On the Necessary Density for Spectrum-blind Nonuniform Sampling[1]

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Abstract — **It is known that in the absence of distortion, the minimum average sampling density for a multiband signal is given by its spectral occupancy [1]. Furthermore, there exist nonuniform sampling patterns of the same average sampling density such that reconstruction is feasible even if the actual spectral support of the multiband signal is unknown [2]. This is called spectrum-blind nonuniform sampling. However, if the samples are distorted, an increased sampling density may lead to superior reconstruction.**

Suppose that a fidelity criterion is imposed on the reconstruction. To satisfy this, it is necessary to sample at an increased density. In this paper, we consider additive noise distortion of the samples, and the fidelity criterion is the probability that the spectral support is correctly reconstructed. In [3], we consider samples distorted by quantization, with a mean-square reconstruction error fidelity criterion.

I. Nonuniform Sampling

Consider a complex-valued length-N sequence $x \in C^N$ with discrete Fourier transform (DFT) $X \in C^N$, where $X(m) = 1/\sqrt{N}\ \sum_{n=0}^{N} x(n)e^{-j\frac{1}{N}2\pi nm}$. Let x be a multiband sequence of spectral occupancy q/N, i.e. let X have (at most) q non-zero components in arbitrary locations, indexed by $\underline{K} = \{k_1, \ldots, k_q\}$, where $k_i \in \{0, \ldots N-1\}$. The spectral occupancy for this vector is $\Omega = q/N$. Define the vector $x_{\underline{c}}$ containing only p of the N components of x, at locations indexed by $\underline{c} = \{c_1, \ldots, c_p\}$. These are the nonuniform samples, with average sampling density $\rho = p/N$. In matrix notation, we can write $x_{\underline{c}} = A_{\underline{c},\underline{K}}S$. Here, S contains the q non-zero components of X, and $A_{\underline{c},\underline{K}}$ is the submatrix of the inverse DFT matrix that is obtained by only retaining the rows with indices in $\underline{c}$ and the columns with indices in $\underline{K}$. We consider the case of distorted samples $y_{\underline{c}} = x_{\underline{c}} + z = A_{\underline{c},\underline{K}}S + z$, where $z \sim \mathcal{N}_c(0, \sigma^2 I)$ is (complex) white Gaussian noise.

II. Necessary Sampling Density

Let the location of the q nonzero components of X be distributed uniformly over all possibilities, and let their (complex) values be distributed as circularly normal, $S \sim \mathcal{N}_c(0, \sigma_S^2 I)$. We define the signal-to-noise ratio (SNR) $\beta = \sigma_S^2/\sigma^2$. It can be shown [4] that for $z = 0$, there exist sampling patterns with sampling density $\rho = \Omega + 1/N$ allowing w.p.1, perfect reconstruction of x from $y_{\underline{c}}$.

We derive a necessary condition for the optimal sampling density for $z \neq 0$. It follows from considering mutual informations. We start by noting that by the data processing lemma, $I(x_{\underline{c}}; y_{\underline{c}}) \geq I((S, \underline{K}); y_{\underline{c}}) = I(\underline{K}; y_{\underline{c}}) + I(S; y_{\underline{c}}|\underline{K})$, which yields

$$\max I(x_{\underline{c}}; y_{\underline{c}}) \geq \max_{\{A_{\underline{c},\underline{K}}\}} \left\{ I(\underline{K}; y_{\underline{c}}) + I(S; y_{\underline{c}}|\underline{K}) \right\}, \quad (1)$$

where first, the max is taken on both sides over all sets $\{A_{\underline{c},\underline{K}}\}$ of matrices satisfying $\left(A_{\underline{c},\underline{k}}A^H_{\underline{c},\underline{k}}\right)_{ii} = \Omega$ (which preserves $E|x_{\underline{c}}(i)|^2 = \Omega\sigma_S^2$); then, on the LHS, the max is taken over all distributions of $x_{\underline{c}}(i)$ for which $E|x_{\underline{c}}(i)|^2 = \Omega\sigma_S^2$ as for the true $x_{\underline{c}}(i)$. The term on the left in Eqn. (1) is simply the capacity of a (complex) additive white Gaussian noise (AWGN) channel with input power constraint $\Omega\sigma_S^2$ and additive noise variance σ^2, thus $\max I(x_{\underline{c}}; y_{\underline{c}}) = p\log_2(1 + \Omega\beta)$.

Next, consider $I(\underline{K}; y_{\underline{c}})$ in Eqn. (1). This is the mutual information across the digital channel from $\underline{K}$ to $y_{\underline{c}}$. A lower bound on the mutual information follows from Fano's inequality, $I(\underline{K}; y_{\underline{c}}) \geq H(\underline{K}) - H_b(P_e) - P_e \log_2\left(\binom{N}{q} - 1\right)$.

Last, consider $I(S; y_{\underline{c}}|\underline{K})$ in Eqn. (1). It is the mutual information across the channel between S and $y_{\underline{c}}$. This is also a Gaussian channel, but its input is not iid. The achieved mutual information is found by averaging over all $\underline{k}$ as $I(S; y_{\underline{c}}|\underline{K}) = E_{\underline{K}} \log_2 \det\left(I_q + \beta A^H_{\underline{c},\underline{K}}A_{\underline{c},\underline{K}}\right)$. For each $\underline{k}$, the maximum over $A_{\underline{c},\underline{k}}$ subject to the aforementioned constraint is achieved (by the geometric-arithmetic mean inequality) by $A_{\underline{c},\underline{k}}$ that has orthogonal columns, yielding $I(S; y_{\underline{c}}|\underline{K}) = q\log_2(1 + \beta\rho)$. This proves the following:

Theorem (Necessary Condition). The optimal sampling density $\rho = p/N$ has to satisfy

$$\rho \log_2(1 + \beta\Omega) \geq \frac{1}{N}\left[\log_2\binom{N}{q} - H_b(P_e) - P_e \log_2\left(\binom{N}{q} - 1\right)\right] + \Omega\log_2(1 + \beta\rho)$$

Letting $N \to \infty$ in the theorem, we obtain

$$\rho \log_2(1 + \beta\Omega) \geq \Omega\log_2(1 + \beta\rho) + (1 - P_e)H_b(\Omega), \quad (2)$$

which is sharp in the limit $\beta \to \infty$, because it reduces to $\rho \geq \Omega$. For finite $SNR\ \beta$, $\rho > \Omega$, with the excess density given by (2).

References

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