oscillations m within a period are concerned. We characterize a subharmonic by the ratio n/m.

Suppose $p/u < q/v$ are two successive subharmonics of level k. Then the double staircase of $(k+1)$-st level subharmonics is

$$\frac{p}{u} \to \frac{p+q}{u+v} \to \cdots \to \frac{p+iq}{u+iv} \to \cdots \to \text{chaos} \to \frac{q}{v} \tag{1}$$

$$\frac{p}{u} \leftarrow \text{chaos} \leftarrow \cdots \leftarrow \frac{q+ip}{v+iu} \leftarrow \cdots \leftarrow \frac{q+p}{v+u} \leftarrow \frac{q}{v}. \tag{2}$$

Note that these sequences of ratios are monotonic.

This law coincides with the law proposed in [1] concerning the period n, except that the levels are interpreted differently. On the other hand, our conjecture concerning the number of oscillations m is new.

V. Locking Frequency

In Fig. 6, the ratio n/m is plotted against f_s/f_0. In principle all levels are included. However, the steps of the higher levels are too narrow to be represented faithfully and the dotted lines simply indicate their presence. Note that the resulting curve is monotically increasing, because of the monotonic sequences of subharmonics at each level. This kind of bizarre curve reminds the devil's staircase of [3]. However, the presence of chaos implies holes in the staircase.

It is striking from Fig. 6 that

$$n/m \approx f_s/f_0. \tag{3}$$

Therefore, the locking frequency of the driven oscillator f_s/n is determined by some rational approximation n/m of f_s/f_0.

VI. Conclusion

The driven second-order oscillator of [1] has been simulated using a piecewise linear model for the nonlinear resistor. The simulations have confirmed the extremely complicated pattern of subharmonic and chaotic solutions already observed in [1]. The higher precision of the simulations have allowed to go further down in the fine structure of these phenomena and to confirm their regularity conjectured in [1]. A more detailed law for the succession of subharmonics has been found. This law also permits to relate the locking frequency of the driven oscillator to some rational approximation of the ratio between the driving frequency and the free-running frequency of the oscillator.

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Invertibility of Linear Periodically Time-Varying Filters

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***Abstract*—Conditions under which a linear periodically time-varying (LPTV) filter can be inverted are given. Such an inversion is useful in multirate signal processing, for example for aliasing cancellation purposes.**

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I. Introduction

Among time-varying filters, the ones with periodic variations [4] are both important and simple to analyze. Examples of periodically time-varying filters are multirate filters, where sampling rate changes by N lead to a periodicity of N [1]. The simplicity of analysis stems from the fact that N time-invariant impulse responses characterize completely a periodically time-varying filter [2], [6] of period N.

In the following we will be concerned with the problem of inverting a periodically time-varying filter, that is, we will try to find its input signal given the output signal and the filter characteristic (which we assume to be known). Note that the invertibility of linear periodically time-varying filters has been addressed in [5] where a particular class of these filters has been derived so that they are automatically invertible.

A typical application of the proposed method is aliasing cancellation in multirate systems. Because of the inherent periodicity of such systems, the output often contains not only a filtered version of the input, but also aliased and thus undesired versions of it. These aliased versions can be eliminated at the output with the techniques described below.

II. Linear Periodically Time-Varying Filters

Described briefly is the analysis framework that will be used in the following (see [6] for details). The z-transforms of all signals and filter impulse responses are assumed to exist. A linear periodically time-varying (LPTV) filter with period N is completely defined by N impulse responses at times $0,1,\cdots,N-1$. Note that there are several other ways to describe LPTV systems, but that they are equivalent within a linear transformation. We call $T_{pi}(z)$ the z-transform of the response to a unit pulse at time i. Now, the input signal $x(n)$ with z-transform $X(z)$ can be decomposed into so-called polyphase components [1] with z-transforms $X_{pi}(z)$ given by

$$X_{pi}(z) = z^{-i} \sum_{n=-\infty}^{\infty} x(nN+i) z^{-nN}. \tag{1a}$$

Note that the signal is equal to the sum of the polyphase components. The output of a LPTV filter can be expressed as the superposition of the N impulse responses, each exited by the adequate polyphase component of the input signal. Using (1), the output $y(n)$ of the LPTV filter, with z-transform $Y(z)$, can be written as

$$Y(z) = \left[t_p(z)\right]^T \cdot x_p(z) \tag{2a}$$

with:

$$t_p(z) = \left[T_{p0}(z) \quad T_{p1}(z) \cdots T_{pN-1}(z)\right]^T \tag{2b}$$

$$x_p(z) = \left[X_{p0}(z) \quad X_{p1}(z) \cdots X_{pN-1}(z)\right]^T. \tag{2c}$$

Because the LPTV filter is varying with a period N, the output will contain a filtered version of the input as well as of modulated versions of the inpout (with modulation by the Nth roots of unity) [6]. Therefore, another useful decomposition of the input is the modulation decomposition, which is given in vector form by

$$x_m(z) = \left[X(z) \quad X(Wz) \cdots X(W^{N-1}z)\right]^T, \qquad W = e^{-j(2\pi/N)}. \tag{3}$$

The polyphase and the modulation decomposition are related by [6]:

$$x_p(z) = \frac{1}{N} F \cdot x_m(z) \tag{4}$$

where F is the usual discrete Fourier transform matrix with the i, jth element equal to W^{ij} (rows and columns are enumerated starting from 0). From (4), (2a) becomes

$$Y(z) = \frac{1}{N}\left[t_p(z)\right]^T \cdot F \cdot x_m(z). \tag{5}$$

Let us define

$$t_m(z) = \left[T_{m0}(z) \quad T_{m1}(z) \cdots T_{mN-1}(z)\right]^T = (1/N) F \cdot t_p(z). \tag{6}$$

Then, (5) can be rewritten similarly to (2a) as

$$Y(z) = \left[t_m(z)\right]^T \cdot x_m(z). \tag{7}$$

III. Invertibility Conditions

The first step in order to invert a periodically time-varying filter consists in deriving a postfilter such that the time-variance is annihilated. The approach taken is to expand the output signal $y(n)$ into its modulation components. The modulation expansion of $Y(z)$ is

$$y_m(z) = \left[Y(z) \quad Y(Wz) \cdots Y(W^{N-1}z)\right]^T. \tag{8}$$

Now, (8) can be written as

$$y_m(z) = T_m(z) \cdot x_m(z) \tag{8a}$$

where

$$T_m(z) = \begin{pmatrix} T_{m0}(z) & T_{m1}(z) & \cdots & T_{mN-1}(z) \\ T_{mN-1}(Wz) & T_{m0}(Wz) & \cdots & T_{mN-2}(Wz) \\ \cdot & \cdot & \ddots & \cdot \\ T_{m1}(W^{N-1}z) & T_{m2}(W^{N-1}z) & \cdots & T_{m0}(W^{N-1}z) \end{pmatrix}. \tag{8b}$$

The elements of the matrix $T_m(z)$ are all time-invariant. Note that the matrix $T_m(z)$ is obtained by circularly shifting the first line and modulating the components by the appropriate Nth root of unity (W^l in the lth line). From the expanded output $y_m(z)$, one obtains a corrected output by applying N filters $R_i(z)$ to the components of $y_m(z)$ and summing their outputs:

$$Y'(z) = \left[R_0(z) R_1(z) \cdots R_{N-1}(z)\right] \cdot y_m(z). \tag{9}$$

That is, $Y(z)$ is filtered by a LPTV filter of period N in order to obtain $Y'(z)$. Now, it is desired to find the $R_i(z)$ ($i = 0 \cdots N-1$) in such a way that the total equivalent filter from the input $X(z)$ to the output $Y'(z)$ is a simple delay. The condition when this can be done is shown in the following theorem.

Theorem: A linear periodically time-varying filter (with period N) can be inverted if and only if the determinant of the matrix $T_m(z)$ has no zeros on the unit circle.

The fact that this condition is sufficient is shown by construction. Take the $R_i(z)$ from the first line of the cofactor matrix of $T_m(z)$:

$$\left[R_0(z) \cdots R_{N-1}\right] = [1 \quad 0 \cdots 0] \cdot \operatorname{cofac}\left[T_m(z)\right]. \tag{10}$$

Then, the corrected output $Y'(z)$ equals

$$\begin{aligned} Y'(z) &= \left[R_0(z) \cdots R_{N-1}(z)\right] \cdot T_m(z) \cdot x_m(z) \\ &= \operatorname{Det}\left[T_m(z)\right] \cdot X(z). \end{aligned} \tag{11}$$

Since all filters appearing in $T_m(z)$ are time-invariant, the determinant of $T_m(z)$ is time-invariant. Thus $Y'(z)$ is a time-invariant function of $X(z)$, showing that the time-variance has been cancelled using the correction filters $R_i(z)$ as defined in (10). Now, since there are no zeros on the unit circle, the determinant of $T_m(z)$ can be cancelled, by a causal or a non-causal filter depending on the zero locations.

The necessity of the determinant having no zeros on the unit circle is proven in two steps. First, it is shown that the rank of $T_m(z)$ has to be equal to N, otherwise the periodic time-variance cannot be cancelled.

Corollary: If the rank of $T_m(z)$ is smaller than N, then it is impossible to annihilate the time-variance of the system.

From the construction above, it is clear that the time-variance can be corrected when the rank is equal N (see (10)–(11)). However, if the rank is smaller than N, the determinant of $T_m(z)$ is zero for all z, and we show by contradiction that the time-variance cannot be annihilated anymore. Assume that there exist filters $R_i(z)$ such that

$$\left[R_0(z) \cdots R_{N-1}\right] \cdot T_m(z) = \left[\alpha(z) \quad 0 \cdots 0\right] \tag{12}$$

and that the rank of $T_m(z)$ is smaller than N.

Now, the $R_i(z)$ can be expanded into a matrix $R_m(z)$ that has the same form as $T_m(z)$ (see (8b)). As can be verified, the product of $R_m(z)$ with $T_m(z)$, assuming that (12) holds, is diagonal and has the following form:

$$R_m(z) \cdot T_m(z) = \begin{pmatrix} \alpha(z) & 0 & \cdots & 0 \\ 0 & \alpha(Wz) & \cdot & 0 \\ \cdot & \cdot & \ddots & \cdot \\ 0 & \cdots & 0 & \alpha(W^{N-1}z) \end{pmatrix}. \tag{13}$$

The rank of a matrix product is upperbounded by the minimum rank of the terms of the product [3]. The right side matrix in (13) has rank N, which is in contradiction with our assumption that $T_m(z)$ has rank smaller than N. It follows that when the rank of $T_m(z)$ is smaller then N, then there is no vector $[R_0(z) \cdots R_{N-1}(z)]$ such that (12) is verified, and therefore the time-variance cannot be cancelled.

The significance of $T_m(z)$ having a rank smaller than N is the following: $T_m(z)$ covers a subspace smaller than N, therefore there exist signals $x_z(n)$ with z-transform $X_z(z)$ having the following property:

$$T_m(z) \cdot \begin{pmatrix} X_z(z) \\ X_z(Wz) \\ \vdots \\ X_z(W^{N-1}z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{14}$$

That is, certain signals simply disappear when going through the LPTV filter. Therefore, with the above corollary, we have shown the necessity of the rank of $T_m(z)$ being equal to N. Now, from (11), we see that the transmission from input to output contains the determinant of $T_m(z)$ once the time-variance is cancelled. Therefore and in order to annihilate the determinant, it is necessary that it does not contain zeros on the unit circle. This concludes the necessity of the proof of the theorem.

Note that the expansion of $Y(z)$ into N components (see (8)) is necessary and sufficient. Expansion into fewer components will not permit a time-variance of period N to be corrected, and expansion into more than N components is unnecessary. The analysis above can be done using the polyphase expansion of signals and filters instead of the modulation expansion that we used. This would yield the same results since the representations are equivalent within a linear transformation.

Examples of systems with rank smaller than N, as well as of a sub-band coding system where aliasing in the output can be cancelled with a LPTV post-filter, can be found in [7].

IV. Conclusion

The invertibility of linear periodically time-varying (LPTV) filters has been addressed. A necessary and sufficient condition under which a LPTV filter can be inverted has been shown using a LPTV post-filter. Such techniques find application, for example, in aliasing cancellation in multirate systems.

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2-D Quadratic Filter Implementation by a General-Purpose Nonlinear Module

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Abstract **—In this letter it is shown that the general-purpose filtering module proposed in [3] can be also used for the implementation of 2-D quadratic filters.**

It has been recently shown that a 2-D finite-support quadratic digital filter can be implemented with a set of parallel branches which operate simultaneously and independently on a common input array [1], [2]; each branch contains a 2-D FIR linear filter, according to a suitable decomposition of the coefficient matrix. In fact, a 2-D quadratic filter with an $N_1 \times N_2$ support is described by

$$y(n_1, n_2) = X^T H X \tag{1}$$

where X is the $N_1 N_2 \times 1$ vector of the input samples arranged in a well-defined order [1], [2], and accordingly H is a symmetric $N_1 N_2 \times N_1 N_2$ matrix formed with the filter coefficients. Since the matrix H can be decomposed into a finite sum of rank one symmetric matrices, we can write

$$H = \sum_{j=1}^{r} q_j R_j R_j^T \tag{2}$$

where r is the rank of $H (r \leqslant N_1 N_2)$, q_j's are real scalar values, and R_j's are $N_1 N_2 \times 1$ vectors. Thus by substituting (2) in (1) we obtain

$$y(n_1, n_2) = \sum_{j=1}^{r} q_j \left[X^T R_j \right] \left[R_j^T X \right] = \sum_{j=1}^{r} q_j y_j^2(n_1, n_2) \tag{3}$$

with

$$y_j(n_1, n_2) = R_j^T X = \sum_{i=1}^{N_1 N_2} r_{j,i} x_i \tag{4}$$

where $r_{j,i}$ and x_i indicate the ith component of the vectors R_j and X, respectively. The final structure is shown in Fig. 1.

On the other hand, it has been shown in [3] that a general filter module (Fig. 2) can be used for the implementation of various classes of image processing operations. Aim of this letter is to show that the quadratic filter of Fig. 1 can be implemented by means of r modules as those introduced in [3]. In fact, for the jth

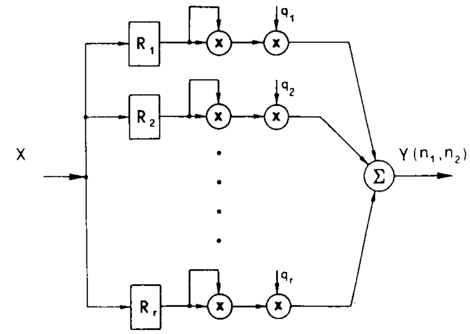

Fig. 1. Implementation of a 2-D quadratic filter by matrix decomposition.

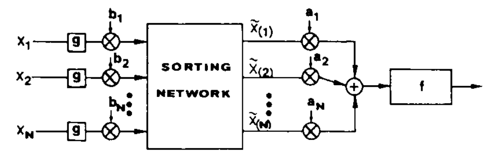

Fig. 2. General-purpose nonlinear module.

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