WAVELET EXTREMA AND ZERO-CROSSINGS REPRESENTATIONS: PROPERTIES AND CONSISTENT RECONSTRUCTION

Zoran Cvetković [*] *and Martin Vetterli* [†]

[*]Department of EE, Columbia University, New York, NY 10027, USA.
[†]EECS Department, UC Berkeley, Berkeley, CA 94720, USA.

ABSTRACT

Properties of nonsubsampled FIR filter banks, used for wavelet extrema and zero-crossings representations, are investigated. Conditions under which iterated nonsubsampled filter banks implement frame or tight frame operators are given and relations with continuous time domain are established. Algorithms for consistent reconstruction of signals from wavelet extrema or zero crossings representations are proposed. These algorithms are characterized by a low computational complexity and a simple implementation. It is also shown that the wavelet extrema representation of a signal and the wavelet zero-crossings representation of its difference provide equivalent information on the signal.

1. INTRODUCTION

The introduction of wavelet extrema and zero crossings representations [1], [2] was motivated by their ability to extract information on signals' sharp variations. Besides, the representation of a signal by zero crossings, or modulus maxima of its wavelet transform can be also viewed as a sampling of the multiscale wavelet transform at points which have some physical importance, providing at the same time shift invariant signal description.

The analysis presented here is based on the discrete time model introduced by Berman et al. [3]. A particular type of discrete wavelet transform used for wavelet extrema or zero crossings representations is implemented by octave band nonsubsampled iterated filter banks. These filter banks under certain conditions perform regular sampling of the dyadic wavelet transform [2] of continuous time signals, which justifies their use and establishes relations with the underlying continuous time framework. Their properties are analyzed in the next section.

This paper also presents new and improved algorithms for the reconstruction from wavelet extrema or zero crossings representations. The set of all signals sharing the same representation, called *reconstruction set*, in general consists of more than a single signal [3], [4].The aim is here to devise techniques for the reconstruction which fully utilize the information which is embedded in the representation. Practically, that means: techniques for finding a signal which can not be distinguished from the original based on the representation, which is any signal in the *reconstruction set*. Such a reconstruction strategy is called *consistent reconstruction* [5]. The first reconstruction schemes, based on alternating projections onto closed convex sets, were proposed by Mallat et al. [1], [2]. Their algorithm [2] for signal reconstruction from wavelet transform modulus maxima, although performing well in experiments, with $O(NlogN)$ complexity per iteration (for length N signals) uses projection operators which can not guarantee convergence. Besides, even in the case it converges, their algorithm does not necessarily recover a signal from the reconstruction set. In the wavelet zero-crossings case, Mallat's algorithm [1] converges to a point in the reconstruction set, but requires $O(Nlog^2N)$ operations per iteration. An alternative reconstruction procedure, by Berman et al. [3], uses the gradient descent algorithm to minimize an appropriate cost function. The algorithms, presented here, use also the method of alternating projections onto closed convex sets, but have the following attractive features: (1) consistent reconstruction, (2) guaranteed convergence, (3) simple and easy implementation and (4) $O(JN)$ operations per iteration for length N signals and the wavelet transform across J scales.

Interpretation of the reconstruction sets as the intersections of conveniently chosen convex sets,introduced here, besides leading to better reconstruction schemes, shows a duality between the two representations. The wavelet extrema representation and wavelet zero-crossings representation of a signal first difference actually provide equivalent characterizations of signals in $\ell^2(\mathbf{Z})$, which is shown in the last section.

Notations The Fourier transforms of some $\phi(x) \in L^2(\mathbf{R})$ and $f(n) \in \ell^2(\mathbf{Z})$ will be written as $\hat{\phi}(\omega)$ and $\hat{f}(e^{j\omega})$ respectively.

2. NONSUBSAMPLED FILTER BANKS

The discrete wavelet transform for wavelet maxima or zero crossings representation is the one implemented by an octave band nonsubsampled filter bank as shown in Figure 1a. Let us denote by $F_0(z)$, $F_1(z)$, ... , $F_J(z)$ the time reversed versions of equivalent filters from the input to outputs of the filter bank, based on $H_0(z)$ and $H_1(z)$ prototype analysis filters, and implementing the discrete wavelet transform across J scales. The discrete wavelet transform operator, denoted as $\mathbf{W}$ in the following, actually calculates inner products of an input signal with integer translates of impulse responses of $F_0(z)$,..., $F_J(z)$. Condition on the prototype filters for a stable reconstruction of signals in $\ell^2(\mathbf{Z})$

[*]Work supported in part by the National Science Foundation under grant ECD-88-11111.

[†]Work supported in part by the National Science Foundation under grants ECD-88-11111 and MIP-90-14189.

0-7803-1775-0/94 $3.00 © 1994 IEEE

from the wavelet transform is given by the following proposition, which is proven in [6].

Proposition 1 **W** *is a frame operator in* $\ell^2(\mathbf{Z})$ *if and only if the prototype filters* $H_0(z)$ *and* $H_1(z)$ *have no zeros in common on the unit circle.*

If the frame condition of the above proposition is satisfied, an inverse operator of **W** is the Hilbert adjoint operator of the dual of **W**, which we will denote by $\tilde{\mathbf{W}}^*$. The dual frame consists of integer translates of impulse responses of filters $F_0^d(z) = F_0(z)/D(z)$, $F_1^d(z) = F_1(z)/D(z)$, ... , $F_J^d(z) = F_J(z)/D(z)$, where

$$D(z) = \sum_{i=0}^{J} F_i(z)F_i(z^{-1}).$$

A filter bank implementation of the $\tilde{\mathbf{W}}^*$ operator is shown in Figure 1b. The inverse $\mathbf{W}^{-1}$, however is not unique. For any pair of filters $G_0(z)$ and $G_1(z)$ satisfying

$$H_0(z)G_0(z) + H_1(z)G_1(z) = 1,$$

perfect reconstruction of an input signal can be achieved by the synthesis filter bank based on the synthesis filters $G_0(z)$ and $G_1(z)$, and having the structure dual to the analysis one, as shown in Figure 1c. Obviously an FIR solution for $G_0(z)$ and $G_1(z)$ exists if and only the prototype analysis filters have no zeros in common [7]. However, the $\tilde{\mathbf{W}}^*$ operator is usually preferable since it is the only inverse which projects to zero signals in the orthogonal complement of the range of **W**, thus ensuring stable reconstruction. Conditions which enable an FIR implementation of $\tilde{\mathbf{W}}^*$, or equivalently finite length dual frame vectors, are investigated next. Having the dual frame with finite length vectors is equivalent to $D(z)$ being equal to a constant, or without loss of generality $D(z) = 1$. Necessary and sufficient condition for this to hold is given by the following proposition [7].

Proposition 2 $D(z) = 1$ *if and only if the prototype filters are power complementary:* $H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1}) = 1$.

If $H_0(z)$ and $H_1(z)$ are power complementary [8], $D(z) = 1$ and the frame vectors are identical to their duals. In that case **W** is a tight frame operator. The reconstruction is then performed by the synthesis filter bank (see Figure 1c) with the synthesis filters which are time reversed versions of the analysis ones, $G_0(z) = H_0(z^{-1})$ and $G_1(z) = H_1(z^{-1})$. This proves the following proposition.

Proposition 3 **W** *is a tight frame operator in* $\ell^2(\mathbf{Z})$ *if and only if the prototype filters are power complementary.*

These results extend immediately to an arbitrary nonsubsampled filter bank tree, i.e. a filter structure obtained by growing a two channel FIR filter bank tree in the following way. We start from a single input double output system consisting of prototype filters $H_0(z)$ and $H_1(z)$, which is the $0-th$ stage of the tree, and continue growing the tree in such a way that at a stage i from the filter $H_j(z^{2^i})$ only the two filters $H_0(z^{2^{i+1}})$ and $H_1(z^{2^{i+1}})$ can grow. In this general case Proposition 1 still holds, as well as the condition

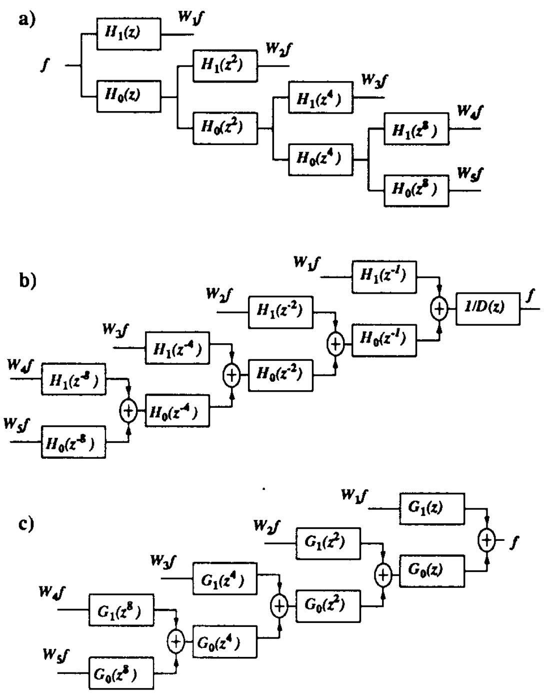

Figure 1. *Nonsubsampled filter banks for the discrete wavelet transform. a) an analysis filter bank implementing the* **W** *operator across 4 scales; b) implementation of the Hilbert adjoint of the dual of* **W**; *c) a synthesis filter bank for that shown in a).*

for an FIR synthesis, and frame vectors are time reversed integer translates of impulse responses of all equivalent filters from the input to an output of the filter bank tree. We will call such frames in $\ell^2(\mathbf{Z})$ *filter bank type frames*. Conditions for the tightness of these frames are given by the following theorem [6].

Theorem 1 *For the filter bank type frames in* $\ell^2(\mathbf{Z})$ *following statements are equivalent:*
1) both the frame vectors and the dual frame vectors have finite lengths for arbitrary filter bank tree;
2) the frame is tight for arbitrary filter bank tree;
3) the prototype analysis filters are power complementary.

Relations to continuous time framework are established if the lowpass prototype filter $H_0(z)$ is regular, or in other words if the infinite product

$$\lim_{n\to\infty} \prod_{s=1}^{\infty} H_0(e^{j2^{-s}\omega}) = \hat{\phi}(\omega)$$

converges to a function $\hat{\phi}(\omega)$ which is the Fourier transform of some continuous function $\phi(x) \in L^2(\mathbf{R})$. With a nonsubsampled filter bank tree with N outputs we may associate a set of N functions $\psi_k(x)$ in $L^2(\mathbf{R})$ which are given in the

Fourier domain as

$$\hat{\psi}_k(\omega) = F_k(e^{j\omega})\hat{\phi}(\omega),$$

where $F_k(e^{j\omega})$ is an equivalent transfer function from the input to an output k. Assume at the input a sequence $f(n)$ obtained by sampling some signal $f_c(x) \in L^2(\mathbf{R})$ prefiltered by $\phi(x)$:

$$f(n) = \int_{-\infty}^{+\infty} f_c(x)\phi(n-x)dx.$$

At an output k of the filter bank a sequence $f_k(n)$ is obtained having Fourier transform

$$\hat{f}_k(e^{j\omega}) = F_k(e^{j\omega}) \sum_{k\in\mathbf{Z}} \hat{f}_c(\omega + 2k\pi)\hat{\phi}(\omega + 2k\pi),$$

meaning that the filter bank performs sampling of convolutions of the signal with ψ_k's. In the case of an octave band filter bank [6] $\psi_k(x)$ $k = 1, ..., N-1$ are

$$\psi_k(x) = \frac{1}{2^k}\psi\left(\frac{x}{2^k}\right),$$

where $\psi(x)$ is a wavelet given by

$$\hat{\psi}(\omega) = H_1\left(e^{j\frac{\omega}{2}}\right)\hat{\phi}\left(\frac{\omega}{2}\right).$$

The filter bank then implements regular sampling of a dyadic wavelet transform [1]. This relation was also proven in [9], however for the completeness of the presentation we gave this alternative direct proof.

3. RECONSTRUCTION

The discrete wavelet transform of some $f \in \ell^2(\mathbf{Z})$, when computed across J scales, consists of $J+1$ signals $W_j f \in \ell^2(\mathbf{Z})$ $j = 1, \dots J+1$, obtained at outputs of the filter bank, as shown in Figure 1a. We shall use here following convention. Signals in $\ell^2(\mathbf{Z})$ will be denoted by lower case letters, their wavelet transforms by corresponding upper case letters and wavelet transform components by superscripted upper case letters, as in: $F = \mathbf{W}f$, $F^j = W_j f$. Note that the range of the discrete wavelet transform is a subspace of $\ell^2(\mathbf{I})$, where $\mathbf{I}$ stands for $\mathbf{Z} \times \{1, 2, \dots J+1\}$. Signals in $\ell^2(\mathbf{I})$ will be denoted by upper case letters, and their $J+1$ components in $\ell^2(\mathbf{Z})$ by the same letters with superscripts as $F = \{F^1, F^2, \dots, F^{J+1}\}$.

For the definition of wavelet extrema and zero crossings representations [3] we introduce an X operator which gives points of local extrema of a sequence:

$$\begin{aligned} Xf = \{k : & f(k+1) \le f(k),\ f(k-1) \le f(k), \\ \text{or } & f(k+1) \ge f(k),\ f(k-1) \ge f(k)\}, \end{aligned}$$

and an operator M extracting values of a sequence at its local extrema $Mf = \{f(k), k \in Xf\}$.

Wavelet extrema representation of a signal f is defined as:

$$E_e f = \{XW_j f, MW_j f\ j = 1, \dots, J+1\}.$$

Let Z denote an operator which provides zero crossings of a sequence, $Zf = \{k : f(k) \cdot f(k-1) \le 0\}$, and U denote an operator which gives integral values of a sequence between all consecutive zero crossings,

$$Uf = \left\{ \begin{array}{c} Uf(k) : Uf(k) = \sum_{j=z_{k-1}}^{z_k - 1} f(j), \\ z_k - k\text{th zero of } f,\ k = 1, 2, \dots, |Zf|+1 \end{array} \right\}.$$

It is assumed here that the points $-\infty$ and $+\infty$ are zero-crossings denoted by z_0 and $z_{|Zf|+1}$, and that $f \in \ell^1(\mathbf{Z})$ so that Uf is well defined. Using this notation, the wavelet zero-crossings representation of an f is defined as

$$E_z f = \{ZW_j f, UW_j f,\ j = 1, \dots, J+1\}.$$

Reconstruction algorithms proposed here recover a signal from the reconstruction set of wavelet extrema or wavelet zero-crossings representation of original signal. Actually, reconstruction procedures take place in the transform domain and the reconstructed signal is then obtained using the inverse wavelet transform.

The closure of the reconstruction set of some $F = \mathbf{W}f$ from the local extrema information on $W_j f$, $j = 1, \dots, J+1$ can be represented as the intersection:

$$\Phi_e^c(F) = \mathcal{V} \bigcap \mathcal{E} \bigcap \left(\bigcap_{i,j} \mathcal{C}_{i,j} \right)$$

of the sets:
- $\mathcal{V}$ - the range of the wavelet transform;
- $\mathcal{E}$ - the set of all $G \in \ell^2(\mathbf{I})$ which have same values as F at all points which are local extrema of F;
- $\mathcal{C}_{i,j}$ - the set of all $G \in \ell^2(\mathbf{I})$ sch that G^i has the same sign of slope as F^i at the point j:

$$\mathcal{C}_{i,j} = \left\{ \begin{array}{c} G : G \in \ell^2(\mathbf{I}), \\ (F^i(j) - F^i(j+1))(G^i(j) - G^i(j+1)) \ge 0 \end{array} \right\}.$$

Note that the sets $\mathcal{C}_{i,j}$ are defined only for those points where F is strictly increasing or decreasing, i.e. such that $F^i(j) \ne F^i(j+1)$.

Obviously, $\mathcal{V}$ is a subspace of $\ell^2(\mathbf{I})$ and $\mathcal{E}$ and $\mathcal{C}_{i,j}$'s are closed convex sets, therefore alternating projections of any initial point $F_0 \in \ell^2(\mathbf{I})$ onto $\mathcal{V}$, $\mathcal{E}$ and all the $\mathcal{C}_{i,j}$'s will converge to a point in their intersection [10] - the reconstruction set $\Phi_e^c(F)$. Hence, the reconstructed signal F_r can be obtained as:

$$F_r = \lim_{n\to\infty} \left(P_{\mathcal{V}} P_{\mathcal{E}} \prod_{i,j} P_{\mathcal{C}_{i,j}} \right)^n F_0 \qquad (1)$$

where $P_{\mathcal{V}}, P_{\mathcal{E}}$, $P_{\mathcal{C}_{i,j}}$ denote the projection operators onto $\mathcal{V}$, $\mathcal{E}$, $\mathcal{C}_{i,j}$ respectively.

In the wavelet zero-crossings case, the closure of the reconstruction set of $F = \mathbf{W}f$, from the wavelet zero crossings representation of f is

$$\Phi_z^c(F) = \mathcal{V} \bigcap \mathcal{U} \bigcap \left(\bigcap_{i,j} \mathcal{Z}_{i,j} \right),$$

where:
• $\mathcal{U}$ - the set of all sequences $G \in \ell^2(\mathbf{I})$ such that for F^i and G^i have the same integral values between any two adjacent zero crossings of F^i, $i = 1, \ldots, J+1$;
• $\mathcal{Z}_{i,j}$ - the set of all sequences $G \in \ell^2(\mathbf{I})$ such that G^i has the same sign as F^i at point j. $\mathcal{Z}_{i,j}$ sets are defined only for the nonzero points of F, $F^i(j) \neq 0$.

Since the sets $\mathcal{Z}$ and $\mathcal{U}$ are also closed and convex, a point F_r in the reconstruction set can be reached as:

$$F_r = \lim_{n\to\infty} \left(P_{\mathcal{V}} P_{\mathcal{U}} \prod_{i,j} P_{\mathcal{Z}_{i,j}} \right)^n F_0.$$

where $P_{\mathcal{U}}$ and $P_{\mathcal{Z}_{i,j}}$ are the projection operators onto $\mathcal{U}$ and $\mathcal{Z}_{i,j}$ respectively, and F_0 is an arbitrary starting point.

Implementation of these projection operators is discussed in details in [7] and is very simple. The numerical complexity of the algorithms is $O(JN)$ operations per iteration for a signal of length N and is mainly due to the $P_{\mathcal{V}}$ operator. This operator is the composition $P_{\mathcal{V}} = \mathbf{W}\tilde{\mathbf{W}}^*$ and has an exact FIR implementation if and only if $\mathbf{W}$ is a tight frame operator. However in experiments whose results we reported in [7] we obtained stable reconstruction with around 30dB SNR in ten iterations of the algorithms, with $\mathbf{W}\mathbf{W}^{-1}$ used instead of $P_{\mathcal{V}}$ for some inverse of $\mathbf{W}$ other than $\tilde{\mathbf{W}}^*$.

4. RELATION BETWEEN WAVELET EXTREMA AND WAVELET ZERO-CROSSINGS REPRESENTATIONS

Consider the extrema representation $R_e f$ of some signal $f \in \ell^2(\mathbf{Z})$, defined as $R_e f = \{Xf, Mf\}$, and the zero-crossings representation $R_z \Delta f$ of its difference Δf,

$$\Delta f(n) = f(n+1) - f(n), \tag{2}$$

defined as $R_z \Delta f = \{Z\Delta f, U\Delta f\}$. Obviously local extrema of f coincide with zero-crossings of Δf: $Xf = Z\Delta f$. Mf and $U\Delta f$ also provide equivalent information on the signal f, as shown below. If z_k now denotes index (location) of kth zero-crossing of Δf, $k = 1, 2, ..., |Z\Delta f|$, the following holds:

$$\begin{aligned} Mf &= \left\{ f(z_1), f(z_2), \ldots, f(z_{|Z\Delta f|}) \right\}, \\ U\Delta f &= \{ (f(z_1) - f(-\infty)), (f(z_2) - f(z_1)), \ldots \\ &\qquad \ldots, (f(+\infty) - f(z_{|Z\Delta f|})) \}. \end{aligned}$$

Since in the most cases of practical importance $f(-\infty) = 0$ and $f(+\infty) = 0$, information contained in Mf and $U\Delta f$ are equivalent, i.e. one uniquely determines the other. An immediate consequence is the following

Proposition 4 *The first order difference (as defined by (2)) of any signal in the reconstruction set of extrema representation of some $f \in \ell^2(\mathbf{Z})$, is in the reconstruction set of zero crossings representation of Δf. Conversely, any signal in the reconstruction set of zero-crossings representation of Δf is the first difference of some signal in the reconstruction set of extrema representation of f.*

Extension of these considerations to wavelet extrema and zero-crossings representations are straightforward, meaning that wavelet extrema representation of a signal and wavelet zero-crossings representation of its first difference provide equivalent characterizations of the signal. This is stated by the following theorem [7] which gives the summary of this section.

Theorem 2 *Consider an arbitrary signal $f \in \ell^2(\mathbf{Z})$ and its difference Δf. Any signal in the reconstruction set of Δf, from its wavelet zero-crossings representation, is the first difference of some signal in the reconstruction set of f, from its wavelet extrema representation and vice versa.*

5. CONCLUSION

This paper investigated properties of the nonsubsampled iterated filter banks used in wavelet transform extrema or zero-crossings based signals processing tasks. Algorithms for consistent signal reconstruction from wavelet extrema or zero crossings representation were proposed having low numerical complexity and simple implementation. It was also shown that wavelet extrema representation of a signal and wavelet zero-crossings representation of its first difference are equivalent.

REFERENCES

[1] S. Mallat,"Zero Crossings of a Wavelet Transform", IEEE Trans. on Information Theory, July 1991.

[2] S. Mallat and S. Zhong,"Characterization of Signals from Multiscale Edges", Technical Report, New York University, November 1991.

[3] Z. Berman and J. S. Baras,"A Theory of Adaptive Quasi Linear Representations", Technical Report, University of Maryland, January 1992.

[4] Y. Meyer,"Un contre-exemple à la conjecture de Marr et à celle de S. Mallat", Preprint, 1991.

[5] N. T. Thao and M. Vetterli, "Deterministic Analysis of Oversampled A/D Conversion and Decoding Improvement Based on Consistent Estimates", IEEE Trans. on Signal Processing, to appear in March 1994.

[6] Z. Cvetković and M. Vetterli, "Nonsubsampled Filter Banks: Frames for $\ell^2(\mathbf{Z})$ and Continuous Time Signal Processing", technical report CU/CTR/TR 352-93-32, Center for Telecommunications Research, Columbia University, New York, December 1993.

[7] Z. Cvetković and M. Vetterli,"Consistent Reconstruction of Signals from Wavelet Extrema and Zero Crossings Representations", submitted to IEEE Trans. SP, October 1993.

[8] P. P. Vaidyanathan, "Multirate Systems and Filter Banks", PTR Prentice-Hall Inc. 1993.

[9] M. J. Shensa," The Discrete Wavelet Transform: Wedding the Á Trous and Mallat Algorithms", IEEE Trans. on Signal Processing, Vol.40, No.10, October 1992, pp.2464-2482.

[10] D. C. Youla,"Mathematical Theory of Image Restoration by the Method of Convex Projections", Chapter in "Image Recovery - Theory and Application", Academic Press, Inc. 1987, H. Stark ed.