On Source/Channel Codes Of Finite Block Length

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Abstract — **For certain fortunate choices of source/channel pairs, all sophisticated coding is in vain: for them, a code of block length one is sufficient to achieve optimal performance [1]. Is the set of "fortunate choices" larger if we allow for codes of block length M? For a certain class of discrete memoryless source/channel pairs, we can prove that the answer is negative as long as M is finite.**

I. Definitions

Consider a discrete memoryless source with alphabet $\mathcal{S}$ which is reconstructed in an alphabet $\hat{\mathcal{S}}$. The source is specified by a probability mass function (pmf) p_S and a distortion measure $d : \mathcal{S} \times \hat{\mathcal{S}} \to \mathbb{R}_0^+$. Consequently, we denote the source by (p_S, d). A discrete memoryless channel with input alphabet $\mathcal{X}$ and output alphabet $\mathcal{Y}$ is specified by a conditional pmf $p_{Y|X}$ and an input cost function $\rho : \mathcal{X} \to \mathbb{R}_0^+$. By analogy to the source, we denote the channel by $(p_{Y|X}, \rho)$.

II. Single-letter Codes

A single-letter source/channel code is a pair of functions (f, g), where $f : \mathcal{S} \to \mathcal{X}$ and $g : \mathcal{Y} \to \hat{\mathcal{S}}$. If the source (p_S, d) is transmitted across the channel $(p_{Y|X}, \rho)$ using the code (f, g), the achieved distortion is $\Delta = Ed(S, \hat{S})$ and the cost needed to achieve this is $\Gamma = E\rho(X)$. This constitutes an optimal communication system only if Δ could not be achieved at lower cost, and Γ does not permit to achieve lower distortion. These two conditions are relatively difficult to verify in general. They require the computation of rate-distortion and capacity-cost functions. However, optimality can also be verified by the following conditions:

Theorem 1. *For a discrete memoryless source (p_S, d), a discrete memoryless channel $(p_{Y|X}, \rho)$ and a single-letter code (f, g), suppose that $I(S; \hat{S}) > 0$ and $I(X;Y) < C_0$.*[2] *This is an optimal communication system if and only if for some constants $c_1 > 0, \rho_0, c_2 > 0$, and a function $d_0(s)$,*

$$d(s, \hat{s}) = -c_1 \log p(s|\hat{s}) + d_0(s) \quad (1)$$

$$\rho(x) = c_2 D(p_{Y|X}(\cdot|x) || p_Y(\cdot)) + \rho_0 \quad (2)$$

and $I(S; \hat{S}) = I(X; Y)$.

This is a slight generalization of the result presented in [2]. For a proof, see [1].

Define the set $\mathbb{S}$ of all discrete memoryless source/channel pairs for which there exists a single-letter code that performs optimally, i.e.

$$\mathbb{S} = \left\{ (p_S, d, p_{Y|X}, \rho) \,\middle|\, \exists (f, g) : \text{optimal} \right\}.$$

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[2]The theorem also holds for *discrete-time* memoryless systems, and a similar theorem holds in case $I(X; Y) = C_0$ [1].

In $\mathbb{S}$, we find for example the Bernoulli(1/2) source with Hamming distortion and the binary symmetric channel (with unconstrained input).

III. Codes Of Finite Block Length

A natural extension of these results is to consider source/channel codes of block length M, namely a pair of functions $(f^{(M)}, g^{(M)})$, where $f^{(M)} : \mathcal{S}^M \to \mathcal{X}^M$ and $g^{(M)} : \mathcal{Y}^M \to \hat{\mathcal{S}}^M$. Since all alphabets are discrete, $(f^{(M)}, g^{(M)})$ can be interpreted as a *single-letter* code for an appropriate source over the alphabet $\mathcal{S}^M$ and the corresponding extended channel with input alphabet $\mathcal{X}^M$ and output alphabet $\mathcal{Y}^M$. Then, we can apply Theorem 1.

An interesting question is as follows: by allowing codes of finite block length M rather than only single-letter codes, will the set $\mathbb{S}$ grow? This question is not answered directly by Theorem 1. Define the set $\mathbb{S}^{(M)}$ of all source/channel pairs for which there exists a source/channel code of block length M that performs optimally. In general, one would expect that this set is larger than $\mathbb{S}$. However, we can prove that for a subset of all discrete memoryless source/channel pairs, it is true that $\mathbb{S}^{(M)} = \mathbb{S}$. In particular, we can make the following statement:

Proposition 2. *Consider only discrete memoryless source/channel pairs that satisfy the following: all alphabets are of the same cardinality, $p(s) > 0$ for all s, the channel transition probability matrix is invertible and the distortion measure is such that the matrix with entries $2^{-d(s,\hat{s})}$ is invertible. Then,*

$$\mathbb{S}^{(M)} = \mathbb{S}.$$

Outline of proof. For a given source pmf p_S, channel conditional pmf $p_{Y|X}$ and code $(f^{(M)}, g^{(M)})$, we compute $d^{(M)}$ and $\rho^{(M)}$ as in Theorem 1. But since by assumption, the source is constrained to be *memoryless*, $d^{(M)}$ must *split* additively, i.e. $d^{(M)}(s^{(M)}, \hat{s}^{(M)}) = \sum_{i=1}^{M} d(s_i, \hat{s}_i)$. By analogy, since the channel also has to be memoryless, $\rho^{(M)}$ must split in a similar fashion. One can show that the assumptions made in the proposition together with the fact that $d^{(M)}$ and $\rho^{(M)}$ have to split additively imply that there is also a source/channel code of block length 1 that achieves optimal performance. See [1].

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References

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