Power spectra related to UWB communications

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***Abstract*— Power spectra of signals related to random spikes are of interest in communications. In the present paper, we give a closed form formula for the power spectrum of a random stream of spikes, where the positions of spikes form a renewal process and the sequence of amplitudes is a correlated time series.**

I. Introduction

Signals related to random spikes arise in many fields of interest. Here we shall consider application to communication systems where the information is coded into streams of pulses (pulse modulation), as in ultra-wide band communications (UWB) [1], [2].

Computing the power spectrum of a spike train provides an important tool of analysis. For instance, spectral design (occupancy and composition) play a key role in the design of many communication systems: this is particularly true for UWB where signals are transmitted with extremely low spectral content while maintaining the average power level required for reliable communications [3].

Previous works have contributed to the exact spectral computation of signals related to a stream of spikes. In the specific case of communication systems, Win [1], [3] computed the power spectral density for a general time-hopping, pulse position modulated signal in the presence of clock jitter, while [4] provides the power spectral density of the family of pulse interval modulated signals.

In the above mentioned works, results have been obtained using the classical approach that uses the direct Fourier relation between the correlation and the power spectral density.

Here we consider a particular case of random stream of "modulated" spikes, where the random positions of the spikes are modeled as a stationary renewal point process, and their amplitudes as a w.s.s. time series. We adopt a point process approach obtaining results that are simple and tractable and that permit detailed analysis of the condition of validity. Moreover, they are easy to understand and the contribution of the various features of the model appears explicitly.

A first result gives the power spectrum of such a "modulated" renewal process. Then we extend our computation to the power spectrum of a "modulated" renewal process filtered by a random function. As we shall see, this latter results allows to generalize the power spectrum computation by taking into account other random quantities (*e.g.* a clock jitter).

Despite the specificity of a renewal point process, the results we have obtained provide a basic tool for the generalization of the mentioned previous works [3], [4].

II. Signal Model

We consider the following model, depicted in Figure 1

$$\Delta_{N_a}(t) = \sum_{n=-\infty}^{+\infty} a_n \delta(t - t_n) \tag{1}$$

i.e. a modulated random Dirac comb, where a_n is a correlated w.s.s. time series, with correlation function $R_a(k) = \mathrm{E}[a_n a_{n+k}]$ and expected value m_a, $\delta(t)$ is the Dirac pseudo-function, and t_n is a sequence of renewal random times, modeled by a renewal point process [5]. In particular, the sequence of random times is ordered as follows

$$\ldots < t_{-1} < t_0 \leq 0 < t_1 < \ldots \tag{2}$$

and it is assumed independent of the time series a_n. Call λ the average number of points per unit of time (average intensity, or rate).

In the following, we will refer to the set of random times $N = \{t_n, -\infty < n < +\infty\}$ as a point process. Moreover, we will use the notation $N((a,b])$ to indicate the number of random times falling in the open interval $(a,b]$. Similarly, we can represent the signal $\Delta_{N_a}(t)$ as the set of random times and the modulating time series $N_a = \{(t_n, a_n), -\infty < n < +\infty\}$. We shall call the latter a *modulated point process*.

The characteristic of a stationary renewal point process is that the inter-arrivals $s_n = t_n - t_{n-1}$, are all independent, and for $n \neq 1$ they are also identically distributed, say with distribution $F_s(s)$ (and characteristic function $g_s(u) = \mathrm{E}\left[e^{ius}\right]$).

III. Power Spectra

We shall need a convenient definition of the spectrum of a random Dirac comb, which is not a *bona fide* w.s.s. process, and for which therefore we cannot use the usual Cramér power spectral measure. The natural extension of the latter

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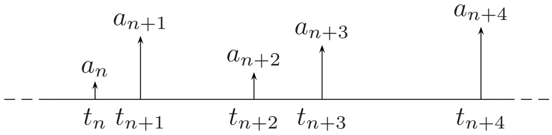


Fig. 1. A sequence of random spikes modulated by a time series

is the Bartlett power spectrum (Bartlett [6], Daley and Vere-Jones [5]).

Definition 3.1: Let N be a stationary point process, with ordered sequence of random times as in (2) and average intensity λ, such that $\mathrm{E}\left[N\left((a,b]\right)^2\right] < \infty$ for every bounded interval $(a,b] \subset$. The measure μ_N on is called the Bartlett spectral measure of N if

$$\mathrm{cov}\left(\sum_{n=-\infty}^{+\infty} \varphi(t_n), \sum_{n=-\infty}^{+\infty} \psi(t_n)\right) = \int_{m} \widehat{\varphi}(\nu)\widehat{\psi}^*(\nu)\mu_N(d\nu) \quad (3)$$

(where $\widehat{}$ denotes Fourier transformation) for all $\varphi, \psi \in \mathrm{L}^1 \cap \mathrm{L}^2$ such that

$$\mathrm{E}\left[\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} \left|\varphi\left(t_n\right)\right|\left|\psi^*\left(t_m\right)\right|\right] < \infty$$

(for details on the hypothesis for the definition see [5], [6]). $\diamondsuit$

(Throughout the text we will use the notation $\mu\left(d\nu\right) = d\mu\left(v\right)$ to denote the measure w.r.t. which the integral is computed).

Let us now consider the point process modulated by the time series of equation (1). According to the above definition, we aim to compute μ_{N_a} such that

$$\mathrm{cov}\left(\sum_{n=-\infty}^{+\infty} a_n\varphi(t_n), \sum_{n=-\infty}^{+\infty} a_n\psi(t_n)\right) = \int_{m} \widehat{\varphi}(\nu)\widehat{\psi}^*(\nu)\mu_{N_a}(d\nu) \quad (4)$$

We have the following result [7]:

Theorem 3.1: Let N_a be the modulated renewal point process defined above. Assume that N satisfies the condition for the existence of its Bartlet spectrum (definition 3.1). Suppose moreover that the renewal point process N has a renewal function that admits a density, *i.e.*

$$\mathrm{E}\left[N\left([0,t]\right)\right] = \int_0^t u\left(s\right)ds$$

such that

$$\int_0^\infty \left|u\left(t\right) - \lambda\right| dt < \infty$$

and that the time series has a summable covariance function

$$\sum_{k=-\infty}^{+\infty} \left|R_a\left(k\right) - \left|m_a\right|^2\right| < \infty \quad (5)$$

Then, the power spectral density of the point process N modulated by the time series a is given by

$$f_{N_a}\left(\nu\right) = \lambda 2\mathrm{Re}\left\{\sum_{k\geq 0} g_s^k\left(-2\pi\nu\right)R_a\left(k\right)\right\} - \lambda R_a\left(0\right) - \lambda^2 m_a^2 \delta\left(\nu\right). \quad (6)$$

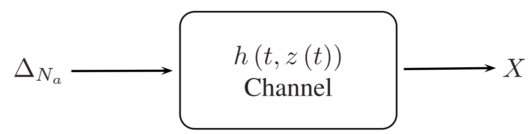
IV. Filtered random stream of spikes

Let us consider an convolutional filter with impulse response h. We include some randomness in the filter, *e.g.* a filter response that can take into account random losses of spikes, random displacements or changes of amplitude, etc. as we shall see in the example section.

More precisely we consider an impulse response $h\left(t,z\right)$, where z is some random element, depending on the position of the spikes.

The result of filtering of a random stream of spikes with such an random impulse response is known as shot noise with random excitation, and the corresponding power spectrum has been computed [8].

We are interested in computing the spectrum when the input of the filter is the modulated renewal stream of spikes presented above. This is an interesting situation when the random impulse response models a fading channel and the modulated stream of spike some pulse modulation transmission, as depicted below



Now, the output is

$$X\left(t\right) = \sum_{n=-\infty}^{+\infty} a_n h\left(t - t_n, z_n\right) \quad (7)$$

where $h\left(t,\cdot\right)$ is the impulse response of some convolutional filter. We want to compute μ_X such that is satisfies equation (8). Then, following the approach presented in [8], and in particular the fundamental isometry formula thereof, we obtain the following result [7]:

Theorem 4.1: Let N and a_n satisfy the conditions of Theorem 3.1. Then the Cramér power spectral measure of the shot noise of equation (7) is given by

$$\mu_X(d\nu) = \left|\mathrm{E}\left[\widehat{h}(\nu,z)\right]\right|^2 \mu_{N_a}(d\nu) + \lambda R_a\left(0\right)\mathrm{Var}\left(\widehat{h}(\nu,z)\right)d\nu$$

where $\widehat{}$ denotes Fourier transformation, and μ_{N_a} is given by equation (6).

Remark that the formula for the power spectral measure of a shot noise with random excitation presented in [8] has the following form

$$\mu_X(d\nu) = \left|\mathrm{E}\left[\widehat{h}(\nu,z)\right]\right|^2 \mu_N(d\nu) + \lambda\mathrm{Var}\left(\widehat{h}(\nu,z)\right)d\nu$$

$$\text{cov}\left(X\left(t\right),X\left(t+s\right)\right)=\text{cov}\left(\sum_{n=-\infty}^{+\infty}a_n h\left(t-t_n,z_n\right),\sum_{n=-\infty}^{+\infty}a_n h\left(t+s-t_n,z_n\right)\right)=\int e^{-i2\pi\nu s}\mu_X\left(d\nu\right) \quad (8)$$

$$\begin{aligned}\widetilde{f}_{N_a}\left(\nu\right)&=\left|\psi_z(\nu)\right|^2\frac{1}{T}\left(f_a\left(T\nu\right)+\frac{1}{T}\left|m_a\right|^2\sum_{n\neq0}\delta\left(\nu-\frac{n}{T}\right)\right)+\frac{1}{T}\left(1-\left|\psi_z(\nu)\right|^2\right)\\&=\frac{1}{T}+\frac{1}{T}\left|\psi_z(\nu)\right|^2\left(f_a\left(T\nu\right)-1\right)+\frac{1}{T}\left|\psi_z(\nu)\right|^2\left|m_a\right|^2\sum_{n\neq0}\delta\left(\nu-\frac{n}{T}\right)\end{aligned} \quad (9)$$

$$\begin{aligned}2\text{Re}\left(\sum_{k\geq0}g_s^k\left(-2\pi\nu\right)R_a\left(k\right)\right)-R_a\left(0\right)&=\sum_{k=-\infty}^{\infty}e^{-2\pi\nu Tk}\left(R_a\left(k\right)-\left|m_a\right|^2\right)+\sum_{k=-\infty}^{\infty}e^{-2\pi\nu Tk}\left|m_a\right|^2\\&=f_a\left(T\nu\right)+\frac{1}{T}\left|m_a\right|^2\sum_{n=-\infty}^{\infty}\delta\left(\nu-\frac{n}{T}\right)\end{aligned} \quad (10)$$

Therefore, a direct comparison puts in evidence the contribution of the time series modulation to the powers spectrum.

V. Examples

We now consider some applications of the presented results.

Example 5.1: Pulse amplitude modulation. In pulse amplitude modulation the spikes are T-equally spaced. In this particular case, the inter-arrivals S_n, $n \neq 1$ are constant and equal to T (the first arrival time t_1 is uniformly distributed over $[0,T]$ so that the corresponding point process is stationary). The average intensity is then $\lambda = 1/T$.

Now

$$g_s^k\left(-2\pi\nu\right)=e^{-2\pi\nu Tk}$$

and therefore, we obtain the expression in equation (10), where f_a is the power spectral density of the time series. Finally we obtain the well known result

$$f_{N_a}\left(\nu\right)=\frac{1}{T}\left(f_a\left(T\nu\right)+\frac{1}{T}\left|m_a\right|^2\sum_{n\neq0}\delta\left(\nu-\frac{n}{T}\right)\right).$$

Remark that the second term is the power spectral pseudo density of a Dirac comb.

Example 5.2: Clock jitter and pulse position-amplitude modulation. Consider now that the stream of modulated spikes (with power spectrum given by (6)) is affected by a clock jitter z, *i.e.* random displacements. Call ψ_z the characteristic function of such displacements. Then, by replacing $h\left(t-t_n,z_n\right)$ with $h\left(t_n+z_n\right)$ in equation (8), and by considering the definition of power spectrum for a modulated stream of spikes, given by equation (4), we obtain the power spectrum in the presence of jitter

$$\widetilde{f}_{N_a}\left(\nu\right)=\left|\psi_z(\nu)\right|^2 f_{N_a}(\nu)+\lambda\left(1-\left|\psi_z(\nu)\right|^2\right)$$

When the spikes are equally spaced, using the result of the previous example we obtain the expression in equation (9).

Remark that the sequence of random displacement z_n can represent the coded information. In this case we have a pulse position and amplitude modulation, widely used in UWB transmissions, and the above expression give its spectrum (a known result [3]).

VI. Concluding remarks

Stream of random spikes are aptly modeled using a point process approach. In particular, the latter provides powerful tools for power spectra computation. The resulting formula are easily generalized to take into account additional random events: this allows for various extensions of published results. Our rigorous approach to spectrum computation requires some condition on the signal. Note in particular that condition (5) excludes long range dependence.

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