Towards a Quantization Theorem

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Analog to digital conversion consists in discretizing an analog signal in time and in amplitude. Shannon's well known theorem guarantees that, when a bandlimited signal is sampled in time at the Nyquist rate or above, no information is lost. It also gives the analytical expression of reconstruction of the bandlimited signal from its samples. Results on reconstruction where also obtained by Logan [1] when the analog signal is discretized in amplitude. Under certain mathematical assumptions, he showed that an octave band signal is uniquely defined by its zero crossings, up to a multiplicative constant. This is indeed the situation when the analog signal is discretized in amplitude into two regions: positive amplitudes and negative amplitudes.

However, none of these results deal with simultaneous discretization in time and in amplitude. In the case of Shannon's theorem, the samples are assumed to keep their continuous amplitudes, and in Logan's theorem, the zero crossings are supposed to be known by their exact continuous valued instants. In practical A/D conversion (ADC), one uses the fact that, after a bandlimited signal is sampled under the Nyquist conditions, amplitude quantization gives an approximate version of the sampled signal, including an error called quantization error. In the oversampled situation, where the sampling frequency is higher than the Nyquist rate, the quantization error can be reduced by lowpass filtering the quantized signal. However, this approach uses only linear and time invariant processing.

Our motivation is to study the exact information available about an analog signal when it is discretized both in time and in amplitude, and derive theoretical laws of reconstruction. We deal with simple ADC where samples are quantized individually, as well as with "noise-shaping" ADC [2] where an integrated and dithered version of the sequence of samples is quantized. Unlike in the situation of Shannon's theorem or Logan's theorem where reconstruction is unique, we show that when a signal is discretized both in time and in amplitude, there is a whole set of possible reconstructions: the set of all analog signals giving the same discrete signal. Nevertheless, this set is uniquely defined by and can be derived from the knowledge of the discrete signal: we call it the signal cell corresponding to the discrete signal.

This property has a consequence on the interpretation of ADC. In general, ADC amounts to dividing the considered space of analog input signals into a partition of such signal cells. We show that these cells are necessarily convex and as a consequence derive the following law of reconstruction: *"the reconstruction of an analog signal given by its discretized version should be chosen in the corresponding signal cell"*. We show in figure 1(a) the input space partition defined by a single-bit single-loop $\Sigma\Delta$ modulator in the case where input signals are zero phase sinusoids with dc component, with the oversampling rate $R = 4$.

Working with the assumption that the analog input signals are bandlimited and periodic on the time window of discretization, we derived upper bounds to the error remaining in such a theoretical reconstruction, and compared them with the performances obtained in classical reconstruction (linear filtering of the quantized signal). We show that, under certain conditions on the quantization threshold crossings of the analog signal, our theoretical reconstruction yields a mean squared error (MSE) at least inversely proportional to R^2, where R is the oversampling rate. We recall that the MSE in classical reconstruction is only inversely proportional to R. Finding MSE upper bounds in the case of n^{th} order noise-shaping ADC is more difficult. However, with the assumption that the error signal generated by the quantizer is composed of uncorrelated and uniformly distributed samples, we show that the MSE of our theoretical reconstruction is inversely proportional to R^{2n+2}, instead of R^{2n+1} in classical reconstruction. This represents

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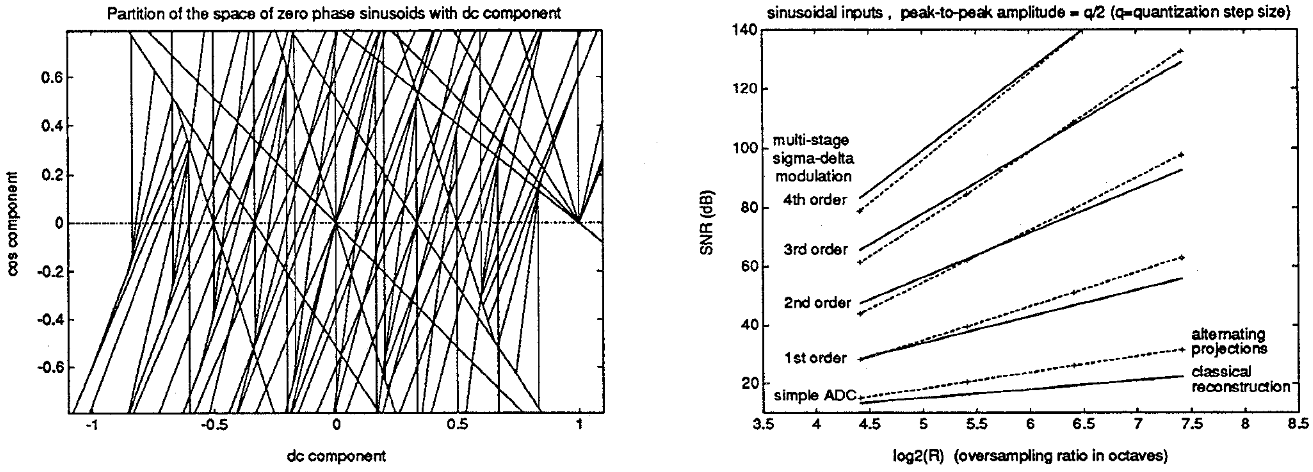
an improvement of the signal to noise ratio (SNR) slope of 3 dB per octave of oversampling. Numerical experiments using algorithms based on the principle of alternating convex projections were performed and verified the SNR slope improvement of 3 dB/octave (see figure 1(b)).

The ultimate goal of our investigation is to derive what could be called a "quantization theorem", that is, a result indicating how much information is contained in the discrete time and discrete amplitude version of a bandlimited signal. In the case of simple ADC, the result was derived [6] under a certain assumption about the input signal. In the case of noise-shaping ADC, we obtained upper bounds on the error [6] and we conjecture that the lower bounds are of the same order (under a certain assumption about the quantization noise). These initial results on a quantization theorem, together with experimental results, already indicate that the reconstruction from a discrete version can be much better than previously thought.



(a) (b)

Figure 1: (a) Partition defined by a single-bit single-loop $\Sigma\Delta$ modulator on the space of zero phase sinusoidal input signals with dc component, with the oversampling rate $R = 4$. The partition restricted to constant inputs (broken horizontal line) was studied by Hein and Zakhor [3]. (b) SNR of signal reconstruction versus oversampling rate, with classical and alternating projection method, for simple ADC and 1^{st} to 4^{th} order single-bit multi-stage $\Sigma\Delta$ modulation with sinusoidal input signals.

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