Power spectral measure and reconstruction error of randomly sampled signals

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Abstract — We say that a signal is randomly sampled when the samples are taken at random instants of time. The study of random sampling and randomly sampled signals is motivated both by practical and theoretical interests. The first ones include spectral analysis (estimation of spectra from a finite number of samples) and quality of service (signal reconstruction), and the second ones include statistical analysis of reconstruction methods.

The present paper focuses on the computation of the (theoretical) spectrum of randomly sampled signals and on the computation of the reconstruction error. Using a point process approach, we obtain general formulas for spatial random sampling, providing powerful tools for the analysis and the processing of randomly sampled signals.

I. INTRODUCTION

Random sampling of a continuous time random signal $X(t)$, $t \in \mathbb{R}$, yields a sequence of samples

$$X(T_n), \quad n \in \mathbb{Z} \quad (1)$$

where T_n , $n \in \mathbb{Z}$, is the sequence of points (times of events) of a point process.

At the extremities of the spectrum of randomness, we find the completely random sampling, or Poisson sampling, where T_n , $n \in \mathbb{Z}$, is a homogeneous Poisson process [1], and the regular sampling, where $T_n = nT$, T^{-1} being the sampling frequency.

Random sampling is in most cases not deliberate. Regular sampling can become random due to jitter (lack of synchronization) or to thinning (loss of samples). Communication systems provide several examples of such a situation. It can, however, be inherent to the sampling procedure, as in laser velocimetry [2], where a sample is collected only at the passage of a grain of matter through the laser beam.

In the following, the signal $X(t)$, $t \in \mathbb{R}$, is called the **sampled signal**, the point process T_n , $n \in \mathbb{Z}$, the **sampler**, the sequence (1) is the **sample sequence**, and the process

$$Y(t) = \sum_{n \in \mathbb{Z}} X(T_n) \delta(t - T_n), \quad (2)$$

where $\delta(t)$ is the Dirac pulse, is called the **sample comb**.

Two theoretical questions arise. The first one is related to spectral analysis, the second one to signal reconstruction.

1. What is the relation between the spectrum of the sample comb (or the sample sequence) to that of the sampled signal?
2. To what extent can we recover the sampled signal from the sampled comb (or the sample sequence)?

Several works have contributed in answering such questions. Early investigation on random sampling [3, 4] were mostly motivated by the searching of alias-free sampling schemes, *i.e.* sampling schemes leading to a one-to-one relation between the spectrum of the sample comb to that of the sampled signal. The aim was to find a sampling framework without the band-limited requirement of the classical Shannon's sampling theorem [5].

The first attempts of a detailed analysis of randomly sampled signals were based on the modeling of the sample comb using the Dirac (pseudo) process δ . Beutler and Leneman [6, 7, 8] obtained formulas for the moments of the sample comb that lead to the expression of the correlation of the sample comb as a function of the correlation of the sampled signal. Leneman and Lewis [9] investigated the reconstruction error for several interpolators of the random samples. Such results depend on the sampling scheme through statistics related to the intervals between successive points of the sampler.

Results on the spectral analysis of randomly sampled signals have been obtained by Masry [10, 11]. Using a point process approach, the spectrum of the sample sequence is expressed as a function of the spectrum of the sampled signal and of the second order quantities of the point process. Then, by reformulating the alias-free concept, alias-free sampling schemes are proved to lead to a consistent spectral estimator.

However, the mentioned works are limited to the sampling on the line ($\mathbb{R}$) and most of the results depend on the sampling scheme through formulas that cannot be extended to $\mathbb{R}^m$, $m \geq 2$.

In the present paper we consider a point process approach to random sampling. With such a formalism, random sampling is formulated in the general spatial case. By modeling the sample comb as a point process, and by using the notion of Bartlett spectrum, we give a gen-

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eral formula for the reconstruction error in terms of the spectrum of the sampled signal and the spectrum of the sample comb.

The sampled signal and the sampler are assumed stationary (or at least wide-sense stationary for the sampled process). We treat the case where the sampled signal and the general sampler are independent, and the case where the rate of a Poisson sampler depends on the sampled signal.

II. SPATIAL RANDOM SAMPLING

We formulate random sampling in the general spatial case. Here the sampled signal is a wide-sense stationary process

$$X(t), \quad t = (t_1, \dots, t_m) \in \mathbb{R}^m$$

with mean m_X , autocovariance function $C_X(\tau) = E[X(t + \tau)X(t)^*] - |m_X|^2$, and power spectral measure μ_X , where

$$C_X(\tau) = \int_{\mathbb{R}^m} e^{2i\pi\langle \nu, \tau \rangle} \mu_X(d\nu)$$

with $\nu = (\nu_1, \dots, \nu_m)$ and $\langle \nu, \tau \rangle = \nu_1\tau_1 + \dots + \nu_m\tau_m$. Moreover, $\{X(t)\}_{t \in \mathbb{R}^m}$ admits the Cramér-Khinchin decomposition $\{Z_X(A)\}_{A \subseteq \mathcal{B}^m}$ [12] ($\mathcal{B}^m =$ Borel σ -field on $\mathbb{R}^m$).

$$X(t) = \int_{\mathbb{R}^m} e^{2i\pi\langle \nu, t \rangle} Z_X(d\nu) + m_X$$

where the integral thereof is a Wiener integral.

The sampler is now a stationary point process on $\mathbb{R}^m$, denoted N . We denote by $N(C)$ the number of sampling points in $C \subset \mathbb{R}^m$. N can be viewed as a random measure $N(\cdot)$ on $\mathbb{R}^m$. It can also be viewed as a random subset of $\mathbb{R}^m$ [13]. In this case $N(C) = |N \cap C|$ where $|A|$ is the cardinal of $A \subseteq \mathbb{R}^m$. The average intensity λ of the sampler is, by definition, the average number of sampling time per unit time, and the sampling frequency is then

$$\nu_s = \lambda$$

We shall consider integrals of the type

$$\int_{\mathbb{R}^m} \varphi(t) N(dt)$$

which are in fact sums $\sum_{s \in N} \varphi(s)$.

Our result will be expressed in terms of the **Bartlett spectrum** [1] of the sampler N , *i.e.* a measure μ_N on $\mathbb{R}^m$ such that for all functions $\varphi: \mathbb{R}^m \rightarrow \mathbb{C}$ that are in $L^1_{\mathbb{C}}(\mathbb{R}^m) \cap L^2_{\mathbb{C}}(\mathbb{R}^m) (= L^1 \cap L^2)$ we have

$$\text{Var} \left(\int_{\mathbb{R}^m} \varphi(t) N(dt) \right) = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_N(d\nu) \quad (3)$$

where $\widehat{\varphi}$ is the Fourier transform of φ .

Bartlett spectra are available in a number of situations
EXAMPLE II.1 N is a Cox process [1]. For a Cox process with a stationary conditional intensity $\{\lambda(t)\}_{t \in \mathbb{R}^m}$ having a Cramér spectral measure μ_λ ,

$$\mu_N(d\nu) = \mu_\lambda(d\nu) + \lambda d\nu \quad (4)$$

When $\mu_\lambda \equiv 0$ we have a homogeneous Poisson process [1], and

$$\mu_N(d\nu) = \lambda d\nu$$

◇

EXAMPLE II.2 N is a regular grid on $\mathbb{R}^2$

$$N = \{U_1 + kT_1, U_2 + lT_2; k, l \in \mathbb{Z}\}$$

where U_1, U_2 are independent and uniformly distributed on $[0, T_1]$ and $[0, T_2]$ (in order that the process be stationary, the sequence has to be randomly shifted). By mean of the Poisson summation formula, from (3) we obtain (see [14])

$$\mu_N(d\nu) = \frac{1}{T_1^2 T_2^2} \sum_{(n_1, n_2) \neq (0,0)} \delta_{\frac{n_1}{T_1}}(d\nu_1) \delta_{\frac{n_2}{T_2}}(d\nu_2) \quad (5)$$

◇

EXAMPLE II.3 N is obtained by i.i.d. displacements (jitter) of the points of the point process $\tilde{N}$ with Bartlett spectrum $\mu_{\tilde{N}}$.

Let $Q(z)$, $z \in \mathbb{R}^m$, be the probability measure of the displacements Z . Let

$$\psi_Z(\nu) = \int_{\mathbb{R}^m} e^{2i\pi\langle \nu, z \rangle} Q(dz)$$

Then [1],

$$\mu_N(d\nu) = |\psi_Z(\nu)|^2 \mu_{\tilde{N}}(d\nu) + \lambda \left(1 - |\psi_Z(\nu)|^2\right) d\nu \quad (6)$$

◇

EXAMPLE II.4 N is obtained by i.i.d. thinning (losses) of the points of the point process $\tilde{N}$ with Bartlett spectrum $\mu_{\tilde{N}}$ (and intensity $\tilde{\lambda}$).

Let p the probability of losing a sample, and $q = 1 - p$. Then,

$$\mu_N(d\nu) = q^2 \mu_{\tilde{N}}(d\nu) + \tilde{\lambda} p q d\nu \quad (7)$$

◇

The sample “brush” is the pseudo process

$$Y(t) = \sum_{s \in N} X(s) \delta(t - s)$$

that can be identified with the measure $\sum_{s \in N} X(s) \delta_s(\cdot)$. To express its spectrum, we consider the **generalized Cramér-Khinchin spectrum**, *i.e.* a measure $\mu_Y(d\nu)$ such that, for any $\varphi(t) \in L^1 \cap L^2$,

$$\text{Var} \left(\int_{\mathbb{R}^m} \varphi(t) Y(t) dt \right) = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Y(d\nu) \quad (8)$$

The reconstruction of the signal $X(t)$ is obtained by filtering the sample comb $\{Y(t)\}$

$$\int_{\mathbb{R}^m} \varphi(t - s) Y(s) ds$$

where $\varphi \in L^1 \cap L^2$. The difference ϵ between $X(t)$ and its approximation, *i.e.* the reconstruction error, is measured by

$$\epsilon = E \left[\left| \int_{\mathbb{R}^m} \varphi(t - u) Y(u) du - X(t) \right|^2 \right]$$

III. SPECTRA OF THE SAMPLE COMB AND RECONSTRUCTION ERROR WHEN THE SAMPLER IS INDEPENDENT FROM THE SIGNAL

Let N be a wide-sense stationary simple point process on $\mathbb{R}^m$ with intensity $\lambda < \infty$, and Bartlett spectrum μ_N . We have the following:

Theorem III.1 *Suppose that $\{X(t)\}$ and N are independent. Then, the generalized process*

$$Y(t) = \sum_{s \in N} X(s) \delta(t - s)$$

admits the extended Cramér-Khinchin power spectral measure

$$\mu_Y(d\nu) = (\mu_N * \mu_X)(d\nu) + \lambda^2 \mu_X(d\nu) + |m_X|^2 \mu_N(d\nu) \quad (9)$$

where $*$ denotes convolution. Moreover, reconstructing the signal $\{X(t)\}_{t \in \mathbb{R}}$ by filtering the sample sequence $\{Y(t)\}_{t \in \mathbb{R}}$ with a filter $\varphi \in L^1 \cap L^2$ gives the following error

$$\epsilon = \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 \mu_Y(d\nu) + |m_X|^2 |\lambda \widehat{\varphi}(0) - 1|^2 - \lambda \int_{\mathbb{R}^m} 2\operatorname{Re}\{\widehat{\varphi}(\nu)\} \mu_X(d\nu) + \int_{\mathbb{R}^m} \mu_X(d\nu) \quad (10)$$

Proof See [14]. $\square$

Examples of the power spectral measure and the reconstruction error can be obtained from the above results for different sampling schemes through the knowledge of μ_N . Of particular interest is the Poisson case and the effect of thinning. For notation ease, we consider a centered signal, i.e. $m_X = 0$.

EXAMPLE III.1 N is a Cox process. From the results on the Bartlett spectrum of a Cox process (Example II.1) we have

$$\mu_Y(d\nu) = (\mu_\lambda * \mu_X)(d\nu) + \lambda^2 \mu_X(d\nu) + \lambda C_X(0) d\nu \quad (11)$$

and, when the process is a homogeneous Poisson process ($\mu_\lambda \equiv 0$)

$$\mu_Y(d\nu) = \lambda^2 \mu_X(d\nu) + \lambda C_X(0) d\nu \quad (12)$$

In this last case the reconstruction error is

$$\epsilon = \int_{\mathbb{R}^m} |\lambda \widehat{\varphi}(\nu) - 1|^2 \mu_X(d\nu) + \lambda C_X(0) \int_{\mathbb{R}^m} |\widehat{\varphi}(\nu)|^2 d\nu$$

Let us now consider the “classical” sampling framework for a centered signal $\{X(t)\}_{t \in \mathbb{R}}$ (in the univariate case), i.e. the signal has a spectral measure μ_X with finite support S of length $2B = l(S)$, and

$$\widehat{\varphi}(\nu) = \begin{cases} \frac{1}{\lambda} & \text{on } S \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

Then, we have

$$\epsilon = \lambda C_X(0) \int_{\mathbb{R}} |\widehat{\varphi}(\nu)|^2 d\nu = \lambda C_X(0) \int_{\mathbb{R}} \frac{1}{\lambda^2} 1_S(\nu) d\nu$$

that is

$$\epsilon = C_X(0) \frac{2B}{\lambda}$$

Therefore, sampling at the Nyquist rate $\lambda = 2B$ gives very poor performances, not better than the estimate based on no observation at all. This does not mean, however, that below the rate $\lambda = 2B$, there is no information (or in a sense as the result suggests “negative information”) concerning the process itself contained in its samples. A better choice of a filter would indeed give a linear estimate with error less than $\sigma^2 = C_X(0)$. For instance, if we let $\widehat{\varphi}$ be real, we find for the error

$$\epsilon = \int_{\mathbb{R}} \left[(\lambda \widehat{\nu} - 1)^2 f_X(\nu) + \lambda^2 \sigma^2 \widehat{\varphi}(\nu)^2 \right] d\nu$$

where it is assumed that $\{X(t)\}_{t \in \mathbb{R}}$ has the power spectral density $f_X(\nu)$. At the minimum,

$$\epsilon = \sigma^2 \left(1 - \int_{\mathbb{R}} \frac{\lambda \widetilde{f}_X(\nu)}{1 + \lambda \widetilde{f}_X(\nu)} \widetilde{f}_X(\nu) d\nu \right)$$

where $\widetilde{f}_X(\nu)$ is the normalized power spectral density

$$\widetilde{f}_X(\nu) = \frac{f_X(\nu)}{\int_{\mathbb{R}} f_X(\nu') d\nu'} = \frac{f_X(\nu)}{\sigma^2}$$

Therefore

$$\epsilon = \sigma^2 (1 - \rho)$$

where

$$\rho = \int_{\mathbb{R}} \frac{\lambda \widetilde{f}_X(\nu)}{1 + \lambda \widetilde{f}_X(\nu)} \widetilde{f}_X(\nu) d\nu$$

can be interpreted as the correlation coefficient between $X(t)$ and N for fixed t . $\diamond$

EXAMPLE III.2 Effect of the thinning on μ_Y and ϵ . Let $\mu_{\widetilde{Y}}$ denote the spectrum of the sample comb obtained by sampling the signal with the sampler $\widetilde{N}$. When the former is “randomly thinned” (as described in example II.4) the spectrum of the resulting sample comb is

$$\mu_Y(d\nu) = q^2 \mu_{\widetilde{Y}}(d\nu) + \widetilde{\lambda} p q C_X(0) d\nu \quad (14)$$

Let $\widetilde{\epsilon}$ be the reconstruction error when the sample brush is characterized by $\mu_{\widetilde{Y}}$ and the sampler by $\mu_{\widetilde{N}}$. Then, from the effect of thinning on μ_Y , the reconstruction error becomes

$$\epsilon = q^2 \widetilde{\epsilon} + (1 - q^2) C_X(0) + \widetilde{\lambda} p q C_X(0) \int_{\mathbb{R}} |\widehat{\varphi}(\nu)|^2 d\nu + \widetilde{\lambda} q (1 - q) \int_{\mathbb{R}^m} 2\operatorname{Re}(\widehat{\varphi}(\nu)) \mu_X(d\nu) \quad (15)$$

Remark that when there the probability of a loss is zero, i.e. $q = 1$, we have $\epsilon = \widetilde{\epsilon}$, while when all the samples are lost, i.e. $q = 0$, we have $\epsilon = C_X(0) = \sigma_X^2$, which is

indeed the error of the estimate based on no observation at all. $\diamond$

Of course, the results of the classical sampling theory can be obtained as a particular case. Indeed, when the sampling is uniform and on the line (for a centered signal, for notation ease) from (9) we obtain the aliased spectrum

$$\mu_Y(d\nu) = \frac{1}{T^2} \sum_{n \in \mathbb{Z}} \mu_X\left(d\nu - \frac{n}{T}\right) \quad (16)$$

Moreover, in the classical sampling situation where the signal $\{X(t)\}_{t \in \mathbb{R}}$ has a spectral measure μ_X with finite support S , and the filter $\varphi \in L^1 \cap L^2$ verifies (13), with $\lambda = 1/T$, the reconstruction is perfect. Then

$$X(t) = \int_{\mathbb{R}} \varphi(t-s) X(s) N(ds) = \sum_{n \in \mathbb{Z}} X(T_n) \varphi(t - T_n)$$

which is the usual reconstruction formula [5].

IV. SPECTRA OF THE SAMPLE COMB AND RECONSTRUCTION ERROR FOR A POISSON SAMPLER WITH RATE DEPENDING ON THE PROCESS

The model for the sampler is now a Cox process [1] on $\mathbb{R}^m$ with the conditional (w.r.t. X) intensity of the form

$$\lambda(t) = \lambda(t, X)$$

For instance, in the univariate case, $\lambda(t) = |X(t)|^2$, $\lambda(t) = \left|\dot{X}(t)\right|^2$ where $\dot{X}$ is the derivative at t of $t \rightarrow X(t)$. More complicated functionals can be considered.

Theorem IV.1 Assume that $\mathbb{E} \left[|X(t)|^2 \lambda(t, X)^2 \right] < \infty$, $\forall t \in \mathbb{R}^m$, and that $\{\lambda(t)\}_{t \in \mathbb{R}^m}$ is a locally integrable process, that is

$$\int_C \lambda(t) < \infty, \quad a.s.$$

for all bounded Borel sets C . Let μ_Z be the extended Cramér spectral measure of the stationary process $Z(t) = X(t) \lambda(t)$. Then, the generalized process

$$Y(t) = \sum_{s \in N} X(s) \delta(t - s)$$

admits the extended Cramér-Khinchin power spectral measure

$$\mu_Y(d\nu) = \mu_Z(d\nu) + \overline{X^2 \lambda} d\nu \quad (17)$$

where $\mathbb{E} \left[|X(t)|^2 \lambda(t) \right] = \overline{X^2 \lambda}$ (independent of t). Moreover, reconstructing the signal by filtering the sample sequence $\{Y(t)\}_{t \in \mathbb{R}}$ with a filter $\varphi \in L^1 \cap L^2$ gives the following error

$$\begin{aligned} \epsilon &= \int_{\mathbb{R}^m} |\hat{\varphi}(\nu)|^2 \mu_Z(d\nu) + \overline{X^2 \lambda} \int_{\mathbb{R}^m} |\hat{\varphi}(\nu)|^2 d\nu \\ &\quad + \int_{\mathbb{R}^m} 2\mathcal{Re} \{ \hat{\varphi}(\nu) \} \mu_{XZ}(d\nu) \\ &\quad + C_X(0) + |m_X - m_Z \hat{\varphi}^*(0)|^2 \end{aligned}$$

Proof See [14]. $\square$

In the particular case of $X(t) \equiv 1$, we recover the formula for the Bartlett spectrum of the Cox process (4), while when $\lambda(t) = X(t)$ we have

$$\mu_Y(d\nu) = \mu_{X^2}(d\nu) + \mathbb{E} [X^3(0)] d\nu$$

V. CONCLUDING REMARKS

Using a point process approach and the notion of Bartlett spectrum, we have obtained formulas for the power spectral measure of the sample comb and the reconstruction error, which are fundamental tools for Spectral Analysis and for the Statistical Analysis of Signal Reconstruction. Such formulas are valid in the general spatial case.

REFERENCES

- [1] D. J. Daley and D. Vere-Jones, *An introduction to the Theory of Point Processes*. Springer-Verlag New York, 1988.
- [2] M. Gaster and J. B. Roberts, "Spectral analysis of randomly sampled signals," *J. Inst. Maths Applics*, vol. 15, pp. 195–216, 1975.
- [3] H. S. Shapiro and R. A. Silverman, "Alias-free sampling of random noise," *SIAM J. Appl. Math.*, vol. 8, no. 2, June 1960.
- [4] F. J. Beutler, "Alias-free randomly timed sampling of stochastic processes," *IEEE Trans. Inform. Theory*, vol. IT-16, no. 2, pp. 147–152, 1970.
- [5] C. E. Shannon, "A mathematical theory of communication," *Bell System Tech. J.*, vol. 27, pp. 379–423, 623–656, 1948.
- [6] F. J. Beutler and O. A. Z. Leneman, "Random sampling of random processes: Stationary point processes," *Information and Control*, vol. 9, pp. 325–346, 1966.
- [7] O. A. Z. Leneman, "Random sampling of random processes: Impulse processes," *Information and Control*, vol. 9, pp. 347–363, 1966.
- [8] F. J. Beutler, "The spectral analysis of impulse processes," *Information and Control*, vol. 12, pp. 236–258, 1968.
- [9] O. A. Z. Leneman and J. B. Lewis, "Random sampling of random processes: Mean-square comparison of various interpolators," *IEEE Trans. Automat. Control*, vol. AC-11, no. 3, pp. 396–403, July 1966.
- [10] E. Masry, "Poisson sampling and spectral estimation of continuous-time processes," *IEEE Trans. Inform. Theory*, vol. IT-24, no. 2, pp. 173–183, March 1978.
- [11] —, "Alias-free sampling: An alternative conceptualization and its applications," *IEEE Trans. Inform. Theory*, vol. IT-24, no. 3, pp. 317–324, May 1978.
- [12] D. Dacunha-Castelle and M. Duflo, *Probability and Statistics*. Springer-Verlag New York, 1986, vol. 2.
- [13] M. N. M. van Lieshout, *Markov Point Processes and their applications*. Imperial College Press, 2000.
- [14] P. Brémaud, L. Massoulié, and A. Ridolfi, "Power spectra of random spike fields & related processes," submitted, 2002.