TIME-VARYING FILTER BANKS AND MULTIWAVELETS

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ABSTRACT

A recent wavelet construction by Geronimo, Hardin and Massopust uses more than one wavelet and scaling function [3]. Strang and Strela gave a filter bank interpretation of that result [6], as well as a condition for moment properties of the resulting wavelets [7]. In this note, we are concerned with the regularity of the resulting iterated filter bank scheme, that is, a matrix extension of the classic result by Daubechies on iterated filters [1]. We show in particular:

(i) the relation between time-varying filter banks and multiwavelets,

(ii) the construction of multiwavelets as limits of iterated time-varying filter banks,

(iii) a necessary condition for the convergence of the iterated matrix product and

(iv) an exploration of examples of multiwavelets as iterations of time-varying filter banks.

1. INTRODUCTION

Wavelet constructions from iterated filter banks, as pioneered by I.Daubechies [1, 2], have become a standard way to derive orthogonal and biorthogonal wavelet bases, see e.g. [4]. The underlying filter banks are well studied, and thus, the design procedure is well understood. Because of the structure of the problem, certain solutions are ruled out (e.g. orthogonal FIR linear phase filter banks). By relaxing the requirement of time-invariance, it is easy to see that new solutions are possible. Such filter banks are related to matrix two-scale equations for multiwavelets [3, 6, 7]. It is thus of interest to study the iteration of time-varying filter banks and their relation to multiwavelets. Note that time-varying filter banks can be seen as time-invariant filter banks with more channels (e.g. 2 channel filter banks with time-variance of period 2 are also 4 channel filter banks). The outline of the paper is as follows. We review material on time-varying filter banks and their analysis in Section 2. Then, Section 3. indicates the construction of multiwavelets from time-varying filter banks, and Section 4. studies the infinite matrix product that is central in this construction. A necessary condition for convergence is given. Finally, Section 5. investigates examples of multiwavelet constructions from time-varying filter banks.

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2. TIME-VARYING FILTER BANKS

It is known that a linear periodically time-varying (LPTV) filter can be described by a multi-input multi-output (MIMO) transfer function relating input and output polyphase components [5]. We concentrate for simplicity on the case when there are only two different impulse responses (period = 2), noting that the general case is similar (except for the size of the transfer matrices). Thus, a period 2 periodically time-varying filter has a polyphase transfer function given in z-transform domain by:

$$\mathbf{H(z)} = \begin{pmatrix} H_{00}(z) & H_{01}(z) \\ H_{10}(z) & H_{11}(z) \end{pmatrix}. \quad (1)$$

In the above, $H_{i,j}(z)$ denotes the z-transform of the impulse response in phase i of the output to an impulse in phase j of the input ($i, j \in \{0,1\}$). That is, both input and output sequences are decomposed into even and odd indexed subsequences, and the LPTV filter of period 2 is now a MIMO linear time-invariant (LTI) with respect to these subsequences.

We are specifically interested in time-varying interpolation filters, that is, an upsampling function (typically by 2) followed by a LPTV interpolation filter. In time domain, the resulting operator (we consider the case of two alternating impulse responses for simplicity) is given by

$$\mathbf{T} = \begin{pmatrix} \ddots & & & & \\ & g[0] & 0 & \dots & \\ & g[1] & 0 & \dots & \\ & g[2] & h[0] & 0 & \dots \\ & g[3] & h[1] & 0 & \dots \\ & g[4] & h[2] & g[0] & 0 \\ & g[5] & h[3] & g[1] & 0 \end{pmatrix}. \quad (2)$$

where $g[k]$ and $h[k]$ are the two interpolation filter impulse responses. Clearly, when $\mathbf{T}$ is applied to a sequence $x[n]$, $x[2n]$ and $x[2n+1]$ lead to impulses $g[k-4n]$ and $h[k-4n-2]$, respectively. That is, even and odd indexed samples lead to different responses, as to be expected. In z-transform domain, write sequences in terms of even and odd indexed subsequences, or polyphase components, as

$$X(z) = X_0(z^2) + z^{-1}X_1(z^2), \quad (3)$$
$$Y(z) = Y_0(z^2) + z^{-1}Y_1(z^2), \quad (4)$$

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for the input and output, as well as the filters

$$G(z) = G_0(z^2) + z^{-1}G_1(z^2), \quad (5)$$
$$H(z) = H_0(z^2) + z^{-1}H_1(z^2). \quad (6)$$

Then, the polyphase components of $Y(z)$ can be written in term of the polyphase components of the input $X(z)$ as

$$\begin{pmatrix} Y_0(z) \\ Y_1(z) \end{pmatrix} = \begin{pmatrix} G_0(z) & z^{-1}H_0(z) \\ G_1(z) & z^{-1}H_1(z) \end{pmatrix} \cdot \begin{pmatrix} X_0(z^2) \\ X_1(z^2) \end{pmatrix}. \quad (7)$$

Call the above matrix $\mathbf{T}(z)$. Its size is given by the number of different impulse responses, or period. Note that the z^{-1} in front of $H_i(z)$ comes from the definition of $h[k]$ is (2), which is shifted by 2 with respect to $g[k]$. In the special case when the filter is time-invariant ($h[k] = g[k-2]$), $\mathbf{T}(z)$ is

$$\mathbf{T}(z) = \begin{pmatrix} G_0(z) \\ G_1(z) \end{pmatrix} \cdot \begin{pmatrix} 1 & z^{-1} \end{pmatrix}. \quad (8)$$

Now, it is easy to study iterated LPTV interpolators. Calling the n-times cascade transfer matrix $\mathbf{T}^{(n)}(z)$

$$\mathbf{T}^{(n)}(z) = \prod_{i=0}^{n-1} \mathbf{T}(z^{2^i}), \quad (9)$$

we get the output, after n-times upsampling and interpolation, as

$$Y^{(n)}(z) = \begin{pmatrix} 1 & z^{-1} \end{pmatrix} \mathbf{T}^{(n)}(z^2) \begin{pmatrix} X_0(z^4) \\ X_1(z^4) \end{pmatrix}. \quad (10)$$

Note that there are two impulse responses, given by

$$G^{(n)}(z) = T_{00}(z^2) + z^{-1}T_{10}(z^2), \quad (11)$$
$$H^{(n)}(z) = T_{01}(z^2) + z^{-1}T_{11}(z^2). \quad (12)$$

As usual, we are particularly interested in the case when the operator $\mathbf{T}$ in (2) is unitary, or $\mathbf{T}^T\mathbf{T} = \mathbf{I}$. Then $\mathbf{T}(z)$ satisfies

$$\tilde{\mathbf{T}}(z) \cdot \mathbf{T}(z) + \tilde{\mathbf{T}}(-z) \cdot \mathbf{T}(-z) = 2 \cdot \mathbf{I} \quad (13)$$

where $\tilde{\mathbf{T}}(z) = \mathbf{T}^T(z^{-1})$ (we assume real filter coefficients). The above is a matrix version of the usual Smith-Barnwell condition for orthonormal filter banks [8, 9]. If $\mathbf{T}$ is unitary, so is $\mathbf{T}^{(n)}$. The important point is that $g^{(n)}[k]$ and $h^{(n)}[k]$ (the impulse responses of $G^{(n)}(z)$ and $H^{(n)}(z)$ from (11)) are of unit norm and orthogonal with respect to shifts by $2^{(n+1)}$.

Note that in the above, we concentrated on the lowpass channel of a time-varying filter bank. For a unitary transformation, we need also a time-varying highpass channel that is orthogonal to the time-varying lowpass, as well as to its own translates. However, for all discussions concerning regularity or iteration, the lowpass channel is the key element (since that is the channel involved in the infinite iteration, while the highpass channel is only applied once).

3. MULTIWAVELETS

As in the usual construction of wavelets from iterated filter banks, we can associate continuous-time functions to the impulse response of the iterated (time-varying) filter. The two impulse responses are given by $g^{(n)}[k]$ and $h^{(n)}[k]$, and we associate

$$\phi_g^{(n)}(t) = 2^{n/2} \cdot g^{(n)}[k] \quad k/2^n \le t \le (k+1)/2^n, \quad (14)$$
$$\phi_h^{(n)}(t) = 2^{n/2} \cdot h^{(n)}[k] \quad k/2^n \le t \le (k+1)/2^n, \quad (15)$$

and translates by even integers will be orthonormal, e.g.

$$< \phi_g^{(n)}(t), \phi_g^{(n)}(t-2l) >= \delta[l], \quad (16)$$

for any finite n.

First, however, we will concentrate on the matrix product itself, from which, under suitable conditions, $\phi_g^{(n)}(t)$ and $\phi_h^{(n)}(t)$ can be obtained.

Associate 4 functions, or one for each entry of $\mathbf{T}^{(n)}(z)$. Actually, for notational convenience, it will be easier to work with the transpose of $\mathbf{T}^{(n)}(z)$, and this using a normalized transfer matrix on the unit circle $\mathbf{M}(\omega)$

$$\mathbf{M}(\omega) = 1/\sqrt{2} \cdot \mathbf{T}^T(e^{j\omega}) = 1/\sqrt{2} \begin{pmatrix} G_0(e^{j\omega}) & G_1(e^{j\omega}) \\ e^{-j\omega}H_0(e^{j\omega}) & e^{-j\omega}H_1(e^{j\omega}) \end{pmatrix} \quad (17)$$

Then, associating piecewise constant approximations with intervals of length $1/2^n$ leads to

$$\mathbf{\Phi}^{(n)}(\omega) = \mathbf{M}(\omega/2) \cdot \mathbf{M}(\omega/4) \ldots \mathbf{M}(\omega/2^n) \cdot \Theta(\omega) \quad (18)$$

where $\Theta(\omega)$ is the normalized interpolation function

$$\Theta(\omega) = e^{-j\omega/2^{n+1}} \cdot \frac{\sin(\omega/2^{n+1})}{\omega/2^{n+1}}. \quad (19)$$

Note that $\mathbf{\Phi}^{(n)}(\omega)$ satisfies

$$\|\Phi_{i0}^{(n)}(\omega)\|_2^2 + \|\Phi_{i1}^{(n)}(\omega)\|_2^2 = 1 \quad (20)$$

given that $G(z)$ and $H(z)$ are orthonormal filters (and so are $G^{(n)}(z)$ and $H^{(n)}(z)$ in (11-12)).

Also, $\Theta(\omega) \to 1$ for any finite ω and large n, and can thus be ignored. In the following, we will be interested in the limit

$$\mathbf{\Phi}(\omega) = \lim_{n\to\infty} \mathbf{\Phi}^{(n)}(\omega) \quad (21)$$
$$= \prod_{i=1}^{\infty} \mathbf{M}(\omega/2^i). \quad (22)$$

Note that

$$\mathbf{\Phi}(\omega) = \mathbf{M}(\omega/2) \cdot \mathbf{\Phi}(\omega/2) \quad (23)$$

or

$$\begin{pmatrix} \phi_{00}(\omega) & \phi_{01}(\omega) \\ \phi_{10}(\omega) & \phi_{11}(\omega) \end{pmatrix} = \begin{pmatrix} M_{00}(\frac{\omega}{2}) & M_{01}(\frac{\omega}{2}) \\ M_{10}(\frac{\omega}{2}) & M_{11}(\frac{\omega}{2}) \end{pmatrix} \begin{pmatrix} \phi_{00}(\frac{\omega}{2}) & \phi_{01}(\frac{\omega}{2}) \\ \phi_{10}(\frac{\omega}{2}) & \phi_{11}(\frac{\omega}{2}) \end{pmatrix} \quad (24)$$

that is, a matrix two scale equation. In time domain, this leads to

$$\phi(t) = 2 \cdot \sum_k \mathbf{M}[k] \cdot \phi(2t-k). \qquad (25)$$

4. PROPERTIES OF THE INFINITE PRODUCT

In the time invariant case, we know that $M(\omega)$ which is a trigonometric polynomial, satisfies the following two necessary constraints (i) $M(0) = 1$ and (ii) $M(\pi) = 0$. What are the equivalent conditions in the matrix iteration case? Call $D(\omega)$ the determinant of $\mathbf{M}(\omega)$, and $\{\lambda_0(\omega), \lambda_1(\omega)\}$ the eigenvalues of $\mathbf{M}(\omega)$.

First, $\mathbf{\Phi}(0)$ has to be finite, and thus, neither eigenvalue of $\mathbf{M}(0)$ can be larger than 1 in absolute value. If both are smaller than 1 in absolute value, $\mathbf{\Phi}(0)$ will be the zero matrix, which contradicts the requirement that it represents scaling functions, or lowpass filters. Thus, either $|\lambda_0(0)| = |\lambda_1(0)| = 1$ or $|\lambda_0(0)| = 1$ and $|\lambda_1(0)| < 1$. For convergence of the infinite product at $\omega = 0$, it is further necessary that eigenvalues of absolute value 1 are actually equal to 1, since otherwise, at least one of the entries will not be a Cauchy sequence. Thus, for pointwise convergence at $\omega = 0$, $\mathbf{M}(0)$ has either (i) $\lambda_0(0) = \lambda_1(0) = 1$, that is $\mathbf{M}(0) = \mathbf{I}$ or (ii) $\lambda_0(0) = 1$ and $|\lambda_1(0)| < 1$.

Let us now investigate conditions on $\mathbf{M}(\pi)$. We assume that the infinite product converges pointwise, and want to see what condition it imposes on $\mathbf{M}(\omega)$. Write

$$\mathbf{M}(\omega) = \mathbf{M}_e(2\omega) + e^{-j\omega}\mathbf{M}_o(2\omega) \qquad (26)$$

where $\mathbf{M}_e(2\omega)$ and $\mathbf{M}_o(2\omega)$ correspond to even and odd polyphase components of $\mathbf{M}(\omega)$. Also, call $\mathbf{M}^{(n)}(\omega)$ the n-times iteration (this is, up to rescaling and tranposition, equal to $\mathbf{T}^{(n)}(z)$ on the unit circle). Then

$$\begin{aligned} \mathbf{M}^{(n)}(\omega) &= \mathbf{M}(2^{n-1}\omega) \cdot \mathbf{M}(2^{n-2}\omega) \ldots \mathbf{M}(\omega) \\ &= \mathbf{M}^{(n-1)}(2\omega)[\mathbf{M}_e(2\omega) + e^{-j\omega}\mathbf{M}_o(2\omega)] \end{aligned} \qquad (27)$$

Consider the even and odd polyphase components of $\mathbf{M}^{(n)}(\omega)$,

$$\mathbf{M}_e^{(n)}(\omega) = \mathbf{M}^{(n-1)}(\omega) \cdot \mathbf{M}_e(\omega), \qquad (28)$$

$$\mathbf{M}_o^{(n)}(\omega) = \mathbf{M}^{(n-1)}(\omega) \cdot \mathbf{M}_o(\omega). \qquad (29)$$

Associate piecewise constant approximations with unit elements of length $1/2^n$ in the usual manner, and take the limit as $n \to \infty$. That is, ω is divided by 2^n. Then, $\mathbf{M}_e^{(n)}(\omega/2^n)$ goes towards $\mathbf{\Phi}(\omega/2)$, as do $\mathbf{M}_o^{(n)}(\omega/2^n)$ and $\mathbf{M}^{(n-1)}(\omega)$. On the other hand, $\mathbf{M}_e(\omega/2^n)$ goes towards $\mathbf{M}_e(0)$, and $\mathbf{M}_o(\omega/2^n)$ towards $\mathbf{M}_o(0)$ for any finite ω. Therefore, (28-29) become

$$\mathbf{\Phi}(\omega/2) = \mathbf{\Phi}(\omega/2) \cdot \mathbf{M}_e(0) \qquad (30)$$

$$\mathbf{\Phi}(\omega/2) = \mathbf{\Phi}(\omega/2) \cdot \mathbf{M}_o(0) \qquad (31)$$

and we get

$$\mathbf{\Phi}(\omega/2) \cdot \mathbf{M}_e(0) = \mathbf{\Phi}(\omega/2) \cdot \mathbf{M}_o(0). \qquad (32)$$

There are two cases:

(i) $\mathbf{\Phi}(\omega)$ has full rank for some ω

$$\mathbf{M}_e(0) = \mathbf{M}_o(0) \Leftrightarrow \mathbf{M}(\pi) = \mathbf{0}. \qquad (33)$$

(ii) $\mathbf{\Phi}(\omega)$ has rank 1 for some ω

$$\mathbf{\Phi}(\omega) \cdot [\mathbf{M}_e(0) - \mathbf{M}_o(0)] = \mathbf{\Phi}(\omega) \cdot \mathbf{M}(\pi) = \mathbf{0}. \qquad (34)$$

Consider case (i). $\mathbf{M}(\omega)$ satisifies the matrix Smith-Barnwell condition (13) which we can rewrite as (after transposition and renormalisation)

$$\mathbf{M}(\omega)\mathbf{M}^T(-\omega) + \mathbf{M}(\omega+\pi)\mathbf{M}^T(-\omega+\pi) = \mathbf{I}. \qquad (35)$$

At $\omega = 0$, since $\mathbf{M}(\pi) = \mathbf{0}$, we get

$$\mathbf{M}(0)\mathbf{M}^T(0) = \mathbf{I}. \qquad (36)$$

that is, $\mathbf{M}(0)$ is unitary, or orthonormal since we assume real filters. That is, it is a rotation matrix, and in order for $\mathbf{M}^{(n)}(0)$ to converge, $\mathbf{M}(0)$ has to be the identity.

Consider case (ii) and (34) at $\omega = 0$,

$$\mathbf{\Phi}(0) \cdot \mathbf{M}(\pi) = \mathbf{0}. \qquad (37)$$

Thus, $\mathbf{\Phi}(0)$ is of rank 1, and its rows are colinear with the left eigenvector $\mathbf{r}_0$ attached to the eigenvalue $\lambda_0(0) = 1$ (since $\mathbf{\Phi}(0) = \lim_{n\to\infty} \mathbf{M}^{(n)}(0)$). Therefore, a necessary condition is

$$\mathbf{r}_0 \cdot \mathbf{M}(\pi) = \mathbf{0}. \qquad (38)$$

We can summarize our findings so far.

Proposition 3.1
Given an infinite matrix product of size 2 by 2

$$\mathbf{\Phi}(\omega) = \prod_{i=1}^{\infty} \mathbf{M}(\omega/2^i) \qquad (39)$$

where $\mathbf{M}(\omega)$ satisfies a matrix Smith-Barnwell condition (35), a necessary condition for convergence to a scaling matrix $\mathbf{\Phi}(\omega)$ such that $\mathbf{\Phi}(0)$ is non-zero and bounded is

(i) $\mathbf{M}(0) = \mathbf{I}$, $\mathbf{M}(\pi) = \mathbf{0}$ (note: $\mathbf{\Phi}(\omega)$ has rank 2)

(ii) $\mathbf{M}(0)$ has eigenvalue $\lambda_0(0) = 1$ and $|\lambda_1(0)| < 1$, $\mathbf{M}(\pi)$ has rank 1 and satisfies $\mathbf{r}_0 \cdot \mathbf{M}(\pi) = \mathbf{0}$ (note: $\mathbf{\Phi}(\omega)$ has rank 1)

Examples

Case (i)

$$\mathbf{M}(\omega) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} D_N(\omega) & 0 \\ 0 & D_M(\omega) \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad (40)$$

where $D_N(\omega)$ is an N-th order Daubechies filter scaled such that $|D_N(\omega)|^2 + |D_N(\omega+\pi)|^2 = 1$ (and a similar relation for $D_M(\omega)$). Clearly, (35) is satisfied. Note that $\mathbf{M}(0) = \mathbf{I}$ and $\mathbf{M}(\pi) = \mathbf{0}$ (both properties follow from properties of the Daubechies filters). The infinite iteration goes to

$$\mathbf{\Phi}(\omega) = 1/2 \begin{pmatrix} \Phi_N(\omega) + \Phi_M(\omega) & \Phi_N(\omega) - \Phi_M(\omega) \\ \Phi_N(\omega) - \Phi_M(\omega) & \Phi_N(\omega) + \Phi_M(\omega) \end{pmatrix} \qquad (41)$$

where $\Phi_N(\omega)$ and $\Phi_M(\omega)$ are the Nth and Mth order Daubechies scaling functions. Note that $\mathbf{\Phi}(\omega)$ has rank 2 almost everywhere.

Case (ii) (see (8))

$$\mathbf{M}(\omega) = \begin{pmatrix} 1 \\ e^{-j\omega} \end{pmatrix} \cdot \begin{pmatrix} M_0(\omega) & M_1(\omega) \end{pmatrix} \quad (42)$$

where $M_0(\omega)$ and $M_1(\omega)$ are the polyphase components of an Nth order Daubechies filter. Note that

$$\mathbf{M}(0) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{M}(\pi) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \quad (43)$$

$\mathbf{M}(0)$ has eigenvalues $\{1, 0\}$ with left eigenvectors $\{1, 1\}$ and $\{1, -1\}$ and thus, the left eigenvector with eigenvalue 1 is indeed orthogonal to $\mathbf{M}(\pi)$. The infinite product then becomes

$$\mathbf{\Phi}(\omega) = 1/2 \begin{pmatrix} \Phi_N(\omega/2) & \Phi_N(\omega/2) \\ e^{-j\omega/2}\Phi_N(\omega/2) & e^{-j\omega/2}\Phi_N(\omega/2) \end{pmatrix} \quad (44)$$

as to be expected, since this is a time-invariant case with Daubechies filters. This is clearly a rank 1 matrix.

Note that the rank 1 case is important in the multiwavelet constructions that will be seen below. That is, $\mathbf{\Phi}(\omega)$ will be of the form

$$\begin{pmatrix} \phi_{00}(\omega) & \phi_{01}(\omega) \\ \phi_{10}(\omega) & \phi_{11}(\omega) \end{pmatrix} = \begin{pmatrix} \phi_a(\omega) & c \cdot \phi_a(\omega) \\ \phi_b(\omega) & c \cdot \phi_b(\omega) \end{pmatrix}, \quad (45)$$

where c follows from the dominant eigenvector of $\mathbf{M}(0)$.

5. MULTIWAVELETS FROM ITERATED LPTV FILTER BANKS

An orthogonal time-varying filter bank with the two lowpass filters $g[n] = 1/5 \cdot [3/\sqrt{2}, 4, 3/\sqrt{2}]$ and $h[n] = 1/10 \cdot [-1/2, -3/\sqrt{2}, 9/2, 10/\sqrt{2}, 9/2, -3/\sqrt{2}, -1/2]$ is given in [6] and corresponds to the coefficients (with renormalization) of the multiwavelet two scale equation in [3]. The resulting matrix $\mathbf{M}(\omega)$ has entries

$$M_{00}(\omega) = \frac{3}{10}(1 + e^{-j\omega}), \quad M_{01}(\omega) = \frac{2\sqrt{2}}{5} \quad (46)$$

$$M_{10}(\omega) = \frac{1}{20\sqrt{2}}(-1 + 9e^{-j\omega} + 9e^{-j2\omega} - e^{-j3\omega}) \quad (47)$$

$$M_{11}(\omega) = \frac{1}{20}(-3 + 10e^{-j\omega} - 3e^{-j2\omega}) \quad (48)$$

The matrix $\mathbf{M}(0)$ has eigenvalues 1 and $-1/5$, and the left eigenvector with eigenvalue 1 is $r_0 = [\sqrt{2}, 1]$. The matrix $\mathbf{M}(\pi)$ has rank 1, and $r_0 \cdot \mathbf{M}(\pi) = 0$, thus verifying the necessary condition. The iterated product actually converges to continuous functions ϕ_a and ϕ_b, as shown in Figure 1 (the value of c is (45) is $1/\sqrt{2}$).

Note that because of the constant c in (45), the two polyphase components of the iterated filters $G^{(n)}(z)$ and $H^{(n)}(z)$ converge to functions of the same shape, but with a different multiplicative scaling factor c. Thus, before merging the two polyphase components, this factor has to be corrected, otherwise the impulse responses $g^{(n)}[k]$ and $h^{(n)}[k]$ oscillate at the maximum frequency. This problem does not appear if $\mathbf{M}(0)$ has dominant left eigenvector $[1, 1]$ or $c = 1$.

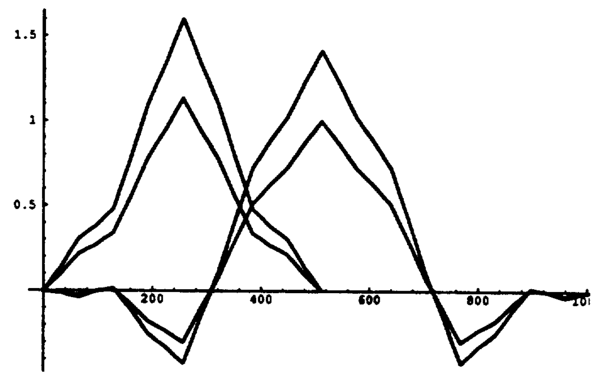

Figure 1: Iteration of a regular 2 by 2 matrix. The matrix in (46-48) converges to four symmetric functions, two of which are simply scales of the two others.

We can use the method of invariant cycles of the mapping $\omega \rightarrow 2\omega \pmod{2\pi}$ to find lower bounds on regularity [2]. These are now based on the eigenvalues of the matrix products in the cycle. For the matrix above and $\omega = 2\pi/3$, we have the invariant cycle $\{2\pi/3, 4\pi/3\}$. The eigenvalues of $\mathbf{M}(e^{j2\pi/3}) \cdot \mathbf{M}(e^{j4\pi/3})$ are 0.01 and 0.0625, which can be used to show that the scaling functions are nondifferentiable by lower bounding the decay of the Fourier transform.

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