SET THEORETIC COMPRESSION WITH AN APPLICATION TO IMAGE CODING

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ABSTRACT

We discuss the concept of set theoretic compression where the input signal is implicitly encoded by the specification of a set which contains it, rather than an estimate which approximates it. This approach assumes the reconstruction of an estimate from the encoded set information only at the decoder side. We explain the motivations of this approach for high signal compression and encoding simplification, and the implication of more complex decoding. We then present the tools to support the approach. We finally show a demonstration of this approach in a particular application of image coding.

1. INTRODUCTION AND CONCEPT

In signal coding, one thinks of the encoded version of an input signal as an approximation of this signal. Signal compression is achieved by forcing the samples of this approximated signal to have values on a finite set of quantized levels. In this paper, we introduce a different approach to signal coding, called *set theoretic coding*. With this approach, the role of the encoder is to give a digital description of a set of signals which contains the input signal. In other words, the information which is encoded is a set, not an estimate. The reconstruction of an estimate of the true signal is left to the decoder, given the set information. This new approach has two contributions: (i) it gives an interpretation of signal coding which is actually valid for all coding schemes; (ii) it allows new types of coding schemes and indicates new ways to achieve signal compression.

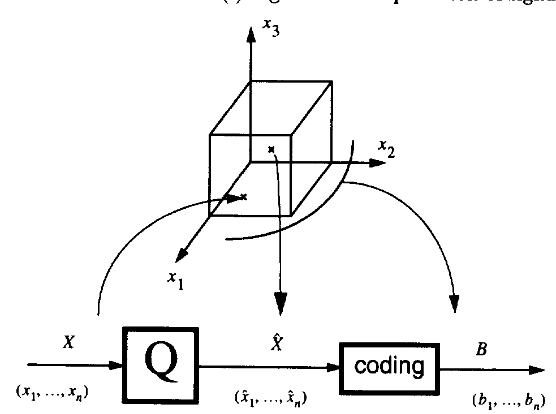

Figure 1: Set theoretic presentation of the simple quantization of an n-point sequence.

In Section 2 and Section 3, we develop these two points respectively. In Section 4, we discuss the application of set theoretic coding to image compression. In Section 5, we give in the area of block DCT coding of images a first demonstration of coding scheme derived from this new approach. In this paper, importance is given to the conceptual discussion of set theoretic coding. However, with the demonstration of Section 5, we also give two appendices containing mathematical descriptions and analytical manipulations including a theorem.

2. SET THEORETIC INTERPRETATION OF CLASSICAL CODING

In fact, the set theoretic view gives an interpretation of the actual information which is encoded in any coding scheme. Figure 1 shows an example of coding system which consists of the direct amplitude quantization of an n-point sequence $X = (x_1, x_2, ..., x_n)$. In the traditional view, the output binary sequence $B = (b_1, ..., b_n)$ gives an encoded description of an approximated version of X, that is the sequence of the quantized values $\hat{X} = (\hat{x}_1, ..., \hat{x}_n)$. In fact, the precise and complete information available from B is that the samples $x_1, ..., x_n$ of X belong by necessity to certain quantization intervals of $\mathbf{R}$, implying that X must belong to a certain hypercube $\mathcal{C}(B)$ of $\mathbf{R}^N$ (see Figure 1). This is nothing but the set theoretic description of encoding. The hypercube is the encoded set. The corresponding decoding part traditionally consists of reconstruction the quantized signal $\hat{X}$ from the binary sequence B. In fact, $\hat{X}$ is nothing but the geometric center of the hypercube $\mathcal{C}(B)$. In the set theoretic view, the traditional decoder appears to pick the geometric center of the encoded set as a reconstruction of X. This reconstruction has the property to minimize the expectation of the mean squared error in the case where the input signals have a uniform probability distribution.

The set theoretic approach is also applicable to coding systems where a linear transform of the input is quantized, instead of the input itself. This is the case of block DCT coding, whose principle is symbolized in Figure 2. The output binary sequence B gives precisely the information that the DCT transform $Y = DCT[X]$ of X belongs to a hypercube of $\mathbf{R}^N$. This implies that X must belong to the set obtained by inverse DCT transform of the hypercube (see Figure 2). It also appears that the traditional decoder picks the geometric center of $\mathcal{C}(B)$ as reconstruction of X. Indeed, this decoder usually takes the inverse DCT transform $\hat{X}$ of $\hat{Y}$ which is the quantized version of Y. As in the previous case of simple quantization, $\hat{Y}$ is the geometric center of the hypercube which contains Y. Because the DCT transform is unitary, it conserves the distance. Therefore, $\mathcal{C}(B)$ is also a hypercube (of same size but rotated) and $\hat{X}$ is necessarily its center (see Figure 2).

Usually, when the quantization resolution is fine enough, this choice of reconstruction is satisfactory and the set theoretic analysis can be omitted. However, in the case of coarse resolution, typical of high signal compression, significant artifacts start to appear between this proposed estimate and the true signal. In the example of block DCT coding with coarse quantization, it appears that choosing the center of the encoded hypercube as estimate yields blocking artifacts which don't exist in the original signal. This is a situation where one should start using the complete set information available. This approach has already been considered in several recent papers [1, 2, 3] where the search for a different estimate in the hypercube with less blocking artifacts is proposed. In [1, 2], the reconstructed image is the element of the hypercube which minimizes a certain highpass energy function. In [3], two reconstruction schemes are proposed which both give an estimate within the encoded hypercube. The first scheme gives an estimate of the hypercube which minimizes the distance with a certain "estimated" set of smooth images[1]. The second scheme gives the estimate of the hypercube which minimizes a certain regularization energy.

All these improved reconstruction schemes show the potential of using the full encoded set information.

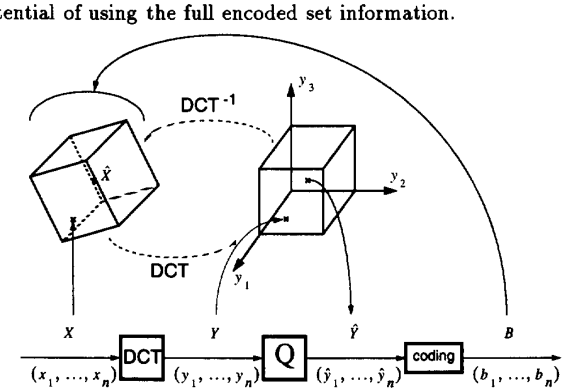

Figure 2: Set theoretic presentation of the DCT encoding of an n-point sequence.

3. DESIGN OF NEW CODING SCHEMES

Choosing an estimate different from the center of the encoded set is a first step for coding innovation. A second step for more radical innovation is to change the definition of the set itself. As seen in the previous paragraph, the encoded set resulting from a traditional coding scheme may include some undesirable estimates, such as its very center in the example of block DCT coding. Improvement can be achieved if the set to be encoded is directly designed so that all its elements are satisfactory estimates of the true signal with respect to the considered artifacts.

At first sight, this may imply more complex encoding. However, one should keep in mind that encoding a set is less constrained than encoding an estimate since a set only gives an implicit information about the input signal. As a result, more freedom and more possibilities are given to the encoding part. The price to pay may be a more complex decoding part since an estimate has to be retrieved from the implicit set information.

The advantage of implicit set encoding is particularly important when dealing with input images containing several features of various perceptual nature, such as texture, edges and low frequency variations. While it is in general difficult to encode directly an estimate which reproduces simultaneously all the desired features, a set of estimates satisfying these features can be simply and implicitly defined as intersection of the sets characterizing each feature respectively. Then, using separate subencoders for each feature is sufficient to give implicitly the description of the intersection set. One big advantage is that there is no need to perform combined processing of the subencoders' outputs, and the subencoders themselves can be designed separately, using techniques specific to the corresponding feature (think of the difference between encoding edges and encoding texture).

The work of combination is only performed at the decoder side. It simply consists of retrieving an estimate which belongs simultaneously to the sets described by the subencoders' outputs respectively. While feature combination is performed in conventional coding systems using linear combination, overlapping or segmentation techniques, feature combination in set theoretic coding is of logic nature since the intersection operation is based on the logic operation "and". For the design of the decoder, tools which have been developed in the field of *set theoretic estimation* [4] can be typically used to retrieve estimates from set theoretic information. For example, when the sets involved in the intersection are convex, an element of the intersection can be reconstructed thanks to the classical algorithm of *alternating projections onto convex sets* (POCS) [5, 4]. This algorithm has been used in [3].

Another advantage of the set theoretic design is that the mappings involved in the encoding process need not be limited to linear transformations. For example, as will be seen in Section 5, non-linear functions such as quadratic functions can be used to encode a set.

4. RATE-DISTORSION RELATION AND IMAGE COMPRESSION

In set theoretic coding, the link between the input signal and the reconstructed signal lies in the encoded set which contains both of these signals by design. Therefore, a measure of quality of the set encoding is the maximum distor-

[1] The minimum distance is zero in the case where the estimated set turns out to have a non-empty intersection with the hypercube.

sion existing between any two elements of the set, or, the size of the set with respect to the distorsion measure. Thus, the classical notion of rate-distorsion relation still exists. It is the relation between the number of bits needed to encode the set and the size of the set. Obviously, the smaller the size is, the more bits will be required.

Now, a potential contribution of set theoretic coding to image compression is its ability to deal with distorsion measures of perceptual nature. Indeed, sets can be more directly constructed according to functions derived from perception models, which may be non-linear or may be empirical. Therefore, a better image compression may be achieved because, for the same bit rate, the encoded set is better adapted to the desired perceptual features and thus, leads to reconstructed images of higher subjective quality. An example of such situation will be given in the next section. In general, better image compression may be achieved by a more refined bit allocation with respect to the desired perceptual features.

5. AN APPLICATION TO IMAGE CODING

In this section, we show a demonstration of the set theoretic approach applied to block DCT coding. While this encoding scheme manages to preserve a good image quality within each block, even at relatively low bit rate, the missing information about the transition of the original image across the block boundaries becomes critical. At the given total bit rate of 0.41 bit/pixel, we propose to reshape the encoded DCT set by giving less priority to the image quality within each block and include some information about the block boundary transitions. We do this by allowing some increase in the size of the DCT hypercube set leading to a new set $\mathcal{C}_0$, and then taking its intersection with another set $\mathcal{S}_0$ containing some information about the block boundary transitions of the original image. In the present experiment, we split the bit rate of 0.41 bits/pixel into 0.39 and 0.02 bits/pixel to encode $\mathcal{C}_0$ and $\mathcal{S}_0$ respectively. This image block boundary information is acquired using a quadratic function applied to the neighboring region of each block boundary, inspired from [3]. The detailed description of this function and the resulting definition of $\mathcal{S}_0$ are given in Appendix I. Assuming that Figure 3(a) represents the DCT hypercube set, Figure 3(b) represents the reshaped encoded set $\mathcal{C}_0 \cap \mathcal{S}_0$.

This reshaping operation implies the use of an extra encoder computing the quadratic function. The detailed description of this encoder is given in Appendix I. Figure 5 shows the experimental results obtained on a part of *Lenna* image (Figure 4). Figure 5(a) corresponds to the classical DCT encoding and decoding. Figure 5(b) shows an estimate of the reshaped set, obtained by alternating projections between the two encoded sets. The details of these projections are given in Appendix II. While some slight degradation may be observed inside certain blocks of Figure 5(b) compared to Figure 5(a) (ringing artifacts for example), the reduction of blocking artifacts in Figure 5(b) has a predominant visual impact , thus leading to a globally more "pleasing" image. Philosophically speaking, improvements can be achieved for the same bit rate by a more balanced encoding of the features according to their visual importance.

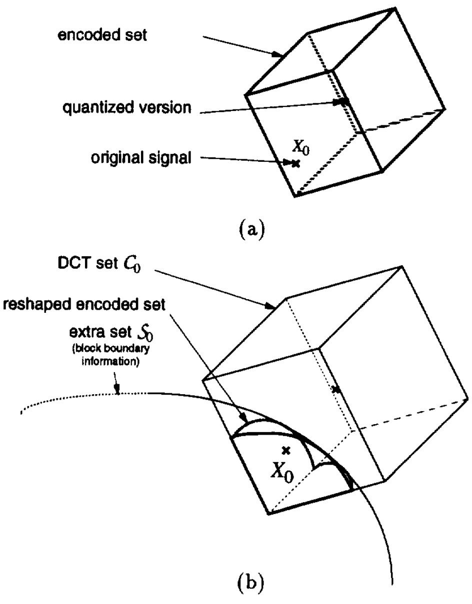

Figure 3: Representation of encoded sets. (a) Classical encoding (ex. block DCT). (b) Encoded set reshaped by intersection ($\mathcal{C}_0 \cap \mathcal{S}_0$).

Further improvements may be obtained by working on the decoding part as in [1, 2, 3]. Indeed, in the above experiment, the alternating projections were used to extract at least one estimate from the encoded set $\mathcal{C}_0 \cap \mathcal{S}_0$. Techniques derived from [1, 2, 3] can be used to find a better estimate within the encoded set $\mathcal{C}_0 \cap \mathcal{S}_0$, with regularization properties for example.

6. CONCLUSION

The previous section gives a concrete illustration of coding manipulations and improvements resulting from the set theoretic approach. A conventional encoder (block DCT) was used as initial illustration. However, the purpose of the set theoretic approach is to motivate the design of a new class of coding schemes for image compression. The above example already shows the potential of this approach for the encoding of perceptual features and the resulting application to image compression.

7. APPENDIX I : DEFINITION THE SET $\mathcal{S}_0$ AND ENCODING

For the sake of simplicity, let us assume that the image X is only composed of two 8×8 blocks, with a common vertical block boundary, as shown in Figure 6. To evaluate the degree of discontinuity of the image X at this boundary, we propose first to look at the evolution of the pixel values along each of the 8 lines. Precisely, for the ith line, we extract the row B_i of the 8 pixel values neighboring the boundary (see Figure 6), and we calculate the weighted average

$B_i \cdot U^T$, where U is a predefined row vector of eight weights. These weights should be chosen in order to amplify the discontinuities specifically localized at the boundary. We have chosen $U = [1, 2, 3, 4, -4, -3, -2, -1]$. Then, we evaluate the global image discontinuity accross the vertical boundary by calculating the energy $\mathcal{E}(X) = \sum_{i=1}^{8}(B_i \cdot U^T)^2$. A similar but simpler version of boundary discontinuity function was previously introduced in [3] using the weighting vector $U = [0, 0, 0, 1, -1, 0, 0, 0]$.

For a given input signal X_0, if $\mathcal{E}(X_0) \leq Q$ where Q is a known value, we know for sure that X_0 belongs to the convex set[2] $\mathcal{S} = \{X \ / \ \mathcal{E}(X) \leq Q\}$. To encode the information of the original image boundary discontinuity, we propose to calculate $\mathcal{E}(X_0)$, quantize this value into Q_0 such that $Q_0 \geq \mathcal{E}(X_0)$ and take Q_0 as output of the encoder. The output Q_0 gives the information that X_0 necessarily belongs to the set of estimate $\mathcal{S}_0 = \{X \ / \ \mathcal{E}(X) \leq Q_0\}$. In the general case of an image of any size, $\mathcal{S}_0$ will be characterized by a complete set of quantized scalar values which correspond to each block boundary, including the vertical and horizontal boundaries.

8. APPENDIX II : DESCRIPTION OF THE CONVEX PROJECTIONS ON $\mathcal{C}_0$ AND $\mathcal{S}_0$

The convex projection on the hypercube $\mathcal{C}_0$ has already been derived in [1, 3]. We will only point out that, because the DCT transform is unitary, and thus conserves distance, X' is the convex projection of X on $\mathcal{C}_0$ if and only if $Y' = \mathrm{DCT}[X']$ is the convex projection of $Y = \mathrm{DCT}[X]$ on the set $\mathrm{DCT}[\mathcal{C}_0]$ (see Figure 7). If we call P the operator of convex projection on the set $\mathrm{DCT}[\mathcal{C}_0]$, this implies that $X' = (\mathrm{DCT}^{-1} \circ \mathrm{P} \circ \mathrm{DCT})[X]$ (see Figure 7). Now, P has a trivial implementation because the set $\mathrm{DCT}[\mathcal{C}_0]$ is a hypercube parallel to the canonical axes. The convex projection

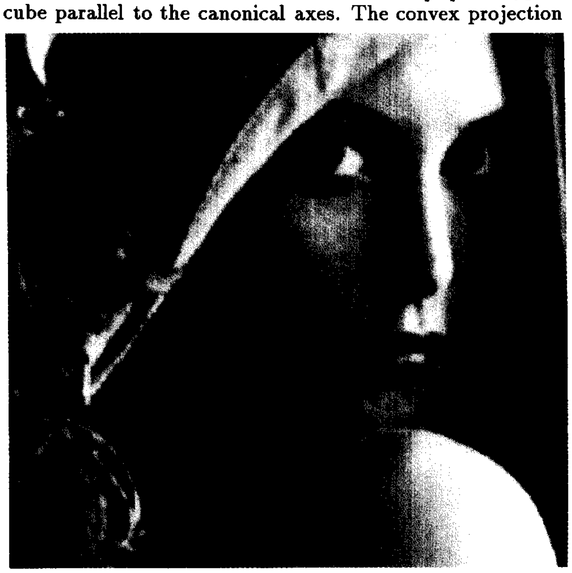

Figure 4: Original image.

[2]The convexity of $\mathcal{S}$ is due to the convexity property of $\mathcal{E}(X)$.

(a)

(b)

Figure 5: Encoding and decoding of the image of Figure 4 at the bit rate of 0.41 bit/pixel. (a) Conventional DCT method. (b) Estimate picked from the reshaped enoded set (see Figure 3(b)).

on the set $\mathcal{S}_0$ is derived as follows. Suppose that the images have the size as in Figure 6. For an image X, let us call the boundary matrix of X, the 8×8 submatrix B whose row vectors are $B_1, ..., B_8$.

Theorem 8.1 *If $X \notin \mathcal{S}_0$, the convex projection of X on $\mathcal{S}_0$ is obtained by adding to B the matrix $\left(\sqrt{\frac{Q_0}{\mathcal{E}(X)}} - 1\right) B \cdot M$, where M is the fixed matrix defined by $M = \frac{U^T \cdot U}{U \cdot U^T}$.*

Sketch of the proof: Whether Y is a vector or a matrix, let us denote the squared sum of its elements by $\|Y\|^2$. By definition, the convex projection of X on $\mathcal{S}_0$ is the element X' which minimizes $\|X' - X\|^2$ subject to the constraint $X' \in \mathcal{S}_0$ equivalent to $\mathcal{E}(X') \leq Q_0$. In fact, because $X \notin \mathcal{S}_0$, the solution X' to this problem necessarily satisfies the equality constraint $\mathcal{E}(X') = Q_0$. It is obvious that only the pixels of the boundary matrix B of X are involved in this problem. Consequently, the solution X' differs from X only by its boundary matrix B', and the problem is to minimize $\|B' - B\|^2$ subject to $\mathcal{E}(X') = Q_0$. Let B'_i be the row vectors of the matrix B'. We have $\|B' - B\|^2 = \sum_{i=1}^{8} \|B'_i - B_i\|^2$. One should first show that, in order for the matrix B' to be the solution to the minimization problem, the row vectors $B'_i - B_i$ must be parallel to U for all $i = 1, ..., 8$. The hint is to see that adding to B'_i any component perpendicular to U will not affect the function $\mathcal{E}(X')$ involved in the equality constraint. Then, we can introduce the notations $B'_i - B_i = \lambda_i U$, $\beta_i = B_i \cdot U^T$ and define the row vectors $\vec{\lambda} = [\lambda_1, ..., \lambda_8]$ and $\vec{\beta} = [\beta_1, ..., \beta_8]$. Because we have

$$\|B' - B\|^2 = \sum_{i=1}^{8} \|B'_i - B_i\|^2 = \sum_{i=1}^{8} \lambda_i^2 \|U\|^2 = \|U\|^2 \|\vec{\lambda}\|^2$$

and

$$\begin{aligned}\mathcal{E}(X') &= \sum_{i=1}^{8} (B'_i \cdot U^T)^2 = \sum_{i=1}^{8} (B_i \cdot U^T + \lambda_i U \cdot U^T)^2 \\ &= \sum_{i=1}^{8} (\beta_i + \lambda_i \|U\|^2)^2 = \left\|\vec{\beta} + \|U\|^2 \vec{\lambda}\right\|^2,\end{aligned}$$

the problem is to minimize $\|\vec{\lambda}\|^2$ subject to $\left\|\vec{\beta} + \|U\|^2\vec{\lambda}\right\|^2 = Q_0$. It can be easily shown that the solution to this problem

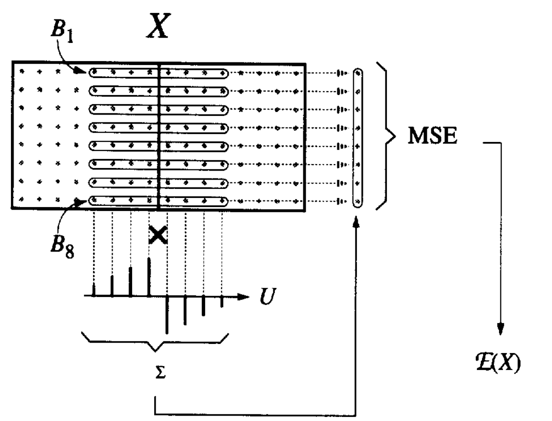

Figure 6: Calculation of the "energy" $\mathcal{E}(X)$ of discontinuity accross the vertical boundary between two blocks.

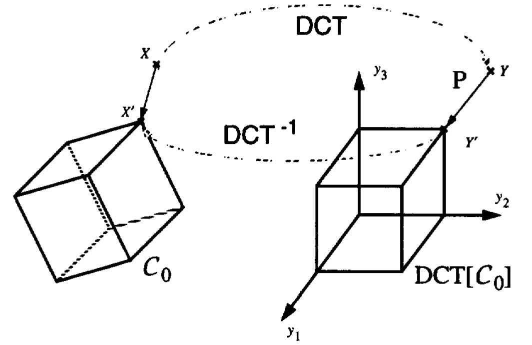

Figure 7: Convex projection on the DCT encoded set $\mathcal{C}_0$.

is $\vec{\lambda} = \left(\frac{\sqrt{Q_0}}{\|\vec{\beta}\|} - 1\right) \frac{\vec{\beta}}{\|U\|^2}$ which implies that $\lambda_i = \left(\sqrt{\frac{Q_0}{\mathcal{E}(X)}} - 1\right) B_i \cdot \frac{U^T}{\|U\|^2}$ for all $i = 1, ..., 8$. From $B'_i - B_i = \lambda_i U$, we obtain $B'_i = B_i + \left(\sqrt{\frac{Q_0}{\mathcal{E}(X)}} - 1\right) B_i \cdot \frac{U^T \cdot U}{\|U\|^2}$ which implies that $B' = B + \left(\sqrt{\frac{Q_0}{\mathcal{E}(X)}} - 1\right) B \cdot M$ □

For an image of any size, the global projection on $\mathcal{S}_0$ consists of locally and successively perform the operation of Theorem 8.1 on each block boundary of the image estimate, using the respective encoded value Q_0 and the respective energy function $\mathcal{E}(X)$.

9. REFERENCES

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