HIGH ORDER BALANCED MULTIWAVELETS

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ABSTRACT

In this paper, we study the issue of regularity for multiwavelets. We generalize here the concept of balancing for higher degree discrete-time polynomial signals and link it to a very natural factorization of the lowpass refinement mask that is the counterpart of the well-known zeros at \( \pi \) condition for wavelets. This enables us to clarify the subtle relations between approximation power, smoothness and balancing order. Using these new results, we are also able to construct a family of orthogonal multiwavelets with symmetries and compact support that is indexed by the order of balancing. More details (filters coefficients, drawings of the whole family, frequency responses, ...) can be obtained on the [website] at http://1ca www.epfl.ch/lebrun

1. INTRODUCTION

In the usual framework of wavelets, the two concepts of reproduction of continuous-time polynomials (approximation theory issue) and preservation/cancelation of discrete-time polynomial signals (subband coding and compression issue) are highly correlated since they have been proved to be equivalent to the same condition on the number of zeros at \( \pi \) in the factorization of the lowpass filter. The situation is different for multiformats. In [7, 8], interested in the subband coding issue in general and the problem of processing one dimensional signals with multiformats in particular, we introduced the concept of balanced multiwavelets that has since inspired many other papers [6, 11, 12]. The aim of this concept was to avoid the artificial step of prefiltering in multiwavelet based systems. Here, we will prove that the notion of balancing order is in fact central to the whole issue of regularity for multiwavelets.

2. MULTIWAVELETS

Generalizing the wavelet case, one can allow a multiresolution analysis \( \{V_n\}_{n \in \mathbb{Z}} \) of \( L^2(\mathbb{R}) \) to be generated by a finite number of scaling functions \( \phi_0(t), \phi_1(t), \ldots, \phi_{r-1}(t) \) and their integer translates. Then, the multiscaling function \( \phi(t) := [\phi_0(t), \ldots, \phi_{r-1}(t)]^T \) verifies a 2-scale equation

\[
\phi(t) = \sum_k M[k] \phi(2t - k)
\]

where \( \{M[k]\}_k \) is a sequence of \( r \times r \) matrices of real coefficients. The multiresolution analysis structure gives \( V_1 = V_0 \oplus W_0 \) where \( W_0 \) is the orthogonal complement of \( V_0 \) in \( V_1 \). We can construct an orthonormal basis of \( W_0 \) generated by \( \psi_0(t), \psi_1(t), \ldots, \psi_{r-1}(t) \) and their integer translates with \( \psi(t) := [\psi_0(t), \ldots, \psi_{r-1}(t)]^T \) derived by

\[
\psi(t) = \sum_k N[k] \phi(2t - k)
\]

where \( \{N[k]\}_k \) is a sequence of \( r \times r \) matrices of real coefficients obtained by completion of \( \{M[k]\}_k \). Introducing the refinement masks \( M(z) := \frac{1}{2} \sum_n M[n] z^{-n} \) and \( N(z) := \frac{1}{2} \sum_n N[n] z^{-n} \), the equations (1) and (2) translate in Fourier domain into

\[
\Phi(2\omega) = M(e^{j2\omega}) \Phi(\omega) \quad \text{and} \quad \Psi(2\omega) = N(e^{j2\omega}) \Phi(\omega)
\]

We can then derive the behavior of the multiscaling function by iterating the first product above. If this iterated matrix product converges, we get in the limit

\[
\Phi(\omega) = M_\infty(\omega) \Phi(0) = \prod_{i=1}^{\infty} M(2^{-2i}) \Phi(0)
\]

For simplicity and without loss of generality, we will now concentrate on the case \( r = 2 \). Furthermore, we will assume that the sequences \( \{M[k]\}_k \) and \( \{N[k]\}_k \) are finite and thus that \( \phi(t) \) and \( \psi(t) \) have compact support. We then recall some result obtained in [1] about the convergence of the iterated matrix product \( M_\infty(\omega) \). For \( M(2) \) satisfying a matrix Smith-Barnwell orthogonality condition

\[
M(2)M^T(2^{-1}) + M(-2)M^T(-2^{-1}) = I
\]

a necessary condition for uniform convergence of the iterated product to a scaling matrix \( M_\infty(\omega) \) such that \( M_\infty(0) \) is non-zero and bounded is either

(i) \( M(1) = I, M(-1) = 0 \) (note that \( M_\infty(\omega) \) has rank 2)

(ii) \( M(1) \) has eigenvalue \( \lambda_0(1) = 1 \) and \( |\lambda_1(1)| < 1 \), \( M(-1) \) has rank 1 and satisfies \( r_0 M(-1) = 0 \) where \( r_0 \) is a left


3. HIGH ORDER BALANCING

In [7, 8], we showed that if the components \( m_0(z) \) and \( m_1(z) \) of the lowpass branch have different spectral behavior, e.g. lowpass behavior for one, highpass for the other, it then leads to unbalanced channels that mix the coarse resolution and details coefficients and create strong oscillations. One expect then some class of smooth signals to be preserved by the lowpass branch and cancelled by the highpass.

3.1. Balancing

We define the band-Toeplitz matrix corresponding to the lowpass analysis

\[
L := \begin{bmatrix}
\end{bmatrix}
\]  

(12)

and in the same way, we define \( H \) the band-Toeplitz matrix corresponding to the highpass analysis. So, we want \( u_1 := [\ldots, 1, 1, 1, 1, \ldots]^T \) to be an eigenvalue of the lowpass branch, hence we introduce

**Definition 3.1.** An orthonormal multiwavelet system is said to be balanced iff the lowpass synthesis operator \( L^T \) preserve \([\ldots, 1, 1, 1, 1, \ldots]^T\) i.e. \( L^T u_1 = u_1 \).

By the orthonormality relations

\[
\begin{bmatrix}
L^T & H^T \\
\end{bmatrix}
\begin{bmatrix}
\phi_l \phi_r \\
\end{bmatrix} = I \quad \text{and} \quad \begin{bmatrix}
\phi_l \\
\phi_r \\
\end{bmatrix}
\begin{bmatrix}
L^T & H^T \\
\end{bmatrix} = I
\]

we get \( L^T L + H^T H = I \), \( LL^T = I \), \( LH^T = 0 \) and \( HH^T = I \).

Then \( L^T u_1 = u_1 \) implies \( Lu_1 = u_1 \) and so \( Hu_1 = 0 \) i.e. \( u_1 \) is cancelled by the highpass branch. Now, we can state

**Theorem 3.1.** The following conditions are equivalent

B0. \( L^T u_1 = u_1 \).

B1. [1, 1] is a left eigenvector of \( M(1) \) for \( \lambda_0(1) = 1 \).

B2. \( \Phi(0) = [1, 1]^T \).

B3'. \( m_0(z) + m_1(z) \) has zeros on the unit circle at \( j, -1, -j \).

B4. One can factorize \( M(z) = \frac{1}{2} T(z^2) M_0(z) T^{-1}(z) \) with

\[
T(z) := \begin{bmatrix}
1 & -1 \\
-1 & 1 \\
\end{bmatrix}
\]

and \( M_0(1) [1] = [1] \).

**Proof.** The equivalences [B0⇒B1⇒B2⇒B3⇒B0] were proved in [8], and [B1⇒B4] is a direct consequence of Theorem 4.1 in [9]. Assuming B4, we get

\[
m_0(z) + m_1(z) = [1 \quad 1] M(z^2) \begin{bmatrix}
1 \\
\frac{1}{z-1} \\
\end{bmatrix}
\]

\[=
\frac{1}{2} \begin{bmatrix}
1 & 1 \\
1 & -1 \\
\end{bmatrix}
M_0(z^2) \begin{bmatrix}
1 \\
\frac{1}{z-1} \\
\end{bmatrix}
\]

\[=
\frac{1}{2} \begin{bmatrix}
1 \\
1 \\
\end{bmatrix}
M_0(z^2) \begin{bmatrix}
1 \\
\frac{1}{z-1} \\
\end{bmatrix}
\]

and this is condition B3.
3.2. High Order Balancing

Definition 3.2 An orthonormal multiwavelet system is said to be balanced of order \( p \) if the signals
\[
u_n := [-\ldots, (-2)^n, -1^n, 0^n, 1^n, 2^n, \ldots]^T
\]
with \( n = 0, \ldots, p - 1 \) are preserved by the operator \( L^T \) i.e.
\[
L^T u_n = 2^{-p} u_n \quad \text{for } n = 0, \ldots, p - 1
\]
Similarly to the previous case, \( L^T u_n = 2^{-n} u_n \) implies \( L u_n = 2^n u_n \) and \( H u_n = 0 \) for \( n = 0, \ldots, p - 1 \). The polynomial structure of the signal is captured up to degree \( p - 1 \) by the lowpass branch coefficients. We then get

Theorem 3.2 The following conditions are equivalent

\( B_{0p} \): \( L^T u_n = 2^{-n} u_n \) for \( n = 0, \ldots, p - 1 \).

\( B_{3p} \): Defining \( \alpha^{(n)}(z) := u_1^{(n)}(z)/u_0^{(n)}(z) \) where \( u_0^{(n)}(z) \) and \( u_1^{(n)}(z) \) are the formal series \( u_0^{(n)}(z) := \sum_{k \in \mathbb{Z}} (2k + i)^n z^{-k} \), we impose \( m_0(z) + \alpha^{(p)}(z^4)m_1(z) \) to have zeros of order \( p \) at \( j, -1, -j \).

\( B_{4p} \): \( M(z) \) can be factorized as
\[
M(z) = \frac{1}{2^p} T^p(z) M_{p-1}(z) T^{-p}(z)
\]
with \( M_{p-1}(1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and \( T(z) \) defined as before.

Proof. \( [B_{0p} \Rightarrow B_{3p}] \): \( M(z) \) satisfies the conditions of Theorem 2.1 in [10] with \( y_n := [0^n, 2^{n-p}] \) for \( n = 0, \ldots, p - 1 \). Then, applying Corollary 4.3 from [10], we get the factorization
\[
M(z) = \frac{1}{2^p} C_0(z) \ldots C_{p-1}(z^2) M_{p-1}(z) C_{p-1}(z) \ldots C_0(z)
\]
with \( C_n(z) := \begin{bmatrix} a_n^{-1} & -a_n^{-1} \\ -z^{-1} b_n & b_n^{-1} \end{bmatrix} \) and the FIR refinement mask \( M_{p-1}(z) \) verifying \( M_{p-1}(1)r_{p-1} = r_{p-1} \) where \( r_{p-1} := [a_n, b_n] = 2^{-p}[1, 1] \) obtained recursively from \( y_n \) for \( n = 0, \ldots, p - 1 \). Thus \( C_n(z) = 2^n T(z) \) and \( M_{p-1}(1)[1, 1]^T = [1, 1]^T \).

\( [B_{4p} \Rightarrow B_{3p}] \): This is proved by induction on \( p \) [we refer to these results as the classical notions of regularity: approximation power and smoothness.

4. REGULARITY

Now, one may wonder how these new results relate to the classical notions of approximation power and smoothness.

4.1. Approximation Power and Balancing Order

One says that \( \phi(t) \) has approximation power \( m \) if one can exactly decompose polynomials \( 1, t, t^2, \ldots, t^{m-1} \) using only \( \phi_0, \phi_1 \) and their integer translates, i.e. for \( n = 0, \ldots, p - 1 \), we have \( t^n = \sum_k \chi_k[k] \phi(t - k) \). Then, assuming that \( \phi(t) \) is balanced of order \( p \), we get that \( M(z) \) factorizes as in (13), so applying \( p \) times Theorem 2.6. from [10], we get that \( \phi(t) \) has at least an approximation power of \( p \).

Proposition 4.1 If an orthonormal multiwavelet system is balanced of order \( p \), then the associated multiscaling function \( \phi(t) \) has an approximation power of at least \( p \).

We can notice that the reciprocity is false: the DGHM [2] multiscaling function has an approximation power of 2 but is not even balanced [8].

4.2. Smoothness and Balancing Order

From the previous proposition, and some results from [10] (Corollary 2.10.), showing links between the approximation power and the smoothness of the multiscaling function (number of continuous derivatives or Sobolev exponent \( s \) i.e. \( \int |\Phi(\omega)|^2(1 + |\omega|^2)^s d\omega < \infty \)), we get the following result
Figure 4: Order 3 balanced orthogonal multiwavelet: the scaling functions are flipped around 3, the wavelets are symmetric/antisymmetric, the length is 7 taps (2x2) and an estimate of the smoothness using Proposition 4.2 gives the Sobolev exponent $s = 1.71$.

Proposition 4.2 If an orthonormal multiwavelet system has balancing order $p$ and the spectral radius of $M_{p-1}(z)$ in the factorization (13) verifies $\rho \left( M_{p-1}(1) \right) < 2$, then defining

$$
\gamma_k := \frac{1}{k} \log_2 \rho \left( M_{p-1}(e^{-j\omega k-1}) \ldots M_{p-1}(e^{-j\omega_0}) \right)
$$

with $\{\omega_0, \ldots, \omega_{k-1}\}$ invariant cycles of $\omega \mapsto 2\omega \pmod{2\pi}$, and $\gamma := \inf_k \gamma_k$, we get that $\phi(t)$ is at most $[p - \gamma - \frac{1}{2}]$ times continuously differentiable (and has at most Sobolev exponent $s = p - \gamma$).

Idea of proof. To characterize the smoothness, we are interested in the decay as $N \to \infty$ of $\Phi(e^{j\omega})$ for $\omega \in [0, 2\pi]$. From the convergence (4), we find the truncated products $M_N(\omega) := \prod_{i=0}^{N-1} M(e^{-j\omega 2^i})$, then evaluating these on the invariant cycle $\{\omega_0, \ldots, \omega_{k-1}\}$, we get

$$
M_{kN}(2^{kN}\omega_0) = \prod_{i=1}^{kN} M(e^{-j\omega_{i-1}2^i}) \left( M(e^{-j\omega_{k-1}}) \ldots M(e^{-j\omega_0}) \right)^N
$$

then we study the asymptotic behavior of this product by looking at the eigenvalues of $M(e^{-j\omega_{k-1}}) \ldots M(e^{-j\omega_0}) = \prod_{k} \lambda_k U_k U_k^T$, where $\lambda_k = \text{diag}(\lambda_{k}^{(0)}, \lambda_{k}^{(1)})$. Then if $\rho(\lambda_k) = \max \{ |\lambda_{k}^{(0)}|, |\lambda_{k}^{(1)}| \} \geq 2^{-k}$ then the scaling functions cannot have Sobolev exponent of more than $i$ and so cannot be more than $[i - 1/2]$ times continuously differentiable. Applying this to the factorization (13), we get the upper bounds on smoothness.

Using results from the Perron-Frobenius theory [5], one can also find lower-bounds and prove that $s = p - \gamma$ is a good estimate of the Sobolev exponents of $\phi(t)$ and $\psi(t)$ [men]. For example in the case of the Haar multiwavelet, with $\omega_0 = 2\pi/3$, $\lambda_0 = 0$, $\lambda_1 = 1/2$, it then proves that the scaling functions cannot be continuous. In the case of the DGHM multiwavelet, $\lambda_0 = 1/10^\alpha$, $\lambda_1 = 1/10^\alpha$, it proves that the scaling functions may be at most $C^1$. They are in fact Lipschitz.

5. CONSTRUCTION OF HIGH ORDER BALANCED MULTIWAVELETS

Using the results above, we are now able to construct a Daubechies like family of multiwavelets. Namely, by imposing the number of $T(z^2) \ldots 1^{-1}(z)$ in the factorization (13), we force the order of balancing. Then, we design $M_{p-1}(z)$ by imposing conditions of orthonormality (5) on $M(z)$, flipping property on $m_0(z), m_1(z)$ (i.e. $m_1(z) = z^{-2L+1} m_0(z)$) and linear phase on $n_0(z)$ and $n_1(z)$. Using a Gröbner bases approach and the program Singular [9], we have been able to construct all the multiwavelets of compact support $C \in [0, 6]$ with flipped scaling functions and symmetric/antisymmetric wavelets for order 2 and 3 of balancing [men]. Fig. 3 and Fig. 4 show some examples of high order balanced multiwavelets with these properties.

6. CONCLUSION

By introducing the concept of high order balancing, we have clarified the issue of general design of multiwavelets. We have proved that this concept was equivalent to a natural counterpart of the zeros at $\pi$ condition. With these results, we made it possible to design general families of high order balanced multiwavelets with the required properties for practical signal processing (preservation/cancellation of discrete-time polynomial signals in the lowpass/highpass subbands, FIR, linear phase and orthogonality). Multiwavelets are eventually asserting themselves as convincing alternative tools for digital signal processing.

7. REFERENCES


