

# Error-Rate Dependence of Non-Bandlimited Signals with Finite Rate of Innovation

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*Abstract* — Recent results in sampling theory [1] showed that perfect reconstruction of non-bandlimited signals with finite rate of innovation can be achieved performing uniform sampling at or above the rate of innovation. We study analog-to-digital (A/D) conversion of these signals, introducing two types of oversampling and consistent reconstruction.

In this work, we consider periodic streams of  $K$  Diracs, that is,  $x(t) = \sum_{k \in \mathcal{Z}} c_k \delta(t - t_k) = \sum_{m \in \mathcal{Z}} X[m] e^{j(2\pi m t)/\tau}$  with  $X[m] = \frac{1}{\tau} \sum_{k=0}^{K-1} c_k e^{-j(2\pi m t_k)/\tau}$  and period  $\tau$ , where  $t_{k+K} = t_k + \tau$ ,  $c_{k+K} = c_k$ ,  $\forall k \in \mathcal{Z}$  and  $\delta(t)$  denotes a Dirac delta function. The signal has  $2K/\tau$  degrees of freedom per unit of time. Time positions  $\{t_k\}_{k=0}^{K-1}$  and weights  $\{c_k\}_{k=0}^{K-1}$  can be perfectly reconstructed by first applying a sinc sampling kernel  $h_B(t) = B \text{sinc}(Bt)$  with bandwidth  $[-B\pi, B\pi]$ , thus obtaining  $y(t) = x(t) * h_B(t)$ , and then taking the  $N \geq 2M + 1$  uniform samples  $y_n = \sum_{m=-M}^M X[m] e^{j2\pi m n T/\tau}$  with  $T = \tau/N$ , such that  $B\tau = 2M + 1 \geq 2K + 1$ . After computing  $2K + 1$  Fourier coefficients from  $y_n$ , we apply annihilating filter method. From the roots of annihilating filter  $\{u_k = e^{-j2\pi t_k/\tau}\}_{k=0}^{K-1}$  we get the  $K$  time positions  $\{t_k\}_{k=0}^{K-1}$ , while the weights  $\{c_k\}_{k=0}^{K-1}$  can then be directly computed.

We overcome the error in amplitude of the samples  $\{y_n\}_{n=0}^{K-1}$ , introduced due to the quantization, by performing two types of oversampling. The first one, *oversampling in time*, consists of taking more samples of  $y(t)$  than necessary, so that  $N > 2M + 1$ , with oversampling ratio  $R_t = N/2M + 1$ . In the second one, *oversampling in frequency*, we extend the bandwidth  $B = 2M + 1$  so that it is greater than the rate of innovation, that is,  $M > K$ , with oversampling ratio  $R_f = (2M + 1)/(2K + 1)$ .

We also introduce the concept of consistent reconstruction for these types of signals. The idea is to exploit all the *a priori* knowledge of the original signal and the quantization process itself. We first define the three sets of constraints on which we have to project. **Set  $\mathcal{S}_1$**  is defined by the quantization operation and consists of the quantization bins in which the samples  $\{y_n\}_{n=0}^{N-1}$  lie. **Set  $\mathcal{S}_2$**  is the set of continuous-time periodic signals bandlimited to  $[-B\pi, B\pi]$  to which  $y(t)$  belongs.

Based on this, satisfying these two sets we provide a first level of accuracy, *weak consistency*, which we achieve by iterating projections  $\mathbf{P}_1$  and  $\mathbf{P}_2$ .

**Def. 1** A reconstruction  $\hat{x}(t)$  satisfies *weak consistency (WC)* iff it is obtained from a signal  $\hat{y}(t)$  such that: a) the samples  $\{\hat{y}_n\}_{n=0}^{N-1}$  lie in the same quantization bins as the original ones,  $\{\hat{y}_n\}_{n=0}^{N-1} \in \mathcal{S}_2$ , b)  $\hat{y}(t) \in \mathcal{S}_1$ .

**Proj.  $\mathbf{P}_1$**  : For every estimate  $\hat{y}_n^i$ ,  $\hat{y}_n^{i+1} = \mathbf{P}_1(\hat{y}_n^i)$  is given by: a)  $\hat{y}_n^{i+1} = \hat{y}_n^i$  if  $\hat{y}_n^i \in \mathcal{S}_1$ , 2) else,  $\hat{y}_n^{i+1}$  is set to the bound of

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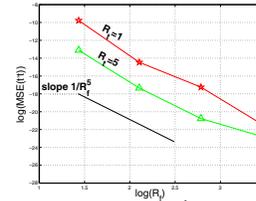


Figure 1: Dependence of  $MSE(t, \hat{t})$  on the factors  $R_t$  and  $R_f$ . the quantization interval in  $\mathcal{S}_1$  closest to  $\hat{y}_n^i$ .

**Proj.  $\mathbf{P}_2$**  : Given an estimate  $\hat{y}^i(t)$ , the new estimate  $\hat{y}^{i+1}(t) = \mathbf{P}_2(\hat{y}^i(t))$  is obtained by low-pass filtering  $\hat{y}^i(t)$ , that is  $\hat{y}^{i+1}(t) = \hat{y}^i(t) * h_B(t)$ . The particular structure of the signal  $x(t)$  defines the third set which, together with previous two sets, is used to enforce a stronger sense of consistency. The **Set  $\mathcal{S}_3$**  is the set of Fourier coefficients that originate from a periodic stream of Diracs,  $X[m] = \frac{1}{\tau} \sum_{k=0}^{K-1} c_k e^{-j2\pi m t_k/\tau}$ .

**Def. 2** A reconstruction  $\hat{x}(t)$  satisfies *strong consistency (SC)* iff: a) it satisfies *weak consistency*, b)  $\hat{y}_n = h_b(t) * \hat{x}(t)|_{nT}$  where  $\hat{x}(t)$  is a periodic stream of  $K$  Diracs.

**Proj.  $\mathbf{P}_3$**  : Given a set of estimated Fourier coefficients  $\hat{\mathbf{X}}^i$ , the projection  $\mathbf{P}_3$  provides  $\{(\hat{c}_k^{i+1}, \hat{t}_k^{i+1})\}_{k=0}^{K-1}$  and a set of Fourier coefficients  $\hat{\mathbf{X}}^{i+1}$  such that  $\hat{\mathbf{X}}^{i+1}[m] = \frac{1}{\tau} \sum_{k=0}^{K-1} \hat{c}_k^{i+1} e^{-j2\pi m \hat{t}_k^{i+1}/\tau}$ .

**Theorem 1** Given  $x(t)$ , for any reconstruction  $\hat{x}(t)$  obtained using  $\mathbf{P}_3$  and which satisfies *WC*, there exist  $\xi \geq 1$  such that if  $R_t, R_f \geq \xi$ , there is a constant  $c > 0$  which depends only on  $x(t)$  and not on  $R_t$  and  $R_f$ , and  $MSE(t, \hat{t}) \leq \frac{c}{R_t^3 R_f^2}$ . (see [3])

For the method that achieve *SC* the experimental results show, a performance of  $MSE(t, \hat{t}) = O(1/R_t^2 R_f^5)$  for time positions (Fig. 1), with parameters:  $K = 2$ ,  $\tau = 10$ ,  $t_k \in (0, \tau]$ ,  $c_k \in [-1, 1]$ .

We also compare two types of encoding, the traditional one, pulse-code modulation encoding (PCM) and the alternative one, based on threshold crossing encoding (TC) [2], and investigate in the dependence of the bit rate on the oversampling factors  $R_t$  and  $R_f$ , and the quantization step size  $\Delta$ . The following table, shows the theoretical results for the bit rate and also both theoretical and experimental results for the MSE of time positions.

	Bit rate (b)	MSE-WC	MSE-SC
TC	$O(\log_2 R_t)$	$O(1/R_t^2)$	$O(1/R_t^2)$
	$O(R_f \log_2 R_f)$	$O(1/R_f^5)$	$O(1/R_f^5)$
	$O(1/\Delta)$	$O(\Delta^2)$	$O(\Delta^2)$
PCM	$O(R_t)$	$O(1/R_t^2)$	$O(1/R_t^2)$
	$O(R_f \log_2 R_f)$	$O(1/R_f^5)$	$O(1/R_f^5)$
	$O(\log_2(1/\Delta))$	$O(\Delta^2)$	$O(\Delta^2)$

Notice that oversampling in time provide the error-rate dependence ( $O(2^{-2\alpha b})$ ) that can be obtain by decreasing the step size ( $O(2^{-2\beta b})$ ).

## REFERENCES

- [1] M. Vetterli, P. Marziliano, and T. Blu "Sampling signals with finite rate of innovation" *IEEE Trans. on Signal Proc.*, vol. 50, no. 6, pp. 1417-1428, June 2002.
- [2] N. T. Thao and M. Vetterli, "Deterministic Analysis of Oversampled A/D Conversion and Decoding Improvement Based on Consistent Estimates" *IEEE Trans. on Signal Proc.*, vol. 42, no. 3, pp. 519-531, 1994.
- [3] I. Jovanović and B. Beferull-Lozano, "Oversampled A/D Conversion of Non-Bandlimited Signals with Finite Rate of Innovation", To be submitted to *IEEE Trans. on Signal Proc.*