To Code Or Not To Code

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Abstract — **The theory and practice of digital communication during the past 50 years has been strongly influenced by Shannon's separation theorem [1]. While it is conceptually and practically appealing to separate source from channel coding, either step requires infinite delay in general for optimal performance. On the other extreme is uncoded transmission, which has no delay but is suboptimal in general. In this paper, necessary and sufficient conditions for the optimality of uncoded transmission are shown. These conditions allow the construction of arbitrary examples of optimal uncoded transmission (beyond the well-known Gaussian example).**

I. Previous and Basic Results

We consider a discrete-time memoryless source represented by the random variable $S \in \mathcal{S}$. The source output S is applied directly to a memoryless channel.[2] The channel output $Y \in \mathcal{Y}$ is our estimate of the source with respect to a distortion measure $d(s, y)$. The source is specified by a probability density (or mass) function $p(s)$ and a distortion measure $d(s, y)$. The channel is specified by a conditional probability density (or mass) function $W(y|s)$ and a channel input cost function $\rho(s)$. Therefore, uncoded transmission achieves (average) distortion $\Delta = Ed(S, Y)$ and (average) input cost $\Gamma = E\rho(S)$.

Definition. Uncoded transmission of the source (p, d) across the channel (W, ρ) is optimal if: *(i)* Δ is the minimum distortion achievable when the maximum input cost is Γ; and *(ii)* Γ is the minimum input cost to achieve distortion at most Δ.

Let $R(D)$ be the rate-distortion function of the source, and $D(R)$ the distortion-rate function. Correspondingly, let $C(P)$ be the capacity-cost function of the channel, and $P(C)$ the cost-capacity function. From the separation theorem, we have the following *Fact:* Uncoded transmission of the source (p, d) across the channel (W, ρ) is optimal if and only if *(i)* $\Delta = D(C(\Gamma))$, and *(ii)* $\Gamma = P(R(\Delta))$.

These two conditions are cumbersome to work with. For most cases of interest, we can find simpler necessary and sufficient conditions. However, let us first exclude certain special cases. Let C_0 denote the capacity of the unconstrained channel (W, ρ), i.e. $C_0 = C(P \to \infty)$.

Condition A. The source (p, d) and the channel (W, ρ) satisfy condition A if *(i)* in case $I(p, W) = 0$, W is the unique achiever of zero mutual information, and *(ii)* in case $I(p, W) = C_0$, p is the unique achiever of C_0.

The condition ensures that $D(R(\cdot))$ and $P(C(\cdot))$ are the identity functions, respectively.

Lemma 1. Granted condition A, uncoded transmission of the source (p, d) across the channel (W, ρ) is optimal if and only if $R(\Delta) = C(\Gamma)$.

This Lemma follows essentially from [2]; however, Condition A was not mentioned there. On a more intuitive level, Lemma 1 implies the following:

Lemma 2. Granted condition A, uncoded transmission of the source (p, d) across the channel (W, ρ) is optimal if and only if *(i)* the source p achieves capacity on the channel (W, ρ) (at input cost Γ), and *(ii)* the channel W achieves the rate-distortion function of the source (p, d) (at distortion Δ).

Unfortunately, in order to compute rate-distortion and capacity-cost functions, we have to resort to numerical methods in general. Thus, neither Lemma 1 nor Lemma 2 give an explicit way to verify whether or not for a given source and channel uncoded transmission is optimal and to construct examples of such source/channel pairs.

II. Main Result

Proposition. Uncoded transmission of the source (p, d) across the channel (W, ρ) for which $0 < I(p, W) < C_0$ is optimal if and only if

$$\begin{aligned} \rho(s) &= c_1 D(W(\cdot|s) || p_Y) + \rho_0 \\ d(s, y) &= -c_2 \log_2 \frac{W(y|s)}{p_Y(y)} + d_0(s), \end{aligned}$$

for some constants $c_1 > 0$, $c_2 > 0$, ρ_0 and an arbitrary function $d_0(s)$, where $D(\cdot||\cdot)$ is the Kullback-Leibler distance and $p_Y(y) = EW(y|S)$ is the pdf of Y.

A proof of this proposition can be found in [3]. A similar result can be obtained for the case $I(p, W) = C_0$. Note that the proposition allows to construct essentially all occurrences of optimal uncoded transmission.

Universality of uncoded transmission. The most interesting applications of uncoded transmission are cases where the separation theorem does not hold, e.g. non-ergodic channels or multi-user communication. Consider a broadcast scenario with one source and many (different) channels. If it turns out that the above proposition is satisfied for the source and each channel individually, then uncoded (broadcast) transmission is (globally) optimal. In this example, uncoded transmission exhibits a property of universality, whereas the performance of any separation-based coding scheme is strictly suboptimal.

References

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[2]For the framework of this paper, we assume that the channel input alphabet is also $\mathcal{S}$. The extension to arbitrary memoryless encoders and decoders will be presented at a later stage.