A GENERAL THEORY FOR LOCAL COSINE BASES WITH MULTIPLE OVERLAPPING

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ABSTRACT

Cosine modulated filter banks are a well-known signal processing tool whose applicative field ranges from coding, to filtering, to spectral estimation. Because of their peculiar structure (the impulse responses are obtained by modulating a prototype window with trigonometric functions) they are easy to design and have a low computation complexity. Their continuous-time counterpart, local cosine bases, play an important role in the construction of Lemarié-Meyer wavelets.

We propose a unified approach to both discrete and continuous time cosine modulated filter banks. The resulting theory offers a single general framework that makes clear the deep similarity between the two cases.

1 INTRODUCTION

Cosine modulated filter banks (CMFB) are filter banks whose impulse responses are obtained by modulating a window with trigonometric functions [1], [2].

Because of this, they are easy to design and a fast algorithm exists. Moreover, the fact that they can be interpreted as a "smooth DCT," suggests that they could achieve good signal compression, but without the artifacts due to the discontinuous nature of the DCT (e.g., blocking effects).

Their continuous-time counterpart has been introduced by Coifman and Meyer [3] as a smoothed short-time Fourier transform. Such a device has been used by Auscher, Weiss and Wickerhauser in [4] to construct the Lemarié and Meyer wavelet [5].

Recently, a generalization of local cosine bases (both in continuous and discrete time) has been presented in [6].

The simplest case of cosine modulated filter banks is when there is overlapping only between two consecutive windows. While the theory of the discrete time case covers also the case of multiple overlapping [1], in continuous time generally only a single overlapping is allowed [4], [6].

The goal of this paper is to show that, despite their apparent differences, there is a deep similarity between continuous and discrete time. Such an objective will be fulfilled by presenting a general theory, within the same spirit of the one presented in [6], but with no limitation on overlapping.

The obtained results are general enough to be used in multiple dimensions and also in certain cases of non stationary filtering like, for example, when changing the window shape and/or the sampling period.

This paper has four sections. In Section 2 we give a brief summary of known results about CMFB and local cosine bases. In Section 3 it is shown that such results are part of a more general framework. An example of window design is also presented. Section 4 gives the conclusions.

2 SUMMARY OF CONTINUOUS AND DISCRETE TIME LOCAL COSINE BASES

2.1 The continuous-time case

In [4] Auscher and al. give an interesting interpretation of the continuous time local cosine bases.

In the local cosine basis scenario, the input signal is multiplied by a (possibly smooth) window $w(x)$ window with support $[-1,1]$[1] and the result is expressed as sum of some suitable cosine functions [4].

In order to cover the whole real line, window $w(x)$ is translated by integer steps.

In [4] it is shown that to window $w(x)$ is associated the vector space of functions $f(x)$ that can be written as $f(x) = w(x)S(x)$ where $S(x)$ satisfies

$$\begin{aligned} \chi_{[0,1]}S(\sigma_{1/2}(x)) &= \chi_{[0,1]}S(x) \\ \chi_{[-1,0]}S(\sigma_{-1/2}(x)) &= -\chi_{[-1,0]}S(x), \end{aligned} \tag{1}$$

[1] There is no loss of generality in fixing the window support in continuous time because one can always translate and/or scale the window.

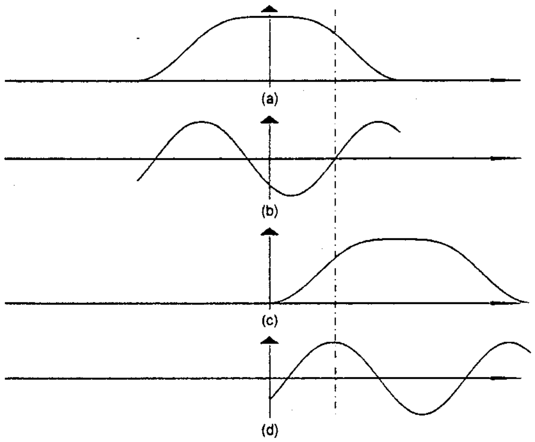

Fig. 1. Graphical proof of orthogonality. (a) Window $w(x)$. (b) Signal $S(x)$ symmetric verifying (1). (c) Window in (a) translated of one step. (d) Signal $S_1(x)$ antisymmetric around 1/2.

where $\chi_I(x)$ is the characteristic function of $I \subset \mathbb{R}$ and $\sigma_y(x) \triangleq 2y - x$ is the symmetry around $y \in \mathbb{R}$. In other words, $S(x)$ is symmetric in $[0, 1]$ and antisymmetric in $[-1, 0]$.

Structure (1) descends from the fact that functions $f(x)$ can be expressed as a linear combination of cosines multiplied by window $w(x)$. Since all the cosine functions enjoy symmetries (1) [4], such a property is clearly inherited by their linear combination.

When the window $w(x)$ is translated by one step along the real line, the corresponding vector space is translated, too. The new vector space consists of functions that can be written as $w(x-1)S_1(x)$ where $S_1(x)$ is antisymmetric around 1/2.3/2.

In [4] it is shown that the orthogonality of the vector spaces corresponding to $w(x)$ and $w(x-1)$ descends from the two different symmetries around 1/2. A graphical explanation of such a fact is shown in Figure 1. Figures 1(a) and 1(d) show the window $w(x)$ and the corresponding symmetric function $S(x)$, Figures 1(c) and 1(d) show the translated window $w(x-1)$ and function $S_1(x)$. When computing the scalar product of $w(x)S(x)$ and $w(x-1)S_1(x)$ one multiplies the four signals of Figure 1 together and integrate the result over $\mathbb{R}$. Now, the product of the two windows symmetric around 1/2, while the product of $S(x)$ with $S_1(x)$ is antisymmetric around the same point. Therefore, the overall product is antisymmetric and integrating it gives zero.

2.2 Discrete time case

In the discrete time case, one has an N-channel filter bank having the impulse response of the i-th channel equal to

$$h_i[n] = w[n] \cos_k(n) \quad (2)$$

with $\cos_k(n) \triangleq (1/\sqrt{N}) \cos(\pi(2k+1)(2n-N+1)/4N)$. Orthogonality properties of discrete time CMFB are usually proved via an algebraic approach showing that the matrix relative to the filter bank (2) is orthogonal [1], [2].

However, the discrete time case can be also attacked with a technique similar to the continuous one by observing that if we feed each channel of the synthesis filter bank with an impulse, the corresponding output signal will be a linear combination of functions (2).

Since it is possible to verify for cosines in (2) symmetries similar to (1), one can repeat the reasoning made in continuous time and prove the known properties of the discrete time case. Note that now x in (1) belongs to the discrete set $\mathbb{Z}$.

3 GENERALIZATION OF LOCAL COSINE BASES TO MULTIPLE OVERLAPPING

Why is it then that the theory in [4] or [6] does not work in the case of multiple overlapping?

The problem resides in the fact that the characteristic functions used in (1) limit the "influence zone" of symmetries $\sigma_{\pm 1/2}$, leaving undetermined the function $S(x)$ outside $[-1, 1]$. In Figure 2(a) one can see a function $S(x)$ verifying (1). The part depicted with the continuous line, can be arbitrarily chosen, while the part drawn with dashed line is determined by the symmetries around $\pm 1/2$. Nothing is known about $S(x)$ outside the interval $[-1, 1]$. Such an indetermination does not matter in the case of single overlapping because $w(x)$ is zero outside $[-1, 1]$.

However, if multiple overlapping is considered, the window support is larger and the indetermination on $S(x)$ assume a greater importance.

The observation that allows one to avoid such a limitation is that the cosines enjoy symmetries (1) even if the multiplication by the characteristic functions is dropped. [2]

Such an "extended" symmetry is, of course, inherited also by their linear combination and in the spirit of [6], one could define the corresponding vector space in a way similar to (1), but without characteristic functions.

To show how orthogonality works in this case, we present, as an example, a graphical study of

[2] It is worth noting that without the characteristic functions, symmetries (1) "interfere" one another, giving rise to an infinite set of symmetries. As an example, in Figure 2(b) one can see the effect of dropping the characteristic function in (1). The dashed line shows a new "symmetry" obtained by applying both $\sigma_{-1/2}$ and $\sigma_{1/2}$.

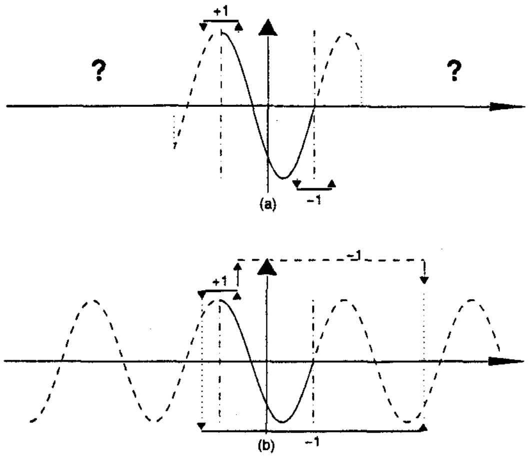

Fig. 2. Effect of the characteristic functions in (1). (a) Function $S(x)$ satisfying (1). (b) If the characteristic functions are not used, the two symmetries interfere one another and $S(x)$ is not free anymore outside $[-1, 1]$.

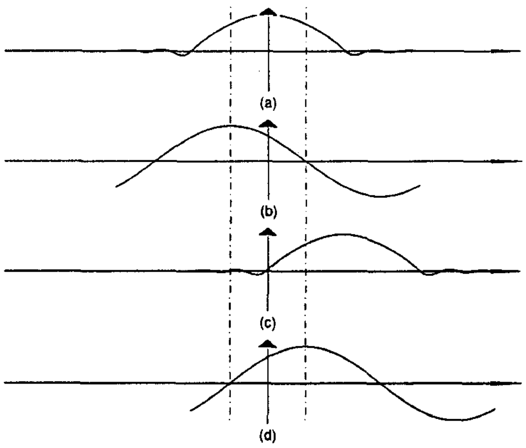

Fig. 3. Multiple overlapping and odd translation. (a) Window $w(x)$. (b) Symmetric signal $S(x)$. (c) and (d) Signals in (a) and (b) translated of one step.

continuous–time local cosine bases with *multiple overlapping*.

In Figure 3(a) one can see a symmetric window $w(x)$ designed for multiple overlapping, while in Figure 3(b) it is displayed a function $S(x)$ satisfying (1), but without the characteristic functions. Figures 3(c) and (d) show the functions in (a) and (b) translated of one step.

Again, multiplying the windows in Figures 3(a) and (c) gives a symmetric function, while multiplying the signals in (b) and (d) gives an antisymmetric one. The overall product is therefore antisymmetric and its area is zero.

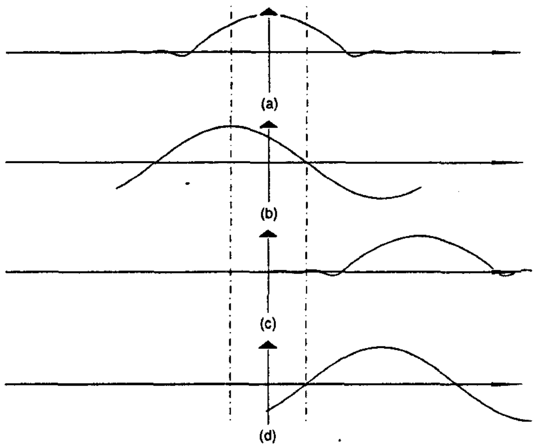

Fig. 4. Multiple overlapping and even translation. (a) Window $w(x)$. (b) Symmetric signal $S(x)$. (c) and (d) Signals in (a) and (b) translated of two steps.

It is possible to show that the same reasoning holds for every *odd* translation, that is, orthogonality for odd translations is granted by window symmetry $w(x) = w(-x)$.

The case of even translations is depicted in Figure 4. Figures 3(a) to 3(d) show, respectively, the window $w(x)$, the corresponding symmetric function $S(x)$, the translated window $w(x-2)$ and the corresponding function $S_2(x)$. Now both $S(x)$ and $S_2(x)$ are antisymmetric around $1/2$, their product, therefore, is symmetric and a cancelation similar to the one in Figure 3 is not possible.

It is possible to prove that orthogonality is granted if the window satisfies *self–orthogonality* conditions

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} w(x+2k)w(x+2k+2n)\\&+ w(\sigma_{1/2}(x)+2k)w(\sigma_{1/2}(x)+2k+2n) = \delta(n).\end{aligned} \tag{3}$$

By defining sequence $v_x[2k] = w(x+2k)$, $v_x[2k+1] = w(\sigma_{1/2}(x)+2k)$, one can restate equation (3) by saying that sequence $v_x[n]$ must be sequence orthogonal with respect to its even translations.

It is possible to give an intuitive explanation of (3) with the help of Figure 5 where one can see the product $R(x) = S(x)S_2(x)$. Such a product is clearly symmetric around $\pm 1/2$ and periodic with period 2. Because of this, for every x_0, the values $R(t)$, $t \in T_{x_0} = (x_0 + 2\mathbb{Z} \cup (1-x_0) + 2\mathbb{Z})$ are equal one another. In Figure 5 the instants belonging to $T_{0.27}$ are shown with dashed lines.

It is such a property that causes the irregular sampling that gives rise to signal $v_x[n]$, since the values of the window product $w(x)w(x-2)$ relative to such

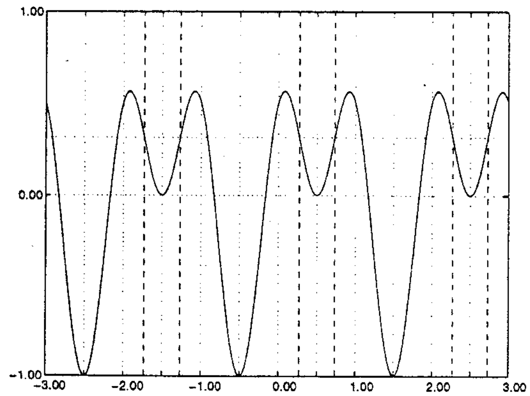

Fig. 5. Graphical explanation of the self–orthogonality condition. The product of two functions enjoying the same symmetries is symmetric both around $\pm 1/2$ and periodic with period 2. Because of this, the samples of the product $w(x)w(x-2)$ relative to the instants shown with dashed lines "see" the same value. Note that the set of such instants can be written as $x_0 + 2\mathbb{Z} \cup (1-x_0) + 2\mathbb{Z}$. In the figure $x_0 = 0.27$.

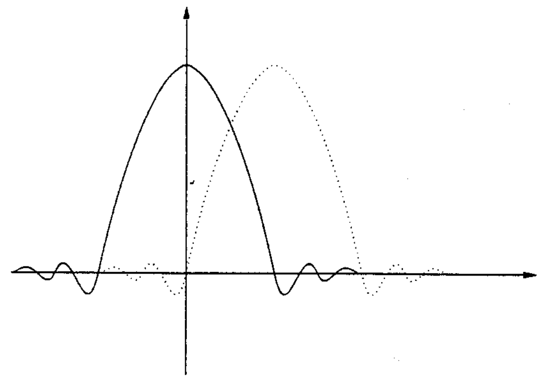

Fig. 6. Continuous line: example of a continuous time smooth (derivable) window designed for multiple overlapping. Dotted line: first translation of the designed window.

instants "see" the same value $R(x_0)$. Because of this, when computing the scalar product between $w(x)S(x)$ and $w(x-2)S_2(x)$ the integral is aware only of the value (3) multiplied by $R(x_0)$.

This, of course, is not a proof. A more formal proof is possible, but it would be rather obscure.

It is interesting to observe that, even in the continuous–time case, orthogonality is expressed by discrete–time conditions and one can exploit that for designing an arbitrarily smooth window.

The idea is that sequence $v_x[n]$, being a signal orthogonal with respect to its even translations, it can be considered as a branch of a two-channel perfect reconstruction filter bank. Since such filter bank can be parameterized with a set of angles $a_1, \ldots, a_N$ [1], [2], we can express the samples of $v_x[n]$ as function of angles $a_1, \ldots, a_N$.

In order to obtain a continuous time window one can consider every a_i as a function of x and it is possible to prove that conditions on smoothness of $w(x)$ can be transformed into boundary conditions upon functions $a_i(x)$ and their derivatives.

An example of continuous and derivable window for multiple overlapping, designed with the presented technique, is shown in Figure 6, together with a translated version of it, drawn with dotted line.

4 CONCLUSIONS

A novel theory of local cosine bases has been presented. The theory embrace both the continuous and the discrete time case, it is general enough to hold also in a multidimensional case and it does not depend on the window length nor on type of overlapping.

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