LOWER BOUND ON THE MEAN SQUARED ERROR IN OVERSAMPLED QUANTIZATION OF PERIODIC SIGNALS

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Abstract - **We analyze optimal reconstruction in oversampled A/D conversion, in the case where the input signals are bandlimited and periodic. We show that the reconstruction MSE cannot decrease with the oversampling ratio faster than $\mathcal{O}(R^{(-2)})$.**

1. Introduction

Oversampled analog-to-digital (A/D) conversion is a technique which permits high conversion resolution while using coarse quantization. Classically, by lowpass filtering the quantized oversampled signal, it is possible to reduce the quantization error power in proportion to the oversampling ratio R [1]. In other words, the reconstruction mean squared error (MSE) is in $\mathcal{O}(R^{-1})$. This means that 1/2 bit of resolution can be gained for every octave of oversampling. It was recently found that this error reduction is not optimal. Under certain conditions, it was shown [2] on periodic bandlimited signals that an upper bound on the MSE of optimal reconstruction is in $\mathcal{O}(R^{-2})$ instead of $\mathcal{O}(R^{-1})$, implying the trade-off of 1 bit per octave of oversampling. We prove on the same type of signals that the order $\mathcal{O}(R^{-2})$ is the theoretical limit of reconstruction as an MSE lower bound. The proof is based on a vector quantization approach with an analysis of partition cell density.

2. Vector quantization approach

We derive this MSE lower bound on a class of bandlimited signals which can be put in a finite dimensional space. We assume that the input signals $x(t)$ are bandlimited by f_m and are periodic with period T. These signals indeed belong to a space of finite dimension W, equal to the integer part of Tf_m+1. They can be considered as W dimensional vectors $\vec{x}$. Then, the operation of sampling $x(t)$ N times uniformly in $[0,T]$ and quantizing the samples uniformly amounts to mapping the associated vector $\vec{x}$ to an index which consists of an N-tuple of integers. Thus, an oversampled A/D converter can be considered as the encoder of a vector quantizer [3]. Such an encoder naturally generates a partition of the vector space.

3. Hyperplane wave partition

We show that the partition generated by an oversampled A/D converter has the following properties: (i) the cells are formed by splitting the vector space by non-interrupted hyperplanes, (ii) the hyperplanes can only have N possible directions, (iii) the hyperplanes having the same direction are equally spaced with each other. We call the partitions having such properties "hyperplane wave partitions". Using these properties, we show that the number of cells included in a bounded region of the space of diameter L has an upper bound as follows:

$$M \leq \binom{N}{W}\left(\frac{L}{d}+2\right)^{W}, \tag{1}$$

where d is the minimum distance between any two parallel hyperplanes. In the case of the A/D conversion partition, we show that $d \geq \frac{q}{\sqrt{2W}}$.

4. Reconstruction MSE lower bound

With the vector quantization approach, optimal reconstruction consists of taking the centroid of the cell corresponding to the index output by the encoder. It was shown by Zador [4] that for an asymptotically large number M of cells, the optimal reconstruction MSE can be lower bounded as a function of M as:

$$MSE_{opt} \geq C(W,\mathrm{p}) \cdot M^{-2/W}, \tag{2}$$

where $C(W,\mathrm{p})$ is a coefficient which only depends on the dimension W of the input space and the probability distribution p of the input vectors. Using the upper bound on M previously obtained and the relationship $N = R \cdot W$ between the number of samples and the oversampling ratio, we derive

$$MSE_{opt} \geq C'(W,\mathrm{p},L,q) \cdot \frac{1}{R^2}, \tag{3}$$

where $C'(W,\mathrm{p},L,q)$ is a coefficient which is independent of the oversampling ratio R. This shows that the asymptotic behavior of the reconstruction MSE versus R is lower bounded by $\mathcal{O}(R^{-2})$.

References

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