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Reduction of the MSE in R-times Oversampled A/D Conversion from $\mathcal{O}(1/R)$ to $\mathcal{O}(1/R^2)$

Nguyen T. Thao and Martin Vetterli

***Abstract*—In oversampled analog-to-digital conversion, the usual reconstruction method using lowpass filtering leads to a mean squared error (MSE) inversely proportional to the oversampling ratio R. In this correspondence, we prove, under certain assumptions and with periodic analog input signals, that optimal reconstruction achieves an MSE with an oversampling ratio dependence order of at least $\mathcal{O}(1/R^2)$. That is, an MSE slope of -6 dB per octave of oversampling is obtained, rather than the conventional -3 dB/octave slope of classical schemes.**

I. INTRODUCTION

Analog-to-digital conversion consists of discretizing an analog signal in time and in amplitude. Shannon's well-known sampling theorem [1] guarantees that when a bandlimited signal is sampled only in time at the Nyquist rate or above, no information is lost. It also gives an analytical expression for the reconstruction of the bandlimited signal from its samples. Results on reconstruction were also obtained by Logan [2] when the analog signal is discretized only in amplitude. Under certain assumptions, he showed that an octave band signal is uniquely defined by its zero crossings, up to a multiplicative constant. This corresponds to amplitude quantization to regions having positive and negative values. However, in practical A/D conversion, analog signals are discretized both in time and amplitude (see Fig. 1). Few analytical results have been derived about the reconstruction of an analog signal from its discrete-time discrete-amplitude (DTDA) version.

Manuscript received June 26, 1992; revised February 23, 1993. The associate editor coordinating the review of this paper and approving it for publication was Dr. Barry Sullivan. This work was supported in part by the National Science Foundation under Grant ECD-88-11111.

The authors are with the Department of Electrical Engineering, and Center for Telecommunications Research, Columbia University, New York, NY 10027-6699.

IEEE Log Number 9213281.

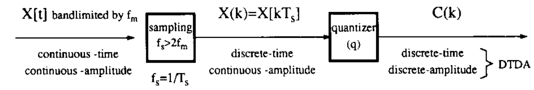

Fig. 1. Discretization scheme of a bandlimited analog signal with maximum frequency f_m.

Of course, if the signal is sampled in time at the Nyquist rate and uniformly quantized with a step size q, then the quantization error is given by $q^2/12$. A more interesting scenario results when the samples are taken above the Nyquist rate, i.e., when oversampling occurs. The classical reconstruction method consists in lowpass filtering the quantized signal, thus preserving the original bandlimited signal but reducing the power of the quantization error signal in proportion to the oversampling ratio R (under certain assumptions [3], [4]).

However, the following insight indicates that the classical reconstruction method may not be optimal in the mean squared error (MSE) sense. Halving the amplitude quantization step size will reduce the quantization error by a factor of 4 in the MSE sense, but halving the sampling period will only reduce the quantization error by 2. This inhomogeneity in the time and amplitude dimensions is counterintuitive. We will show that optimal reconstruction leads to homogeneity, that is, halving either amplitude or time quantization leads to a reduction of quantization error by a factor of 4.

The suboptimality in classical reconstruction stems from the fact that a *requantization* of the lowpass filtered reconstruction does not in general lead to the same quantized signal [5], [6]. That is, the DTDA version of the original signal is different from the DTDA version of the lowpass filtered reconstruction. It was shown in [5] and [6] that an estimate which does not reproduce the DTDA version of the original signal can be automatically improved in terms of MSE. Therefore, by necessity, any *optimal* reconstruction scheme should at least provide an estimate which reproduces the DTDA version of the original signal.

In this correspondence, we analyze the MSE of an estimate given by an *optimal* reconstruction scheme, with the assumption that the analog signals are periodic in the time interval in which they are coded. Assuming that the original signal has a minimum number of quantization threshold crossings (QTC's), we show that the MSE is at least inversely proportional to R^2 instead of R, for R high enough.

This result is the consequence of an analysis of the information present in the DTDA signal. After defining the mathematical context of our derivations in Section II, we show in Section III that when the oversampling ratio is high enough, the DTDA signal gives the location of the analog signal's QTC's with a time uncertainty equal to the sampling period. As shown in Section IV, this implies the $\mathcal{O}(1/R^2)$ behavior of the MSE.

II. MATHEMATICAL CONTEXT AND NOTATIONS

As mentioned in the Introduction, we consider that the bandlimited signals are sampled and quantized on the time interval $[0, T]$ and are T-periodic. We designate such signals using boldface italic capital letters, like $\boldsymbol{X}$. We denote the value of $\boldsymbol{X}$ at time t by $X[t]$. Bandlimited and T-periodic real signals necessarily have a finite

Fourier expansion as follows:

$$X[t] = \sum_{i=-M}^{M} X_i e^{j2\pi i(t/T)}, \qquad \text{with } X_{-i} = \overline{X}_i \in C$$
$$(\text{where } j = \sqrt{-1}). \tag{1}$$

The number of nonzero low-frequency components is $W = 2M + 1$, and characterizes the bandwidth of $\boldsymbol{X}$. We will simply say that $\boldsymbol{X}$ has a bandwidth equal to W. The space of bandlimited signals defined in (1) is called $\mathcal{V}$. Given a bandwidth W, because of the Hermitian symmetry of the coefficients $(X_i)_{-M \le i \le M}$, $\mathcal{V}$ is a real space of dimension W. According to Parseval's equality

$$\|\boldsymbol{X}\| = \left(\frac{1}{T}\int_0^T |X[t]|^2 dt\right)^{1/2} = \left(\sum_{i=-M}^{M} |X_i|^2\right)^{1/2} = \|\vec{X}\| \tag{2}$$

where $\vec{X} = [X_{-M} \cdots X_M]^T \in \boldsymbol{C}^W$ is called the associated vector of $\boldsymbol{X}$. The distance between two signals $\boldsymbol{X}$ and $\boldsymbol{X}'$ of $\mathcal{V}$ will be measured by their MSE defined by $MSE(\boldsymbol{X}, \boldsymbol{X}') = \|\boldsymbol{X}' - \boldsymbol{X}\|^2 = \|\vec{X}' - \vec{X}\|^2$. When a signal $\boldsymbol{X} \in \mathcal{V}$ is given by its associated vector $\vec{X}$, thanks to (1), the values of $\boldsymbol{X}$ at W instants $t_1, \cdots, t_W \in [0, T]$ can be obtained from $\vec{X}$ as

$$[X[t_1] \cdots X[t_W]]^T = \mathcal{M}(t_1, \cdots, t_W) \cdot \vec{X} \tag{3}$$

where $\mathcal{M}(t_1, \cdots, t_W)$ is the $W \times W$ Vandermonde matrix $[e^{j2\pi i(t_k/T)}]_{\substack{1 \le k \le W \\ -M \le i \le M}}$. This matrix has the following property (proved in the Appendix).

Property 2.1: Let $S = \{(t_1, \cdots, t_W) \in]0, T]^W / \forall i \ne j, t_i \ne t_j\}$. Then for all $(t_1, \cdots, t_W) \in S$, $\mathcal{M}(t_1, \cdots, t_W)$ is invertible. Moreover, given a real number $\delta > 0$, $[\mathcal{M}(t_1, \cdots, t_W)]^{-1}$ is bounded on the set $S_\delta = \{(t_1, \cdots, t_W) \in [\delta, T]^W / \forall i = 2, \cdots, W, t_i \ge t_{i-1} + \delta\}$.

One of the consequences is the following property.

Property 2.2: Let $\boldsymbol{X} \in \mathcal{V}$ and $t_1, \cdots, t_W \in]0, T]$ be W distinct instants. Then, $\boldsymbol{X}$ is uniquely defined by $X[t_1], \cdots, X[t_W]$.

Proof: For this choice of instants $t_1, \cdots, t_W$, $\mathcal{M}(t_1, \cdots, t_W)$ is invertible, according to Property 2.1. Therefore, $\vec{X}$ is uniquely defined by $X[t_1], \cdots, X[t_W]$ from (3). So is $\boldsymbol{X}$. □

When a signal of $\mathcal{V}$ is sampled, N designates the number of samples in the time window $[0, T]$. The sampling period is then $T_s = T/N$, and the oversampling ratio R is related to N by $R = N/W$. According to the discretization scheme of Fig. 1, the DTDA version of $\boldsymbol{X}$ is a sequence $\boldsymbol{C}$ of discrete values $C(k)$. The discretization mapping from $\boldsymbol{X}$ to $\boldsymbol{C}$ is symbolized as $\boldsymbol{C} = Q[\boldsymbol{X}]$.

III. Information in the Discrete-Time Discrete-Amplitude Signal

Consider an analog signal $\boldsymbol{X}$ as a 2-D graph with time and amplitude as the dimensions (Fig. 2(a)). Sampling corresponds to a discretization of the time axis, while quantization corresponds to a discretization of the amplitude axis which indicates in what amplitude interval (or quantization interval) the signal is, at a given sampling instant. This is shown in Fig. 2(a) as the shaded vertical segments. The sequence $\boldsymbol{C} = Q[\boldsymbol{X}]$ can be represented by the sequence of these vertical segments.

Conversely, start from the DTDA signal $\boldsymbol{C}$ of Fig. 2(a) and consider any analog signal $\boldsymbol{X}$ such that $Q[\boldsymbol{X}] = \boldsymbol{C}$. When two consecutive vertical segments of $\boldsymbol{C}$ show an amplitude jump, like between the sampling instants 4 and 5, $\boldsymbol{C}$ indicates that $\boldsymbol{X}$ has a QTC between these two instants. The localization of this QTC is represented in Fig. 2(b) by a horizontal segment. If n is the number of vertical segment jumps, $\boldsymbol{C}$ gives the localization of n QTC's in the form of a sequence $(l_i, I_i)_{1 \le i \le n}$, where l_i is the level of the ith QTC, and I_i is the sampling interval when the ith QTC occurs.

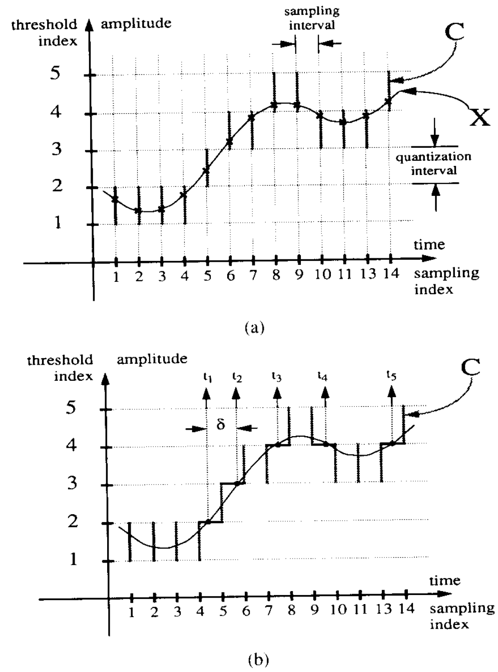

Fig. 2. Information in the discrete-time discrete-amplitude (DTDA) signal: (a) Derivation of $\boldsymbol{C} = Q[\boldsymbol{X}]$ from $\boldsymbol{X}$ (sequence of shaded vertical segments); (b) QTC information provided by $\boldsymbol{C}$ about any $\boldsymbol{X}$ such that $Q[\boldsymbol{X}] = \boldsymbol{C}$. From the knowledge of $\boldsymbol{C}$, it is derived that the analog signal has a QTC in each horizontal segment (represented in dark).

Consider now a fixed analog signal $\boldsymbol{X}$. For this signal, various sampling rates can be considered. It is possible to choose the oversampling ratio R high enough so that the sampling period T_s is less than the minimum distance between any two QTC's of $\boldsymbol{X}$. In this situation, $\boldsymbol{X}$ has no more than one QTC per sampling interval. Then, every QTC of $\boldsymbol{X}$ is necessarily indicated by the sequence $\boldsymbol{C} = Q[\boldsymbol{X}]$. Indeed, since $\boldsymbol{X}$ crosses a quantization threshold at most once in a sampling interval, the two vertical segments of $\boldsymbol{C}$, respectively, preceding and following a QTC, are necessarily at different amplitude levels.

We have shown that, when the oversampling ratio is high enough, the sequence $\boldsymbol{C} = Q[\boldsymbol{X}]$ uniquely characterizes the set of QTC's of $\boldsymbol{X}$, with time precision T_s. Rigorously speaking, if n is the total number of QTC's and $\boldsymbol{X}$ is sampled at a high enough rate, then 1) the sequence $\boldsymbol{C} = Q[\boldsymbol{X}]$ gives the level l_i and the sampling interval I_i of the ith QTC of $\boldsymbol{X}$, for any $i = 1, \cdots, n$; and 2) any other analog signal $\boldsymbol{X}'$ such that $Q[\boldsymbol{X}'] = \boldsymbol{C}$ necessarily has a QTC at the level l_i and in the sampling interval I_i for each $i = 1, \cdots, n$.

IV. MSE Upper Bound

The previous section showed that the sequence $\boldsymbol{C} = Q[\boldsymbol{X}]$ gives the location of the QTC's of $\boldsymbol{X}$ with absolute precision in the amplitude levels, and a time uncertainty equal to the sampling period. Theoretically speaking, if the oversampling ratio R was infinite, we would also have infinite precision in the time specification. This amounts to knowing the amplitude values of $\boldsymbol{X}$ at n particular instants, where n is the number of QTC's. Suppose, moreover, that

$\boldsymbol{X} \in \mathcal{V}$. From Property 2.2, we know that the values of $\boldsymbol{X}$ at W arbitrary known instants uniquely define $\boldsymbol{X}$. Suppose that $\boldsymbol{X}$ has W QTC's. Then, in the limit of R going to infinity, the sequence $\boldsymbol{C} = Q[\boldsymbol{X}]$ would uniquely define $\boldsymbol{X}$. Now, when R is finite this is no longer true. But, according to Section III, if R is high enough, any other signal $\boldsymbol{X}'$ such that $Q[\boldsymbol{X}'] = \boldsymbol{C}$ has W QTC's at the same levels as those of $\boldsymbol{X}$, respectively, and at instants which differ by less than the sampling period $T_s = T/N = TW/R$. If, moreover, $\boldsymbol{X}' \in \mathcal{V}$, we show in the next theorem that this QTC property on $\boldsymbol{X}'$ implies the MSE upper bound $MSE(\boldsymbol{X}, \boldsymbol{X}') \leq c/R^2$, where c is a positive constant which depends only on $\boldsymbol{X}$.

Theorem 4.1: Assume that a signal $\boldsymbol{X} \in \mathcal{V}$ (where $\mathcal{V}$ is the space of signals defined in (1)) and has more than W quantization threshold crossings[1] in $]0, T]$. There exist two constants $c > 0$ and $R_0 \geq 1$, which only depend on $\boldsymbol{X}$ such that: $\forall R \geq R_0, \forall \boldsymbol{X}' \in \mathcal{V}$,

$$\boldsymbol{X} \text{ and } \boldsymbol{X}' \text{ produce the same DTDA signal} \Rightarrow MSE(\boldsymbol{X}, \boldsymbol{X}') \leq \frac{c}{R^2}.$$

Proof: $\boldsymbol{X}$ may have more than W QTC's, but let us arbitrarily choose W of them, call $0 < t_1 < \cdots < t_W \leq T$ their instants, and $(l_1, \cdots, l_W)$ their levels. We have $X[t_i] = l_i$ for all $i = 1, \cdots, W$. Using the convention $t_0 = 0$, let δ be the minimum distance between $t_0, \cdots, t_W$, that is, $\delta = \min_{1 \leq i \leq W}(t_i - t_{i-1})$ (see Fig. 2(b)). Let us choose R large enough so that the sampling period $T_s = T/N$ is smaller than $\delta/3$, and call $(I_1, \cdots, I_W)$ the sampling intervals containing $(t_1, \cdots, t_W)$, respectively. Since the sampling intervals have a length equal to T_s, we have: $\forall (t'_1, \cdots, t'_W) \in I_1 \times \cdots \times I_W$,

$$\forall i = 1, \cdots, W, \qquad |t'_i - t_i| < T_s = \frac{T}{N}. \tag{4}$$

From the condition $T_s \leq \delta/3$, we have the following constraint: $\forall (t'_1, \cdots, t'_W) \in I_1 \times \cdots \times I_W$,

$$\forall i = 1, \cdots, W, \qquad t'_i \geq t'_{i-1} + \frac{\delta}{3}, \quad \text{with the convention } t'_0 = 0 \tag{5}$$

since

$$\begin{aligned} t'_i - t'_{i-1} &= (t'_i - t_i) + (t_i - t_{i-1}) + (t_{i-1} - t'_{i-1}) \\ &\geq t_i - t_{i-1} - |t'_i - t_i| - |t_{i-1} - t'_{i-1}| \\ &\geq \delta - T_s - T_s \geq \frac{\delta}{3}. \end{aligned}$$

Therefore, $I_1 \times \cdots \times I_W \subset S_{\delta/3}$. Let $\boldsymbol{X}'$ be an element of $\mathcal{V}$ which produces the same DTDA signal as that of $\boldsymbol{X}$ at the oversampling ratio $R = N/W$. According to Section III, $\boldsymbol{X}'$ necessarily has QTC's at the levels $(l_1, \cdots, l_W)$ in the sampling intervals $(I_1, \cdots, I_W)$. Therefore, $\exists (t'_1, \cdots, t'_W) \in I_1 \times \cdots \times I_W, \forall i = 1, \cdots, W, X'[t'_i] = l_i$. This W-tuple $(t'_1, \cdots, t'_W)$ necessarily verifies (4) and (5). Using (1) and (3), the fact that $X'[t'_i] = X[t_i] = l_i$ for every $i = 1, \cdots, W$ can be written

$$\mathcal{M}(t'_1, \cdots, t'_W) \cdot \vec{X}' = \mathcal{M}(t_1, \cdots, t_W) \cdot \vec{X} = [l_1 \cdots l_W]^T. \tag{6}$$

Subtracting $\mathcal{M}(t_1, \cdots, t_W) \cdot \vec{X}'$ in the two first members of this equation, we find

$$\begin{aligned} (\mathcal{M}(t'_1, \cdots, t'_W) - \mathcal{M}(t_1, \cdots, t_W)) \cdot \vec{X}' \\ = -\mathcal{M}(t_1, \cdots, t_W) \cdot (\vec{X}' - \vec{X}). \end{aligned} \tag{7}$$

Because of (4), $|t'_i - t_i|$ is upper bounded by T/N and thus goes to zero when N goes to infinity. Therefore, in the limit of N going to ∞, we have[2]

$$\begin{aligned} \mathcal{M}(t'_1, \cdots, t'_W) - \mathcal{M}(t_1, \cdots, t_W) \\ \underset{N \to \infty}{\simeq} \sum_{i=1}^{W} (t'_i - t_i) \frac{\partial \mathcal{M}}{\partial t_i}(t_1, \cdots, t_W). \end{aligned} \tag{8}$$

According to Property 2.1, the fact that $(t_1, \cdots, t_W)$ and $(t'_1, \cdots, t'_W)$ belong to $I_1 \times \cdots \times I_W \subset S_{\delta/3} \subset S$, implies that $\mathcal{M}(t_1, \cdots, t_W)$ and $\mathcal{M}(t'_1, \cdots, t'_W)$ are invertible. Moreover, $[\mathcal{M}(t'_1, \cdots, t'_W)]^{-1}$ is bounded, which implies that $\vec{X}' = [\mathcal{M}(t'_1, \cdots, t'_W)]^{-1} \cdot [l_1 \cdots l_W]^T$ is bounded (expression obtained from (6)). Using (7) and (8), we can thus write

$$\begin{aligned} \vec{X}' - \vec{X} \underset{N \to \infty}{\simeq} -[\mathcal{M}(t_1, \cdots, t_W)]^{-1} \\ \cdot \sum_{i=1}^{W} (t'_i - t_i) \frac{\partial \mathcal{M}}{\partial t_i}(t_1, \cdots, t_W) \cdot \vec{X}'. \end{aligned} \tag{9}$$

Since $\vec{X}'$ is bounded, the right-hand side goes to zero when N goes to ∞. Therefore, $\vec{X}'$ tends to $\vec{X}$, and (9) is still true when replacing $\vec{X}'$ by $\vec{X}$ in the right-side.[3] We obtain

$$\vec{X}' - \vec{X} \underset{N \to \infty}{\simeq} \sum_{i=1}^{W} (t'_i - t_i) \vec{F}_i \tag{10}$$

where

$$\vec{F}_i = -[\mathcal{M}(t_1, \cdots, t_W)]^{-1} \frac{\partial \mathcal{M}}{\partial t_i}(t_1, \cdots, t_W) \cdot \vec{X}.$$

For each $i = 1, \cdots, W, \vec{F}_i$ is a vector which depends only on the signal $\boldsymbol{X}$. Using the fact that $|t'_i - t_i|$ is upper bounded by T/N, (10) implies that, for N large enough, $\|\vec{X}' - \vec{X}\| \leq T/N \sum_{i=1}^{W} \|\vec{F}_i\|$. Using the definition of $MSE(\boldsymbol{X}, \boldsymbol{X}')$ and the fact that $N = RW$, the proof is completed by taking $c = (T/W \sum_{i=1}^{W} \|\vec{F}_i\|)^2$. □

Remark: The only assumption used in this theorem was the number of QTC's of the input signal. For example, this does not require quantization to be uniform. However, the constant c obtained in the upper bound may depend on how the input signal crosses the thresholds. In particular, c depends on $\vec{F}_i$ which contains the term $(\partial \mathcal{M}/\partial t_i)(t_1, \cdots, t_W) \cdot \vec{X}$. One can verify that this expression contains the slope of $\boldsymbol{X}$ at the ith QTC.

V. Conclusion

Under the assumption of periodicity of the bandlimited input signals and a sufficient number of quantization threshold crossings, it

[1] We will not count as QTC's, points where $X[t]$ reaches a quantization threshold without crossing it.

[2] The symbol $\underset{N \to \infty}{\simeq}$ designates the asymptotic equivalence. We recall that two vectorial functions $f(N)$ and $g(N)$ are said to be asymptotically equivalent when N goes to infinity if and only if $\|f(N) - g(N)\| = o(\|g(N)\|)$.

[3] This is valid provided that $\vec{X} \neq \vec{0}$. This is indeed the case because the QTC requirement on $\boldsymbol{X}$ implies that $\boldsymbol{X}$ is a nonzero signal.

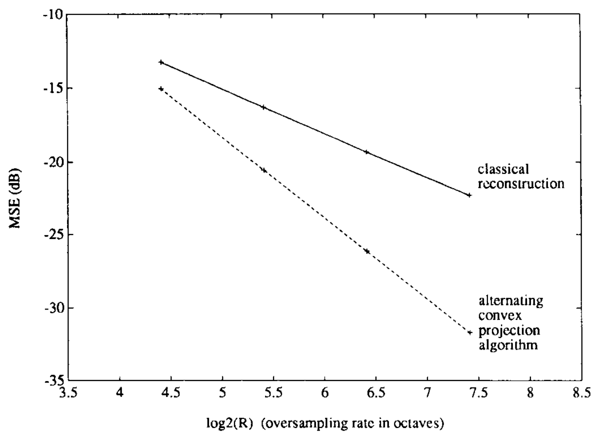

Fig. 3. Experimental results [6] of reconstruction in oversampled A/D conversion. Classical reconstruction and an optimal reconstruction scheme (alternating convex projection algorithm [5], [6]) are performed on the DTDA version of sinusoidal inputs (peak-to-peak amplitude equal to twice the quantization step size). The MSE of reconstruction is measured for different oversampling ratios. The alternating convex projection algorithm [5], [6] achieves approximately an MSE slope of -6 dB/octave of oversampling, instead of -3 dB/octave in classical reconstruction.

was shown that the upper bound on the reconstruction error decreases inversely with the square of the oversampling ratio (Theorem 4.1). This is in contrast with the performance of classical reconstruction, where the MSE decreases inversely with the oversampling ratio only. The condition for achieving such a performance is that the reconstruction produce the same DTDA signal as that of the original signal. Reconstruction algorithms verifying this condition are given in [5] and [6] (alternating convex projections). The experimental results shown in Fig. 3 [6] confirm the result of Theorem 4.1.

APPENDIX
PROOF OF PROPERTY 2.1

$\mathcal{M}(t_1, \cdots, t_W)$ is a Vandermonde matrix. Its determinant is equal to $\prod_{-M\le i\le M} e^{-j2\pi M(t_i/T)}$ times $\prod_{-M\le l<k\le M}(e^{j2\pi(t_k/T)} - e^{j2\pi(t_l/T)})$. For $(t_1, \cdots, t_W) \in S$, such that $t_i \neq t_j$ for all $i \neq j$, one can check that this determinant is nonzero. This proves that $\mathcal{M}(t_1, \cdots, t_W)$ is invertible. Let us write $[\mathcal{M}(t_1, \cdots, t_W)]^{-1} = [M_{ij}(t_1, \cdots, t_W)]_{\substack{1\le i\le W\\ 1\le j\le W}}$. Using the algebraic expression of the inverse of a matrix, it can be seen that $M_{ij}(t_1, \cdots, t_W)$ is a continuous function of $(t_1, \cdots, t_W)$ on the set S. Let $\delta > 0$. S_δ is a subset of S which is compact, since it is closed and bounded. Therefore, every coefficient $M_{ij}(t_1, \cdots, t_W)$ is bounded on S_δ, since the image of a compact subset through a continuous function is compact and thus bounded. $[\mathcal{M}(t_1, \cdots, t_W)]^{-1}$ is then bounded on S_δ. □

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A Mildly Weaker Sufficient Condition in IIR Adaptive Filtering

Majid Nayeri

***Abstract*— The cross-covariance matrix of two stable autoregressive (AR) sequences is considered. A mildly weaker condition is identified that ensures the nonsingularity of this matrix. As one consequence of this result, a weaker sufficient condition is obtained that would guarantee the unimodality of the mean-square output error surface of an IIR adaptive filter with white noise excitation.**

I. INTRODUCTION

The two most popular approaches to filtering, identification, prediction, estimation, etc., are the equation error method and the output error method. Goodwin and Sin [1], and Ljung and Soderstrom [2] have treated the equation error based algorithms thoroughly from the perspective of convergence and applications. An attractive feature of the equation error is its unique minimum mean square equation error (MSEE) solution regardless of the linear model and the properties of the input. However, this property is not shared with the output error method in general when the model is an infinite impulse response (IIR) filter. But, there are sufficient conditions which guarantee the uniqueness of the minimum mean square output error (MSOE) solution in the identification setting [3], where the model (adaptive filter) can characterize the plant (unknown) completely.

The goal of this paper is to present a weaker sufficient condition than what was presented in [3] when the input is white noise and the model is constrained to the minimal parameterization for identifying the plant. In this paper, first a cross-correlation matrix is introduced in section II where some of its properties are outlined. In section III, these properties are used to extract the weaker sufficient condition for the uniqueness of the IIR identifier which would minimize the MSOE.

II. A CROSS-CORRELATION MATRIX

Consider the $m \times m$ matrix $\boldsymbol{P}$ defined by

$$\boldsymbol{P}(A, C, x, m) = E[\phi_m(n)\psi_m^T(n)] \tag{1}$$

Manuscript received July 6, 1992; revised March 15, 1993. The associate editor coordinating the review of this paper was Dr. James Zeidler. This work was supported in part by the Office of Naval research under Grant N00014-91-J-1329 and in part by the National Science Foundation under Grant MIP-9016734.
The author is with the Department of Electrical Engineering, Michigan State University, East Lansing, MI 48825-1226.
IEEE Log Number 9213282.