Defining $H(x) = \log Z(H) + H(x)/T$, also a Walsh expression of degree $d$, and applying Theorem 1 with $H$ and $g(x) = \exp(-x)$ yields that all information on $\pi$ is contained in the correlations of degree $\leq d$. This is equivalent to the statement that a $dth$-order Boltzmann machine without hidden units is incapable of capturing correlations of degree $> d$, a well-known "folk-theorem," unproven until now.

REFERENCES


The Commutativity of Up/Downsampling in Two Dimensions

Jelena Kovačević and Martin Vetterli

Abstract—It is shown under which conditions up- and downsampling can commute in two dimensions. This is the generalization to arbitrary two-dimensional lattices of the result that one-dimensional up- and downsampling commute if their sampling rates are coprime [1]. To the authors' knowledge, no two dimensions the problem has been open until now. When the two-dimensional sampling is separable, the extension of the result is trivial. The interesting case appears when the two-dimensional sampling is represented by arbitrary lattices. Thus conditions under which the commutativity can be achieved are more complex and are closely related to the notion of the greatest common sublattice of the sampling lattices in question. In this correspondence, after some preliminaries, we state and prove a theorem solving the problem of commutativity in two dimensions. Some illustrative examples are given in Figs. 3(a)–3(c).

One of the possible applications of the result would be in building multirate filter banks with rational sampling rate changes [2]. In [2] a direct method for designing filter banks with arbitrary rational sampling rate changes was given, which as its key element uses the result on commutativity. There was shown that, in order to avoid designing one filter bank which

would divide the spectrum into a number of parts and the other one which would resynthesize the appropriate subspaces so as to get fractional parts, one had to interchange upsample with a downsampler in each branch. Note that throughout the paper $\gcd(a, b)$ will denote the greatest common multiple of $a$ and $b$ and $\sgcd(a, b)$ the greatest common divisor; bold letters will denote vectors and matrices.

II. SOME RESULTS FROM THE THEORY OF LATTICES

This section presents some basic concepts from lattice theory [3].

Definition 1: Let $a_1, a_2$ be two linearly independent real vectors in two-dimensional real Euclidean space $\mathbb{R}^2$. A lattice $\Lambda$ in $\mathbb{R}^2$ is the set of all linear combinations of $a_1, a_2$ with integer coefficients:

$$\Lambda = \{\lambda_1 a_1 + \lambda_2 a_2, \lambda_1, \lambda_2 \in \mathbb{Z}\}.$$  

(1)

If $D$ is a matrix with columns $a_1, a_2$, then a lattice is the set of all vectors generated by $D \cdot n$, $n \in \mathbb{Z}^2$. Since the elements of $D$ belong to $\mathbb{Z}$, which is a principal ideal ring, unimodular matrices would be all those with determinant equal to $\pm 1$ [4]. In what follows all matrices involved will be integer matrices. Note that a basis for a lattice is not uniquely determined since $D'V$ with unimodular $V$ is again a basis for $\Lambda$, while $d(\Lambda) = \det(D)$ is unique, and physically represents the reciprocal of the sampling density [5]. Thus, for example, $\mathbb{Z}^2$ is the lattice generated by a $2 \times 2$ identity matrix $I$ that corresponds to the standard orthonormal basis. Since only $d(\mathbb{Z}^2)$ is unique, it follows that $\mathbb{Z}^2$ can be generated by any unimodular $D$.

Definition 2: If every point of lattice $\Lambda$ is also a point of lattice $M$, then we say that $\Lambda$ is a sublattice of $M$.

The determinant of $\Lambda$ is then an integer multiple of the determinant of $M$.

Definition 3: Let $\Lambda_1$ and $\Lambda_2$ be lattices. The greatest common sublattice of $\Lambda_1$ and $\Lambda_2$, denoted by $\gcd(\Lambda_1, \Lambda_2)$, is the set of all points belonging to both $\Lambda_1$ and $\Lambda_2$ [6], i.e.,

$$\gcd(\Lambda_1, \Lambda_2) = \Lambda_1 \cap \Lambda_2.$$  

(2)

Then it is obvious that $d(\gcd(\Lambda_1, \Lambda_2)) = \gcd(d(\Lambda_1), d(\Lambda_2))$. It is often useful to choose a basis so as to have a simple form of $D$. It can be shown [3] that $D$ can always be uniquely represented as

$$D = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$  

(3)

where $a > 0$, $d > 0$, and $0 \leq b < a$. This representation will be used throughout this correspondence. A unit cell $U$ will denote a set of points belonging to the parallelogram formed by the two basis vectors $a_1, a_2$. Note that it contains exactly $\det(D)$ points.
Fig. 1 shows an example of a lattice together with its unit cell. Also, in what follows a rectangular lattice is the one with $b = 0$ in (3) while a quadratic one is rectangular with $a = d$. Now, we state a proposition showing how to change a basis for $\mathbb{Z}^2$ so as to transform any lattice into a rectangular one.

**Proposition 1:** Any lattice generated by $D$ can be represented as a rectangular lattice in $\mathbb{Z}^2$ generated by some unimodular $I_1$.

**Proof:** A matrix whose elements belong to a principal ideal domain possesses a Smith normal form [4], i.e., it can be written as

$$D = I_1 \Lambda I_2,$$  \hspace{1cm} (4)

where $I_1$ and $I_2$ are unimodular and $\Lambda$ is a diagonal matrix. Rewrite (4) as

$$D I_2^{-1} = I_1 \Lambda.$$  \hspace{1cm} (5)

Since $I_2$ is unimodular, $I_2^{-1}$ is unimodular as well, and thus $D I_2^{-1}$ still represents the same lattice. Now (5) is exactly what we want since the desired rectangular lattice is represented by $I_1$ and $\Lambda$ is the matrix containing the nonstandard basis for $\mathbb{Z}^2$. $\Box$

III. THE COMMUTATIVITY OF UP/DOWNSAMPLING

Let us briefly recall the one-dimensional result [1]: Upsampling by a factor of $N_1$ and downsampling by a factor of $N_2$ can be interchanged if and only if $N_1$ and $N_2$ are relatively prime. Going back to the original problem, let us consider Figs. 2(a) and 2(b). $D_1$ and $D_2$ are matrices corresponding to the downsampling and upsampling lattices $\Lambda_1$ and $\Lambda_2$, respectively. Note that $U_1$ will denote the unit cell of the first lattice. In the case where downsampling by $D_1$ comes first, the Fourier transform of the signal at the output can be expressed as [5]:

$$Y(\Omega) = X_D(D_1^T \Omega) = \frac{1}{N_1} \sum_{k = 0}^{N_1 - 1} X \left( D_1^{-1} D_2^T \Omega - 2\pi k \right).$$  \hspace{1cm} (6)

where $X_D$ is the signal after downsampling, $\Omega$ is a two-dimensional frequency vector, and $N_1 = \text{det}(D_1)$. If $D_2$ came first instead, we would have

$$Y(\Omega) = \frac{1}{N_1} \sum_{k = 0}^{N_1 - 1} X_U \left( D_1^{-1} \Omega - 2\pi k \right).$$  \hspace{1cm} (7)

where $X_U$ is the signal after upsampling. In order to have $Y(\Omega) = Y(\Omega)$, we must first ensure that the resulting matrices next to $\Omega$ are the same, i.e., $D_2$ and $D_1^{-1}$ have to commute. This leads to the following combinations of possible up/downsampling matrices (assuming $D_1$ and $D_2$ are of the form (3) with corresponding subscripts):

1. $b_1 = 0 \land a_1 = d_1$, i.e., $D_1$ is quadratic and $D_2$ is arbitrary,
2. $b_1 = 0 \land b_2 = 0 \land a_1 = d_1$, i.e., $D_1$ is rectangular and $D_2$ is rectangular or quadratic,
3. $b_1 = 0 \land d_1 = a_2 + d_1 (d_2 - a_2) / b_1$, i.e., $D_1$ is arbitrary nonrectangular and $D_2$ is arbitrary with the given constraint on $d_2$.

Next we have to find when the set of vectors generated in (6) with $k \in U_1$ is equivalent to the set of vectors generated in (7) with again $k \in U_1$. Calling these two sets $A$ and $B$, we have

$$A = \left\{ e^{-2\pi i k (D_2)^{T} \Omega} \right\},$$  \hspace{1cm} (8)

$$B = \left\{ e^{-2\pi i k (D_1)^{T} \Omega} \right\}.$$  \hspace{1cm} (9)

Equivalence of these sets is exactly the possibility of interchanging an upsampler with a downsampler. Note that $A$ contains exactly $N_1$ distinct elements, i.e.,

$$\forall k, n \in U_1, \quad k = n \leftrightarrow e^{-2\pi i k (D_1)^{T} \Omega} = e^{-2\pi i k (D_2)^{T} \Omega}.$$  \hspace{1cm} (\ast \ast)

Now we are ready to state the following theorem.

**Theorem 1:** Assuming $(D_1)^{T}$ and $(D_2)^{T}$ commute, an upsampler and a downsampler are interchangeable (i.e., the sets $A$ and $B$ just defined are equivalent) iff the determinant of the greatest common sublattice of $\Lambda_1$ and $\Lambda_2$ equals the product of the determinants of $\Lambda_1$ and $\Lambda_2$, i.e.,

$$\forall k, n \in U_1, \quad k = n \leftrightarrow e^{-2\pi i k (D_1)^{T} \Omega} = e^{-2\pi i k (D_2)^{T} \Omega}.$$  \hspace{1cm} (\ast \ast)

The proof of the Theorem is given in the Appendix A.

As an illustration, three examples are given in Figs. 3(a)–3(c). The first one shows what happens if the matrices $(D_1)^{T}$ and $(D_2)^{T}$ do not commute. The second one is an instance where the matrices commute but the greatest common sublattice does not satisfy the condition given in Theorem 1. In Fig. 3(c) a pair of lattices where the interchange is possible is given. Note that, for the sake of simplicity, the members of the sets $A$ and $B$ in the examples are actually the angles in the exponent in (8) and (9). In each of the examples, the sets $A$ and $B$ show explicitly whether the interchange is possible or not.

To conclude, let us see how Theorem 1 reduces to the one-dimensional result [1]. The downsampling matrix $D_1$ reduces to a single coefficient $N_1$, the upsampling matrix $D_2$ reduces to $N_2$, and the greatest common sublattice of the two is actually the one corresponding to sampling by $C = \text{lcm}(N_1, N_2)$. Thus (\ast \ast) reduces to $C = N_1 N_2$, which is equivalent to $N_1$ and $N_2$ being relatively prime. Therefore, we have the following lemma.

**Lemma 1:** An upsampler and a downsampler are interchangeable iff $N_1$ and $N_2$ are relatively prime, i.e.,

$$\forall k, n \in \{0 \cdots N_1 - 1\}, \quad k = n \leftrightarrow e^{-2\pi i k N_1 / N_2} \neq e^{-2\pi i k N_2 / N_1}$$

$$C = \text{lcm}(N_1, N_2) = N_1 N_2,$$

which is exactly the statement in [1].

IV. CONCLUSION

We have shown how to interchange an upsampler and a downsampler in two dimensions. The result holds for arbitrary sampling lattices and is a generalization of the one-dimensional result that an upsampler commutes with a downsampler iff their rates are coprime.
Fig. 3. (a) Downsampling lattice is “quincuncial” upsampling one is “hexagonal.” Matrix \( D_2 \) and inverse of \( D_1 \) do not commute and thus cannot be interchanged. (b) Downsampling lattice is “quincuncial,” upsampling one represents separable sampling by two in both dimensions. Now matrices commute, but greatest common sublattice of the two is the same as upsampling one and thus this pair cannot be interchanged. (c) Downsampling lattice represents separable sampling by three in both dimensions, upsampling one is “quincuncial.” Matrices commute and greatest common sublattice satisfies conditions of the Theorem 1. Hence, in this case, up- and downsampler can be interchanged.

**APPENDIX A**

**PROOF OF THEOREM 1**

We are going to prove the theorem for each of the three cases stated at the beginning of Section III.

**A. Case** \( b_1 = 0 \land a_1 = d_1 \)

Since now \( D_1 = a_1 I \) and \( D_2 \) is general, the matrix corresponding to the greatest common sublattice of \( A_1 \) and \( A_2 \) is of the form

\[
M = \begin{bmatrix}
\text{lcm}(a_1, a_2) & kb_2 + la_2 \\
0 & kd_2
\end{bmatrix}.
\]

where \( k \) is the smallest integer such that \( a_1 \) divides \( kd_2 \), and \( a_1 \) divides \( kb_2 + la_2 \). Therefore \( k \leq a_1 \) since we can always choose \( k = a_1 \), satisfying the above stated conditions. We are going to consider two distinct cases: when \( a_1 \) and \( a_2 \) are and are not relatively prime.

1) \( \gcd(a_1, a_2) = 1 \): Since \( a_1 \) and \( a_2 \) are relatively prime, the first entry in matrix \( M \) reduces to \( a_1 a_2 \) and

\[
\det(M) = a_1 a_2, \quad kd_2 \leq a_1^2 a_2 d_2.
\]

We can now find \( k \) from

\[
k d_2 = p \text{lcm}(a_1, a_2), \quad k = p \frac{a_1}{\gcd(a_1, d_2)}, \quad p \in \mathbb{Z}.
\]

Now minimum \( k \) can be chosen as above with \( p = 1 \). Since \( a_1 \) and \( a_2 \) are relatively prime, a unique \( l \) can always be found such that \( a_1 \) divides \( kb_2 + la_2 \). Thus \( M \) is completely determined and

\[
p_{a-1} = n_{a} = 2^j, \quad j \text{ is a nonnegative integer}. \quad \text{To prove this,}
\]

we only have to show that under this assumption, there exists a source distribution such that \( R_1 = R_{\text{RMS}} \). It is easy to see that

\[
H_b \left( p_{a_k, a_{k-1}} \right) - H_b \left( p_{a_k, a_{k-1}} \right) + \left[ p_{a_{k-1}, 1} + p_{a_{k-1}, 1} \right]
\]

\[
\left[
\begin{array}{c}
p_{a_{k-1}, 1} \\
p_{a_{k-1}, 1}
\end{array}
\right]
\]
(11) reduces to
\[ \det(M) = \frac{\det(D_1) \det(D_2)}{\gcd(a_1, d_2)}. \]  
\[ (i, ii) \Rightarrow (\ast): \text{Let us suppose that } (\ast) \text{ does not hold, i.e.,} \]
there exist different \( k \) and \( n \) from \( U \) producing the same members of the set \( B \), i.e.,
\[ D_k(D_1)^{-1} m = p \quad m = k - n, \quad p \in \mathcal{G}^2. \]
Since \( D_k \) and \( (D_1)^{-1} \) commute, we can rewrite this as
\[ D_k^T(m_1) = D_p(p_1). \]
or
\[ p_1 = m_1 a_2 \quad p_2 = m_2 a_1 + m_1 b_2. \]
Since \( \gcd(a_1, a_2) = 1 \), \( m_1 < a_1 \), and \( p_1 \) is integer, it follows that
\[ m_1 \text{ has to be zero}. \]
Hence \( p_2 = m_2 a_1 / a_2 \). Knowing that \( m_2 \neq 0 \), \( m_2 < a_2 \), and \( p_2 \) is integer, we conclude that \( \gcd(a_1, a_2) > 1 \) and therefore \( \det(M) < \det(D_1) \det(D_2) \), which is in contradiction with our assumption.
\[ (ii) \Rightarrow (\ast): \text{Suppose that } (\ast) \text{ does not hold, i.e.,} \]
\[ \det(M) < \det(D_1) \det(D_2). \]
It follows then that \( \gcd(a_1, a_2) > 1 \).
Hence we can choose \( m_1 \) and \( m_2 \) as
\[ m_1 = 0, \quad m_2 = \frac{a_1}{\gcd(a_1, a_2)} < a_1. \]
yielding
\[ p_1 = 0, \quad p_2 = \frac{a_2}{\gcd(a_1, a_2)}. \]
again a contradiction.

2) \( \gcd(a_1, a_2) > 1 \): Now, obviously,
\[ \det(M) = \text{lcm}(a_1, a_2) k d_2 a_1 k d_2 \text{lcm}(a_1, a_2) \]
\[ < \det(D_1) \det(D_2), \]
meaning that \( (\ast) \) is never satisfied. To prove the theorem in this case, we have to prove that \( (\ast) \) can never be satisfied as well.
If \( \gcd(a_1, a_2) > 1 \), we can always choose \( m_1 = 0 \) and \( m_2 = a_1 / \gcd(a_1, a_2) < a_1 \), showing that \( (\ast) \) does not hold.
If \( \gcd(a_1, a_2) = 1 \), we can always uniquely choose \( t_1 \) and \( t_2 \) such that \( t_1 a_1 + t_2 a_2 = 1 \). Hence the choice
\[ m_1 = \frac{a_1}{\gcd(a_1, a_2)} < a_1, \quad m_2 = -m_1 b_2 t_2 (\text{mod } a_1), \]
if \( b_2 \neq 0 \) or
\[ m_1 = \frac{a_1}{\gcd(a_1, a_2)} < a_1, \quad m_2 = 0 \]
if \( b_2 = 0 \), shows that there always exists a choice of \( m_1 \) and \( m_2 \) such that \( (\ast) \) does not hold.

B. Case \( b_1 = 0 \land b_2 = 0 \land a_i \neq d_i \)

Since both matrices are rectangular, \( \det(D_1) = a_1 d_1 \),
\[ \det(D_2) = a_2 d_2, \quad \text{and } \gcd(a_1, a_2) \text{ is also a rectangular lattice} \]
with \( \text{lcm}(a_1, a_2) \) and \( \text{lcm}(d_1, d_2) \) on the diagonal. Since \( \text{lcm}(a, b) = ab / \gcd(a, b) \), we get
\[ \det(gcd(L_1, L_2)) = \frac{\det(D_1) \det(D_2)}{\gcd(a_1, a_2) \gcd(d_1, d_2)}. \]
\[ (\ast) \text{ is equivalent to } \det(gcd(L_1, L_2)) = \det(D_1) \det(D_2) \text{ satisfying} \]
\( (\ast) \). The converse is equally easily proven for \( (\ast) \) holds then both \( \gcd(a_1, a_2) = 1 \) and \( \gcd(d_1, d_2) = 1 \), yielding \( (\ast) \).

C. Case \( b_1 \neq 0 \land d_2 = a_2 + b_2 (d_1 - a_1) / b_1 \)

In this case we have an arbitrary nonrectangular matrix \( D_1 \)
and thus we can apply Proposition 1 to transform it into a rectangular matrix in \( \mathcal{G}^2 \) generated by a nonstandard basis. Since commutativity is preserved, this instance reduces trivially to one of the previous two, which concludes the proof of the theorem.

\[ \square \]

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References


**Correction to "On Universal Hypotheses Testing Via Large Deviations"**

Ofer Zeitouni and Michael Gutman

In the above paper, there is a mistake in the proof of Theorem 1 (the discrete case). The proof of the Theorem as written holds true only when all coordinates of \( P_i \) are strictly positive (i.e., when \( \Sigma = \text{supp} P_i \)).

When some of the coordinates of \( P_i \) are zero, it is clear that all empirical measures associated with \( P_i \) have \( \text{supp} \mu_x \subseteq \text{supp} P_i \), and the blow up of \( \Omega \) defined in (3) must take place only in the subset of \( \Sigma \) that belongs to \( \text{supp} P_i \). Thus, a priori, define \( \hat{\Sigma} = \text{supp} P_i \), and restrict all probability measures to \( M(\hat{\Sigma}) \), letting \( \mu_x \) be associated with \( P_i \) if \( \text{supp} \mu_x \subseteq \hat{\Sigma} \). The blow up \( \Omega_\delta \) of (3) will now be defined w.r.t. \( M(\hat{\Sigma}) \), and the rest of the proof is unchanged.

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