in one dimension and their sizes are $(2k + 1) \times (2k + 1)$ and $(2l + 1) \times (2l + 1)$, where k and l are not both odd or both even at the same time [11]. Thus a polyphase matrix has the following form:

$$\mathbf{H}_p(z_1, z_2) \;=\; \begin{pmatrix} \displaystyle\sum_{i=0}^{k}\sum_{j=0}^{k} a_{ij} z_1^{-i} z_2^{-j} & \displaystyle\sum_{i=0}^{k-1}\sum_{j=0}^{k-1} b_{ij} z_1^{-i} z_2^{-j} \\ \displaystyle\sum_{i=0}^{l}\sum_{j=0}^{l} c_{ij} z_1^{-i} z_2^{-j} & \displaystyle\sum_{i=0}^{l-1}\sum_{j=0}^{l-1} d_{ij} z_1^{-i} z_2^{-j} \end{pmatrix} \tag{8}$$

with $k+l$ odd. Note that since the polynomials involved are persymmetric in each of the above sums we have pairs of identical coefficients, for example $a_{ij} = a_{k-i,k-j}$. Since the determinant has to be a monomial we must have a nonzero coefficient next to $z_1^{-\frac{(k+l-1)}{2}} z_2^{-\frac{(k+l-1)}{2}}$, while all the other ones ($\frac{(k+l)^2-1}{2}$ of them) must equal to zero.
The smallest useful example in this class was found to be (the polyphase components are given):

$$H_{00}(z_1, z_2) \;=\; b(1 + z_1^{-1} z_2^{-1}) + z_1^{-1} + z_2^{-1}, \tag{9}$$

$$H_{01}(z_1, z_2) \;=\; a, \tag{10}$$

$$\begin{aligned} H_{10}(z_1, z_2) \;&=\; b\frac{c}{a}(1 + z_1^{-2} z_2^{-2}) + d z_1^{-1} z_2^{-1} + \\ &+ (b + \frac{c}{a})(z_1^{-1} + z_2^{-1})(1 + z_1^{-1} z_2^{-1}) + z_1^{-2} + z_2^{-2}, \end{aligned} \tag{11}$$

$$H_{11}(z_1, z_2) \;=\; c(1 + z_1^{-1} z_2^{-1}) + a(z_1^{-1} + z_2^{-1}), \tag{12}$$

yielding the following filters:

$$\begin{pmatrix} & 1 & \\ b & a & b \\ & 1 & \end{pmatrix} \quad \begin{pmatrix} & & 1 & & \\ & b + c/a & a & b + c/a & \\ bc/a & c & d & c & bc/a \\ & b + c/a & a & b + c/a & \\ & & 1 & & \end{pmatrix}. \tag{13}$$

The determinant of the above generated basic block is $(a(d-2) - 2bc) z_1^{-1} z_2^{-1}$ corresponding to a z_1^{-2} delay in the upsampled domain. Obviously $a(d-2) - 2bc \neq 0$. Cascades of the polyphase matrices in eqs.(9)-(12) will generate $(2k-1) \times (2k-1)$ and $(2k+1) \times (2k+1)$ size filters retaining

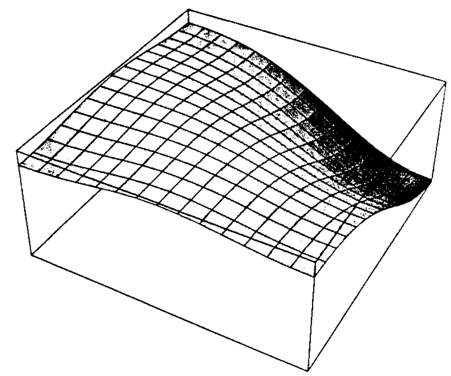

Figure 1: Magnitude of the frequency response of a 5×5 linear phase diamond shaped filter with $a = -4$ and $d = -28$.

the same symmetry properties and perfect reconstruction, and this for $k \geq 2$. This can be shown by induction where the initial step is obvious by inspection. For the inductive step assume that the polyphase components H_{00}, H_{01}, H_{10} and H_{11} are of the following sizes in the polyphase domain: $k \times k$, $(k-1) \times (k-1)$, $(k+1) \times (k+1)$ and $k \times k$ respectively which actually corresponds to $(2k-1) \times (2k-1)$ and $(2k+1) \times (2k+1)$ size filters. Then multiply the corresponding polyphase matrix from the right by the one obtained from eq. (13). By the previously mentioned properties of the persymmetric polynomials it follows that the resulting matrix will contain four persymmetric polyphase components of sizes $(k+1) \times (k+1)$, $k \times k$, $(k+2) \times (k+2)$ and $(k+1) \times (k+1)$ corresponding to $(2k+1) \times (2k+1)$ and $(2k+3) \times (2k+3)$ size linear phase filters □.

By choosing $b = 1$ and $c = a$ one obtains additional circular symmetry. The determinant of a polyphase matrix thus becomes $a(d-4)z_1^{-1}z_2^{-1}$ excluding values of $a = 0$ and $d = 4$ as possible. It is worth noting that (5×5) and (7×7) filters obtained in the next step of the cascade are general solutions for the linear phase filters with circular symmetry and the above size. This can be verified by comparing the corresponding coefficients in the polyphase matrix obtained by the circular cascade and the one which is the general solution to the problem. Special attention has to be paid to the forbidden values which turn out to be the same in both cases, meaning that all the solutions to the general case can be obtained by the cascade. A useful example is obtained by substituting $a = -4$ and $d = -28$ in eq.(13). These filters were successfully applied to HDTV representation and coding [3]. The magnitude of the frequency response of the lowpass filter is given in Figure 1.

Having shown that it is possible to generate cascades of odd-length, different size, linear phase diamond shaped filters let us now relax the shape restriction and show how to obtain cascades of linear phase, nearly diamond shaped filters, but this time of the same size:

$$\mathbf{H}_p(z_1, z_2) = \mathbf{W}_2 \prod_{i=K}^{0} \mathbf{D}_1(z_1, z_2)\mathbf{U}_{i1} \cdot \mathbf{D}_2(z_1, z_2)\mathbf{U}_{i2}, \quad (14)$$

where $\mathbf{W}_2$ is a Walsh-Hadamard matrix of size 2, $\mathbf{D}_1$ is a diagonal matrix with 1 and z_1^{-1} on the diagonal, $\mathbf{D}_2$ is a diagonal matrix with 1 and z_2^{-1} on the diagonal, and $\mathbf{U}_{i1,2}$ are matrices of the form:

$$\mathbf{U}_{i1,2} = \begin{pmatrix} 1 & a_{i1,2} \\ a_{i1,2} & 1 \end{pmatrix}. \quad (15)$$

The proof that the resulting cascade is indeed linear phase is by induction using the testing condition given by eq.(7). The initial step with $k = 0$ is trivial since the matrix is Walsh-Hadamard. For the inductive step assume that (7) holds for some $\mathbf{H}_p$. We want to prove that it holds for $\mathbf{H}'_p = \mathbf{H}_p \cdot \mathbf{D}_1 \cdot \mathbf{U}_1 \cdot \mathbf{D}_2 \cdot \mathbf{U}_2$ as well. To see this let us start from the right hand side of the eq.(7):

$$\begin{aligned} & z_1^{-w_1} z_2^{-w_2} \mathbf{S} \cdot \mathbf{H}'_{p-} \mathbf{J} = \\ &= z_1^{-w_1} z_2^{-w_2} \mathbf{S} \cdot \mathbf{H}_{p-} \mathbf{D}_{1-} \mathbf{U}_1 \mathbf{D}_{2-} \mathbf{U}_2 \mathbf{J}, \\ &= z_1^{-w_1} z_2^{-w_2} z_1^{w_1-1} z_2^{w_2-1} \mathbf{S} \cdot \mathbf{S} \cdot \mathbf{H}_{p+} \mathbf{J} \mathbf{D}_{1-} \mathbf{U}_1 \mathbf{D}_{2-} \mathbf{U}_2 \mathbf{J}, \\ &= z_1^{-1} z_2^{-1} \mathbf{H}_{p+} \mathbf{J} \mathbf{D}_{1-} \mathbf{U}_1 \mathbf{D}_{2-} \mathbf{U}_2 \mathbf{J}, \\ &= \mathbf{H}_{p+} \mathbf{D}_{1+} \mathbf{U}_1 \mathbf{D}_{2+} \mathbf{U}_2, \\ &= \mathbf{H}'_{p+}. \end{aligned} \quad (16)$$

In the above proof $\mathbf{S} = diag(1, -1)$. We also used the fact that the $\mathbf{U}_i$'s are persymmetric and the properties mentioned in the previous section.□

The smallest possible filters generated by the cascade are given by:

$$\begin{pmatrix} & 1 & a_2 & \\ a_1 & a_1 a_2 & a_1 a_2 & a_1 \\ & a_2 & 1 & \end{pmatrix} \quad \begin{pmatrix} & 1 & a_2 & \\ -a_1 & -a_1 a_2 & a_1 a_2 & a_1 \\ & -a_2 & -1 & \end{pmatrix} \quad (17)$$

Though we know that the cascade structure given is not complete the first pair of useful filters given above covers the whole space of linear phase 4×3 filters. For the proof we refer to [11].

3.2 The Paraunitary Case

It can be shown that in this case a general polyphase matrix will be of the following form [11]:

$$\mathbf{H}_p(z_1, z_2) = \begin{pmatrix} H_{00+} & H_{01+} \\ -z_1^{-k} z_2^{-l} H_{01-} & z_1^{-k} z_2^{-l} H_{00-} \end{pmatrix}, \quad (18)$$

which shows that the paraunitary solutions possess some important structural properties.

(a) The filter $H_1(z_1, z_2)$ is completely specified by the filter $H_0(z_1, z_2)$ by modulation with $(-1)^{n_1+n_2}$ and reversal.

(b) The two polyphase components of $H_0(z_1, z_2)$ are of the same size. This requirement automatically excludes the possibility of having paraunitary filters of the desired perfect diamond shape.

(c) Finally, in order to achieve perfect reconstruction the polyphase components of $H_0(z_1, z_2)$ have to meet:

$$H_{00_+}H_{00_-} + H_{01_+}H_{01_-} = 1. \tag{19}$$

Equation (19) can now be met through an optimization. Instead, we derive cascade structures which produce automatically paraunitary filter banks:

$$\mathbf{H}_p(z_1, z_2) = \mathbf{U}_{K-1} \prod_{i=K-2}^{0} \mathbf{D}_i(z_1, z_2)\mathbf{U}_i, \tag{20}$$

where $\mathbf{D}(z_1, z_2)$ is a diagonal matrix of delays and the $\mathbf{U}_i$'s are unitary matrices. It is easy to verify that $\mathbf{H}_p(z_1, z_2)$ in (20) is paraunitary since all the blocks are unitary but completeness of the structure is not guaranteed, unlike the one-dimensional case.
Similarly to the linear phase case we can generate nearly diamond shaped paraunitary filter banks by substituting the corresponding $\mathbf{U}_i$'s in eq. (14) by unitary matrices:

$$\mathbf{U}_{i1,2} = \begin{pmatrix} 1 & -a_{i1,2} \\ a_{i1,2} & 1 \end{pmatrix}. \tag{21}$$

The smallest example in this class is (just the first filter is given):

$$h_0(n_1, n_2) = \begin{pmatrix} & 1 & -a_2 & \\ -a_0a_1 & a_0a_1a_2 & -a_1a_2 & -a_1 \\ & -a_0a_2 & -a_0 & \end{pmatrix}. \tag{22}$$

Starting from a general paraunitary system of the above size, two solutions were obtained both of which can be generated by the first step of the proposed structure. The first one is as given while the second one is produced when the diagonal matrices of delays are interchanged. This in turn shows that at least for the paraunitary filters of size 4×3 the above structure generates a complete solution. The reconstruction filters are the same (within reversal).
Note that in the two-channel case linear phase and paraunitariness requirements are mutually exclusive.

4 Analysis of the Hexagonal Case

The sampling process is now described by a matrix $\mathbf{D_h} = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}$ with the corresponding sampling factor $N = 4$. Similarly to the quincunx case, let us propose a cascade structure which would generate four filters (two symmetric ones and two antisymmetric ones) guaranteeing perfect reconstruction, linear phase and/or paraunitariness. The size of the filters' region of support in each subsequent step would be $2k \times 2(2k-1)$ and the shape is close to hexagonal. Concentrating first on the linear phase solution:

$$\mathbf{H}_p(z_1, z_2) = \mathbf{W}\mathbf{H}_4 \prod_{i=K}^{0} \begin{pmatrix} 1 & & & \\ & z_1^{-1} & & \\ & & z_2^{-1} & \\ & & & z_1^{-1}z_2^{-1} \end{pmatrix} \mathbf{U}_i. \tag{23}$$

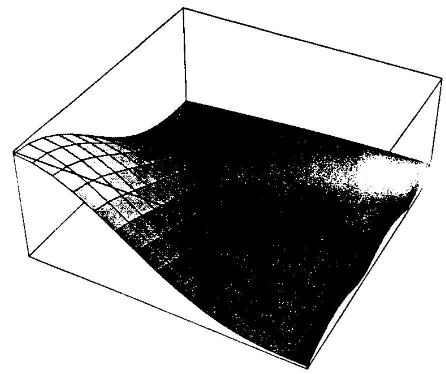

Figure 2: Magnitude of the frequency response of the lowpass filter in the first step of the hexagonal cascade with $a = 2$, $f = 1$, $d = \frac{1}{2}$ and $e = -\frac{1}{2}$.

In the previous equation $\mathbf{U}_i$'s are persymmetric matrices:

$$\mathbf{U}_i = \begin{pmatrix} 1 & a & b & c \\ d & e & f & g \\ g & f & e & d \\ c & b & a & 1 \end{pmatrix}, \tag{24}$$

with the determinant equal to:

$$\begin{aligned} \det(\mathbf{U}) &= ((c+1)(f+e) - (a+b)(g+d)) \cdot \\ &\cdot ((c-1)(f-e) + (a-b)(g-d)), \end{aligned} \tag{25}$$

which has to be nonzero in order to avoid singular blocks. To verify that the cascade produces indeed linear phase filters we can use the same approach as in the proof of the quincunx case. However the symmetry matrix is of the following form $\mathbf{S} = diag(1, -1, -1, 1)$ since the first and the fourth filters are symmetric while the other two are antisymmetric. The smallest useful example would be (just the first filter is given, the other three are similar except for the signs):

$$h_0(n_1, n_2) = \begin{pmatrix} & & 1 & b & & \\ d & f & a & c & g & e \\ e & g & c & a & f & d \\ & & b & 1 & & \end{pmatrix}. \tag{26}$$

Imposing some additional symmetry with $b = 1$, $c = a$, $g = f$ we get the filter:

$$h_0(n_1, n_2) = \begin{pmatrix} & & 1 & 1 & & \\ d & f & a & a & f & e \\ e & f & a & a & f & d \\ & & 1 & 1 & & \end{pmatrix}. \tag{27}$$

The forbidden values are $a \neq \pm 1$, $d \neq e$ and $d + e \neq 2f$. The magnitude of the frequency response of the lowpass filter with $a = 2$, $f = 1$, $d = \frac{1}{2}$ and $e = -\frac{1}{2}$ is given in Figure 2. Note that this is just initial design and thus the quality of the highpass filters is not exceptional.
Unlike the quincunx case however, now we have the option

of obtaining paraunitary and linear phase solutions at the same time. To achieve it the matrix $\mathbf{U}_i$ in addition to being persymmetric has to be unitary. A general 4×4 real unitary matrix can be factorized using 6 rotations. One of the possible solutions for the matrix $\mathbf{U}_i$ would then be:

$$\mathbf{U}_i = \begin{pmatrix} 1 & -a & -b & -ab \\ a & 1 & -ab & b \\ b & -ab & 1 & a \\ -ab & -b & -a & 1 \end{pmatrix}, \tag{28}$$

which is obviously both unitary (within scaling) and persymmetric with two free parameters. The first filter in the smallest useful example is then:

$$h_0(n_1, n_2) = \begin{pmatrix} & & 1 & -b & & \\ a & -ab & -a & -ab & b & 1 \\ 1 & b & -ab & -a & -ab & a \\ & & -b & 1 & & \end{pmatrix}. \tag{29}$$

5 Applications

Recently a technique has been proposed for subband decomposition of a progressively scanned video signal into an interlaced one and "deinterlacing" information [3], [11]. After passing the above signals through a synthesis bank the original progressive one can be perfectly recovered. The results and concepts presented in section 3 are a good tool to provide such a splitting which is useful in dealing with questions such as compatibility and coding. Some of the filters presented have already been used for HDTV with promising results. Also, an interlaced signal can be split into a lowpass progressive one and a highpass "interlacing" signal. Such a scheme would be useful for coding purposes, since the highpass version will be relatively easy to code, and the lowpass version is well suited for motion based processing owing to its progressive nature.
As already pointed out the theory presented in section 2 was developed in a general framework of nonseparable extensions of the one-dimensional concepts both in terms of sampling structures and the filters involved. A problem arising with separable sampling structures is that some of the subbands obtained contain mixed orientations. A possible remedy would be to consider the hexagonal sampling lattice which produces subbands with pure directional orientation [12]. The cascade structure developed in section 4 can be useful to obtain such a splitting while having low computational complexity and allowing filters which are both linear phase and paraunitary.

6 Conclusion and Further Work

In this paper new results in the design of multidimensional filter banks were presented with particular attention to quincunx and hexagonal sampling cases. Useful cascade structures were derived addressing such design issues as linear phase and paraunitary filters while guaranteeing perfect reconstruction. Some applications of hexagonal and quincunx filter banks were pointed out. Further work includes careful design of the filters obtained by the proposed cascades. For example in the hexagonal case all four filters must have good orientational selectivity. More work has to be done in finding other possible design structures.

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