CONSISTENCY IN QUANTIZED MATCHING PURSUIT

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ABSTRACT

This paper explores the effects of coefficient quantization in applying the matching pursuit algorithm to source coding of vectors in $\mathbb{R}^N$. By considering the issue of *consistency*, we find that even though matching pursuit is designed to produce a linear combination to estimate a given source vector, optimal reconstruction in the presence of coefficient quantization requires a nonlinear algorithm. Such an algorithm was implemented and was experimentally confirmed to have superior reconstruction properties in comparison to the standard linear reconstruction. The improvement depends on the source, dictionary and operating point; in some cases the MSE was lessened by as much as a factor of five.

1. MATCHING PURSUIT

Matching pursuit is an algorithm for finding linear combinations that approximate a given signal vector. It was introduced to the signal processing community in the context of time-frequency analysis by Mallat and Zhang [1]. Mallat and his students have uncovered many of its properties [2].

Let $\mathcal{D} = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ span $\mathbb{R}^N$. Also impose the additional constraint that $\|\varphi_k\| = 1$ for all k. We will call $\mathcal{D}$ our *dictionary* of vectors. Matching pursuit is an algorithm to represent $f \in H$ by a linear combination of elements of $\mathcal{D}$. Furthermore, matching pursuit is an iterative scheme that at each step attempts to approximate f as closely as possible in a greedy manner. If the dictionary is highly redundant, we expect that after a few iterations we will have an efficient approximate representation of f.

In the first step of the algorithm, k_0 is selected such that $|\langle \varphi_{k_0}, f\rangle|$ is maximized. Then f can be written as its projection onto φ_{k_0} and a residue $R_1 f$,

$$f = \langle \varphi_{k_0}, f\rangle \varphi_{k_0} + R_1 f.$$

The algorithm is iterated by treating $R_1 f$ as the vector to be best approximated by a multiple of φ_{k_1}. At step $p+1$, k_p is chosen to maximize $|\langle \varphi_{k_p}, R_p f\rangle|$ and

$$R_{p+1} f = R_p f - \langle \varphi_{k_p}, R_p f\rangle \varphi_{k_p}.$$

Identifying $R_0 f = f$, we can write

$$f = \sum_{i=0}^{n-1} \langle \varphi_{k_i}, R_i f\rangle \varphi_{k_i} + R_n f. \tag{1}$$

Hereafter we will denote $\langle \varphi_{k_i}, R_i f\rangle$ by α_i.

The reader is referred to [2] for details on the convergence of matching pursuit and other properties. Note that the output of a matching pursuit expansion is not only the coefficients $(\alpha_0, \alpha_1, \ldots)$, but also the indices $(k_0, k_1, \ldots)$. For storage and transmission purposes, the indices must be accounted for.

2. QUANTIZED MATCHING PURSUIT

Although matching pursuit has been applied to low bit rate compression problems [3, 4, 5], which inherently require coarse coefficient quantization, little work has been done to understand the qualitative effects of coefficient quantization in matching pursuit. In this section we explore some of these effects. The issue of consistency in these expansions is explored in §2.2. In §2.3, a detailed example on the application of matching pursuit to quantize an $\mathbb{R}^2$-valued source is presented. This serves to illustrate the concepts from §2.2 and demonstrate the potential for improved reconstruction using consistency.

2.1. Discussion

Coefficients are quantized in any computer implementation of matching pursuit. When the quantization is fine, it is generally safe to neglect it. If the quantization is coarse, as it must be for moderate to low bit rate compression applications, the effects of quantization may be significant.

Define *quantized matching pursuit* to be matching pursuit with non-negligible quantization of the coefficients. We will denote the quantized coefficients by $\hat{\alpha}_i = q(\alpha_i)$, where q is a (scalar) quantization function. Note that quantization destroys the orthogonality of the projection and residual.

We are assuming that the quantization of α_i occurs before the residual $R_{i+1} f$ is calculated, and that the quantized version is used in determining the residual so that quantization errors do not propagate to subsequent iterations. Since $\hat{\alpha}_i$ must be determined before α_{i+1}, it is implicit in this assumption is that the coefficient quantization is scalar.

For any particular application, there are several design problems: a dictionary must be chosen, scalar quantizers must be designed, and the number of iterations (or a stopping criterion) must be set. In principle, these could be jointly optimized for a given source distribution, distortion measure, and rate measure. In practice, this is an overly broad problem.

2.2. Consistency

Let $Q : X \to Y$ be a quantization function. We say that $\hat{x} \in X$ is a *consistent estimate* of $x \in X$, or a *consistent reconstruction*, if $Q(\hat{x}) = Q(x)$ [6]. In words, we would say that an estimate is consistent if it has the same quantized

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version as the original; it is "consistent" with the observation of $Q(x)$. Consistency depends only on the deterministic properties of Q, and not on statistical properties of the X-valued source.

Reconstructions from a matching pursuit representation are generally computed by using the quantized coefficients in (1), giving

$$\hat{f} = \sum_{i=0}^{p-1} \hat{\alpha}_i \varphi_{k_i}.$$

The shortcoming of this reconstruction is that it disregards the effects of quantization; hence it can produce inconsistent estimates. We will see that a matching pursuit representation implicitly contains many linear constraints and that inconsistency is not uncommon.

Suppose p iterations of matching pursuit are performed with the dictionary $\mathcal{D}$. The output of the (quantized) matching pursuit algorithm is

$$\{k_0, \hat{\alpha}_0, k_1, \hat{\alpha}_1, \ldots, k_{p-1}, \hat{\alpha}_{p-1}\}. \tag{2}$$

(There is nothing consistent or inconsistent about this set.) Denote the output of matching pursuit (with the same dictionary and quantizers) applied to $\hat{f}$ by

$$\{k_0', \hat{\alpha}_0', k_1', \hat{\alpha}_1', \ldots, k_{p-1}', \hat{\alpha}_{p-1}'\}.$$

If

$$k_i = k_i' \text{ and } \hat{\alpha}_i = \hat{\alpha}_i' \tag{3}$$

for $i = 0, 1, \ldots, p-1$, we say that $\hat{f}$ is a *strictly consistent* estimate. If (3) holds except possibly that $k_i \neq k_i'$ for some i for which $\hat{\alpha}_i = \hat{\alpha}_i' = 0$, we say that $\hat{f}$ is a *loosely consistent* estimate. The second definition is included because a reasonable coding scheme might discard k_i if $\hat{\alpha}_i = 0$.

The crucial point is that there is more information in (2), along with $\mathcal{D}$ and knowledge of the workings of matching pursuit, than there is in $\hat{f}$. In particular, (2) gives a set of linear inequality constraints that defines a partition cell in which f lies. $\hat{f}$ is an estimate of f that does not necessarily lie in this cell.

Let us now list the complete set of constraints implied by (2). For notational convenience, we assume uniform scalar quantization of the coefficients with stepsize Δ and midpoint reconstruction. The selection of k_0 implies

$$|\langle \varphi_{k_0}, f \rangle| \geq |\langle \varphi, f \rangle|, \quad \forall\, \varphi \in \mathcal{D}. \tag{4}$$

For each element of $\mathcal{D} \setminus \{\varphi_{k_0}\}$, (4) specifies a pair of half-space constraints with boundary planes passing through the origin. An example of such a constraint in $\mathbb{R}^2$ is shown in Figure 1. If φ_{k_0} is the vector with the solid arrowhead (chosen from all of the marked vectors), the source vector must like in the hatched area. For $N > 2$, the intersection of these constraints is two infinite convex polyhedral cones situated symmetrically with their apexes at the origin. The value of $\hat{\alpha}_0$ gives the constraint

$$\langle \varphi_{k_0}, f \rangle \in \left[\hat{\alpha}_0 - \frac{\Delta}{2}, \hat{\alpha}_0 + \frac{\Delta}{2}\right]. \tag{5}$$

This specifies a pair of planes, perpendicular to φ_{k_0}, between which f must lie. Constraints (4) and (5) are illustrated in Figure 2 for $\mathbb{R}^3$. The vector with the solid arrowhead was chosen among all the marked dictionary vectors as φ_{k_0}. Then the quantization of α_0 implies that the source vector lies in the volume shown.

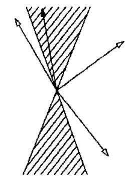


Figure 1. *Illustration of consistency constraint (4).*

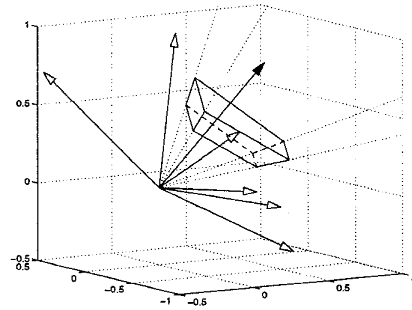


Figure 2. *Illustration of consistency constraints (4) & (5).*

At the $(i-1)$st step, the selection of k_i gives the constraints

$$\left|\left\langle \varphi_{k_i}, f - \sum_{\ell=0}^{i-1} \hat{\alpha}_\ell \varphi_{k_\ell} \right\rangle\right| \geq \left|\left\langle \varphi, f - \sum_{\ell=0}^{i-1} \hat{\alpha}_\ell \varphi_{k_\ell} \right\rangle\right|,$$

$\forall\, \varphi \in \mathcal{D}$. This defines $M-1$ pairs of linear half-space constraints with boundaries passing through $\sum_{\ell=0}^{i-1} \hat{\alpha}_\ell \varphi_{k_\ell}$. As before, these define two infinite pyramids situated symmetrically with their apexes at $\sum_{\ell=0}^{i-1} \hat{\alpha}_\ell \varphi_{k_\ell}$. Then $\hat{\alpha}_i$ gives

$$\left\langle \varphi_{k_i}, f - \sum_{\ell=0}^{i-1} \hat{\alpha}_\ell \varphi_{k_\ell} \right\rangle \in \left[\hat{\alpha}_i - \frac{\Delta}{2}, \hat{\alpha}_i + \frac{\Delta}{2}\right].$$

This again specifies a pair of planes, now perpendicular to φ_{k_i}, between which f must lie.

By being explicit about the constraints as above, we see that, except in the case that $0 \in \left[\hat{\alpha}_i - \frac{\Delta}{2}, \hat{\alpha}_i + \frac{\Delta}{2}\right]$ for some i, the partition cell defined by (2) is convex.[1] Thus by using an appropriate projection operator, one can find a strictly consistent estimate from any initial estimate. The partition cells are intersections of cells of the form shown in Figure 2.

Experiments were performed to assess how the probability of an inconsistent estimate depends on $\mathcal{D}$, r, and Δ. The loose sense of consistency was used in all the experiments.

The first set of experiments involved quantizing an $\mathbb{R}^2$-valued source with the $\mathcal{N}(0, I)$ distribution. With dictionaries formed by taking linearly independent subsets of $2M$-th roots of unity, M was varied between 2 and 256 while Δ was varied between $10^{-1.9}$ and $10^{0.3}$. Figure 3 shows the probability of inconsistency as a function of M and Δ. The probability of inconsistency is significant! The surface is rather complicated, but we can identify two trends: the probability of inconsistency goes to zero as M is increased and as $\Delta \to 0$.

[1] The "hourglass" cell that results from $0 \in \left[\hat{\alpha}_i - \frac{\Delta}{2}, \hat{\alpha}_i + \frac{\Delta}{2}\right]$ does not pose a problem in reconstruction.

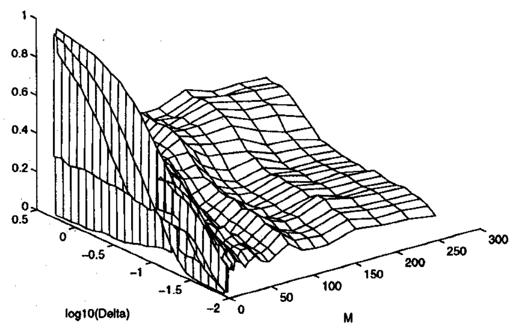


Figure 3. *Probability of inconsistent reconstruction for an $\mathbb{R}^2$-valued source as a function of M and Δ.*

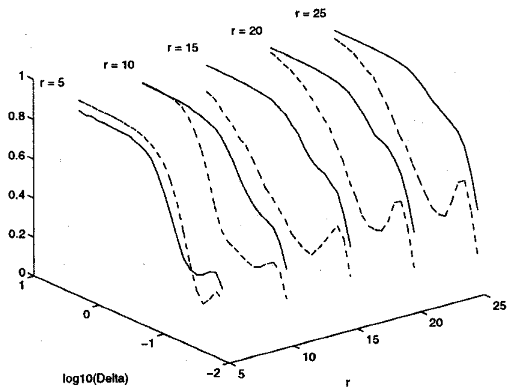


Figure 4. *Probabilities of inconsistent reconstruction for an $\mathbb{R}^5$-valued source. Dictionaries correspond to oversampled A/D conversion (solid) or maximally space points on the unit sphere (dashed).*

To explore the dependence on $\mathcal{D}$, experiments were performed for quantizing an $\mathbb{R}^5$-valued source with the $\mathcal{N}(0, I)$ distribution. The consistency of reconstruction was checked for two iteration expansions. Dictionary sizes of $M = 25$, 50, 75, 100, and 125 were used. The results are shown in Figure 4. The solid curves were generated with dictionaries corresponding to oversampled A/D conversion [7]. The dashed curves were generated using dictionaries of maximally spaced points [8]. For both types of dictionaries, the probability of inconsistency goes to one for very coarse quantization and goes to zero as $\Delta \to 0$. The qualitative difference between the curves indicates that there are complicated geometric factors involved.

2.3. An Example in $\mathbb{R}^2$

Consider quantization of an $\mathbb{R}^2$-valued source. Assume that two iterations will be performed with the four element dictionary

$$\mathcal{D} = \left\{ \left[\cos \frac{(2k-1)\pi}{8} \quad \sin \frac{(2k-1)\pi}{8} \right]^T \right\}_{k=1}^{4}.$$

Even if the distribution of the source is known, it is difficult to find analytical expressions for optimal quantizers. Since we wish to use fixed, untrained quantizers, we will use uniform quantizers for α_0 and α_1. Since it will generally be true that $\varphi_{k_0} \perp \varphi_{k_1}$, it makes sense for the quantization step sizes for α_0 and α_1 to be equal.

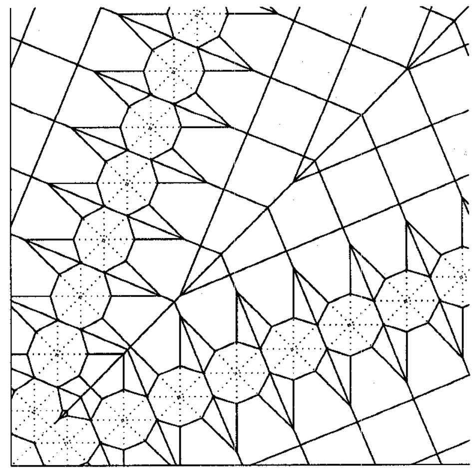


Figure 5. *Partitioning of first quadrant of $\mathbb{R}^2$ by matching pursuit with four element dictionary.*

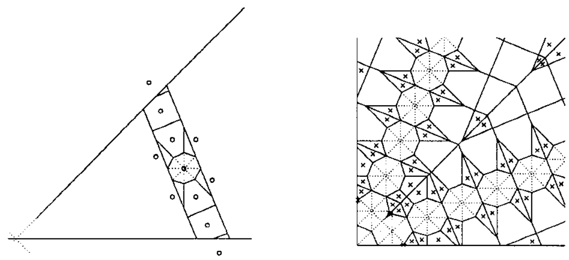
The partitions generated by matching pursuit are very intricate. Figure 5 shows the partitioning of the first quadrant when zero is a quantizer reconstruction value, *i.e.* the quantizer reconstruction points are $\{m\Delta\}_{m\in\mathbb{Z}}$ and decision points are $\{(m+\frac{1}{2})\Delta\}_{m\in\mathbb{Z}}$ for some quantization stepsize Δ.[2] The dotted lines show boundaries that are created by choice of k_0 (k_1) but, depending on the reconstruction method, might not be important because $\hat{\alpha}_0 = 0$ ($\hat{\alpha}_1 = 0$). In this partition, most of the cells are squares, but there are also some smaller cells. Unless the source distribution happens to have high density in the smaller cells, the smaller cells are inefficient in a rate-distortion sense. The fraction of cells that are not square $\to 0$ as $\Delta \to 0$.

This quantization of $\mathbb{R}^2$ gives concrete examples of inconsistency. The reconstruction points were not marked on the partition diagram because the correspondence between cells and reconstruction points would not have been clear. Figure 6(a) depicts parts of these partitions with linear reconstruction points marked with circles. These show that linear matching pursuit reconstructions are not always consistent. Figure 6(b) is a copy of Figure 5 with cells that lead to inconsistent linear reconstructions marked with $\times$'s.

3. IMPROVED RECONSTRUCTION USING CONSISTENCY

In this section we present experimental evidence of the rate-distortion improvement obtained by using a consistent reconstruction algorithm. The consistent reconstruction algorithm is based on the method of alternating projections [10] and the following facts:

[2] The partition is somewhat different when the quantizer has different decision points, *e.g.* $\{(m+\frac{1}{2})\Delta\}_{m\in\mathbb{Z}}$ [9]. The ensuing conclusions are qualitatively unchanged.



(a) (b)

Figure 6. *(a) Partition of Figure 5 with linear reconstruction points marked. (b) Partition of Figure 5 with regions leading to inconsistent reconstructions marked.*

1. Given $\hat{f}$ and φ such that $|\langle\varphi,\hat{f}\rangle| > |\langle\varphi_{k_0},\hat{f}\rangle|$, let $\tilde{\varphi}_{k_0} = \mathrm{sgn}(\langle\varphi_{k_0},\hat{f}\rangle)\varphi_{k_0}$ and $\tilde{\varphi} = \mathrm{sgn}(\langle\varphi,\hat{f}\rangle)\varphi$. Then $\hat{f} - \langle\tilde{\varphi}_{k_0} - \tilde{\varphi}, \hat{f}\rangle(\tilde{\varphi}_{k_0} - \tilde{\varphi})$ is the orthogonal projection onto the set described by (4).
2. Given $\hat{f}$ such that $\langle\varphi_{k_0},\hat{f}\rangle > \hat{\alpha}_0 + \frac{\Delta}{2} \geq 0$, $\hat{f} - (\langle\varphi_{k_0},\hat{f}\rangle - \hat{\alpha}_0 - \frac{\Delta}{2})\varphi_{k_0}$ is the orthogonal projection onto the set described by (5). Similar expressions hold for other cases.

The experiments involved quantization of a zero mean i.i.d. Gaussian source. Dictionaries were formed from [8]. Source vectors were generated by forming blocks of N samples. Rate was measured by summing the (scalar) sample entropies of $k_0, k_1, \ldots, k_{p-1}$ and $\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_{p-1}$, where p is the number of iterations of the algorithm. Figure 7(a) gives simulation results obtained with $N = 3$ and $M = 7$. The $\times$'s, which are connected by dashed lines, are $D(R)$ points resulting from using linear reconstruction. The o's, connected by solid lines, are $D(R)$ points obtained with consistent reconstruction. Traversing each curve from left to right corresponds to varying Δ from $10^{-0.2}$ to $10^{-1.6}$. Since consistency is not an issue for a single-iteration expansion, the curves coincide for $p = 1$. The peak improvement due to consistent reconstruction is a reduction in MSE by more than a factor of five. Figure 7(b) shows similar results for $N = 4$ and $M = 11$.

ACKNOWLEDGEMENT

The authors would like to thank Dr. D. Taubman for asking them whether consistency is an issue in matching pursuit.

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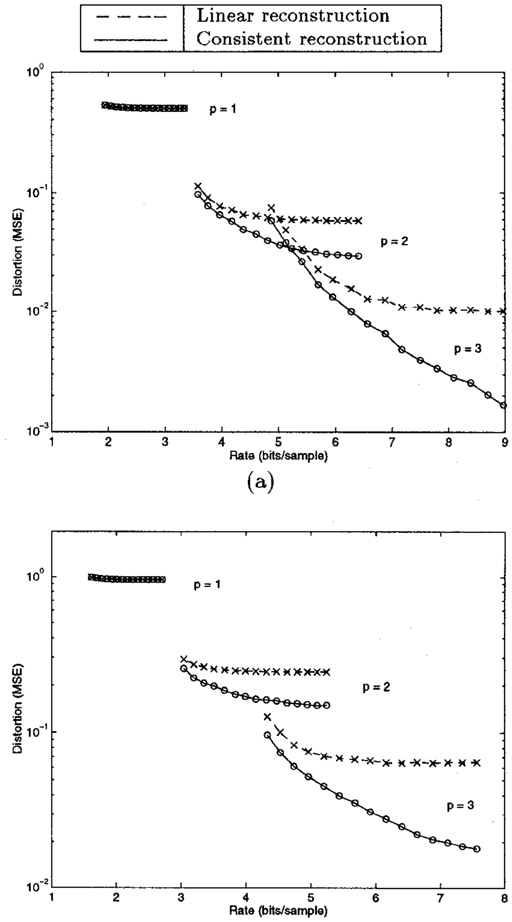

Figure 7. *Simulation results: (a) $N = 3$, $M = 7$; (b) $N = 4$, $M = 11$.*

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