To appear in Proc. *European Signal Processing Conference (EUSIPCO'2000)*, September 5-8, 2000, Tampere, Finland.

FRACTIONAL DERIVATIVES, SPLINES AND TOMOGRAPHY

Michael Unser, Stefan Horbelt and Thierry Blu
Biomedical Imaging Group, Swiss Federal Institute of Technology Lausanne
CH-1015 Lausanne EPFL, Switzerland

Email: {Michael.Unser, Stefan.Horbelt, Thierry.Blu}@epfl.ch
URL: http://bigwww.epfl.ch/

ABSTRACT

We develop a spline calculus for dealing with fractional derivatives. After a brief review of fractional splines, we present the main formulas for computing the fractional derivatives of the underlying basis functions. In particular, we show that the γth fractional derivative of a B-spline of degree α (not necessarily integer) is given by the γth fractional difference of a B-spline of degree $\alpha-\gamma$. We use these results to derive an improved version the filtered backprojection algorithm for tomographic reconstruction. The projection data is first interpolated with splines; the continuous model is then used explicitly for an exact implementation of the filtering and backprojection steps.

1. INTRODUCTION

Splines are made up of polynomials and are essentially as easy to manipulate. One operation that is especially simple to implement is differentiation. It has the same effect on splines as it has on polynomials: it reduces the degree by one. The derivative of a B-spline of degree n is given by

$$D\beta^n(x) = \Delta\beta^{n-1}(x) = \beta^{n-1}(x+\tfrac{1}{2}) - \beta^{n-1}(x-\tfrac{1}{2})$$

where Δ denotes the central finite difference operator. The implication of this differentiation formula is that one can calculate spline derivatives simply by applying finite differences to the B-spline coefficients of the representation. Thus, with splines, one has an exact equivalence between finite differences and differentiation and not just an approximate one as is usually the case in numerical analysis. This is a property that can be exploited advantageously for implementing differential signal processing operators [6].

Our purpose in this paper is to consider more general forms of differentiation (fractional derivatives) and to develop the corresponding spline calculus. The main difficulty with fractional derivatives is that the derivatives of polynomials (or splines) are no-longer polynomial when the order of differentiation in non-integer. This forces us to consider the enlarged family of fractional splines [7]; these are reviewed in Section 2. In Section 3, we present the differentiation rules for the fractional splines and show that this family is closed under fractional differentiation: specifically, the γth derivative of a fractional spline of degree α is a fractional spline of degree $\alpha-\gamma$, where α and γ are not necessarily integer. Finally, in Section 4, we indicate how these results are useful for improving the implementation of the filtered backprojection (FBP) algorithm for tomographic reconstruction [4, 5].

2. FRACTIONAL SPLINES

In this section, we define the fractional splines and summarize the main properties of their basic constituents: the fractional B-splines. For more details, refer to [7].

2.1 Power functions

The purest examples of fractional splines of degree α are the one-sided and rectified power functions, x_+^α and $|x|_*^\alpha$, which both exhibit one singularity of order α (Hölder exponent) at the origin. The one-sided power function is defined by:

$$x_+^\alpha = \begin{cases} x^\alpha & x \geq 0 \\ 0, & \text{otherwise} \end{cases}. \qquad (1)$$

For $\alpha \notin N$, its Fourier transform is $\Gamma(\alpha+1)/(j\omega)^{\alpha+1}$.

The second symmetric type, $|x|_*^\alpha$, is defined as the function whose Fourier transform is $\Gamma(\alpha+1)/|\omega|^{\alpha+1}$. For α non-even, it is a (rectified) power function; otherwise, it has an additional logarithmic factor:

$$x_*^\alpha = \begin{cases} \dfrac{|x|^\alpha}{-2\sin(\frac{\pi}{2}\alpha)}, & \alpha \text{ not even} \\ \dfrac{x^{2n}\log x}{(-1)^{1+n}\pi}, & \alpha = 2n \text{ (even)} \end{cases} \qquad (2)$$

2.2 Causal fractional B-splines

By analogy with the classical B-splines, one constructs the fractional causal B-splines by taking the $(\alpha+1)$-fractional difference of the one-sided power function

$$\beta_+^\alpha(x) := \frac{\Delta_+^{\alpha+1} x_+^\alpha}{\Gamma(\alpha+1)} = \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{+\infty} (-1)^k \binom{\alpha+1}{k} (x-k)_+^\alpha \qquad (3)$$

where $\Gamma(u+1) = \int x^u e^{-x} dx$ is Euler's gamma function. $\Delta_+^{\alpha+1}$ is the $(\alpha+1)$-fractional difference operator; it is a *convolu-tion* operator whose transfer function is

$$\hat{\Delta}_+^{\alpha+1}(\omega) = (1-e^{-j\omega})^{\alpha+1} = \sum_{k=0}^{+\infty} (-1)^k \binom{\alpha+1}{k} e^{-j\omega k}. \qquad (4)$$

The fractional B-splines are in L_2 for $\alpha > -\frac{1}{2}$. They are compactly supported for α integer; otherwise, they decay like $|x|^{-(\alpha+2)}$ (cf. [7], Theorem 3.1). The Fourier domain equivalent of (3) is

$$\hat{\beta}_+^\alpha(\omega) = \left(\frac{1-e^{-j\omega}}{j\omega}\right)^{\alpha+1} \qquad (5)$$

2.3 Symmetric fractional B-splines

We construct the symmetric B-splines by taking $(\alpha+1)$-symmetric fractional differences of the rectified power function:

$$\beta_*^\alpha(x) := \frac{\Delta_*^{\alpha+1} x_*^\alpha}{\Gamma(\alpha+1)} = \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{+\infty} (-1)^k \left|\begin{matrix}\alpha+1 \\ k\end{matrix}\right| |x-k|_*^\alpha \qquad (6)$$

where $\Delta_*^\alpha \xleftrightarrow{Fourier} |1-e^{-j\omega}|^\alpha$ is the symmetric fractional difference operator. Similar to their causal counterparts, these functions are not compactly supported either unless n is odd, in which case they coincide with the traditional polynomial B-splines. When α is not odd, they decay like $|x|^{-(\alpha+2)}$ and their asymptotic form is available [7]. The Fourier counterpart of (6) is simply

$$\hat{\beta}_*^\alpha(\omega) = \left|\frac{\sin(\omega/2)}{\omega/2}\right|^{\alpha+1}. \qquad (7)$$

Note that the expansion coefficients on the right hand side of (3) and (6) are generalized versions of the binomials. They are both compatible with the following extended definition:

$$\binom{u}{v} = \frac{\Gamma(u+1)}{\Gamma(v+1)\Gamma(u-v+1)} \qquad (8)$$

where the gamma function replaces the factorials encountered in the standard formula when u and v are both integer. The coefficients in (6) are a re-centered version given by

$$\left|\begin{matrix} r \\ k \end{matrix}\right| = \binom{r}{k+\frac{r}{2}} \qquad (9)$$

2.4 Fractional splines

In most general terms, fractional splines may be defined as linear combinations of shifted fractional power functions or fractional B-splines. As in the polynomial case, it is usually more advantageous to use the second type of representation. The fractional B-splines have all the good properties of the conventional B-splines, except that they lack compact support when α is not an integer. In particular, they form a Riesz basis which ensures that that B-spline representation is stable numerically. Thus, if we consider the basic integer grid, we may represent a fractional spline signal by its B-spline expansion

$$s(x) = \sum_{k \in Z} c(k) \beta^\alpha (x-k) \qquad (10)$$

where we use the generic notation $\beta^\alpha(x)$ to specify any one of the fractional B-splines ($\beta_+^\alpha(x)$, or $\beta_*^\alpha(x)$). What this means is that a fractional spline signal $s(x)$ with knots at the integers is unambiguously characterized through its B-spline coefficients $c(k)$, $k \in Z$ (discrete/continuous representa-tion). The representation is one-to-one—there is exactly one coefficient $c(k)$ by sample value $s(k)$. Note that this spline representation is compatible which the traditional model used in signal processing for it can be shown that the signal (10) converges to a bandlimited function as the order of the spline increases [1].

3. FRACTIONAL DIFFERENTIATION

3.1. Fractional derivatives

We consider two versions of fractional derivatives which can be defined in the Fourier domain. The first type, which is compatible with Liouville's definition [2], is given by

$$D^\gamma f(x) \xleftrightarrow{Fourier} (j\omega)^\gamma \hat{f}(\omega) \qquad (11)$$

where $\hat{f}(\omega) = \int f(x) e^{-j\omega x} dx$ denotes the Fourier transform of $f(x)$ and where $z^\gamma = |z|^\gamma e^{j\gamma \arg(z)}$ with $j = \sqrt{-1}$ and $\arg(z) \in [-\pi, \pi[$.

The second type of derivative, which is a symmetrized version of first, is defined by

$$D_*^\gamma f(x) \xleftrightarrow{Fourier} |\omega|^\gamma \hat{f}(\omega) \qquad (12)$$

Note that the first type agrees with the usual definition of the derivative when α is integer, while the second one only does when α is even.

3.2 Differentiation rules

The general B-spline differentiation rules are

$$D^\gamma \beta_+^\alpha(x) = \Delta_+^\gamma \beta_+^{\alpha-\gamma}(x) \qquad (13)$$

$$D_*^\gamma \beta_*^\alpha(x) = \Delta_*^\gamma \beta_*^{\alpha-\gamma}(x) \qquad (14)$$

where D^γ and D_*^γ are defined by (11) and (12), respectively. This is established easily in the Fourier domain. For instance, to obtain (14), we substitute (7) in (12) and rewrite the Fourier transform of $D_*^\gamma \beta_*^\alpha(x)$ as

$$|\omega|^\gamma \left|\frac{\sin(\omega/2)}{\omega/2}\right|^{\alpha+1} = \underbrace{\left|\frac{\sin(\omega/2)}{2}\right|^\gamma}_{|1-e^{j\omega}|^\gamma} \cdot \underbrace{\left|\frac{\sin(\omega/2)}{\omega/2}\right|^{\alpha+1-\gamma}}_{\hat{\beta}_*^{\alpha-\gamma}(\omega)}$$

We now briefly indicate how these rules can be applied to obtain the fractional derivative of the spline signal in (10). Taking the fractional derivative and interchanging the order of summation, we get

$$\begin{aligned} D^\alpha s(x) &= \sum_{k\in Z} c(k) \Delta^\gamma \beta^{\alpha-\gamma}(x-k) \\ &= \sum_{k\in Z} \underbrace{(\Delta^\gamma * c)(k)}_{d(k)} \beta^{\alpha-\gamma}(x-k) \end{aligned} \qquad (15)$$

where we have moved the fractional difference operator into the discrete domain. Thus, the B-spline coefficients $d(k)$ of $D^\alpha s(x)$ are obtained by convolving the $c(k)$'s with the digital filter Δ^γ whose frequency response is $(1-e^{-j\omega})^\gamma$ or $|1-e^{-j\omega}|^\gamma$, depending on the type of derivative.

4. FRACTIONAL SPLINES AND TOMOGRAPHY

The mathematical basis for the standard filtered backprojection tomographic reconstruction algorithm is the following identity $\forall f \in L_2(R^2)$ (cf. [3])

$$f(x,y) = R^* K R f(x,y) = R^* K\{p_\theta(t)\} \qquad (16)$$

with $t = (x,y)\cdot\vec{\theta}$ where $\vec{\theta} = (\cos\theta, \sin\theta) \in S$ is the unit vector that specifies the direction of the projection; $p_\theta(t) = R_\theta f(t) = \iint_{R^2} f(\vec{x})\delta(\vec{x}\cdot\vec{\theta} - t)d\vec{x}$ is the *Radon transform* of f and R^* is the so-called *backprojection* operator; it is the adjoint of the Radon or *projection* operator R. The right hand side of (16) provides the filtered backprojection solution for the recovery of the function $f(x,y)$ from its projection data $p_\theta(t)$.

The algorithm proceeds in two steps. First, each projection $p_\theta(t)$ is filtered continuously with the ramp or Ram-Lak filter [4]; the crucial observation here is that the *filtering operator* K is proportional to our fractional derivative $D_* \leftrightarrow |\omega|$; i.e., $K = (2\pi)^{-1} D_*$. Second, the filtered projections are projected back onto the image and averaged according to the formula

$$R^* K\{p_\theta(t)\} = \frac{1}{2\pi}\int_0^\pi D_* p_\theta(t) d\theta \cong \frac{1}{2N}\sum_{i=1}^N D_* p_{\theta_i}(t) \qquad (17)$$

with $t = (x,y)\cdot\vec{\theta}$. The reconstruction formula (16) is exact provided that one treats the projection data $p_\theta(t)$ as a continuum both in terms of t and θ. In practice, however, one has only access to a finite number of projections at the angles θ_i, and the continuous average in (17) is usually replaced by the discrete one on the right. The error can be assumed to be negligible provided that the number of projections N is sufficient.

In our method, we assume that the projection data at angle θ is a fractional spline of degree α:

$$p_\theta(t) = R_\theta f(t) = \sum_{k\in Z} c(k)\beta_*^\alpha(t-k) \qquad (18)$$

After symmetric differentiation (ramp filter), we find that

$$D_* p_\theta(t) = \sum_{k\in Z} d(k)\beta_*^{\alpha-1}(t-k) \qquad (19)$$

where the $d(k)$ are obtained by applying the symmetric finite differences to the $c(k)$ (cf. (15)). Thus, we have an explicit continuous representation of the filtered projection which can then be directly plugged into (17).

In practice, we are given the sampled values of the projection $p_\theta(k)$ and the first step is to determined the B-spline coefficients $c(k)$ such that the spline model interpolates these values exactly. This can be done by digital filtering. Combining both filters together (interpolation and ramp-filter), we get

$$d(k) = (h_* * p_\theta)(k) \qquad (20)$$

where h_* is the digital filter whose transfer function is

$$h_*(k) \xleftrightarrow{Fourier} \frac{|1-e^{j\omega}|}{B_*^\alpha(e^{j\omega})} = \frac{\sin(\omega/2)/2}{\sum_{n\in Z} |\mathrm{sinc}(\omega/2\pi - n)|^{\alpha+1}} \qquad (21)$$

In our implementation, we select α even (typ. $\alpha = 2$ or **4**) such that the basis functions in (19) are polynomial B-splines that are compactly supported. This allows us to use the spline model (19) to our full advantage in the backprojection part of the algorithm (Eq. (17)). The digital filtering part of the algorithm (20) is implemented in the Fourier domain since the filter h_* has infinite support. The interesting aspect of the algorithm is that, once we have selected the spline model (18), all other aspects of the computation are exact. In particular, the discretization of the ramp filter is achieved implicitly through (21).

Some experimental results are presented in Fig. 1; to facilitate the comparison, we give the reconstruction errors amplified by a factor of four. The test image (Fig. 1d) is of size 128×128 and its Radon transform was computed over 256 equidistant angles. Fig. 1a displays the reconstruction error for the standard algorithm (Shepp-Logan filter [5]) with linear interpolation for the backprojection. The PSNR is 26.89dB; switching to the Ram-Lak filter improves this measure to 28.03dB. The reconstruction error for the proposed FBP algorithm with $\alpha=2$ is shown in Fig. 1b (PSNR=29.80). The results are slightly better (smaller magnitude of the error) than the standard approach (Fig. 1a or Ram-Lak filter), even though the backprojection was implemented using the same piecewise linear interpolation model. This suggest that the use of a consistent design, where the ramp filter is discretized in accordance with the underlying signal model, is helpful. The best results (PSNR=30.37) were obtained with $\alpha=4$ (cf. Fig. 1c); in this case, the backprojection was implemented using cubic B-spline basis functions. Here we suspect that the main reason for the improvement is the use of a higher order interpolation model, especially in the backprojection part of the procedure.

5. CONCLUSION

The fractional splines offer the same conceptual ease for dealing with fractional derivatives as the polynomial splines do with derivatives. In the B-spline domain, fractional differentiation gets translated into simple fractional finite differences. This spline calculus provides a general tool for the discretization and implementation of fractional derivative operators.

The Ram-Lak filter, which plays a crucial role in tomography, corresponds to our symmetric differential operator $D_* \longleftrightarrow |\omega|$. It is an non-local operator that can be implemented exactly provided that one has a spline representation of the projection data. We have proposed a modification of the standard FBP algorithm that takes advantage of this property. We have found that working with splines is also beneficial for the back-projection part of the reconstruction process.

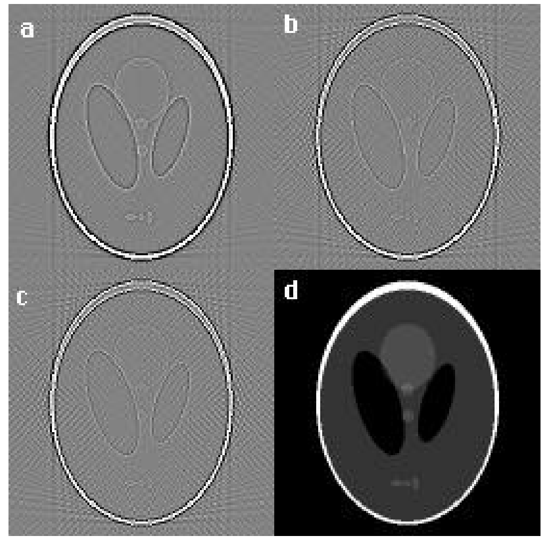

Fig. 1: Reconstruction errors for the various algorithms. (a) standard reconstruction using the Shepp-Logan filter, (b) fractional spline reconstruction with $\alpha=2$, (c) fractional spline reconstruction with $\alpha=4$, (d) test image (Shepp-Logan phantom).

References

[1] A. Aldroubi and M. Unser, "Sampling procedures in function spaces and asymptotic equivalence with Shannon's sampling theory," *Numer. Funct. Anal. and Optimiz.*, vol. 15, no. 1&2, pp. 1-21, 1994.

[2] J. Liouville, "Sur une formule pour les différentielles à indices quelconques à l'occasion d'un Mémoire de M. Tortolini," *J. Math. Pures Appl.*, vol. 20, pp. unnumbered, 1855.

[3] D. Ludwig, "The Radon transform on Euclidean space," *Comm. on pure and applied mathematics*, vol. XIX, pp. 49-81, 1966.

[4] G.N. Ramachandran and A.V. Lakshminarayanan, "Three dimensional reconstructions from radiographs and electron micrographs: Application of convolution instead of Fourier transform," *Proc. Nat. Acad. Sci.*, vol. 68, pp. 2236-2240, 1971.

[5] L.A. Shepp and B.F. Logan, "The Fourier reconstruction of a head section," *IEEE Trans. Nucl. Sci.*, vol. NS-21, pp. 21-43, 1974.

[6] M. Unser, A. Aldroubi and M. Eden, "B-spline signal processing: Part II—efficient design and applications," *IEEE Trans. Signal Processing*, vol. 41, no. 2, pp. 834-848, 1993.

[7] M. Unser and T. Blu, "Fractional splines and wavelets," *SIAM Review*, vol. 42, no. 1, pp. 43-67, 2000.