PERFECT RECONSTRUCTION FILTER BANKS WITH RATIONAL SAMPLING RATE CHANGES

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Abstract

This paper presents a general, direct method for designing perfect reconstruction filter banks with rational sampling rate changes, an open problem until now. Such filter banks have N branches, each one having a sampling factor of p_i/q_i and their sum equals to one. A design example showing the advantage of using the direct over the indirect method is given. Due to recent results pointing to the relationship between filter banks and wavelet theory, the regularity question is addressed as well, and a regular filter is shown for a dilation factor of $\frac{3}{2}$.

1 Introduction

The most studied case of filter banks is the one with integer sampling rate changes. However, if one wants to analyze the signal into unequal subbands, rational sampling rates have to be allowed (see figure 1). Then, each channel would have a sampling factor p_i/q_i and their sum equals to one (so as to preserve the sampling density). Although it is known how to solve this problem in practice, since one just has to divide the spectrum in $Q = lcm(q_i)$ parts and then resynthesize the appropriate subspectrums, this approach, being indirect, is suboptimal in terms of computational complexity and filter quality. Previous work in this area was aimed only at aliasing cancellation [1] or solutions that are built through tree splitting. It is worth noting that the tree splitting schemes when only the lowpass signal is subdivided lead to a division of the frequency region into parts of size $\frac{1}{2^i}$ (in case of subsampling by 2) and are similar to the so-called wavelet transform [2].

In this paper we present a general, direct method for designing perfect reconstruction filter banks with arbitrary rational sampling rate changes. Note that this is a continuation of previous work [3] where the direct design method was presented for a subclass of solutions. Section 2 presents the direct method and the tools that enable it. Section 3 gives a design example and compares it to the solution obtained with the indirect method. Section 4 examines the $(\frac{2}{3}, \frac{1}{3})$ case from the wavelet theory point of view.

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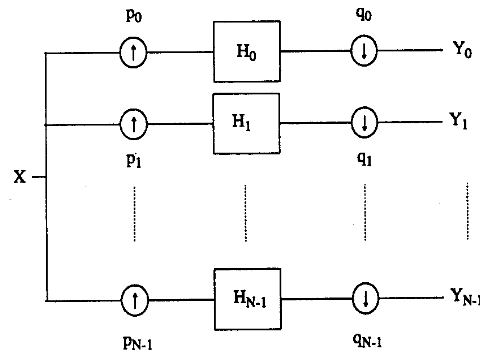

Figure 1: Filter bank with rational sampling rate changes.

2 A Direct Design Method

In this section we want to show how a filter bank with arbitrary rational sampling rate changes can be designed directly. Let us start from the critically sampled filter bank as given in figure 1. We assume that the common factors between each (p_i, q_i) have been cancelled so that they are relatively prime.

First, we show how to transform a single branch with upsampling by p and downsampling by q using a p-channel analysis bank with sampling by q and an inverse polyphase transform of size p. The way to do it is given in figure 2a. The filter in the i-th branch is given by $H_i'(z) = z^{d_i} H_{t_i}(z)$ where $d_i = \lfloor \frac{qi}{p} \rfloor$, $t_i = qi \bmod p$ ($\lfloor x \rfloor$ denotes the biggest integer not greater than x) and $H_0, \ldots H_{p-1}$ are the polyphase components of H with respect to p. It should be noted that for $p = 1$ there is no transform, *i.e.* $H_0' = z^{d_0} H_{t_0} = z^0 H_0 = H$ since the only polyphase component with respect to $p = 1$ is the filter itself. The proof that the two representations are equivalent is given in the Appendix.

Next, we show how to express a single branch with downsampling by q using a p-channel analysis bank with sampling by $Q = pq$ and an inverse polyphase transform of size p. This is given in figure 2b. Note that the filter in the i-th branch is just a shifted version of the original filter, $H_i(z) = z^{iq} H(z)$. The proof that the two representations are equivalent is given in the Appendix.

Finally, figure 3 shows how by using the above trans-

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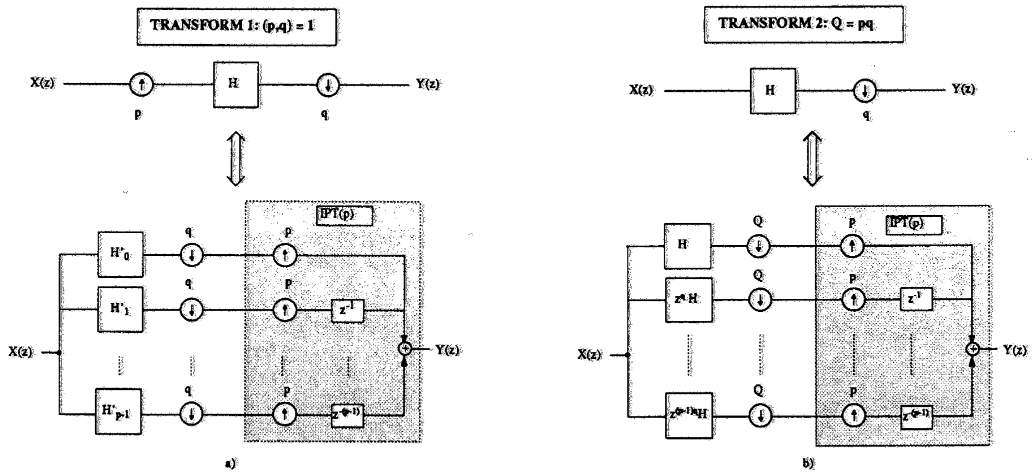

Figure 2: a) Transform 1: expressing a single branch with upsampling by p and downsampling by q using a p-channel analysis bank with sampling by q and an inverse polyphase transform of size p. All the filters involved are just shifted polyphase components of the original filter. For $p = 1$ there is no transform. b) Transform 2: expressing a single branch with downsampling by q using a p-channel analysis bank with sampling by $Q = pq$ and an inverse polyphase transform of size p. All the filters involved are just shifted versions of the original filter.

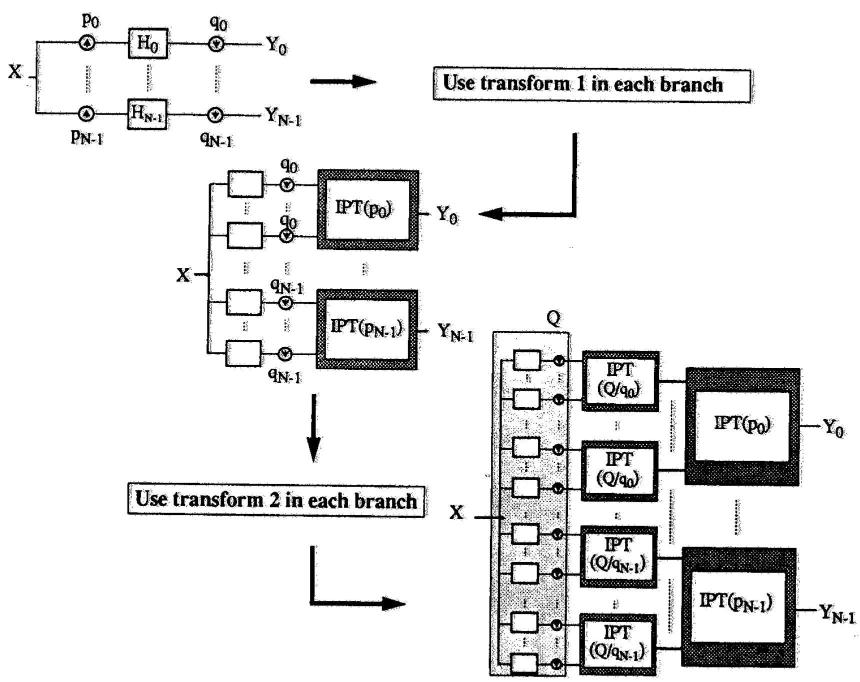

Figure 3: To transform any bank we first apply transform 1 and then transform 2 in each branch. As a result an analysis bank with $Q = \mathrm{lcm}(q_0, \ldots, q_{N-1})$ branches is obtained.

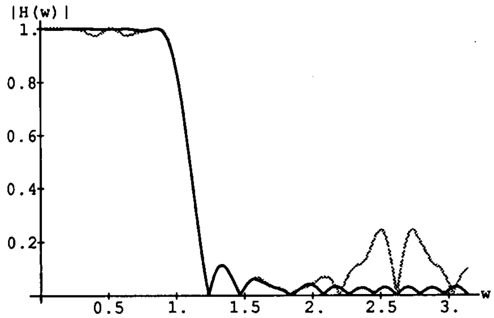

Figure 4: Magnitudes of the frequency responses of the lowpass filters designed using the indirect (gray plot) and direct method (black plot). Note the improvement obtained by using the direct design method (the passband is flattened and the value in the stopband has been reduced).

forms we design a filter bank from figure 1. First we apply transform 1 in each branch which yields an analysis bank with $n = \sum_{i=0}^{N-1} p_i$ branches and sampling rates $q_0, \ldots, q_{N-1}$. Now if $Q = \mathrm{lcm}(q_0, \ldots, q_{N-1})$ we apply transform 2 in each branch to obtain an analysis bank with $n = \sum_{i=0}^{N-1} p_i \cdot \frac{Q}{q_i} = Q \sum_{i=0}^{N-1} \frac{p_i}{q_i} = Q$ branches and sampling by Q and this one we know perfectly how to design!

It is worth noting here the difference between the indirect and the direct method. In the indirect one we design the two stages of the analysis bank separately and moreover we have no idea what kind of characteristics the equivalent filters ($H_0, \ldots, H_{N-1}$ from figure 1) are going to have since we do not know how these filters are related to the filters in the analyzing and resynthesyzing banks. Using a direct method however, allows us to design any filter bank with rational sampling rates, having at the same time complete control over the desired characteristics of the filters $H_0, \ldots, H_{N-1}$.

3 Design Example

As a simple example consider a bank with sampling by $\frac{2}{3}$ and $\frac{1}{3}$. Let us first construct the system indirectly, *i.e.* we design a 3-channel analysis bank and then resynthesyze the first two branches using a 2-channel synthesis bank. We use optimized paraunitary examples that appeared in [4] and [5]. The 3-channel bank contains filters of length 15 and the 2-channel one filters of length 8 with lattice coefficients $a_1 = -2.638026$, $a_2 = 0.7154463$, $a_3 = -0.2598479$ and $a_4 = 0.06388361$. As a result we obtain a lowpass filter of length $14 \cdot 2 + 7 \cdot 3 + 1 = 50$ and a highpass filter which is the third filter from the 3-channel bank. The magnitude response of the lowpass filter is given by the gray plot in figure 4.

Now instead of this method we first obtain the equivalent lowpass filter and then minimize its error in the stopband directly. In order to do that we use the 4 lattice parameters from the 2-channel bank as the minimization variables. We do not touch the 3-channel bank so as not to ruin the highpass filter. The obtained optimized lattice coefficients are $a_1 = -0.371151$, $a_2 = 2.732850$, $a_3 = 1.056070$ and $a_4 = 0.664108$. The magnitude response of the resulting filter is given by the black plot in figure 4. As can be seen from there the improvement is obvious: the passband has been flattened and the stopband has been greatly reduced.

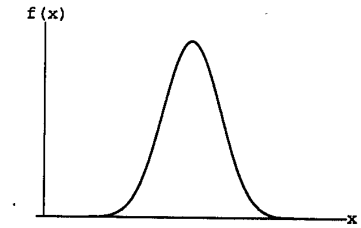

Figure 5: Fifth iteration of the filter $H(z) = (1+z^{-1})^3(1+z^{-1}+z^{-2})^3$ converging to a continuous function $f(x)$.

4 Wavelets With $\frac{3}{2}$ Dilation Factor

In this section we address some of the questions that arise when looking at the filter bank problem from the wavelet theory point of view restricting ourselves at the same time to a representative case, namely sampling by $\frac{2}{3}$ and $\frac{1}{3}$. Let us point out that to establish correspondence between filter banks and wavelets we split the branch with the lowpass filter using the same filter bank. Repeating this procedure to infinity, the *wavelet* and the *scaling function* can be identified, scaling function as the equivalent filter in the path going through all lowpass branches, and wavelet in the same path except that in the last stage we go through the highpass branch. For more details see [6, 7].

When constructing wavelets of compact support, one would like them to be continuous functions and this can be achieved when the lowpass filter meets a so-called *regularity* condition. For sampling by 2, Daubechies in [6] gives a sufficient condition for the iterated filter to converge to a continuous function. It basically states that for a filter to be regular we have to impose a sufficient number of zeroes at π (aliasing frequency) and attenuate enough the remaining factor. Following the same reasoning we conjecture that in this case a filter having sufficient number of zeroes at π, $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ would be regular. To corroborate this statement we construct a filter having three zeroes at each location, *i.e.* $H(z) = (1+z^{-1})^3(1+z^{-1}+z^{-2})^3$. Figure 5 shows graphically how the iterated filter converges to a continuous function. To complete the perfect reconstruction system we give one of the possible highpass filters as $H_1(z) = (z^{-1}-1)^2(1+20z^{-1}+\frac{4287}{67}z^{-2}+20z^{-3}+z^{-4})$. Note that the synthesis part of this system would give rise to nonregular filters. Thus, we have constructed a biorthogonal basis with regular analysis. The next logical step would be to see whether we can construct an orthonormal basis with compactly supported wavelets. Unfortunately it turns out that this is not possible (for proof see [8]).

5 Conclusion

In this paper the solution to the problem of designing perfect reconstruction filter banks with arbitrary rational sampling rates is given. A design example showing the advantage of using this method over the indirect one is presented. And finally, the case with $(\frac{2}{3}, \frac{1}{3})$ sampling was examined from the wavelet theory point of view. We conjectured how to construct regular filters and gave an example.

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A. Appendix

Here we sketch the proofs for the equivalence of representations in figure 2. For more details see [9]. Refer to 2a. Since by assumption $\gcd(p,q) = 1$, in each branch of the bank in the lower part of the figure we can interchange upsampler with the downsampler and then move filters and delays across samplers (for more details refer to [3]). The equivalent filter obtained in this manner will be denoted by H'. We want to show that it equals H. Thus:

$$H'(z) \;=\; \sum_{i=0}^{p-1} z^{-(iq - p\lfloor\frac{qi}{p}\rfloor)} H_{qi \bmod p}(z^p) \;=\; H(z), \qquad (1)$$

since $(iq - p\lfloor\frac{qi}{p}\rfloor) = qi \bmod p$ and for q and p coprime $(qi \bmod p)$ covers the whole set of integers from $\{0, \ldots, p-1\}$.□

Now we prove the equivalence of the two representations in figure 2b. Let us denote the output in the upper figure by Y_1 and in the lower one by Y_2. Also in the lower figure the output of the i-th branch after the delays will be denoted by Y_i'. We want to show that Y_1 and Y_2 are the same. In the proofs the following two facts are going to be used:

$$W_{N_1 N_2}^{N_1} \;=\; W_{N_2}, \quad \sum_{i=0}^{p-1} W_p^{ik} = \begin{cases} p & \text{if } k \bmod p = 0 \\ 0 & \text{if } k \bmod p \neq 0 \end{cases} \qquad (2)$$

where W_N denotes the N-th root of unity. The output after filtering and downsampling by q can be written as:

$$Y_1(z) \;=\; \frac{1}{q} \sum_{k=0}^{q-1} H(W_q^k z^{\frac{1}{q}}) X(W_q^k z^{\frac{1}{q}}).$$

In the lower figure, the output of the i-th branch before the adder can be expressed as:

$$Y_i'(z) \;=\; \frac{1}{Q} \sum_{k=0}^{Q-1} W_p^{ki} H(W_Q^k z^{\frac{1}{q}}) X(W_Q^k z^{\frac{1}{q}}). \qquad (3)$$

Summing all of them up we get:

$$Y_2(z) \;=\; \frac{1}{Q} \sum_{k=0}^{Q-1} H(W_Q^k z^{\frac{1}{q}}) X(W_Q^k z^{\frac{1}{q}}) \sum_{i=0}^{p-1} W_p^{ki}. \qquad (4)$$

Using (2) it becomes obvious that in the previous equation $Y_2(z)$ is non-zero only when $k \bmod p = 0$ and this when k goes from 0 to $(Q-1)$ happens exactly q times, for $k = 0$, $k = p$, $k = 2p$, ..., $k = (q-1)p$. Thus we can rewrite the previous equation as (with $k = tp$):

$$Y_2(z) \;=\; \frac{1}{q} \sum_{t=0}^{q-1} H(W_q^t z^{\frac{1}{q}}) X(W_q^t z^{\frac{1}{q}}) \;=\; Y_1(z).\square \qquad (5)$$

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