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# FILTER BANKS ALLOWING PERFECT RECONSTRUCTION

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**Abstract.** Splitting a signal into $N$ filtered channels subsampled by $N$ is an important problem in digital signal processing. A fundamental property of such a system is that the original signal can be perfectly recovered from the subsampled channels. It is shown that this can always be done, and that FIR solutions exist. This is done by mapping the $NM$-dimensional nonlinear problem (where $N$ is the number of channels and $M$ the length of the FIR filters) into an $M$-dimensional linear problem. For $N=2$, a general class of FIR solutions is derived, together with methods to find filters. The dual problem of mixing $N$ signals into one channel upsampled by $N$ is also addressed. Several applications are proposed. All results are obtained by looking at the $N$ filter bank as a true $N$ channel system, rather than $N$ separate channels.

**Zusammenfassung.** Die Aufspaltung eines Signals in $N$ Kanäle, die jeweils $N$-fach unterabgetastet werden, ist ein wichtiges Problem der Signalverarbeitung. Eine grundlegende Eigenschaft solch eines Systems ist es, daß das Originalsignal unverzerrt aus den unterabgetasteten Kanalsignalen rekonstruiert werden kann. Es wird gezeigt, daß dies stets gelingt und daß es FIR-Lösungen gibt. Dazu dient eine Abbildung des $NM$-dimensionalen, nicht-linearen Entwurfsproblems (wobei $N$ die Kanalzahl und $M$ die Filterlänge ist) auf ein $M$-dimensionales, lineares Problem. Für den Fall $N=2$ wird eine allgemeingültige Klasse von FIR-Lösungen hergeleitet. Zudem werden Verfahren zum Auffinden der Koeffizienten angegeben. Das duale Problem wird ebenfalls behandelt: Es besteht darin, $N$ Einzelsignale nach $N$-facher Erhöhung der Datenrate zu einem Gesamtsignal zu vermischen. Verschiedene Anwendungen werden vorgeschlagen. Alle Ergebnisse beruhen darauf, daß nicht jeder einzelne der $N$ Kanäle, sondern die Filterbank wirklich als $N$-Kanal-System betrachtet wird.

**Résumé.** La séparation d'un signal en $N$ canaux sous-échantillonnés d'un facteur $N$ est un problème important en traitement numérique des signaux. Une propriété fondamentale d'un tel système est que le signal original puisse être reconstruit parfaitement à partir des canaux sous-échantillonnés. Il est montré dans la suite que ceci est toujours possible, et que des solutions RIF existent. Celles-ci sont obtenues en transformant un problème non linéaire de dimension $NM$ (où $N$ est le nombre de canaux et $M$ la longueur des filtres RIF) en un problème linéaire de dimension $M$. Pour $N=2$, on dérive une classe générale de solutions RIF, ainsi que des méthodes pour trouver des filtres. Le problème dual du multiplexage de $N$ signaux en un signal de fréquence d'échantillonnage $N$ fois plus élevée est également considéré. Plusieurs applications sont proposées. Tous les résultats sont obtenus en considérant le problème des bancs de $N$ filtres comme un problème à $N$ dimensions plutôt que $N$ problèmes séparés.

**Keywords.** Filter banks, multirate systems, decimation, interpolation, quadrature mirror filters.

## 1. Introduction

Let us first briefly state the basic problem we want to solve. Suppose an infinite sequence of samples $x(n)$. This sequence is filtered into $N$ sequences $y_0(n) \ldots y_{N-1}(n)$ (with linear, time invariant filters). The sequences $y_i(n)$ are subsampled by a factor $N$, that is only every $N$th sample is kept, or $y_i'(n) = y_i(Nn)$. Now, the problem is to recover $x(n)$ from the subsampled sequences $y_i'(n)$ (see Fig. 1).

Obviously, there is the same number of samples per unit of time in $x(n)$ and in all the $y_i'(n)$ together, thus a solution should exist. Nevertheless, there were not many practical solutions to the problem up to

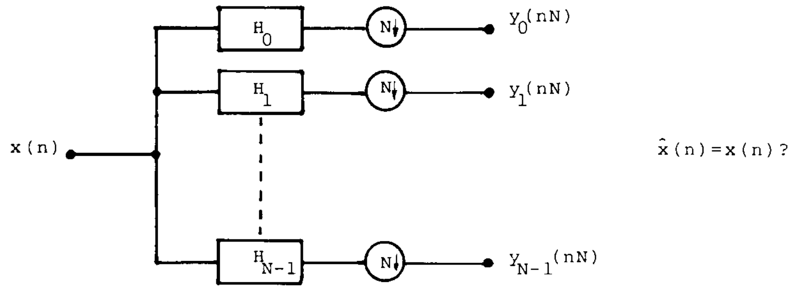


Fig. 1. General problem statement. Can the initial signal be recovered from the subsampled channel signals?

recently. On the one hand, the downsampling/upsampling process creates aliased versions of the original signal (unless perfectly sharp bandpass filters are used before subsampling, leading to infinitely long filters) which have to be cancelled in the reconstruction process, but, on the other hand, the original signal appears filtered at the output (and this filtering has to be cancelled, which might be impossible for stability reasons).

The work on subsampled filter banks was initiated by the introduction of the quadrature mirror filter concept [5, 6, 7, 8]. The two channel QMF filter bank, while not solving the problem perfectly, annihilates the aliasing perfectly, a very useful feature in speech processing [3]. The work on efficient implementation of filter banks started with the computation of the transmultiplexers by polyphase networks and FFTs [1]. The merging of the two approaches was first proposed by Nussbaumer [13] and was further investigated by many authors [12, 15, 16, 20, 30]. The first perfect FIR solution for the two channel subsampled filter bank was proposed by Smith and Barnwell [22] and Wackersreuther [29]. The matrix notation we developed to address the general case [28] was also independently introduced by Ramstad [19] and Smith and Barnwell [23]. Thorough treatments of the filter bank problems were done by Vary and co-authors [9, 25, 26].

The main results appearing below are briefly stated hereafter:

- emphasis is put on FIR analysis and FIR synthesis filters, because IIR solutions lead to implicit pole/zero cancellations,
- a general class of FIR solutions for $N=2$ is derived,
- linear phase solutions are shown to exist,
- two new methods to generate FIR filters that will satisfy the perfect reconstruction requirement for $N=2$ are developed,
- for $N>2$, the $NM$-dimensional nonlinear problem (with $M$ being the filter length) that has to be solved to find FIR solutions is shown to reduce to an $M$-dimensional linear problem,
- solutions are shown to exist and a method to find them is given,
- it is shown that aliasing can always be cancelled,
- the case where the $N$ filters are derived from a single prototype filter by frequency shifting is shown to only have an IIR solution,
- the dual problem of mixing $N$ signals onto a single channel upsampled by $N$ is solved.

All these results (except the aliasing cancellation property and one design procedure [29]) are, to our knowledge, original. Most results are obtained by using a general polyphase representation of the filters appearing in a subsampled filter bank and by reducing the general problem to the analysis of the determinant of the polyphase filter matrix. While a matrix notation was also used in [19, 23], this generalized polyphase representation seems to be original. As applications, a subband coder incorporating linear prediction, a scrambling scheme for analog signals, noncritically subsampled filter banks and filter banks on finite fields are proposed.

The outline of the paper is the following: Section 2 states the general problem and two known but unpractical solutions. Section 3 thoroughly investigates the two channel case (which is simple yet important in practice). Section 4 looks at the general case, and shows that FIR solutions for both the analysis and synthesis can exist. Section 5 looks at the dual problem which is closely related to the initial one. Section 6, finally, proposes a couple of applications. Appendix A shows that only FIR analysis and synthesis does not produce implicit pole/zero cancellation and Appendix B gives some more results on the two channel case.

In the following, all signals and filters are assumed to be complex unless specified otherwise, for simplicity and generality reasons only. Note as well that both filters and matrices (or vectors) use the numbering starting from 0 to $N-1$ ($N$ being the dimension), which is unusual for linear algebra, but makes notations more coherent and simple.

## 2. The problem and two obvious solutions

In Fig. 1, the problem addressed below is stated pictorially: the signal $x(n)$ with $z$-transform $X(z)$ [17, 18] is filtered into $N$ channels, which are then subsampled by $N$. Can the original signal be recovered from the $N$ subsampled channels? In order to solve the problem stated in Fig. 1, we consider the system depicted in Fig. 2. There, the operations in the analysis part have been matched by equivalent operations

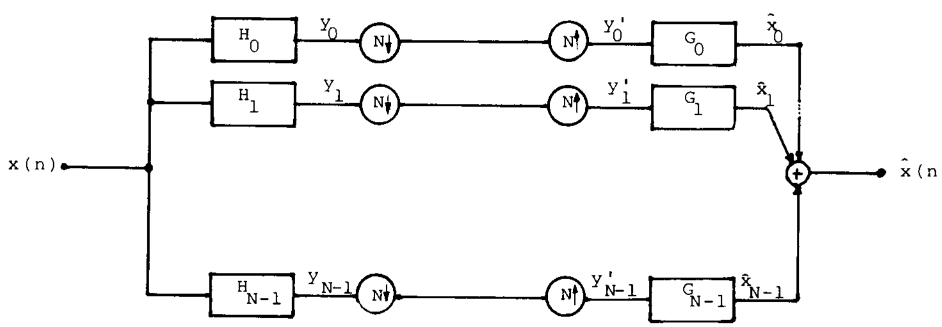


Fig. 2. Solution to the general problem involving upsampling and interpolation.

in the synthesis part in order to recover the initial signal. The function $N\downarrow$ means subsampling by $N$, that is replacing the sequence $x(n)$ by the sequence $x'(n)=x(nN)$. In the $z$-domain, this can be shown to be equal to [4, 21]

$$X'(z)=(1/N)\sum_{k=0}^{N-1} X(W^k z^{1/N}), \quad W=\mathrm{e}^{-\mathrm{j}2\pi/N}. \tag{1}$$

The function $N\uparrow$ means upsampling by $N$, which corresponds to replacing $x'(n)$ by $x''(n)=x'(n)$ for $n=lN$, and zero otherwise. This leads to the following $z$-transform:

$$X''(z)=X'(z^N). \tag{2}$$

Actually, the cascading of subsampling and upsampling by the factor $N$ corresponds to the modulation of the sequence $x(n)$ by the function $l(n)$ given by

$$l(n)=(1/N)\sum_{k=0}^{N-1} W^{-nk}, \tag{3}$$

which gives the following $z$-transform for $x''(n)=x(n)l(n)$:

$$X''(z)=(1/N)\sum_{k=0}^{N-1} X(W^k z). \tag{4}$$

Thus, downsampling and subsequent upsampling of a signal produces an output containing the signal itself as well as $N-1$ aliased versions (which are undesired).

In order to reproduce the input signal exactly at the output in the system of Fig. 2, one can use perfect bandpass filters (infinitely sharp and nonoverlapping) with a bandwidth of $1/N$ (assume a normalized sampling frequency of 1). Then, subsampling by $N$ is allowed (since no spectral overlapping occurs). After upsampling, interpolation with the perfect bandpass and subsequent addition of the bands reproduce the input signal perfectly as can readily be verified. The only problem is that the required filters have to be infinitely long, otherwise the reconstructed signal is an approximated version of the original signal only, and, in particular, aliased versions of the original signal will appear in the output signal. In the above approach, one tried to verify the sampling theorem before subsampling (thus requiring perfect bandpass filters), and thus the problem was approached on one channel at a time basis. No use was made of the fact that all channels are computed simultaneously.

Another solution [27], where the simultaneity of the process in the $N$ channels is used, appears when the analysis filters are of length $N$ (equal to the number of filters). Then, the vector of the subsampled signals at time $n$ can be seen to be equal to the product of a matrix $\boldsymbol{H}$ with a vector $\boldsymbol{x}$ containing the $N$ last samples of the input signal ($[x(n), x(n-1), \ldots, x(n-N+1)]$). The rows of the matrix $\boldsymbol{H}$ are obtained from the coefficients of the input filters. Similarly, the $N$ reconstructed outputs ($[\hat{x}(n), \hat{x}(n-1), \ldots, \hat{x}(n-N+1)]$) are equal to the product of a matrix $\boldsymbol{G}$ with the vector of the subsampled signals. The matrix $\boldsymbol{G}$ has its columns equal to the coefficients of the synthesis filters (which are also of length $N$). Then, if $\boldsymbol{H}$ is invertible (that is, the $N$ analysis filters are linearly independent) and $\boldsymbol{G}=\boldsymbol{H}^{-1}$, it can be shown that the reconstruction is perfect. The problem here is that length $N$ filters in an $N$ channel filter bank are in general too short for practical applications.

Thus, the question that will be addressed next is: are there length $M$ FIR filters, $N<M<\infty$, that will allow perfect reconstruction?

At this point, a remark is already appropriate: the first solution above divided the $N$ channel problem into $N$ separate problems of down-upsampling by $N$ and required therefore infinitely long filters. The second solution simply solves the problem by looking at all the channels at the same time, and while not satisfactory because of the short filter length, it is nevertheless perfect. Thus, the $N$ channel filter bank problem should always be considered as a whole, '$N$-dimensional problem', which has to be solved as such.

## 3. The general two channel case

The two channel case is depicted in Fig. 3. For channel 0, the following holds (using (4) and the convolution property of the $z$-transform):

$$Y_0(z)=H_0(z)X(z), \tag{5}$$

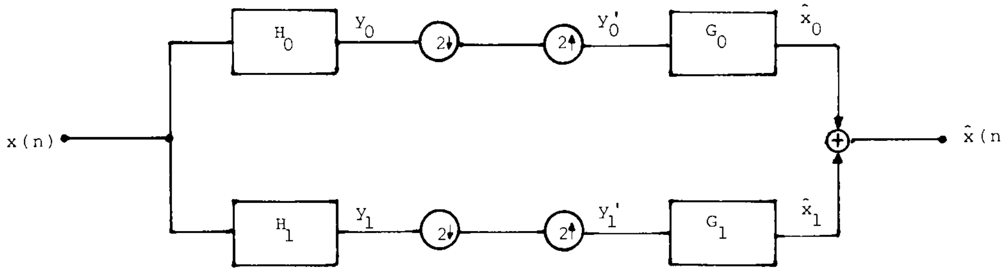


Fig. 3. Two channel case.

$$Y_0'(z)=\tfrac{1}{2}[H_0(z)X(z)+H_0(-z)X(-z)], \tag{6}$$

$$\hat{X}_0(z)=\tfrac{1}{2}[H_0(z)X(z)+H_0(-z)X(-z)]G_0(z). \tag{7}$$

Similar relations hold for channel 1. Thus, $\hat{X}(z)$ can be written as

$$\hat{X}(z)=\tfrac{1}{2}([H_0(z)G_0(z)+H_1(z)G_1(z)]X(z)+[H_0(-z)G_0(z)+H_1(-z)G_1(z)]X(-z)). \tag{8}$$

The reconstructed signal is therefore a function of the original signal $X(z)$, plus a function of the modulated signal $X(-z)$, as shown in Fig. 4 and in equation (9):

$$\hat{X}(z)=F_0(z)X(z)+F_1(z)X(-z), \tag{9}$$

where

$$F_0(z)=\tfrac{1}{2}[H_0(z)G_0(z)+H_1(z)G_1(z)], \tag{10}$$

$$F_1(z)=\tfrac{1}{2}[H_0(-z)G_0(z)+H_1(-z)G_1(z)]. \tag{11}$$

A necessary and sufficient condition for perfect reconstruction (since $F_0(z)$ and $F_1(z)$ are linear and time invariant) is that $F_0(z)$ is a pure delay and $F_1(z)$ is equal to zero.

In matrix notation, this is equivalent to

$$\begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix}=\frac{1}{2}\begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix}\cdot\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix}=\begin{bmatrix} z^{-k} \\ 0 \end{bmatrix} \tag{12}$$

or

$$\boldsymbol{f}(z)=\boldsymbol{M}(z)\boldsymbol{g}(z)=[z^{-k}\ \ 0]^{\mathrm{T}}, \tag{13}$$

where the meanings of $\boldsymbol{f}$, $\boldsymbol{M}$, and $\boldsymbol{g}$ in (13) are obvious from (12). Now, setting

$$\boldsymbol{g}(z)=\boldsymbol{M}^{-1}(z)[z^{-k}\ \ 0]^{\mathrm{T}} \tag{14}$$

will solve the given problem of exact reconstruction. This is shown in Fig. 5 and the resulting output is

$$\hat{X}(z)=[\boldsymbol{f}(z)]^{\mathrm{T}}\boldsymbol{x}(z)=z^{-k}X(z), \tag{15}$$

where

$$\boldsymbol{x}(z)=[X(z)\ \ X(-z)]^{\mathrm{T}}. \tag{16}$$

While (14) gives a perfect solution, the problem of existence (causality, type of synthesis filters) has yet

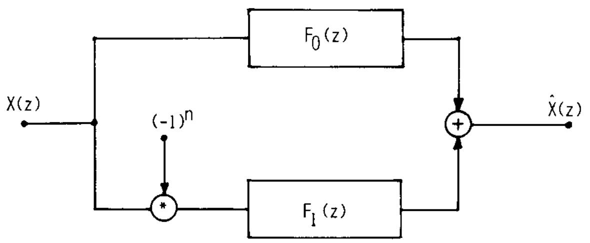


Fig. 4. Equivalent two channel system, where the output is a linear function of the original and the modulated input.

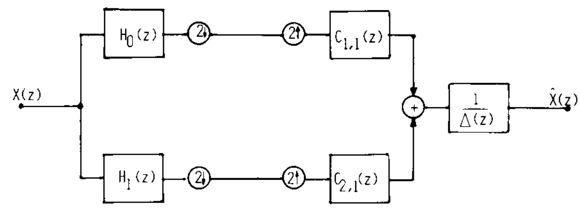


Fig. 5. Two channel system with perfect reconstruction.

to be addressed. Consider now the inverse of the filter matrix $\boldsymbol{M}(z)$:

$$\boldsymbol{M}^{-1}(z)=(1/\Delta(z))\boldsymbol{C}(z), \tag{17}$$

where

$$\Delta(z)=\tfrac{1}{4}[H_0(z)H_1(-z)-H_0(-z)H_1(z)] \tag{18}$$

is the determinant of $\boldsymbol{M}(z)$ and

$$\boldsymbol{C}(z)=\frac{1}{2}\begin{bmatrix} H_1(-z) & -H_1(z) \\ -H_0(-z) & H_0(z) \end{bmatrix} \tag{19}$$

is the cofactor matrix of $\boldsymbol{M}(z)$.

We will consider separately $1/\Delta(z)$ and $\boldsymbol{C}(z)$, because $1/\Delta(z)$ can be seen as a common post-filter for all synthesis filters (and can thus be applied after summation) but also because $1/\Delta(z)$ and $\boldsymbol{C}(z)$ lead in general to different types of filters.

Thus, choose the synthesis filters as

$$\boldsymbol{g}(z)=\boldsymbol{C}(z)[1\ \ 0]^{\mathrm{T}} \tag{20}$$

or $G_0(z)=\tfrac{1}{2}H_1(-z)$ and $G_1(z)=-\tfrac{1}{2}H_0(-z)$. Then, we have the following transmission vector:

$$\boldsymbol{f}(z)=\boldsymbol{M}(z)\boldsymbol{C}(z)[1\ \ 0]^{\mathrm{T}}=[\Delta(z)\ \ 0]^{\mathrm{T}}, \tag{21}$$

and the following reconstructed signal:

$$\hat{X}(z)=\Delta(z)X(z). \tag{22}$$

The following general remark can be made.

**Remark.** Choosing the synthesis filters as the first column elements from the cofactor matrix $\boldsymbol{C}(z)$ of the analysis filter matrix $\boldsymbol{M}(z)$ leads to the following properties:

(a) The aliasing is perfectly cancelled.

(b) The input/ouput transfer function is equal to the determinant $\Delta(z)$ of the analysis filter matrix.

Several possibilities are now open in order to achieve a perfect input/output transfer function.

(i) Use a post-filter equal to $1/\Delta(z)$. As shown in Appendix A, this means implicit pole/zero cancellation, which can lead to numerical problems (besides the fact that care has to be taken when choosing the analysis filters so that $1/\Delta(z)$ is a stable filter).

(ii) Use a post-filter $1/\Delta'(z)$ such that $\Delta(z)/\Delta'(z)$ is an all-pass filter. This leads to perfect amplitude reconstruction but to phase distortion.

(iii) Choose the analysis filters in such a way that $\Delta(z)$ is a pure delay, that is, a monomial in $z$ or $z^{-1}$ (a monomial is a polynomial having a single nonzero coefficient). This achieves perfect reconstruction within a delay, and does not have the problems of (i) or (ii).

In the following, we will look for solutions of the third kind, since they meet all our requirements. As shown in Appendix A, only FIR analysis and synthesis filters lead to perfect reconstruction without implicit pole/zero cancellation. Thus, only FIR analysis filters are considered below.

The power of the method shown so far is that the whole problem of filtering/decimation allowing perfect reconstruction has been reduced to investigate properties of the determinant of the filter matrix $\boldsymbol{M}(z)$ from (13).

Assume that $H_0(z)$ and $H_1(z)$ are FIR filters of length $M_0$ and $M_1$, respectively. We define $P(z)$ by

$$P(z) = H_0(z)H_1(-z) = \sum_{i=0}^{M_0+M_1-2} p_i z^{-i}. \tag{23}$$

Then, the determinant $\Delta(z)$ is simply given by

$$\Delta(z) = \tfrac{1}{4}[P(z) - P(-z)] = 2 \sum_{i=0}^{M_s-2} p_{2i+1} z^{-2i-1}, \tag{24}$$

where $M_s = \frac{1}{2}(M_0 + M_1)$ when $M_0 + M_1$ is even and $M_s = \frac{1}{2}(M_0 + M_1 + 1)$ when $M_0 + M_1$ is odd.

Now, if the $p_{2i+1}$ are all zero but one (equal to 2), and the $p_{2i}$ are arbitrary, then the reconstruction will be perfect using the FIR synthesis filters given by (20).

Three methods are now possible in order to derive FIR analysis filters that will allow perfect synthesis with FIR filters. They are simply different ways to meet the requirement that equation (24) should reduce to a monomial.

**Method 1.** The first method is outlined below:

(a) Take a polynomial $P(z)$ satisfying the following conditions:

- degree $= M_0 + M_1 - 2 = M_s - 2$.
- $P_{2i}$ arbitrary,
- $P_{2i+1} = \begin{cases} 0, & i \neq k, \\ 2, & i = k. \end{cases}$

(b) Factorize $P(z)$ into its $M_s - 2$ factors containing one zero each:

$$P(z) = d_0 d_1 \prod_{i=0}^{M_s-3} (z^{-1} + \alpha_i), \tag{25}$$

where $d_0$ and $d_1$ are scalar normalizing factors.

(c) Divide the set of zeros into two sets, and this arbitrarily:

$$P(z) = d_0 \prod_{i=0}^{M_0-2} (z^{-1} + \alpha_i)\, d_1 \prod_{l=M_0-1}^{M_s-3} (z^{-1} + \alpha_l). \tag{26}$$

(d) Set $H_0(z)$ and $H_1(z)$ equal to

$$H_0(z) = d_0 \prod_{i=0}^{M_0-2} (z^{-1} + \alpha_i), \qquad H_1(z) = d_1 \prod_{l=M_0-1}^{M_s-3} (-z^{-1} + \alpha_l). \tag{27), (28}$$

From this construction, we have

$$\Delta(z)=\tfrac{1}{4}[H_0(z)H_1(-z)-H_0(-z)H_1(z)]=\tfrac{1}{4}[P(z)-P(-z)]=z^{-2k-1}, \tag{29}$$

and thus, using $G_0(z)=\frac{1}{2}H_1(-z)$ and $G_1(z)=-\frac{1}{2}H_0(-z)$ as in (20), we find $\hat{X}(z)=X(z)z^{-2k-1}$ at the output.

That this construction can produce reasonable filters is shown in a simple example. Take a perfect halfband filter [18] whose $z$-transform is given by

$$H(z)=2\sum_{i=-\infty}^{\infty}\sin(\tfrac{1}{2}i\pi)/(\tfrac{1}{2}i\pi)z^{-i}. \tag{30}$$

Truncate it below $-2k-1$ and above $2k+1$, apply a window if desired and delay it by $z^{-2k-1}$. The resulting filter $H'(z)$, given as

$$H'(z)=2\sum_{i=0}^{4k+2}\sin(\tfrac{1}{2}(i-2k-1)\pi)/(\tfrac{1}{2}(i-2k-1)\pi)z^{-i}, \tag{31}$$

has the required property described in (a), namely

$$H'(z)-H'(-z)=4z^{-2k-1}. \tag{32}$$

Factorizing $H'(z)$ into its $4k+2$ zeros, grouping conjugate pairs, and distributing the pairs and zeros among $H_0(z)$ and $H_1(-z)$ will produce a low-pass ($H_0$) and a high-pass ($H_1$). This produces a perfect system in the sense described above.

**Method 2.** Another simple method is to choose $H_0(z)$ and then derive $H_1(z)$ by solving the following equation:

$$\Delta(z)=\tfrac{1}{4}[H_0(z)H_1(-z)-H_0(-z)H_1(z)]=z^{-2k-1}, \tag{33}$$

where $k$ can be arbitrarily chosen within the range $0,\ldots,M-2$ (if we assume that both $H_0(z)$ and $H_1(z)$ are length $M$ filters). Then, (33) leads to a set of $M-1$ equations of the following form:

$$M-1\left\{\overbrace{\begin{bmatrix} h_{0,1} & h_{0,0} & 0 & 0 & 0 & \cdots & & & 0 \\ h_{0,3} & h_{0,2} & h_{0,1} & h_{0,0} & 0 & \cdots & & & 0 \\ 0 & & & & & & & & \\ \vdots & & & & & & & & \vdots \\ & & & & & & & & 0 \\ 0 & & & & & & h_{0,M-2} & h_{0,M-3} & h_{0,M-4} \\ 0 & & & \cdots & & & 0 & h_{0,M-1} & h_{0,M-2} \end{bmatrix}}^{M}\right. \cdot \begin{bmatrix} h_{1,0} \\ -h_{1,1} \\ \vdots \\ h_{1,k} \\ \vdots \\ h_{1,M-2} \\ -h_{1,M-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 2 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \tag{34}$$

where $h_{i,j}$ means the $j$th coefficient of the $i$th filter (here, we assume both $k$ and $M$ to be even). Choosing one of the coefficients of $H_1$, for example $h_{1,0}$ equal to 1, leads to an equivalent set of $M-1$ equations for the $M-1$ unknowns $h_{1,1}, \ldots, h_{1,M-1}$:

$$M-1\left\{\overbrace{\begin{bmatrix} h_{0,0} & 0 & 0 & 0 & \cdots & & 0 \\ h_{0,2} & h_{0,1} & h_{0,0} & 0 & \cdots & & 0 \\ 0 & & & & & & \\ \vdots & & & & & & \vdots \\ & & & & & & \\ 0 & \cdots & & h_{0,M-2} & h_{0,M-3} & & h_{0,M-4} \\ 0 & \cdots & 0 & 0 & h_{0,M-1} & & h_{0,M-2} \end{bmatrix}}^{M-1}\right. \cdot \begin{bmatrix} -h_{1,1} \\ h_{1,2} \\ \vdots \\ h_{1,k} \\ \vdots \\ h_{1,M-2} \\ -h_{1,M-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 2 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} h_{0,1} \\ h_{0,3} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}. \tag{35}$$

This system is in general solvable. If not, one has to change one or more coefficients of the filter $H_0(z)$ until the determinant of the matrix in (35) differs from zero.

Since the choice of the arbitrary values in (34) to get (35) can be delicate, it is possible to use a different method to solve the system in (34). If one has an a priori knowledge of what $H_1(z)$ should be, one can add an equation and thus get $M$ equations for the $M$ unknowns of $H_1(z)$. For example, if $H_1(z)$ is a high-pass filter, one can place a zero at $z=1$ by adding an equation stating that the sum of the $h_{1,i}$'s should be zero. Of course, this supplementary equation should be linearly independent of the previous ones, otherwise the system becomes unsolvable.

**Method 3.** Yet another method is to choose the filters $H_0(z)$ and $H_1(z)$ in a class of fiters that will automatically satisfy the determinant constraint. One such class, proposed in [22, 29], consists in choosing $H_0(z)$ such that its autocorrelation function has zero even-index samples (except the zeroth term of course). Assuming $H_0(z)$ of length $M$ and setting $H_1(z)=z^{-M+1}H_0(-z^{-1})$ will satisfy the determinant constraint automatically as can be verified.

To conclude this section on two channel FIR filter banks, we show in Appendix B the case where one of the channels is delayed by $z^{-1}$. Then, the determinant of $\boldsymbol{M}(z)$ is equal to

$$\Delta(z)=\tfrac{1}{4}[H_0(z)H_1(-z)+H_0(-z)H_1(z)]. \tag{36}$$

Thus, all results can be carried over by simply noting that $\Delta(z)$ is a polynomial with even-index samples only, which corresponds to the previously analysed determinant delayed by one sample. Still, by using the powerful determinant analysis, we show also in Appendix B that there exists a linear phase solution for the two channel case, but only for even-length filters, where $H_0(z)$ and $H_1(z)$ have different symmetry.

## 4. The $N$ channel case

While the two channel case described above could be treated without using matrix notation, the $N$ channel case calls for a matrix approach. As will be seen, the representation form of the filter bank problem is of crucial importance, and this for the following three reasons:

(a) understanding of the system and physical interpretation,

(b) analysis of the computational complexity,

(c) numerical properties of the system.

Thus, we will introduce different representations of the problem, namely:

(a) the modulated filter matrix $\boldsymbol{M}(z)$, which is the initial representation of the problem of perfect reconstruction with aliasing cancellation,

(b) the polyphase filter matrix $\boldsymbol{P}(z)$, which gives physical insight into the problem and allows a simpler mathematical treatment,

(c) the diagonal filter matrix $\boldsymbol{D}(z)$, which appears when the filter bank is obtained from a single prototype filter by modulation and permits a treatment of the computational complexity.

The various representations are obtained with multiplication by Fourier matrices and can be seen as basis changes.

Assume a system as depicted in Fig. 2. Similarly to equations (5)-(7), we get, for the $i$th channel,

$$\hat{X}_i = (1/N)\left[\sum_{k=0}^{N-1} H_i(W^k z)X(W^k z)\right]G_i(z), \quad W = \mathrm{e}^{-\mathrm{j}2\pi/N}. \tag{37}$$

Now, the reconstructed signal $\hat{X}(z)$ is a linear combination of $X(z)$ and its $N-1$ aliased components $X(W^k z)$, or, similarly to (9),

$$\hat{X}(z) = \sum_{k=0}^{N-1} F_k(z)X(W^k z), \tag{38}$$

where

$$F_i(z) = (1/N)\sum_{l=0}^{N-1} H_l(W^i z)G_l(z). \tag{39}$$

Again, we want $F_0(z)$ to be equal to a perfect delay and $F_i(z)$, $i \neq 0$, to be equal to zero. In matrix notation, this leads to the following system:

$$\begin{bmatrix} F_0(z) \\ F_1(z) \\ \vdots \\ F_{N-1}(z) \end{bmatrix} = (1/N)\begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{N-1}(z) \\ H_0(Wz) & H_1(Wz) & \cdots & H_{N-1}(Wz) \\ \vdots & \vdots & & \vdots \\ H_0(W^{N-1}z) & & \cdots & H_{N-1}(W^{N-1}z) \end{bmatrix} \cdot \begin{bmatrix} G_0(z) \\ G_1(z) \\ \vdots \\ G_{N-1}(z) \end{bmatrix} = \begin{bmatrix} z^{-k} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{40}$$

or

$$\boldsymbol{F}(z) = \boldsymbol{M}(z)\boldsymbol{g}(z) = \boldsymbol{u}(z), \tag{41}$$

with

$$\boldsymbol{f}_i(z) = F_i(z), \quad \boldsymbol{M}_{i,j}(z) = (1/N)H_j(W^i z), \quad \boldsymbol{g}_i(z) = G_i(z), \quad \boldsymbol{u}(z) = [z^{-k}\ 0 \ldots 0].$$

Again, we assume the $H_i(z)$ to be FIR filters, and we choose $\boldsymbol{g}(z)$ to be

$$\boldsymbol{g}(z) = \boldsymbol{C}(z)[1\ 0 \ldots 0]^{\mathrm{T}}, \tag{42}$$

where $\boldsymbol{C}(z)$ is the cofactor matrix of $\boldsymbol{M}(z)$. Then, the reconstructed signal is equal to, similarly to (22),

$$\hat{X}(z) = \Delta(z)X(z), \tag{43}$$

where $\Delta$ is the determinant of the filter matrix $\boldsymbol{M}(z)$.

Similarly to (22) and Fig. 5, the determinant can be cancelled at the output, thus yielding perfect reconstruction. Note that this implies implicit pole/zero cancellation. If this is undesired (for the reasons explained in Appendix A), we require that both the analysis and synthesis filters be FIR, and, therefore, we want the determinant to reduce to a pure delay.

In order to analyse the determinant of the modulated filter matrix $\boldsymbol{M}(z)$, we premultiply $\boldsymbol{M}(z)$ by the Fourier matrix $\boldsymbol{F}$ and we obtain the polyphase filter matrix representation

$$\boldsymbol{P}(z)=\boldsymbol{F}\boldsymbol{M}(z), \tag{44}$$

where:

$$\boldsymbol{P}_{ij}(z)=z^{-j}H_{ij}(z^N), \qquad \boldsymbol{F}_{n,m}=W^{nm}, \tag{45}$$

and $H_{ij}(z^N)$ is the $j$th polyphase component of the $i$th filter, that is, the filter

$$H_{ij}(z^N)=h_{i,j}+h_{i,j+N}z^{-N}+h_{i,j+2N}z^{-2N}+\cdots. \tag{46}$$

The $i$th filter $H_i(z)$ is related to its polyphase components in the following way:

$$H_i(z)=\sum_{j=0}^{N-1} z^{-j}H_{i,j}(z^N) \tag{47}$$

and, reciprocally,

$$H_{i,j}(z^N)=z^j(1/N)\sum_{k=0}^{N-1} H_i(W^{jk}z). \tag{48}$$

Fig. 6 shows a pictorial representation of the polyphase filter interpretation of the filtering/subsampling process, and an equivalent figure could be drawn for the upsampling/filtering process.

Now, the determinants $\Delta_{\mathrm{p}}(z)$ of $\boldsymbol{P}(z)$ and $\Delta(z)$ of $\boldsymbol{M}(z)$ are equal, except for a constant factor given by the determinant $\Delta_{\mathrm{f}}$ of $\boldsymbol{F}$:

$$\Delta_{\mathrm{p}}(z)=\Delta_{\mathrm{f}}\Delta(z). \tag{49}$$

Similarly, the inverses and cofactor matrices are related:

$$[\boldsymbol{P}(z)]^{-1}=[\boldsymbol{M}(z)]^{-1}\boldsymbol{F}^{-1}, \qquad \boldsymbol{C}_{\mathrm{p}}(z)=\frac{\Delta_{\mathrm{p}}(z)}{\Delta(z)}\boldsymbol{C}(z)\boldsymbol{F}^{-1}. \tag{50), (51}$$

Now, we choose the reconstruction filters as

$$\boldsymbol{g}(z)=\boldsymbol{C}_{\mathrm{p}}(z)[1\ \ 1\ \ 1\dots1]^{\mathrm{T}}. \tag{52}$$

Then, the transmission vector $\boldsymbol{F}(z)$ becomes

$$\begin{aligned}\boldsymbol{f}(z)&=\boldsymbol{M}(z)\boldsymbol{C}_{\mathrm{p}}(z)[1\ \ 1\ \ 1\dots1]^{\mathrm{T}}=\Delta_{\mathrm{p}}(z)\boldsymbol{M}(z)(1/\Delta(z))\boldsymbol{C}(z)\boldsymbol{F}^{-1}[1\ \ 1\ \ 1\dots1]^{\mathrm{T}}\\&=[\Delta_{\mathrm{p}}(z)\ \ 0\ \ 0\dots0]^{\mathrm{T}},\end{aligned} \tag{53}$$

and the reconstructed signal is equal to

$$\hat{X}(z)=\Delta_{\mathrm{p}}(z)X(z). \tag{54}$$

Therefore, we can consider the polyphase filter matrix alone, thus avoiding the matrix $\boldsymbol{M}(z)$ which is redundant (each filter coefficient appearing $N$ times) and in general complex.

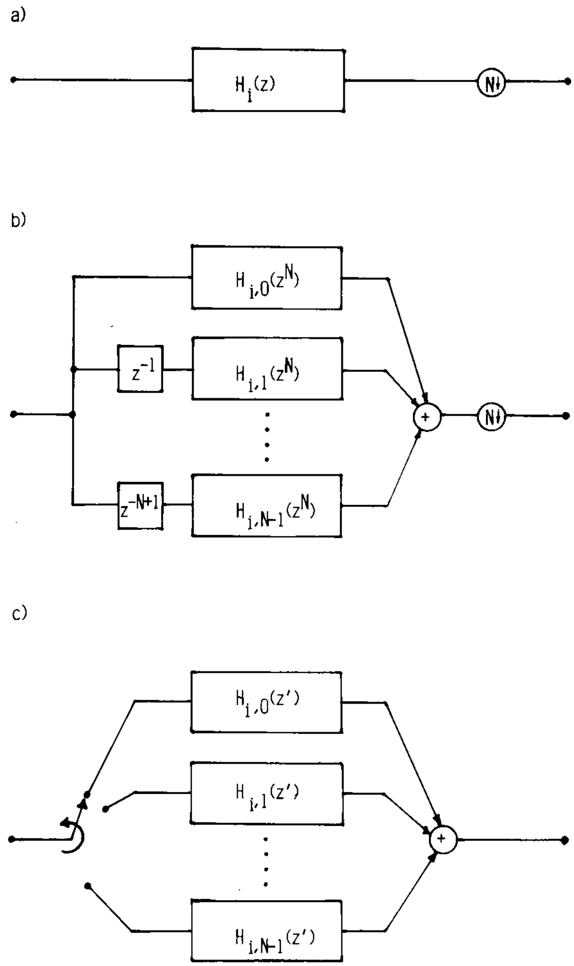


Fig. 6. Equivalence between subsampled FIR filtering and polyphase representation. (a) Initial FIR filter. (b) Equivalent FIR filter. (c) Equivalent polyphase filters.

Consider the determinant of the polyphase filter matrix $\boldsymbol{P}(z)$. Instead of the usual recursive determinant formula, we use the one which consists of the sum of all possible products where not two terms are taken from the same row or column [24]. The determinant of an $N \times N$ matrix $\boldsymbol{A}$ becomes the sum of $N!$ terms:

$$\det[\boldsymbol{A}] = \sum_{\theta} (a_{0,\alpha} a_{1,\beta} \cdots a_{N-1,\gamma}) \det[K_\theta], \tag{55}$$

where $K_\theta$ is a permutation matrix indicating which terms appear in the $\theta$th product. The determinant of $K_\theta$ is either 1 or $-1$ and $\alpha, \beta, \ldots, \gamma$ are all different (because there are no repetitions). Using this formula for the determinant, we can write $\Delta_{\mathrm{p}}(z)$ as follows:

$$\Delta_{\mathrm{p}}(z) = z^{-N(N-1)/2} \sum_{\theta} (H_{\alpha,0}(z^N) H_{\beta,1}(z^N) \cdots H_{\gamma,N-1}(z^N)) \det[K_\theta]. \tag{56}$$

The factor $z^{-N(N-1)/2}$ is obtained because we take an element from each row of the matrix $\boldsymbol{P}(z)$: an element from the $i$th row has an associated delay of $z^{-i}$ (from (45)) and the product of $N$ elements from different rows produces a delay equal to $1 \times z^{-1} \cdots z^{-N+1}$, that is, $z^{-N(N-1)/2}$. The product of the $N$ polyphase filters in (56) is a polynomial in $z^{-N}$, since each polyphase filter is already a polynomial in $z^{-N}$. As can be verified, the lowest power of this product is zero, and the highest is $N(M-N)$ when all $N$ filters are of length $M$. Thus, the determinant of the polyphase filter matrix $\boldsymbol{P}(z)$ has at most $M-N+1$ nonzero coefficients, because only coefficients with indexes multiple of $N$ can be different from zero. Therefore, we can rewrite $\Delta_p(z)$ in the following manner:

$$\Delta_p(z) = d_0 z^{-N(N-1)/2} + d_1 z^{-N(N-1)/2-N} + \cdots + d_{M-N} z^{-NM+N(N+1)/2}. \tag{57}$$

Now, the requirement that the overall transmission should be a delay means that the $M-N+1$ coefficients of $\Delta_p(z)$ should all be zero but one. Therefore, $M-N+1$ equations have to be satisfied, but we have $N$ filters with $M$ coefficients each, that is, $NM$ unknowns. Note that the $z$-transforms of the filters are multiplied with one another, that is, the equations to be solved are nonlinear. In order to reduce these nonlinear equations to a linear system of equations of dimension $M-N+1$ with $M-N+1$ unknowns, one can perform the following steps:

(a) Choose the coefficients appearing in $N-1$ columns of $\boldsymbol{P}(z)$, that is, choose $N-1$ filters (actually, one could also choose $N-1$ rows, that is $N-1$ polyphase components of the $N$ filters).

(b) Choose $N-1$ coefficients of the last filter (or of the remaining polyphase components).

In this manner, there are only $M-N+1$ unknowns left, and it can be seen that the corresponding equations are now linear. Note that the resulting system is not always solvable (depending on the a priori coefficients), but that these singularities are rather scarce.

As an example, we derive a filter bank for $N=3$ and $M=7$. For simplicity, we take the following analysis filters:

$$H_0(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6}, \tag{58}$$

$$H_1(z) = 1 - z^{-1} + z^{-2} - z^{-3} + z^{-4} - z^{-5} + z^{-6}, \tag{59}$$

$$H_2(z) = 1 + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} + h_4 z^{-4} + h_5 z^{-5} + z^{-6}. \tag{60}$$

One can verify that the determinant of the equivalent polyphase filter matrix $\boldsymbol{P}(z)$, given as

$$\boldsymbol{P}(z) = \begin{bmatrix} H_{0,0}(z^3) & H_{1,0}(z^3) & H_{2,0}(z^3) \\ z^{-1}H_{0,1}(z^3) & z^{-1}H_{1,1}(z^3) & z^{-1}H_{2,1}(z) \\ z^{-2}H_{0,2}(z^3) & z^{-2}H_{1,2}(z^3) & z^{-2}H_{2,2}(z^3) \end{bmatrix}, \tag{61}$$

is equal to

$$\Delta_p(z) = \tfrac{2}{3}[(h_4-1)z^{-15} + (h_1-h_3)z^{-12} + (h_3-h_5)z^{-6} + (1-h_2)z^{-3}]. \tag{62}$$

Setting $h_1=1$, $h_2=-1$, $h_3=1$, $h_4=1$, and $h_5=1$, reduces (62) to

$$\Delta_p(z) = \tfrac{4}{3}z^{-3}. \tag{63}$$

From the cofactor matrix $C_p(z)$ we obtain the following synthesis filters (where unnecessary delays were cancelled):

$$G_0(z) = \tfrac{1}{6}[1 + z^{-1} - z^{-4} - z^{-5} + z^{-7} + z^{-8} - z^{-10}], \tag{64}$$

$$G_1(z) = \tfrac{1}{6}[-z^{-1} + z^{-2} - z^{-4} + z^{-5} - z^{-7}], \tag{65}$$

$$G_2(z) = \tfrac{1}{6}[-1 + z^{-2} - z^{-8} + z^{-10}]. \tag{66}$$

Using these filters in a system like the one depicted in Fig. 2 leads to zero aliasing transmission, and the reconstructed signal is equal to

$$\hat{X}(z)=z^{-2}X(z). \tag{67}$$

Note that, in this example, $H_0(z)$ and $H_1(z)$ are 'reasonable', since we could choose them, but that $H_3(z)$ is dictated by solving (62) to a monomial, and can be 'strange'. Also, there is a problem with the size of the synthesis filters. Since they are obtained from the cofactor matrix, their length is upperbounded by $(M-1)(N-1)+1$ in the worst case, that is, in general much longer than the analysis filters. Only in the case $N=2$ they are guaranteed to be of the same length as the analysis filters.

In practice, both for the computational ease and for application reasons, one would like to have filters which are modulated versions of a single prototype filter $H_p(z)$:

$$h_{i,n}=W^{in}h_{p,n}, \qquad H_i(z)=H_p(W^i z). \qquad (68),(69)$$

Thus, if $H(z)$ is a lowpass filter (with a bandwidth of the order of $1/N$), then the $H_i(z)$ are bandpass filters displaced by $i/N$.

Unfortunately, when the filters are chosen as in (68) and (69), the determinant cannot be a pure delay unless $N=M$ (note that in the solution from [22, 29] for $N=2$, the second filter is modulated but also time reversed, and thus the method cannot readily be generalized to $N>2$).

Consider the polyphase filter matrix in the case when the filters are modulated. Then, $\boldsymbol{P}(z)$ has the following form:

$$\boldsymbol{P}_{n,m}(z)=W^{-nm}z^{-n}H_{p,n}(z^N), \tag{70}$$

where $H_{p,n}(z^N)$ is the $n$th polyphase component of the prototype filter $H_p(z)$. Postmultiplying $\boldsymbol{P}(z)$ by the Fourier matrix and dividing by $N$ yields the diagonal filter matrix $D(z)$, or, expressed in terms of $\boldsymbol{M}(z)$,

$$\boldsymbol{D}(z)=(1/N)\boldsymbol{F}\boldsymbol{M}(z)\boldsymbol{F}=\begin{bmatrix} H_{p,0}(z^N) & & & 0 \\ & z^{-1}H_{p,1}(z^N) & & \\ 0 & & & z^{-N+1}H_{p,N-1}(z^N)\end{bmatrix}. \tag{71}$$

Now, the determinants of $\boldsymbol{D}(z)$ and $\boldsymbol{M}(z)$ are equal except for a sign factor:

$$\Delta_d(z)=\begin{cases} \Delta(z), & \lfloor\frac{1}{2}(N+1)\rfloor \text{ odd}, \\ -\Delta(z), & \lfloor\frac{1}{2}(N+1)\rfloor \text{ even}, \end{cases} \tag{72}$$

as can be verified by evaluating the determinant of the Fourier matrix. Now, $\Delta_d(z)$ is equal to

$$\Delta_d(z)=z^{-N(N-1)/2}\prod_{i=0}^{N-1}H_{p,i}(z^N), \tag{73}$$

and, in order for $\Delta_d(z)$ to be a monomial, all the factors of the product in (73) have to be monomials. Thus, $H_p(z)$ cannot have more than $N$ nonzero coefficients (actually, it requires exactly one nonzero coefficient in each polyphase filter, otherwise $\Delta_d(z)$ is either zero or not a monomial).

If we allow infinite impulse response (IIR) synthesis filters, we can invert $\boldsymbol{D}(z)$ and use the following synthesis filters:

$$\begin{aligned} g(z)&=z^{-N+1}[\boldsymbol{M}(z)]^{-1}[1\ 0\dots0]^{\mathrm{T}} \\ &=z^{-N+1}\boldsymbol{F}[[H_{p,0}(z^N)]^{-1},[z^{-1}H_{p,1}(z^N)]^{-1},\dots,[z^{-N+1}H_{p,N-1}(z^N)]^{-1}]^{\mathrm{T}}, \end{aligned} \tag{74}$$

from where the reconstructed signal $\hat{X}(z)$ follows to

$$\hat{X}(z) = z^{-N+1} X(z). \tag{75}$$

Note that, in (74), it is required that $[\boldsymbol{D}(z)]^{-1}$ is stable, or that each polyphase filter has its zeroes strictly within the unit circle.

Of course, in the case of modulated filters, both the analysis and the synthesis filter bank can be computed with reduced computational complexity, by using the polyphase/FFT approach first introduced in [1]. In the analysis filter bank, one has to compute the following result:

$$\begin{bmatrix} Y_0(z) \\ Y_1(z) \\ \vdots \\ Y_{N-1}(z) \end{bmatrix} = \begin{bmatrix} H_{p,0}(z^N) & z^{-1}H_{p,1}(z^N) & z^{-N+1}H_{p,N-1}(z^N) \\ H_{p,0}(z^N) & W^{-1}z^{-1}H_{p,1}(z^N) & W^{-N+1}z^{-N+1}H_{p,N-1}(z^N) \\ \vdots & \vdots & \vdots \\ H_{p,0}(z^N) & \cdots & \end{bmatrix} \cdot \begin{bmatrix} X(z) \\ X(z) \\ \vdots \\ X(z) \end{bmatrix}, \tag{76}$$

which is equal to

$$\begin{bmatrix} Y_0(z) \\ Y_1(z) \\ \vdots \\ Y_{N-1}(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W & \cdots & W^{N-1} \\ \vdots & \vdots & & \vdots \\ 1 & W^{N-1} & \cdots & \end{bmatrix} \cdot \begin{bmatrix} H_{p,0}(z^N)X(z) \\ z^{-1}H_{p,1}(z^N)X(z) \\ \vdots \\ z^{-N+1}H_{p,N-1}(z^N)X(z) \end{bmatrix}, \tag{77}$$

or, assuming that the prototype filter length is a multiple of $N$ such that $M = kN$, the computational load per set of subsampled output samples is equivalent to the evaluation of $N$ polyphase filters of length $k$ as well as an FFT of size $N$.

In the case of the synthesis filter bank, the output $\hat{X}(z)$ is obtained from the channel signals $Y_i'(z)$ in the following way:

$$\begin{aligned} \hat{X}(z) &= [\boldsymbol{g}(z)]^{\mathrm{T}} [Y_0'(z) \;\; Y_1'(z) \;\; \ldots \;\; Y_{N-1}'(z)]^{\mathrm{T}} \\ &= z^{-N+1}[1 \;\; 1 \ldots 1][\boldsymbol{D}(z)]^{-1}\boldsymbol{F}[Y_0(z) \;\; \ldots \;\; Y_{N-1}(z)]^{\mathrm{T}}, \end{aligned} \tag{78}$$

that is, with a Fourier transform and $N$ all-pole filters. Note that the $Y_i'(z)$ have only every $N$th sample different from zero, thus the synthesis requires the same order of complexity as the analysis. Fig. 7 shows an $N$ channel system with modulated filters computed by Fourier transforms.

If one does not want IIR solutions, one can evaluate the cofactor matrix in order to get FIR synthesis filters and neglect the determinant. Note that the synthesis filters are also modulated in that case, thus allowing efficient implementations (even if they are longer than the analysis filters). Of course, approximations can be made, and especially products of filters which are nonadjacent can be regarded as being zero. This approach is known as 'pseudo-QMF' filter banks [12, 13, 15, 20, 30] and allows both efficient implementation and quasi-QMF behaviour. While that kind of filters requires further investigation, the above framework is certainly helpful.

Concluding this section, we recall that the proposed approach always allows the cancelling of the aliasing in a subsampled filter bank (by choosing the synthesis filters accordingly to the cofactor matrix) and that FIR analysis/synthesis systems can be found for $N > 2$ and $M > N$. Finally, the modulated filter bank case does not have an FIR solution, but has an efficient analysis and synthesis implementation.

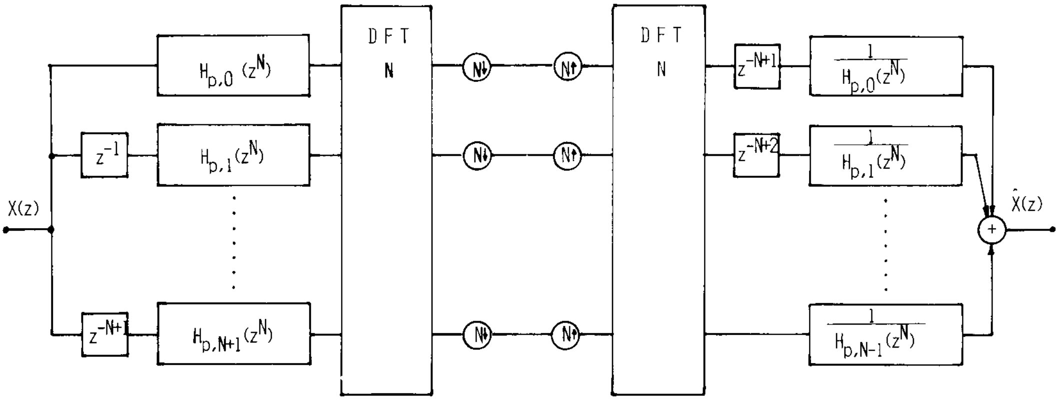


Fig. 7. Modulated filter bank implemented with polyphase filters and DFTs.

## 5. The dual problem

In the previous sections, we have considered the problem of separating a signal (at sampling frequency $f$) into $N$ channel signals (at sampling frequency $f/N$) by filtering, in such a way that the original can be perfectly reconstructed.

In the following, we look at the dual problem which consists of multiplexing $N$ signals (with initial sampling frequency $f$) into one signal (at sampling frequency $Nf$) by filtering, in a way that permits the perfect reconstruction of the $N$ original signals (Fig. 8). Note that the two obvious solutions from Section 2 hold as well here. Both when the filters are perfect bandpass or length $M = N$ FIR filters, the signal can be reconstructed perfectly. Again, the interesting case appears for $M > N$, but not perfectly bandpass.

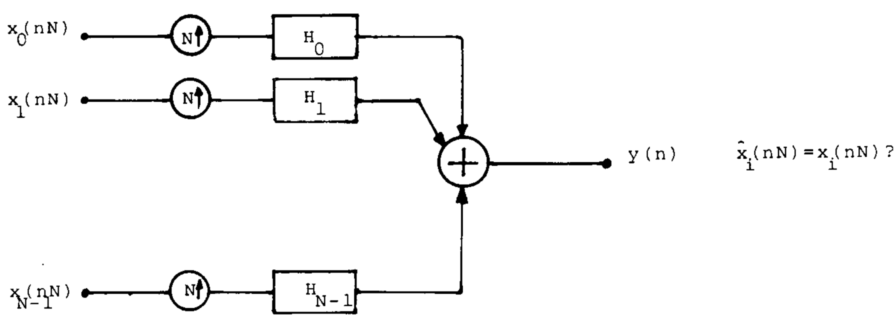


Fig. 8. General dual problem statement. Can the initial signals be recovered from the upsampled channel?

The case $N = 2$ is investigated below. Assume a system as depicted in Fig. 9. Then, using (2) and then (1), the signals in Fig. 9 equal

$$X_i'(z) = X_i(z^2), \quad i = 0, 1, \tag{79}$$

$$Y(z) = H_0(z)X_0(z^2) + H_1(z)X_1(z^2), \tag{80}$$

$$\hat{X}_i'(z) = G_i(z)Y(z), \quad i = 0, 1, \tag{81}$$

$$\hat{X}_i(z) = \tfrac{1}{2}[\hat{X}_i'(z^{1/2}) + \hat{X}_i'(-z^{1/2})], \tag{82}$$

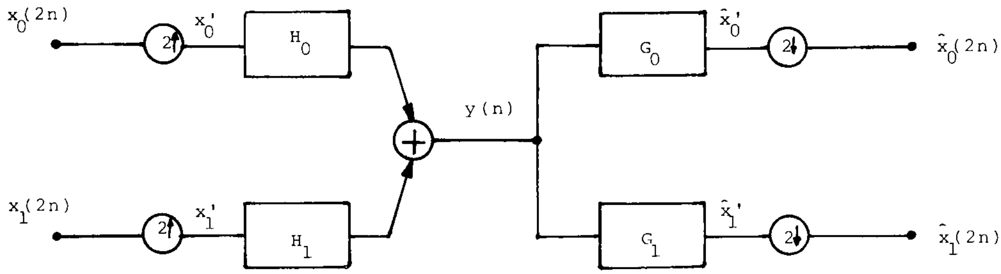


Fig. 9. Solution to the two channel dual problem.

or, explicitly,

$$\hat{X}_i(z)=\tfrac{1}{2}(G_i(z^{1/2})[H_0(z^{1/2})X_0(z)+H_1(z^{1/2})X_1(z)] + G_i(-z^{1/2})[H_0(-z^{1/2})X_0(z)+H_1(-z^{1/2})X_1(z)]),\quad i=0,1. \tag{83}$$

Thus, a necessary and sufficient condition in order to recover $\hat{X}_i(z)$ equal to $X_i(z)$ is the following:

$$\tfrac{1}{2}[G_i(z^{1/2})H_j(z^{1/2})+G_i(-z^{1/2})H_j(-z^{1/2})]=\begin{cases} z^{-k}, & i=j,\\ 0, & i\neq j.\end{cases} \tag{84}$$

Below, for reasons similar to those mentioned in Section 3 and Appendix A, we consider the case where all involved filters are FIR. Introducing the following notation:

$$T_{i,j}(z)=H_i(z)G_j(z),\quad i,j=0,1, \tag{85}$$

it is easy to see that (84) is equivalent to say,

$$\tfrac{1}{2}[T_{i,j}(z)+T_{i,j}(-z)]=\begin{cases} z^{-2k}, & i=j,\\ 0, & i\neq j,\end{cases} \tag{86}$$

or simply that:
- $T_{i,i}(z)$ has a single even-index coefficient different from zero, while having arbitrary odd-index ones,
- $T_{i,j}(z)$, $i\neq j$, has all even-index coefficients equal to zero, while having arbitrary odd-index ones.

This has an obvious interpretation: $X_i'(z)$ has only information on the even-index terms of its $z$-transform (being zero otherwise because of the upsampling) and all odd-index terms of $\hat{X}_i'(z)$ are disregarded due to the subsampling. Thus, if the transmission from input $i$ to output $i$ has a single even coefficient and the transmission from $i$ to $j$ has no even coefficient different from zero, then an impulse appearing (on even time) at the input of filter $i$ will be transmitted to output $i$ only, and this without distortion. Thus, the signal at input $i$ is perfectly reconstructed at output $i$, disregarding a delay.

Calling $H_{ie}(z)$ and $H_{io}(z)$ the polynomials incorporating the even and odd parts of $H_i(z)$, that is,

$$H_{ie}(z)=\tfrac{1}{2}[H_i(z)+H_i(-z)],\qquad H_{io}(z)=\tfrac{1}{2}[H_i(z)-H_i(-z)], \tag{87),(88}$$

and noting that (86) includes only the even part of $T_{i,j}(z)$, we rewrite (86) as follows:

$$\tfrac{1}{2}[H_{ie}(z)G_{je}(z)+H_{io}(z)G_{jo}(z)]=\begin{cases} z^{-2k}, & i=j,\\ 0, & i\neq j.\end{cases} \tag{89}$$

This leads to the following matrix equation:

$$\frac{1}{2}\begin{bmatrix} H_{0e}(z) & H_{0o}(z) & 0 & 0 \\ H_{1e}(z) & H_{1o}(z) & 0 & 0 \\ 0 & 0 & H_{0e}(z) & H_{0o}(z) \\ 0 & 0 & H_{1e}(z) & H_{1o}(z) \end{bmatrix} \cdot \begin{bmatrix} G_{0e}(z) \\ G_{0e}(z) \\ G_{1e}(z) \\ G_{1o}(z) \end{bmatrix} = \begin{bmatrix} z^{-2k} \\ 0 \\ 0 \\ z^{-2k} \end{bmatrix}. \tag{90}$$

Then, it is sufficient to invert the $2 \times 2$ nonzero blocks in (90). For example, $G_0(z)$ is obtained from

$$\begin{bmatrix} G_{0e}(z) \\ G_{0o}(z) \end{bmatrix} = [1/(2\Delta(z))] \begin{bmatrix} H_{1o}(z) & -H_{0o}(z) \\ -H_{1e}(z) & H_{0e}(z) \end{bmatrix} \cdot \begin{bmatrix} z^{-2k} \\ 0 \end{bmatrix}, \tag{91}$$

where

$$\Delta(z) = \tfrac{1}{4}[H_{0e}(z)H_{1o}(z) - H_{0o}(z)H_{1e}(z)]. \tag{92}$$

Now, if $\Delta$ is a delay (actually an odd delay since $\Delta$ is an odd function of $z$) and the output filters are chosen accordingly to (90) and (91), then both $\hat{X}_0(z)$ and $\hat{X}_1(z)$ are equal to $X_0(z)$ and $X_1(z)$ within a delay.

Note that the matrix in (91) is the cofactor matrix of the $2 \times 2$ block in (90). A remark is again appropriate here. Choosing

$$[G_{0e}(z) \;\; G_{0o}(z)]^{\mathrm{T}} = [H_{1o}(z) \;\; -H_{1e}(z)]^{\mathrm{T}} z^{-1}, \tag{93}$$

$$[G_{1e}(z) \;\; G_{1o}(z)]^{\mathrm{T}} = [-H_{0o}(z) \;\; H_{0e}(z)]^{\mathrm{T}} z^{-1} \tag{94}$$

(or respectively as the first and second column of the cofactor matrix) leads to the following properties of the system in Fig. 9 (note the $z^{-1}$ factor in order to be in phase with the subsampling):

- there is no crosstalk from one input channel to another output channel,
- the transmission of a channel to its output is equal to the determinant of the submatrix in (90) times $z^{-1}$.

The above method can be extended to the case $N > 2$. Due to the upsampling and subsampling by $N$, one is only interested in the transmission at every multiple of $N$. This leads, through relations similar to (86)-(90), to the inversion of a matrix of the following form:

$$M_{\mathrm{d}}(z) = (1/N) \begin{bmatrix} H_{0,0}(z) & H_{0,1}(z) & \dots & H_{0,N-1}(z) \\ H_{1,0}(z) & & & \\ \vdots & \vdots & & \vdots \\ H_{N-1,0}(z) & & \dots & \end{bmatrix}, \tag{95}$$

where

$$H_{i,j}(z) = h_{i,j} z^{-j} + h_{i,j+N} z^{-j-N} + \cdots. \tag{96}$$

Again, one can choose the synthesis filters as elements of the cofactor matrix (thus eliminating the cross over), and the determinant can be reduced to a delay, thus allowing perfect reconstruction.

Obviously, the dual and the initial problem are closely related, since both require the inversion of a polyphase filter matrix. Therefore, the same analysis filters can be used for the initial and the dual problem, and the synthesis filters in the dual problem can be deduced from the ones of the initial problem by some scaling and shifting.

## 6. Applications

The purpose of this section is not to show impletation results, but rather to point to potential applications.

### *6.1. Subband coder incorporating an LPC filter*

It was shown in Section 3 that, in the two channel case, one could choose one filter, $H_0(z)$, and derive the other, $H_1(z)$, by solving a linear system of equations given in (35). An interesting choice for $H_0(z)$ is the following:

$$H_0(z) = 1 - L(z), \tag{97}$$

where

$$L(z) = l_1 z^{-1} + l_2 z^{-2} + \cdots + l_{N-1} z^{-N+1}. \tag{98}$$

$L(z)$ is the filter whose current output is the best linear prediction of the current sample value from the previous $N-1$ samples [11]. The output of $H_0(z)$ is the so-called residual, that is, loosely stated, the 'unpredictable' part of the signal.

Now, $H_1(z)$ is evaluated by solving (35). In some sense, the output of $H_1(z)$ is the complementary signal with regard to the residual. Both signals are then subsampled by 2 and the synthesis filters are chosen according to (20). The system is depicted in Fig. 10. This approach is currently under investigation, but sufficient results are not yet available to judge the power of the method. The idea is simply to split a signal into its residual and complementary parts, in a way similar to the splitting into low and high pass components with a conventional QMF bank. It is hoped that the matching of the filters to the signal (rather than having fixed filters as in a conventional system) will improve the coding gain without destroying the quality associated with subband coders. Of course, the process can be iterated, especially on the complementary signal, since the subsampled residual might be crudely quantized.

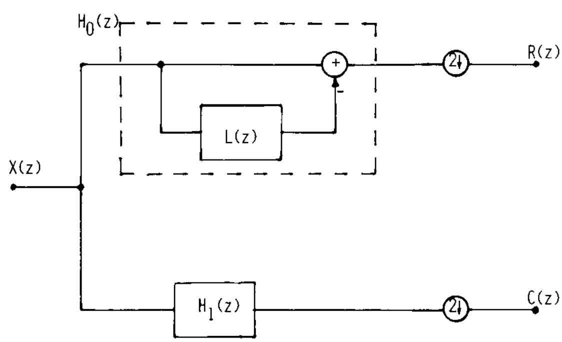


Fig. 10. Example of a subband coder using an LPC filter. $R(z)$ is the subsampled residual and $C(z)$ is the complementary signal.

### *6.2. Scrambling of analog signals*

When scrambling analog signals, the bandwidth should not grow. Therefore, methods have been proposed where the analog signal is filtered into bands which are then interchanged. This can be done in the analog or in the digital domain. Using the filterbanks developed so far, which allow separation or mixing of signals permitting exact recovery, one can devise such a scrambling system. Since the filters are not restricted to bandpass filters, complex functions can be used, which can make decryption even more difficult. Of course, since the transmitted signal is analog and will be distorted, the reconstruction will not be perfect.

The system, depicted in Fig. 11, works as follows. The signal is first split into $N$ subsampled channels using filters as developed in Section 3. Then, the channels are permuted and mixed together as described in Section 5. Note that the splitting and the mixing filters could be similar, but are different in general. At the receiver, the mixing is undone first, then the channels are backpermuted, and, finally, the initial splitting is undone as well, thus reproducing the original. In Fig. 11, the case for $N=4$ is shown (for practical purposes, $N$ should be bigger, of course), using two stages of two filters each. The number of possible permutations is

$$N_p = N! \tag{99}$$

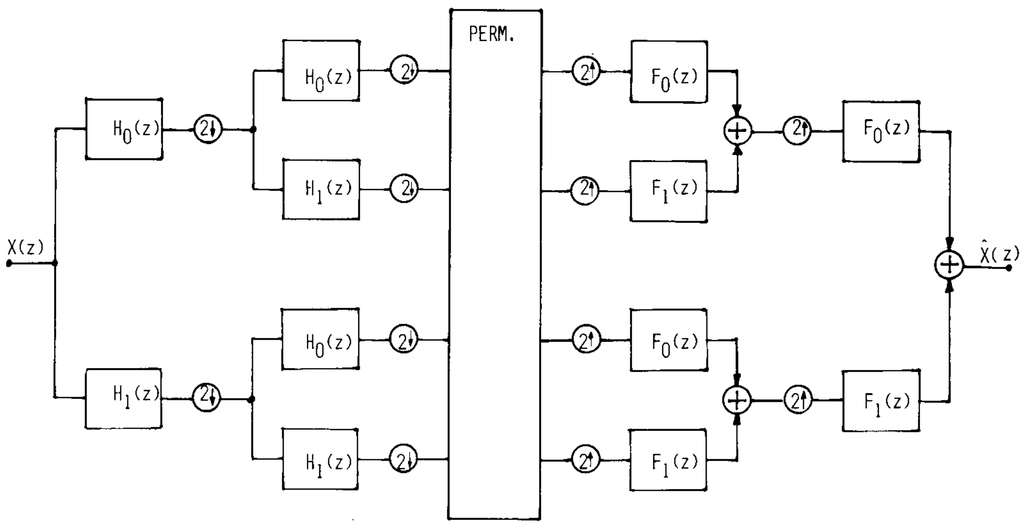


Fig. 11. Example of scrambling. After downsampling and permutation, the signals are recombined to form the channel signal.

## 6.3. *Filter banks with noncritical subsampling*

A filter bank is critically subsampled when the output of an $N$ *channel* bank is subsampled by $N$. Noncritical subsampling is the case when the subsampling factor $N'$ is smaller than $N$, the number of channels. In the latter case, we obtain a rectangular system of $N'$ equations involving $N$ filters instead of the square system of (40). Obviously, since there are less contraints, solutions will exist as well (ad absurdum, one could take $N-N'$ filters equal to zero, and solve (40) of dimension $N'$). The new degrees of freedom can be used to meet new, self-imposed constraints. Take as a simple example a two channel filter bank without subsampling. Then, the only equation to be met is

$$H_0(z)G_0(z)+H_1(z)G_1(z)=z^{-k}. \tag{100}$$

Choosing, for example, $H_0(z)=H(z)$ and $H_1(z)=H(-z)$, where $H(z)$ is a length $M$ FIR filter, as well as $G_0(z)=G(z)$ and $G_1(z)=G(-z)$, $G(z)$ also being FIR, leads to the following condition:

$$H(z)G(z)+H(-z)G(-z)=z^{-2k}. \tag{101}$$

This condition can be met using Method 1 of Section 3, for example. Note that the two analysis filters used here would not work in the subsampled case, but are possible here because there is no aliasing cancellation to be met. The point here is to simply show that the approach from the previous sections is quite general and useful for $N'$ going from 1 to $N$.

### 6.4. Fractional sampling rate change

An interesting application of the initial together with the dual system is the fractional sampling rate change. Assume, for example, five signals with sampling frequency $f$ that should be multiplexed onto three channels with sampling frequency $\frac{5}{3}f$. Again, the number of samples per unit of time is the same in the five input channels and in the three output channels. One solution is to first multiplex the five channels onto one channel with sampling frequency $f' = 5f$, and then to demultiplex this channel into three channels, which yields the desired sampling frequency $f'' = \frac{5}{3}f$. Such a system is depicted in Fig. 12. If the filters $H_i(z)$ and $F_i(z)$ meet the requirement of determinants being monomials, a perfect FIR reconstruction of the original signals can be achieved.

### 6.5. Filter banks on finite fields

If one really wants 'perfect' reconstruction, for example in error detection applications, one can resort to arithmetic on finite fields [2]. It is not difficult to generalize the methods shown so far to filter banks on finite fields. To keep things simple, we look at arithmetic modulo a prime number $p$, that is, to arithmetic over GF($p$). Then, all filter coefficients and signal values belong to GF($p$). The $z$-transform is defined as usual. For simplicity, we only look at the two channel case. In order to express the down-upsampling by $N$, we require an $N$th root of unity in GF($p$), which means that $p$ must be strictly greater than $N$. Calling $\alpha$ in GF($p$) the element such that $\alpha^2 = 1$ and $\beta$ such that $\beta + \beta = 1$, we express the down/upsampling by 2 as

$$X'(z) = \beta[X(z) + X(\alpha z)]. \tag{102}$$

Thus, similarly to (12), we have to invert a matrix $\boldsymbol{M}(z)$ equal to

$$\boldsymbol{M}(z) = \begin{bmatrix} H_0(z) & H_1(z) \\ H_0(\alpha z) & H_1(\alpha z) \end{bmatrix}. \tag{103}$$

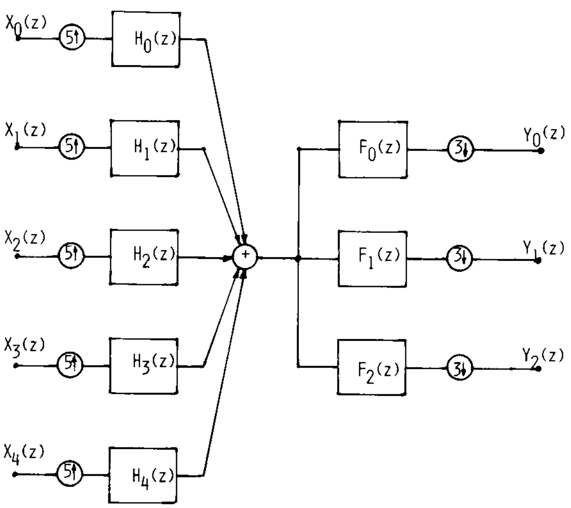


Fig. 12. Example of a fractional sampling rate change. The inputs have sampling frequency $f$ and the outputs a sampling frequency $\frac{5}{3}f$.

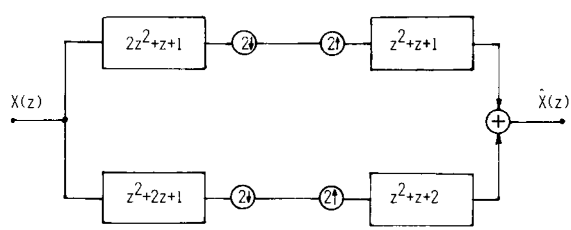


Fig. 13. Two channel system over GF(3).

Its determinant has odd terms only. Thus, if we want it to be a monomial, we start with a polynomial having arbitrary even terms but only a single nonzero odd term. Then, we factor it into two polynomials which can be used as filters $H_0(z)$ and $H_1(z)$. Of course, the factorization is done in GF($p$). Such a simple system is given in Fig. 13, where all computations are done in GF(3).

Note here that implicit pole/zero cancellation (Appendix A) does not cause problems as in the case of systems over the field of real or complex numbers. Thus, one could cancel the determinant with an all-pole filter as well. The generalization for $N$ greater than 2 (when an $N$th root of unity exists in GF($p$)) follows the same lines as in Section 4, where $W$ is now the $N$th root of unity in GF($p$).

## 7. Conclusion

The problem of subsampled filter banks has been addressed by developing an analysis framework. Using a powerful matrix notation, several new results were obtained, especially for FIR analysis and synthesis filters. In that case, the $N \times M$ nonlinear design problem (where $N$ is the number of channels, and $M$ the filter length) was shown to reduce to an $M$-dimensional linear problem. This is obtained by analysing the matrix inversion (where the matrix elements are polynomials) and, in particular, the determinant of the matrix.

For $N = 2$, the class of FIR solutions allowing perfect reconstruction has been demonstrated, which includes the solutions known so far [22, 29], but also new solutions (for example, a solution with two linear phase analysis filters). Two new methods are given in order to find actual filters. The original QMF filters [5, 6, 7, 8] are shown to be the solution when the synthesis filters are obtained from the cofactor matrix (but neglecting the determinant), and this method can be generalized to arbitrary $N$, thus allowing perfect aliasing cancellation. Furthermore, for $N > 2$, it is shown that the determinant can be reduced to a pure delay (by solving a linear system of equations of size $M$), thus allowing perfect reconstruction with FIR analysis and synthesis filters. Actual solutions have been shown, but note that while there is no reason to doubt the existence of a solution for arbitrary $N$ and $M$, the conjecture has yet to be proven.

The case where the filters are obtained by modulation from a single prototype filter is shown to only have an IIR perfect synthesis solution, but the efficient polyphase/FFT analysis [1] is extended to a perfect and efficient synthesis as well.

The dual problem of multiplexing $N$ signals onto a single channel upsampled by $N$ was addressed as well and shown to be equivalent to the initial problem. Finally, several applications are proposed, among others a new speech coding scheme and filter banks on finite fields.

In conclusion, looking at the $N$ channel filter bank problem as a global, $N$ channel system has proven to be fruitful. The design of $N$ 'simultaneous' filters is therefore quite different from conventional, single filter design.

## Acknowledgment

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## Appendix A

Below, we prove that a perfect analysis/synthesis system without implicit pole/zero cancellation requires both FIR analysis and FIR synthesis filters.

First we define what we mean by implicit or explicit pole zero cancellation. By explicit pole/zero cancellation, we mean numerator/denominator simplications done on the transfer function of a single filter. This is done before implementing a given filter.

By implicit pole/zero cancellation, we mean numerator/denominator simplifications between two cascaded or two parallel filters, but where the two filters are physically separated. A cascade example is given below, where $H_0(z)$ and $H_1(z)$ are two separate filters (appearing, for example, in the analysis and the synthesis part respectively):

$$H_0(z)H_1(z) = [(A(z)B(z))/C(z)][D(z)/(E(z)B(z))] = (A(z)D(z))/(C(z)E(z)). \tag{A.1}$$

Therefore, $B(z)$ was implicitly cancelled between the two filters. A parallel example would be the following:

$$\begin{aligned} H_0(z)+H_1(z) &= (A(z)B(z))/(C(z)[A(z)+D(z)]) + (D(z)B(z))/(C(z)[A(z)+D(z)]) \\ &= (A(z)B(z)+D(z)B(z))/(C(z)[A(z)+D(z)]) = B(z)/C(z). \end{aligned} \tag{A.2}$$

Here, the factor $[A(z)+D(z)]$ was implicitly cancelled between the two filters. While explicit pole/zero cancellation is obviously always permitted, implicit cancellation has two problems associated with it.

The first one is internal stability, that is, a transfer function that is externally stable (from input to output) may have an unstable part inside [10]. This problem can be avoided by careful analysis.

The second and more fundamental problem for practical realizations is the precision problem. While, theoretically, the pole/zero cancellation is realized, in a physical system with finite precision arithmetic the cancellation will in general not be done perfectly. Note that the effect is nonlinear and thus difficult to track and quantify. While our purpose is not to solve this particular problem, we will try to avoid the implicit cancellations in the following.

It will be shown below that perfect reconstruction without implicit pole/zero cancellation can only be achieved when both the analysis and the synthesis filters are FIR.

Consider the input/output transmission $F_0(z)$ from (13):

$$F_0(z) = \tfrac{1}{4}\{[H_1(-z)/\Delta(z)]H_0(z) - [H_0(-z)/\Delta(z)]H_1(z)\}. \tag{A.3}$$

Explicit pole/zero cancellation can be done, for example, between $H_1(-z)$ and $\Delta(z)$ in (A.3), but cancellation between the two summands in (A.3) are implicit ones since they are physically separated filters. Similarly, cancellations between $H_0(z)$ and $\Delta(z)$ in the first summand are implicit as well.

Assume now that all explicit cancellations have been made in (A.3). Then we do not want implicit cancellations between the two summands and we want the transmission $F_0(z)$ to have no poles (except at infinity or zero, which means time shifts only). Thus, the two summands have to be FIR. Now, we do not want implicit cancellations between the factors of the summands. Therefore, both factors need to be

FIR, which means that $H_0(z)$ has to be FIR (because it is a factor of the first summand) and that $H_1(z)$ has to be FIR (because it is a factor of the second summand).

Now, the fact of analysis filters being FIR leads to the requirement that the synthesis filters have to be FIR as well (otherwise, there would be implicit cancellations between the analysis and the synthesis part of the system). This is achieved when the determinant constraint is fulfilled ($\Delta(z)$ being a monomial) because then perfect reconstruction is guaranteed with FIR synthesis filters.

## Appendix B

Further properties of two channel systems are explored below. First we look at the delayed channel case, and then we prove that linear phase systems exist only for even length filters where the two filters have different symmetry.

### *B.1. Delayed channel case*

Assume that channel 1 is delayed by $z^{-1}$ at the input and that channel 0 is delayed by $z^{-1}$ at the output. Then, $\hat{X}_0(z)$ and $\hat{X}_1(z)$ satisfy the following equalities:

$$\hat{X}_0(z) = \tfrac{1}{2}[H_0(z)X(z) + H_0(-z)X(-z)]G_0(z)z^{-1}, \tag{B.1}$$

$$\hat{X}_1(z) = \tfrac{1}{2}[z^{-1}H_1(z)X(z) - z^{-1}H_1(-z)X(-z)]G_1(z). \tag{B.2}$$

Thus, $F_0(z)$ and $F_1(z)$ are equal to

$$\begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} H_0(z) & z^{-1}H_1(z) \\ H_0(-z) & -z^{-1}H_1(-z) \end{bmatrix} \cdot \begin{bmatrix} z^{-1}G_0(z) \\ G_1(z) \end{bmatrix}, \tag{B.3}$$

and the determinant equals

$$\Delta(z) = \tfrac{1}{4}[H_0(z)H_1(-z) + H_0(-z)H_1(z)]z^{-1}. \tag{B.4}$$

$\Delta(z)$ is a function with odd powers of $z^{-1}$ when both $H_0(z)$ and $H_1(z)$ are FIR filters.

Consider now the term $H_0(z)H_1(-z)$. If this polynomial has arbitrary odd-index terms but only one nonzero even-index term, then $\Delta(z)$ is a pure delay and the signal can be reconstructed with FIR synthesis filters as well. This condition is equivalent to the one in Section 3, since it simply means that $H_0(z)H_1(-z)$ is shifted by one sample.

### *B.2. Linear phase solutions*

The perfect FIR solution for the two channel case given in [22, 29] leads to minimum phase filters, but the question remains: are there linear phase solutions as well? This question is interesting because one often wants channel signals which are in phase, typically in subband coding applications. Below, we look at the nondelayed case with two filters of the same length.

A linear phase FIR filter has either a symmetric or an antisymmetric impulse response [18]. We introduce the function sym$[H(z)]$ which is defined as follows:

$$\text{sym}[H(z)] = \begin{cases} 1 & \text{if } H(z) \text{ has a symmetric impulse response,} \\ -1 & \text{if } H(z) \text{ has an antisymmetric impulse response,} \\ 0 & \text{if } H(z) \text{ has no symmetry in its impulse response.} \end{cases}$$

This function has the following properties:

$$\mathrm{sym}[H(z)G(z)] = \mathrm{sym}[H(z)]\,\mathrm{sym}[G(z)], \tag{B.5}$$

$$\mathrm{sym}[H(-z)] = \begin{cases} \mathrm{sym}[H(z)] & \text{if } H(z) \text{ is of odd length,} \\ -\mathrm{sym}[H(z)] & \text{if } H(z) \text{ is of even length.} \end{cases} \tag{B.6}$$

When two filters have the same length and the same symmetry, then this symmetry is preserved by addition of the filters; in other words:

$$\mathrm{sym}[H(z)+G(z)] = \mathrm{sym}[H(z)]\mathrm{sym}[G(z)]\tfrac{1}{2}(\mathrm{sym}[H(z)+\mathrm{sym}[G(z)]). \tag{B.7}$$

When the lengths are different, the result has no symmetry.

Using this function, we analysis the determinant $\Delta(z)$ given by (17) when both $H_0(z)$ and $H_1(z)$ are length $M$ linear phase filters. Look at $H_0(z)H_1(-z)$. When $M$ is odd, because of (B.5) and (B.6) we have

$$\mathrm{sym}[H_0(z)H_1(-z)] = \mathrm{sym}[H_0(z)]\,\mathrm{sym}[H_1(z)]. \tag{B.8}$$

Since the product filter has odd length as well, we have

$$\mathrm{sym}[H_0(z)H_1(-z)] = \mathrm{sym}[H_0(-z)H_1(z)], \tag{B.9}$$

and therefore,

$$\mathrm{sym}[\Delta(z)] = \mathrm{sym}[H_0(z)]\,\mathrm{sym}[H_1(z)]. \tag{B.10}$$

Now, the center of symmetry of $\Delta$ is the coefficient with index $M-1$, that is, an even number. Since $\Delta(z)$ is an odd function of $z^{-1}$, the coefficient of $z^{-M+1}$ is zero. All the nonzero coefficients of $\Delta(z)$ appear therefore twice, that is, $\Delta(z)$ can never be a monomial.

When $M$ is even,

$$\mathrm{sym}[H_0(z)H_1(-z)] = (-1)\,\mathrm{sym}[H_0(z)]\,\mathrm{sym}[H_1(z)]. \tag{B.11}$$

since the product filter is of odd length,

$$\mathrm{sym}[H_0(z)H_1(-z)] = \mathrm{sym}[H_0(-z)H_1(z)] \tag{B.12}$$

or

$$\mathrm{sym}[\Delta(z)] = \mathrm{sym}[H_0(z)H_1(-z)] = (-1)\,\mathrm{sym}[H_0(z)]\,\mathrm{sym}[H_1(z)]. \tag{B.13}$$

The center of symmetry is the coefficient with index $M-1$, an odd number. All other coefficients of $\Delta(z)$ except the center one appear twice. Thus, in order for $\Delta(z)$ to be a monomial, the $(M-1)$st coefficient is the only one that can and must be different from zero. Thus, $\Delta(z)$ should have even symmetry, that is, $H_0(z)$ and $H_1(z)$ should have different symmetry.

In conclusion, one can see that minimum phase filters are not the only solution to the two channel perfect decomposition scheme. It is possible to derive a two channel system with perfect reconstruction and using linear phase filters. This is achieved when:

- $M$ is even,
- $H_0(z)$ and $H_1(z)$ are of different symmetry.

Obviously, the same type of analysis can be applied to other cases as well (delayed channels, filters of different lengths, etc.) in order to find the set of possible solutions.

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