New Results on Multidimensional Filter Banks and Wavelets *

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ABSTRACT

New results on multidimensional filter banks and wavelets are presented. We concentrate on the two-dimensional case with sampling by 2 in each dimension and show how, by using an appropriate cascade structure, nonseparable filters being both orthogonal and linear phase can be obtained, a construction not possible when one-dimensional systems are used separately. Condition for having at least one zero of the lowpass filter at all aliasing frequencies (necessary for convergence) is given. Finally, a first design example of two-dimensional orthonormal wavelets with symmetries is presented, together with the demonstration of continuity.

I. INTRODUCTION

During the past decade, the field of filter banks, or subband coding, has established itself firmly as one of the very successful methods for compressing signals ranging from speech to images to video [1, 2]. At the same time, and from another field – applied mathematics, the theory of wavelets emerged as a powerful tool for providing time-frequency localized expansions of signals [3]. Recently, it has been shown that the two – filter banks and wavelets, are closely connected, in that one can use iterated filter banks to obtain continuous-time wavelet bases [3, 4], as well as see filter banks as a "discrete wavelet transform" [5].

While most of these developments concentrated on one-dimensional signals, and the multidimensional case was handled via the tensor product (applying one-dimensional techniques separately along the dimensions, *e.g.* for images), some of the more recent efforts concentrated on the "true" multidimensional case, both from the filter bank and the wavelet aspects [6, 7, 8, 9]. By true we mean that both nonseparable sampling and filtering are allowed. Although the true multidimensional approach suffers from some drawbacks (*e.g.* higher computational complexity, analysis becomes more involved), it offers a few important advantages. For example, using nonseparable filters leads to more degrees of freedom in design, and consequently better filters. Then, nonseparable sampling opens a possibility of having schemes better adapted, for example, to the human visual system. Finally, some solutions, previously impossible, can be achieved using true multidimensional systems.

*Work supported in part by the National Science Foundation under grants ECD-88-11111 and MIP-90-14189.

One such instance is the design of linear phase and orthonormal filters when the sampling is separable by two in two dimensions (a case of practical interest in image compression). It is well-known that in the one-dimensional two-channel case, the above requirements are mutually exclusive (assuming filters have real coefficients and are FIR). This, in turn implies, that the same restriction would hold if the system were used separately in each dimension. However, by using nonseparable filters, one can actually find solutions producing filters being both linear phase and orthonormal [10]. The aim of this paper is to show how, by using these solutions, one can construct two-dimensional orthonormal bases of wavelets with symmetries. We offer the first example of a two-dimensional compactly supported wavelet basis with the above properties.

II. SEPARABLE SAMPLING BY 2 IN 2 DIMENSIONS

We have mentioned earlier that a "true" multidimensional treatment will imply nonseparable sampling or nonseparable filtering. Here, we will concentrate on a separable sampling lattice widely used in image compression, while the filters themselves will be nonseparable.

Let us thus examine the system given in Fig. 1(a) with the sampling lattice as in Fig. 1(b). It is obvious from the figure that this lattice is separable (*i.e.* the sampling could be performed first along the rows and then along the columns). Since the sampling density is 4 (*i.e.* one out of every 4 samples is kept while all others are discarded), the corresponding critically sampled filter bank has 4 channels. The analysis filters are denoted by $H_0(z_1,z_2),\ldots,H_3(z_1,z_2)$, and they are nonseparable. Their synthesis counterparts are denoted by $G_0(z_1,z_2),\ldots,G_3(z_1,z_2)$. Then, the input/output relationship is given by

$$\begin{aligned} Y(z_1,z_2) &= \frac{1}{4}\begin{pmatrix} 1 & z_1^{-1} & z_2^{-1} & z_1^{-1}z_2^{-1} \end{pmatrix} \cdot \\ &\cdot \mathbf{G}_p(z_1^2,z_2^2)\;\mathbf{H}_p(z_1^2,z_2^2)\;\mathbf{x}_p(z_1^2,z_2^2). \end{aligned} \qquad (1)$$

In the above, the matrices $\mathbf{H}_p$, $\mathbf{G}_p$, are the so-called analysis/synthesis polyphase matrices containing the polyphase components of analysis/synthesis filters, respectively (polyphase components are the filters' impulse responses with respect to the 4 cosets of the sampling lattice). Similarly, $\mathbf{x}_p$ contains the 4 polyphase components of the input signal.

To obtain perfect reconstruction (*i.e.* the output signal is the perfect replica of the input, possibly scaled and delayed), one has to ensure that

$$\mathbf{G}_p(z_1^2,z_2^2)\;\mathbf{H}_p(z_1^2,z_2^2) = c\cdot z_1^{-k_1}z_2^{-k_2}\mathbf{I}. \qquad (2)$$

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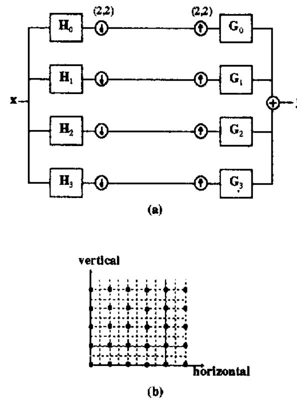

Figure 1: Filter bank with sampling by 2 in 2 dimensions. (a) Schematic representation. (b) Sampling lattice.

For FIR filters, the necessary and sufficient condition is that the determinant of the analysis polyphase matrix be a delay [7, 8]. An easy way to achieve that is to build $\mathbf{H}_p$ using cascade structures since then the perfect reconstruction property is structurally guaranteed, and moreover, some other properties (*e.g.* linear phase) can be easily imposed.

III. LINEAR PHASE AND ORTHOGONAL CASCADE

A. Cascade Structure

The way the first linear phase cascade structure (in one dimension) was constructed in [11], was by using a linear phase testing condition. This condition was later extended to two and more dimensions, and was presented in a more general form as well (not only in the polyphase domain) [7, 8]. The way it could be written for this particular case is [10]

$$\mathbf{H}_p(z_1, z_2) \;=\; (z_1 z_2)^{-k} \cdot \mathbf{a}\; \mathbf{H}_p(z_1^{-1}, z_2^{-1})\; \mathbf{J}, \qquad (3)$$

where $\mathbf{a}$ is a diagonal symmetry matrix containing ± 1's along the diagonal (1 for a symmetric filter and -1 for an antisymmetric filter) and $\mathbf{J}$ is an antidiagonal matrix.

Using the above, in [10], a cascade structure was presented generating four linear phase and orthogonal filters of the same size, where two of them are symmetric and two are antisymmetric

$$\mathbf{H}_p(z_1, z_2) \;=\; \mathbf{H}_{p0} \prod_{i=1}^{k} \mathbf{D}(z_1, z_2)\mathbf{U}_i, \qquad (4)$$

and $\mathbf{H}_{p0}$ was chosen to be the matrix representing the Walsh-Hadamard transform of size 4, $\mathbf{D}$ is the diagonal matrix of delays $(1 \;\; z_1^{-1} \;\; z_2^{-1} \;\; z_1^{-1}z_2^{-1})$, and $\mathbf{U}_i$ are scalar persymmetric (*i.e.* they satisfy $\mathbf{U}_i = \mathbf{J}\mathbf{U}_i\mathbf{J}$) matrices of the following form:

$$\mathbf{U}_i \;=\; \begin{pmatrix} a_i & c_i & b_i & d_i \\ c_i & -a_i & -d_i & b_i \\ b_i & -d_i & -a_i & c_i \\ d_i & b_i & c_i & a_i \end{pmatrix}, \qquad (5)$$

and are required to be unitary. Note that with different signs in the above matrix, the cascade would produce non-orthogonal linear phase filters, but with added symmetries. However, it turns out that without simplifications, the above cascade is very difficult to use for constructing wavelets.

B. Factorization

It has been observed that the number of channels (or sampling density in critically sampled filter banks) is responsible for the fact that many results look similar (at least algebraically) irrespective of the number of dimensions. Thus, for example, the above cascade was developed having in mind the one in [11]. By the same token, we are going to use some results developed for the one-dimensional case to simplify our cascade [12].

Note first that the matrix $\mathbf{U}_i$ can be expressed as follows:

$$\begin{aligned} \mathbf{U}_i &= \begin{pmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{J}\mathbf{B}_i\mathbf{J} & \mathbf{J}\mathbf{A}_i\mathbf{J} \end{pmatrix}, \\ &= \underbrace{\begin{pmatrix} \mathbf{I} & \\ & \mathbf{J} \end{pmatrix}}_{\mathbf{P}} \begin{pmatrix} \mathbf{A}_i & \mathbf{B}_i\mathbf{J} \\ \mathbf{B}_i\mathbf{J} & \mathbf{A}_i \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{I} & \\ & \mathbf{J} \end{pmatrix}}_{\mathbf{P}}, \\ &= \mathbf{P} \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{pmatrix}}_{\mathbf{W}} \underbrace{\begin{pmatrix} \mathbf{r}_{2i} & \\ & \mathbf{r}_{2i+1} \end{pmatrix}}_{\mathbf{R}_i} \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{pmatrix}}_{\mathbf{W}} \mathbf{P}, \end{aligned}$$

where the notation is following [12]. Then for the above matrix to be unitary, $\mathbf{r}_{2i}$ and $\mathbf{r}_{2i+1}$ have to be unitary as well, that is, each $\mathbf{r}_i$ is characterized by one degree of freedom ($\binom{N/2}{2}$ in the general case of N channels, where N is even). Also, the starting matrix $\mathbf{H}_{p_0}$ has to be unitary and satisfy the testing condition (3), and thus

$$\mathbf{H}_{p0} \;=\; \mathbf{R}_0\; \mathbf{W}\; \mathbf{P}. \qquad (6)$$

Therefore, the whole structure can be written as

$$\mathbf{H}_p(z_1, z_2) \;=\; \mathbf{R}_0\mathbf{W}\mathbf{P} \prod_{i=1}^{k} \mathbf{D}(z_1, z_2)\mathbf{P}\mathbf{W}\mathbf{R}_i\mathbf{W}\mathbf{P}, \qquad (7)$$

where each matrix $\mathbf{r}_i$ in $\mathbf{R}_i$ is one Givens rotation

$$\mathbf{r}_i \;=\; \begin{pmatrix} \cos\alpha_i & \sin\alpha_i \\ -\sin\alpha_i & \cos\alpha_i \end{pmatrix}. \qquad (8)$$

The filters obtained in this manner are going to be orthogonal, have linear phase and will be of size $2(k+1) \times 2(k+1)$. The number of degrees of freedom will then be $2(k+1)$. Note also that, as in [12], the above cascade is valid for all filter banks with an even number of channels. However, it is not clear whether it is complete, even in the sense of [12].

IV. WAVELET DESIGN AND REGULARITY

A way to obtain wavelet bases is by iterating filter banks, as shown in Fig. 2. Then, a continuous-time, piecewise constant function at any point in the system is constructed from the equivalent discrete-time filter. By iterating to infinity, one can identify the scaling function, as the result of iterating the lowpass branch, and wavelets, as the result of going through all lowpass iterations and one highpass branch. To ensure that the resulting functions converge and are in L_2, it has been shown in [8, 13] that at least one zero at all aliasing frequencies is necessary.

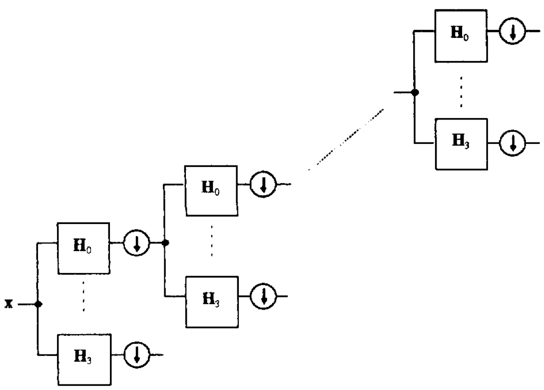

Figure 2: Filter bank with iterated lowpass branch used for constructing continuous-time wavelet bases.

A. One Zero Is Necessary

Let us therefore try to impose one zero at all aliasing frequencies, that is, at $(z_1, z_2) = (-1,-1)$, $(z_1, z_2) = (1,-1)$, $(z_1, z_2) = (-1, 1)$. First note that to obtain filters, one has to upsample the polyphase matrix, that is

$$\begin{pmatrix} H_0(z_1, z_2) \\ H_1(z_1, z_2) \\ H_2(z_1, z_2) \\ H_3(z_1, z_2) \end{pmatrix} = \mathbf{H}_p(z_1^2, z_2^2) \begin{pmatrix} 1 \\ z_1^{-1} \\ z_2^{-1} \\ z_1^{-1} z_2^{-1} \end{pmatrix}. \quad (9)$$

Thus, for $z_{1,2} = \pm 1$, the diagonal matrix of delays in (7), $\mathbf{D}(z_1, z_2)$, becomes an identity matrix, and bearing in mind that $\mathbf{P}^2 = \mathbf{W}^2 = \mathbf{I}$,

$$\mathbf{H}_p(z_1^2, z_2^2)\,|_{z_{1,2}=\pm 1} = \mathbf{R}_0\mathbf{W}\mathbf{P}\prod_{i=1}^{k}\mathbf{P}\mathbf{W}\mathbf{R}_i\mathbf{W}\mathbf{P} \quad (10)$$

$$= (\prod_{i=0}^{k}\mathbf{R}_i)\mathbf{W}\mathbf{P}. \quad (11)$$

But

$$\prod_{i=0}^{k}\mathbf{R}_i = \begin{pmatrix} \prod_{i=0}^{k}\mathbf{r}_{2i} & \\ & \prod_{i=0}^{k}\mathbf{r}_{2i+1} \end{pmatrix}, \quad (12)$$

and

$$\prod_{i=0}^{k}\mathbf{r}_{2i} = \begin{pmatrix} \cos\sum_i \alpha_{2i} & \sin\sum_i \alpha_{2i} \\ -\sin\sum_i \alpha_{2i} & \cos\sum_i \alpha_{2i} \end{pmatrix}, \quad (13)$$

and similarly for $\prod_{i=0}^{k}\mathbf{r}_{2i+1}$. Calling $a_i = \sum_i \alpha_{2i}$ and $b_i = \sum_i \alpha_{2i+1}$, we finally obtain

$$\mathbf{H}_p(z_1^2, z_2^2)\,|_{z_{1,2}=\pm 1} = \begin{pmatrix} \cos a_i & \sin a_i & \sin a_i & \cos a_i \\ -\sin a_i & \cos a_i & \cos a_i & -\sin a_i \\ \cos b_i & \sin b_i & -\sin b_i & -\cos b_i \\ -\sin b_i & \cos b_i & -\cos b_i & \sin b_i \end{pmatrix}.$$

Now the condition that $H_0(-1,-1) = 0$ translates to

$$\cos a_i = \sin a_i. \quad (14)$$

Bearing also in mind that $H_0(1, 1) = 2$ [8], we get that

$$a_i = \sum_i \alpha_{2i} = 2n\pi + \frac{\pi}{4}, \quad (15)$$

that is, the sum of all even angles has to equal $2n\pi + \pi/4$. In a similar manner, we get that $H_0(1,-1) = H_0(-1, 1) = 0$ by construction. Thus, it is sufficient for the sum of all even angles to satisfy (15), and the lowpass filter will have a zero at all three aliasing frequencies. This is similar to the condition in the one-dimensional two-channel case, where the sum of all angles has to be $\pi/4$ [14].

B. Design Example

Having presented a cascade structurally producing filters being both orthogonal and linear phase, let us now give a design example leading to a continuous-time orthonormal wavelet basis characterized by a scaling function $\varphi(t_1, t_2)$ and three "mother" wavelets $\psi_i(t_1, t_2)$, $i = 1, 2, 3$, where both the scaling function and the wavelets are symmetric/antisymmetric.

We start by using (7) with $k = 2$ leading to filters of size 6×6, and requiring the lowpass filter to have a second-order zero at all three aliasing frequencies, that is

$$H_0(z_1, z_2) = 0, \quad \frac{\partial}{\partial z_{1,2}} H_0(z_1, z_2) = 0, \quad (16)$$

for $(z_1, z_2) = (-1,-1)$, $(z_1, z_2) = (1,-1)$, $(z_1, z_2) = (-1, 1)$. Upon solving the set of nonlinear equations, one gets the following solution:

$$\alpha_0 = \frac{\pi}{4}, \quad \alpha_1 = \pi - \arcsin\frac{1}{4}, \quad (17)$$

$$\alpha_2 = 0, \quad \alpha_3 = 2\arcsin\frac{1}{4} - \pi, \quad (18)$$

$$\alpha_4 = 0, \quad \alpha_5 = -\frac{\pi}{2} - \arcsin\frac{1}{4}. \quad (19)$$

It is obvious from the above that the even angles indeed sum up to $\pi/4$ as required by (15).

C. Regularity Estimates

Once the solution is found, one has to verify that the function it converges to will be at least continuous. A fast way of estimating it is by monitoring the behavior of the largest first-order differences of the iterates. For this solution, the maximum first-order differences decrease with an almost constant rate. However, this is only an indicator. In [15], the author develops a method for checking the regularity of a two-dimensional filter. Since the theory behind it is quite involved, here we just outline the process, for more details, the reader is referred to [15]. First, one computes the iterate and then finite differences in two directions. This is followed by identifying all polyphase components of these finite differences, and by finding estimates for each one of them. Finally, the maximum of all the above estimates – ρ – is found. Then the lower bound is [15]

$$s = -\log_{\sqrt{2}}\rho, \quad (20)$$

which in our case is found in iteration 6 and equals to $s = 0.4011$ which tells us that the function is at least continuous (it is continuous when $s > 0$). The fourth iteration of the scaling function and the corresponding wavelets is given in Fig. 3 (frontal view is given so as to make the symmetries obvious).

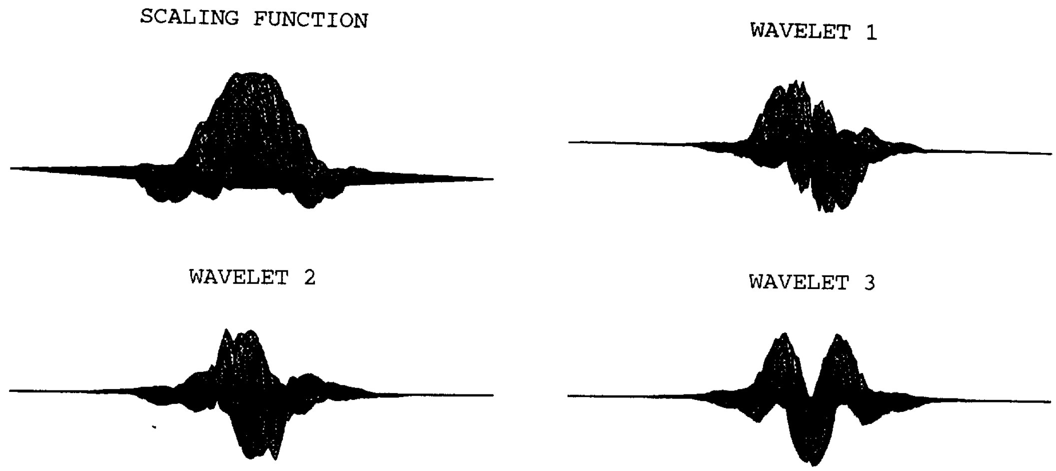

Figure 3: Fourth iteration of the scaling function and wavelets obtained using cascade (7) with $k = 2$ and angles as in (17)-(19).

V. CONCLUSION

We have shown how one can obtain two-dimensional nonseparable filters/wavelets being both orthogonal and linear phase, construction which was not possible when one-dimensional systems were used separately. We use a particular cascade structure, which structurally guarantees that the filters obtained will be both orthogonal and linear phase. Owing to the necessity of at least one zero of the lowpass filter at all aliasing frequencies (for convergence purposes), we found the requirement on the angles in the cascade so that the condition is satisfied. Finally, a design example of orthonormal wavelets with symmetries is given together with the demonstration of continuity.

ACKNOWLEDGMENT

The authors would like to thank Dr. Villemoes for his help in determining the regularity of the filters, as well as for the software he supplied.

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