# OVERSAMPLED FIR FILTER BANKS AND FRAMES IN $\ell^2(\mathbf{Z})$

*Martin Vetterli* *Zoran Cvetković*

Department of Electrical Engineering and Computer Sciences,
University of California at Berkeley, California, USA

## ABSTRACT

Perfect reconstruction FIR filter banks are equivalent to a particular class of frames in $\ell^2(\mathbf{Z})$. These frames are the subject of this paper. Necessary and sufficient conditions on a filter bank to implement a frame or a tight frame decomposition are given, as well as the necessary and sufficient condition for the perfect reconstruction using FIR filters. Complete parameterizations of FIR filter banks satisfying these conditions are also given. Further, we study the condition under which the frame dual to the frame associated to an FIR filter bank is also FIR, and give a parameterization of a class of filter banks having this property.

## 1. INTRODUCTION

The idea of time-frequency localized representations goes back to the 1940's and the work of Gabor [3] who proposed signal decompositions in terms of modulated Gaussians. It was aimed at overcoming the major drawback of the two traditional signal descriptions, one in time and the other in frequency, which both achieve infinitely fine resolutions in their respective domains but no resolution in the complementary domains. On the other hand, expansions based on modulated Gaussians, which achieve the lower bound on the uncertainty in the joint time-frequency domain, should facilitate descriptions with good resolution in both time and frequency. Expansions with respect to different kinds of time-frequency localized waveforms have been subsequently used in physics, geophysics and signal processing. However, not before the 1980's have they received a thorough and rigorous treatment.

The theory of expansions into time-frequency localized atoms in $L^2(\mathbf{R})$ has been developed beyond the orthogonal or biorthogonal case, focusing on redundant representations based on Weyl-Heisenberg and wavelet frames [4]. One of the primary reasons for studying overcomplete expansions is that the requirement for orthogonality or linear independence imposes considerable constraints which can sometimes be in conflict with other design specifications. Ironically enough, perhaps the most striking example is the fact that Gabor analysis with orthonormal bases with good resolution in both time and frequency is not possible.

In parallel, these expansions were studied in the discrete-time domain, $\ell^2(\mathbf{Z})$, in the framework of filter banks and subband coding schemes. However, this research has been confined mainly to orthonormal and biorthonormal bases which are equivalent to critically sampled filter banks [1, 2]. This paper studies expansions in $\ell^2(\mathbf{Z})$ which are equivalent to oversampled FIR filter banks. The issues investigated here are 1) necessary and sufficient conditions on a filter bank to implement a frame or a tight frame decomposition, 2) the feasibility of perfect reconstruction using FIR filters after an FIR analysis, and 3) parameterizations of interesting classes of perfect reconstruction FIR oversampled filter banks.

**Notation** For a sequence $\varphi$, $\tilde{\varphi}$ will denote the complex conjugate of the time-reversed version of $\varphi$. When used with a matrix whose entries are rational functions of the complex variable $z$, $\tilde{\mathbf{H}}(z)$ will denote the matrix obtained from $\mathbf{H}(z)$ by transposing it, changing all coefficients of the rational functions by their complex conjugates, and substituting $z$ by $z^{-1}$. If $\tilde{\mathbf{H}}(z) = \mathbf{H}(z)$ we say that $\mathbf{H}(z)$ is *parahermitian*. A polynomial matrix $\mathbf{H}(z)$, such that $\det\mathbf{H}(z)$ is a nonzero constant, is called *unimodular matrix*. In this paper we shall use the term *polynomial* for *Laurent polynomials* in general. Complex conjugate transpose of a vector $\mathbf{v}$ will be denoted as $\mathbf{v}^*$.

## 2. OVERSAMPLED FILTER BANKS AND FRAME EXPANSIONS

The theory of filter banks [1, 2] provides a convenient framework for both the study and the implementation of an important class of signal decompositions in $\ell^2(\mathbf{Z})$, i.e. those which underlie signal analysis through a sliding window using a selected set of prototype waveforms. In general, they have the form

$$x[n] = \sum_{i=0}^{K-1} \sum_{j=-\infty}^{\infty} c_{i,j}\varphi_{i,j}[n], \tag{1}$$

where vectors $\varphi_{i,j}[n]$ denote translated versions of $K$ prototype waveforms, $\varphi_{i,j}[n] = \varphi_i[n - jN]$. Expansions of the type given in (1) can provide a stable representation of any signal in $\ell^2(\mathbf{Z})$ if and only if the family $\Phi$,

$$\Phi = \{\varphi_{i,j} : \varphi_{i,j}[n] = \varphi_i[n - jN]\}_{i=0,1,\ldots,K-1, j\in\mathbf{Z}} \tag{2}$$

constitutes a frame, i.e. if for any $x \in \ell^2(\mathbf{Z})$

$$A\|x\|^2 \le \sum_{i=0}^{K-1} \sum_{j=-\infty}^{\infty} |< x, \varphi_{i,j} >|^2 \le B\|x\|^2, \tag{3}$$

for some constants $0 < A \leq B < \infty$, which are called *frame bounds*. If $\Phi$ is a frame, then there exists an analysis frame

$$\Psi = \{\psi_{i,j} : \psi_{i,j}[n] = \psi_i[n - jN]\}_{i=0,1,\ldots,K-1, j\in \mathbf{Z}} \quad (4)$$

such that the coefficients of the expansion in (1) can be calculated as inner products with its vectors, that is

$$x[n] = \sum_{i=0}^{K-1} \sum_{j=-\infty}^{\infty} < x, \psi_{i,j} > \varphi_{i,j}[n]. \quad (5)$$

For a given frame $\Phi$, corresponding analysis frame $\Psi$ in (5) is not unique. One particular solution is the frame dual to $\Phi$ [6]. If the frame bounds are equal, $A = B$, we say that the frame is tight. It can be shown that, under this condition, the frame $\Phi$ is up to a multiplicative factor equal to its dual [5] so that the expansion formula (1) gets the form reminiscent of orthonormal expansions,

$$x[n] = \frac{1}{A} \sum_{i=0}^{K-1} \sum_{j=-\infty}^{\infty} < x, \varphi_{i,j} > \varphi_{i,j}[n]. \quad (6)$$

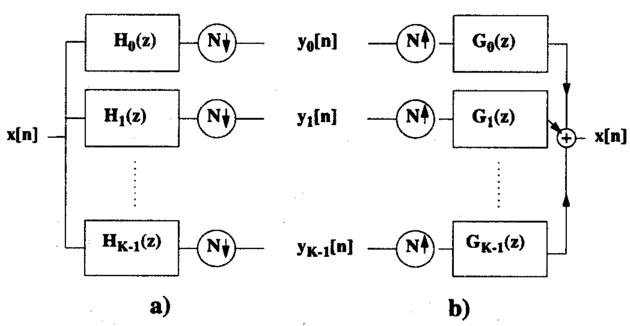


**Figure 1. Oversampled filter bank, $N < K$. a) Analysis filter bank. b) Synthesis filter bank.**

Vector families of this type are equivalent to filter banks. A $K$ channel analysis filter bank with subsampling by factor $N$ in the channels, shown in Figure 1a, computes inner products $< x, \varphi_{i,j} >$ if the impulse responses of the analysis filters $H_0(z)$, $H_1(z)$, ... , $H_{K-1}(z)$ are selected as $h_i[n] = \tilde{\varphi}_i[n]$, $i = 0, 1, ..., K-1$. If the vector family $\Phi$ associated with an analysis filter bank in this way is a frame, we say that the filter bank *implements a frame decomposition* and refer to $\Phi$ as its *associated frame*, or a *filter bank frame*. On the other hand, a synthesis filter bank as the one in Figure 1b, implements the reconstruction formula

$$\hat{x}[n] = \sum_{i=0}^{K-1} \sum_{j=-\infty}^{\infty} y_i[j]\varphi_{i,j}[n] \quad (7)$$

provided that the impulse responses of the filters $G_0(z)$, ... , $G_{K-1}(z)$ are equal to the waveforms $\varphi_i$, that is $g_i[n] = \varphi_i[n]$. In the following we consider filter bank frames that consist of finite-length vectors, with $K \geq N$, that is oversampled FIR filter banks. Note that a filter bank can implement a frame decomposition only if $K \geq N$.

## 3. FRAME CONDITIONS

Frame conditions on an oversampled filter bank will be expressed in terms of properties of its polyphase analysis matrix. In the case of a $K$ channel filter bank with subsampling by factor $N$, the polyphase analysis matrix $\mathbf{H_p}(z)$ is defined as

$$\left[\mathbf{H_p}(z)\right]_{i,j} = H_{i,j}(z), \quad i = 0, ..., K-1, \; j = 0, ..., N-1, \quad (8)$$

where

$$H_{ij}(z) = \sum_{n=-\infty}^{+\infty} h_i[nN - j]z^{-n}, \quad (9)$$

represents the $j$-th polyphase component of $H_i(z)$.

The necessary and sufficient conditions on a filter bank to implement a frame or a tight frame decomposition are given by the following theorems.

**Theorem 1** *[11] An FIR filter bank implements a frame decomposition in $\ell^2(\mathbf{Z})$ if and only if its polyphase analysis matrix is of full rank on the unit circle.*

**Theorem 2** *[11] A filter bank implements a tight frame decomposition in $\ell^2(\mathbf{Z})$ if and only if its polyphase analysis matrix is paraunitary, $\tilde{\mathbf{H}}_\mathbf{p}(z)\mathbf{H_p}(z) = c\mathbf{I}$*

Equivalently, a filter bank implements a frame decomposition if and only if there exists a matrix $\mathbf{G_p}(z)$ of stable, rational, not necessarily causal functions such that

$$\mathbf{G_p}(z)\mathbf{H_p}(z) = c\mathbf{I}. \quad (10)$$

The matrix $\mathbf{G_p}(z)$ is called the *polyphase synthesis matrix* and its entries, $\left[\mathbf{G_p}(z)\right]_{i,j} = G_{ij}(z)$, are the polyphase components of the synthesis filter bank (see Figure 1b),

$$G_i(z) = \sum_{i=0}^{N-1} z^{-j}G_{ij}(z^N), \quad (11)$$

which can be used for perfect reconstruction from the decomposition obtained from the analysis filter bank. Solution for $\mathbf{G_p}(z)$ of the polyphase equation (10), and hence the synthesis filter bank is not unique if $K > N$. One solution for $\mathbf{G_p}(z)$ is the pseudoinverse of $\mathbf{H_p}(z)$, which is given by

$$\mathbf{H}^+(z) = \left(\tilde{\mathbf{H}}_\mathbf{p}(z)\mathbf{H_p}(z)\right)^{-1} \tilde{\mathbf{H}}_\mathbf{p}(z). \quad (12)$$

The frame associated with $\mathbf{G_p}(z) = \mathbf{H}^+(z)$ is the frame dual to the frame associated with the analysis filter bank [5]. This further means that reconstruction from noisy subband components using this filter bank projects to zero the noise component which is orthogonal to the range of the frame expansion of the analysis filter bank. Hence it is important to investigate when both a filter bank frame and its dual consist of finite-length vectors.

**Theorem 3** *[11] For a frame associated with an FIR filter bank, with the polyphase analysis matrix $\mathbf{H_p}(z)$, its dual frame consists of finite-length vectors if and only if $\tilde{\mathbf{H}}_\mathbf{p}(z)\mathbf{H_p}(z)$ is unimodular.*

Note that Theorem 3 does not preclude the existence of an FIR perfect reconstruction synthesis filter bank even if $\tilde{\mathbf{H}}_\mathbf{p}(z)\mathbf{H_p}(z)$ is not unimodular.

## 4. PARAMETERIZATIONS OF FIR FILTER BANK FRAMES

The parameterization of filter bank frames which is given here is based on the Smith form [7] of corresponding polyphase analysis matrices. Any polynomial matrix $\mathbf{H_p}(z)$, of dimension $K \times N$ ($K \geq N$), can be decomposed as the product

$$\mathbf{H_p}(z) = \mathbf{R}(z)\mathbf{D}(z)\mathbf{C}(z), \tag{13}$$

where $\mathbf{R}(z)$ and $\mathbf{C}(z)$ are unimodular matrices of dimensions $K \times K$ and $N \times N$, respectively, while $\mathbf{D}(z)$ is a diagonal $K \times N$ polynomial matrix

$$\mathbf{D}(z) = \begin{bmatrix} d_1(z) & 0 & \dots & 0 \\ 0 & d_2(z) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_N(z) \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}. \tag{14}$$

The unimodular matrices can be chosen so that the polynomials $d_i(z)$ are monic and that $d_i(z)$ is a factor of $d_{i+1}(z)$. The matrix $\mathbf{D}(z)$ is called the *Smith form* of $\mathbf{H_p}(z)$. The unimodular matrices $\mathbf{R}(z)$ and $\mathbf{C}(z)$ are products of finitely many elementary matrices

$$\mathbf{R}(z) = \mathbf{R}_1(z)\mathbf{R}_2(z)\cdots\mathbf{R}_m(z)$$

$$\mathbf{C}(z) = \mathbf{C}_1(z)\mathbf{C}_2(z)\cdots\mathbf{C}_n(z).$$

Elementary matrices $\mathbf{R}_i(z)$, $\mathbf{C}_j(z)$ correspond to elementary row (column) operations, and have one of the following forms:

- a permutation matrix, i.e. the identity matrix with two rows permuted;
- a diagonal matrix with elements on the diagonal equal to unity, except for one which is equal to a nonzero scalar;
- a matrix with ones on the main diagonal and a single non-zero entry off the diagonal, which is a polynomial $\alpha(z)$.

An example of the three types of elementary $4 \times 4$ matrices is given below.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha(z) & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

A complete parameterization of FIR filter bank frames follows directly from the Smith form, and is stated by the following proposition.

**Proposition 1** *[11] An oversampled FIR filter bank implements a frame decomposition in $\ell^2(\mathbf{Z})$ if and only if the polynomials on the diagonal of the Smith form of its polyphase analysis matrix have no zeros on the unit circle.*

A parameterization of FIR filter bank frames having dual frames which are also FIR is given by the following proposition which is a corollary of Theorem 3.

**Proposition 2** *[11] Consider an oversampled FIR filter bank with the polyphase analysis matrix $\mathbf{H_p}(z)$. $\tilde{\mathbf{H}}_\mathbf{p}(z)\mathbf{H_p}(z)$ is unimodular if $\mathbf{H_p}(z)$ has the following form:*

$$\mathbf{H_p}(z) = \mathbf{H}_0\mathbf{R}(z)\mathbf{D}(z)\mathbf{C}(z), \tag{15}$$

*where $\mathbf{H}_0$ is a $K \times N$ ($K \geq N$) matrix of scalars, such that $\tilde{\mathbf{H}}_0\mathbf{H}_0 = c\mathbf{I}$, $\mathbf{R}(z)$ and $\mathbf{C}(z)$ are products of finitely many elementary matrices, and $\mathbf{D}(z)$ is a $N \times N$ diagonal matrix of polynomials, with nonzero monomials on the diagonal.*

*On the other hand, any unimodular parahermitian matrix of polynomials, $\mathbf{P}(z)$, which is positive definite on the unit circle, can be factored as*

$$\mathbf{P}(z) = \tilde{\mathbf{H}}_\mathbf{p}(z)\mathbf{H_p}(z),$$

*where $\mathbf{H_p}(z)$ is of the form given in (15).*

A necessary and sufficient condition for an FIR synthesis is given by the following proposition, which also implicitly gives a complete parameterization of FIR oversampled filter banks in this class.

**Proposition 3** *[11] Perfect reconstruction with FIR filters after analysis by an oversampled FIR filter bank is possible if and only if polynomials on the diagonal of the Smith form of the polyphase analysis matrix are monomials.*

As it was established in the previous section, tight filter bank frames are equivalent to paraunitary polynomial matrices. A $K \times N$ paraunitary matrix ($K > N$) can always be embedded into a $K \times K$ paraunitary matrix [8]. The parameterization of the rectangular paraunitary polyphase matrices, that is filter bank tight frames in $\ell^2(\mathbf{Z})$, which we give in the following proposition, follows directly from one of the factorizations of square paraunitary matrices studied by Vaidyanathan [1].

**Proposition 4** *[11] A $K \times N$ ($K \geq N$) polynomial matrix $\mathbf{H_p}(z)$ is paraunitary if and only if it has the decomposition*

$$\mathbf{H_p}(z) = \mathbf{V}_M(z)\mathbf{V}_{M-1}(z)\cdots\mathbf{V}_1(z)\mathbf{H}_0. \tag{16}$$

*The building blocks, $\mathbf{V}_i(z)$, have the following form,*

$$\mathbf{V}_i(z) = \mathbf{I} - \mathbf{v}_i\mathbf{v}_i^* + z^{-1}\mathbf{v}_i\mathbf{v}_i^*, \tag{17}$$

*where $\mathbf{v}_i$ denotes a unit norm vector, while $\mathbf{H}_0$ is a $K \times N$ matrix of scalars such that $\tilde{\mathbf{H}}_0\mathbf{H}_0 = c\mathbf{I}$.*

## 5. CONCLUSION

In this paper properties of oversampled FIR filter banks are studied. Necessary and sufficient conditions on a filter bank to implement a frame or a tight frame decomposition in $\ell^2(\mathbf{Z})$ were given in terms of properties of the corresponding polyphase analysis matrix. Complete parameterizations of filter bank frames and tight frames were also given. A necessary and sufficient condition for the feasibility of perfect reconstruction with FIR filters was also established, as well as a necessary and sufficient condition for an FIR filter bank frame to have an FIR dual. Filter banks in these two classes were also parameterized.

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## REFERENCES

[1] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice-Hall, Englewood Cliffs, New Jersey, 1993.

[2] M. Vetterli and J. Kovačević, *Wavelets and Subband Coding*, Prentice-Hall, Englewood Cliffs, New Jersey, 1995

[3] D. Gabor, Theory of Communications, *J. IEE*, Vol.93 (III), 1946, pp.429-457.

[4] I. Daubechies, The Wavelet Transform, Time-Frequency Localization and Signal Analysis, *IEEE Trans. on Information Theory*, Vol.36, No.5, September 1990, pp.961-1005.

[5] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Series in Appl. Math, SIAM, 1992.

[6] R. J. Duffin and A. C. Schaeffer, A Class of Nonharmonic Fourier Series, *Trans. Amer. Math. Soc*, Vol.72, March 1952, pp.341-366.

[7] H. J. S. Smith, On Systems of Linear Indeterminate Equations and Congruences, *Philos. Trans. Royal Soc. London*, Vol.151, 1861, pp.293-326.

[8] H. Park, *A Computational Theory of Laurent Polynomial Rings and Multidimensional FIR Systems*, Ph. D. Dissertation, University of California at Berkeley, May 1995.

[9] R. Balian, Un principe d'incertitude fort en théorie du signal on mécanique quantique, *C. R. Acad. Sc. Paris*, vol.292, série 2, 1981.

[10] F. Low, *Complete Sets of Wave Packets*, in *A Passion for Physics - Essays in Honor of Geoffrey Chew*, pp.17-22, Singapore: World Scientific, 1985.

[11] Z. Cvetković and M. Vetterli, Oversampled Filter Banks, *submitted to IEEE Trans. Signal Processing*, December 1994.