Orthogonalization of Compactly Supported Wavelet Bases[1]

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The relation between orthonormal bases of compactly supported wavelets, and Finite Impulse Response (FIR) filter banks has been firmly established [2]. Briefly put, one can use a filter bank to generate the basis functions for a multiresolution analysis scheme. The essential requirements for such a scheme, as introduced by Mallat and Meyer, are a nested set of function spaces V_j, $n; \epsilon Z$ and a function $g(x)$ such that the set $\{g(x-n), n; \epsilon Z\}$ is a basis for V_0.

Of course many different bases for the same space will exist in general. The result we present is that if $g(x)$ is supported on an interval, then we can easily construct an alternative orthogonal basis for V_0, using realizable Infinite Impulse Response (IIR) filters.

Theorem 1 *If the set $\{g(x-k), k \epsilon Z\}$ forms a non-orthogonal basis for V_0, obeys a two-scale difference equation, and $g(x)$ is compactly supported, then it is always possible to find an orthonormal basis $\{\phi(x-k), k \epsilon Z\}$, where:*

$$\Phi(w) = \prod_{i=1}^{\infty} 2^{-1/2} H_0(e^{jw/2^i}), \tag{1}$$

and where $H_0(e^{jw})$ is a rational function of e^{jw}.

The proof is in [3]. The idea is easily illustrated. Suppose that $L(e^{jw})$ is the FIR filter that generates the compactly supported basis $g(x)$, that is:

$$G(w) = \prod_{i=1}^{\infty} 2^{-1/2} L(e^{jw/2^i}). \tag{2}$$

It turns out that the IIR filter used to generate the orthogonal basis has the form:

$$H_0(e^{jw}) = \frac{L(e^{jw}) \cdot E(e^{jw})}{E(e^{j2w})}, \tag{3}$$

for some FIR filter $E(e^{jw})$. Using this in the Fourier domain description of the scaling function, we find that because of (3) successive numerators and denominators of the product cancel:

$$\begin{aligned} \Phi(w) = \prod_{i=1}^{\infty} 2^{-1/2} H_0(w/2^i) &= \prod_{i=1}^{\infty} 2^{-1/2} L(w/2^i) \cdot \prod_{i=1}^{\infty} \frac{E(e^{jw/2^i})}{E(e^{j2w/2^i})} \\ &= G(w) \cdot \frac{E(e^{jw/2})}{E(e^{jw})} \cdot \frac{E(e^{jw/4})}{E(e^{jw/2})} \cdots = G(w) \cdot \frac{E(e^{j0})}{E(e^{jw})}. \end{aligned} \tag{4}$$

So the infinite product for $H_0(e^{jw})$ necessarily converges since that for $L(e^{jw})$ does. In the time domain $\phi(x)$ is a linear combination of integer shifts of the function $g(x)$. This also means that we do not have to separately make regularity estimates for $\Phi(w)$ if the regularity of $G(w)$ is known, since they will be precisely equal.

The nested structure of the multiresolution analysis scheme means that having an orthonormal basis for V_0 gives orthonormal bases for all of the V_j spaces; thus we get an orthogonal wavelet

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basis. The implication is that any non-orthogonal wavelet basis, which is compactly supported, can be replaced by an orthogonal one which is generated by realizable IIR filters. An important special case is to find bases for the spline function spaces. First note that a compactly supported non-orthogonal basis is indeed available for the N-th order spline space. This is given by the N-th order B-spline function, which is defined by: $g(x) = s(x) * s(x) \cdots s(x)$, where there are $N-1$ convolutions, and $s(x)$ is the characteristic function of the interval $[0,1)$. So the basis is the set $\{g(x-k), k \epsilon Z\}$. To get an orthogonal basis from this we apply Theorem 1. The construction is straightforward, and the details are in [3]. That the wavelet and scaling function are indeed splines is most easily seen for the $N = 2$ case, which is shown in Figure 1.

Bases can thus be generated for spline spaces of any order. These are alternatives to the bases of Battle and Lemarié [1, 5], but have the advantage of being based on realizable filters. In the spline case our solutions are equivalent to those of Stromberg [4], but this is of course just one example; Theorem 1 can be applied to any space that has a compactly supported basis.

References

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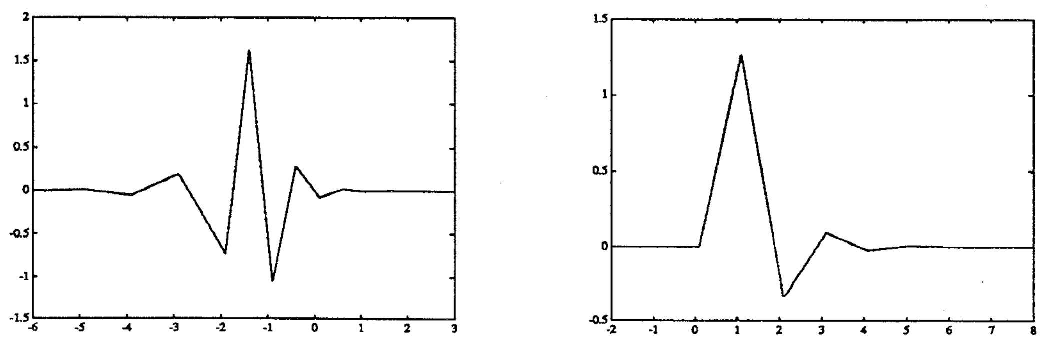

(a) (b)

Figure 1: Orthogonal IIR wavelet basis for piecewise linear spline space. (a) Wavelet (b) Scaling function.