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Wavelets and Recursive Filter Banks

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Abstract—Recent work has shown that perfect reconstruction filter banks can be used to derive continuous-time bases of wavelets; the case of finite impulse response filters, which lead to compactly supported wavelets, has been examined in detail. In this paper we show that infinite impulse response filters lead to more general wavelets of infinite support. We give a complete constructive method which yields all orthogonal two channel filter banks, where the filters have rational transfer functions, and show how these can be used to generate orthonormal wavelet bases. A family of orthonormal wavelets, which shares with those of Daubechies the property of having a maximum number of disappearing moments, is shown to be generated by the halfband Butterworth filters. When there is an odd number of zeros at π we show that closed forms for the filters are available without need for factorization. A still larger class of orthonormal wavelet bases having the same moment properties is presented, and contains the Daubechies and Butterworth filters as the boundary cases. We then show that it is possible to have both linear phase and orthogonality in the infinite impulse response case, and give a constructive method. We show how compactly supported bases may be orthogonalized, and construct bases for the spline function spaces. These are alternatives to those of Battle and Lemarié, but have the advantage of being based on filter banks where the filters have rational transfer functions and are thus realizable. Design examples are presented throughout.

I. INTRODUCTION

THE subject of wavelets has been studied by applied mathematicians for a number of years, as representing an alternative to traditional Fourier based analysis techniques. Considerable interest has been shown by the signal processing community more recently owing, in large measure, to the influence of pivotal papers by Mallat [1]–[3] and Daubechies [4]. These demonstrate the strong link between the subject of wavelets and of multirate filter banks. Briefly put, multirate filter banks give the structures required to generate important cases of wavelets and the wavelet transform.

Among the most celebrated wavelet bases are those of Meyer [5], Battle [6], and Lemarié [7], and these can be

realized using orthonormal multirate filter banks; however, the filters involved are not rational, and the corresponding wavelets cannot be computed exactly, so they are limited from a signal processing view. More interesting are the compactly supported wavelets of Daubechies [4]. These are based orthogonal finite impulse response (FIR) filter banks, which have in fact been under study for some time [8]–[10].

Our principal interest in this paper is orthogonal filter banks and their relation to wavelet bases. We consider infinite impulse response (IIR) filter banks, which have been less studied, and which allow greater freedom than their FIR counterparts. The relation between FIR filter banks and wavelets was examined in [11].

The essential contribution of the paper consists of new results on orthogonal IIR filter banks which allow us to thoroughly examine the structure of possible solutions, and present new designs. The connection with wavelets allows us to use these designs to get novel orthogonal wavelets, which are based on structures that are computable with finite complexity. Thus we present filters that are of interest in their own right, but which also allow us to generate wavelet bases which are in some senses comparable, and in others superior to those already published.

The summary of the paper is as follows. We present a succinct review of the relation between orthonormal wavelet bases and filter banks having orthogonality properties in Section II. We recall that designing a certain class of orthonormal wavelet bases is related to the simpler problem of designing orthogonal filter banks, provided that the filters satisfy certain regularity conditions. Since this material has been reviewed in a number of papers our treatment is limited to the essential points. Readers unfamiliar with the subject might consult [8], [12], [13] for additional coverage of filter banks, and [14], [4], [5], [11], [15] for treatments of the connection with wavelets.

In Section III we present a constructive method to find all orthogonal filter banks, where the filters have rational transfer functions. In certain cases of considerable interest, we actually get closed form expressions for the filters; so that no factorization or approximation is necessary. This contrasts sharply with the FIR case, and seems to be the first closed form for a nontrivial implementable wavelet.

Section IV recalls that wavelets with moment properties are derived from filter banks where the filter frequency responses are maximally flat. We construct the

whole family of maximally flat filters for orthogonal filter banks, and show that the Butterworth halfband filters and the Daubechies solutions are included as special cases.

Section V illustrates that linear phase and orthogonality are not mutually exclusive properties for IIR filter banks, as they were in the FIR case. Filters of considerable interest can be designed that lead to orthogonal wavelets with symmetry.

We show in Section VI that if a compactly supported basis for one of the spaces in the multiresolution analysis structure exists, then we can always generate an orthogonal basis from realizable IIR filters. A special case is the Nth order spline function space; so we construct bases which have the advantage over those of Battle and Lemarié that they are realizable.

Certain results have been presented in preliminary form in [16]–[18].

Notation

The set of real numbers will be represented by R, the set of integers by Z. The inner product over the space of square-summable sequences $l^2(Z)$ is: $\langle a(n), b(n) \rangle = \sum_{n=-\infty}^{\infty} a^*(n) b(n)$, where $a(n), b(n) \in l^2(Z)$, and superscript * denotes complex conjugation. Generally we shall deal with sequences and functions that are real. We define $\|a(n)\|_2^2 = \langle a(n), a(n) \rangle$. The z transform of a sequence is defined by $H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$. The discrete time Fourier transform is $H(e^{jw}) = H(z)|_{z=e^{jw}}$. Similarly over the space of square-integrable functions $L^2(R)$ we have the inner product: $\langle f(x), g(x) \rangle = \int_{-\infty}^{\infty} f^*(x) g(x) \, dx$, where $f(x), g(x) \in L^2(R)$. The squared norm is given by $\|f(x)\|_2^2 = \langle f(x), f(x) \rangle$. For continuous-time functions we will use subscripts to denote affine variable changes where the scales are powers of 2 as follows $f_{jk}(x) = 2^{-j/2} \cdot f(2^{-j}x - k)$.

Our main interest is with filters that have z transforms that can be written as $H(z) = z^k A(z)/B(z)$ for some $A(z)$ and $B(z)$ which are polynomials in z^{-1}. Since we deal with both causal and anticausal filters we shall often have positive and negative powers of z^{-1}. A function that has terms in both z and z^{-1} is not a polynomial, but we refer to it as an FIR function provided that it has a finite number of terms. The following shorthand notation for a causal FIR function of length N is used: $\sum_{n=0}^{N-1} a_n z^{-n} = (a_0, a_1, a_2, \cdots a_{N-1})$.

In the following we shall refer to any symmetric or antisymmetric filter that has a central term as having whole sample symmetry (WSS) or whole sample antisymmetry (WSA), and one that does not have a central term as having half sample symmetry or half sample antisymmetry (HSS or HSA). In the case of FIR filters, WSS and WSA correspond to filters of odd length, and are often referred to as Type I and Type III filters, respectively [19]; whereas HSS and HSA imply filters of even length, called Type II and Type IV, respectively. Some of the basic properties of symmetric sequences that we will need are reviewed in the Appendix subsection A.

II. WAVELETS AND FILTER BANKS

A. Multiresolution Signal Processing

The material of this section can also be found in [4], [2], [11], [15]. Two texts give very comprehensive treatments [5], [20]; a more tutorial approach is given in [21].

The axiomatic description of a multiresolution analysis scheme, as introduced by Mallat [1], [3] and Meyer [5] is that we should have

i) A succession of spaces:

$$\cdots V_2 \subset V_1 \subset V_0 \subset V_{-1} \cdots, \qquad (1)$$

where the union of all the V_j's is $L^2(R)$, and the intersection of all of the spaces contains only the origin,

ii) $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j-1}$,

iii) $\exists \phi(x) \in V_0$ such that the set $\phi(x - n), n \in Z$ constitutes an orthonormal basis[1] for V_0.

It follows that the set $\{\phi_{jk}(x) = 2^{-j/2} \cdot \phi(2^{-j}x - k), k \in Z\}$ is an orthonormal basis for V_j.

Next, let W_j be the orthogonal complement of V_j in V_{j-1}; that is, $x \in V_j, y \in W_j \Leftrightarrow \langle x, y \rangle = 0$ and $V_{j-1} = V_j \oplus W_j$. Obviously $V_j \subset V_{j-1}$ and $W_j \subset V_{j-1}$, so that the basis functions of V_j and W_j ($\phi_{jk}(x)$ and $\psi_{jk}(x)$, respectively) can be written as a linear combination of the basis functions of $V_{j-1} (\phi_{j-1k}(x))$. This gives the relations:

$$\phi(x) = 2^{1/2} \cdot \sum_{n=-\infty}^{\infty} h_0(n) \cdot \phi(2x - n) \qquad (2)$$

$$\psi(x) = 2^{1/2} \cdot \sum_{n=-\infty}^{\infty} h_1(n) \cdot \phi(2x - n) \qquad (3)$$

where infinitely many of the $h_0(n)$ and $h_1(n)$ may differ from zero. Since (2) relates $\phi(x)$ and $\phi(2x)$ it is called a two-scale difference equation. Note that $\phi(x)$ and $\psi(x)$ are called the scaling function and wavelet, respectively.

Because V_j and W_j are orthogonal we find

$$\langle \phi(x), \psi(x - k) \rangle = 0 = \langle h_0(n), h_1(n - 2k) \rangle. \quad (4)$$

Thus the two sequences $h_0(n)$ and $h_1(n)$ must be orthogonal with respect to shifts by two.

Equally, by imposing the constraint that the bases for V_j and W_j be orthogonal:

$$\langle \phi(x), \phi(x - k) \rangle = \delta_k = \langle \psi(x), \psi(x - k) \rangle \quad (5)$$

we find that

$$\langle h_0(n), h_0(n - 2k) \rangle = \delta_k = \langle h_1(n), h_1(n - 2k) \rangle. \quad (6)$$

B. Wavelets Derived from Filter Banks

We have seen above that the multiresolution analysis scheme with orthogonal basis functions satisfying (2) and (3) implies certain restrictions on the related sequences $h_0(n)$ and $h_1(n)$; that is, (4) and (6) must hold. Also, since

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[1]Actually it is sufficient to have a basis, which can then be orthogonalized.

$\langle \phi(2x - k), \phi(x) \rangle = 2^{-1/2} \cdot h_0(k)$, and $\langle \psi(2x - k), \phi(x) \rangle = 2^{-1/2} h_1(k)$; it is obvious that once the basis functions $\phi(x)$ and $\psi(x)$ are known the related filters are easily found. However, it is not yet obvious how functions satisfying the desired constraints may be found.

1) Limit Functions of Filter Banks: A way to construct $\phi(x)$ and $\psi(x)$ from the associated discrete sequences was first shown by Daubechies in [4]. Essentially it entails considering the limit of a sequence of functions $f^{(i)}(x)$ which are piecewise constant on intervals of length $1/2^i$. The value of the constant is equal to the coefficient of a filter found by cascading i copies of the filter $H_0(z)$ followed by a subsampler [11], [21].

Assuming for the moment that the limit exists, and that the filters $h_0(n)$, $h_1(n)$ satisfy the orthogonality constraints (4) and (6), we have as $i \to \infty$

$$f^{(\infty)}(x) = 2^{1/2} \cdot \sum_{m=-\infty}^{\infty} h_0(m) f^{(\infty)}(2x - m). \quad (7)$$

Taking the Fourier transform:

$$F^{(\infty)}(w) = 2^{-1/2} \cdot H_0(e^{jw/2}) \cdot F^{(\infty)}(w/2). \quad (8)$$

Now define $M_0(e^{jw}) = 2^{-1/2} H_0(e^{jw})$, so that

$$F^{(\infty)}(w) = M_0(e^{jw/2}) F^{(\infty)}(w/2)$$

$$= \prod_{l=1}^{\infty} M_0(e^{jw/2^l}) \cdot F^{(\infty)}(0). \quad (9)$$

Now also consider the related function:

$$G^{(\infty)}(w) = 2^{-1/2} \cdot H_1(e^{jw/2}) \cdot F^{(\infty)}(w/2).$$

so that

$$g^{(\infty)}(x) = 2^{1/2} \cdot \sum_{m=-\infty}^{\infty} h_1(m) f^{(\infty)}(2x - m). \quad (10)$$

On comparing (7) and (2), (10) and (3) we now see that $f^{(\infty)}(x)$, and $g^{(\infty)}(x)$ satisfy the two scale difference equations required of the orthonormal wavelet construction. It can be verified that the orthogonality relations of the functions $f^{(\infty)}(x)$, and $g^{(\infty)}(x)$ follow from the orthogonality of the filters [4], [11].

Thus the problem of finding the basis functions for the wavelet scheme is reduced to one of finding appropriate pairs of sequences $h_0(n)$ and $h_1(n)$. For much of the rest of the paper, we will concern ourselves with this.

2) Regularity: That the infinite product (9) converges as $i \to \infty$ cannot be taken for granted. Cases where convergence fails altogether, or where the product converges to a discontinuous function are easily found. We would like some guarantees about the convergence of (9) and the continuity of the functions $\phi(x)$ and $\psi(x)$, when they exist. Exactly such a criterion is derived in [4], and is reviewed below.

First factor $M_0(z)$ into its roots at $z = -1$ (if there is not at least one then the infinite product cannot converge [4], [22], [23]) and a remainder function $K(z)$, in the fol-

lowing way:

$$M_0(z) = [(1 + z^{-1})/2]^N K(z).$$

Note that it can be shown that $K(1) = 1$ from the definitions; i.e., (6) gives $H_0^2(1) = 2$, so that $M_0(1) = 1$. Now call B the supremum of $|K(z)|$ on the unit circle: $B = \sup_{w \in [0, 2\pi]} |K(e^{jw})|$. Then the following sufficient, but not necessary, test from [4] can be used.

Proposition 2.1 (Daubechies [4]): If $B < 2^{N-1}$, and

$$\sum_{n=-\infty}^{\infty} |k(n)|^2 |n|^\epsilon < \infty, \quad \text{for some } \epsilon > 0 \quad (11)$$

then the piecewise constant function $f^{(i)}(x)$ defined in (9) converges pointwise to a continuous function $f^{(\infty)}(x)$.

Since we will consider filters with a finite number of poles and zeros, it is clear that their impulse responses (which may be one or two sided) have exponential decay. Thus (11) is always satisfied, and $B < 2^{N-1}$ is the only check we need perform.

In the FIR case the wavelets generated have compact support; the wavelets generated by IIR filter banks, however, are supported on the whole real line, but will have exponential decay just as the filters themselves do [24].

C. Filter Banks

We now put the connection between filter banks and wavelets to work. Our interest in this paper is orthogonal wavelet bases, hence we restrict our attention to orthogonal filter banks. More general perfect reconstruction filter banks give rise to biorthogonal systems just as in the FIR case [11]. We assume some familiarity with the basic properties of multirate operations; these are detailed for example in [8], [12], [13].

1) Perfect Reconstruction: The structure shown in Fig. 1 is a maximally decimated two channel multirate filter bank. If $\hat{X}(z) = X(z)$ the filter bank has the perfect reconstruction property, and we refer to it as a PRFB. We now make the following choice for the synthesis filters:

$$[G_0(z), G_1(z)] = [H_0(z^{-1}), H_1(z^{-1})] \quad (12)$$

and choose $H_1(z) = z^{2k-1} H_0(-z^{-1})$. It is easily shown that the output $\hat{X}(z)$ of the overall analysis/synthesis system is then given by

$$\hat{X}(z) = \frac{1}{2}[G_0(z) G_1(z)] \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} \begin{bmatrix} X(z) \\ X(-z) \end{bmatrix} \quad (13)$$

$$= \frac{1}{2}[H_0(z) H_0(z^{-1}) + H_0(-z) H_0(-z^{-1})] \cdot X(z). \quad (14)$$

So for this arrangement of the filters it is clear that we get perfect reconstruction provided:

$$H_0(z) H_0(z^{-1}) + H_0(-z) H_0(-z^{-1}) = 2.$$

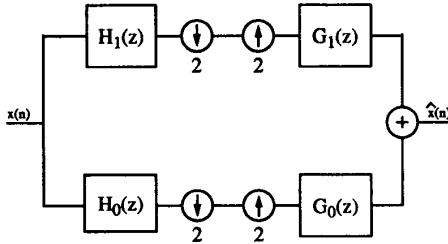The importance of this construction is established by the next lemma.

Fig. 1. Maximally decimated two channel multirate filter bank.

We first introduce additional notation that we will need. The 2×2 matrix in (13) is called $H_m(z)$, the modulation matrix of the system or the aliasing component matrix [8]. The following polyphase notation for the filters is standard [8], [12], [13]:

$$H_i(z) = H_{i0}(z^2) + z^{-1} H_{i1}(z^2)$$

that is, $h_{i0}(n)$ contains the even-indexed coefficients of the filter $h_i(n)$, while $h_{i1}(n)$ contains the odd ones. Thus

$$\begin{bmatrix} H_{00}(z^2) & H_{01}(z^2) \\ H_{10}(z^2) & H_{11}(z^2) \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}. \quad (15)$$

The matrix on the left-hand side is called the polyphase matrix $H_p(z^2)$.

Lemma 2.2: The following are equivalent:

a) $[H_m(z^{-1})]^T \cdot H_m(z) = 2 \cdot I$,
b) $[H_p(z^{-1})]^T \cdot H_p(z) = I$,
c) $H_0(z) H_0(z^{-1}) + H_0(-z) H_0(-z^{-1}) = 2$ and $H_1(z) = z^{2k-1} H_0(-z^{-1}) A(z^2)$, where $A(z)$ is all pass,
d) $\langle h_i(n), h_i(n - 2k) \rangle = \delta_k$ and $\langle h_0(n), h_1(n - 2k) \rangle = 0 \ \forall \ k \in Z$.

A proof can be found in [25]. It is also proved that the choice of synthesis filters (12) is unique for the orthogonal construction.

Because of the impulse response relations in d) we shall refer to any filter bank satisfying the conditions of Lemma 2.2 as orthogonal; in the filter bank literature the terms orthogonal, paraunitary, and lossless are often used interchangeably [26]. Observe that in Lemma 2.2c) we use functions of the form $H_i(z) H_i(z^{-1})$, which are called *autocorrelation* or *positive real* functions. It deserves mention that the study of lossless systems and positive real functions has a long history in both circuit theory and signal processing [26]–[29]. When we wish to impose orthogonality on the filter bank to be used, we shall use whichever of the equivalent conditions of Lemma 2.2 is most convenient.

Note that the particular solution given after (12), $H_1(z) = z^{2k-1} H_0(z^{-1})$, did not contain the all pass introduced in c). This is the solution where FIR filters are used, since the only FIR all-pass function is a delay.

Note also that if we define $P(z) = H_0(z) H_0(z^{-1})$ then c) requires that in addition to being an autocorrelation, $P(z)$ satisfies

$$P(z) + P(-z) = 2. \quad (16)$$

Since this condition plays an important role in what follows, we will refer to any function having this property as *valid*. Much of the focus of the paper will be in designing autocorrelation functions that are valid. We shall be interested only in valid functions that are rational, so that they can be factored into rational filters $H_0(z)$ and $H_0(z^{-1})$. These filters can then be implemented using recursive difference equations [19], whereas filters that do not have rational transfer functions have no finite complexity physical implementation.

III. ORTHOGONAL IIR FILTER BANKS

We have already seen that constructing an orthogonal filter bank can be reduced to the task of finding a function $P(z)$ which is a valid autocorrelation; that is, a function that satisfies (16) and can be factored as $P(z) = H(z) H(z^{-1})$. We first establish a preliminary result on the form of valid rational functions.

Lemma 3.1: If a valid function $P(z)$ has no common factors between the numerator and denominator, then the denominator is one of the two upsampled polyphase components of the numerator.

Proof: We can write

$$P(z) = \sum_{n = -\infty}^{\infty} [p(2n) + p(2n + 1)z^{-1}] z^{-2n}$$

so the constraint gives

$$P(z) + P(-z) = 2 \cdot \sum_{n = -\infty}^{\infty} p(2n) z^{-2n}$$

$$= 2 \Rightarrow p(2n) = \delta_n. \quad (17)$$

Clearly,

$$P(z) = 1 + \sum_{n = -\infty}^{\infty} p(2n + 1) z^{-2n - 1}$$

$$= 1 + z^{-1} F(z^2). \quad (18)$$

If $F(z^2)$ has no common factors between its numerator and denominator, then they must each be functions of z^2, possibly multiplied by some delay z^k. That is, $F(z^2) = z^k N(z^2)/(z^k D(z^2))$. So we have

$$P(z) = \frac{z^{-k} D(z^2) + z^{-(k+1)} N(z^2)}{z^{-k} D(z^2)}.$$

Thus the denominator is the first polyphase component if k is even, and the second if k is odd. The numerator and denominator of $P(z)$ are coprime if and only if $N(z)$ and $D(z)$ are. $\qquad \Box$

A. Structure of the Solutions

Clearly Lemma 3.1 gives a simple method to design a rational function $P(z)$ which is valid. Lemma 2.2 then

shows that this can be used to give an orthogonal filter bank if this function is an autocorrelation, that is it can be factored as $P(z) = H_0(z)H_0(z^{-1})$, since the essential requirement of Lemma 2.2c) is that $H_0(z)H_0(z^{-1})$ be valid. The next theorem puts these parts together and shows how to design valid autocorrelation functions, and hence orthogonal filter banks. Its utility is that it is constructive and complete.

Theorem 3.2: All orthogonal rational two channel filter banks can be formed as follows:

i) Choosing an arbitrary polynomial $R(z)$, form

$$P(z) = \frac{2 \cdot R(z)R(z^{-1})}{R(z)R(z^{-1}) + R(-z)R(-z^{-1})}. \qquad (19)$$

ii) Factor as $P(z) = H(z)H(z^{-1})$.
iii) Form the filter $H_0(z) = A_0(z)H(z)$, where $A_0(z)$ is an arbitrary all pass.
iv) Choose $H_1(z) = z^{2k-1}H_0(-z^{-1})A_1(z^2)$, where $A_1(z)$ is again an arbitrary all pass.
v) Choose $G_0(z) = H_0(z^{-1})$, and $G_1(z) = H_1(z^{-1})$.

Proof: From Lemma 2.2c) it is necessary and sufficient to find a valid rational autocorrelation function $P(z)$; since once this is factored as $P(z) = H_0(z)H_0(z^{-1})$ then $H_1(z)$ is specified by Lemma 2.2c), and $G_0(z)$ and $G_1(z)$ by (12).

We show first that (19) always gives a valid, rational autocorrelation. It is valid, since

$$P(z) + P(-z) = \frac{2 \cdot R(z)R(z^{-1}) + R(-z)R(-z^{-1})}{R(z)R(z^{-1}) + R(-z)R(-z^{-1})}$$

$$= 2.$$

It is clearly rational, $R(z)$ being a polynomial. The numerator of (19) is an autocorrelation; so is the denominator, since it is the sum of two autocorrelations $R(z)R(z^{-1})$ and $R(-z)R(-z^{-1})$. Hence $P(z)$ itself is an autocorrelation and can be factored

$$P(z) = H(z)H(z^{-1}) = H_0(z)H_0(z^{-1}) \cdot A_0(z)A_0(z^{-1})$$

for some $H(z)$ and an arbitrary rational all-pass $A_0(z)$.

Next we show that any valid rational autocorrelation can be written as in (19) for some polynomial $R(z)$.

First, any common factors between the numerator and denominator of the given function can be cancelled; the result is clearly still a valid rational autocorrelation. So it can be written

$$P(z) = \frac{R(z)R(z^{-1})}{B(z)B(z^{-1})}$$

for some polynomials $R(z)$ and $B(z)$. Now we can use Lemma 3.1 to get that the denominator, $B(z)B(z^{-1})$, is one of the unsampled polyphase components of the numerator:

$$D_0(z^2) = [R(z)R(z^{-1}) + R(-z)R(-z^{-1})]/2$$

or

$$D_1(z^2) = [R(z)R(z^{-1}) - R(-z)R(-z^{-1})]/2.$$

Note that $R(z)R(z^{-1})$ is always of odd length and is symmetric. It follows that one of its upsampled polyphase components, $D_0(z^2)$, is whole sample symmetric (WSS), while $D_1(z^2)$ is half sample symmetric (HSS). Since half sample symmetric polynomials always have at least one zero at $z = -1$ (see Appendix subsection A), $D_1(z^2)$ is not a suitable choice for the denominator, as we wish to avoid poles on the unit circle. We therefore have that

$$P(z) = \frac{2 \cdot R(z)R(z^{-1})}{R(z)R(z^{-1}) + R(-z)R(-z^{-1})} \qquad \square \quad (20)$$

Remarks:

1) If $H_0(z)$ is causal then $H_1(z)$ will be anticausal; similarly for $G_0(z)$ and $G_1(z)$. This implies that IIR orthogonal filter banks can only be realized over finite length signals; in this case the filters are implemented as the sum or product of causal and anticausal components [30]–[32].

2) The introduction of the all-pass factors $A_0(z)$ and $A_1(z)$ affect only the phase of the filters to be implemented, and not their magnitudes. Equally, in the factorization required by step ii) there is considerable choice for the phase of the filters; $H(z)$ could be minimum phase or maximum phase or mixed phase. The magnitude of course does not change. Irrational orthogonal factorizations of a rational $P(z)$ function are also possible. We give an example in Section IV-D.

3) The theorem shows that if $R(z)$ ranges over the polynomials then (20) is complete for rational $P(z)$ functions. If $R(z)$ is chosen to be any function, rational or not, it is clear by inspection that (20) will still be a valid autocorrelation, but not in general rational. Completeness is less obvious in this case.

4) If $R(z)R(z^{-1})$ is itself valid, that is $R(z)R(z^{-1}) + R(-z)R(-z^{-1}) = 2$, and $A_0(z)$ and $A_1(z)$ are both chosen to be delays, then all of the filters specified by Theorem 3.2 are FIR. The synthesis filters are always time reversed versions of the analysis filters, just as in the orthogonal FIR case [11]. All of the FIR orthogonal filter banks can be implemented in a paraunitary lattice structure [10]; a similar result is true for IIR orthogonal filter banks [33], so an efficient and numerically robust implementation is always available.

5) Examples of orthogonal IIR filter banks have been noted before; the earliest examples appear to be in [30] and [34]. They were also studied in [33].

B. Closed Form Factorization

Theorem 3.2 establishes the importance of valid rational functions which are autocorrelations. Numerical factorization poses certain difficulties, however. This is certainly a problem in the FIR case; for example, even when $P(z)$ is known exactly, the accuracy with which the coefficients of $H_0(z)$ can be determined is dependent on the numerical robustness of the root extraction procedure.

We now show that in the special case where $R(z)$ is symmetric and of even length, a closed form factorization is available. The requirement that $R(z)$ be symmetric is

very reasonable, since the numerator has to control the stopband of the filter $H(z)$ and typically has all of its zeros on the unit circle; if this is so, then $R(z)$ is symmetric provided that it is real. For example, all of the digital Butterworth, Chebyshev, and elliptic filters have symmetric numerators.

Consider a causal symmetric FIR function $R(z)$ of even length $N + 1$. Using the relationship between the polyphase components given in fact A.1 in the Appendix: $R_1(z) = R_0(z^{-1})z^{-(N-1)/2}$ we can simplify

$$R(z) = R_0(z^2) + z^{-1}R_1(z^2) = R_0(z^2) + z^{-N}R_0(z^{-2}).$$

(21)

This gives

$$R(z)R(z^{-1}) = [R_0(z^2) + z^{-N}R_0(z^{-2})]$$
$$\cdot [R_0(z^{-2}) + z^N R_0(z^2)]$$
$$= 2R_0(z^2)R_0(z^{-2}) + [z^{-N}R_0(z^{-2})R_0(z^{-2})$$
$$+ z^N R_0(z^2)R_0(z^2)].$$

Clearly, since N is odd:

$$D_0(z^2) = [R(z)R(z^{-1}) + R(-z)R(z^{-1})]/2$$
$$= 2R_0(z^2)R_0(z^{-2}).$$

And hence

$$P(z) = \frac{R(z)R(z^{-1})}{2R_0(z^2)R_0(z^{-2})}.$$

It is now obvious that one possible choice for factorizing $P(z)$ is

$$H(z) = \frac{R(z)}{\sqrt{2}\, R_0(z^2)}.$$

(22)

Since $R(z)$ and $R_0(z^2)$ are known exactly, this is a closed form; so $H(z)$ is directly available. Example 4.1 below illustrates this. The importance of this result can be seen by noting that the coefficients of the wavelet expansion can be obtained exactly, since they do not depend on any numerical procedure to find the transfer functions $H_0(z)$ and $H_1(z)$. This appears to be the first closed form for the filters used to generate a nontrivial realizable wavelet.

Remarks:

1) Observe that $H(z)$ can be written

$$H(z) = 2^{-1/2} \cdot (1 + z^{-N}A(z^2))$$

(23)

where $A(z) = R_0(z^{-1})/R_0(z)$ is an $(N-1)/2$-th order all pass. The other analysis and synthesis filters have similar expressions, and thus can be implemented very efficiently. It is worth pointing out that the filters in this particular case are themselves valid.

2) Note that $H(z)$ in (22) will in general have poles both inside and outside the unit circle. Since causal and anticausal parts must be implemented separately if the filter is to be stable it will be necessary to know the factors

of $R_0(z)$. This is easily achieved if in the design procedure it is written in terms of its roots rather than in direct form

$$R_0(z) = \prod_i (1 - \alpha_i z^{-1}).$$

In this case the causal and anticausal parts of $H_0(z)$ can be identified and implemented without factorization.

IV. Wavelets with Moment Properties

According to Proposition 2.1 the limits of iterated orthogonal digital filter banks can be used to derive wavelet bases. The sufficient condition to guarantee continuity of the wavelets was that the iterated low-pass filter, that is $H_0(z)$, should contain an adequate number of zeros at $z = -1$. It is for this reason that in the design of compactly supported wavelet bases [4], [35], [11] the emphasis was placed on using filters that have a maximum number of zeros at $z = -1$. In addition, a zero of order N at $z = -1$ in $H_0(z)$ implies N vanishing moments for the wavelet [4]

$$\int_{-\infty}^{\infty} x^k \psi(x)\, dx = 0 \qquad k = 0, 1, \cdots, N - 1. \quad (24)$$

It can be shown that having a maximum number of zeros at $z = -1$, implies a maximally flat characteristic for the filters involved [4], [25], [36]. This implies that both the wavelet and the filter spectrum have considerable smoothness, which may be advantageous in certain contexts.

Our procedure to design orthogonal filters amounts then to the following:

i) Choosing $B_{2N}(z) = (1 + z^{-1})^N(1 + z)^N$ for some N,

i) Finding least degree positive real $F(z) = F_N(z)/F_D(z)$ such that

$$P(z) = B_{2N}(z)F(z)$$
$$= \frac{(1 + z^{-1})^N(1 + z)^N F_N(z)}{F_D(z)}$$

is valid, and

iii) Factoring $P(z) = H_0(z)H_0(z^{-1})$.

Of course in [4] only FIR solutions were of interest; so the solutions had $F_D(z) = 1$. In other words, the multiplicative factor $F(z)$ required to make $B_{2N}(z)F(z)$ valid had only zeros. In the next subsection we examine the opposite extreme, where $F(z)$ is all pole, i.e., $F_N(z) = 1$. These in fact give rise to the Butterworth halfband filters.

In Section IV-B we examine solutions intermediate between the Daubechies ($F(z)$ all zero), and Butterworth ($F(z)$ all pole); that is, where $F(z)$ is still of minimal degree, but has some combination of poles and zeros.

A. Butterworth Wavelets

Using Theorem 3.2, constructing regular IIR filter banks that lead to infinitely supported wavelets is very simple. Following Daubechies and the FIR case, if we again place a maximum number of zeros at $z = -1$ then

we simply choose $R(z) = (1 + z^{-1})^N$. This gives

$$P(z) = \frac{2 \cdot (1 + z^{-1})^N (1 + z)^N}{(z^{-1} + 2 + z)^N + (-z^{-1} + 2 - z)^N}$$

$$= H_0(z) H_0(z^{-1}). \tag{25}$$

These filters are the IIR counterparts of the FIR filters given in [4] in that they generate wavelets with regularity that increases linearly with the degree N of the zero at $z = -1$. That these filters indeed satisfy the requirements of Proposition 2.1 can be verified numerically (see Section IV-C).

These are in fact the Nth order halfband digital Butterworth filters [19]. That these particular filters satisfy the conditions for perfect reconstruction was also pointed out in [32], [37], and their use for the construction of wavelets in [38], [39]. The Butterworth filters are known to be the maximally flat IIR filters of a given order.

We propose these Butterworth wavelets as alternatives to the compactly supported examples of [4]; they enjoy exactly the same moment properties, but achieve much better filtering action for the same complexity, and are considerably smoother. An additional advantage is that since $R(z)$ is symmetric we can make use of the closed form factorization of Section III-B if we choose N to be odd. So in this case we can explicitly write

$$H_0(z) = \frac{\displaystyle\sum_{k=0}^{N} \binom{N}{k} z^{-k}}{\sqrt{2} \cdot \displaystyle\sum_{l=0}^{(N-1)/2} \binom{N}{2l} z^{-2l}} \tag{26}$$

and the other filters follow from Theorem 3.2.

Example 4.1: Take $R(z) = (1 + z^{-1})^N$ as above and $N = 7$, so that we can use the closed form factorization, hence

$$P(z) = \frac{(1, 14, 91, 364, 1001, 2002, 3003, 3432, 3003, 2002, 1001, 364, 91, 14, 1) \cdot z^7}{14z^6 + 364z^4 + 202z^2 + 3432 + 2002z^{-2} + 364z^{-4} + 14z^{-6}}$$

$$= \frac{E(z)E(z^{-1})}{F(z)F(z^{-1})}$$

where

$$\frac{E(z)}{F(z)} = \frac{(1 + 7z^{-1} + 21z^{-2} + 35z^{-3} + 35z^{-4} + 21z^{-5} + 7z^{-6} + z^{-7})}{\sqrt{2} \cdot (1 + 21z^{-2} + 35z^{-4} + 7z^{-6})}$$

So using the description of the filters in Theorem 3.2, with the simplest case $A_0(z) = A_1(z) = 1$ and $k = 0$ we find

$$H_0(z) = \frac{(1 + 7z^{-1} + 21z^{-2} + 35z^{-3} + 35z^{-4} + 21z^{-5} + 7z^{-6} + z^{-7})}{\sqrt{2} \cdot (1 + 21z^{-2} + 35z^{-4} + 7z^{-6})}$$

$$H_1(z) = z^{-1} \frac{(1 - 7z^1 + 21z^2 - 35z^3 + 35z^4 - 21z^5 + 7z^6 - z^7)}{\sqrt{2} \cdot (1 + 21z^2 + 35z^4 + 7z^6)}$$

$$G_0(z) = H_0(z^{-1}) \qquad G_1(z) = H_1(z^{-1}).$$

In the notation of Proposition 2.1, $B = 8 < 2^6$ so that for this choice of $H_0(z)$ the left-hand side of (7) converges to a continuous function. The wavelet, scaling function, and their spectra are shown in Fig. 2.

Remarks:

1) As mentioned in Section III-B, if the poles of the closed form lie both inside and outside the unit circle, one will still need the factors for implementation. For the halfband Butterworth filters a closed form for the pole positions themselves is easily derived [19], [40]. For example, in the case where N is odd the poles of $H_0(z)$ are

$$z_k = \pm j \left[\tan \frac{(2k + 1)\pi}{2N} \right]^{-1}$$

$$k = 0, 1, \cdots, (N - 3)/2.$$

For the $N = 7$ case above the poles are thus $\{\pm j4.38128, \pm j1.25396, \pm j0.48157\}$. Thus $H_0(z)$ can be implemented as the product of a stable left-sided and a stable right-sided filter $H_0(z) = H_L(z) H_R(z)$, where

$$H_L(z) = \frac{(1 + z^{-1})^4}{(1 + 4.38128^2 z^{-2})(1 + 1.25396^2 z^{-2})}$$

$$H_R(z) = \frac{(1 + z^{-1})^3}{\sqrt{2} \cdot (1 + 0.48157^2 z^{-2})}.$$

In the Butterworth case where N is even we can still get a closed form for the $2N$ poles of $P(z)$, even though the closed form factorization (22) is not used. The poles are at

$$z_k = \pm j \left[\tan \frac{(2k + 1)\pi}{4N} \right]^{-1} \qquad k = 0, 1, \cdots, N - 1.$$

2) All of the wavelet plots in this paper were produced using six iterations of the "graphical recursion" [4]. The time axis is scaled such that the advertised orthogonality
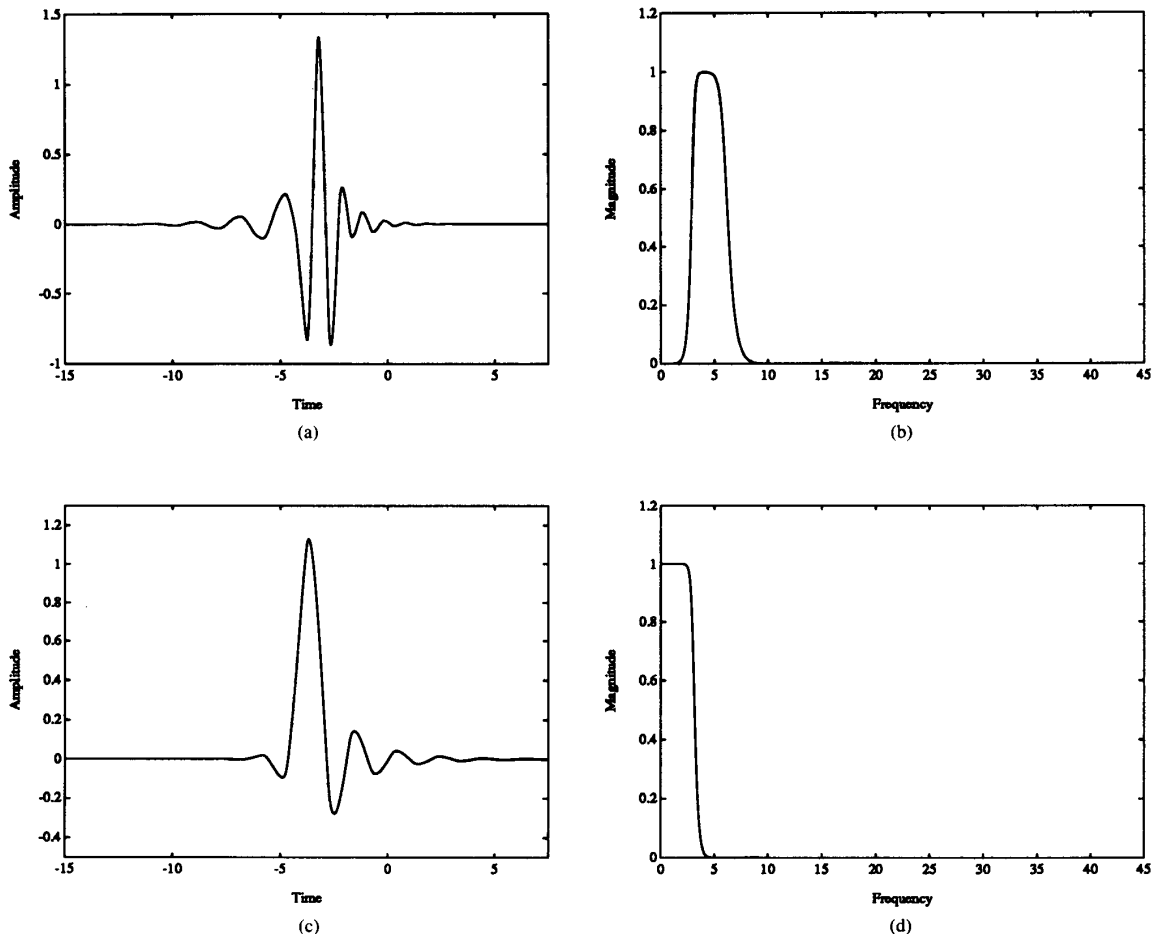
Fig. 2. Example of Butterworth orthogonal wavelet; here $N = 7$, and the closed form factorization has been used. (a) The wavelet. (b) Spectrum of the wavelet. (c) Scaling function. (d) Spectrum of the scaling function.

properties with respect to integer shifts are indeed satisfied. The frequency axes are scaled such that the Fourier domain version of the orthogonality condition (derived by applying the Poisson summation to (5)) is satisfied.

B. Intermediate Solutions

At the beginning of the section we pointed out that in the construction of wavelets with a certain number of vanishing moments, the essence of the design was finding a minimal degree $F(z) = F_N(z)/F_D(z)$, such that $P(z) = B_{2N}(z)F(z)$ was valid. We now explore examples between the extremes of the Daubechies $(F_D(z) = 1)$ and the Butterworth $(F_N(z) = 1)$ cases.

First note that when $P(z)$ is a rational autocorrelation, both numerator and denominator will be of odd length, and symmetric. As pointed out in the proof of Theorem 3.2 the denominator is in fact the upsampled whole sample symmetric (WSS) polyphase component of the numerator. There are two cases:

- A symmetric FIR function of length $4k + 1$ has an upsampled WSS polyphase component of length $4k + 1$.

- A symmetric FIR function of length $4k + 3$ has an upsampled WSS polyphase component of length $4k + 1$.

To find a solution where $P(z)$ has less poles than in the Butterworth case we must find a function $F_N(z)/F_D(z)$ where $F_D(z)$ is of length $4(k - p) + 1$ for some $0 < p < k$ and $F_N(z)$ is of minimal degree such that

$$P(z) = \frac{(1 + z^{-1})^N(1 + z)^N F_N(z)}{F_D(z)}$$

is valid. In the Daubechies case we fixed $F_D(z) = 1$ and found the minimal degree $F_N(z)$, and in the Butterworth we fixed $F_N(z) = 1$ and found the minimal degree $F_D(z)$. For the intermediate cases we fix the length of $F_D(z)$ as $4(k - p) + 1$ for some $0 < p < k$ and then find the minimal degree $F_N(z)$. For a given binomial factor $(1 + z^{-1})^N(1 + z)^N$ the total number of poles and zeros of $F(z)$ will not necessarily be the same for the Daubechies, intermediate and Butterworth solutions, although, in fact, it will never vary by more than two.

Note that $F_D(z)$ is the WSS component of $(1 + z^{-1})^N(1 + z)^N F_N(z)$, but is to be of lower degree than the WSS component of $(1 + z^{-1})^N(1 + z)^N$. Thus it is apparent that

some of the terms of $(1 + z^{-1})^N (1 + z)^N F_N(z)$ must be zero, and the WSS component must not contain the end terms. This last condition implies that we must have that $(1 + z^{-1})^N (1 + z)^N F_N(z)$ is of length $4k + 3$; since otherwise, if it is of length $4k + 1$, the WSS component is also of length $4k + 1$, and contains the end terms. It is convenient to treat separately the two cases for N even and odd.

$N = 2k + 1$ odd: The length of the denominator is $4(k - p) + 1$. If we try $F_N(z)$ of length $4p + 1$ then the length of the numerator, $(1 + z^{-1})^N (1 + z)^N F_N(z)$, is $4(k + p) + 3$, and that of its WSS component $4(k + p) + 1$. The difference between the length of the WSS component of $(1 + z^{-1})^N (1 + z)^N F_N(z)$, and the length of $F_D(z)$ is hence $4 \cdot 2p$. Since the WSS component of the numerator is symmetric, and a function of z^2, setting one pair of its end terms to zero in fact decreases its length by 4. If this

We have used d to indicate elements of the HSS component of the numerator, and x for elements of the WSS component. Clearly if the indicated end terms of the WSS component equal zero, then the denominator will be of length 9. It is easily seen that the conditions to set the end terms to zero are

$$14 + a = 0$$

$$364 + 92a + 14b = 0$$

$\Rightarrow (a, b) = (-14, 66)$. Thus $F_N(z) = (z^2 - 14z + 66 - 14z^{-1} + z^{-2})$. Here $B = 8.011 < 2^6$ and again convergence to a continuous function is guaranteed. The wavelet, scaling function, and spectra are shown in Fig. 3.

The second intermediate solution is for $p = 2$, so that $F_N(z)$ is of length 9, and we want the WSS component of the numerator to have $2p = 4$ pairs of end terms set to zero:

	z^{11}	z^{10}	z^9	z^8	z^7	z^6	z^5	z^4	z^3	z^2
$z^4 B_{14}(z)$	1	14	91	364	1001	2002	3003	3432	3003	$\cdots$
$az^3 B_{14}(z)$		1	14	91	364	1001	2002	3003	3432	$\cdots$
$bz^2 B_{14}(z)$			1	14	91	364	1001	2002	3003	$\cdots$
$cz^1 B_{14}(z)$				1	14	91	364	1001	2002	$\cdots$
$dz B_{14}(z)$					1	14	91	364	1001	$\cdots$
$cz^{-1} B_{14}(z)$						1	14	91	364	$\cdots$
$bz^{-2} B_{14}(z)$							1	14	91	$\cdots$
$az^{-3} B_{14}(z)$								1	14	$\cdots$
$z^{-4} B_{14}(z)$									1	$\cdots$
$B_{14}(z) F_N(z)$	d	x $\uparrow$	d	x $\uparrow$	d	x $\uparrow$	d	x $\uparrow$	d	$\cdots$

can be done $2p$ times then the WSS component of the numerator, and the denominator will be of the same length. Note that $2p$ is also the number of independent elements in $F_N(z)$. In fact, the solution is found by solving a $2p \times 2p$ system of linear equations.

$N = 2k$ even: The length of the denominator again is $4(k - p) + 1$. Now if we try $F_N(z)$ of length $4(p - 1) + 3$, the length of the numerator is $4(k + p - 1) + 3$, and that of its WSS component $4(k + p - 1) + 1$. The difference between the length of the WSS component of the numerator, and the length of $F_D(z)$ is hence $4 \cdot (2p - 1)$. Again $2p - 1$ is the number of independent elements in $F_N(z)$. In this case the solution is found by solving a set of $(2p - 1) \times (2p - 1)$ linear equations.

Example 4.2: $N = 7$. Note that $N = 2k + 1$ where $k = 3$. There are thus two intermediate solutions for $p = 1, 2$. Taking the $p = 1$ case first, note that $F_N(z)$ is of length 5, and we wish to set $2p = 2$ pairs of end terms of the WSS component of the numerator to zero. The situation is illustrated below:

The conditions to set the indicated end terms to zero now are

$$14 + a = 0$$

$$364 + 91a + 14b + c = 0$$

$$2002 + 1001a + 364b + 92c + 14d = 0$$

$$3432 + 3004a + 2016b + 1092c + 364d = 0.$$

The solution is $(a, b, c, d) = (-14, 592/7, -274, 3218/7)$. Here $B = 8.011 < 2^6$ and convergence to a continuous function is assured. The wavelet, scaling function and spectra are very similar to those for $p = 1$.

Note: The denominator is of length $4(k - p) + 1$ in these intermediate solutions, with $0 < p < k$. For $p = 0$ we would get the Butterworth solution, and for $p = k$ the Daubechies'. For $k = 0, 1$, that is $N = 1, 2, 3$, there are obviously no intermediate solutions.

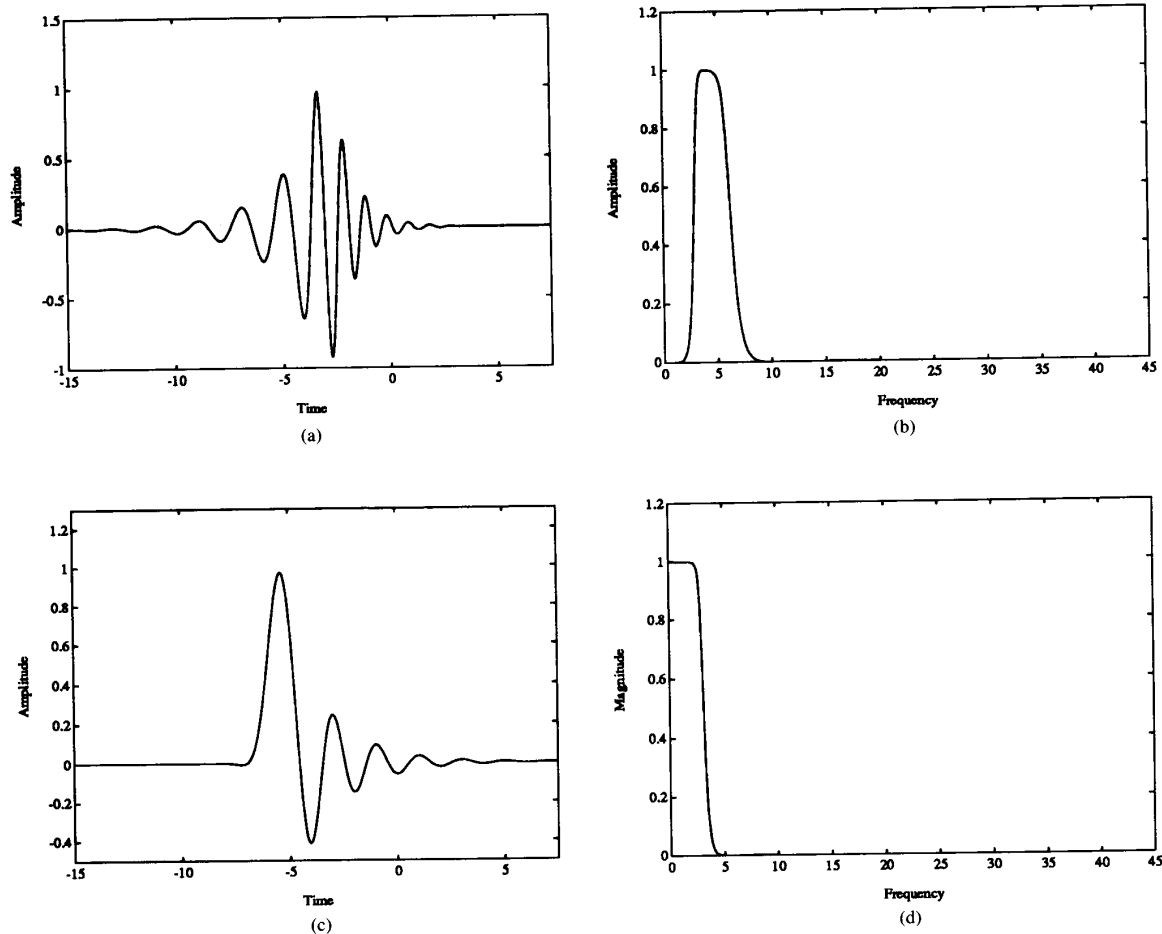|	z^9	z^8	z^7	z^6	z^5	z^4	z^3	z^2	z	z^0
$z^2 B_{14}(z)$	1	14	91	364	1001	2002	3003	3432	3003	2002	$\cdots$
$az B_{14}(z)$		1	14	91	364	1001	2002	3003	3432	3003	$\cdots$
$b B_{14}(z)$			1	14	91	364	1001	2002	3003	3432	$\cdots$
$az^{-1} B_{14}(z)$				1	14	91	364	1001	2002	3003	$\cdots$
$z^{-2} B_{14}(z)$					1	14	91	364	1001	2002	$\cdots$
$B_{14}(z) F_N(z)$	d	x $\uparrow$	d	x $\uparrow$	d	x	d	x	d	x	$\cdots$

Fig. 3. Example of intermediate solution orthogonal wavelet; this is the $N = 7$, $p = 1$ solution. (a) The wavelet. (b) Spectrum of the wavelet. (c) Scaling function. (d) Spectrum of the scaling function.

C. Tabulating the $P(z)$ Functions

A point we wish to emphasize is that in all of the design techniques discussed above it was the construction of $P(z)$ that was central. This was the case for the Daubechies' designs of [4], and the Butterworth and intermediate designs of Sections IV-A and B. Once $P(z)$ is determined the magnitude spectrum of $H_0(z)$ is fixed irrespective of the all-pass factors $A_0(z)$ and $A_1(z)$ of Theorem 3.2, and the factorization chosen.

If we desire filters that are maximally flat, or equivalently, wavelets that have a maximum number of vanishing moments then we design a $P(z)$ with the maximum number of zeros at $z = -1$. Those minimum degree $P(z)$'s with this property are easily listed, and this has been done in Table I for the cases $N = 1, 2, \cdots, 5$. The table exhausts the minimal degree maximally flat $P(z)$ autocorrelation functions for these orders. For each of the $P(z)$ functions, except the $N = 1$ case, the filters satisfy the conditions of Proposition 2.1; hence convergence is guaranteed. For each of the $P(z)$ functions we have included an estimate of r such that the corresponding wavelet is in

C^r; i.e., if $r > 1.0$ it is continuous, with a continuous derivative, etc. The estimation methods used were taken from [20], [23], [41].

For comparison purposes the graph of the $N = 7$ Daubechies wavelet and scaling function are given in Fig. 4.

D. Irrational Factorizations

Theorem 3.2 demonstrates how to calculate all valid rational autocorrelation functions. For implementation reasons we have been interested only in orthogonal rational factorizations. It is nonetheless possible to take an irrational factorization of a rational $P(z)$ function and use it to derive an orthonormal wavelet basis. For example, if we take $P(z) = H_0(z) H_0(z^{-1})$ where $H_0(z) = \sqrt{P(z)}$ we end up with linear phase filters. That $H_0(z)$ is necessarily irrational, where one of the $P(z)$ functions designed in this section is used, is guaranteed by Lemma 5.1 below. For example, if we use the Butterworth $N = 7$ case, as in example 4.1, we get the wavelet and scaling function shown in Fig. 5. The magnitude spectra plots are of course

TABLE I

The Various $P(z)$ Solutions for a Given Number of Zeros at $z = -1$. Daubechies, Intermediate, and Butterworth Solutions for $N = 1, \cdots 5$ Are Shown

Solution	$P(z)$	Regularity
$N = 1$ Haar	$(1 + z)(1 + z^{-1}) \cdot 2^{-1}$	$r = 0$
$N = 2$ Daubechies	$(1 + z)^2(1 + z^{-1})^2 \cdot (-1, 4, -1) \cdot z2^{-4}$	$r > 0.5 - \epsilon$
Butterworth	$\dfrac{(1 + z)^2(1 + z^{-1})^2 z^{-2}}{(1, 0, 6, 0, 1)}$	$r > 0.5$
$N = 3$ Daubechies	$(1 + z)^3(1 + z^{-1})^3 \cdot (3, -18, 38, -18, 3) \cdot z^2 2^{-8}$	$r > 0.9150$
Butterworth	$\dfrac{(1 + z)^3(1 + z^{-1})^3 \cdot z^{-2}}{(6, 0, 20, 0, 6)}$	$r > 1.0$
$N = 4$ Daubechies	$(1 + z)^4(1 + z^{-1})^4 \cdot (-5, 40, -131, 208, -131, 40, -5) \cdot z^3 2^{-12}$	$r > 1.2750$
Intermediate	$\dfrac{(1 + z)^4(1 + z^{-1})^4 \cdot (1, -8, 1) \cdot z^{-1}}{(160, 0, 448, 0, 160)}$	$r > 1.497$
Butterworth	$\dfrac{(1 + z)^4(1 + z^{-1})^4 z^{-4}}{(1, 0, 28, 0, 70, 0, 28, 0, 1)}$	$r > 1.5$
$N = 5$ Daubechies	$(1 + z)^5(1 + z^{-1})^5 \cdot (35, -350, 1520, -3650, 5018, -3650, 1520, -350, 35) \cdot z^4 2^{-16}$	$r > 1.5960$
Intermediate	$\dfrac{(1 + z)^5(1 + z^{-1})^5 \cdot (1, -10, 34, -10, 1)}{(1792, 0, 4608, 0, 1792)}$	$r > 1.9991$
Butterworth	$\dfrac{(1 + z)^5(1 + z^{-1})^5 z^{-4}}{(10, 0, 120, 0, 252, 0, 120, 0, 10)}$	$r > 2.0$

identical to those in Fig. 2, since these are independent of the factorization chosen. It is worth pointing out that the wavelet is very similar to an orthonormal wavelet constructed by Meyer, but based on irrational filters [5].

Clearly Theorem 3.2 generates all orthogonal filter banks where $P(z)$ is rational, even if the filters themselves are not so.

E. Wavelets with Fewer Moments

In [4] a complete characterization of all orthonormal wavelets based on FIR filter banks was given. Theorem 3.2 provides such a characterization for the IIR case, since it explicitly allows us to construct the whole family of orthonormal wavelets with a given number of disappearing moments. This was done above in the Butterworth, intermediate, and Daubechies cases for a maximum number of disappearing moments. A characterization of the family with N disappearing moments, whether N is maximal or not, is given by the filters found from Theorem 3.2 where $R(z)$ contains a factor $(1 + z^{-1})^N$. The resulting wavelets (assuming convergence) are then guaranteed to be orthonormal and have $\int x^k \psi(x) \, dx = 0$ $k = 0, 1, \cdots N - 1$.

V. Linear Phase Orthogonal IIR Solutions

In [4], [35], [11] it was pointed out that it is not possible to generate a nontrivial basis of real finite length wavelets which are orthonormal and symmetric. In fact, the only solution is the Haar basis, which is not continuous. If we were prepared to consider complex FIR filters it would be possible [42], but filters with complex coefficients are generally of less interest.

We have not thus far addressed the possibility of achieving linear phase with orthogonal rational IIR filters. We first consider the possibility that one of the maximally flat $P(z)$ functions already derived might factor $P(z) = H(z)H(z^{-1})$, where $H(z)$ is a rational linear phase filter. The next lemma proves that this is never possible for the Daubechies, intermediate, or Butterworth $P(z)$ functions of any order. In other words, if we desire linear phase filters the solutions presented so far will not serve.

After considering once more the structure of orthogonal IIR solutions, however, we see how the linear phase condition can be structurally imposed, and use this to generate designs. While the filters never have as many zeros at $z = -1$ as those of Section IV, they give wavelets that are very smooth. This result was presented in preliminary form in [16], [17].
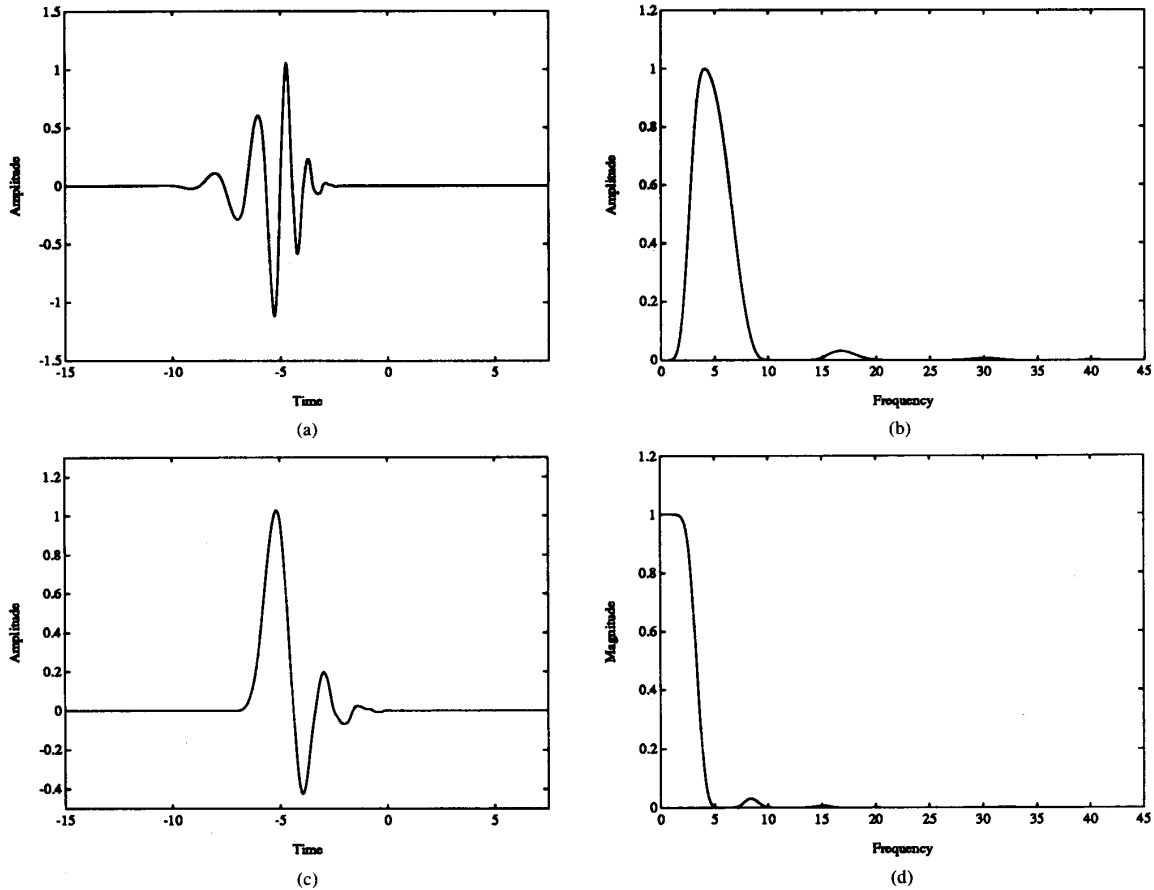
Fig. 4. Example of Daubechies orthogonal wavelet; this is the $N = 7$ case. (a) The wavelet. (b) Spectrum of the wavelet. (c) Scaling function. (d) Spectrum of the scaling function.
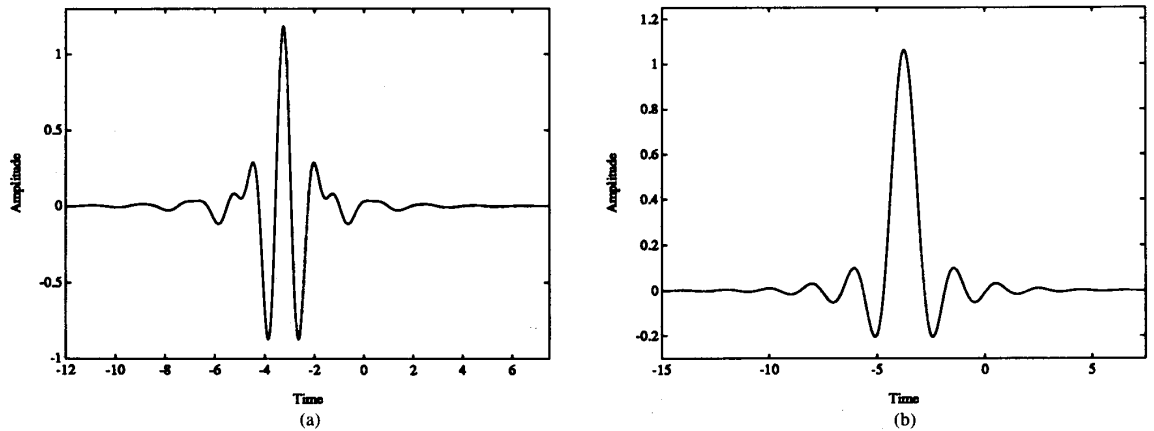

Fig. 5. Orthogonal basis from irrational factorization of Butterworth case $N = 7$. The magnitude spectra are as in Fig. 2. (a) The wavelet. (b) The scaling function.

A. Structure of Linear Phase Orthogonal Solutions

We first show that none of the particular orthogonal solutions presented so far can be used if rational filters are required.

Lemma 5.1: The Daubechies, intermediate, and Butterworth solutions to the equation

$$P(z) + P(-z) = 2$$

can never be factored as $P(z) = H(z)H(z^{-1})$ where $H(z)$ is a rational linear phase filter.

The proof is in Appendix subsection B.

The above result is not unexpected; all of these designs were found by merely ensuring that Lemma 2.2c) was satisfied. If we wish in addition to guarantee linear phase we shall have to impose this structurally before we begin the design. We find it more convenient to work with the equivalent condition Lemma 2.2b). We first recall a preliminary result on the structure of orthogonal polyphase matrices [4], [26].

Lemma 5.2: An orthogonal polyphase matrix is necessarily of the form:

$$H_p(z) = \begin{bmatrix} H_{00}(z) & H_{01}(z) \\ -H_{01}(z^{-1})\Delta_p(z) & H_{00}(z^{-1})\Delta_p(z) \end{bmatrix} \quad (27)$$

where

$$H_{00}(z)H_{00}(z^{-1}) + H_{01}(z)H_{01}(z^{-1}) = 1 = \Delta_p(z)\Delta_p(z^{-1}) \quad (28)$$

and $\Delta_p(z) = \det H_p(z)$ is an all-pass function.

Proof: Lemma 2.2b) gives immediately

$$\begin{bmatrix} H_{00}(z^{-1}) & H_{10}(z^{-1}) \\ H_{01}(z^{-1}) & H_{11}(z^{-1}) \end{bmatrix} = \frac{1}{\Delta_p(z)} \begin{bmatrix} H_{11}(z) & -H_{01}(z) \\ -H_{10}(z) & H_{00}(z) \end{bmatrix}$$

which leads to

$$H_{11}(z) = H_{00}(z^{-1})\Delta_p(z) = H_{00}(z^{-1})/\Delta_p(z^{-1})$$

from which follows that $\Delta_p(z^{-1}) = [\Delta_p(z)]^{-1}$, that is, $\Delta_p(z)$ is an all-pass filter [26]. Also

$$H_{10}(z) = -H_{01}(z^{-1})\Delta_p(z). \qquad \square$$

We have seen before the linear phase filters are of two types, those that have half sample symmetry or antisymmetry (HSS or HSA) and those that have whole sample symmetry or antisymmetry (WSS or WSA); again we find it convenient to treat them separately.

B. Half Sample Symmetric Case

If linear phase filters are half sample symmetric or antisymmetric then the polyphase components are related as in Fact A.1. We can use this to force linear phase on the polyphase filter matrix (27).

Lemma 5.3: In an orthogonal filter bank, where the filters are half sample symmetric, it is necessary and sufficient that the polyphase matrix be of the form

$$H_p(z) = \begin{bmatrix} A(z) & z^{-l}A(z^{-1}) \\ -z^{l-n}A(z) & z^{-n}Z(z^{-1}) \end{bmatrix} \quad (29)$$

where $A(z)A(z^{-1}) = 1$.

Proof: One of the filters must be HSS while the other is HSA, since these always have at least one zero each at $z = -1$ and $z = 1$, respectively, and, because of (28), the filters must have no zeros in common.

Hence if $H_0(z) = H_{00}(z^2) + z^{-1}H_{01}(z^2)$ is HSS then $H_{00}(z) = z^l H_{01}(z^{-1})$ for some l. Similarly, $H_{10}(z) =$

$-z^m H_{11}(z^{-1})$ for some m. The HSS polyphase matrix is

$$H_p(z) = \begin{bmatrix} H_{00}(z) & z^{-l}H_{00}(z^{-1}) \\ H_{10}(z) & -z^{-m}H_{10}(z^{-1}) \end{bmatrix}. \quad (30)$$

On equating (27) and (30) we get $H_{01}(z) = z^{-l}H_{00}(z)$, $H_{10}(z) = -H_{00}(z)\Delta_p(z)z^l$, and

$$-z^{-m}H_{10}(z^{-1}) = z^{-m-l}H_{00}(z^{-1})\Delta_p(z^{-1})$$

$$= H_{00}(z^{-1})\Delta_p(z).$$

Now the fact that $\Delta_p(z) = 1/\Delta_p(z^{-1})$ gives $\Delta_p^2(z) = z^{-m-l}$, so that $\Delta_p(z)$ is a delay z^{-n}, and $2n = m + l$. This is the desired result. $\qquad \square$

For example, choosing $l = n = 0$, we get

$$H_0(z) = A(z^2) + z^{-1}A(z^{-2}) \quad (31)$$

$$H_1(z) = -A(z^2) + z^{-1}A(z^{-2}). \quad (32)$$

In order to force some regularity we might wish to design $H_0(z)$ to have again the maximum possible number of zeros at $z = -1$. This can be done by solving a fairly simple set of nonlinear equations. Taking the filters in (31) and (32) and the simple all-pass section

$$A(z) = \frac{1 + az^{-1} + bz^{-2}}{b + az^{-1} + z^{-2}}$$

with $a = 6$, $b = 15/7$ we get that $H_0(z)$ contains five zeros at $z = -1$, has a reasonable lowpass response and gives a wavelet that is very smooth. We find that $B = 4.0506$ and an estimate of the regularity gives $r > 1.9819$. The wavelet and its spectrum are shown in Fig. 6.

C. Whole Sample Symmetric Case

Next suppose $H_0(z)$ is to be whole sample symmetric (WSS). In this case one of the polyphase components must be half sample symmetric, the other whole sample symmetric, and both must be either symmetric or antisymmetric. Since antisymmetric filters always have a zero at $z = 1$ the latter case can never satisfy (28).

It is also implied by (28) that the denominators of $H_{00}(z)$ and $H_{01}(z)$ are equal, so we must solve

$$N_{00}(z)N_{00}(z^{-1}) + N_{01}(z)N_{01}(z^{-1}) = D(z)D(z^{-1})$$

where $N_{00}(z)$ and $N_{01}(z)$ are the numerators and $D(z)$ is the common denominator.

Since a rational IIR filter is symmetric if and only if both numerator and denominator are, we need consider only the symmetry of $N_{00}(z)$, $N_{01}(z)$, and $D(z)$. There are four cases that give that $H_{00}(z)$ and $H_{01}(z)$ have the whole/half sample symmetries described above. One can verify that these are that $D(z)$, $N_{00}(z)$, and $N_{01}(z)$ are all symmetric and have lengths that are respectively (odd, odd, even), (odd, even, odd), (even, even, odd), and (even, odd, even). The last two cases, where $D(z)$ has even length, are immediately ruled out, since a symmetric even length FIR function implies at least one zero on the unit circle.
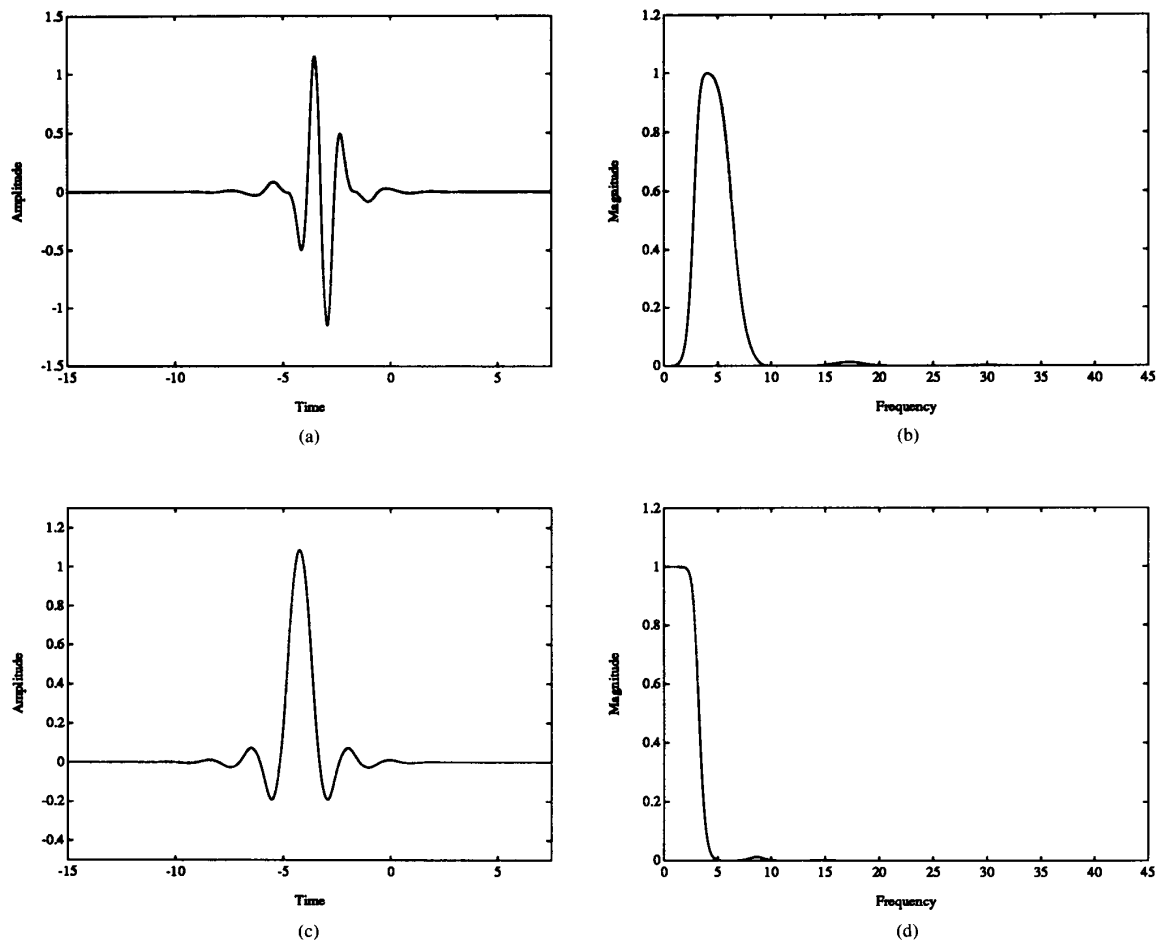
Fig. 6. Example of linear phase orthogonal wavelet. (a) The wavelet (antisymmetric). (b) Spectrum of the wavelet. (c) The scaling function (symmetric). (d) Spectrum of the scaling function.

For example for the (odd, odd, even) case $H_{00}(z)$ has whole sample symmetry, $H_{01}(z)$ has half sample symmetry, and the polyphase matrix is lossless and gives filters that have whole sample symmetry.

Finding good solutions is not as easy as in the HSS case, since the method is not constructive. However, examples can be constructed by solving a set of nonlinear equations. Consider the small example: $N_{00}(z) = a + bz^{-1} + az^{-2}$, $N_{01}(z) = c + cz^{-1}$, and $D(z) = a + dz^{-1} + az^{-2}$. The values $(a, b, c, d) = ((5 + 4\sqrt{2})/14, 1, (12 + 4\sqrt{2})/14, (21 + 24\sqrt{2} + 16 \cdot 2^{3/2})/49)$ gives a solution such that the lowpass filter $H_0(z)$ has two zeros at $z = -1$. An estimate of its regularity gives $r > 0.5$.

VI. ORTHOGONALIZATION OF WAVELET BASES

One of the interesting wavelet bases is that derived by Battle and Lemarié [6], [7], which has the property of being a basis for the spline function spaces.

The B-spline functions obviously form a basis for this space, but are not orthogonal with respect to integer shifts; in the language of Section II-A we have a basis for V_0,

but not an orthogonal one. The condition for orthogonality can also be written in the Fourier domain using the Poisson summation formula [20], [25]:

$$\langle \phi(x), \phi(x - n) \rangle = \delta_n \Leftrightarrow \sum_{k = -\infty}^{\infty} |\Phi(w + 2\pi k)|^2 = 1.$$

(33)

Now assume that we have a non-orthogonal basis for a multiresolution analysis, given by a function $g(x)$ and its integer translates. Then it is easy to see one way that the orthogonalization of the nonorthogonal basis $g(x - n)$ may be performed in the Fourier domain

$$\Phi(w) = \frac{G(w)}{\sqrt{\sum_{k = -\infty}^{\infty} |G(w + 2\pi k)|^2}}.$$

(34)

Clearly, $\Phi(w)$ satisfies the Fourier domain orthogonality condition (33), and the rest of the multiresolution analysis machinery follows; this is precisely the procedure fol-

lowed by Battle and Lemarié [6], [7]. The sequence $h_0(n)$ associated with the two scale difference equation (2) for $\phi(x)$ is, however, not given by a rational function. Hence the filter bank implementation is not realizable, by which we mean that there is no finite complexity recursive implementation of the filters. Often for such nonrealizable filters a truncated version of the infinite impulse response is taken, so that an approximate FIR implementation is used; see for example [2].

A. Orthogonalizing Continuous-Time Bases with Recursive Filters

There are many different orthonormal bases that span the same space; the ones derived by Battle and Lemarié are by no means the only ones for the spline spaces. We next show that if there is a compactly supported wavelet basis for V_0, then it is always possible to find an orthonormal basis, which is infinitely supported, but for which the filters involved are rational and thus realizable. As a special case we shall construct realizable bases for the spline spaces, which are alternatives to those of Battle and Lemarié.

Theorem 6.1: If the set $\{g(x - k), k \in Z\}$ forms a nonorthogonal basis for V_0, obeys a two-scale difference equation, and $g(x)$ is compactly supported, then it is always possible to find an orthonormal basis $\{\phi(x - k), k \in Z\}$, where

$$\Phi(w) = \prod_{i=1}^{\infty} 2^{-1/2} H_0(e^{jw/2^i}) \quad (35)$$

and where $H_0(e^{jw})$ is a rational function of e^{jw}.

Proof: The proof is constructive. The normalizing function used in (34) (i.e., the denominator of the right-hand side) is 2π-periodic, and can be written as a discrete-time Fourier transform:

$$\sum_{k=-\infty}^{\infty} |G(w + 2\pi k)|^2 = \sum_n c_n e^{-jwn} = C(e^{jw}). \quad (36)$$

It can be shown [20] that the Fourier coefficients are obtained from

$$c_n = \int_{-\infty}^{\infty} g(x) g(x - n) \, dx. \quad (37)$$

Since $g(x)$ is compactly supported, it is obvious that only finitely many of the c_n are nonzero. Equally, since the c_n are the Fourier coefficients of a positive real function, it is clear (from the Riesz factorization lemma [20]) that we can always factor

$$C(z) = E(z) E(z^{-1}). \quad (38)$$

Note that $C(z)$ cannot have zeros on the unit circle, because of the fact that $g(x - n)$'s form a Riesz basis [20]. The choice

$$\Phi(w) = \frac{G(w)}{E(e^{jw})} \quad (39)$$

clearly satisfies (33); so that the $\phi(x - k)$ are orthogonal. Since $E(e^{jw})$ is 2π-periodic we get

$$\phi(x) = \sum_k f(k) g(x - k)$$

where $F(e^{jw}) = 1/E(e^{jw})$. That is, $\phi(x)$ is a linear combination of shifted versions of $g(x)$; hence the span of $\{\phi(x - k) k \in Z\}$ is also V_0.

So we now have that both the sets $\{g(x - k), k \in Z\}$ and $\{\phi(x - k), k \in Z\}$ form bases for V_0. But $g(x)$ obeys the two scale difference equation (2); so for some $l(n)$:

$$g(x) = 2^{1/2} \cdot \sum_{n=-\infty}^{\infty} l(n) \cdot g(2x - n)$$

$$\Rightarrow G(w) = L(e^{jw/2}) \cdot G(w/2). \quad (40)$$

However, since the two sets span the same space, we can always write the function $\phi(x)$ as a linear combination of the functions $g(x - k)$:

$$\phi(x) = 2^{1/2} \cdot \sum_{n=-\infty}^{\infty} \alpha(n) \cdot g(x - n) \quad (41)$$

so that by substituting in the expression for $g(x)$ from (40) we get that for some sequence $h_0(n)$

$$\phi(x) = 2^{1/2} \cdot \sum_{n=-\infty}^{\infty} h_0(n) \cdot \phi(2x - n)$$

$$\Rightarrow \Phi(w) = H_0(e^{jw/2}) \cdot \Phi(w/2). \quad (42)$$

Thus $\phi(x)$ satisfies a two scale difference equation also.

Substituting (39) into the Fourier version of (42) we get

$$\frac{G(w)}{E(e^{jw})} = \frac{H_0(e^{jw/2}) \cdot G(w/2)}{E(e^{jw/2})}.$$

Comparing this with (40) gives the relation

$$H_0(e^{jw}) = \frac{L(e^{jw}) \cdot E(e^{jw})}{E(e^{j2w})}. \quad (43)$$

Note that $L(e^{jw})$ is an FIR function, since when it is iterated in (40) it gives $g(x)$, which is compactly supported. Equally $E(e^{jw})$ is FIR, since it is one of the factor of $C(e^{jw})$. Hence $H_0(e^{jw})$ is a rational function of e^{jw}, and corresponds to a filter that can be implemented recursively. □

Since $\phi(x)$ gives an orthogonal basis for V_0 we see from Section II-A that $h_0(n)$ and $h_1(n)$, (given by $H_1(z) = z^{2k-1} H_0(z^{-1})$), satisfy (6) and (4). In other words, the conditions of Lemma 2.2d) hold and we have an orthogonal filter bank, with rational filters, that generates our basis for V_0. It follows from Theorem 3.2 that the function

$$P(z) = \frac{L(z) L(z^{-1}) \cdot E(z) E(z^{-1})}{E(z^2) E(z^{-2})}$$
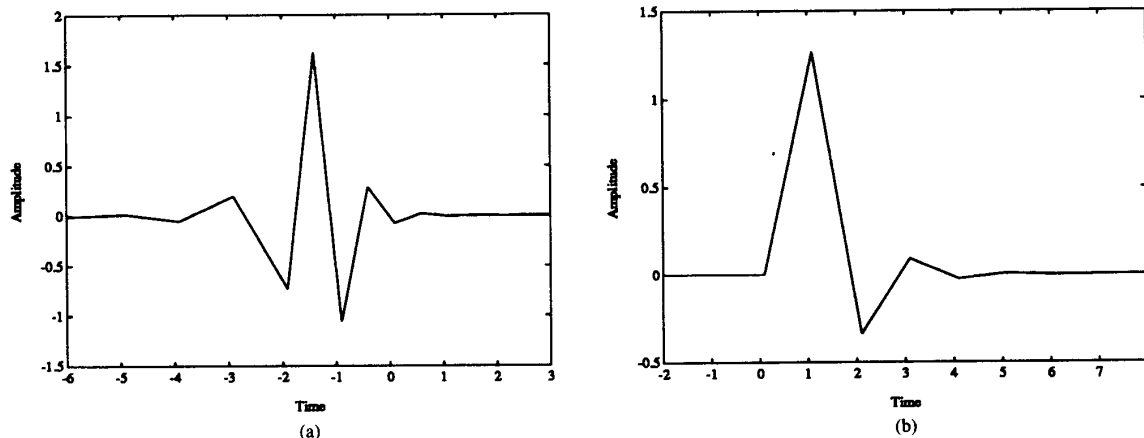
is valid.

Fig. 7. Orthogonal basis for the piecewise linear spline space constructed from realizable filters. (a) The wavelet. (b) The scaling function.

Note that because of (43) successive numerators and denominators of the product cancel:

$$\Phi(w) = \prod_{i=1}^{\infty} H_0(w/2^i) = \prod_{i=1}^{\infty} L(w/2^i) \cdot \prod_{i=1}^{\infty} \frac{E(e^{jw/2^i})}{E(e^{j2w/2^i})}$$

$$= G(w) \cdot \frac{E(e^{jw/2})}{E(e^{jw})} \cdot \frac{E(e^{jw/4})}{E(e^{jw/2})} \cdots$$

$$= G(w) \cdot \frac{E(e^{j0})}{E(e^{jw})}. \tag{44}$$

So the infinite product for $H_0(e^{jw})$ converges since that for $L(e^{jw})$ does. This also means that we do not have to separately make regularity estimates for $\Phi(w)$ if the regularity of $G(w)$ is known, since $\phi(x)$ is a linear combination of integer shifts of the function $g(x)$, and thus has the same regularity.

B. Bases for the Spline Spaces Using Recursive Filter Banks

An application of the above result is to find bases for the spline spaces. First note that the Nth order B-spline function, which is defined by: $g(x) = s(x) * s(x) \cdots s(x)$, where there are $N - 1$ convolutions, and $s(x)$ is the characteristic function of the interval $[0, 1)$, is compactly supported. Further, the set $\{g(x - k), k \in Z\}$ is a basis for the Nth order spline function space. To get an orthogonal basis from this we apply Theorem 6.1.

Note that the Fourier transform of the B-spline $g(x)$ can be written [20]

$$G(w) = \prod_{i=1}^{\infty} (1 + e^{-jw2^{-i}})^N.$$

In other words, $L(e^{jw}) = (1 + e^{-jw})^N/2^N$.

The coefficients of $E(z)E(z^{-1})$ are found from (37), that is, by evaluating $\langle g(x), g(x - n) \rangle$ [20], [43]. Those of $E(x)$ are then obtained by spectral factorization (38). Thus we end up with

$$H_0(z) = (1 + z^{-1})^N E(z)/(2^N \cdot E(z^2)). \tag{45}$$

Successive terms in the infinite product cancel, as in (44), and we get

$$\Phi(w) = G(w) \cdot \frac{E(e^{j0})}{E(e^{jw})}.$$

Hence

$$\phi(x) = \sum_{k=-\infty}^{\infty} f(k)g(x - k)$$

where $F(e^{jw}) = 1/E(e^{jw})$ is an all-pole filter, that is, $\phi(x)$ is a linear combination of splines.

Finding polynomial solutions $E(z)$ such that $H_0(z)$ in (45) gives an orthogonal filter bank that was also done by Strömberg [44]. This solution was also noted by Unser and Aldroubi [45], and also Argenti et al. [46].

That the wavelet and scaling function are indeed splines is most easily seen for the $N = 2$ case where they are piecewise linear. The wavelet and scaling function are shown in Fig. 7.

The relation between the wavelet basis proposed here, and those of Battle and Lemarié is readily seen if we consider the associated function $P(z)$:

$$P(z) = \frac{(1 + z^{-1})^N(1 + z)^N \cdot E(z)E(z^{-1})}{2^{2N} \cdot E(z^2)E(z^{-2})}. \tag{46}$$

Observe that if we factor it as $P(z) = \sqrt{P(z)} \cdot \sqrt{P(z)}$ and use $H_0(z) = \sqrt{P(z)}$ in (35) the cancellation property between successive numerators and denominators still holds, and we end up with

$$\Phi_{BL}(w) = G(w) \cdot \sqrt{\frac{E^2(e^{j0})}{E(e^{jw})E(e^{-jw})}}$$

which is the same as the form in (34) when $E(e^{j0}) = 1$.

In other words, the different orthonormal bases here correspond to different factorizations of $P(z)$. Note, however, that it is not in general true that different orthogonal factorizations of $P(z)$ give rise to wavelet bases that span the same space. For example the choice $H_0(z) = (1 +$

TABLE II
THE VARIOUS $P(z)$ SOLUTIONS FOR A GIVEN NUMBER OF ZEROS AT $z = -1$. SPLINE SOLUTIONS FOR $N = 1, \cdots 5$ ARE SHOWN

Solution	$P(z)$	Regularity
$N = 1$	$(1 + z)(1 + z^{-1}) \cdot 2^{-1}$	$r = 0$
$N = 2$	$\dfrac{(1 + z)^2(1 + z^{-1})^2 \cdot (1, 4, 1) \cdot z^{-1}2^{-3}}{(1, 0, 4, 0, 1)}$	$r = 1.0$
$N = 3$	$\dfrac{(1 + z)^3(1 + z^{-1})^3 \cdot (1, 26, 66, 26, 1) \cdot z^{-2}2^{-5}}{(1, 0, 26, 0, 66, 0, 26, 0, 1)}$	$r = 2.0$
$N = 4$	$\dfrac{(1 + z)^4(1 + z^{-1})^4 \cdot (1, 120, 1191, 2416, 1191, 120, 1) \cdot z^{-3}2^{-7}}{(1, 0, 120, 0, 1191, 0, 2416, 0, 1191, 0, 120, 0, 1)}$	$r = 3.0$
$N = 5$	$\dfrac{(1 + z)^5(1 + z^{-1})^5 \cdot (1, 502, 14608, 88234, 156190, 88234, 14608, 502, 1) \cdot z^{-4}2^{-9}}{(1, 0, 502, 0, 14608, 0, 88234, 0, 156190, 0, 88234, 0, 14608, 0, 502, 0, 1)}$	$r = 4.0$

$z^{-1})^N E(z)/E(z^{-2})$ gives an orthogonal basis, but we do not get the cancellations in the infinite product, and the wavelets do not span the spline spaces. In Table II we have listed the $P(z)$ functions used to get the orthogonal filters to generate the first five spline spaces. It can clearly be seen that they have the form given in (46). Note also that the regularity of the wavelet for the Nth order spline space is precisely $N - 1$.

It is clear that the filters that generate the Battle–Lemarié wavelets have linear phase; however, they are not rational for any order, as is proved by the next Lemma.

Lemma 6.2: The spline solutions to the equation

$$P(z) + P(-z) = 2$$

can never be factored as $P(z) = H(z)H(z^{-1})$ where $H(z)$ is a rational linear phase stable filter.

The proof is in Appendix subsection C. Note that in this case unstable solutions are possible, i.e., where $H(z)$ has poles on the unit circle, whereas no solutions at all were possible for the cases covered in Lemma 5.1.

VII. CONCLUSION

We have examined in detail the structure of orthogonal two channel filter banks, and their relation with orthonormal bases of wavelets. Also we discuss the two channel case only; the N channel case is more complex. We placed particular stress on filters that have a maximum number of zeros at π, since these maximally flat filters give rise to wavelets that have a large number of disappearing moments and are very smooth. The Daubechies, Butterworth, and intermediate solutions were of this form. The filters that were used to realize bases for the spline spaces also had a large, but not maximum, number of zeros at π.

It should also be pointed out that while in this paper we have been interested exclusively with orthogonal filter banks it is of course possible to factor any of the $P(z)$ functions we have presented in a nonorthogonal fashion. This was essentially the procedure followed in [35], [11], where linear phase factorizations of the Daubechies' $P(z)$ were taken. As noted in [47], [11], however, it can be

difficult in the FIR case to get filters with flat spectra when linear phase is desired. This is also true when IIR filters are involved. In other words, it is quite difficult to factor any of the recursive $P(z)$'s listed in Tables I and II to obtain linear phase rational filters which still have acceptable response. Of course, it is always possible, as we saw for the Butterworth case in Section IV-D and Battle–Lemarié case in Section VI-B, to factor any of these $P(z)$'s in a linear phase orthogonal fashion, but where the filters involved are irrational.

An important consideration that is often encountered in the design of wavelets, or of the filter banks that generate them, is the necessity of satisfying competing design constraints. This makes it necessary to clearly understand whether desired properties are mutually exclusive. We have attempted in this paper to give a fairly comprehensive treatment of orthogonal two channel filter banks. A natural question is to wonder how the solutions described relate to others, and which constraints can be simultaneously satisfied. In Fig. 8 we have attempted to do this with a Venn diagram illustrating various perfect reconstruction solutions. As we have emphasized, the properties of the filter bank depend on $P(z)$ and the factorization chosen. For the three properties most often desired, viz., FIR filters, orthogonality, and linear phase, the necessary and sufficient conditions (in addition to that for perfect reconstruction (16)) are:

1) *Orthogonality:* $P(z)$ is an autocorrelation; $H_0(z)$ and $G_0(z)$ are its spectral factors.

2) *Linear phase:* $P(z)$ is linear phase, and $H_0(z)$ and $G_0(z)$ are its linear phase factors.

3) *Finite support:* $P(z)$ is FIR, and $H_0(z)$ and $G_0(z)$ are its FIR factors.

Obviously in none of the cases is the factorization unique. If $P(z)$ is indeed an autocorrelation, for example, various spectral factorizations exist.

Perfect reconstruction solutions, with the constraint that $P(z)$ be rational with real coefficients, must satisfy (16), and the structure of such solutions was given in Lemma 3.1. Such general solutions, which do not necessarily have additional properties, were given in [8].
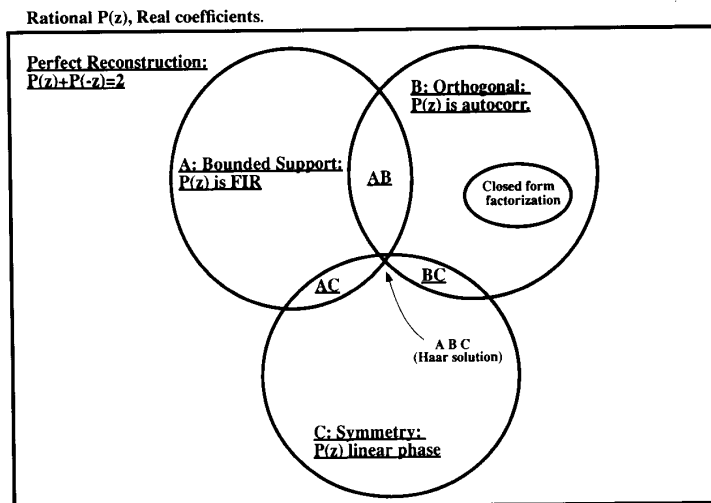
Rational P(z), Real coefficients.

Fig. 8. Two channel perfect reconstruction filter banks. The Venn diagram illustrates which competing constraints can be simultaneously satisfied. The sets A, B, C contain FIR, orthogonal, and linear phase solutions, respectively. Solutions in the intersection $A \cap B$ are examined in [8]-[10], [4]; those in the intersection $A \cap C$ are detailed in [13], [47], [49], [11], [35]; solutions in $B \cap C$ are constructed in Section V. The intersection $A \cap B \cap C$ contains only trivial solutions.

TABLE III
PROPERTIES WHICH ARE SIMULTANEOUSLY ACHIEVABLE FOR TWO-CHANNEL FILTER BANKS, AND COMMENTS ON THE SOLUTIONS.
A "1" IN A PARTICULAR BOX INDICATES THAT THE SOLUTION NECESSARILY HAS THE CORRESPONDING PROPERTY

Orthogonal	Linear Phase	FIR	Real	Rational	Solutions
1	1	1	1	1	Haar basis (1910)
0	1	1	1	1	Biorthogonal solutions [35], [11], [13], [47], [49]
1	0	1	1	1	Daubechies [4], paraunitary solutions [8], [10]
1	1	1	0	1	Complex factorizations [42]
1	1	0	1	1	Linear phase IIR solutions Section V
1	1	0	1	0	Battle–Lemarié bases [6], [7], Meyer bases [5]
1	0	0	1	1	Characterized by Theorem 3.2

The solutions of set A, where all of the filters involved are FIR, were studied in [8], [13]. Set B contains all orthogonal solutions, and has been the main focus of this paper. A complete characterization of this set was given in Theorem 3.2. A very different characterization, based on lattice structures is given in [33]. Particular cases of orthogonal solutions were also given in [32]. Set C contains the solutions where all filters are linear phase, first examined in [13].

The earliest examples of perfect reconstruction solutions [48], [9] were orthogonal and FIR, i.e., they were in $A \cap B$. A constructive parametrization of $A \cap B$ was given in [10]. The construction and characterization of examples which converge to wavelets was first done in [4]. Filter banks with FIR linear phase filters (i.e., $A \cap C$) were first given in [13], and also studied in terms of lattices in [47], [49]. The construction of wavelet examples is given in [35] and [11]. Filter banks which are linear phase and orthogonal were constructed in Section V, and first presented in [16], [17].

That there exist only trivial solutions which are linear phase, orthogonal, and FIR is indicated by the intersection $A \cap B \cap C$; the only solutions are two tap filters [4], [26], [11].

Fig. 8 illustrates the filter bank solutions; if the filters are regular then they will lead to wavelets. Of the dyadic wavelet bases known to the authors the only ones based on filters where $P(z)$ is not rational are those of Meyer [5], and the only ones where the filter coefficients are complex are those of Lawton [42]. For the case of the Battle–Lemarié wavelets, while the filters themselves are not rational, the $P(z)$ function is, hence the filters would belong to $B \cap C$ in the figure. The alternatives presented in Section VI-B would belong to $B/(A \cup C)$, since they are orthogonal, but are neither linear phase nor FIR.

Table III attempts to clarify some of the competing constraints by tabulating which of the properties—orthogonality, linear phase, FIR, real coefficients, and rational transfer function—are simultaneously attainable and comments on the solutions.

APPENDIX

A. Filters with Symmetry

Fact A.1: For a symmetric discrete sequence $R(z) = R_0(z^2) + z^{-1}R_1(z^2)$ the following relations between the polyphase components hold:

    i) $R(z)$ WSS: $R_0(z^{-1}) = R_0(z)$, $z^2 R_1(z^{-1}) = R_1(z)$,

    ii) $R(z)$ HSS: $R_1(z^{-1}) = R_0(z)$,

    iii) $R(z)$ WSA: $R_0(z^{-1}) = -R_0(z)$, $z^2 R_1(z^{-1}) = -R_1(z)$,

    iv) $R(z)$ HSA: $R_1(z^{-1}) = -R_0(z)$.

Proof:

i): WSS $\Rightarrow R(z) = R(z^{-1})$. So: $R_0(z^2) + z^{-1}R_1(z^2) = R_0(z^{-2}) + zR_1(z^{-2})$. Equating even and odd powers of z^{-1} we find $R_0(z^{-1}) = R_0(z)$, $z^2 R_1(z^{-1}) = R_1(z)$.

ii): HSS $\Rightarrow R(z) = z^{-1}R(z^{-1})$. So $R_0(z^2) + z^{-1}R_1(z^2) = z^{-1}R_0(z^{-2}) + R_1(z^{-2})$. Equating even and odd powers of z gives $R_1(z^{-1}) = R_0(z)$.

The other properties follow by similar analysis. $\square$

It follows immediately that an HSS filter always has a zero at $z = -1$, and a HSA filter always has one at $z = 1$.

Fact A.2: For a rational IIR filter that has linear phase, let N_1 be the length of the numerator, and N_2 the length of the denominator, then if $N_1 - N_2$ is odd the filter is WSS or WSA, and if $N_1 - N_2$ is even it is HSS or HSA.

Proof: If $H(z) = N(z)/D(z)$ then

$$H(e^{jw}) = \left| \frac{N(e^{jw})}{D(e^{jw})} \right| \cdot e^{j(\phi_N(w) - \phi_D(w))}$$

where $\phi_N(w)$ and $\phi_D(w)$ are the phases of the numerator and denominator, respectively. Clearly $H(z)$ will have linear phase if and only if both numerator and denominator do.

Since $N(z)$ and $D(z)$ are linear phase FIR functions we have $N(z) = z^l N(z^{-1})$ where l is even if N_1 is odd, and l is odd if N_2 is even. Also $D(z) = z^m D(z^{-1})$, with similar constraints for m. Hence

$$H(z) = \frac{N(z)}{D(z)} = z^{l-m} \cdot \frac{N(z^{-1})}{D(z^{-1})}.$$

Now $l - m$ is even if N_1 and N_2 are both even or both odd (i.e., $N_1 - N_2$ is even), and is odd otherwise (i.e., $N_1 - N_2$ is odd). Using Fact A.1 $l - m$ even implies that $H(z)$ is WSS or WSA, and $l - m$ odd implies that it is HSS or HSA. $\square$

B. Proof of Lemma 5.1

Proof: If $H(z)$ is linear phase then

$$H(z) = \frac{z^k C(z) C(z^{-1})}{D(z^2) D(z^{-2})},$$

for some integer delay z^k, and

$$P(z) = H(z)H(z^{-1}) = \left[\frac{C(z) C(z^{-1})}{D(z^2) D(z^{-2})} \right]^2.$$

Hence every pole and every zero must be double.

1) Butterworth Case: In the Butterworth case $P(z)$ can be written

$$P(z) = \frac{(1 + z^{-1})^N (1 + z)^N}{(z^{-1} + 2 + z)^N + (-z^{-1} + 2 - z)^N}$$

$$= \frac{(1 + z^{-1})^{2N}}{(1 + z^{-1})^{2N} + (1 - z^{-1})^{2N}(-1)^N}.$$

Note that the denominator, $W(z)$, is a polyphase component of $(1 + z^{-1})^{2N}$ following Lemma 3.1. If all poles of $P(z)$ are to double we must have

$$W(z_0) = 0 \Rightarrow \left. \frac{dW(z)}{d(z^{-1})} \right|_{z_0} = 0.$$

But

$$\left. \frac{dW(z)}{d(z^{-1})} \right| = 2N \cdot (1 + z^{-1})^{2N-1} - 2N$$

$$\cdot (-1)^N \cdot (1 - z^{-1})^{2N-1}$$

is a polyphase component of $(1 + z^{-1})^{2N-1} = B_0(z^2) + z^{-1}B_1(z^2)$. So the polyphase components of two successive binomials must share a zero. Consider

$$(1 + z^{-1})^{2N} = (1 + z^{-1}) \cdot (B_0(z^2) + z^{-1}B_1(z^2))$$

$$= (B_0(z^2) + z^{-2}B_1(z^2)) + z^{-1}(B_0(z^2)$$

$$+ B_1(z^2)).$$

If the polyphase component of $(1 + z^{-1})^{2N}$, which is $(B_0(z^2) + z^{-2}B_1(z^2))$, and that of $(1 + z^{-1})^{2N-1}$ share a zero, then clearly $B_1(z^2)$ must contain this zero also. This would imply that $B_0(z)$ and $B_1(z)$ are not coprime; this is a contradiction, however, the polyphase components of $(1 + z^{-1})^N$ are known to be coprime for all N [11].

2) Intermediate Cases: Here we shall make use of Fact A.2 to show that the solutions have to be half sample symmetric or antisymmetric if they are to have linear phase, and then show that they do not satisfy the form of Lemma 5.3.

a) Consider the $N = 2k + 1$ case: since the numerator and denominator of $P(z)$ have length $4(k + p) + 3$ and $4(k - p) + 1$ respectively, the numerator and denominator of $H(z)$ should have lengths $N_1 = 2(k + p) + 2$ and $N_2 = 2(k - p) + 1$. Hence $N_1 - N_2 = 4p + 1$, which is odd, and $H(z)$ must be HSS or HSA by Fact A.2. But by Lemma 5.3 for $H(z)$ to be HSS its polyphase components must be all-pass filters. Each of the polyphase components have numerator and denominator of lengths $2(k + p) + 1$ and $2(k - p) + 1$, respectively; hence they cannot be all pass if $p \neq 0$.

b) Consider $N = 2k$: here $N_1 = 2(k - p + 1) + 2$ and $N_2 = 2(k - p) + 1$, so $N_1 - N_2 = 2(2p - 1) + 1$ which is again odd. So again $H(z)$ must be HSS or HSA, and the polyphase components must be all passes. As before examining the lengths of the numerator and denominator of H_{00} and H_{01} rules this out. The lengths are $2(k + p - 1) + 1$ and $2(k - p) + 1$, respectively.

3) Daubechies Case: The filters are always of even length, and hence either HSS of HSA if linear phase. Hence their polyphase components must be all passes; but since the only FIR all pass is a delay, the only solutions are those of length two. This was already proved in [4], using a different argument. ☐

C. Proof of Lemma 6.2

Proof: Again all poles and zeros of $P(z)$ must be double if the filters have linear phase. Recall that in this case we require the $P(z)$ with the form given in (46) to be valid. Suppose indeed that every pole and zero were double, then we could write, for some $D(z)$:

$$P(z) = \frac{(1 + z^{-1})^{2N} z^N \cdot (D(z)D(z^{-1}))^2}{2^{2N} \cdot (D(z^2)D(z^{-2}))^2}. \quad (47)$$

Since the numerator is a polyphase component of the denominator

$$1/2 [(1 + z^{-1})^{2N} z^N \cdot (D(z)D(z^{-1}))^2 + (1 - z^{-1})^{2N}(-z)^N$$

$$\cdot (D(-z)D(-z^{-1}))^2] = 2^{2N} \cdot (D(z^2)D(z^{-2}))^2.$$

Evaluate at $z = 1$ to get

$$2^{2N-1}(D(1)D(1))^2 = 2^N (D(1)D(1))^2$$

which is clearly a contradiction unless $D(1) = 0$. $D(1) = 0$, however, implies poles on the unit circle; for rational solutions to exist they must be unstable. ☐

Note rational linear phase solutions which are unstable do exist in the spline case. For example, when $N = 2$:

$$P(z) = \frac{(1, 4, 6, 4, 1)(1, -4, 6, -4, 1)}{(1, 0, -4, 0, 6, 0, -4, 0, 1)}$$

$$= \left[\frac{(1, 2, 1)(1, -2, 1)}{(1, 0, -2, 0, 1)} \right]^2.$$

The denominator in this case has double roots at $z = 1$ and $z = -1$, as does the numerator.

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