Thèse n° 10 968

Probabilistic and Deterministic Wellposedness for Low Regularity Dispersive Equations

Présentée le 1er juillet 2024

Faculté des sciences de base
Chaire des équations différentielles partielles
Programme doctoral en mathématiques

pour l'obtention du grade de Docteur ès Sciences

par

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Acceptée sur proposition du jury

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2024

Acknowledgements

I sincerely thank my doctoral advisor Joachim Krieger for making this work possible through his constant support and encouragement. I gained valuable mathematical insight from our frequent discussions, and I am particularly grateful for his support in helping me to develop connections with mathematicians beyond EPFL.
I thank Katharina Schratz for hosting me at Sorbonne University and introducing me to the numerical analysis of dispersive PDEs, as well as for financing my visit through her ERC grant LAHACODE. Thank you also to Frédéric Rousset for many interesting discussions during my stay.
Thank you to Gigliola Staffilani, Jonas Lührmann, Georgios Moschidis and Xue-Mei Li for taking the time to examine my thesis.
I thank Anne Kolly for her patience in answering my emails and for her support during my time as PhD representative. I am also grateful to Rachel Bordelais for helping me manage all the logistical aspects of my PhD.
Thank you to my undergraduate Directors of Studies Ana Khukhro and Holly Krieger, who gave me the confidence to pursue a PhD.
Thanks to my extended office mates Yang, Dylan, Jaimé and Tobias, and all the other excellent people I met at EPFL. A special mention goes to Liyan who has been a constant friend to me.
Lastly, thank you to my parents and to Joanna. Thank you to Jeremy.

Abstract

This thesis concerns the well-posedness of nonlinear dispersive equations in the low regularity setting. We will present two results on global existence for such equations with data at or below the scaling regularity.

In chapter 1 we take a probabilistic perspective to study the energy-critical nonlinear Schrödinger equation in dimensions $d > 6$. We prove that the Cauchy problem is almost surely globally well-posed with scattering for super-critical initial data in $H^s(\mathbb{R}^d)$ whenever $s > \max\{\frac{4d-1}{3(2d-1)}, \frac{d^2+6d-4}{(2d-1)(d+2)}\}$. The randomisation is based on a decomposition of the data in physical space, frequency space and the angular variable. This extends previously known results in dimension 4 and the main difficulty in the generalisation to high dimensions is the non-smoothness of the nonlinearity. The work of this chapter is taken from the publication [Mar23].

Chapter 2 concerns the half-wave maps equation, a nonlocal geometric equation arising in the continuum dynamics of Haldane-Shashtry and Calogero-Moser spin systems. We will prove that in three dimensions the equation is "weakly" globally well-posed (in the sense of [Tao01a]) for angularly regular data which is small in a critical Besov space, partially generalising known results in dimensions $d \geq 4$. The main difficulty in moving to three dimensions is the loss of a key $L_t^2 L_x^\infty$ Strichartz estimate. We overcome this by using Sterbenz's improved Strichartz estimates [Ste05] in conjunction with commuting vector fields to develop trilinear estimates in weighted Strichartz spaces which avoid the use of the $L_t^2 L_x^\infty$ endpoint. This work is taken from [Mar24].

Keywords: Dispersive partial differential equations, well-posedness, global existence, random initial data, nonlinear Schrödinger equation, half-wave maps equation.

Résumé

Cette thèse concerne le caractère bien-posé des équations dispersives non linéaires à régularité faible. Nous présentons deux résultats sur l'existence globale pour de telles équations avec des données à régularité critique ou sur-critique.
Dans le chapitre 1, nous prenons une perspective probabiliste pour étudier l'équation de Schrödinger non linéaire à énergie critique en dimension $d > 6$. Nous démontrons que le problème de Cauchy est presque sûrement bien posé avec scattering pour des données initiales sur-critiques dans $H^s(\mathbb{R}^d)$ pourvu que $s > \max\{\frac{4d-1}{3(2d-1)}, \frac{d^2+6d-4}{(2d-1)(d+2)}\}$. La randomisation se base sur une décomposition à la fois en espace physique, en fréquence et en variable angulaire. Ceci étend des résultats connus en dimension 4 et la principale difficulté dans la généralisation aux hautes dimensions est la nature non lisse de la non-linéarité. Le travail de ce chapitre est tiré de la publication [Mar23].
Le chapitre 2 concerne le « half-wave maps equation », une équation non-locale géométrique qui survient dans la dynamique du continuum des systèmes de type Haldane-Shashtry et Calogero-Moser. Nous prouvons qu'en dimension trois, l'équation est « faiblement » bien posée (dans le sens de [Tao01a]) pour des données initiales angulairement régulières qui sont petites dans un espace de Besov critique, ce qui généralise partiellement des résultats connus en dimension $d \geq 4$. La principale difficulté du passage à trois dimensions est la perte de l'essentielle estimation de Strichartz en $L_t^2 L_x^\infty$. Nous surmontons ce problème en utilisant les estimations de Strichartz améliorées de Sterbenz [Ste05] conjointement avec des champs de vecteurs commutant. Ceci nous permet de développer des estimations trilinéaires dans des espaces de Strichartz avec poids en évitant l'usage de l'espace $L_t^2 L_x^\infty$. Ce travail est tiré de [Mar24].

Mots-clés : Équations aux dérivées partielles dispersives, bien posé, existence globale, données initiales aléatoires, équation de Schrödinger non linéaire, half-wave maps.

Contents

Introduction

This thesis concerns the existence of solutions to nonlinear dispersive equations at low regularity. A classical example of such an equation is the nonlinear Schrödinger equation

$$\begin{cases} (i\partial_t + \Delta)u = \sigma\, u|u|^{p-1} \\ u(0,x) = u_0(x) \end{cases} \qquad (p > 1,\ \sigma = \pm 1,\ \ u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}). \tag{NLS}$$

which will be the focus of the first chapter of this thesis, where we will discuss the work contained in [Mar23]. The second chapter will concern the more recently introduced half-wave maps equation and the result in [Mar24].

We begin this introduction with a discussion of the general principles and background concerning well-posedness of nonlinear dispersive equations. For the sake of concreteness, we will restrict our attention to (NLS), as many of the principles that apply to this equation are also relevant to the half-wave maps equation. Later in the introduction (Sections 0.0.1 and 0.0.2), we will present more specific background on the main topics of this thesis, leaving detailed descriptions of the results and outlines of the arguments to the relevant chapters.

We refer to Section 0.1.1 for standard notation and definitions used in this introduction.

The main concern of this thesis is the question of global existence of solutions, i.e.

- For which classes of initial data u_0 do solutions to (NLS) exist for all time?

This question is non-vacuous, as can be seen by the physically motivated example of the three dimensional *focusing* cubic NLS (i.e. $d = 3$, $p = 3$, $\sigma = -1$),[1] which is known to have solutions which blow up in finite time for arbitrarily smooth initial data [Gla77].

Before developing this discussion further, we should clarify what is meant by a *solution* of (NLS). Since we will mainly be interested in low regularity solutions, which may not even afford the two derivatives required for the equation to make sense classically, we adopt the following more general concept of solution for initial data belonging to a Sobolev space $H^s(\mathbb{R}^d)$.

[1] Here, "focusing" refers to the choice $\sigma = -1$ in the nonlinearity. The choice $\sigma = +1$ corresponds to the "defocusing" equation.

Definition 1 (Solution)**.** *Let $s \in \mathbb{R}$, $0 \in I \subset \mathbb{R}$. For $u_0 \in H^s(\mathbb{R}^d)$, $u \in C^0_{t,loc}H^s_x(I \times \mathbb{R}^d) \cap X$ is called a* strong solution *of* (NLS) *if it solves the Duhamel formula*

$$u(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}(\sigma\, u|u|^{p-1})(s)ds \qquad \text{for all } t \in I.$$

Here X is some function space which ensures that the nonlinearity $u|u|^{p-1}$ and the Duhamel integral make sense as distributions. The operator $e^{it\Delta}$ denotes the solution operator for the free (linear) Schrödinger equation.

Before we consider global solutions, let us first understand for which classes of initial data we expect to have local solutions. We will seek well-posed solutions in the following sense.

Definition 2 (Well-posedness)**.** *Let $s \in \mathbb{R}$. We say that* (NLS) *is locally well-posed in $H^s(\mathbb{R}^d)$ if*

1. *Existence: For any $u_0 \in H^s(\mathbb{R}^d)$, there exists an interval $0 \in I \subset \mathbb{R}$ and a strong solution $u \in C(I, H^s(\mathbb{R}^d))$ to* (NLS) *with data u_0.*
2. *Uniqueness: There exists a space X as in the previous definition such that u is the unique solution to* (NLS) *in the space $C(I, H^s(\mathbb{R}^d)) \cap X$.*
3. *Continuous dependence: If $u_{0,k} \to u_0$ in $H^s(\mathbb{R}^d)$, then the corresponding solutions u_k converge to u in $C(I, H^s(\mathbb{R}^d))$.*

This issue of well-posedness turns out to be intimately related to the scaling symmetry of the equation. For (NLS), one may verify that for any solution u, the rescaled function

$$u_\lambda(t,x) := \lambda^{\frac{2}{p-1}}u(\lambda^2 t, \lambda x)$$

remains a solution for any $\lambda \neq 0$. This rescaling leaves the homogeneous Sobolev space $\dot{H}^{s_c}$,

$$s_c := \frac{d}{2} - \frac{2}{p-1},$$

invariant in the sense that

$$\|u(0,\cdot)\|_{\dot{H}^{s_c}} = \|u_\lambda(0,\cdot)\|_{\dot{H}^{s_c}}$$

It was shown in the seminal work of Cazenave and Weissler [CW90], that (NLS) is locally well-posed in H^s whenever $s \in [\max\{s_c, 0\}, n/2)$ and the nonlinearity is sufficiently regular.[2,3] In this thesis we will primarily be concerned with low regularity solutions in the regime $s_c > 0$, so the threshold will generally be interpreted as $s \geq s_c$.

[2] In particular, we require that either p be an odd integer so that the nonlinearity $u|u|^{p-1}$ is a polynomial, or else $\lfloor s \rfloor < p-1$. The assumption $s < n/2$ can also often be discarded, see for example Proposition 3.8 [Tao06].

[3] The continuity of the data-solution map shown in [CFH11].

In the *subcritical* case, $s > s_c$, the arguments of [CW90] show that we in fact have the following lower bound on the time of existence of the solution:

$$T \gtrsim_{s,d,p} \|u_0\|_{\dot{H}^s}^{-\beta}, \qquad \beta = \frac{2}{s - s_c}. \tag{0.0.1}$$

In particular, if the local solution only exists up to some maximal time T_+, we must have

$$\lim_{t \nearrow T_+} \|u(t)\|_{\dot{H}^s} = +\infty. \tag{0.0.2}$$

The proof of local well-posedness involves a contraction mapping (also known as Picard iteration) argument in Strichartz spaces.[4] This is based on the principle that over small time scales the nonlinear forcing can be treated as a perturbation to the underlying linear equation.

In low regularity spaces H^s with $s < \max\{s_c, 0\}$, various ill-posedness results are known. In the focusing case, $\sigma = -1$, the existence of finite-time blow-up solutions implies by rescaling that one can obtain data converging to 0 in H^s which blow up in arbitrarily short time. Even without the focusing assumption, it was shown in [CCT03] that the solution map $u_0 \mapsto u_0(t)$ fails to be uniformly continuous at $u_0 = 0$. For further examples of ill-posedness see for instance [KPV01, AC09] and the appendix of [BGT05].

We now turn to the question of global well-posedness, for which the theory is far less complete. Observe first that in the critical case $s = s_c$, the lower bound (0.0.1) might lead us to expect that for small initial data ($\|u_0\|_{\dot{H}^{s_c}} \ll 1$) all solutions should be global. This is indeed a direct consequence of the contraction mapping argument of [CW90] and we say that there is "small data-global well-posedness" in the critical space $\dot{H}^{s_c}$.

In Chapter 2, we will use an approach along these lines to construct global solutions to the half-wave maps equation with small initial data. See Section 0.0.2 for further details.

When it comes to constructing global solutions for large data, iterative arguments are less fruitful. Indeed, it is clear that in such cases the nonlinearity cannot be viewed as perturbative over large time scales. However, we can sometimes exploit conservation laws to obtain global control on solutions. For example, the nonlinear Schrödinger equation admits the conserved mass

$$M(t) := \int_{\mathbb{R}^d} |u(t,x)|^2 dx,$$

and it follows from (0.0.2) that (NLS) admits global solutions for arbitrary L^2 data provided the equation is *mass-subcritical*, i.e. $s_c < 0$. In the case $s_c \geq 0$ with the focusing choice of nonlinearity, we have already seen that finite time blow-up is possible for large data [Gla77].

To understand the situation for $s_c \geq 0$ in the defocusing case, we introduce a second

[4] See Section 0.1.3.

conserved quantity, the energy

$$E(t) := \frac{1}{2}\int |\nabla u(t,x)|^2 dx + \sigma\frac{1}{p+1}\int |u(t,x)|^{p+1} dx.$$

Restricting henceforth to $\sigma = +1$, the energy controls the $\dot{H}^1$ norm and it again follows from (0.0.2) that (NLS) admits global solutions for arbitrary H^1 data provided the equation is *energy-subcritical*, i.e. $s_c < 1$. For H^s data below the energy space, i.e. $s_c < s < 1$, global wellposedness is still sometimes achievable using Bourgain's "high-low" method [Bou98]. This involves exploiting a nonlinear smoothing effect on the high frequency portion of the solution to show that

$$u - e^{it\Delta}u_0 \in C_{\mathrm{loc}}(\mathbb{R}, H^1(\mathbb{R}^d)),$$

Since $e^{it\Delta}$ preserves the regularity of u_0, this prevents u from blowing up in the H^s norm and leads to global wellposedness.[5] Results for higher regularity solutions are often possible by persistence of regularity (see Proposition 3.11, [Tao06]).

We next consider energy-critical equations ($s_c = 1$), for which

$$p = \frac{d+2}{d-2}.$$

In this case the lower bound (0.0.1) fails in H^1, and conservation laws are insufficient to deduce global well-posedness. Nonetheless, following initial results for radial data due to Bourgain [Bou99] and Grillakis [Gri00], the global existence of finite energy solutions to the defocusing NLS was established by Colliander-Keel-Staffilani-Takaoka-Tao [CKS$^+$08] in dimension three. The problem in four dimensions and higher was settled by Ryckman and Visan [RV07] and Visan [Vis07] respectively.

In addition to global well-posedness, the results [CKS$^+$08, RV07, Vis07] also yield precise information on the asymptotic behaviour of solutions in terms of *scattering*: there exist states $u_\pm \in \dot{H}^1$ such that

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta}u_\pm\|_{\dot{H}^1} = 0. \tag{0.0.3}$$

Lastly, for energy-supercritical equations ($s_c > 1$), the issue of global well-posedness is least tractable due to the lack of relevant conserved quantities. For certain (still defocusing) equations in this regime, finite time blow up was shown relatively recently in the celebrated work [MRRS22].[6]

[5] Of course, if $s_c < 0$ one could also use the L^2 conservation and obtain global solutions in L^2. One could then deduce global existence in H^s for some $0 < s < 1$ by persistence of regularity for a smooth nonlinearity.

[6] We also draw attention to the works [SY20, Sy21] establishing almost sure global well-posedness for certain energy supercritical nonlinear Schrödinger equations with energy subcritical initial data ($s > 1$) in the periodic setting. Probabilistic results of this type are discussed in detail in the next section.

This concludes the general part of the introduction, bringing us reasonably up to speed on the state of the "deterministic" theory of (NLS). Similar theories relating well-posedness to the invariances of the equation and conserved quantities are also available for other semilinear dispersive equations, such as the nonlinear wave [LS95, Sog08] and KdV [KPV96, Bou93] equations. In the next two sections we will give additional background relevant to the main chapters of this thesis. The first section concerns the extension of the above results for NLS into the low-regularity supercritical setting by probabilistic methods, and the second concerns the analogous theory for the recently introduced "half-wave maps" equation.

0.0.1 Supercritical Cauchy theory for NLS: a probabilistic approach.

In the first chapter of this thesis we consider whether the well-posedness theory discussed above can be improved for *generic* initial data. For instance,

- For data chosen at random from a *supercritical* Sobolev space ($s < s_c$), can we still expect local (global!?) solutions?
- In spaces where conservation laws are not available, might we still find global solutions for "most" initial data?

Note that a positive answer to the first question above would be in contrast to the ill-posedness shown in [CCT03, KPV01, AC09, BGT05], while a positive result for the second question would be an extension of the developments in [Bou98].

The study of (NLS) from a "probabilisitic" perspective was pioneered by Lebowitz-Rose-Speer [LRS88] in their work on invariant measures. These measures are supported on low regularity Sobolev spaces and so provide a means of interpreting such spaces as probability spaces. Moreover, the flow of (NLS) can be seen to be volume-preserving with respect to these "Gibbs measures" (akin to Liouville's theorem for finite dimensional Hamiltonian systems), which can serve as a useful substitute for conservation laws when considering global results. This was first observed by Bourgain [Bou94], who used the invariance to prove almost sure global well-posedness of (NLS) below the energy space in the 1-D periodic setting.[7] In [Bou96] he also showed almost sure well-posedness for *supercritical* initial data by taking advantage of improved integrability estimates for the randomised free evolution. However, the latter results are only valid for a modified "Wick-ordered" NLS.

The notion of an invariant measure is clearly highly valuable, however there are often significant barriers to constructing one. On the one hand, in high dimensions the Gibbs measure proposed in [LRS88] is supported on functions of such low regularity that there

[7] On the torus, one has similar heuristics for the well-posedness of (NLS), however the reality is more complicated due to the lack of dispersion.

are issues even defining it.[8] On the other hand, the construction relies on the existence of an orthonormal basis of eigenfunctions of the Laplacian. Since our primary interest is in the Euclidean domain $\mathbb{R}^d$, such a construction cannot easily be applied and we must seek other methods of randomising the initial data.[9]

A commonly used randomisation procedure is the so-called *Wiener randomisation*, which naturally generalises the construction in [LRS88]. For a compact domain this method is due to Burq-Tzvetkov [BT08] (in the context of the nonlinear wave equation) and allows for the construction of a large family of measures on any Sobolev space H^s, albeit non-invariant.[10] See [BT13] for more details on the interpretation of these measures and their measure-theoretic properties. Here we will present the analogous procedure in the Euclidean setting, introduced by Lührmann-Mendelson [LM14] and Bényi-Oh-Pocovnicu [BOP14] based on a similar randomisation in [ZF12]. In the periodic case, the randomisation relies on the discrete Fourier decomposition of the initial data, so the first step is to define an analogous decomposition in the Euclidean setting. We introduce smooth, radial cut-offs $\psi : \mathbb{R}^d \to [0, 1]$ equal to 1 on $\{x \in \mathbb{R}^d : |x| \leq \sqrt{d}\}$ and vanishing for $|x| \geq 2\sqrt{d}$, then define[11]

$$\psi_j(x) := \frac{\psi(x-j)}{\sum_{k\in\mathbb{Z}^d} \psi(x-k)} \qquad (j \in \mathbb{Z}^d).$$

Observe that the ψ_j form a partition of unity on $\mathbb{R}^d$ and denote by P_j the Fourier multiplier

$$P_j u_0 := \mathcal{F}^{-1}(\psi_j(\cdot)\mathcal{F}(u_0)(\cdot)) \qquad \Longrightarrow \qquad u_0 = \sum_{j\in\mathbb{Z}^d} P_j u_0.$$

The next ingredient is a family of independent identically distributed Gaussian random variables $(g_j)_j : \Omega \to \mathbb{R}$ of mean zero on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.[12] We then consider the probability measure μ_{u_0} on $H^s(\mathbb{R}^d)$ induced by the random function

$$\Omega \ni \omega \mapsto \sum_{j\in\mathbb{Z}^d} g_j(\omega) P_j u_0 =: u_0^\omega. \tag{0.0.4}$$

One may verify that $u_0^\omega \in H^s(\mathbb{R}^d)\backslash H^{s+\epsilon}(\mathbb{R}^d)$ almost surely for any $\epsilon > 0$, so the randomisation does not regularise on the level of Sobolev spaces (see Lemma B.1,

[8] Under a radial assumption, Gibbs measures were constructed in higher (three) dimensions in [Tzv06, Tzv07, BT07].

[9] An invariant measure was constructed for (NLS) on the real line in [CdS15], however the probabilistic techniques are more involved than in the periodic setting. See also [dS14] concerning the Klein-Gordon equation. Invariant measures have also been constructed for the nonlinear Schrödinger equation with an external potential, for which an orthonormal basis of eigenfunctions can be found. The results can sometimes be mapped back to (NLS) by transformations, see for instance [BTT13, Den10, PRT14] and [dS13] for a similar procedure in the context of the nonlinear wave equation.

[10] Importantly, we can construct these measures on spaces of arbitrary regularity, whereas the invariant measures of [LRS88, Bou94, Bou96] force the regularity we must work in.

[11] The factor of $\sqrt{d}$ is just to ensure that every point of $\mathbb{R}^d$ is in the support of some ψ_j.

[12] It is possible to consider more general families of random variables, see for instance [BT13].

[BT08]). Contrary to the periodic case, the measure μ_{u_0} does not have dense support in $H^s(\mathbb{R}^d)$, however it still allows us to generate results for large sets of initial data.

The key property of the above randomisation procedure is that, while it does not improve the regularity of the evolution $e^{it\Delta}u_0^\omega$ (which almost surely retains the regularity of u_0) it does improve the integrability. This is based on Bernstein's inequality

$$\|P_j u\|_{L_x^q} \lesssim \|P_j u\|_{L_x^p} \qquad (\infty \geq q \geq p \geq 1).$$

Note there is no loss of derivatives in this estimate due to our working on unit scales. To see the significance of this, we remark that many probabilistic well-posedness results rely on decomposing the desired solution u^ω (with random data u_0^ω) into the rough free evolution $e^{it\Delta}u_0^\omega$ and the nonlinear portion $v^\omega := u^\omega - e^{it\Delta}u_0^\omega$. Then v^ω must satisfy

$$\begin{cases} (i\partial_t + \Delta)v^\omega = \sigma\,(v^\omega + e^{it\Delta}u_0^\omega)|v^\omega + e^{it\Delta}u_0^\omega|^{p-1} \\ v^\omega(0,\cdot) = 0 \end{cases} \tag{0.0.5}$$

(in the case of (NLS)). By showing that $e^{it\Delta}u_0^\omega$ has improved integrability properties (compared to those attainable by standard Strichartz estimates), it can often be shown by direct iteration that (0.0.5) is well-posed in a *subcritical* Sobolev space, often H^1, even when u_0 is very rough. This idea is closely related to the high-low argument of [Bou98].

In Chapter 1, we will present an almost-sure global well-posedness and scattering result for a class of energy critical nonlinear Schrödinger equations with supercritical data, using a careful refinement of the randomisation above (see Section 1.1.2). It is therefore important to understand how randomising initial data can lead to improved global results in the absence of an invariant measure.

Unless stated otherwise, henceforth all results discussed concern defocusing equations on the Euclidean space with respect to the Wiener randomisation (0.0.4). When we say a result holds "almost surely in H^s", we mean

"For all $u_0 \in H^s$ the result holds almost surely with respect to the measure μ_{u_0} on H^s, i.e. with data u_0^ω for almost every $\omega \in \Omega$."

One natural approach for obtaining global results is to use Bourgain's high-low method in the probabilistic setting. This involves treating the smoother low frequency part of the solution deterministically, and appealing to probabilistic methods to show almost-sure smoothing for the high frequency portion. This approach was first used in [CO12] to prove almost sure global well-posedness of the (Wick-ordered) cubic NLS in mass-supercritical Sobolev spaces H^s with $-\frac{1}{12} < s < 0$. This result is on the torus with respect to a natural probability measure on the negative Sobolev spaces.[13] The same method was applied in

[13]The argument of [CO12] is valid in both the focusing and defocusing settings, since when working in negative Sobolev spaces we rely on the conserved mass rather than energy.

the Euclidean setting in [LM14] for a range of energy-subcritical wave equations, proving almost sure global well-posedness in a range of scaling supercritical Sobolev spaces. In this energy subcritical setting, global well-posedness can also be achieved by finding probabilistic a priori energy bounds on the nonlinear evolution v^ω (see (0.0.5)), as in [BT13, LM16]. See also [SX16].

As in the deterministic case, the issue of probabilistic global well-posedness is more delicate for energy-critical equations. The first result in this direction for NLS appeared in [BOP15a], where Bényi-Oh-Pocovnicu proved *conditional* almost-sure global well-posedness for the 4-D energy critical NLS with supercritical data. The condition is an a priori energy bound for the nonlinear evolution v^ω, which is needed to treat (0.0.5) as a perturbation of the usual energy critical NLS in H^1 (and then appealing to the deterministic results for that equation). In [OOP17] such energy estimates were obtained, yielding almost-sure global well-posedness in dimensions 5 and 6 (where there is the additional challenge of handling a non-polynomial nonlinearity). See also in [Poc17, OP16] for results on the nonlinear wave equation (NLW) and [OP17] for analogous results in the periodic setting.[14]

The results discussed above all relied on the deterministic theory for the energy-critical NLS, for which finite energy solutions are also known to scatter at large times. It is therefore reasonable to ask whether some supercritical data might also lead to almost-sure scattering. The results of the previous paragraph do not imply this since the a priori energy bounds involved have explicit time dependence, which is a known obstruction to scattering. We note however that for *small* initial data, scattering on large sets in supercritical Sobolev spaces was shown in [LM14, Poc17].

The first large data probabilistic scattering result for an energy critical equation was obtained by Dodson-Lührmann-Mendelson in [DLM20] for the 4-D cubic NLW with "radial data".[15] The proof again relies on energy critical perturbation theory, with the key addition of a Morawetz estimate adapted to the forced equation (0.0.5) to find a *global* energy bound via a "double bootstrap" argument.[16] The use of this estimate requires almost sure spatial decay on the randomised free evolution, which was obtained for radial data by means of a radially averaged Sobolev estimate,

$$\left\| |x|^{\frac{3}{2}} \left(\sum_{j\in\mathbb{Z}^4} |P_j f(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^\infty_x(\mathbb{R}^4)} \lesssim_s \|f\|_{H^s_{\mathrm{rad}}(\mathbb{R}^4)} \qquad (s>0, \quad f \text{ radial}).$$

One may compare this to the usual radial Sobolev estimate, $\||x|^{\frac{3}{2}} f\|_{L^\infty_x(\mathbb{R}^4)} \lesssim \|f\|_{H^1_{\mathrm{rad}}(\mathbb{R}^4)}$,

[14] We remark that global results for NLW are generally simpler due to the smoothing effect of the wave propagator and the presence of time derivatives in the conserved energy, which significantly simplifies estimates of the energy increment.

[15] This means that the data u_0 generating the measure (0.0.4) are radial, however this does *not* imply that the measure μ_{u_0} is supported on radial functions.

[16] Note that Morawetz estimates also play an important role in the deterministic scattering theory of (NLS).

and observe the important distinction that even when f is radial the square sum $(\sum_{j\in\mathbb{Z}^4} |P_j f(x)|^2)^{\frac{1}{2}}$ may not be.

The methods of [DLM20] were adapted to the energy-critical NLS in [KMV19] (still in four dimensions with radial data), invoking local smoothing estimates and a modified Morawetz inequality to obtain improved integrability for the derivatives of v^ω and $e^{it\Delta}u_0^\omega$. The regularity threshold for almost sure scattering was then lowered from $\frac{5}{6}$ to $\frac{1}{2}$ in [DLM19] by working in lateral function spaces, previously introduced in the context of the derivative nonlinear Schrödinger equation and Schrödinger maps problems.

In [Spi21], Spitz removed the radial assumption by using a modified randomisation procedure involving a decomposition into spherical harmonics and on unit scales in physical space, see Section 1.1.2. This randomisation allows access to a much wider range of spacetime bounds for the free evolution and its derivatives without appealing to Morawetz or local smoothing estimates.

The goal of Chapter 1 is to extend the results of [Spi21] to energy-critical equations in high dimensions with non-algebraic nonlinearities. The work is taken from [Mar23].

0.0.2 Small Data Global Regularity for the Half-Wave Maps Equation.

We now introduce the second major topic of this thesis, namely the well-posedness of the half-wave maps equation,

$$\begin{cases} \partial_t \phi = \phi \times (-\Delta)^{1/2}\phi \\ \phi(0,\cdot) = \phi_0 \end{cases} \qquad (\phi : \mathbb{R}\times\mathbb{R}^d \to \mathbb{S}^2). \tag{HWM}$$

This is a geometric equation, where we view the sphere $\mathbb{S}^2$ as embedded into $\mathbb{R}^3$ so that the cross product on the right hand side makes sense (note in particular that then $\partial_t\phi \perp T_\phi\mathbb{S}^2$). The operator $(-\Delta)^{1/2}$ is defined (for sufficiently regular functions) via its action in Fourier space,

$$\mathcal{F}((-\Delta)^{1/2}\phi)(\xi) := |\xi|\,\mathcal{F}(\phi)(\xi),$$

and may be interpreted as a nonlocal spatial derivative.

(HWM) is a relatively recently introduced equation [ZS15], and even the local well-posedness theory is not yet fully developed. The equation admits the scaling invariance

$$u(t,x) \quad \rightsquigarrow \quad u(\lambda t, \lambda x)$$

from which we deduce the critical exponent is $s_c = d/2$. The equation also admits the positive definite conserved quantities

$$\text{Mass:} \qquad M(t) = \int_{\mathbb{R}^d} |u(t,x)|^2 dx$$

$$\text{Energy:} \qquad E(t) = \int_{\mathbb{R}^d} |(-\Delta)^{\frac{1}{4}} u(t,x)|^2 dx$$

Given the previous discussions on (NLS), we are thus led to search for local well-posedness in H^s with $s > d/2$, and small data-global well-posedness in $\dot{H}^{d/2}$. The energy space is now $\dot{H}^{1/2}$ so the equation is energy-critical in dimension 1, and supercritical in all higher dimensions. In particular, we expect global results to be challenging for $d \geq 2$.

The half-wave maps equation was first derived in one dimension [ZS15] upon taking the classical then continuum limit of a Haldane-Shashtry spin chain. See also [GL18]. The equation was further shown in [LS20] to arise in the continuum limit of the completely integrable (classical) Calegero-Moser spin systems. It is therefore unsurprising that the one-dimensional equation is completely integrable in the sense that it admits a Lax Pair [GL18], and in this setting there has been significant interest in special solutions of the equation. Indeed, soliton solutions were first studied numerically and analytically in [ZS15], with further investigations of multi-solitons in [BKL20, Mat22] and a complete classification of the finite-energy travelling solitary waves in [LS18].

When it comes to the question of well-posedness, the one-dimensional problem turns out to be the most delicate. One way to see this is from the wave-like structure of the equation (see (0.0.6) below), which is less useful in one dimension where waves do not disperse and standard techniques cannot be applied (see Section 0.1.3). So far, results on the existence of solutions in one dimension include [Liu23] establishing the global existence of weak solutions with large data in $\dot{H}^1 \cap \dot{H}^{1/2}$, and the more recent work [Ohl23] concerning the global existence of a particular family of rational solutions.

We now discuss the known well-posedness results for (HWM) in high dimensions, starting with the observation of Krieger and Sire [KS17] that the quasilinear half-wave maps equation can reduced to the semilinear problem[17]

$$\begin{aligned}\Box\phi \equiv (\partial_t^2 - \Delta)\phi = &- \phi\, \partial^\alpha\phi \cdot \partial_\alpha\phi \\ &+ \Pi_{\phi^\perp}[((-\Delta)^{1/2}\phi)(\phi \cdot (-\Delta)^{1/2}\phi)] \\ &+ \phi \times [(-\Delta)^{1/2}(\phi \times (-\Delta)^{1/2}\phi) - (\phi \times (-\Delta)\phi)] \end{aligned} \tag{0.0.6}$$

upon differentiating in time.[18] Here $\Pi_{\phi^\perp}$ denotes the projection onto the orthogonal complement of ϕ and we sum over $\alpha = 0, \ldots, 3$ with respect to the Minkowski metric so that

$$\partial^\alpha\phi \cdot \partial_\alpha\phi = -|\partial_t\phi|^2 + |\nabla_x\phi|^2.$$

The passage from (HWM) to (0.0.6) relies heavily on the property $\phi \cdot \phi = 1$.

The formulation (0.0.6) shows that (HWM) is intimately linked to the *wave maps*

[17] Despite the presence of the Laplacian in the nonlinearity of (0.0.6), the equation behaves like a semilinear equation due to the cancellation structure of the final term.

[18] In passing from (HWM) to the second order equation (0.0.6) we fix the initial velocity $\partial_t\phi(0,\cdot) \equiv \phi_0 \times (-\Delta)^{1/2}\phi_0$.

equation,

$$\begin{cases} \Box\phi = -\phi\ \partial^\alpha\phi \cdot \partial_\alpha\phi \\ (\phi, \partial_t\phi)(0,\cdot) = (\phi_0, \phi_1) \end{cases} \tag{WM}$$

which has been the subject of intense study during late 1990s and early 2000s. In particular, the local and small data-global well-posedness of (WM) is well understood. In preparation for our discussion on half-wave maps, we start by reviewing the relevant results for wave maps below. Henceforth we use the notation $\phi[t] := (\phi(t), \partial_t\phi(t))$.

Interlude on the Theory of Wave Maps.

In the scaling subcritical case, $s > d/2$, the wave maps equation is known to be locally well-posed in both the Sobolev and Besov spaces. This was first proved in dimension 3 by Klainerman-Machedon in [KM96b] and then extended to all dimensions greater than or equal to 2 by Klainerman-Selberg in [KS97], the special one dimensional result appearing in [MNT10].[19,20] It is important to note that these results are specific to the nonlinearity of (WM). Indeed, Lindblad [Lin93] showed ill-posedness of the 3-D quadratic derivative nonlinear wave equation

$$\Box\phi = (\partial_t\phi)^2 \tag{0.0.7}$$

in $H^{2-\epsilon} \times H^{1-\epsilon}$ for any $\epsilon > 0$, even though the equation scales in the same way as (WM).[21]

The particular structure of (WM) which allows for improved wellposedness is the presence of the "null form"

$$\partial^\alpha\phi \cdot \partial_\alpha\phi = \frac{1}{2}(\Box(\phi\cdot\phi) - 2\phi\cdot\Box\phi), \tag{0.0.8}$$

which relates the nonlinearity to the linear operator of the equation and kills off resonant interactions. This plays a fundamental role in the analysis of the wave maps equation, and is exploited in [KM96b, KS97] via the identity (0.0.8) by iterating in the $X^{s,\theta}$ spaces, dating back to Bourgain [Bou93] (see Section 2.10).

The scale-invariant case $s = d/2$ is far more delicate. In fact, the equation is known to be *ill-posed* in the critical Sobolev space $\dot{H}^{d/2}$ insofar as the solution operator not being uniformly continuous. This was shown in [DG04] via a family of solutions contained in geodesics on the target manifold $\mathbb{S}^2$ (see also [Tao00] for the case $d = 1$). Nonetheless, Tataru showed in [Tat98, Tat01] that small data-global wellposedness holds in the critical Besov space $\dot{B}^{d/2}_{2,1}$ provided $d \geq 2$ (again the null structure plays an important role

[19]The references cited concern the Sobolev case, however it is possible to adapt them to the Besov case (see Section 2.10).

[20]The one-dimensional case is special owing to the fact that the 1-D wave equation is not dispersive. Unlike for (HWM), however, the one dimensional wave maps equation is *not* energy-critical. Rather the conserved energy for (WM) scales like $\dot{H}^1$ and the equation is energy-critical in dimension 2.

[21]In three dimensions, the space $H^{2+\epsilon} \times H^{1+\epsilon}$ is the minimum regularity attainable by appealing only to Strichartz estimates. See section 0.1.3.

here).[22] The Besov space (defined in (2.1.15)) is a stronger version of the Sobolev space satisfying the embedding
$$\dot{B}^{d/2}_{2,1} \hookrightarrow L^\infty(\mathbb{R}^d)$$
which just fails for $\dot{H}^{d/2}$. This is particularly useful when dealing with geometric problems such as (WM) since it renders the problem local with respect to the target manifold. Of Tataru's results [Tat98, Tat01], the high dimensional case ($d \geq 4$) is simpler, proved via a contraction mapping argument in modified versions of the $X^{s,\theta}$ spaces. This method breaks down in low dimensions and the argument for $d = 2$, 3 in [Tat01] relies on the construction of intricate function spaces involving a decomposition in Fourier space with respect to angular sectors on the characteristic lightcone.

Despite the ill-posedness results, Tao was still able to prove a certain notion of "weak wellposedness" for wave maps in the critical Sobolev space $\dot{H}^{d/2}$ (see [Tao01a, Tao01b] for $d \geq 5$ and $d = 2, 3, 4$ respectively). Precisely, he proved that for any $\phi[0] \in H^s \times H^{s-1}$, $s > d/2$, the subcritical local solution $\phi[t]$ provided by [KM96b, KS97] can be continued globally provided that the critical norm of the data is sufficiently small:
$$\|\phi[0]\|_{\dot{H}^{d/2} \times \dot{H}^{d/2-1}} < \epsilon \ll 1. \tag{0.0.9}$$
Moreover, for $|s - d/2| \ll 1$ he obtained the uniform bounds
$$\|\phi[t]\|_{L^\infty(\mathbb{R};\, H^s \times H^{s-1})} \lesssim \|\phi[0]\|_{H^s \times H^{s-1}}.$$
It follows that all smooth, compactly supported initial data which are sufficiently small in $\dot{H}^{d/2} \times \dot{H}^{d/2-1}$ lead to a global solution, however in accordance with the ill-posedness results there is no claim of continuous dependence.[23]

Tao's argument is based on a methodical study of the frequency interactions in the nonlinearity with two key novel ingredients. Since the lack of uniform continuity implies that Picard iteration will surely fail, Tao introduced a new bootstrap argument based on the concept of *frequency envelopes*. This new technology provides a means of tracking the transfer of energy among different frequencies in order to maintain control of the solution in the more regular H^s norms, leading to global existence. The second new ingredient is a co-ordinate transformation on the sphere which eliminates certain difficult frequency interactions. In the high dimensional case $d \geq 5$ this transformation allows Tao to close the argument using only Strichartz spaces, rather than spaces adapted to the null structure as in [KM96b, KS97, Tat98, Tat01]. In low dimensions $d = 2, 3, 4$ the argument is significantly more involved and the null structure again plays an essential role. The methods of [Tao01a] play an important role in the argument in Chapter 2.

This concludes our interlude on the wave maps equation, and we now return to our main

[22] In one dimension, there is again a failure of uniform continuity for the solution operator in the critical Besov space [Tao00].

[23] See [Tat05] for a weaker notion of continuous dependence for critical wave maps.

discussion on half-wave maps. To this point, the well-posedness theory of (HWM) in high dimensions is limited to the scaling-critical case.[24] This was first investigated by Krieger and Sire [KS17] who proved small data-global wellposedness in the critical Besov space for $d \geq 5$. This was extended to four dimensions in [KK21]. The key idea of Krieger and Sire was to use the reformulation (0.0.6) to study (HWM) as a semilinear wave equation. Naturally, the high dimensional results of Tataru for critical wave maps [Tat98] played an important role in their analysis. Note that beyond the explicit connection to wave maps via the first forcing term, the entire nonlinearity of (0.0.6) is heuristically of wave-maps type

$$\phi \, \nabla\phi \, \nabla\phi,$$

provided one can account for the action of the nonlinear projection operator and cancellations in the nonlocal derivatives. The difficulty is that the new forcing terms lack the null structure that was so crucial in the analysis of (WM). It turns out that this can be compensated by the in fact stronger geometric structure of these terms, which in essence comes down to the property

$$\phi \cdot \nabla\phi = 0 \tag{0.0.10}$$

for functions into the sphere. By rewriting this identity in terms of frequency cancellations, Krieger-Sire were able to handle the *half*-wave maps contributions to the nonlinearity of (0.0.6) entirely in Strichartz spaces, incorporating the methods from [Tat98] to handle the wave maps terms. A delicate Fourier analysis was required to exploit (0.0.10) in the context of the nonlocal derivatives.

In the critical Sobolev space, Liu [Liu21] showed weak well-posedness in the sense of [Tao01a] by incorporating the methods of [KS17] into Tao's argument. We also mention that the issue of uniqueness was addressed in [EFS22], and the half-wave maps equation into manifolds other than the sphere has been studied in [GL18, Liu21], both in the context of hyperbolic space.

In Chapter 2, we extend the results of [KS17, KK21] to three dimensions under an additional assumption of angular regularity on the initial data. As for the wave maps equation, (HWM) is increasingly complicated in lower dimensions and in passing from four to three dimensions we in particular lose the key endpoint $L_t^2 L_x^\infty$ Strichartz estimate. To overcome this, we adapt the methods of [Tao01a] and [KS17] by exploiting improved Strichartz estimates for functions with angular regularity [Ste05] and incorporating the full range of commuting vector fields for the wave operator. A more detailed discussion of our methods can be found in the introduction of Chapter 2. This work is taken from the preprint [Mar24] (submitted to Ars Inveniendi Analytica).

[24] The subcritical case $s > d/2$ is discussed in Section 2.10 in Chapter 2. Due to the loss of regularity in passing from the first order half-wave maps equation to the second order equation (0.0.6), a standard subcritical well-posedness result was not achieved.

0.1 Preliminaries

0.1.1 Notation

In this section we present some of the general notation used throughout this thesis. Topic-specific notation is defined in the respective chapters.

Throughout, $C_{\alpha_1,...,\alpha_n}$ denotes a constant depending only on the parameters $\alpha_1, ..., \alpha_n$ whose precise value may change line-to-line. We write $X \lesssim_{\alpha_1,...,\alpha_n} Y$ to mean $X \leq C_{\alpha_1,...,\alpha_n} Y$ and $X \sim_\alpha Y$ to mean $X \lesssim_\alpha Y$ and $Y \lesssim_\alpha X$.

Unless stated otherwise, whenever $p \in [1, \infty]$, $p' \in [1, \infty]$ denotes its conjugate exponent such that

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

We will frequently work in the Lebesgue spaces L^p_x with norms

$$\|f\|_{L^p_x} \equiv \|f\|_p = \left(\int_{\mathbb{R}^d} |f(x)|^p \, dx \right)^{1/p}$$

and for spacetime functions $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ we also use the mixed spaces $L^p_t L^q_x$ where

$$\|f\|_{L^p_t L^q_x(I \times \mathbb{R}^d)} \equiv \|f\|_{L^p_t L^q_x(I)} \equiv \|f\|_{p,q[I]} := \| \|f\|_{L^q_x(\mathbb{R}^d)} \|_{L^p_t(I)}.$$

For the Fourier transform we use the notation

$$\mathcal{F}(f)(\xi) \equiv \hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) dx$$

which allows use to define the inhomogeneous Sobolev spaces H^s, $s \in \mathbb{R}$, by

$$\|f\|_{H^s} := \|(1 + |\xi|^2)^{s/2} \hat{f}(\xi)\|_{L^2_\xi}.$$

Whenever we say that a function space is "defined by the norm $\| \cdot \|$", we mean that the space is the closure of the Schwarz functions with respect to the given norm. In this vein we define the homogeneous Sobolev spaces $\dot{H}^s$ via

$$\|f\|_{\dot{H}^s} := \| |\xi|^s \hat{f}(\xi)\|_{L^2_\xi}.$$

This is well-defined as a space of distributions if and only if $s < d/2$.[25]

Next we introduce the free evolution operators for the Schrödinger and wave equations. Denote

$$e^{it\Delta} f := \mathcal{F}^{-1}(e^{-it|\xi|^2} \hat{f}(\xi)) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi - it|\xi|^2} \hat{f}(\xi) d\xi \tag{0.1.1}$$

[25] In contrast, the critical homogeneous Besov spaces we will work with in Chapter 2 do embed into distributions.

the free solution to the linear Schrödinger equation $(i\partial_t + \Delta)u = 0$ with sufficiently regular data $u(0, x) = f(x)$. For the wave equation we denote

$$W_t(f, g) \equiv \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|}g \tag{0.1.2}$$

the free solution to $\Box u = 0$ with data $(u, \partial_t u)(0, x) = (f(x), g(x))$. We interpret the right hand side above using the Euler formulae and the definitions

$$e^{\pm it|\nabla|}f := \mathcal{F}^{-1}(e^{\pm it|\xi|}\hat{f}(\xi)) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi \pm it|\xi|}\hat{f}(\xi)d\xi. \tag{0.1.3}$$

with a similar formula for $\frac{e^{\pm it|\nabla|}}{|\nabla|}g$. We will sometimes write $\sqrt{-\Delta}$ instead of $|\nabla|$.

0.1.2 The Littlewood-Paley decomposition

Littlewood-Paley theory is a core tool in modern harmonic analysis. Let $\chi : \mathbb{R}^d \to [0, 1]$ be a smooth function supported in $\{|\xi| \leq 2\}$ and equal to 1 on $\{|\xi| \leq 1\}$. For $M \in 2^{\mathbb{Z}}$ set

$$\chi_M(\xi) := \chi(\xi/M) - \chi(2\xi/M) \tag{0.1.4}$$

so that χ is supported in $\{\frac{M}{2} \leq |\xi| \leq 2M\}$ and

$$\sum_{M\in 2^{\mathbb{Z}}} \chi_M(\xi) = 1$$

for all $\xi \neq 0$. For any $f \in \mathcal{S}'(\mathbb{R}^d)$ define the Littlewood-Paley multiplier

$$P_M f := \mathcal{F}^{-1}(\chi_M \cdot \hat{f}). \tag{0.1.5}$$

We will also write $\widetilde{\chi}_M$ for $\sum_{|\log(N/M)|\leq C} \chi_N$ and $\widetilde{P}_M f$ or $f_{\sim M}$ for $\mathcal{F}^{-1}(\widetilde{\chi}_M \cdot \hat{f})$ whenever C is any fixed constant up to 100. Similarly, $f_{<M} := \sum_{N<M} f_N$ and so on.

A key property of these multipliers is that they are uniformly bounded on all L^p spaces:

$$\|P_N f\|_{L^p_x} \lesssim_p \|f\|_{L^p_x} \qquad (1 \leq p \leq \infty).$$

Moreover, if we define Fourier multipliers $|\nabla|^s$ by

$$|\nabla|^s f := \mathcal{F}^{-1}(|\xi|^s \hat{f}(\xi))$$

we have

$$\||\nabla|^s P_N f\|_{L^p(\mathbb{R}^d)} \sim_{p,d} N^s \|P_N f\|_{L^p(\mathbb{R}^d)} \qquad (1 \leq p \leq \infty).$$

In the case $s = 1$, we also have the key Riesz estimate which allows us to exchange nonlocal derivatives for true derivatives,

$$\||\nabla|P_N f\|_{L^p(\mathbb{R}^d)} \sim_{p,d} \|\nabla P_N f\|_{L^p(\mathbb{R}^d)} \qquad (1 \leq p \leq \infty).$$

In fact, the Riesz estimate holds even in the absence of the projection P_N provided we restrict the range to $1 < p < \infty$.

We end this section with the following fundamental inequality relating the norm of a function to that of its Littlewood-Paley square sum.

Lemma 0.1.1 (Littlewood-Paley Inequality [LP31, Ste70a])**.** *For any $1 < p < \infty$ it holds*

$$\|f\|_{L^p(\mathbb{R}^d)} \sim_{p,d} \left\| \left(\sum_{N \in 2^{\mathbb{Z}}} |P_N f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)}$$

Warning: *In Chapter 2, in order to be consistent with other papers on the topic we adopt the different notation $P_k f$ to mean $P_{2^k} f$ with $k \in \mathbb{Z}$.*

0.1.3 Strichartz estimates for wave and Schrödinger equations

In this section we present the key dispersive estimates for the Schrödinger and wave equations. By *dispersive*, we mean that the equations admit wave-like solutions travelling at different velocities depending on the *wavenumber*.[26] For example, considering (0.1.1)-(0.1.3) we find the plane wave solutions

$$\begin{aligned} &\text{Schrödinger:} && e^{i(x\cdot k - t|k|^2)} = e^{i|k|(x\cdot\hat{k} - t|k|)} \\ &\text{Wave:} && e^{i(x\cdot k \pm t|k|)} = e^{i|k|(x\cdot\hat{k} \pm t)} \end{aligned}$$

for fixed $k \in \mathbb{R}^d$. We see that in the Schrödinger case different waves even travel at different *speeds*, $|k|$, however in the wave case they only travel in different directions, making this is a somewhat degenerate case of dispersion.

The dispersion leads to pointwise decay of solutions with sufficiently decaying initial data (so that plane wave solutions are precluded). The estimates below can be proved using the theory of oscillatory integrals:

$$\text{Schrödinger:} \qquad \|e^{it\Delta} f\|_{L^\infty_x} \lesssim_d |t|^{-d/2} \|f\|_{L^1_x} \tag{0.1.6}$$

$$\text{Wave:} \qquad \|e^{\pm it|\nabla|}(P_1 f)\|_{L^\infty_x} \lesssim_d |t|^{-(d-1)/2} \|P_1 f\|_{L^1_x} \tag{0.1.7}$$

Note the stronger decay for the Schrödinger equation, and the necessary inclusion of the Littlewood-Paley projection for the wave estimate. We have stronger decay in higher dimensions where there are “more directions for the waves to disperse”. Interpolating the

[26] The *wavenumber* k indicates the direction, $\hat{k}$, and wavelength $2\pi/|k|$ of a wave.

decay estimates above with the Plancherel identities

$$\|e^{it\Delta}P_1 f\|_{L^2_x} = \|P_1 f\|_{L^2_x} \qquad \|e^{\pm it|\nabla|}f\|_{L^2_x} = \|f\|_{L^2_x}$$

one can deduce decay estimates in all L^p_x spaces, $2 \leq p \leq \infty$.

We now turn to the time-averaged *Strichartz* estimates. These are obtained from the estimates above using functional analytic techniques involving a TT^* argument and the Hardy-Littlewood-Sobolev inequality. We obtain estimates for solutions to inhomogeneous equations via Duhamel's formula, which in the Schrödinger case reads as

$$u = \int_{t_0}^{t} e^{i(t-s)\Delta}F(s)ds$$

for the solution to the inhomogeneous equation $(i\partial_t + \Delta)u = F$, $u(0) = 0$.

Theorem 0.1.2 (Strichartz estimates for the Schrödinger equation)**.** *Let $d \geq 1$. We call $2 \leq q, r \leq \infty$ a Schrödinger-admissible pair if*

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \qquad (q, r, d) \neq (2, \infty, 2). \tag{0.1.8}$$

Let (q, r) and $(\tilde{q}, \tilde{r})$ be Schrödinger-admissible pairs and denote by $\tilde{q}'$ and $\tilde{r}'$ the conjugate exponents of $\tilde{q}$ and $\tilde{r}$. Let $I \subset \mathbb{R}$ be a time interval containing t_0. It holds

$$\|e^{i(t-t_0)\Delta}f\|_{L^q_t L^r_x(I\times\mathbb{R}^d)} \lesssim_{d,r,q} \|f\|_{L^2(\mathbb{R}^d)} \tag{0.1.9}$$

$$\left\|\int_I e^{-is\Delta}F(s)ds\right\|_{L^2(\mathbb{R}^d)} \lesssim_{d,\tilde{r},\tilde{q}} \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x(I\times\mathbb{R}^d)} \tag{0.1.10}$$

$$\left\|\int_{t_0}^{t} e^{i(t-s)\Delta}F(s)ds\right\|_{L^q_t L^r_x(I\times\mathbb{R}^d)} \lesssim_{d,r,q,\tilde{r},\tilde{q}} \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x(I\times\mathbb{R}^d)} \tag{0.1.11}$$

For the wave equation, we only have estimates in dimensions $d \geq 2$.

Theorem 0.1.3 (Strichartz estimates for the wave equation)**.** *Let $d \geq 2$. We call $2 \leq q, r \leq \infty$ a wave-admissible pair if*

$$\frac{2}{q} + \frac{n-1}{r} < \frac{n-1}{2}, \qquad (q, r, d) \neq (2, \infty, 3).$$

Set

$$s(d, q, r) = \frac{d}{2} - \frac{1}{q} - \frac{d}{r}$$

Let (q, r) and $(\tilde{q}, \tilde{r})$ be wave-admissible pairs satisfying the scaling condition

$$s(p, q, d) + s(\tilde{q}, \tilde{p}, d) = 1$$

and denote $\tilde{q}'$, $\tilde{r}'$ the conjugate exponents of $\tilde{q}$, $\tilde{r}$. Let $I \subset \mathbb{R}$ be a time interval containing

t_0. *For* $N \in 2^{\mathbb{Z}}$ *it holds*

$$\|e^{\pm i(t-t_0)|\nabla|} P_N f\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} \lesssim_{d,r,q} \|P_N f\|_{\dot{H}^s(\mathbb{R}^d)} \tag{0.1.12}$$

If moreover q, $\tilde{q} \neq \infty$, *we may remove the projection* P_N *in the estimate above, and there is the inhomogeneous estimate*

$$\left\| \int_{t_0}^{t} \frac{\sin((t-s)|\nabla|)}{|\nabla|} F(s)ds \right\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} \lesssim_{d,r,q,\tilde{r},\tilde{q}} \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I\times\mathbb{R}^d)} \tag{0.1.13}$$

In the two main chapters of this thesis we will consider different settings in which the above estimates can be improved. In the first chapter this will be by randomisation, and in the second chapter for functions with angular regularity.

0.1.4 Spherical harmonics

In this section we describe some basic properties of the spherical harmonics which will be of use to us in the coming chapters. We refer to Chapter IV of [SW16] for further details.

For each $k \geq 0$ we denote by E_k the space of (surface) spherical harmonics of degree k, that is to say the kth eigenspace of the spherical Laplacian Δ_{sph} with corresponding eigenvalue $-k(k+d-2)$. Δ_{sph} is the angular component of the usual Laplacian, so that

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{d-1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\Delta_{\text{sph}} f$$

in spherical coordinates. For example, in dimension three we have

$$\Delta_{\text{sph}} f = \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right)$$

for co-ordinates $(x_1, x_2, x_3) = r(\sin\theta\,\cos\varphi, \sin\theta\,\sin\varphi, \sin\theta)$. We also have the expression

$$\Delta_{\text{sph}} = \sum_{\substack{i,j=1,\dots,d \\ i<j}} \Omega_{ij}^2$$

where Ω_{ij} are the angular derivatives

$$\Omega_{ij} := x_i\partial_j - x_j\partial_i.$$

One may also characterise E_k as the restrictions to the unit sphere of homogeneous harmonic polynomials of degree k.

Each E_k is a finite dimensional vector space of dimension

$$N_k = \binom{d+k-1}{k} - \binom{d+k-3}{k-2}.$$

The spaces $(E_k)_{k\geq 0}$ are mutually orthogonal and span $L^2(\mathbb{S}^{d-1})$ so we may construct an orthonormal Hilbert basis

$$B = (b_{k,l})_{k\in\mathbb{N}, l=1,\dots,N_k}$$

for $L^2(\mathbb{S}^{d-1})$, with each $b_{k,l} \in E_k$. In particular, for any $f \in L^2(\mathbb{R}^d)$ and almost every $r > 0$ we may uniquely express

$$f(r\theta) = \sum_{k\geq 0}\sum_{l=1}^{N_k} c_{k,l}(r)b_{k,l}(\theta), \qquad c_{k,l}(r) := \int_{\mathbb{S}^{d-1}} f(r,\theta)\bar{b}_{k,l}(\theta)d\theta.$$

It follows that $c_{k,l} \in L^2(r^{d-1}dr)$ and

$$\|f\|^2_{L^2(\mathbb{R}^d)} = \sum_{k\geq 0}\sum_{l=1}^{N_k} \|c_{k,l}\|^2_{L^2(r^{d-1}dr)}$$

An interesting and important property of the spherical harmonics is that they are invariant under the action of the Fourier transform, that is to say that for each $k \geq 0$ there is a map $T_k : L^2(r^{d-1}dr) \mapsto L^2(r^{d-1}dr)$ such that

$$\mathcal{F}(c(r)b_{k,l}(\theta))(\rho,\omega) = T_k(c)(\rho)b_{k,l}(\omega).$$

Here we use coordinates $x = r\theta$ in physical space and $\xi = \rho\omega$ in Fourier space, with θ, $\omega \in \mathbb{S}^{d-1}$. Precisely,

$$(T_k c)(r) = (2\pi)^{\frac{d}{2}} i^{-k} r^{-\frac{d-2}{2}} \int_0^\infty c(s)J_{\nu(k)}(rs)s^{\frac{d}{2}}ds \tag{0.1.14}$$

with $\nu(k) := \frac{d+2k-2}{2}$ and

$$J_\mu(r) := \frac{(r/2)^\mu}{\Gamma\left(\frac{2\mu+1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^1 e^{irs}(1-s^2)^{\frac{2\mu-1}{2}}ds \qquad (\mu > -1/2)$$

the Bessel function of the first kind.

Since the operators $e^{-it\Delta}$ and $e^{\pm it\sqrt{-\Delta}}$ are given by radial multipliers we find in particular that the spaces E_k are preserved by the linear evolutions of the Schrödinger and wave equations, i.e.

$$\begin{aligned} e^{it\Delta}(c(r)b_{k,l}(\theta)) &= T_k^{-1}(e^{-it|\cdot|^2}T_k(c))(r)b_{k,l}(\theta) \\ e^{\pm it|\nabla|}(c(r)b_{k,l}(\theta)) &= T_k^{-1}(e^{\pm it|\cdot|}T_k(c))(r)b_{k,l}(\theta) \end{aligned} \tag{0.1.15}$$

Warning: *Again for consistency with other works we use different notation for the spherical harmonics in Chapter 2. Precisely we write* Y_l^i *(*$l \geq 0$*,* $i = 1, \ldots, N_l$*) in place of* $b_{l,i}$.

1 Almost Sure Scattering of the Energy-Critical NLS in $d > 6$.

The content of this chapter is taken from [Mar23], published in *Communications on Pure and Applied Analysis*, with minor modifications for consistency with the rest of the thesis.

1.1 Introduction

We consider the defocusing energy-critical nonlinear Schrödinger equation (NLS) in dimension $d > 6$

$$\begin{cases} (i\partial_t + \Delta)u = u|u|^{\frac{4}{d-2}} \\ u(0) = f \in H^s(\mathbb{R}^d) \end{cases} \qquad (0 < s < 1) \tag{1.1.1}$$

Here "defocusing" refers to the plus sign in front of the nonlinearity and "energy-critical" refers to the fact that the conserved energy

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(t)|^2 dx + \frac{d-2}{2d}\int_{\mathbb{R}^d} |u(t)|^{\frac{2d}{d-2}} dx \tag{1.1.2}$$

is invariant with respect to the scaling symmetry $u(t,x) \mapsto \lambda^{\frac{d-2}{2}} u(\lambda^2 t, \lambda x)$. Since the energy scales like the $\dot{H}^1$ norm of u, we say the equation has scaling regularity 1.

As discussed in the main introduction, it was shown in [Vis07] that equation (1.1.1) is globally well-posed with scattering for initial data in the energy space $\dot{H}^1$, however for $s < 1$ this is not in general true [CCT03]. The goal of this chapter is to investigate the global wellposedness of (1.1.1) below the critical regularity $s = 1$ by randomising the initial data, generalising known results in dimension four [KMV19, DLM20, Spi21] to high dimensions $d > 6$. We show that for all $s \in (s_d, 1)$, where s_d is a constant depending only on the dimension, the equation is almost surely globally well-posed with respect to a particular randomisation in $H^s(\mathbb{R}^d)$. We moreover establish almost sure scattering in $\dot{H}^s(\mathbb{R}^d)$ both forwards and backwards in time. The randomisation is based on a decomposition of the initial data in physical space, Fourier space and the angular variable as in [Spi21].

The main difficulty we encounter in moving to high dimensions is the non-smoothness of the nonlinearity $u|u|^{\frac{4}{d-2}}$. To deal with this, we use an adapted version of the work of Tao-Visan [TV05] in Section 1.5 to study the stability of the energy-critical NLS which is needed to prove a conditional scattering result, since in high dimensions the nonlinearity is not twice differentiable and standard stability techniques are insufficient. We also prove local wellposedness via a regularisation argument (Section 1.4), allowing us to work with higher regularity solutions when proving the scattering condition is satisfied. This is necessary due to the lack of persistence of regularity for the high dimensional equation (1.1.1). The regularisation we use effectively renders the nonlinearity energy-subcritical, allowing us to use persistence of H^2 regularity as in [Caz03]. This is sufficient to perform computations involving the energy in Section 1.6.

The many-fold randomisation procedure we consider in this work was introduced by Spitz in [Spi21], however each sub-randomisation had previously been used with success. In particular, the randomisation with respect to a unit-scale frequency decomposition, also known as the *Wiener randomisation* (0.0.4), has been extensively applied to nonlinear Schrödinger and wave equations, among others, since its simultaneous introduction by Lührmann-Mendelson [LM14] and Bényi-Oh-Pocovnicu [BOP15a, BOP15b], see also [ZF12]. Randomisation in the angular variable was introduced by Burq-Krieger in [BK21] in the context of a wave maps type equation, and randomisation in physical space has had applications to the final state problem of the NLS and other dispersive equations, see for example [NY18, Mur19]. The randomisation we use also involves a dyadic frequency decomposition, however unlike its unit-scale counterpart, randomisation with respect to this decomposition alone has not proved useful since it does not entail any improved integrability.

1.1.1 Main Result

We now state our main result. We will define the randomisation of the initial data fully in the next section, however loosely speaking, for any function $f \in H^s(\mathbb{R}^d)$, its randomisation over a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is an H^s-valued random variable

$$\Omega \ni \omega \mapsto f^\omega \in H^s(\mathbb{R}^d).$$

Theorem 1.1.1. *Let $d > 6$, $s_d := \max\{\frac{4d-1}{3(2d-1)}, \frac{d^2+6d-4}{(2d-1)(d+2)}\} < s < 1$. Let $f \in H^s(\mathbb{R}^d)$ and f^ω denote the randomisation of f (defined in Section 1.1.2). Then there exists $\Sigma \subset \Omega$ with $\mathbb{P}(\Sigma) = 1$ such that for every $\omega \in \Sigma$ there exists a unique global solution*

$$u(t) \in e^{it\Delta} f^\omega + C(\mathbb{R}; H^1(\mathbb{R}^d))$$

to the defocusing energy-critical nonlinear Schrödinger equation with initial data f^ω

$$\begin{cases} (i\partial_t + \Delta)u = u|u|^{\frac{4}{d-2}} \\ u(0) = f^\omega \end{cases} \tag{1.1.3}$$

Moreover, this solution scatters both forwards and backwards in time, i.e. there exist u_+, $u_- \in \dot{H}^s(\mathbb{R}^d)$ such that

$$\lim_{t \to \pm\infty} \|u(t) - e^{it\Delta} u_\pm\|_{\dot{H}^1(\mathbb{R}^d)} = 0$$

Observe that $s_d = \frac{d^2+6d-4}{(2d-1)(d+2)}$ if and only if $d \leq 10$.

Remark 1.1.2. By a solution to equation (1.1.1), we mean a solution to the Duhamel formulation of the equation

$$u(t) = e^{it\Delta} f - i \int_0^t e^{i(t-s)\Delta} (u|u|^{\frac{4}{d-2}})(s) ds$$

in an appropriate function space.

Remark 1.1.3. In Theorem 1.1.1 uniqueness holds in the sense that upon writing the solution u in the form

$$u(t) = e^{it\Delta} f^\omega + v(t) \tag{1.1.4}$$

with $v \in C(\mathbb{R}; H^1(\mathbb{R}^d)) \bigcap W(I)$, where the space $W(I)$ will be defined shortly (see Section 1.3), the function v is unique.

Remark 1.1.4. By writing a solution u of (1.1.3) in the form (1.1.4) we find that v must satisfy the forced equation

$$\begin{cases} (i\partial_t + \Delta)v = (F + v)|F + v|^{\frac{4}{d-2}} \\ v(0) = v_0 \end{cases} \tag{1.1.5}$$

with $F = e^{it\Delta} f^\omega$ and $v_0 = 0$. Thus, it is sufficient to study the wellposedness of (1.1.5) in $H^1(\mathbb{R}^d)$ under some appropriate conditions on F.

Before going into further details we briefly outline the structure of this chapter. In Sections 1.1.2 and 1.1.3 we will introduce the randomisation procedure for f^ω and the regularisation that we will use for the nonlinearity.

After discussing some preliminaries in Section 1.2 we will establish (deterministic) local wellposedness of (1.1.5) in the critical space $\dot{H}^1$ in Section 1.4, under certain conditions on $F = e^{it\Delta} f^\omega$, via a regularisation argument in the space $\dot{W}$ with norm

$$\|v\|_{\dot{W}(I)} := \|\nabla v\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}} (I \times \mathbb{R}^d)}$$

This is the norm used by Tao and Visan to study the energy-critical NLS in [TV05]. The argument will also require the forcing, F, to lie in $\dot{W}(I)$. Setting $F = e^{it\Delta} f^\omega$ this represents a gain in derivatives which we obtain via a randomisation-improved radially averaged Strichartz estimate as in [Spi21] (see Section 1.7).

We remark that $\dot{W}$ is not the optimal space to work in to establish local wellposedness of (1.1.5). Indeed, the requirement that $e^{it\Delta} f^\omega$ also lies in $\dot{W}$ represents a gain of

$$\frac{(d-1)(d-2)}{(2d-1)(d+2)}$$

derivatives on $e^{it\Delta} f^\omega$. However, when used at its endpoint the randomisation-improved radially averaged Strichartz estimate allows us to gain up to $\frac{d-1}{2d-1}$ derivatives and our method can be extended to obtain almost sure local wellposedness for

$$1 - \frac{d-1}{2d-1} = \frac{d}{2d-1} < s < 1$$

We are not able to acheive twice this gain as in [Spi21] due to the non-algebraic nature of the nonlinearity which prevents a more precise analysis of the equation on dyadic scales.

In Section 1.5 we prove a conditional scattering result. The local wellposedness theory of Section 1.4 is accompanied by a scattering criterion: if the solution to (1.1.5) satisfies

$$\|v\|_{\dot{W}(I^*)} < \infty$$

on its maximal interval of existence I^* then the solution is global and scatters as $t \to \pm\infty$. In this section we show that this condition is satisfied provided the energy of v is uniformly bounded on I^*. To this end we develop a perturbation theory in the space $\dot{W}$ to compare solutions of (1.1.5) with those of the "usual" NLS (1.1.3), since by [Vis07] we already have a bound on those solutions in $\dot{W}$ in terms of their energy. Since for $d > 6$ the nonlinearity is not twice differentiable, we cannot develop the perturbation theory in the standard way and instead adapt the work in [TV05] on the stability of high dimensional energy-critical Schrödinger equations.

In this section we again work in the space $\dot{W}$ and again this is not optimal. Improving the result for this section would require further notation and not improve the final restriction on s in Theorem 1.1.1, so we do not present the optimal case.

In Section 1.6 we prove the uniform-in-time energy bound mentioned above, placing the forcing term in spaces with low time integrability as in [Spi21]. We argue via the regularised solutions, since the true solution does not have sufficient regularity to perform the necessary computations (in particular, an explicit differentiation of the energy).

Finally in Section 1.7 we show that $F^\omega := e^{it\Delta} f^\omega$ indeed satisfies all the conditions needed to run the arguments above (almost surely). This follows the same arguments as in [Spi21]. In particular, the randomisation with respect to the angular variable allows us to (almost surely) gain derivatives on the free evolution via improved radial Strichartz estimates (see Proposition 1.7.3), and we no longer need to exploit local smoothing effects as in [DLM19]. The unit-scale randomisation in the physical variable allows us to prove estimates on the gradient of F in spaces of type $L_t^1 L_x^p$ by appealing to the temporal decay of the Schrödinger semi-group (Proposition 1.7.10). This allows us to bound the energy increment in Section 1.6 without appealing to Morawetz estimates. These additions are what enables us to avoid a radial assumption as in [DLM19, KMV19].

1.1.2 Randomisation Procedure

We now describe how to construct the random variable f^ω appearing in the main theorem.

Decomposition in Fourier space, physical space, and the angular variable.

In what follows, let $f \in L^2(\mathbb{R}^d)$.

We first introduce the physical space decomposition. Let $\varphi : \mathbb{R}^d \to [0, 1]$ be a smooth, radially symmetric function with $\varphi(x) = 1$ for $|x| \leq \sqrt{d}$ and $\varphi(x) = 0$ for $|x| \geq 2\sqrt{d}$.

$$\varphi_i(x) := \frac{\varphi(x-i)}{\sum_{k\in\mathbb{Z}^d}\varphi(x-k)} \tag{1.1.6}$$

so that φ_i has support in $\{x : |x-i| \leq 2\sqrt{d}\}$. We then have the unit scale decomposition of f in physical space,

$$f = \sum_{i\in\mathbb{Z}^d}\varphi_i(x)f(x)$$

Note that this representation holds in both the L^2 and the pointwise sense.

We next apply an angular decomposition to each component $\varphi_i f$ using the spherical harmonics defined in Section 0.1.4. First decompose $\varphi_i f$ on dyadic scales in Fourier space, using the Littlewood-Paley multipliers defined in (0.1.5):

$$\varphi_i f = \sum_{M\in 2^{\mathbb{Z}}} P_M(\varphi_i f)$$

It is convenient to rescale $P_M(\varphi_i f)$ to unit frequency by setting

$$g_i^M = (P_M(\varphi_i f))(M^{-1}\cdot)$$

Now recall that there is an orthonormal Hilbert basis

$$B = (b_{k,l})_{k\in\mathbb{N}, l=1,\dots,N_k}$$

of $L^2(\mathbb{S}^{d-1})$ consisting of spherical harmonics. By Theorem 6 of [BL13] (see also Theorem 1 of [BL14] and Theorem 1.1 of [BK21]), there exists a choice of such basis, which we call a *good basis*, such that for any $q \in [2,\infty)$, it holds

$$\|b_{k,l}\|_{L^q(\mathbb{S}^{d-1})} \leq C_{q,d} \tag{1.1.7}$$

for some constant $C_{q,d}$ depending only on the indicated parameters and independent of k, l. Fix a good basis as described and decompose

$$\hat{g}_i^M(\rho\theta) = \sum_{k=0}^{\infty}\sum_{l=1}^{N_k} \hat{c}_{k,l}^{M,i}(\rho) b_{k,l}(\theta)$$

with each $\hat{c}_{k,l}^{M,i}$ supported in $[\frac{1}{2}, 2]$ (by orthogonality). Then using that the spherical harmonics are invariant under the Fourier transform, and in particular the formula (0.1.14), we have

$$g_i^M(r\theta) = (2\pi)^{-d}\mathcal{F}(\widehat{g}_i^M)(-r\theta) = \sum_{k=0}^{\infty}\sum_{l=1}^{N_k} a_k r^{-\frac{d-2}{2}}\left(\int_0^\infty \hat{c}_{k,l}^{M,i}(s)J_{\nu(k)}(rs)s^{\frac{d}{2}}ds\right)b_{k,l}(\theta)$$

for $a_k = (2\pi)^{-\frac{d}{2}} i^k$, using that $b_{k,l}(-\theta) = (-1)^k b_{k,l}(\theta)$. It is useful to observe at this point that

$$\|g_i^M\|^2_{L^2(\mathbb{R}^d)} = \sum_{k=0}^{\infty} \sum_{l=1}^{N_k} \|\hat{c}_{k,l}^{M,i}\|^2_{L^2(\rho^{d-1} d\rho)} \tag{1.1.8}$$

Scaling this back to frequency M we obtain

$$P_M(\varphi_i f)(r\theta) = \sum_{k=0}^{\infty} \sum_{l=1}^{N_k} a_k (Mr)^{-\frac{d-2}{2}} \left(\int_0^\infty \hat{c}_{k,l}^{M,i}(s) J_{\nu(k)}(Mrs) s^{\frac{d}{2}} ds \right) b_{k,l}(\theta) \tag{1.1.9}$$

The final step is to include a unit-scale frequency decomposition. To this end we introduce the operators

$$P_j f := \mathcal{F}^{-1}(\psi_j(\xi) \hat{f}(\xi)) \tag{1.1.10}$$

where $\psi_j(\xi) := \varphi_j(\xi)$ is as in the physical space decomposition. We make this change of notation in order to clarify the distinction between the decompositions in physical and frequency space. Incorporating these projections into (1.1.9) we obtain

$$P_M(\varphi_i f)(r\theta) = \sum_{j \in \mathbb{Z}^d} \sum_{k=0}^{\infty} \sum_{l=1}^{N_k} a_k M^{-\frac{d-2}{2}} P_j \left[r^{-\frac{d-2}{2}} \left(\int_0^\infty \hat{c}_{k,l}^{M,i}(s) J_{\nu(k)}(Mrs) s^{\frac{d}{2}} ds \right) b_{k,l}(\theta) \right]$$

from which

$$f(r\theta) = \sum_{M \in 2^{\mathbb{Z}}} \sum_{i,j \in \mathbb{Z}^d} \sum_{k=0}^{\infty} \sum_{l=1}^{N_k} a_k M^{-\frac{d-2}{2}} P_j \left[r^{-\frac{d-2}{2}} \left(\int_0^\infty \hat{c}_{k,l}^{M,i}(s) J_{\nu(k)}(Mrs) s^{\frac{d}{2}} ds \right) b_{k,l}(\theta) \right]$$

with convergence in $L^2(\mathbb{R}^d)$.

Randomisation with respect to the decomposition.

We now introduce a family

$$(X_{i,j,k,l}^M : M \in 2^{\mathbb{Z}}, i, j \in \mathbb{Z}^d, k \in \mathbb{N}_0, l \in \{1, \dots, N_k\})$$

of independent, mean-zero, real-valued random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with respective distributions $(\mu_{i,j,k,l}^M : M \in 2^{\mathbb{Z}}, i, j \in \mathbb{Z}^d, k \in \mathbb{N}_0, l \in \{1, \dots, N_k\})$ for which there exists a $c > 0$ such that

$$\int_{\mathbb{R}} e^{\gamma x} d\mu_{i,j,k,l}^M(x) \leq e^{c\gamma^2}$$

for all $\gamma \in \mathbb{R}$, $M \in 2^{\mathbb{Z}}$, $i, j \in \mathbb{Z}^d$, $k \in \mathbb{N}_0$, $l \in \{1, \dots, N_k\}$. This is satisfied by independent identically distributed Gaussians for example. We can then define the randomisation

$$\begin{aligned} f^\omega &= \sum_{M\in 2^{\mathbb{Z}}} \sum_{i,j\in\mathbb{Z}^d} \sum_{k=0}^{\infty} \sum_{l=1}^{N_k} X^M_{i,j,k,l}(\omega) P_j f^{M,i}_{k,l} \\ &\equiv \sum_{M,i,j,k,l} X^M_{i,j,k,l}(\omega) a_k M^{-\frac{d-2}{2}} P_j \left[r^{-\frac{d-2}{2}} \left(\int_0^\infty \hat{c}^{M,i}_{k,l}(s) J_{\nu(k)}(Mrs) s^{\frac{d}{2}} ds \right) b_{k,l}(\theta) \right] \end{aligned} \tag{1.1.11}$$

which is well-defined in $L^2(\Omega, L^2(\mathbb{R}^d))$.

Remark 1.1.5. In fact, for $f \in H^s(\mathbb{R}^d)$, the randomisation f^ω also lies in $H^s(\mathbb{R}^d)$ almost surely. In particular, it holds

$$\| \|f^\omega\|_{H^s(\mathbb{R}^d)} \|_{L^2(\Omega)} \lesssim_d \|f\|_{H^s(\mathbb{R}^d)}$$

This can be seen using the fundamental large deviation estimate of Burq and Tvetkov (see Section 1.7), combined with the orthogonality of the decompositions in frequency space and into spherical harmonics, and Corollary 3.3 of [Spi21] to handle the intertwining of the physical space decomposition and the H^s norm. In what follows, we implicitly restrict to a subset $\Sigma \subset \Omega$ of probability one such that $f^\omega \in H^s(\mathbb{R}^d)$ for every $\omega \in \Sigma$.

Remark 1.1.6. It is important to note that the above randomisation does not in general improve the regularity of the data. In particular, choose the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to be the product of spaces $(\Omega_i, \mathcal{A}_i, \mathbb{P}_i)_{i=1,2,3}$ and the random variables to be given by

$$X^M_{i,j,k,l}(\omega) = X_j(\omega_1) X^M_{k,l}(\omega_2) X_i(\omega_3), \qquad \omega = (\omega_1, \omega_2, \omega_3) \in \Omega_1 \times \Omega_2 \times \Omega_3$$

with the X_j, $X^M_{k,l}$, X_i independent identically distributed Bernoulli random variables on Ω_1, Ω_2, Ω_3 respectively taking values ± 1 with equal probability $\frac{1}{2}$. Then one can show that, for $0 < s < 1$, $f \notin H^s(\mathbb{R}^d)$ implies that $f^\omega \notin H^s(\mathbb{R}^d)$ for almost every $\omega \in \Omega$. See Appendix 1.B for further details.

1.1.3 Regularisation of the Nonlinearity

We shall study solutions to (1.1.1) via a regularisation of the nonlinearity $g(u) := u|u|^{\frac{4}{d-2}}$, allowing us to work with H^2 solutions when performing calculations involving the energy later on. This step is not necessary in the lower 4 dimensional settings of [DLM19] and [Spi21] when the nonlinearity in (1.1.1) is algebraic and persistence of regularity allows us to directly construct a solution in H^1 as a limit of solutions in H^2.

Denote $p = \frac{d+2}{d-2}$, so $p - 1 = \frac{4}{d-2}$. For each $n \in \mathbb{N}$ define

$$g_n(u) := u\varphi_n'(|u|^2)$$

for $\varphi_n(x) = n^{p+1}\varphi_1(x/n^2)$. Here $\varphi_1 \in C^\infty((0,\infty)) \cap C([0,\infty))$ with $\varphi_1(0) = 0$ and

$$\varphi_1'(x) = \begin{cases} x^{\frac{p-1}{2}} \text{ for } 0 < x \leq 1 \\ 2^{p-1} \text{ for } x \geq 4 \end{cases} \tag{1.1.12}$$

in such a way that $\varphi_1'(x) \leq x^{\frac{p-1}{2}}$ for all $x \geq 0$. Thus

$$\varphi_n'(x) = \begin{cases} x^{\frac{p-1}{2}} \text{ for } 0 < x \leq n^2 \\ (2n)^{p-1} \text{ for } x \geq (2n)^2 \end{cases} \tag{1.1.13}$$

and $g_n(u) = g(u)$ whenever $|u| \leq n$. Since φ_1'' is compactly supported, we also see that $|\varphi_n''(x)| \lesssim |x|^{\frac{p-3}{2}}$.

Consider the regularised NLS

$$\begin{cases} (i\partial_t + \Delta)u_n = g_n(u_n) \\ u_n(0) = u_{n,0} \in H^2(\mathbb{R}^d) \end{cases} \tag{1.1.14}$$

By Theorem 4.8.1 of [Caz03] we see that (1.1.14) admits a local solution in $C(I, H^2(\mathbb{R}^d)) \cap C^1(I, L^2(\mathbb{R}^d))$ on some neighbourhood I of 0. Here

$$C^1(I, L^2(\mathbb{R}^d)) := \{f \in C(I, L^2(\mathbb{R}^d)) : \partial_t f \in C(I, L^2(\mathbb{R}^d))\} \tag{1.1.15}$$

where $\partial_t f$ is defined as the vector-valued distribution such that

$$\int_I \partial_t \psi(t) f(t,\cdot)dt = -\int_I \psi(t)\partial_t f(t,\cdot)dt$$

for all $\psi \in \mathcal{D}(I)$, with the above integrals evaluated in the Bochner sense.

Theorem 5.3.1 of [Caz03] then shows that this solution exists in H^2 for as long as it exists in H^1, which is for all time since solutions of (1.1.14) have conserved energy

$$E_n(u_n) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^d} \varphi_n(|u_n|^2)dx$$

We thus see that (1.1.14) admits global solutions in $C(\mathbb{R}, H^2(\mathbb{R}^d)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^d))$.

As discussed in Remark 1.1.4, in this work we will actually study the wellposedness of the forced equation (1.1.5) in $H^1(\mathbb{R}^d)$, with the forcing term given by the free evolution of the randomised data: $F = e^{it\Delta} f^\omega$. Thus to obtain H^2 solutions to (1.1.5), we must also regularise the forcing.

Set $F_n = P_{\leq n} F = e^{it\Delta} P_{\leq n} f^\omega$, where

$$P_{\leq n} f^\omega := \sum_{\substack{M \in 2^{\mathbb{Z}} \\ M \leq 2^n}} P_M f^\omega$$

Then by Lemma 4.8.2 of [Caz03], $F_n \in C(\mathbb{R}, H^2(\mathbb{R}^d)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^d))$. Observe that for any $1 \leq a, b \leq \infty$ it holds

$$\|F_n\|_{L_t^a L_x^b(\mathbb{R})} \lesssim_{a,b,d} \|F\|_{L_t^a L_x^b(\mathbb{R})}$$

Fix $v_0 \in H^1(\mathbb{R}^d)$. Setting $u_n := v_n + F_n$ and $u_{n,0} = P_{\leq n}(v_0 + f^\omega)$ in (1.1.14), we thus obtain unique global solutions to the forced NLS

$$\begin{cases} (i\partial_t + \Delta) v_n = g_n(F_n + v_n) \\ v_n(0) = v_{n,0} \in H^2(\mathbb{R}^d) \end{cases}$$

in $C(\mathbb{R}, H^2(\mathbb{R}^d)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^d))$. Here $v_{n,0} := P_{\leq n} v_0 \to v_0$ in $H^1(\mathbb{R}^d)$.

We will show in Section 1.4 that the solutions v_n converge locally to solutions of the non-regularised equation (1.1.5).

1.2 Notation and Preliminaries

1.2.1 Notation

In addition to the notation introduced in Section 0.1.1, we will also need the homogeneous ℓ^2 Besov spaces with

$$\|f\|_{\dot{B}^r_{q,2}(I)} := \left(\sum_{N \in 2^{\mathbb{Z}}} N^{2r} \|P_N f\|^2_{L_x^q(I \times \mathbb{R}^d)} \right)^{\frac{1}{2}}$$

as well as the mixed spacetime Besov spaces with

$$\|f\|_{\dot{B}^r_{p,q,2}(I)} := \left(\sum_{N \in 2^{\mathbb{Z}}} N^{2r} \|P_N f\|^2_{L_t^p L_x^q(I \times \mathbb{R}^d)} \right)^{\frac{1}{2}}$$

Since we shall always be considering the ℓ^2 Besov-type spaces we will sometimes omit the subscript “2”, writing only $\dot{B}^r_{p,q}(I)$.

Throughout this chapter it is always assumed that $d > 6$, and we will often use the notation $p = \frac{d+2}{d-2} \in (1, 2)$ without comment.

1.2.2 Properties of the Nonlinearity

Denote $g(u) := u|u|^{\frac{4}{d-2}}$. We record here some properties of g for future reference. As well as the trivial bound $|g(u)| \leq |u|^{\frac{d+2}{d-2}}$, we have the gradient bounds $|g_z(u)| \lesssim_d |u|^{\frac{4}{d-2}}$, $|g_{\bar{z}}(u)| \lesssim_d |u|^{\frac{4}{d-2}}$. Here $g_z, g_{\bar{z}}$ denote the complex derivatives:

$$g_z(x+iy) \equiv \partial_z g(x+iy) = \frac{1}{2}(\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}), \qquad g_{\bar{z}}(x+iy) = \frac{1}{2}(\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y})$$

for $z = x+iy$, $x, y \in \mathbb{R}$. We also have the difference bound

$$|g(u_1) - g(u_2)| \lesssim_d (|u_1|^{\frac{4}{d-2}} + |u_2|^{\frac{4}{d-2}})|u_1 - u_2| \tag{1.2.1}$$

which follows from the identity

$$g(u_1 + u_2) - g(u_1) = \int_0^1 [g_z(u_1 + \theta u_2)u_2 + g_{\bar{z}}(u_1 + \theta u_2)\bar{u}_2]d\theta \tag{1.2.2}$$

On the other hand, the chain rule

$$\nabla g(u(x)) = g_z(u(x))\nabla u(x) + g_{\bar{z}}(u(x))\nabla \bar{u}(x) \tag{1.2.3}$$

with the bound[1]

$$|g_z(u_1) - g_z(u_2)| \lesssim_d |u_1 - u_2|^{\frac{4}{d-2}} \tag{1.2.4}$$

(and the analogous statement for $g_{\bar{z}}$), implies that

$$|\nabla g(u_1) - \nabla g(u_2)| \lesssim_d |u_1 - u_2|^{\frac{4}{d-2}}|\nabla u_1| + |u_2|^{\frac{4}{d-2}}|\nabla u_1 - \nabla u_2| \tag{1.2.5}$$

Moreover, the above bounds all also hold for g_n with bounds independent of n.

1.2.3 Deterministic Estimates

We first recall some basic estimates related to the Littlewood-Paley inequality (0.1.1), which allows us to easily transfer between the Besov and standard Lebesgue spaces. Combined with the triangle inequality it yields

$$\|f\|_{L_t^p L_x^q} \lesssim_{q,d} \left(\sum_{N \in 2^{\mathbb{Z}}} \|P_N f\|^2_{L_t^p L_x^q} \right)^{\frac{1}{2}} \tag{1.2.6}$$

[1] Note this bound holds since $d > 6$. For $d \leq 6$ we have Lipschitz continuity of g_z, $g_{\bar{z}}$, making some aspects of the problem simpler to study.

for any $2 \le p \le \infty$, $2 \le q < \infty$. We also have the dual estimate

$$\left(\sum_{N \in 2^{\mathbb{Z}}} \|P_N f\|_{L_t^p L_x^q}^2 \right)^{\frac{1}{2}} \lesssim_{q,d} \|f\|_{L_t^p L_x^q} \tag{1.2.7}$$

for $1 \le p \le 2$, $1 < q \le 2$.

1.3 Function Spaces

We now define the function spaces in which we shall place the solution and the forcing in order to obtain local wellposedness.

Let I be an open time interval. We will place the solution v to the forced NLS into the space defined by the norm

$$\|v\|_{W(I)} := \|v\|_{V(I)} + \|\nabla v\|_{V(I)}$$

where

$$\|v\|_{V(I)} := \|v\|_{\frac{2(d+2)}{d-2}, \frac{2d(d+2)}{d^2+4}[I]}$$

We will also denote $\|v\|_{\dot{W}(I)} := \|\nabla v\|_{V(I)}$.

To prove local wellposedness it will be sufficient to place the forcing term F into the same space W. However to obtain the conditional scattering result in Section 1.5 we will need F to lie in the stronger space[2]

$$R(I) := W(I) \cap \dot{B}^{\frac{4}{d+2}}_{d+2, \frac{2(d+2)}{d}}(I) \tag{1.3.1}$$

which is necessary in order to apply the theory developed in [TV05] to study the stability of the forced equation.

Again we will also denote $\dot{R}(I) := \dot{W}(I) \cap \dot{B}^{\frac{4}{d+2}}_{d+2, \frac{2(d+2)}{d}}(I)$.

Observe that the above norms are continuous in their endpoints and are "time-divisible" in the sense that for each of the spaces $S(I)$ just introduced there exists a finite constant $\alpha(S) > 0$ such that

$$\left(\sum_{j=1}^{J} \|v\|_{S(I_j)}^{\alpha(S)} \right)^{\frac{1}{\alpha(S)}} \le \|v\|_{S(I)}$$

whenever I is the disjoint union of consecutive intervals $(I_j)_{j=1}^J$. In particular, $\alpha(W) = \alpha(\dot{W}) = \alpha(V) = \frac{2(d+2)}{d-2}$ and $\alpha(R) = \alpha(\dot{R}) = d + 2$ (see, for example, [DLM19]).

[2] Here we use the classical definition $\|\cdot\|_{X \cap Y} \equiv \|\cdot\|_X + \|\cdot\|_Y$.

We deduce that whenever $\|v\|_{S(I)} < \infty$ for S any of $W, \dot{W}, V, R$ or $\dot{R}$, we may partition I into J consecutive intervals $(I_j)_{j=1}^J$ with disjoint interiors such that

$$\|v\|_{S(I_j)} \leq \epsilon$$

for each $j = 1, ..., J$ and

$$J \leq 2\left(\frac{\|v\|_{S(I)}}{\epsilon}\right)^{\alpha(S)}$$

We end this section with the observation that for $F_n = P_{\leq n}F$ the regularised forcing term as in Section 1.1.3 it holds $\|F_n\|_{S(I)} \lesssim_d \|F\|_{S(I)}$ and

$$\|F_n - F\|_{S(I)} \to 0$$

as $n \to \infty$ for S any of the function spaces $W, \dot{W}, R, \dot{R}$ or V.

1.4 Local wellposedness

In this section we will prove the deterministic local wellposedness of the problem

$$\begin{cases} (i\partial_t + \Delta)v = (F+v)|F+v|^{\frac{4}{d-2}} \\ v(t_0) = v_0 \in H^1(\mathbb{R}^d) \end{cases} \tag{1.4.1}$$

in H^1 under appropriate conditions on the forcing term F. We will construct solutions via the regularised equation

$$\begin{cases} (i\partial_t + \Delta)v_n = g_n(F_n + v_n) \\ v_n(t_0) = v_{n,0} \in H^2(\mathbb{R}^d) \end{cases} \tag{1.4.2}$$

for g_n as in Section 1.1.3.

1.4.1 Linear and Nonlinear Estimates

Let $I \subset \mathbb{R}$ be an interval containing t_0. First observe the following inhomogeneous estimate, which is a direct application of the Strichartz inequality (0.1.2):

$$\|e^{i(t-t_0)\Delta}v_0\|_{\dot{W}(\mathbb{R})} \lesssim_d \|v_0\|_{\dot{H}^1}$$

Then by the inhomogeneous Strichartz estimate (0.1.11) followed by the chain rule (1.2.3) we have

$$\left\|\nabla\int_{t_0}^t e^{i(t-s)\Delta}g_n(u)(s)ds\right\|_{q,r[I]} \lesssim_{d,q,r} \||u|^{\frac{4}{d-2}}\nabla u\|_{2,\frac{2d}{d+2}[I]}$$

for any Strichartz pair (q,r). Moreover, using Hölder's inequality and the Sobolev embedding $\dot{W}^{1,\frac{2d(d+2)}{d^2+4}}(\mathbb{R}^d) \hookrightarrow L^{\frac{2(d+2)}{d-2}}(\mathbb{R}^d)$ we observe that

$$\| |u_1|^{\frac{4}{d-2}} u_2 \|_{2,\frac{2d}{d+2}[I]} \lesssim_d \|u_1\|_{\dot{W}(I)}^{\frac{4}{d-2}} \|u_2\|_{V(I)} \tag{1.4.3}$$

which in particular gives

$$\left\| \int_{t_0}^t e^{i(t-s)\Delta} g_n(u)(s) ds \right\|_{\dot{W}(I)} \lesssim \| |u|^{\frac{4}{d-2}} \nabla u \|_{2,\frac{2d}{d+2}[I]} \lesssim_d \|u\|_{\dot{W}(I)}^p$$

for all $u \in \dot{W}(I)$, recalling $p = \frac{d+2}{d-2}$.

1.4.2 Proof of Local wellposedness and Scattering Condition

Theorem 1.4.1. *There exists $\epsilon_0(d) > 0$ such that the following holds. Let $v_0 \in H^1(\mathbb{R}^d)$ and $F \in W \cap L_t^\infty L_x^{\frac{2d}{d-2}}(\mathbb{R})$. Let $I \ni t_0$ be a sufficiently small time interval such that*

$$\|e^{i(t-t_0)\Delta} v_0\|_{\dot{W}(I)} + \|F\|_{\dot{W}(I)} \le \epsilon \tag{1.4.4}$$

for some $0 < \epsilon < \epsilon_0(d)$. Then there exists a unique solution $v \in C(I, H^1(\mathbb{R}^d)) \cap W(I)$ to (1.4.1) which satisfies

$$\|v\|_{\dot{W}(I)} \le 4\epsilon$$

This solution extends to a maximal interval of existence $I^* := (T_-, T_+)$ in this space. Moreover,

1. *if $T_+ < \infty$, then $\|v\|_{\dot{W}([t_0,T_+))} = \infty$*
2. *if $T_+ = \infty$ and $\|v\|_{\dot{W}([t_0,T_+))} < \infty$, then the solution v scatters forwards in time, i.e. there exists $v_+ \in \dot{H}^1(\mathbb{R}^d)$ with*

$$\lim_{t\to\infty} \|v(t) - e^{it\Delta} v_+\|_{\dot{H}^1} = 0$$

The analogous statements hold for T_-.

On compact subintervals $\tilde{I}$ of I^*, v is obtained as a limit in $L_t^q L_x^r(\tilde{I})$ of solutions v_n to the regularised equation (1.4.2), for any Strichartz pair (q, r).

Proof. Denote by v_n the unique solution in $C(I, H^2(\mathbb{R}^d)) \cap C^1(I, L^2(\mathbb{R}^d))$ to (1.4.2) with initial data $v_{n,0} = P_{\le n} v_0$. We will show that $(v_n)_n$ is Cauchy in $V(I)$. Observe that for any Strichartz pair (q, r) and any $l \ge n$, we have

$$\begin{aligned} \|v_n - v_l\|_{q,r[I]} \lesssim_{q,r,d} & \|v_{n,0} - v_{l,0}\|_{L^2(\mathbb{R}^d)} + \|g_n(F_n + v_n) - g_n(F_l + v_l)\|_{2,\frac{2d}{d+2}[I]} \\ & + \|g_n(F_l + v_l) - g(F_l + v_l)\|_{2,\frac{2d}{d+2}[I]} + \|g(F_l + v_l) - g_l(F_l + v_l)\|_{2,\frac{2d}{d+2}[I]} \end{aligned}$$

We bound each of these terms separately. Firstly, by (1.2.1) applied to g_n and the nonlinear estimate (1.4.3) we have, for any u_1, $u_2 \in W(I)$,

$$\|g_n(u_1) - g_n(u_2)\|_{2,\frac{2d}{d+2}[I]} \lesssim_d (\|u_1\|^{p-1}_{\dot W(I)} + \|u_2\|^{p-1}_{\dot W(I)})\|u_1 - u_2\|_{V(I)}$$

and the analogous bound for g. Next, since $g_n(u) = g(u)$ for $|u| \le n$, we may bound

$$\begin{aligned}
\|g_n(u) - g(u)\|_{2,\frac{2d}{d+2}[I]} &\lesssim_d \||u|^p \mathbb{1}_{|u|\ge n}\|_{2,\frac{2d}{d+2}[I]} \\
&\lesssim_d \|u\|^{p-1}_{\dot W(I)} \|u \cdot \mathbb{1}_{|u|\ge n}\|_{\frac{2(d+2)}{d-2},\frac{2d(d+2)}{d^2+4}[I]} \\
&\lesssim_d \|u\|^{p-1}_{\dot W(I)} \|\mathbb{1}_{|u|\ge n}\|_{\infty,\frac{d(d+2)}{4}[I]} \|u\|_{\frac{2(d+2)}{d-2},\frac{2d}{d-2}[I]} \\
&\lesssim_d \|u\|^{p-1}_{\dot W(I)} \left(\sup_{t\in I} \frac{1}{n^{\frac{2d}{d-2}}} \int_{\mathbb{R}^d} |u|^{\frac{2d}{d-2}} dx\right)^{\frac{4}{d(d+2)}} |I|^{\frac{d-2}{2(d+2)}} \|u\|_{\infty,\frac{2d}{d-2}[I]} \\
&\lesssim_d |I|^{\frac{d-2}{2(d+2)}} n^{-\frac{8}{d^2-4}} \|u\|^{p-1}_{\dot W(I)} \|u\|^{\frac{d^2+4}{d^2-4}}_{\infty,p+1[I]}
\end{aligned}$$

Thus since $l \ge n$, using that $F_n = P_{\le n}F$ and $\dot H^1(\mathbb{R}^d) \hookrightarrow L^{p+1}(\mathbb{R}^d)$, we have

$$\begin{aligned}
\|v_n - v_l\|_{q,r[I]} \lesssim_{q,r,d} &\|v_{n,0} - v_{l,0}\|_{L^2(\mathbb{R}^d)} \\
&+ (\|v_n\|^{p-1}_{\dot W(I)} + \|v_l\|^{p-1}_{\dot W(I)} + \|F\|^{p-1}_{\dot W(I)} + \|F\|^{p-1}_{\dot W(I)})\|v_n - v_l + F_n - F_l\|_{V(I)} \\
&+ |I|^{\frac{d-2}{2(d+2)}} n^{-\frac{8}{d^2-4}} (\|v_l\|^{p-1}_{\dot W(I)} + \|F\|^{p-1}_{\dot W(I)})(\|v_l\|^{\frac{d^2+4}{d^2-4}}_{L^\infty_t \dot H^1(I)} + \|F\|^{\frac{d^2+4}{d^2-4}}_{\infty,p+1[\mathbb{R}]})
\end{aligned} \tag{1.4.5}$$

To proceed, we need a bound on $\|v_n\|_{\dot W(I)}$. By the nonlinear estimates we have, for any $t_0 \in I' \subset I$,

$$\begin{aligned}
\|v_n\|_{\dot W(I')} \le &\|e^{i(t-t_0)\Delta}(v_{n,0} - v_0)\|_{\dot W(I)} + \|e^{i(t-t_0)\Delta} v_0\|_{\dot W(I)} + C_d(\|v_n\|^p_{\dot W(I')} + \|F_n\|^p_{\dot W(I')}) \\
\le &2\epsilon + C_d\|v_n\|^p_{\dot W(I')}
\end{aligned}$$

for n sufficiently large, $\epsilon(d)$ sufficiently small. Taking $\epsilon(d)$ smaller still, a standard continuity argument shows that $\|v_n\|_{\dot W(I)} \le 4\epsilon$.

Lastly we observe that

$$\|v_n\|_{L^\infty_t \dot H^1_x(I)} \lesssim_d \|v_{n,0}\|_{\dot H^1} + \|v_n\|^p_{\dot W(I)} + \|F_n\|^p_{\dot W(I)} \lesssim_d \|v_0\|_{\dot H^1} + \epsilon^p \tag{1.4.6}$$

so the v_n are uniformly bounded in $\dot H^1$ on I, say by $C(v_0, \epsilon, d)$.

Putting the above estimates into (1.4.5) along with the assumption (1.4.4), we see that $(v_n)_n$ is Cauchy in $L^q_t L^r_x$ for any Strichartz pair (q, r). In particular $(v_n)_n$ has a limit $v \in V(I)$, which still satisfies $\|v\|_{\dot W(I)} \le 4\epsilon$ and solves equation (1.4.1).

By standard arguments, one may extend v to a maximal interval of existence (T_-, T_+), such that it is the unique solution to (1.4.1) in $C([\alpha, \beta]; H_x^1(\mathbb{R}^d)) \bigcap W([\alpha, \beta])$ for any $T_- < \alpha < t_0 < \beta < T_+$.

We next prove the blow up criterion. We work forwards in time since the result in the negative time direction is proved in the same way. Suppose that $T_+ < \infty$ and $\|v\|_{\dot{W}([t_0,T_+))} < \infty$. Consider a sequence $t_n \nearrow T_+$. Note that

$$e^{i(t-t_n)\Delta}v(t_n) = e^{i(t-t_0)\Delta}v_0 - i\int_{t_0}^{t_n} e^{i(t-s)\Delta}g(F+v)ds = v(t) + i\int_{t_n}^{t} e^{i(t-s)\Delta}g(F+v)ds$$

Thus by the continuity of the $\dot{W}$ norm, we find

$$\begin{aligned}
&\|e^{i(t-t_n)\Delta}v(t_n)\|_{\dot{W}([t_n,T_+))} + \|F\|_{\dot{W}([t_n,T_+))} \\
\leq &\|v\|_{\dot{W}([t_n,T_+))} + C_d(\|v\|^p_{\dot{W}([t_n,T_+))} + \|F\|^p_{\dot{W}([t_n,T_+))}) + \|F\|_{\dot{W}([t_n,T_+))} \leq \frac{\epsilon}{2}
\end{aligned} \tag{1.4.7}$$

for n sufficiently large. Then since F, $e^{i(t-t_n)\Delta}v(t_n) \in \dot{W}(\mathbb{R})$ we find $\eta > 0$ such that

$$\|e^{i(t-t_n)\Delta}v(t_n)\|_{\dot{W}([t_n,T_++\eta])} + \|F\|_{\dot{W}([t_n,T_++\eta])} \leq \epsilon$$

Therefore by the local wellposedness result we can extend the solution to $T_+ + \eta$, which is a contradiction.

Finally, we turn to scattering. Suppose that $T_+ = \infty$ and $\|v\|_{\dot{W}([t_0,\infty))} < \infty$. Define

$$v_+ = e^{-it_0\Delta}v_0 - i\int_{t_0}^{\infty} e^{-is\Delta}g(F+v)ds$$

Then for any $t > t_0$, the dual Strichartz estimate (0.1.10) gives

$$\begin{aligned}
\|v(t) - e^{it\Delta}v_+\|_{\dot{H}^1(\mathbb{R}^d)} &= \left\| \int_t^{\infty} e^{i(t-s)\Delta}g(F+v)(s)ds \right\|_{\dot{H}^1(\mathbb{R}^d)} \\
&\lesssim_d \|v\|^p_{\dot{W}([t,\infty))} + \|F\|^p_{\dot{W}([t,\infty))} \to 0 \text{ as } t \to \infty
\end{aligned}$$

since $\|v\|_{\dot{W}([t_0,\infty))} < \infty$ and $\|F\|_{\dot{W}([t_0,\infty))} < \infty$. Thus $v_+ \in \dot{H}^1(\mathbb{R}^d)$ and the solution v scatters to v_+ as $t \to +\infty$.

Lastly, the fact that v is the limit of $(v_n)_n$ on compact subintervals of (T_-, T_+) follows by induction of the existence proof over subintervals on which $\|v\|_{\dot{W}}$ and $\|F\|_{\dot{W}}$ are small, using (1.4.7) to obtain (1.4.4) on each interval. The number of such intervals required is controlled due to the time-divisibility of the $\dot{W}$-norm. □

Remark 1.4.2. *Observe that by applying Strichartz's inequality at dyadic scales followed* *by the dual estimate (1.2.7) we obtain*

$$\left(\sum_{N\in 2^{\mathbb{Z}}} N^2\|P_N v\|_{q,r[I]}^2\right)^{\frac{1}{2}} \lesssim \left(\sum_{N\in 2^{\mathbb{Z}}} \|P_N v_0\|_{\dot H^1}^2\right)^{\frac{1}{2}} + \left(\sum_{N\in 2^{\mathbb{Z}}} \|P_N \nabla g(F+v)\|_{2,\frac{2d}{d+2}[I]}^2\right)^{\frac{1}{2}}$$
$$\lesssim \|v_0\|_{\dot H^1} + \|\nabla g(F+v)\|_{2,\frac{2d}{d+2}[I]} < \infty$$

for any Strichartz pair (q,r), $I \subset\subset I^*$.

1.5 Conditional Scattering

In this section we will prove the following theorem, giving a sufficient condition for scattering of the solution to the forced NLS

$$\begin{cases} (i\partial_t + \Delta)v = |F+v|^{\frac{4}{d-2}}(F+v) \\ v(t_0) = v_0 \in H^1(\mathbb{R}^d) \end{cases} \tag{1.5.1}$$

studied in the previous section.

Theorem 1.5.1. *(Conditional Scattering) Let $v_0 \in H^1(\mathbb{R}^d)$, $F \in R \cap L_t^\infty L_x^{\frac{2d}{d-2}}(\mathbb{R})$ (see the definition in 1.3.1). Let $v(t)$ be the solution to (1.5.1) defined on its maximal interval of existence I^*. Suppose moreover that*

$$M := \sup_{t\in I^*} E(v(t)) < \infty$$

Then $I^* = \mathbb{R}$, i.e. $v(t)$ is globally defined, and it holds that

$$\|v\|_{\dot W(\mathbb{R})} \le C(M, \|F\|_{\dot R(\mathbb{R})}, d) \tag{1.5.2}$$

As a result, the solution v scatters in $\dot H^1$ as $t \to \pm\infty$.

Throughout this section v will refer to the solution to (1.5.1) obtained in Theorem 1.4.1, defined on its maximal interval of existence $I^* := (T_-, T_+)$. We first present a lemma bounding the $\dot W(\mathbb{R})$ norm of solutions to the unforced defocusing equation

$$\begin{cases} (i\partial_t + \Delta)u = |u|^{\frac{4}{d-2}}u \\ u(t_0) = u_0 \in \dot H^1(\mathbb{R}^d) \end{cases} \tag{1.5.3}$$

Lemma 1.5.2. *There exists a non-decreasing function $K : [0,\infty) \to [0,\infty)$ with the following property. Let $u_0 \in \dot H^1(\mathbb{R}^d)$ and $t_0 \in \mathbb{R}$. Then there exists a unique global solution $u \in C(\mathbb{R}; \dot H^1(\mathbb{R}^d))$ to the defocusing energy-critical NLS (1.5.3) satisfying*

$$\|u\|_{\dot W(\mathbb{R})} \le K(E(u_0))$$

where

$$E(u_0) := \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u_0|^2 dx + \frac{d-2}{2d}\int_{\mathbb{R}^d} |u_0|^{\frac{2d}{d-2}} dx$$

Proof. The existence of a global solution follows from the work of Visan [Vis07]. Combining Theorem 1.1 and Lemma 3.1 of [Vis07] with the conservation of energy for solutions to (1.5.3),[3] we infer the existence of a non-decreasing function $K : [0,\infty) \to [0,\infty)$ such that the solution $u \in C(\mathbb{R}; \dot{H}^1(\mathbb{R}^d))$ to (1.5.3) satisfies

$$\|u\|_{\dot{S}^1(\mathbb{R}\times\mathbb{R}^d)} \leq K(E(u_0))$$

where $\|u\|_{\dot{S}^1(\mathbb{R}\times\mathbb{R}^d)} := \sup \left(\sum_{N\in 2^{\mathbb{Z}}} N^2 \|P_N u\|^2_{q,r[\mathbb{R}]}\right)^{\frac{1}{2}}$, with the supremum taken over all Strichartz admissible pairs (q, r). Since this norm controls the $\dot{W}$-norm (by the Littlewood-Paley inequality) we have the result. □

Given the blow-up criterion proved in Theorem 1.4.1, to prove global existence and scattering of v it is sufficient to show that

$$\|v\|_{\dot{W}(I^*)} < \infty \tag{1.5.4}$$

With this in mind, and in light of Lemma 1.5.2, we will develop a suitable perturbation theory to compare solutions of (1.5.1) with those of (1.5.3) in $\dot{W}$.

We start with a lemma concerning short-time perturbations.

Lemma 1.5.3. *(Short-time perturbations) Let $I \subset \mathbb{R}$ be a compact time interval containing t_0 and let $u_0, v_0 \in H^1(\mathbb{R}^d)$ with*

$$\|u_0\|_{\dot{H}^1(\mathbb{R}^d)}, \|v_0\|_{\dot{H}^1(\mathbb{R}^d)} \leq E$$

for some $E > 0$. Let u solve the defocusing NLS (1.5.3) with initial data $u(t_0) = u_0$.

Let $F \in R \cap L_t^\infty L_x^{\frac{2d}{d-2}}(\mathbb{R})$. Then there exists a constant $\epsilon_0(E, d) \in (0, 1)$ such that if we further suppose

$$\|u\|_{\dot{W}(I)} \leq \epsilon_0 \tag{1.5.5}$$

$$\left(\sum_N \|P_N e^{i(t-t_0)\Delta}(u_0 - v_0)\|^2_{\dot{W}(I)}\right)^{\frac{1}{2}} \leq \epsilon \tag{1.5.6}$$

$$\|F\|_{\dot{R}(I)} \leq \epsilon \tag{1.5.7}$$

[3] Conservation of energy for solutions to (1.5.3) is well-known. Nonetheless we remark that, as in the next section of this chapter, the formal calculations used to prove it can for example be justified via the regularisation (1.1.14), using the stability theory in Theorem 1.3 of [TV05] to show that solutions of the regularised problem converge locally uniformly to solutions of (1.5.3) in $\dot{H}^1_x$.

for some $0 < \epsilon < \epsilon_0$, then there exists a unique solution $v : I \times \mathbb{R}^d \to \mathbb{C}$ solving the forced equation (1.5.1) with $v(t_0) = v_0$ satisfying

$$\|v - u\|_{\dot{W}(I)} \leq C_{d,1}\epsilon^{\frac{7}{(d-2)^2}} \tag{1.5.8}$$

$$\|\nabla[g(F+v) - g(u)]\|_{2,\frac{2d}{d+2}[I]} \leq C_{d,1}\epsilon^{\frac{28}{(d-2)^3}} \tag{1.5.9}$$

for a constant $C_{d,1} > 1$ depending only on the dimension d.

Proof. In view of the local existence theory, it suffices to prove (1.5.8) and (1.5.9) as a priori estimates. In what follows all spacetime norms are taken over $I \times \mathbb{R}^d$. Define $w := v - u$, which solves

$$\begin{cases} (i\partial_t + \Delta)w = |u+w+F|^{\frac{4}{d-2}}(u+w+F) - |u|^{\frac{4}{d-2}}u \text{ on } I \times \mathbb{R}^d \\ w(t_0) = v_0 - u_0 \end{cases} \tag{1.5.10}$$

with $\|w(t_0)\|_{\dot{H}^1} \lesssim E$. We have

$$\|w\|_{\dot{W}} \lesssim_d \|e^{i(t-t_0)\Delta}w(t_0)\|_{\dot{W}} + \|\nabla[g(u+w+F) - g(u)]\|_{2,\frac{2d}{d+2}} \tag{1.5.11}$$

where by (1.2.5)

$$\begin{aligned} &\|\nabla[g(u+w+F) - g(u)]\|_{2,\frac{2d}{d+2}} \\ \lesssim_d &\||F|^{\frac{4}{d-2}}\nabla F\|_{2,\frac{2d}{d+2}} + \||w|^{\frac{4}{d-2}}\nabla F\|_{2,\frac{2d}{d+2}} + \||F|^{\frac{4}{d-2}}\nabla u\|_{2,\frac{2d}{d+2}} + \||w|^{\frac{4}{d-2}}\nabla u\|_{2,\frac{2d}{d+2}} \\ &+ \||F|^{\frac{4}{d-2}}\nabla w\|_{2,\frac{2d}{d+2}} + \||w|^{\frac{4}{d-2}}\nabla w\|_{2,\frac{2d}{d+2}} + \||u|^{\frac{4}{d-2}}\nabla F\|_{2,\frac{2d}{d+2}} + \||u|^{\frac{4}{d-2}}\nabla w\|_{2,\frac{2d}{d+2}} \end{aligned}$$

Then using the nonlinear bound $\||u_1|^{\frac{4}{d-2}}\nabla u_2\|_{2,\frac{2d}{d+2}} \lesssim_d \|u_1\|_{\dot{W}}^{\frac{4}{d-2}}\|u_2\|_{\dot{W}}$, assumptions (1.5.5)-(1.5.7) and Young's inequality we have

$$\begin{aligned} &\|\nabla[g(u+w+F) - g(u)]\|_{2,\frac{2d}{d+2}} \\ \lesssim_d &\epsilon^{\frac{d+2}{d-2}} + \epsilon^{\frac{4}{d-2}}\epsilon_0 + \epsilon_0^{\frac{4}{d-2}}\epsilon + \epsilon^{\frac{4}{d-2}}\|w\|_{\dot{W}} + \epsilon_0^{\frac{4}{d-2}}\|w\|_{\dot{W}} + \epsilon\|w\|_{\dot{W}}^{\frac{4}{d-2}} \\ &+ \|w\|_{\dot{W}}^{\frac{d+2}{d-2}} + \||w|^{\frac{4}{d-2}}\nabla u\|_{2,\frac{2d}{d+2}} \\ \lesssim_d &\epsilon^{\frac{4}{d-2}} + \epsilon_0^{\frac{4}{d-2}}\|w\|_{\dot{W}} + \||w|^{\frac{4}{d-2}}\nabla u\|_{2,\frac{2d}{d+2}} + \|w\|_{\dot{W}}^{\frac{d+2}{d-2}} \end{aligned} \tag{1.5.12}$$

taking $\epsilon_0(d) < 1$.

It is tempting to also expand the remaining term in $\dot{W}$ and run a continuity argument, however this will produce the term $\epsilon_0\|w\|_{\dot{W}}^{\frac{4}{d-2}}$ on the right hand side which is an issue for $d > 6$ since the power $\frac{4}{d-2}$ is less than 1. We therefore make use of auxiliary spaces X and Y introduced by Visan and Tao in [TV05]. These spaces invoke only $\frac{4}{d+2} < \frac{4}{d-2}$

derivatives instead of a whole derivative as in $\dot{W}$ making it possible to run a standard continuity argument in X. Once we have a bound on $\|w\|_X$ we can use it in (1.5.12) to bound $\|w\|_{\dot{W}}$.

The spaces X and Y are defined by the following norms:

$$\|f\|_X := \left(\sum_N N^{\frac{8}{d+2}} \|P_N f\|^2_{d+2, \frac{2(d+2)}{d}} \right)^{\frac{1}{2}}$$

$$\|h\|_Y := \left(\sum_N N^{\frac{8}{d+2}} \|P_N h\|^2_{\frac{d+2}{3}, \frac{2(d+2)}{d+4}} \right)^{\frac{1}{2}}$$

Note that the space X scales in the same way as $\dot{W}$. Observe also that $\|F\|_X \leq \|F\|_{\dot{R}}$.

To see that w belongs to X we observe the following relation between X and $\dot{W}$: By Bernstein's inequality we have

$$\|f\|_X \lesssim_d \left(\sum_N N^2 \|P_N f\|^2_{d+2, \frac{2d(d+2)}{d^2+2d-4}} \right)^{\frac{1}{2}}$$

We now interpolate the $L_t^{d+2} L_x^{\frac{2d(d+2)}{d^2+2d-4}}$ norm between $L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}$ and $L_t^\infty L_x^2$ yielding

$$\|P_N f\|_{d+2, \frac{2d(d+2)}{d^2+2d-4}} \lesssim \|P_N f\|^{\frac{2}{d-2}}_{\frac{2(d+2)}{d-2}, \frac{2d(d+2)}{d^2+4}} \|P_N f\|^{\frac{d-4}{d-2}}_{\infty, 2}$$

and apply Hölder's inequality for sequences to get

$$\begin{aligned} \|f\|_X &\lesssim_d \left(\sum_N N^2 \|P_N f\|^2_{\frac{2(d+2)}{d-2}, \frac{2d(d+2)}{d^2+4}} \right)^{\frac{2}{2(d-2)}} \left(\sum_N N^2 \|P_N f\|^2_{\infty,2} \right)^{\frac{d-4}{2(d-2)}} \\ &\lesssim_d \left(\sum_N \|P_N f\|^2_{\dot{W}} \right)^{\frac{1}{d-2}} \left(\sum_N \|P_N f\|^2_{L_t^\infty \dot{H}_x^1} \right)^{\frac{d-4}{2(d-2)}} \end{aligned} \tag{1.5.13}$$

Thus by Remark 1.4.2, w indeed belongs to X.

We can use this to bound the remaining term in (1.5.12). Indeed, by the Littlewood-Paley inequality followed by Bernstein's inequality we have

$$\| |w|^{\frac{4}{d-2}} \nabla u\|_{2, \frac{2d}{d+2}} \leq \|w\|_X^{\frac{4}{d-2}} \|\nabla u\|_{\frac{2(d^2-4)}{d^2-12}, \frac{2d(d^2-4)}{d^3-2d^2-4d+24}}$$

where, since $(\frac{2(d^2-4)}{d^2-12}, \frac{2d(d^2-4)}{d^3-2d^2-4d+24})$ is a Strichartz pair, we can use the nonlinear estimate to bound

$$\|\nabla u\|_{\frac{2(d^2-4)}{d^2-12}, \frac{2d(d^2-4)}{d^3-2d^2-4d+24}} \lesssim_d \|u_0\|_{\dot{H}^1(\mathbb{R}^d)} + \|u\|_{\dot{W}}^{\frac{d+2}{d-2}} \lesssim_d E + \epsilon_0^{\frac{d+2}{d-2}} \lesssim_d E$$

for $\epsilon_0(E,d)$ sufficiently small.

Substituting this into (1.5.12) and combining the result with (1.5.11) we have

$$\|w\|_{\dot{W}} \lesssim_d \|e^{i(t-t_0)\Delta}w(t_0)\|_{\dot{W}} + \epsilon^{\frac{4}{d-2}} + \epsilon_0^{\frac{4}{d-2}}\|w\|_{\dot{W}} + E\|w\|_X^{\frac{4}{d-2}} + \|w\|_{\dot{W}}^{\frac{d+2}{d-2}} \tag{1.5.14}$$

So we must show that $\|w\|_X$ is small. This will require two estimates both proved in [TV05], see also [LZ11]. The first (Lemma 3.2, [TV05]) is a Strichartz-type estimate between X and Y:

$$\left\|\int_{t_0}^t e^{i(t-s)\Delta}F(s)ds\right\|_X \lesssim_d \|F\|_Y \tag{1.5.15}$$

and the second (Lemma 3.3, [TV05]) is the nonlinear estimate

$$\|g_z(v)u\|_Y \lesssim_d \|v\|_{\dot{W}}^{\frac{4}{d-2}}\|u\|_X \tag{1.5.16}$$

(with a similar estimate for $g_{\bar{z}}$).

Using (1.5.15) and the fact that w satisfies equation (1.5.10) we immediately obtain

$$\|w\|_X \lesssim_d \|e^{i(t-t_0)\Delta}w(t_0)\|_X + \|g(u+w+F) - g(u)\|_Y \tag{1.5.17}$$

First consider the free evolution term. By (1.5.13) we see

$$\begin{aligned}\|e^{i(t-t_0)\Delta}w(t_0)\|_X &\lesssim_d \left(\sum_N \|P_N e^{i(t-t_0)\Delta}w(t_0)\|_{\dot{W}}^2\right)^{\frac{1}{d-2}} \left(\sum_N \|P_N w(t_0)\|_{\dot{H}^1}^2\right)^{\frac{d-4}{2(d-2)}}\\ &\lesssim_d \epsilon^{\frac{2}{d-2}} E^{\frac{d-4}{d-2}}\end{aligned}$$

We now move onto the second term in (1.5.17). Using (1.2.2), Minkowski's inequality and the nonlinear estimate (1.5.16) we have

$$\begin{aligned}\|g(u+w+F) - g(u)\|_Y &\le \int_0^1 \|u+\theta(F+w)\|_{\dot{W}}^{\frac{4}{d-2}}\|F+w\|_X d\theta\\ &\lesssim_d (\|u\|_{\dot{W}}^{\frac{4}{d-2}} + \|F\|_{\dot{W}}^{\frac{4}{d-2}} + \|v\|_{\dot{W}}^{\frac{4}{d-2}})(\|F\|_X + \|w\|_X)\\ &\lesssim_d (\epsilon_0^{\frac{4}{d-2}} + \epsilon^{\frac{4}{d-2}} + \|v\|_{\dot{W}}^{\frac{4}{d-2}})(\epsilon + \|w\|_X)\end{aligned}$$

where we used that $\|F\|_{\dot{W}} + \|F\|_X = \|F\|_{\dot{R}} \le \epsilon$ in the last line.

To bound $\|v\|_{\dot{W}}$, we first use that v_0 is close to u_0 and that u satisfies the standard NLS (1.5.3) to bound the linear part:

$$\begin{aligned}\|e^{i(t-t_0)\Delta}v_0\|_{\dot{W}} &\lesssim_d \|e^{i(t-t_0)\Delta}(v_0-u_0)\|_{\dot{W}} + \|e^{i(t-t_0)\Delta}u_0\|_{\dot{W}}\\ &\lesssim_d \epsilon + \|u\|_{\dot{W}} + \|u\|_{\dot{W}}^{\frac{d+2}{d-2}}\end{aligned}$$

$$\lesssim_d \epsilon_0$$

We can thus apply the local wellposedness theory to infer that, on the interval I,

$$\|v\|_{\dot{W}} \lesssim_d \epsilon_0$$

Returning to (1.5.17) we thus have

$$\begin{aligned}\|w\|_X \lesssim_d & \epsilon^{\frac{2}{d-2}} E^{\frac{d-4}{d-2}} + (\epsilon_0^{\frac{4}{d-2}} + \epsilon^{\frac{4}{d-2}} + \epsilon_0^{\frac{4}{d-2}})(\epsilon + \|w\|_X) \\ \lesssim_d & \epsilon^{\frac{2}{d-2}} E^{\frac{d-4}{d-2}} + \epsilon + \epsilon_0^{\frac{4}{d-2}} \|w\|_X\end{aligned}$$

Thus choosing ϵ_0 sufficiently small depending on d and E we conclude

$$\|w\|_X \lesssim_d \epsilon^{\frac{2}{d-2}} E^{\frac{d-4}{d-2}}$$

Now that we have bounded $\|w\|_X$, we can return to (1.5.14) to bound $\|w\|_{\dot{W}}$. We have

$$\begin{aligned}\|w\|_{\dot{W}} \lesssim_d & \|e^{i(t-t_0)\Delta} w(t_0)\|_{\dot{W}} + \epsilon^{\frac{4}{d-2}} + \epsilon_0^{\frac{4}{d-2}} \|w\|_{\dot{W}} + E\|w\|_X^{\frac{4}{d-2}} + \|w\|_{\dot{W}}^{\frac{d+2}{d-2}} \\ \lesssim_d & \epsilon + \epsilon^{\frac{4}{d-2}} + \epsilon_0^{\frac{4}{d-2}} \|w\|_{\dot{W}} + E(\epsilon^{\frac{2}{d-2}} E^{\frac{d-4}{d-2}})^{\frac{4}{d-2}} + \|w\|_{\dot{W}}^{\frac{d+2}{d-2}} \\ \lesssim_d & \epsilon^{\frac{7}{(d-2)^2}} + \frac{1}{2}\|w\|_{\dot{W}} + \|w\|_{\dot{W}}^{\frac{d+2}{d-2}}\end{aligned}$$

for $\epsilon_0(E, d)$ sufficiently small. The result (1.5.8) now follows from a standard continuity argument.

Lastly, we show (1.5.9). By (1.5.12)

$$\begin{aligned}\|\nabla[g(F+v) - g(u)]\|_{2,\frac{2d}{d+2}} \lesssim_d & \epsilon^{\frac{4}{d-2}} + \epsilon_0^{\frac{4}{d-2}} \|w\|_{\dot{W}} + \|u\|_{\dot{W}} \|w\|_{\dot{W}}^{\frac{4}{d-2}} + \|w\|_{\dot{W}}^{\frac{d+2}{d-2}} \\ \lesssim_d & \epsilon^{\frac{4}{d-2}} + \|w\|_{\dot{W}} + \|w\|_{\dot{W}}^{\frac{4}{d-2}} + \|w\|_{\dot{W}}^{\frac{d+2}{d-2}}\end{aligned}$$

using the bounds assumed on $\|u\|_{\dot{W}}$ and $\|F\|_{\dot{W}}$. Substituting in the bound just obtained for $\|w\|_{\dot{W}}$ gives the result. □

We now extend this result by removing the smallness assumption on u in the case when u and v have the same initial data.

Lemma 1.5.4. *(Long-time perturbations) Let $I \subset \mathbb{R}$ be a compact time interval with $t_0 \in I$ and let $v_0 \in H^1(\mathbb{R}^d)$ with*

$$E(v_0) \leq E$$

Let $u \in C(I, \dot{H}^1(\mathbb{R}^d))$ be the solution to the defocusing NLS (1.5.3) with initial data

$u(t_0) = v_0$ and

$$\|u\|_{\dot{W}(I)} \leq K$$

for some $K > 0$. Then there exists $\epsilon_1(E, K, d) \in (0,1)$ such that for any $F \in R(I)$ sufficiently small in the sense that

$$\|F\|_{\dot{R}(I)} \leq \epsilon_1,$$

there exists a unique solution $v : I \times \mathbb{R}^d \to \mathbb{C}$ to the forced equation (1.5.1) with initial data $v(t_0) = v_0$ and it holds

$$\|v - u\|_{\dot{W}(I)} \leq 1 \tag{1.5.18}$$

Proof. Without loss of generality assume $t_0 = \inf I$. As in the previous proposition it suffices to prove the bound as an a priori estimate. In order to make use of the short-time perturbation theory, we will induct over intervals on which the $\dot{W}(I)$-norm of u is small. To this end we partition I into consecutive intervals with disjoint interiors $(I_j)_{j=1}^J$ such that

$$\|u\|_{\dot{W}(I_j)} \leq \epsilon_0(2(2E)^{\frac{1}{2}}, d) \tag{1.5.19}$$

for each $j = 1, ..., J$, where ϵ_0 is as in Lemma 1.5.3. By the time-divisibility properties of $\dot{W}$ we are able to do this with

$$J \lesssim \left(\frac{K}{\epsilon_0(2(2E)^{\frac{1}{2}}, d)} \right)^{\frac{2(d+2)}{d-2}} \tag{1.5.20}$$

Denote $I_j = [t_{j-1}, t_j]$. We must check that the conditions of Lemma 1.5.3 are satisfied on this interval.

We first make two observations. Using Strichartz's inequality (0.1.10) we have

$$\begin{aligned} \|v(t_j) - u(t_j)\|_{\dot{H}^1(\mathbb{R}^d)} \simeq & \left\| \int_{t_0}^{t_j} e^{i(t_j - s)\Delta} \nabla[g(F+v) - g(u)](s) ds \right\|_{L^2(\mathbb{R}^d)} \\ \leq & A_{d,1} \|\nabla[g(F+v) - g(u)]\|_{2, \frac{2d}{d+2}[t_0, t_j]} \end{aligned} \tag{1.5.21}$$

Secondly, by the Strichartz estimates (0.1.9) and (0.1.10) followed by the embedding (1.2.7) we obtain

$$\begin{aligned} & \left(\sum_N \|P_N e^{i(t-t_j)\Delta}(v(t_j) - u(t_j))\|^2_{\dot{W}(I_{j+1})} \right)^{\frac{1}{2}} \\ \simeq & \left(\sum_N N^2 \left\| \int_{t_0}^{t_j} e^{i(t-s)\Delta} P_N[g(F+v) - g(u)](s) ds \right\|^2_{\frac{2(d+2)}{d-2}, \frac{2d(d+2)}{d^2+4}[I_{j+1}]} \right)^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&\leq A_{d,2} \left(\sum_N N^2 \|P_N[g(F+v) - g(u)](s)\|^2_{2,\frac{2d}{d+2}[t_0,t_j]} \right)^{\frac{1}{2}} \\
&\leq A'_{d,2} \|\nabla[g(F+v) - g(u)]\|_{2,\frac{2d}{d+2}[t_0,t_j]} \qquad (1.5.22)
\end{aligned}
$$

Set $A_d := \max\{1, A_{d,1}, A'_{d,2}\}$.

We now prove a technical claim that will be useful for the inductive step. In the rest of this proof we denote $\alpha := \frac{28}{(d-2)^3}$ and $C_d := A_d C_{d,1} \geq 1$, with $C_{d,1}$ the constant from Lemma 1.5.3.

Claim 1. *We may take $\epsilon_1(E, K, d) > 0$ sufficiently small such that the following holds: Define a sequence $(\epsilon(j))_{j=1}^{J+1}$ by*

$$
\epsilon(1) = \epsilon_1(E, K, d), \qquad \epsilon(j+1) = C_d \sum_{k=1}^{j} \epsilon(k)^\alpha \text{ for } 1 \leq j \leq J
$$

Then for all $1 \leq j \leq J+1$ it holds

$$
\epsilon_1 \leq \epsilon(j) \leq (2C_d)^{\sum_{k=0}^{j-2} \alpha^k} \epsilon_1^{\alpha^{j-1}} < \min\{\epsilon_0(2(2E)^{\frac{1}{2}}, d), (2E)^{\frac{1}{2}}\}
$$

Proof of claim. The cases $j = 1, 2$ are easily verified (since $\alpha < 1$). Suppose that the claim holds for all $1 \leq j \leq k$ for some $k \leq J$. Then by definition

$$
\epsilon(k+1) = \epsilon(k) + C_d \epsilon(k)^\alpha \leq 2C_d \epsilon(k)^\alpha \leq (2C_d)^{\sum_{k'=0}^{k-1} \alpha^{k'}} \epsilon_1^{\alpha^k}
$$

as required. That $\epsilon(k) \geq \epsilon_1$ is clear since the sequence is increasing, and for $1 \leq k \leq J$,

$$
\epsilon(k+1) \leq (2C_d)^{\sum_{k'=0}^{J-1} \alpha^{k'}} \epsilon_1^{\alpha^J} < \min\{\epsilon_0(2(2E)^{\frac{1}{2}}, d), (2E)^{\frac{1}{2}}\}
$$

for ϵ_1 sufficiently small depending on E, K and d. □

We now prove a second claim in which we reduce the long time perturbation result to the short time result on the intervals I_j. In the rest of this proof we will take $\epsilon_1(E, K, d)$ as in the above claim.

Claim 2. *Under the assumptions of the lemma, for all $1 \leq j \leq J$ it holds*

$$
\|v(t_{j-1})\|_{\dot{H}^1(\mathbb{R}^d)} \leq 2(2E)^{\frac{1}{2}}
$$

$$
\left(\sum_N \|P_N e^{i(t-t_{j-1})\Delta}(u(t_{j-1}) - v(t_{j-1}))\|^2_{\dot{W}(I_j)} \right)^{\frac{1}{2}} \leq \epsilon(j) < \epsilon_0(2(2E)^{\frac{1}{2}}, d)
$$

$$
\|\nabla[g(F+v) - g(u)]\|_{2,\frac{2d}{d+2}[I_j]} \leq C_{d,1} \epsilon(j)^\alpha
$$

$$
\|u - v\|_{\dot{W}(I_j)} \leq C_{d,1} \epsilon(j)^{\frac{7}{(d-2)^2}}
$$

for $\epsilon(j)$ as in the previous claim.

Proof of claim. Recall that for each $j = 1, ..., J$ it holds

$$\|u\|_{\dot{W}(I_j)} \leq \epsilon_0(2(2E)^{\frac{1}{2}}, d)$$
$$\|F\|_{\dot{R}(I_j)} \leq \epsilon_1 < \epsilon_0(2(2E)^{\frac{1}{2}}, d)$$
$$\|u(t_{j-1})\|_{\dot{H}^1} \leq (2E)^{\frac{1}{2}}$$

where we used that u has conserved energy.

For $j = 1$, we have $u(t_0) = v(t_0) = v_0$ so we can immediately apply Lemma 1.5.3 to obtain (using $\epsilon(1) = \epsilon_1$)

$$\|\nabla[g(F+v) - g(u)]\|_{\frac{2d}{d+2}[I_1]} \leq C_{d,1}\epsilon(1)^{\alpha}$$

and

$$\|u - v\|_{\dot{W}(I_1)} \leq C_{d,1}\epsilon(1)^{\frac{7}{(d-2)^2}}$$

Now suppose that the claim holds for all $1 \leq j \leq k$ for some $k \leq J-1$. Then by (1.5.21) we have

$$\begin{aligned}\|u(t_k) - v(t_k)\|_{\dot{H}^1(\mathbb{R}^d)} \leq & A_d \sum_{k'=1}^{k} \|\nabla[g(F+v) - g(u)]\|_{2,\frac{2d}{d+2}[I_{k'}]} \\ \leq & A_d \sum_{k'=1}^{k} C_{d,1}\epsilon(k')^{\alpha} = \epsilon(k+1) < (2E)^{\frac{1}{2}}\end{aligned}$$

and so $\|v(t_k)\|_{\dot{H}^1(\mathbb{R}^d)} < 2(2E)^{\frac{1}{2}}$. Similarly using (1.5.22) we see that

$$\begin{aligned}\left(\sum_N \|P_N e^{i(t-t_k)\Delta}(v(t_k) - u(t_k))\|^2_{\dot{W}(I_{k+1})}\right)^{\frac{1}{2}} \leq & A_d \sum_{k'=1}^{k} \|\nabla[g(F+v) - g(u)]\|_{2,\frac{2d}{d+2}[I_{k'}]} \\ \leq & \epsilon(k+1) < \epsilon_0(2(2E)^{\frac{1}{2}}, d)\end{aligned}$$

Thus since also $\|F\|_{\dot{R}(I_{k+1})} \leq \epsilon_1 \leq \epsilon(k+1)$, we can apply Lemma 1.5.3 on I_{k+1} to obtain

$$\|u - v\|_{\dot{W}(I_{k+1})} \leq C_{d,1}\epsilon(k+1)^{\frac{7}{(d-2)^2}}$$

and

$$\|\nabla[g(F+v) - g(u)]\|_{2,\frac{2d}{d+2}[I_{k+1}]} \leq C_{d,1}\epsilon(k+1)^{\alpha}$$

This completes the proof of the claim. $\square$

We now sum the bounds over all the sub-intervals and use that $\alpha < \frac{7}{(d-2)^2}$ to obtain

$$\|u - v\|_{\dot{W}(I)} \leq C_{d,1} \sum_{j=1}^{J} \epsilon(j)^{\alpha} \leq \epsilon(J+1) < 1$$

□

Using the perturbation theory developed we are now able to prove the conditional scattering result.

Proof of Theorem 1.5.1. By the local wellposedness theory, it remains to prove that

$$\|v\|_{\dot{W}(I^*)} \leq C(M, \|F\|_{\dot{R}(\mathbb{R})}, d)$$

Consider first $[t_0, T_+)$. Partition $[t_0, T_+)$ into J consecutive intervals I_j such that

$$\|F\|_{\dot{R}(I_j)} \leq \epsilon_1(M, K(M), d)$$

where K is the non-decreasing function from Lemma 1.5.2 and ϵ_1 is the constant from Lemma 1.5.4. Due to the time divisibility of the $\dot{R}$-norm, we can do this with

$$J \lesssim \left(\frac{\|F\|_{\dot{R}(\mathbb{R})}}{\epsilon_1(M, K(M), d)} \right)^{d+2}$$

Denote $I_j = [t_{j-1}, t_j]$ for $1 \leq j \leq J$. On each I_j we compare v with the solution u_j to the usual defocusing NLS (1.5.3) with initial data $u(t_{j-1}) = v(t_{j-1})$, satisfying $E(u(t_{j-1})) \leq M$ by assumption. By Lemma 1.5.2 we know that such a solution u_j exists globally in time and satisfies

$$\|u_j\|_{\dot{W}(I_j)} \leq \|u_j\|_{\dot{W}(\mathbb{R})} \leq K(M)$$

We can therefore apply Lemma 1.5.4 on each I_j to see that v satisfies

$$\|v\|_{\dot{W}(I_j)} \leq \|v - u_j\|_{\dot{W}(I_j)} + \|u_j\|_{\dot{W}(I_j)} \leq K(M) + 1$$

Summing the estimates over the intervals I_j and arguing in the same way on $(T_-, t_0]$ implies the result. □

1.6 Energy bound

In this section we prove that the solution v to the forced NLS (1.5.1) does indeed satisfy a uniform in time energy bound on its maximal interval of existence $I^* = (T_-, T_+)$, under assumptions on F which we shall prove to hold almost surely for $F = e^{it\Delta} f^{\omega}$ in the next section. The precise result is the following.

Theorem 1.6.1. *Let $v_0 \in H^1(\mathbb{R}^d)$, $t_0 = 0$. Denote by $v \in C_t^0 H_x^1 \bigcap W(I^*)$ the unique solution to* (1.5.1) *obtained in Theorem 1.4.1, for a forcing term $F \in R \cap L_t^\infty L_x^{\frac{2d}{d-2}}(\mathbb{R})$ which solves the linear Schrödinger equation $(i\partial_t + \Delta)F = 0$ with L^2 initial data and satisfies*

$$F \in L_t^{\frac{1}{\sigma}} L_x^{\frac{2d}{d-4\sigma}}$$

$$\nabla F \in L_t^2 L_x^{\frac{4d-2}{2d-3-\sigma}} \bigcap L_t^2 L_x^{\frac{2d(2d-1)}{2d^2-7d+4+d\sigma}} \bigcap L_t^1 L_x^{\frac{2d}{d-4}} \bigcap L_t^{\frac{d-2}{d-2-4\sigma}} L_x^{\frac{2d(d-2)}{d(d-6)+16\sigma}}(\mathbb{R})$$

for some $\sigma(d)$ sufficiently small. Then it holds

$$\begin{aligned}\sup_{t\in I^*} E(v(t)) \leq{}& (1 + E(v_0) + \|F\|_{\infty,\frac{2d}{d-2}[\mathbb{R}]}^{\frac{2(d-2)}{d-6}} + \|F\|_{\infty,\frac{2d}{d-2}[\mathbb{R}]}^{\frac{2(d+2)}{d-2}}) \\ &\cdot \exp(C_d(\|F\|_{\frac{1}{\sigma},\frac{2d}{d-4\sigma}[\mathbb{R}]}^{\frac{1}{\sigma}} + \|\nabla F\|_{2,\frac{4d-2}{2d-3-\sigma}[\mathbb{R}]}^2 + \|\nabla F\|_{2,\frac{2d(2d-1)}{2d^2-7d+4+d\sigma}[\mathbb{R}]}^2 \\ &+ \|\nabla F\|_{1,\frac{2d}{d-4}[\mathbb{R}]} + \|\nabla F\|_{\frac{d-2}{d-2-4\sigma},\frac{2d(d-2)}{d(d-6)+16\sigma}[\mathbb{R}]}^{\frac{d-2}{d-2-4\sigma}}))\end{aligned} \tag{1.6.1}$$

for some $C_d > 0$.

This will follow from an analogous theorem for the regularised solutions v_n to

$$\begin{cases} (i\partial_t + \Delta)v_n = g_n(F_n + v_n) \\ v_n(0) = v_{n,0} \in H^2(\mathbb{R}^d) \end{cases} \tag{1.6.2}$$

with $g_n(u) = u\varphi_n'(|u|^2)$, $F_n = P_{\leq n}F$ and $v_{n,0} = P_{\leq n}v_0$, as in Section 1.1.3.

Theorem 1.6.2. *Suppose that F satisfies the assumptions of Theorem 1.6.1. Let v_n be the unique global solution to* (1.6.2) *in $C(\mathbb{R}, H^2(\mathbb{R}^d)) \bigcap C^1(\mathbb{R}, L^2(\mathbb{R}^d))$. Then it holds*

$$\begin{aligned}\sup_{t\in\mathbb{R}} E_n(v_n(t)) \leq{}& (1 + E(v_{n,0}) + \|F\|_{\infty,\frac{2d}{d-2}[\mathbb{R}]}^{\frac{2(d-2)}{d-6}} + \|F\|_{\infty,\frac{2d}{d-2}[\mathbb{R}]}^{\frac{2(d+2)}{d-2}}) \\ &\cdot \exp(C_d(\|F\|_{\frac{1}{\sigma},\frac{2d}{d-4\sigma}[\mathbb{R}]}^{\frac{1}{\sigma}} + \|\nabla F\|_{2,\frac{4d-2}{2d-3-\sigma}[\mathbb{R}]}^2 + \|\nabla F\|_{2,\frac{2d(2d-1)}{2d^2-7d+4+d\sigma}[\mathbb{R}]}^2 \\ &+ \|\nabla F\|_{1,\frac{2d}{d-4}[\mathbb{R}]} + \|\nabla F\|_{\frac{d-2}{d-2-4\sigma},\frac{2d(d-2)}{d(d-6)+16\sigma}[\mathbb{R}]}^{\frac{d-2}{d-2-4\sigma}}))\end{aligned} \tag{1.6.3}$$

for some $C_d > 0$. Here

$$E_n(v_n(t)) := \frac{1}{2}\int_{\mathbb{R}^d} |\nabla v_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^d} \varphi_n(|v_n|^2) dx$$

Before proving this theorem, we show how it can be used to deduce Theorem 1.6.1.

Proof of Theorem 1.6.1 given Theorem 1.6.2. Fix a compact subinterval $0 \in I \subset I^*$. Observe that $E(v_{n,0}) \to E(v_0)$. Therefore, denoting M_n the quantity on the right hand side of inequality (1.6.3) and M the right hand side of (1.6.1), we have $M_n \to M$.

Consider the sequence $v_{n,n} := v_n \mathbb{1}_{|v_n| \le n}$. For every $t \in I$ we have

$$\|v_{n,n}(t)\|_{L^{p+1}(\mathbb{R}^d)}^{p+1} = \int_{|v_n| \le n} |v_n(t)|^{p+1} dx \le \frac{p+1}{2} \int_{\mathbb{R}^d} \varphi_n(|v_n(t)|^2) dx \le (p+1) \sup_{n'} M_{n'}$$

and so for each $t \in I$ there exists a subsequence $(v_{n_j,n_j})_j$ weakly converging to v in $L^{p+1}(\mathbb{R}^d)$, from which we deduce

$$\|v(t)\|_{L^{p+1}(\mathbb{R}^d)}^{p+1} \le \liminf_j \frac{p+1}{2} \int_{\mathbb{R}^d} \varphi_{n_j}(|v_{n_j}(t)|^2) dx$$

for every $t \in I$. Similarly, we have $\|v(t)\|_{\dot{H}^1(\mathbb{R}^d)}^2 \le \liminf_j \|v_{n_j}\|_{\dot{H}^1}^2$, so

$$E(v(t)) \le \liminf_j E_{n_j}(v_{n_j}(t)) \le \liminf_j M_{n_j} = M$$

□

We now prove Theorem 1.6.2. The idea of the proof is to work on small intervals on which F is small and use a bootstrap argument to control the energy increment there. By only placing F into spaces with finite time exponents we are able to iterate this finitely many times to obtain a bound over the whole interval. In early papers on this topic [KMV19, DLM19, DLM20], a double bootstrap method was used, simultaneously controlling the solution in weighted L^p spaces via Morawetz inequalities. Thanks to a randomised L_t^1-estimate introduced by Spitz in [Spi21] (see Section 1.7.2), this is not necessary here and we can directly bound the energy increment by placing F into spaces of low time-integrability.

Proof of Theorem 1.6.2. In this proof we will often use the notation p instead of $\frac{d+2}{d-2}$, so any implicit constants depending on p in fact depend only on d. We will show the bound holds on the compact interval $[0,T]$ for any $T > 0$. Since the bound does not depend on T, this clearly extends to $[0,\infty)$, and the argument in the reverse time direction is analogous.

Define the norm

$$\|F\|_Z := \|F\|_{\frac{1}{\sigma}, \frac{2d}{d-4\sigma}} + \|\nabla F\|_{2, \frac{4d-2}{2d-3-\sigma}} + \|\nabla F\|_{2, \frac{2d(2d-1)}{2d^2-7d+4+d\sigma}} + \|\nabla F\|_{1, \frac{2d}{d-4}} + \|\nabla F\|_{\frac{d-2}{d-2-4\sigma}, \frac{2d(d-2)}{d(d-6)+16\sigma}}$$

on any time interval. Partition $[0,T]$ into J consecutive subintervals $I_j := [t_{j-1}, t_j]$, $j = 1, \dots, J$ such that

$$\|F\|_{Z[I_j]} \le \eta$$

for each $j = 1, \dots, J$ and some $\eta < 1$ to be determined which depends only on the dimension d. Note that by the time-divisibility properties of the Z norm, it is possible to do this with

$$J \lesssim_d \|F\|^{\frac{1}{\sigma}}_{\frac{1}{\sigma},\frac{2d}{d-4\sigma}[\mathbb{R}]} + \|\nabla F\|^2_{2,\frac{4d-2}{2d-3-\sigma}[\mathbb{R}]} + \|\nabla F\|^2_{2,\frac{2d(2d-1)}{2d^2-7d+4+d\sigma}[\mathbb{R}]} + \|\nabla F\|_{1,\frac{2d}{d-4}[\mathbb{R}]} + \|\nabla F\|^{\frac{d-2}{d-2-4\sigma}}_{\frac{d-2}{d-2-4\sigma},\frac{2d(d-2)}{d(d-6)+16\sigma}[\mathbb{R}]} \tag{1.6.4}$$

For each $j = 1, \dots, J$ define

$$A_{t_{j-1}}(t_j) := 1 + \sup_{t\in[t_{j-1},t_j]} E_n(v_n(t))$$

In the following calculations all spacetime norms above are taken over $[t_{j-1}, t] \times \mathbb{R}^d$. Since $v_n \in C^1(\mathbb{R}, L^2(\mathbb{R}^d)) \bigcap C(\mathbb{R}, H^2(\mathbb{R}^d))$ one may differentiate $E_n(v_n(t))$ to obtain

$$\partial_t E_n(v_n(t)) = -\mathrm{Re} \int_{\mathbb{R}^d} \partial_t \bar{v}_n(\Delta v_n - v_n\varphi_n'(|v_n|^2))dx$$

which is well-defined since $\partial_t v_n,\ \Delta v_n \in L^2(\mathbb{R}^d)$. Integrating this over $[t_{j-1}, t]$ and performing a calculation similar to that in [KMV19, DLM19] (see Appendix 1.A for details), we obtain

$$\begin{aligned}
&|E_n(v_n(t)) - E_n(v_n(t_{j-1}))| \\
&\le \frac{1}{2} \sup_{[t_{j-1},t]} \|\varphi_n(|F_n + v_n|^2) - \varphi_n(|v_n|^2) - \varphi_n(|F_n|^2)\|_{L^1(\mathbb{R}^d)} && (1.6.5)\\
&\quad + \|\nabla\overline{F_n} \cdot \nabla(g_n(F_n + v_n) - g_n(F_n))\|_{1,1} && (1.6.6)
\end{aligned}$$

First consider (1.6.5). Observe that

$$\Big|\varphi_n(|F_n + v_n|^2) - \varphi_n(|v_n|^2) - \varphi_n(|F_n|^2)\Big| \lesssim_p |F_n|^p|v_n| + |F_n||v_n|^p$$

uniformly in n. Hence by Young's inequality we have

$$\begin{aligned}
&\Big\|\varphi_n(|F_n + v_n|^2) - \varphi_n(|v_n|^2) - \varphi_n(|F_n|^2)\Big\|_{\infty,1} \\
&\lesssim_d \delta\|v_n\|^2_{\infty,p+1} + C_{\delta,d}\|F_n\|^{\frac{2}{2-p}}_{\infty,p+1} + C_{\delta,d}\|F_n\|^{2p}_{\infty,p+1} \\
&\lesssim_d \delta A_{t_{j-1}}(t_j) + C_{\delta,d}\|F_n\|^{\frac{2}{2-p}}_{\infty,p+1} + C_{\delta,d}\|F_n\|^{2p}_{\infty,p+1}
\end{aligned}$$

for any $\delta > 0$, since $\|v_n\|^2_{\infty,p+1} \lesssim \|v_n\|^2_{L^\infty_t \dot{H}^1_x}$.

We now turn to (1.6.6). This time we use (1.2.5) to bound

$$(1.6.6) \lesssim_d \||v_n|^{p-1}|\nabla F_n|^2\|_{1,1} + \||v_n|^{p-1}|\nabla v_n||\nabla F_n|\|_{1,1} + \||F_n|^{p-1}|\nabla F_n||\nabla v_n|\|_{1,1}$$

We can control these terms as follows, using that $A_{t_{j-1}}(t_j) \geq 1$:

$$\begin{aligned}
\||v_n|^{p-1}|\nabla F_n|^2\|_{1,1} \leq &\|v_n\|_{\infty,p+1}^{p-1}\|\nabla F_n\|_{2,\frac{4d-2}{2d-3-\sigma}}\|\nabla F_n\|_{2,\frac{2d(2d-1)}{2d^2-7d+4+d\sigma}} \\
\lesssim_d &A_{t_{j-1}}(t_j)\|\nabla F_n\|_{2,\frac{4d-2}{2d-3-\sigma}}\|\nabla F_n\|_{2,\frac{2d(2d-1)}{2d^2-7d+4+d\sigma}} \\
\||v_n|^{p-1}|\nabla v_n||\nabla F_n|\|_{1,1} \leq &\|v_n\|_{\infty,p+1}^{p-1}\|\nabla v_n\|_{\infty,2}\|\nabla F_n\|_{1,\frac{2d}{d-4}} \\
\lesssim_d &A_{t_{j-1}}(t_j)\|\nabla F_n\|_{1,\frac{2d}{d-4}} \\
\||F_n|^{p-1}|\nabla F_n||\nabla v_n|\|_{1,1} \leq &\|\nabla v_n\|_{\infty,2}\|F_n\|_{\frac{1}{\sigma},\frac{2d}{d-4\sigma}}^{p-1}\|\nabla F_n\|_{\frac{d-2}{d-2-4\sigma},\frac{2d(d-2)}{d(d-6)+16\sigma}} \\
\leq &A_{t_{j-1}}(t_j)\|F_n\|_{\frac{1}{\sigma},\frac{2d}{d-4\sigma}}^{p-1}\|\nabla F_n\|_{\frac{d-2}{d-2-4\sigma},\frac{2d(d-2)}{d(d-6)+16\sigma}}
\end{aligned}$$

Noting that the spaces into which F_n has been placed here are exactly those which make up the Z-norm, we can bound each term by $C_d A_{t_{j-1}}(t_j)\eta$, and so by (1.6.5)-(1.6.6) it holds

$$\begin{aligned}
A_{t_{j-1}}(t_j) \lesssim_d 1 + E_n(v_n(t_{j-1})) + \delta A_{t_{j-1}}(t_j) + C_{\delta,d}\|F_n\|_{\infty,p+1[t_{j-1},t_j]}^{\frac{2}{2-p}} \\
+ C_{\delta,d}\|F_n\|_{\infty,p+1[t_{j-1},t_j]}^{2p} + \eta A_{t_{j-1}}(t_j)
\end{aligned}$$

Choosing $\delta(d)$ and $\eta(d)$ sufficiently small and using that $\|F_n\|_{a,b} \lesssim \|F\|_{a,b}$ for $1 \leq a, b \leq \infty$ (and likewise for ∇F_n), we thus have

$$A_{t_{j-1}}(t_j) \leq C_d^*(1 + E_n(v_n(t_{j-1})) + \|F\|_{\infty,p+1[\mathbb{R}]}^{\frac{2}{2-p}} + \|F\|_{\infty,p+1[\mathbb{R}]}^{2p})$$

for some constant $C_d^* > 1$.

Iterating the results on the consecutive intervals $(I_j)_{j=1}^J$, we obtain

$$A_{t_{j-1}}(t_j) \leq (2C_d^*)^j(1 + E_n(v_n(0)) + \|F\|_{\infty,p+1[\mathbb{R}]}^{\frac{2}{2-p}} + \|F\|_{\infty,p+1[\mathbb{R}]}^{2p})$$

for all $j = 1, \ldots, J$, from which

$$A_0(T) \leq (2C_d^*)^J(1 + E(v_{n,0}) + \|F\|_{\infty,p+1[\mathbb{R}]}^{\frac{2}{2-p}} + \|F\|_{\infty,p+1[\mathbb{R}]}^{2p})$$

where we used that $E_n(v_{n,0}) \leq E(v_{n,0})$. Combining this with (1.6.4) yields the result. $\square$

1.7 Almost sure bounds for the forcing term

In this section we show that the randomised linear evolution $F^\omega := e^{it\Delta}f^\omega$ almost surely satisfies the conditions required for wellposedness and scattering provided the initial data f lies in a Sobolev space of sufficiently high regularity. In particular, we prove the following theorem:

Theorem 1.7.1. *Let* $\max\{\frac{4d-1}{3(2d-1)}, \frac{d^2+6d-4}{(2d-1)(d+2)}\} < s < 1$ *and* $f \in H^s(\mathbb{R}^d)$*. Let* f^ω *denote the randomisation of* f *as in* (1.1.11) *and* $F^\omega := e^{it\Delta} f^\omega$*. Then when* $\sigma(d)$ *is sufficiently small we have*

$$F^\omega \in L_t^\infty L_x^{\frac{2d}{d-2}} \bigcap L_t^{\frac{1}{\sigma}} L_x^{\frac{2d}{d-4\sigma}} \bigcap R(\mathbb{R})$$

$$\nabla F^\omega \in L_t^2 L_x^{\frac{4d-2}{2d-3-\sigma}} \bigcap L_t^2 L_x^{\frac{2d(2d-1)}{2d^2-7d+4+d\sigma}} \bigcap L_t^1 L_x^{\frac{2d}{d-4}} \bigcap L_t^{\frac{d-2}{d-2-4\sigma}} L_x^{\frac{2d(d-2)}{d(d-6)+16\sigma}}(\mathbb{R})$$

for almost every $\omega \in \Omega$.

When combined with the results of the previous section, this completes the proof of Theorem 1.1.1.

The proof of the bounds in the above theorem is split into subsections according to the method used to obtain the almost sure bound. Throughout, we shall make repeated use of the following important generalisation of Khintchine's inequality due to Burq and Tzvetkov [BT08], formulated here as in [BT08].

Lemma 1.7.2. *(Large Deviation Estimate, Lemma 3.1 [BT08]) Let* $(g_k)_{k\in\mathbb{N}}$ *be a sequence of independent, real-valued, zero-mean random variables on a probability space* $(\Omega, \mathcal{A}, \mathbb{P})$ *with distributions* $(\mu_k)_k$ *satisfying*

$$\int_{\mathbb{R}} e^{\gamma x} d\mu_k(x) \leq e^{c\gamma^2} \qquad \forall \gamma \in \mathbb{R}$$

with the constant $c > 0$ *independent of* k, γ*. Then there is a constant* $C > 0$ *such that*

$$\left\| \sum_{k\in\mathbb{N}} c_k g_k \right\|_{L^\beta(\Omega)} \leq C\sqrt{\beta} \left(\sum_{k\in\mathbb{N}} |c_k|^2 \right)^{\frac{1}{2}}$$

for all $(c_k)_k \in \ell^2(\mathbb{N})$ *and* $\beta \in [2, \infty)$*.*

1.7.1 Bounds using randomisation-improved Strichartz.

In this section we will prove that under the conditions of Theorem 1.7.1, we have

$$F^\omega \in L_t^\infty L_x^{\frac{2d}{d-2}} \bigcap L_t^{\frac{1}{\sigma}} L_x^{\frac{2d}{d-4\sigma}} \bigcap R(\mathbb{R}) \tag{1.7.1}$$

and

$$\nabla F^\omega \in L_t^2 L_x^{\frac{4d-2}{2d-3-\sigma}} \bigcap L_t^2 L_x^{\frac{2d(2d-1)}{2d^2-7d+4+d\sigma}} \bigcap L_t^2 L_x^{\frac{2d(d-2)}{d(d-6)+16\sigma}}(\mathbb{R}) \tag{1.7.2}$$

almost surely. Note that it is not claimed in the Theorem that ∇F^ω lies in the final space listed above, but we need it in order to deduce one of the other bounds by interpolation.

These results rely on the following proposition allowing us to gain derivatives on the randomised free evolution, adapted from [Spi21]. Throughout this section, f^ω always refers to the randomisation of f as in (1.1.11) and all spacetime norms are over $\mathbb{R} \times \mathbb{R}^d$.

The key estimate for this section then reads as follows.

Proposition 1.7.3 (See Proposition 3.4(ii), [Spi21]). *Let $(q, p_0) \in [2, \infty)$ satisfy*

$$\frac{1}{q} \leq \left(d - \frac{1}{2}\right)\left(\frac{1}{2} - \frac{1}{p_0}\right) \qquad \text{and} \qquad (q, p_0) \neq \left(2, \frac{4d-2}{2d-3}\right) \tag{1.7.3}$$

Let $p \in [p_0, \infty)$. Then for any $f \in H^s(\mathbb{R}^d)$ with $s \geq 0$, it holds

$$\|e^{it\Delta} f^\omega\|_{L^\beta_\omega \dot{B}^{s+\frac{2}{q}+\frac{d}{p_0}-\frac{d}{2}}_{q,p,2}} \lesssim_{d,q,p,p_0} \sqrt{\beta}\|f\|_{H^s(\mathbb{R}^d)}$$

for all $\beta \in [1, \infty)$.

Observe that the maximum derivative gain by this estimate occurs at the non-allowed endpoint $(2, \frac{4d-2}{2d-3})$, where we would gain

$$\frac{2}{q} + \frac{d}{p_0} - \frac{d}{2} = \frac{d-1}{2d-1}$$

derivatives.

Since the proof of Proposition 1.7.3 is very similar to the $d = 4$ case in [Spi21], we omit it here, however we remark that it relies crucially on a Strichartz estimate in radially averaged spaces due to Guo [Guo16].

The bounds on F^ω without any derivatives are then implied by the following corollary:

Corollary 1.7.4. *Let $q \in [2, \infty)$, $p \in [2, \infty)$ satisfy*

$$\frac{2}{q} + \frac{d}{p} \leq \frac{d}{2} \tag{1.7.4}$$

Let $f \in L^2(\mathbb{R}^d)$, f^ω its randomisation. Then for almost every $\omega \in \Omega$ it holds

$$\|e^{it\Delta} f^\omega\|_{L^q_t L^p_x} < \infty$$

Proof. Let $\beta \geq 2$. By the Littlewood-Paley inequality we have

$$\|e^{it\Delta} f^\omega\|_{L^\beta_\omega L^q_t L^p_x} \lesssim_d \left\| \left(\sum_N \|P_N e^{it\Delta} f^\omega\|_{L^q_t L^p_x} \right)^{\frac{1}{2}} \right\|_{L^\beta_\omega} = \|e^{it\Delta} f^\omega\|_{L^\beta_\omega \dot{B}^0_{q,p,2}}$$

Since (q, p) satisfy (1.7.4), there exists $2 \leq p_0 \leq p$ such that (q, p_0) is a Strichartz pair, i.e.

$$\frac{2}{q} + \frac{d}{p_0} - \frac{d}{2} = 0$$

Since every Strichartz pair satisfies (1.7.3), we are able to immediately apply Proposition 1.7.3 with $s = 0$ to obtain the result. $\square$

It is then immediate that, for $\sigma(d)$ sufficiently small, $F^\omega \in L_t^{\frac{1}{\sigma}} L_x^{\frac{2d}{d-4\sigma}}$ almost surely.

To show that $F^\omega \in L_t^\infty L_x^{\frac{2d}{d-2}}(\mathbb{R})$ for almost every ω requires an endpoint case of Proposition 1.7.3 allowing for $q = \infty$. We prove this as in [KMV19].

Lemma 1.7.5. *Let $s \geq \frac{d-2}{2d-1}$, $f \in H^s(\mathbb{R}^d)$, f^ω its randomisation. Then for almost every $\omega \in \Omega$ it holds*

$$e^{it\Delta} f^\omega \in L_t^\infty L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)$$

Proof. Let $I \subset \mathbb{R}$ with $|I| = \delta$ to be determined. Let t_0, $t \in I$. Then for any $N \in 2^{\mathbb{Z}}$ we have

$$\|P_N e^{it\Delta} f^\omega\|_{L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)} \leq \|P_N e^{it\Delta} f^\omega(t_0)\|_{L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)} + \|\partial_t P_N e^{it\Delta} f^\omega\|_{L_t^1 L_x^{\frac{2d}{d-2}}(I)}$$

and averaging this over $t_0 \in I$ we find

$$\begin{aligned}
\|P_N e^{it\Delta} f^\omega\|_{L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)} \lesssim & \delta^{-1} \|P_N e^{it\Delta} f^\omega\|_{1,\frac{2d}{d-2}[I]} + \|\partial_t P_N e^{it\Delta} f^\omega\|_{1,\frac{2d}{d-2}[I]} \\
\lesssim & \delta^{-\frac{d-2}{2d}} \|P_N e^{it\Delta} f^\omega\|_{\frac{2d}{d-2},\frac{2d}{d-2}[\mathbb{R}]} + \delta^{\frac{d+2}{2d}} \|\partial_t P_N e^{it\Delta} f^\omega\|_{\frac{2d}{d-2},\frac{2d}{d-2}[\mathbb{R}]} \\
\lesssim & \delta^{-\frac{d-2}{2d}} \|P_N e^{it\Delta} f^\omega\|_{\frac{2d}{d-2},\frac{2d}{d-2}[\mathbb{R}]} + \delta^{\frac{d+2}{2d}} N^2 \|P_N e^{it\Delta} f^\omega\|_{\frac{2d}{d-2},\frac{2d}{d-2}[\mathbb{R}]} \\
\lesssim & N^{\frac{d-2}{d}} \|P_N e^{it\Delta} f^\omega\|_{\frac{2d}{d-2},\frac{2d}{d-2}[\mathbb{R}]}
\end{aligned}$$

choosing $\delta = N^{-2}$ in the last line.

Averaging over ω we thus obtain, via the Littlewood-Paley inequality,

$$\begin{aligned}
\|e^{it\Delta} f^\omega\|_{L_\omega^\beta L_t^\infty L_x^{\frac{2d}{d-2}}(\mathbb{R})} \lesssim & \left\| \left(\sum_N \|P_N e^{it\Delta} f^\omega\|^2_{\infty,\frac{2d}{d-2}[\mathbb{R}]} \right)^{\frac{1}{2}} \right\|_{L_\omega^\beta} \\
\lesssim & \left\| \left(\sum_N (N^{\frac{d-2}{d}} \|P_N e^{it\Delta} f^\omega\|_{\frac{2d}{d-2},\frac{2d}{d-2}})^2 \right)^{\frac{1}{2}} \right\|_{L_\omega^\beta} \\
= & \|e^{it\Delta} f^\omega\|_{L_\omega^\beta \dot{B}^{\frac{d-2}{d}}_{\frac{2d}{d-2},\frac{2d}{d-2},2}}
\end{aligned}$$

Applying Proposition 1.7.3 with $s = \frac{d-2}{2d-1}$, $q = p = \frac{2d}{d-2}$ and $p_0 = \frac{2d(2d-1)}{2d^2-3d+4}$ we see that this is bounded by $\sqrt{\beta}\|f\|_{H^s}$. $\square$

For the remaining bounds of (1.7.1)-(1.7.2), we prove another corollary of Proposition 1.7.3 to handle the terms involving ∇F^ω.

Corollary 1.7.6. *Let $q, p \in [2, \infty)$ satisfy (1.7.3), $s > 1 - \left(\frac{d-1}{2d-1}\right)\frac{2}{q} \in (0, 1)$. Then for* *any $f \in H^s(\mathbb{R}^d)$, f^ω its randomisation, we have*

$$\|\nabla e^{it\Delta} f^\omega\|_{L_t^q L_x^p} < \infty$$

for almost every $\omega \in \Omega$.

Proof. Setting $p_0 = \left(\frac{1}{2} - \frac{2}{2d-1} \cdot \frac{1}{q}\right)^{-1} \in (2, \infty)$ for $q \neq 2$, and $p_0 = \frac{4d-2}{2d-3-\delta}$ for $q = 2$, where δ is a small constant, the pair (q, p_0) satisfies (1.7.3) and

$$\frac{2}{q} + \frac{d}{p_0} - \frac{d}{2} = \begin{cases} \left(\frac{d-1}{2d-1}\right) \frac{2}{q} \text{ for } q \neq 2 \\ \frac{d-1}{2d-1} - \frac{\delta}{2} \frac{d}{2d-1} \text{ for } q = 2 \end{cases}$$

Applying Proposition 1.7.3 in combination with the Littlewood-Paley inequality for these parameters yields the result, provided $\delta(d, p, s)$ is sufficiently small. □

Applying this corollary with $q = 2$ we immediately obtain that

$$\nabla F^\omega \in L_t^2 L_x^{\frac{4d-2}{2d-3-\sigma}} \bigcap L_t^2 L_x^{\frac{2d(2d-1)}{2d^2-7d+4+d\sigma}} \bigcap L_t^2 L_x^{\frac{2d(d-2)}{d(d-6)+16\sigma}}(\mathbb{R})$$

completing the proof of (1.7.1)-(1.7.2) since $s > 1 - \left(\frac{d-1}{2d-1}\right)$.

To conclude this section, we show that $\|F^\omega\|_{R(\mathbb{R})} < \infty$ almost everywhere.

Proposition 1.7.7. *Let $s > \frac{d^2+6d-4}{(2d-1)(d+2)}$. Then for almost every $\omega \in \Omega$ we have*

$$F^\omega \in R(\mathbb{R})$$

Proof. Recall

$$\|F^\omega\|_{R(\mathbb{R})} := \|F^\omega\|_{\frac{2(d+2)}{d-2}, \frac{2d(d+2)}{d^2+4}}(\mathbb{R}) + \|\nabla F^\omega\|_{\frac{2(d+2)}{d-2}, \frac{2d(d+2)}{d^2+4}}(\mathbb{R}) + \|F^\omega\|_{\dot{B}^{\frac{4}{d+2}}_{d+2, \frac{2(d+2)}{d}}}(\mathbb{R})$$

For the first terms, we apply Corollaries 1.7.4 and 1.7.6 with $q = \frac{2(d+2)}{d-2}$, $p = \frac{2d(d+2)}{d^2+4}$, which forces the lower bound

$$s > \frac{d^2 + 6d - 4}{(2d-1)(d+2)}$$

For the final term, apply Proposition 1.7.3 with $q = d + 2$, $p = \frac{2(d+2)}{d}$ and $p_0 = \frac{2(2d-1)(d+2)}{2d^2+3d-6} \in [2, p]$ to obtain

$$\|F^\omega\|_{L_\omega^\beta \dot{B}^{\frac{4}{d+2}}_{d+2, \frac{2(d+2)}{d}, 2}} \lesssim \sqrt{\beta} \|f\|_{H^s(\mathbb{R}^d)}$$

for any $s > \frac{2(3d-1)}{(d+2)(2d-1)}$ and $\beta \geq 1$. □

1.7.2 Randomised L_t^1 estimate

In this section we will prove that, under the conditions of Theorem 1.7.1, we have

$$\nabla F^\omega \in L_t^1 L_x^{\frac{2d}{d-4}} \bigcap L_t^{\frac{d-2}{d-2-4\sigma}} L_x^{\frac{2d(d-2)}{d(d-6)+16\sigma}}(\mathbb{R}) \tag{1.7.5}$$

almost surely. Since we have already proved that $\nabla F^\omega \in L_t^2 L_x^{\frac{2d(d-2)}{d(d-6)+16\sigma}}(\mathbb{R})$ almost surely, for the third bound it is sufficient to prove that

$$\nabla F^\omega \in L_t^1 L_x^{\frac{2d(d-2)}{d(d-6)+16\sigma}}(\mathbb{R}) \tag{1.7.6}$$

We thus only need to find estimates in Lebesgue spaces with time exponent 1. Key to such bounds are the following propositions which are generalisations of results of Spitz [Spi21] to high dimensions. The proofs are the same as in the dimension 4 case so we do not present them here, however we remark that it is for these results that the physical space part of the randomisation of f is necessary.

The first result exploits the decay properties of the Schrödinger semi-group to achieve bounds in spaces with low time integrability away from $t = 0$:

Proposition 1.7.8 (Proposition 3.6, [Spi21])**.** *Let $s \geq 0$ and consider $q \in [1, \infty)$, $p \in [2, \infty)$ $\sigma \geq 0$ such that*

$$\sigma < \frac{d}{2} - \frac{1}{q} - \frac{d}{p}$$

Let $f \in H^s(\mathbb{R}^d)$ and f^ω be its randomisation as in (1.1.11)*. Then it holds*

$$\|t^\sigma e^{it\Delta} f^\omega\|_{L_\omega^\beta L_t^q \dot{B}^s_{p,2}([1,\infty))} \lesssim_{d,q,p,\sigma} \sqrt{\beta} \|f\|_{H^s(\mathbb{R}^d)}$$

for all $\beta \geq 1$.

The gain in derivatives needed for (1.7.5) is obtained by interpolating this with the improved Strichartz estimate of Proposition 1.7.3 to obtain the following:

Proposition 1.7.9 (Proposition 3.7, [Spi21])**.** *Let $s > \frac{d+1}{2d-1}$. Then for any $f \in H^s(\mathbb{R}^d)$, f^ω its randomisation, it holds*

$$\|\nabla e^{it\Delta} f^\omega\|_{L_\omega^\beta L_t^1 L_x^\infty(\mathbb{R})} \lesssim_{s,d} \sqrt{\beta} \|f\|_{H^s(\mathbb{R}^d)}$$

for all $\beta \geq 1$.

To prove (1.7.5), we need a more general version of this proposition allowing for a larger range of exponents in the x-variable. The proof is a modification of the proof in [Spi21] of the previous result.

Proposition 1.7.10. *Let $s > \frac{4d-1}{3(2d-1)}$, $\beta \geq 1$. Then for any $f \in H^s(\mathbb{R}^d)$, f^ω its* *randomisation, it holds*

$$\|\nabla e^{it\Delta} f^\omega\|_{L^\beta_\omega L^1_t L^r_x(\mathbb{R})} \lesssim_{d,s,r} \sqrt{\beta}\|f\|_{H^s(\mathbb{R}^d)} \tag{1.7.7}$$

for any $\frac{2d}{d-4} \leq r \leq \infty$.

Proof. We will prove the case $r = \frac{2d}{d-4}$. This is sufficient by interpolation with Proposition 1.7.9.

Observe that we may decompose the left hand side of (1.7.7) as

$$\begin{aligned}\|\nabla e^{it\Delta} f^\omega\|_{L^\beta_\omega L^1_t L_x^{\frac{2d}{d-4}}(\mathbb{R})} &\lesssim \|e^{it\Delta} f^\omega\|_{L^\beta_\omega L^1_t \dot{B}^1_{\frac{2d}{d-4},2}(\mathbb{R})} \\ \lesssim &\|e^{it\Delta} f^\omega\|_{L^\beta_\omega L^1_t \dot{B}^1_{\frac{2d}{d-4},2}(-1,1)} + \|e^{it\Delta} f^\omega\|_{L^\beta_\omega L^1_t \dot{B}^1_{\frac{2d}{d-4},2}(-\infty,-1]} + \|e^{it\Delta} f^\omega\|_{L^\beta_\omega L^1_t \dot{B}^1_{\frac{2d}{d-4},2}[1,\infty)}\end{aligned}$$

We first consider the term over $(-1,1)$. By Hölder's inequality we have

$$\|e^{it\Delta} f^\omega\|_{L^\beta_\omega L^1_t \dot{B}^1_{\frac{2d}{d-4},2}(-1,1)} \lesssim_d \|e^{it\Delta} f^\omega\|_{L^\beta_\omega L^2_t \dot{B}^1_{\frac{2d}{d-4},2}(-1,1)} \lesssim_d \|e^{it\Delta} f^\omega\|_{L^\beta_\omega \dot{B}^{\nu(\delta)+\beta(\delta)}_{2,\frac{2d}{d-4},2}(-1,1)}$$

where

$$\beta(\delta) = \frac{d-1}{2d-1} - \frac{\delta}{2}\frac{d}{2d-1} \qquad \text{and} \qquad \nu(\delta) = \frac{d}{2d-1} + \frac{\delta}{2}\frac{d}{2d-1}$$

for some $0 < \delta(d,s) \ll 1$ to be determined.

We may now apply Proposition 1.7.3 with $q = 2$, $p_0 = \frac{4d-2}{2d-3-\delta}$, $p = \frac{2d}{d-4}$ to obtain

$$\|e^{it\Delta} f^\omega\|_{L^\beta_\omega L^1_t \dot{B}^1_{\frac{2d}{d-4},2}(-1,1)} \lesssim_d \sqrt{\beta}\|f\|_{H^{\nu(\delta)}} \lesssim_d \sqrt{\beta}\|f\|_{H^s}$$

for $\delta(s,d)$ sufficiently small.

Next consider the term over $[1,\infty)$. By Hölder's inequality for sequences we have

$$\|e^{it\Delta} f^\omega\|_{\dot{B}^1_{\frac{2d}{d-4},2}} = \left(\sum_N N^2 \|P_N e^{it\Delta} f^\omega\|^2_{\frac{2d}{d-4}}\right)^{\frac{1}{2}} \leq \|e^{it\Delta} f^\omega\|^\alpha_{\dot{B}^{1+\gamma}_{\frac{2d}{d-4},2}} \|e^{it\Delta} f^\omega\|^{1-\alpha}_{\dot{B}^{1-\frac{\alpha\gamma}{1-\alpha}}_{\frac{2d}{d-4},2}}$$

for $\alpha \in [0,1)$, $\gamma \in [0, \frac{1-\alpha}{\alpha})$ to be determined.

Combining this with Hölder's inequality in time, we have

$$\|e^{it\Delta} f^\omega\|_{L^\beta_\omega L^1_t \dot{B}^1_{\frac{2d}{d-4},2}[1,\infty)} \lesssim_\delta \left\| t^{\frac{1+\delta}{2}} \|e^{it\Delta} f^\omega\|^\alpha_{\dot{B}^{1+\gamma}_{\frac{2d}{d-4},2}} \|e^{it\Delta} f^\omega\|^{1-\alpha}_{\dot{B}^{1-\frac{\alpha\gamma}{1-\alpha}}_{\frac{2d}{d-4},2}} \right\|_{L^\beta_\omega L^2_t[1,\infty)}$$

$$\lesssim_\delta \|e^{it\Delta} f^\omega\|^\alpha_{L^\beta_\omega L^2_t \dot{B}^{1+\gamma}_{\frac{2d}{d-4},2}[1,\infty)} \|t^{\frac{1+\delta}{2(1-\alpha)}} e^{it\Delta} f^\omega\|^{1-\alpha}_{L^\beta_\omega L^2_t \dot{B}^{1-\frac{\alpha\gamma}{1-\alpha}}_{\frac{2d}{d-4},2}[1,\infty)} \tag{1.7.8}$$

We will bound the first term of (1.7.8) using the randomisation-improved Strichartz estimate from Proposition 1.7.3, and the second term using Proposition 1.7.8. Fix

$$\alpha = \frac{2}{3} - \delta \qquad \text{and} \qquad \gamma = \frac{d-1}{3(2d-1)}$$

(chosen to optimise the gain in derivatives in what follows). Applying Proposition 1.7.3 with $q = 2$, $p_0 = \frac{4d-2}{2d-3-\delta}$ and $p = \frac{2d}{d-4}$ we obtain, for $\beta(\delta)$ and $\nu(\delta)$ as before,

$$\begin{aligned}
\|e^{it\Delta} f^\omega\|_{L^\beta_\omega L^2_t \dot{B}^{1+\gamma}_{\frac{2d}{d-4},2}[1,\infty)} &\lesssim_d \|e^{it\Delta} f^\omega\|_{L^\beta_\omega \dot{B}^{\gamma+\nu(\delta)+\beta(\delta)}_{2,\frac{2d}{d-4},2}[1,\infty)} \\
&\lesssim_{d,\delta} \sqrt{\beta}\|f\|_{H^{\gamma+\nu(\delta)}} \\
&\lesssim_{d,\delta} \sqrt{\beta}\|f\|_{H^s}
\end{aligned}$$

since $\gamma + \nu(\delta) = \frac{4d-1}{3(2d-1)} + \frac{\delta}{2}\frac{d}{2d-1} < s$ for δ sufficiently small.

For the second term we apply Proposition 1.7.8 with $q = 2$, $p = \frac{2d}{d-4}$, $\sigma = \frac{1+\delta}{2(1-\alpha)} = \frac{3+3\delta}{2+6\delta}$ to find

$$\|t^{\frac{1+\delta}{2(1-\alpha)}} e^{it\Delta} f^\omega\|_{L^\beta_\omega L^2_t \dot{B}^{1-\frac{\alpha\gamma}{1-\alpha}}_{\frac{2d}{d-4},2}[1,\infty)} \lesssim \sqrt{\beta}\|f\|_{H^s}$$

since $1 - \frac{\alpha\gamma}{1-\alpha} = \frac{4d-1}{3(2d-1)} + O(\delta) < s$ for δ sufficiently small. Returning to (1.7.8) we have

$$\|e^{it\Delta} f^\omega\|_{L^\beta_\omega L^1_t \dot{B}^1_{\frac{2d}{d-4},2}[1,\infty)} \lesssim_d \sqrt{\beta}\|f\|_{H^s}$$

Treating the term over $(-\infty, -1]$ in the same way we obtain the desired result. $\square$

The bounds (1.7.5) (via (1.7.6)) are now immediate, observing that $r = \frac{2d(d-2)}{d(d-6)+16\sigma}$ is greater than $\frac{2d}{d-4}$ for $\sigma(d)$ sufficiently small.

Appendix

1.A Calculation of energy increment (1.6.5)-(1.6.6)

Proposition 1.A.1. *Let $v_n \in C^1(\mathbb{R}, L^2) \cap C(\mathbb{R}, H^2)$ solve* (1.6.2) *for some F_n satisfying the conditions of Theorem 1.6.1. Then for any T_1, $T_2 \in \mathbb{R}$ it holds*

$$E_n(v_n(T_2)) - E_n(v_n(T_1)) = -\frac{1}{2}\left[\int_{\mathbb{R}^d} \varphi_n(|F_n + v_n|^2) - \varphi_n(|v_n|^2) - \varphi_n(|F_n|^2)dx\right]_{T_1}^{T_2}$$
$$- Im \int_{T_1}^{T_2} \int_{\mathbb{R}^d} \nabla \overline{F_n} \cdot \nabla(g_n(F_n + v_n) - g_n(F_n))dxdt$$

Before proving this proposition, we recall without proof the following useful fact:

Let X be a Banach space. Then any $f \in C^1(\mathbb{R}, X)$ is in fact Fréchet differentiable from $\mathbb{R}$ to X with Fréchet derivative $\partial_t f(t, \cdot)$ (see, for example, Section 1.3 [Caz03]).[4]

We will also use the following result to differentiate the nonlinearity:

Lemma 1.A.2. *Let $\psi \in C^2(\mathbb{C}, \mathbb{C})$ with bounded second derivatives. Suppose also $\psi(w) \in L^1(\mathbb{R}^d)$ for all $w \in L^2(\mathbb{R}^d)$. Then the map*

$$H : w \mapsto \int_{\mathbb{R}^d} \psi(w(x))dx$$

is Fréchet differentiable from $L^2(\mathbb{R}^d)$ to $\mathbb{R}$ with derivative

$$DH|_w(h) = \int_{\mathbb{R}^d} (h\partial_z \psi(w) + \bar{h}\partial_{\bar{z}}\psi(w))dx$$

Applying this lemma with $\psi(z) = \varphi_n(|z|^2)$ and using the chain rule we observe that for any $v \in C^1(\mathbb{R}, L^2(\mathbb{R}^d))$ it holds

$$\partial_t \int_{\mathbb{R}^d} \varphi_n(|v|^2)dx = 2\mathrm{Re} \int_{\mathbb{R}^d} \partial_t \overline{v}\ g_n(v)dx \tag{1.A.1}$$

[4] $C^1(\mathbb{R}, X)$ is defined anaologously to 1.1.15.

We can now prove Proposition 1.A.1.

Proof of Proposition 1.A.1. We split the energy into a kinetic and a potential term:

$$KE(v) := \frac{1}{2}\langle \nabla v, \nabla v\rangle_{L^2} \qquad \text{and} \qquad G_n(u) := \frac{1}{2}\int_{\mathbb{R}^d} \varphi_n(|u|^2)dx$$

The map KE is (Fréchet) differentiable from $H^1(\mathbb{R}^d)$ to $\mathbb{R}$, thus if we further suppose that $v \in C^1(\mathbb{R}, H^1)$ we have

$$\frac{d}{dt}KE(v(t)) = \mathrm{Re}\int_{\mathbb{R}^d} \overline{\nabla v}\nabla(\partial_t v)dx = -\mathrm{Re}\int_{\mathbb{R}^d} \partial_t \bar{v}\Delta v dx$$

and the same formula holds for $v_n \in C^1(\mathbb{R}, L^2(\mathbb{R}^d)) \cap C(\mathbb{R}, H^2(\mathbb{R}^d))$ by approximation.

Likewise, by (1.A.1), we have

$$\frac{d}{dt}G_n(v_n(t)) = \mathrm{Re}\int_{\mathbb{R}^d} \partial_t \overline{v_n} g_n(v_n)dx$$

Combining these results and using that v_n satisfies equation (1.6.2) we obtain

$$\begin{aligned}
E_n(v_n(T_2)) - E_n(v_n(T_1)) &= \mathrm{Re}\int_{T_1}^{T_2}\int_{\mathbb{R}^d} \partial_t\overline{v_n}[-\Delta v_n + g_n(v_n)]dxdt \\
&= -\,\mathrm{Re}\int_{T_1}^{T_2}\int_{\mathbb{R}^d} \partial_t\overline{v_n}[g_n(F_n+v_n) - g_n(v_n)]dxdt \\
&= -\,\mathrm{Re}\int_{T_1}^{T_2}\int_{\mathbb{R}^d} (\partial_t(\overline{F_n+v_n})g_n(F_n+v_n) - \partial_t\overline{v_n}g_n(v_n) - \partial_t\overline{F_n}g_n(F_n))dxdt \\
&\quad + \mathrm{Re}\int_{T_1}^{T_2}\int_{\mathbb{R}^d} \partial_t\overline{F_n}(g_n(F_n+v_n) - g_n(F_n))dxdt
\end{aligned}$$

It then remains to use (1.A.1) to rewrite the first integral above, and $\partial_t F_n = i\Delta F_n$ for the second, to obtain

$$\begin{aligned}
&E_n(v_n(T_2)) - E_n(v_n(T_1)) \\
&= -\frac{1}{2}\int_{T_1}^{T_2}\partial_t\int_{\mathbb{R}^d} \varphi_n(|F_n+v_n|^2) - \varphi_n(|v_n|^2) - \varphi_n(|F_n|^2)dxdt \\
&\quad + \mathrm{Im}\int_{T_1}^{T_2}\int_{\mathbb{R}^d} \Delta\overline{F_n}(g_n(F_n+v_n) - g_n(F_n))dxdt \\
&= -\frac{1}{2}\left[\int_{\mathbb{R}^d} \varphi_n(|F_n+v_n|^2) - \varphi_n(|v_n|^2) - \varphi_n(|F_n|^2)dx\right]_{T_1}^{T_2} \\
&\quad - \mathrm{Im}\int_{T_1}^{T_2}\int_{\mathbb{R}^d} \nabla\overline{F_n}\cdot\nabla(g_n(F_n+v_n) - g_n(F_n))dxdt
\end{aligned}$$

where we used the fundamental theorem of calculus and integrated by parts to obtain the final equality. $\square$

1.B Justification of Remark 1.1.6

Here we will outline the proof of the following statement claimed in Remark 1.1.6.

Lemma 1.B.1. *Let $0 < s < 1$. Let $f \in L^2(\mathbb{R}^d)$ and f^ω denote its randomisation* (1.1.11)*. Then if the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and the random variables $X^M_{i,j,k,l} : \Omega \to \mathbb{R}$ are as described in Remark 1.1.6, we have that $f \notin H^s(\mathbb{R}^d)$ implies $f^\omega \notin H^s(\mathbb{R}^d)$ for almost every $\omega \in \Omega$.*

We will follow the method of [BT08] (Appendix B), using the almost-orthogonality of the projections P_j to estimate

$$\begin{aligned}&\int_{\Omega_1\times\Omega_2\times\Omega_3} e^{-\|f^\omega\|^2_{H^s}} d\mathbb{P}(\omega_1,\omega_2,\omega_3)\\ &\leq \int_{\Omega_2\times\Omega_3}\int_{\Omega_1} e^{-C\sum_j |X_j(\omega_1)|^2 \|P_j \sum_{k,l,M,i} X^M_{k,l}(\omega_2)X_i(\omega_3) f^{M,i}_{k,l}\|^2_{H^s}} d\mathbb{P}(\omega_1)d\mathbb{P}(\omega_2,\omega_3)\\ &\leq \int_{\Omega_2\times\Omega_3} e^{-C\|\sum_{k,l,M} X^M_{k,l}(\omega_2)\sum_i X_i(\omega_3) f^{M,i}_{k,l}\|^2_{H^s}} d\mathbb{P}(\omega_2,\omega_3)\end{aligned}$$

where we used that the random variables X_j take values in $\{\pm 1\}$.

We can treat the integral over Ω_2 similarly to find

$$\int_{\Omega_1\times\Omega_2\times\Omega_3} e^{-\|f^\omega\|^2_{H^s}} d\mathbb{P}(\omega_1,\omega_2,\omega_3) \leq \int_{\Omega_3} e^{-C\|\sum_i X_i(\omega_3)\varphi_i f\|^2_{H^s}} d\mathbb{P}(\omega_3)$$

In order to repeat the argument on Ω_3 and complete the proof, it remains to prove

$$\|f\|^2_{H^s} \lesssim_{s,d} \sum_i \|\varphi_i f\|^2_{H^s} \lesssim_{s,d} \|\sum_i X_i(\omega_3)\varphi_i f\|^2_{H^s} \tag{1.B.1}$$

We will only prove the second inequality, the first being similar. Denote $f_i := X_i(\omega) f = \pm f$ and write

$$\|\varphi_i f\|_{\ell^2_i H^s_x(\mathbb{R}^d)} = \|\varphi_i f_i\|_{\ell^2_i H^s_x(\mathbb{R}^d)} \leq \|P_{\leq M_0}\varphi_i f_i\|_{\ell^2_i H^s_x(\mathbb{R}^d)} + \|P_{>M_0}\varphi_i f_i\|_{\ell^2_i H^s_x(\mathbb{R}^d)}$$

for some $M_0 \geq 1$. We first handle the low frequency contributions using the almost-orthogonality of the $(\varphi_i)_i$:

$$\|P_{\leq M_0}\varphi_i f_i\|_{\ell^2_i H^s_x} \lesssim \langle M_0\rangle^s \|\varphi_i f_i\|_{\ell^2_i L^2_x} \lesssim \langle M_0\rangle^s \|\sum_i \varphi_i f_i\|_{L^2}$$

For the high frequency contributions we have to study the commutators $[P_M, \varphi_i]$. We have

$$\|P_{>M_0}\varphi_i f_i\|_{\ell^2_i H^s_x} \lesssim \|M^s P_M \varphi_i f_i\|_{\ell^2_{M>M_0}\ell^2_i L^2_x}$$

$$
\begin{aligned}
&\lesssim \|M^s \varphi_i P_M f_i\|_{\ell^2_{M>M_0}\ell^2_i L^2_x} + \underbrace{\|M^s[P_M,\varphi_i]f_i\|_{\ell^2_{M>M_0}\ell^2_i L^2_x}}_{(A)} \\
&\lesssim \|M^s \sum_i \varphi_i P_M f_i\|_{\ell^2_{M>M_0} L^2_x} + (A) \\
&\lesssim \|P_{>M_0} \sum_i \varphi_i f_i\|_{H^s_x} + \underbrace{\|M^s \sum_i [\varphi_i, P_M] f_i\|_{\ell^2_{M>M_0} L^2_x}}_{(B)} + (A) \\
&\lesssim \| \sum_i \varphi_i f_i\|_{H^s_x} + (A) + (B)
\end{aligned}
$$

so it remains to bound (A) and (B).

Let's study (A) first. Since the $(X_i)_i$ take values in $\{\pm 1\}$ we have that $|[P_M,\varphi_i]f_i| = |[P_M,\varphi_i]f|$ so

$$
\begin{aligned}
(A) &= \|M^s[P_M,\varphi_i]f\|_{\ell^2_{M>M_0}\ell^2_i L^2_x} \\
&\lesssim \|M^s[P_M,\varphi_i]\tilde{\varphi}_i f\|_{\ell^2_{M>M_0}\ell^2_i L^2_x} + \|M^s \varphi_i P_M \sum_{j:|j-i|>8\sqrt{d}} \varphi_j f\|_{\ell^2_{M>M_0}\ell^2_i L^2_x}
\end{aligned} \tag{1.B.2}
$$

for $\tilde{\varphi}_i = \sum_{j:|j-i|\leq 8\sqrt{d}} \varphi_j$, so that $\tilde{\varphi}_i \varphi_i = 1$. To handle the first term, we will use the bound

$$
\|[P_M,\varphi_i]g\|_2 \lesssim_d M^{-1}\|\nabla\varphi_i\|_\infty \|g\|_2 \lesssim_d M^{-1}\|g\|_2 \tag{1.B.3}
$$

which follows from writing the frequency projection as a convolution operator. For the second term we will use a slightly stronger form of Lemma 3.2 from [Spi21], holding for any $D > 0$, $|i-j| \geq 8\sqrt{d}$:

$$
\|\varphi_i P_M \varphi_j g\|_2 \lesssim_{d,D} M^{-D}|i-j|^{-D}\|\varphi_j g\|_2 \tag{1.B.4}
$$

With these results in hand, we are able to bound (A). Using (1.B.3) on the first term and the triangle inequality followed by (1.B.4) on the second term we obtain

$$
\begin{aligned}
(A) &\lesssim \|M^{s-1}\tilde{\varphi}_i f\|_{\ell^2_{M>M_0}\ell^2_i L^2_x} + \|M^{s-D}|i-j|^{-D}\varphi_j f\|_{\ell^2_{M>M_0}\ell^2_i \ell^1_{j:|j-i|>8\sqrt{d}} L^2_x} \\
&\lesssim M_0^{s-1}\|f\|_2 + M_0^{s-D}\left\| \||i-j|^{-D/2}\|_{\ell^2_{j:|j-i|>8\sqrt{d}}} \cdot \||i-j|^{-D/2}\varphi_j f\|_{\ell^2_{j:|j-i|>8\sqrt{d}} L^2_x} \right\|_{\ell^2_i}
\end{aligned}
$$

by the Cauchy-Schwarz inequality. For D sufficiently large, $\||i-j|^{-D/2}\|_{\ell^2_{j:|j-i|>8\sqrt{d}}}$ is finite and we may swap the sums over i and j in the second term to obtain

$$
\begin{aligned}
(A) &\lesssim M_0^{s-1}\|f\|_2 + M_0^{s-D}\||i-j|^{-D/2}\varphi_j f\|_{\ell^2_j \ell^2_{i:|i-j|>8\sqrt{d}} L^2_x} \\
&\lesssim M_0^{s-1}\|f\|_{L^2(\mathbb{R}^d)}
\end{aligned}
$$

We now turn to

$$(B) = \|M^s \sum_i [P_M, \varphi_i] f_i\|_{\ell^2_{M>M_0} L^2_x}$$

This time write

$$[P_M, \varphi_i] = \tilde{\varphi}_i [P_M, \varphi_i] + (1 - \tilde{\varphi}_i) P_M \varphi_i$$

We have

$$\|M^s \sum_i \tilde{\varphi}_i [P_M, \varphi_i] f_i\|_{\ell^2_{M>M_0} L^2_x} \leq \|M^s \tilde{\varphi}_i [P_M, \varphi_i] f_i\|_{\ell^2_{M>M_0} \ell^2_i L^2_x} \leq (A) \lesssim M_0^{s-1} \|f\|_2$$

since the $\tilde{\varphi}_i$ are bounded. On the other hand, using that $f_i = X_i(\omega) f$, we have

$$\begin{aligned}
\|M^s \sum_i (1 - \tilde{\varphi}_i) P_M \varphi_i f_i\|_{\ell^2_{M>M_0} L^2_x} =& \|M^s \sum_i \sum_{j:|j-i|>8\sqrt{d}} \varphi_j P_M \varphi_i f\|_{\ell^2_{M>M_0} L^2_x} \\
=& \| \sum_j \varphi_j \cdot M^s P_M (\sum_{i:|i-j|>8\sqrt{d}} \varphi_i) f\|_{\ell^2_{M>M_0} L^2_x} \\
\lesssim& \|\varphi_j \cdot M^s P_M (\sum_{i:|i-j|>8\sqrt{d}} \varphi_i) f\|_{\ell^2_{M>M_0} \ell^2_j L^2_x}
\end{aligned}$$

by the almost-orthogonality of the projections φ_j. Using the triangle inequality followed by the estimate (1.B.4), we can bound the previous line by

$$\begin{aligned}
&\|M^{s-D} |i-j|^{-D} \varphi_i f\|_{\ell^2_{M>M_0} \ell^2_j \ell^1_{i:|i-j|>8\sqrt{d}} L^2_x} \\
\lesssim& M_0^{s-D} \| |i-j|^{-D} \varphi_i f\|_{\ell^2_j \ell^1_{i:|i-j|>8\sqrt{d}} L^2_x} \\
\lesssim& M_0^{s-D} \| \| |i-j|^{-D/2} \|_{\ell^2_{i:|i-j|>8\sqrt{d}}} \cdot \| |i-j|^{-D/2} \varphi_i f\|_{\ell^2_{i:|i-j|>8\sqrt{d}} L^2_x} \|_{\ell^2_j} \\
\lesssim& M_0^{s-D} \| |i-j|^{-D/2} \varphi_i f\|_{\ell^2_i \ell^2_{j:|j-i|>8\sqrt{d}} L^2_x} \lesssim M_0^{s-1} \|f\|_2
\end{aligned}$$

since $D, M_0 \geq 1$. This completes the estimate for (B).

Combining the estimates we have just found with the bound on the low frequency contributions we conclude that

$$\begin{aligned}
\|\varphi_i f\|_{\ell^2_i H^s_x(\mathbb{R}^d)} \lesssim& \langle M_0 \rangle^s \| \sum_i \varphi_i f_i \|_{L^2(\mathbb{R}^d)} + \| \sum_i \varphi_i f_i \|_{H^s(\mathbb{R}^d)} + M_0^{s-1} \|f\|_{L^2(\mathbb{R}^d)} \\
\lesssim& (1 + \langle M_0 \rangle^s) \| \sum_i X_i(\omega) \varphi_i f \|_{H^s(\mathbb{R}^d)} + M_0^{s-1} \|\varphi_i f\|_{\ell^2_i L^2_x(\mathbb{R}^d)}
\end{aligned}$$

Now, since $\|\varphi_i f\|_{\ell^2_i L^2_x(\mathbb{R}^d)} \lesssim \|\varphi_i f\|_{\ell^2_i H^s_x(\mathbb{R}^d)}$ and $s < 1$, if we take M_0 sufficiently large (depending only on s and d) we may move this term to the left hand side to obtain

$$\left(\sum_i \|\varphi_i f\|^2_{H^s(\mathbb{R}^d)} \right)^{\frac{1}{2}} \lesssim_{s,d} \| \sum_i X_i(\omega) \varphi_i f \|_{H^s(\mathbb{R}^d)}$$

which completes the proof of the second inequality in (1.B.1).

2 Global Solutions to the 3D Half-Wave Maps Equation with Angular Regularity.

The work of this chapter is taken from the preprint [Mar24], submitted to *Ars Inveniendi Analytica*.

2.1 Introduction

This chapter concerns the global existence of solutions to the three dimensional half-wave maps equation

$$\begin{cases} \partial_t \phi = \phi \times (-\Delta)^{1/2}\phi \\ \phi(0,\cdot) = \phi_0 \end{cases} \qquad (\phi : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{S}^2) \tag{2.1.1}$$

in the critical Besov space $\dot{B}^{3/2}_{2,1}$, partially generalising known results in higher dimensions (see Theorem 2.1.1 for the precise statement). The space $\dot{B}^{3/2}_{2,1}$ is critical in the sense that it is invariant with respect to the scaling

$$\phi(t,x) \mapsto \phi_\lambda(t,x) := \phi(\lambda t, \lambda x).$$

As discussed in the main introduction (Section 0.0.2), the half-wave maps equation is known to be globally well-posed for small critical Besov data in high dimensions, thanks to Krieger-Sire [KS17] for $d \geq 5$ and Krieger-Kiesenhofer [KK21] for $d = 4$. We recall the formulation of (2.1.1) as a semilinear wave equation (Section 2, [KS17]),

$$\begin{aligned}(\partial_t^2 - \Delta)\phi = &-\phi\, \partial^\alpha \phi^T \partial_\alpha \phi \\ &+ \Pi_{\phi^\perp}[((-\Delta)^{1/2}\phi)(\phi \cdot (-\Delta)^{1/2}\phi)] \\ &+ \phi \times [(-\Delta)^{1/2}(\phi \times (-\Delta)^{1/2}\phi) - (\phi \times (-\Delta)\phi)]\end{aligned} \tag{2.1.2}$$

where the first term in the forcing corresponds to that of the wave maps equation, and the whole nonlinearity can (very loosely speaking) be written as

$$\phi\, \nabla\phi\nabla\phi.$$

We will now briefly discuss the difficulties in extending the methods of Krieger and Sire to three dimensions. Krieger and Sire used dyadic versions of the $X^{s,\theta}$ spaces to handle the wave maps term in (2.1.2), in an argument relying heavily on the null structure. When it comes to the new half-wave maps terms, it was observed that there is enough geometric structure (see (2.1.13)) to close the argument in the Strichartz spaces $L^p_t L^q_x$, where

$$\frac{2}{p} + \frac{d-1}{q} \leq \frac{d-1}{2}, \qquad d \geq 2,\ (d,p,q) \neq (3,2,\infty).$$

Alas, this range becomes increasingly restrictive in lower dimensions, and already in dimension 4 we lose the $L^2_t L^4_x$ space which was used frequently in [KS17]. This was overcome in [KK21] using a refinement of the methods of [KS17] and the results were extended to $d = 4$. In three dimensions, the range of available estimates becomes smaller still, and in particular we lose the endpoint $L_t^2 L_x^\infty$ [Tao98]. This space plays an essential role in the arguments of [KS17, KK21] and hence it is not clear how to generalise these methods to $d = 3$. On the other hand, if we restrict to radial data, we may appeal to the wider range of *radially admissible* Strichartz spaces, with

$$\frac{1}{p} + \frac{d-1}{q} < \frac{d-1}{2}.$$

We thus recover the $L_t^2 L_x^\infty$ space and the methods of [KK21] can be straightforwardly applied. The standard and radial Strichartz pairs are displayed in Figure 2.1.1.

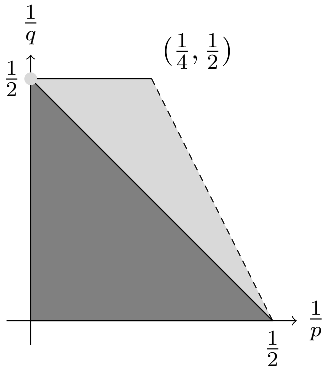

Figure 2.1.1: The admissible Strichartz pairs (p, q) in $d = 3$. The dark gray region depicts the standard admissible pairs, and the light gray region the extended range of radially admissible pairs. The endpoint $L_t^2 L_x^\infty$ is radially but not standard admissible.

In this chapter we will make the weaker assumption that the data is not radial but merely has some angular regularity. In this setting Sterbenz [Ste05] proved modified Strichartz estimates in the full range of radially admissible spaces which we will exploit to obtain the following "weak" small data-global wellposedness result. We introduce the notation

$$\|\langle\Omega\rangle u\| \equiv \|u\| + \max_{i,j} \|\Omega_{ij} u\| \tag{2.1.3}$$

for any norm $\|\cdot\|$ and the angular derivatives Ω_{ij}, see (2.1.7). Here and throughout, $\|\langle\Omega\rangle(x\cdot\nabla)\phi[0]\|$ is taken to mean $\max_{k,l=1,2,3} \|\langle\Omega\rangle(x_k \nabla_l \phi[0])\|$.

Theorem 2.1.1. *Let $\phi_0 : \mathbb{R}^3 \to \mathbb{S}^2$ be a smooth initial datum which is constant outside a compact set. There exists $0 < \epsilon < 1$ such that whenever*

$$\|\langle\Omega\rangle\phi_0\|_{\dot{B}^{3/2}_{2,1}} + \|\langle\Omega\rangle(x\cdot\nabla)\phi_0\|_{\dot{B}^{3/2}_{2,1}} < \epsilon \tag{2.1.4}$$

the problem (2.1.1) admits a global smooth solution. Moreover for any s sufficiently close to $3/2$ it holds

$$\|\phi(t)\|_{\dot{B}^s_{2,1}} \lesssim_s \|\langle\Omega\rangle\phi_0\|_{\dot{B}^s_{2,1}} + \|\langle\Omega\rangle(x\cdot\nabla)\phi_0\|_{\dot{B}^s_{2,1}} \tag{2.1.5}$$

for all $t \in \mathbb{R}$.

The slightly unusual assumptions on the initial data come from our use of commuting vector fields, to be discussed shortly.

We now give some more details on the proof of Theorem 2.1.1. As we have discussed, in low dimensions the analysis of the wave maps equation becomes increasingly reliant on the null structure, and the iteration argument of Tataru [Tat01] involves the development of highly tailored function spaces. We therefore turn to Tao's approach for studying wave maps in the critical Sobolev space [Tao01a] which works entirely in the framework of Strichartz spaces and does not rely so essentially on the null structure. The cost is that we can only obtain a weak wellposedness result as in Theorem 2.1.1.

The argument of [Tao01a] relies on a carefully chosen coordinate transformation which cancels out the most difficult frequency interactions in the nonlinearity. These are the (lowest)∇(low)∇(high) interactions in which one of the differentiated factors appears at low frequency, but not as low as the non-differentiated factor. Admitting this cancellation, the principal difficulty of the present work is dealing with interactions of the form

$$(\text{low})\nabla(\text{lowest})\nabla(\text{high}). \tag{2.1.6}$$

Tao controlled such interactions by placing the terms into $L^2_t L^\infty_x$, $L^2_t L^\infty_x$ and $L^\infty_t L^2_x$ respectively, with no flexibility in the estimate. As we have mentioned, the space $L^2_t L^\infty_x$ is no longer available to us. To overcome this we incorporate into our function spaces a range of commuting vector fields,

$$L_n := x_n \partial_t + t\partial_{x_n} \qquad \text{and} \qquad \Omega_{ij} := x_i \partial_{x_j} - x_j \partial_{x_i} \qquad (n, i, j = 1, 2, 3), \tag{2.1.7}$$

first introduced in the context of global regularity for nonlinear wave equations in [Kla85].[1] By incorporating these into the Strichartz norms, we are able to develop spacetime estimates for terms of the form (2.1.6), gaining decay in time via the Lorentz boosts and in space via the heuristic

$$\phi(x) \simeq \frac{1}{\Omega_{ij}}\Omega_{ij}\phi(x) \simeq \frac{1}{x_i\xi_j - x_j\xi_i}\Omega_{ij}\phi(x) \simeq \frac{1}{|x_{ij}||\xi_{ij}|\sin(\angle(x_{ij},\xi_{ij}))}\Omega_{ij}\phi(x) \tag{2.1.8}$$

Here x, ξ denote the physical and Fourier variables respectively, and x_{ij}, ξ_{ij} their projections onto the $i-j$ plane. Assuming ϕ has angular regularity and can absorb the derivative Ω_{ij}, we therefore gain decay in x whenever the Fourier and physical variables have some angular separation (see Lemma 2.5.2). In practice, we implement this via a simultaneous decomposition of the trilinear term (2.1.6) on angular caps in physical and Fourier space. See Lemma 2.5.3 for the detailed argument.

We remark that it is the use of commuting vector fields which limits our result to Besov

[1] Unlike in [Kla85], our method does not rely on energy estimates so we are able to work at low regularity. Note also that in three dimensions Klainerman's vector field method requires the nonlinearity to satisfy the null condition [Kla86], which is not satisfied by (2.1.2).

rather than Sobolev spaces. The issue is that we occasionally need bounds such as

$$\|\Omega_{ij}\phi\|_{L_t^\infty L_x^\infty}, \|L_n\phi\|_{L_t^\infty L_x^\infty} \lesssim 1,$$

which in the absence of the commuting vector fields would come for free from the fact that the solution lies on the sphere.

Before discussing our methods further, we note here the paper [Ste07] of Sterbenz regarding global regularity of the $(4+1)$-dimensional Yang-Mills equation in Lorentz gauge, which also uses an angular regularity assumption to exploit the improved estimates of [Ste05]. The argument there is based on measuring angular concentration phenomena and avoids the use of the Lorentz boosts in order to recover estimates in $X^{s,\theta}$-based spaces. We also refer to [Hon22] for results on nonlinear wave equations and [HKO24] in the context of a supercritical nonlinear Schrödinger equation.

We now give a brief outline of the main argument and structure of this chapter. We will prove Theorem 2.1.1 via the following small data-global existence result for the differentiated equation (2.1.2). Denote $\phi[t] \equiv (\phi(t), \partial_t\phi(t))$.

Theorem 2.1.2. *Let $\phi[0] := (\phi_0, \phi_1) : \mathbb{R}^3 \to \mathbb{S}^2 \times T\mathbb{S}^2$ be a smooth initial data pair which is constant outside a compact set. There exists $0 < \epsilon < 1$ such that whenever*

$$\|\langle\Omega\rangle\phi[0]\|_{\dot{B}^{3/2}_{2,1}\times\dot{B}^{1/2}_{2,1}} + \|\langle\Omega\rangle(x\cdot\nabla)\phi[0]\|_{\dot{B}^{3/2}_{2,1}\times\dot{B}^{1/2}_{2,1}} < \epsilon \tag{2.1.9}$$

the equation (2.1.2) *with data $\phi[0]$ admits a global smooth solution $\phi[t]$ with*

$$\|\phi[t]\|_{\dot{B}^{s}_{2,1}\times\dot{B}^{s-1}_{2,1}} \lesssim_s \|\langle\Omega\rangle\phi[0]\|_{\dot{B}^{s}_{2,1}\times\dot{B}^{s-1}_{2,1}} + \|\langle\Omega\rangle(x\cdot\nabla)\phi[0]\|_{\dot{B}^{s}_{2,1}\times\dot{B}^{s-1}_{2,1}} \tag{2.1.10}$$

If moreover $\phi_1 = \phi_0 \times (-\Delta)^{1/2}\phi_0$, the global solution solves the half-wave maps equation (2.1.1)*.*

Note that in the case $\phi_1 = \phi_0 \times (-\Delta)^{1/2}\phi_0$, the smallness assumption on ϕ_1 in (2.1.9) is inherited from that on ϕ_0 so this theorem implies Theorem 2.1.1.

The starting point for our proof is the following local existence result, whose proof is postponed to Section 2.10 so as not to distract from the main argument.

Theorem 2.1.3. *There exists $\nu > 0$ such that for any $3/2 < s < 3/2 + \nu$ the following holds. Let $\phi[0] \in B^s_{2,1} \times B^{s-1}_{2,1}$ be a smooth initial data taking values in $\mathbb{S}^2 \times T\mathbb{S}^2$, equal to* *a constant p outside a compact set.*[2,3] *Suppose further that*

$$\|\langle\Omega\rangle\phi[0]\|_{\dot{B}^{3/2}_{2,1}\times\dot{B}^{1/2}_{2,1}} + \|\langle\Omega\rangle(x\cdot\nabla)\phi[0]\|_{\dot{B}^{3/2}_{2,1}\times\dot{B}^{1/2}_{2,1}} < \epsilon$$

for some ϵ sufficiently small. Then there exists $T > 0$ depending only on $\|\phi[0]\|_{B^s_{2,1}\times B^{s-1}_{2,1}}$ and a smooth solution $\phi \in C([0,T], B^s_{2,1}) \cap C^1([0,T], B^{s-1}_{2,1})$ to (2.1.2). *Moreover, $\phi(t) \in \mathbb{S}^2$ for all $t \in [0,T]$.*

If we further have $\phi_1 = \phi_0 \times (-\Delta)^{1/2}\phi_0$, this solution solves the half-wave maps equation (2.1.1) *on its maximal interval of existence.*

Remark 2.1.4. Note the unusual assumption of smallness in a critical norm. This restriction appears somewhat artificial since it is only needed to keep the Picard iterates away from the origin in order to control the projection operator $\Pi_{\phi^\perp}$, which is only a feature of the differentiated equation.

Returning to the main argument, we see from Theorem 2.1.3 that it suffices to find uniform bounds on the solution in subcritical Besov spaces. Following Tao's method of frequency envelopes, we will show (in Section 2.3) that this reduces to proving a priori estimates for the solution in a certain critical space S (defined in Section 2.2.2). Since we are working in a scale invariant setting, it is sufficient to bound the solution at unit frequency, $P_0\phi =: \psi$. Then by straightforward linear estimates (Section 2.2.2), we find that it effectively remains to bound

$$\|\langle\Omega\rangle L\Box\psi\|_{L^1_t L^2_x}$$

Accordingly we will write $F = \mathit{error}$ if $\|\langle\Omega\rangle LF\|_{L^1_t L^2_x}$ is suitably small.[4]

In Section 2.5 we will show that the nonlocal half-wave maps terms in equation (2.1.2) are in fact entirely of the form *error*, as will be discussed further at the end of the introduction, so it remains to consider the wave maps contributions to the nonlinearity. We first (in Section 2.4) discard the frequency interactions in which the non-differentiated factor of ϕ appears at high frequency, which can be dealt with via standard Strichartz estimates. This reduces the equation to[5]

$$\Box\psi = -P_0(2\phi_{\leq -10}\partial_\alpha\phi^T_{\leq -10}\partial^\alpha\phi_{>-10} + \phi_{\leq -10}\partial_\alpha\phi^T_{>-10}\partial^\alpha\phi_{>-10}) + \mathit{error}$$

(where we are now more precise about the meaning of a "low" and "high" frequency

[2] Since ϕ lies on $\mathbb{S}^2$, when we say e.g. $\phi \in B^s_{2,1}$ we really mean that $\phi - p \in B^s_{2,1}$ for p the limit of the initial data at infinity, which is viewed as fixed throughout the chapter.

[3] This assumption is far stronger than necessary, and not preserved under the flow (since the equation is nonlocal). A more suitable assumption for our purposes is actually that $\phi[0]$, $L_n\phi[0]$ and $\Omega_{ij}\phi[0]$ lie in $B^{s'}_{2,1} \times B^{s'-1}_{2,1}$ for every $s' \geq 1$. This property is preserved by the flow (as can be seen by a persistence-of-regularity type argument) and thus leads to a blow-up criterion.

[4] The actual definition of an error term is slightly modified in the main argument for technical reasons, however the reader is advised to ignore this for the time being.

[5] With the whole term localised to unit frequency, (low)∇(low)∇(low) interactions are impossible.

term). In Section 2.6 we discard the (low)∇(high)∇(high) interactions via a normal form transformation using the null structure. We similarly pass the localisation P_0 through the low frequency factors in the remaining forcing term and achieve

$$\Box\Phi = -2\phi_{\le -10}\partial_\alpha\phi^T_{\le -10}\partial^\alpha\psi + error$$

for the transformed variable Φ, so that only the (low)∇(low)∇(high) wave maps interactions remain. We handle such terms using Tao's gauge transformation in Sections 2.7 and 2.8. Precisely, we construct a matrix field U satisfying

$$\partial_\alpha U \simeq -(\phi_{\le -10}\partial_\alpha\phi^T_{\le -10})\, U \tag{2.1.11}$$

such that upon transforming Φ to $w := U^{-1}\Phi$, the remaining forcing terms cancel out and we find

$$\Box w = error$$

We have to be a little careful since at this point we are working with Φ rather than ψ, however this issue is minor.

In showing (2.1.11) (Section 2.8), the (lowest)∇(low)∇(high) terms cancel out exactly, and we finally have to deal with the terms of the form

$$\text{(low)}\nabla\text{(lowest)}\nabla\text{(high)},$$

for which we invoke the arguments involving commuting vector fields already discussed.

Remark 2.1.5. We are able to slightly simplify the gauge transformation from [Tao01a] due to our working in Besov spaces. In particular, we do not need to antisymmetrise the equation in order to obtain almost-orthogonality of the transformation matrix, which is instead automatically a perturbation of the identity.

It remains to discuss how to control the nonlocal terms appearing in the half-wave maps equation. This is the content of Section 2.5. The main difference from the wave maps terms arises in studying interactions which are (morally speaking) of type

$$\text{(low)}\nabla\text{(high)}\nabla\text{(high)} \qquad \text{or} \qquad \text{(lowest)}\nabla\text{(low)}\nabla\text{(high)} \tag{2.1.12}$$

The analogous wave maps source terms were discarded by the normal form and gauge transformations respectively, both of which relied on the structure of the nonlinearity so can no longer be applied. To compensate this we use that the remaining terms of (2.1.2) involve interactions which are loosely speaking of the form

$$\phi \cdot \nabla\phi, \tag{2.1.13}$$

which vanishes for functions on the sphere. As in [KS17], we exploit this cancellation via the following identity which allows us to flip the low frequency factors in (2.1.12) to high frequency, and thus appeal to Strichartz-based methods:

$$P_k(\phi_{<k-10}\cdot\phi_{\geq k-10}) = -\frac{1}{2}P_k(\phi_{\geq k-10}\cdot\phi_{\geq k-10}) \tag{GeId}$$

This is a straightforward consequence of the property $P_k(\phi\cdot\phi) = P_k(1) = 0$. Besides this, the half-wave maps terms present various technical complications due to the nonlocal nature of the operator $(-\Delta)^{1/2}$. This is a particular issue when working with the commuting vector fields which are non-translation invariant.

2.1.1 Notation

We emphasise again that we adopt a different notation from the previous chapter and denote

$$P_k\phi \equiv \phi_k := \mathcal{F}^{-1}(\chi_k(\xi)\hat{\phi}(\xi)) \tag{2.1.14}$$

where χ_k corresponds to the function χ_{2^k} from (0.1.4). Again $\tilde{\chi}_k(\xi) = \sum_{j=k-C}^{k+C}\chi_j(\xi)$, $\tilde{P}_k\phi \equiv \phi_{\sim k} = \mathcal{F}^{-1}(\tilde{\chi}_k(\xi)\hat{\phi}(\xi))$ and so on. To reduce notation, we will often abusively write $j \ll k$ to mean $j \leq k - C$, of course this really means $2^j \ll 2^k$. We have similar interpretations for $j \sim k$, $j \lesssim k$ etc..

Our argument is based in the homogeneous ℓ^1 Besov spaces with norm

$$\|\phi\|_{\dot{B}^s_{2,1}} := \sum_{k\in\mathbb{Z}} 2^{sk}\|\phi_k\|_{L^2_x} \tag{2.1.15}$$

or in the subcritical case (for the proof of local wellposedness) the inhomogeneous spaces

$$\|\phi\|_{B^s_{2,1}} := \sum_{k>0} 2^{sk}\|\phi_k\|_{L^2_x} + \|P_{\leq 0}\phi\|_{L^2_x}$$

In addition to the usual Littlewood-Paley cut-offs we will also need dyadic cut-offs in physical space which we denote $\varphi_\lambda(x)$, $\lambda\in\mathbb{Z}$ (again this inconsistent with the notation of the previous chapter). Here $\varphi_\lambda(x) \equiv \chi_\lambda(x)$ but we adopt a different notation in order to emphasise that the cut-offs are acting in different spaces. We will also use notation such as $\varphi_{\geq\lambda} := \sum_{\lambda'\geq\lambda}\varphi_{\lambda'}$, and denote $\varphi_\lambda(t)$ for the analogous cut-offs in the time variable.

Throughout M should always be interpreted as a very large constant.

2.2 Preliminaries

2.2.1 Angular Derivatives and Commuting Vector Fields

In our argument the Lorentz boosts, L_n, and the angular derivative operators, Ω_{ij}, defined in (2.1.7) will play a key role. Observe that these operators obey the Leibniz rule. We will also need the Riesz transforms R_n defined by $\mathcal{F}(R_n\phi)(\xi) = \frac{\xi_n}{|\xi|}\mathcal{F}(\phi)(\xi)$

$(n = 1, 2, 3)$, which we recall are bounded on L^p_x for $1 < p < \infty$.

One may readily verify that the operators L_n and Ω_{ij} commute with the wave operator $\Box$, and satisfy the relations

$$[L, \partial] = \partial, \qquad [L, (-\Delta)^{1/2}] = R\partial_t, \qquad [L, \Omega] = L \tag{2.2.1}$$

and

$$[\Omega, R] = R, \qquad [\Omega, \partial] = \partial \tag{2.2.2}$$

Here L, R, Ω, and ∂ denote linear combinations of the identity with the operators $(L_n)_{n=1,2,3}$, $(R_n)_{n=1,2,3}$, $(\Omega_{ij})_{i,j=1,2,3}$, $(\partial_\alpha)_{\alpha=0,1,2,3}$ respectively. Note that Ω commutes with any radial Fourier multiplier such as $(-\Delta)^{1/2}$, thanks to the property $\mathcal{F}(\Omega_{ij}\phi) = \Omega_{ij}\mathcal{F}(\phi)$.

Unfortunately, there is a non-trivial commutation relation between the L_n and the Littlewood-Paley operators P_j, which will be a source of some irritation in what follows. Precisely, let $\mathcal{P}_j$ denote a generic operator corresponding to a (not necessarily radial) smooth multiplier $\chi^{(\mathcal{P})}(2^{-j}\xi)$, with $\mathrm{supp}\chi^{(\mathcal{P})} \subset \mathrm{supp}\chi$. It holds

$$[L_n, \mathcal{P}_j] = 2^{-j}\partial_t\mathcal{P}_j \qquad \text{and} \qquad [\Omega, \mathcal{P}_j] = \mathcal{P}_j \tag{2.2.3}$$

for potentially different operators $\mathcal{P}_j$ of the same form on the right hand side.

We now introduce the angular Sobolev spaces which will play an important role in our proof, using the construction in [Ste05].

For a function f on $\mathbb{R}^3$, we define fractional angular derivatives $|\Omega|^s$ as follows. First decompose f into a sum of spherical harmonics:[6]

$$f(r,\theta) = \sum_{l=0}^{\infty}\sum_{i=1}^{N_l} c_l^i(r) Y_l^i(\theta), \qquad c_l^i(r) := \int_{\mathbb{S}^2} f(r,\theta)\overline{Y}_l^i(\theta)d\theta \tag{2.2.4}$$

Since the spherical harmonics are eigenfunctions of the spherical Laplacian Δ_{sph},

$$\Delta_{\mathrm{sph}} Y_l^i = -l(l+1)Y_l^i, \qquad l \geq 0,\ i = 0, \dots, N_l,$$

it follows that a suitable definition of $|\Omega|^s f$ is given by

$$|\Omega|^s f(r,\theta) := \sum_{l=0}^{\infty}\sum_{i=1}^{N_l} [l(l+1)]^{s/2} c_l^i(r) Y_l^i(\theta)$$

so that $|\Omega|^2 = -\Delta_{\mathrm{sph}}$. Note that this vanishes whenever f is a radial function.

[6] Recall that we use the different notation $(Y_l^i)_{i=1,\dots,N_l}$ for an orthonormal basis of the space of spherical harmonics of degree l.

Furthermore, since the decomposition into spherical harmonics is preserved by the wave evolution (see (0.1.15)), it also follows that the fractional angular derivatives commute with the free evolution operators:

$$|\Omega|^s(e^{\pm it\sqrt{-\Delta}}f) = e^{\pm it\sqrt{-\Delta}}(|\Omega|^s f)$$

By incorporating angular regularity into our function spaces, we are able to make use of the following generalised Strichartz estimate which follows from the work of [Ste05].

Theorem 2.2.1 ($n = 3$ Strichartz estimates with angular regularity)**.** *Let (p, q) be a pair which is radially admissible but not standard wave admissible:*

$$\frac{1}{p} + \frac{2}{q} < 1, \qquad\qquad \frac{1}{p} + \frac{1}{q} > \frac{1}{2}$$

Suppose further that $p \neq 2$. Then for all $\eta > 0$ sufficiently small it holds

$$\|e^{\pm it\sqrt{-\Delta}}P_k f\|_{L^p_t L^q_x} \lesssim_\eta 2^{(\frac{3}{2}-\frac{1}{p}-\frac{3}{q})k}(\|P_k f\|_{L^2_x} + \||\Omega|^{s(p,q)}P_k f\|_{L^2_x})$$

for any function f such that the right hand side is finite. Here

$$s(p,q) := \frac{2}{p} + \frac{2}{q} - 1 + \epsilon(p,q;\eta)$$

where $\epsilon(p,q;\eta) \to 0$ as $\eta \to 0$. Note that $s(p,q) \leq \frac{1}{2}$ for (p,q) as given and η sufficiently small.

Proof. By scaling it suffices to consider $k = 0$. We use the notation of [Ste05] and direct the reader to that work for further details. In particular, let $\theta : [0,\infty) \to [0,1]$ be a smooth function equal to 1 on $[1,2]$ and vanishing outside $[1/2,4]$, and set $\theta_N(l) := \theta(N^{-1}l)$ for $N \in 2^{\mathbb{N}}$. For the decomposition of f as in (2.2.4) we then denote

$$f_N := \sum_{l=0}^{\infty}\sum_{i=1}^{N_l}\theta_N(l)c_l^i(r)Y_l^i(\theta) \tag{2.2.5}$$

Let $\eta > 0$. By Proposition 3.4 in [Ste05] we find

$$\|e^{\pm it\sqrt{-\Delta}}P_0 f_N\|_{L^2_t L^{r_\eta}_x} \lesssim_\eta N^{\frac{1}{2}+\eta}\|P_0 f_N\|_{L^2_x}$$

for some $r_\eta \searrow 4$ as $\eta \to 0$. A three-way interpolation of this result with the standard Strichartz estimate

$$\|e^{\pm it\sqrt{-\Delta}}P_0 f_N\|_{L_t^{\frac{2}{1-\eta}} L^\infty_x} \lesssim_\eta \|P_0 f_N\|_{L^2_x}$$

and the energy estimate

$$\|e^{\pm it\sqrt{-\Delta}}P_0 f_N\|_{L^\infty_t L^2_x} \lesssim \|P_0 f_N\|_{L^2_x}$$

yields
$$\|e^{\pm it\sqrt{-\Delta}} P_0 f_N\|_{L^p_t L^q_x} \lesssim_\eta N^{s(p,q)} \|P_0 f_N\|_{L^2_x} \simeq \||\Omega|^{s(p,q)} P_0 f_N\|_{L^2_x}$$
provided we choose η sufficiently small to ensure that the pair (p, q) is covered by the interpolation.

The radial part of the evolution is covered by the radial Strichartz estimate (Theorem 1.3, [Ste05]): denoting $f_0 := c^0_0(r)Y^0_0(\theta) = c^0_0(r)$, the radial part of f, we have

$$\|e^{\pm it\sqrt{-\Delta}} P_0 f_0\|_{L^p_t L^q_x} \lesssim \|P_0 f_0\|_{L^2_x}$$

The result then follows from the Littlewood-Paley-Stein theorem for the sphere (Theorem 2 [Str72], see also [Ste70b].), upon observing that the angular frequency localisation (2.2.5) commutes with the operator $P_0 e^{\pm it\sqrt{-\Delta}}$. □

In practice, we will only work with integer-order angular derivatives so as to use the Leibniz properties discussed previously. For this we must be able to exchange fractional angular derivatives for true derivatives, which is possible thanks to the following result:

Lemma 2.2.2 (Riesz estimate for angular Sobolev spaces (Theorem 3.5.3, [DX13]))**.** *Let* $1 < p < \infty$*. Then*
$$\max_{i,j} \|\Omega_{ij} f\|_{L^p_x} \simeq \||\Omega| f\|_{L^p_x}$$
for any f such that the right hand side is finite.

We also use the following monotonicity property for the angular Sobolev spaces, which can be proved for example using the decay of the corresponding multiplier (see Corollary 1, [Str72]).

Lemma 2.2.3 (Monotonicity of Angular Sobolev Spaces)**.** *Let* $1 < p < \infty$*,* $s > s' > 0$*. It holds*
$$\||\Omega|^{s'} f\|_{L^p_x} \lesssim_{s-s'} \||\Omega|^s f\|_{L^p_x}$$

Combined with Theorem 2.2.1 the previous two lemmas yield the following (defining $\langle\Omega\rangle$ as in (2.1.3)).

Corollary 2.2.4. *Let $\mathcal{Q}$ be any finite set of radially admissible pairs (p, q) with $p \neq 2$. Then it holds*

$$\max_{(p,q)\in\mathcal{Q}} 2^{(\frac{3}{2}-\frac{1}{p}-\frac{3}{q})k} \|e^{\pm it\sqrt{-\Delta}} P_k f\|_{L^p_t L^q_x} \lesssim_{\mathcal{Q}} \|\langle\Omega\rangle P_k f\|_{L^2_x}$$

2.2.2 Function Spaces and Linear Estimates

Our function spaces are an adaptation of the usual Besov-type Strichartz spaces. Henceforth $\mathcal{Q}$ will denote a fixed set of radially admissible exponents as in Corollary 2.2.4 to be determined throughout the proof, but certainly containing $(\infty, 2)$. We then define the norm

$$\|\phi\|_{S([0,T])} = \sum_{k\in\mathbb{Z}} \|\phi_k\|_{S_k([0,T])}$$

with

$$\|\phi_k\|_{S_k([0,T])} := \max_{(p,q)\in\mathcal{Q}} 2^{(\frac{1}{p}+\frac{3}{q}-1)k} \|\langle\Omega\rangle^{1-\delta(p,q)}\nabla_{t,x} P_k\phi\|_{L^p_t L^q_x([0,T]\times\mathbb{R}^3)}$$

Here[7]

$$\delta(p,q) = \begin{cases} 0 \text{ if } \frac{1}{p}+\frac{1}{q} \le \frac{1}{2} \\ 1 \text{ otherwise} \end{cases}$$

We will also work with the vector fields L_n introduced in Section 2.2.1, however rather than incorporating these into the norm we will directly apply them to the solution we are working with.[8] Define

$$\phi^L := L\phi = \begin{pmatrix} \phi \\ L_1\phi \\ L_2\phi \\ L_3\phi \end{pmatrix}, \qquad L := \begin{pmatrix} 1 \\ L_1 \\ L_2 \\ L_3 \end{pmatrix}, \tag{2.2.6}$$

so

$$\|P_k\phi^L\|_{S_k([0,T])} \simeq \max_{n=0,\ldots,3} \max_{(p,q)\in\mathcal{Q}} 2^{(\frac{1}{p}+\frac{3}{q}-1)k} \|\langle\Omega\rangle^{1-\delta(p,q)}\nabla_{t,x} P_k L_n\phi\|_{L^p_t L^q_x([0,T]\times\mathbb{R}^3)}$$

with the convention $L_0 := 1$.

We have the following linear estimate which is a straightforward application of Corollary 2.2.4:

Theorem 2.2.5 (Linear Estimate)**.** *Let ϕ satisfy the linear wave equation $\Box\phi = F$ with initial data $\phi[0] \equiv (\phi(0,\cdot), \partial_t\phi(0,\cdot))$ on the interval $[0,T]$. It holds*

$$\|\phi_k\|_{S_k([0,T])} \lesssim \|\langle\Omega\rangle P_k\phi[0]\|_{\dot H^{3/2}\times\dot H^{1/2}} + \|\langle\Omega\rangle F_k\|_{L^1_t \dot H^{1/2}_x([0,T]\times\mathbb{R}^3)}$$

and as a corollary

$$\begin{aligned} &\|P_k\phi^L\|_{S_k([0,T])} \\ &\lesssim \|\langle\Omega\rangle P_k\phi[0]\|_{\dot H^{3/2}\times\dot H^{1/2}} + \|\langle\Omega\rangle P_k(x\cdot\nabla)\phi[0]\|_{\dot H^{3/2}\times\dot H^{1/2}} + \|\langle\Omega\rangle P_k(x\cdot\Box\phi(0))\|_{\dot H^{1/2}} \\ &\qquad + \|\langle\Omega\rangle P_k F^L\|_{L^1_t\dot H^{1/2}_x([0,T]\times\mathbb{R}^3)} \end{aligned}$$

where F^L is as in (2.2.6)*.*

[7]Note that a more natural choice would be $\delta(p,q) = \max\{\frac{2}{p}+\frac{2}{q}-1+\epsilon(p,q;\eta), 0\}$, however we opt for the weaker norm above so as to encounter only full angular derivatives. Presumably it would be possible to to work with fractional angular derivatives by introducing a paradifferential calculus in the angular variable, see for instance [HKO24, Hon22].

[8]This is for technical reasons to handle the non-trivial commutator $[L_n, P_k]$.

2.2.3 Angular Multipliers

To conclude this section, we introduce the angular multipliers which will play a key role in the main estimates of this work (see Lemma 2.5.2). For fixed $\rho \leq 0$, we introduce a smooth partition of unity on the sphere, $(\sigma_\rho^\beta)_{\beta \in \mathcal{S}_\rho}$, given by

$$\sigma_\rho^\beta(x) := \frac{\sigma(2^{-\rho}\|\hat{x} \times \beta\|)}{\sum_{\beta' \in \mathcal{S}_\rho} \sigma(2^{-\rho}\|\hat{x} \times \beta'\|)} \tag{2.2.7}$$

for $\sigma \in C_c^\infty$ supported on $[0, 101/100]$ and equal to 1 on $[0,1]$. $\mathcal{S}_\rho$ is a set of $\sim 2^{-2\rho}$ points on the sphere such that for every $\hat{x} \in \mathbb{S}^2$ there exists $\beta \in \mathcal{S}_\rho$ such that $\|\hat{x} \times \beta\| \leq 2^\rho$. We choose our functions in such a way as to ensure the almost-orthogonality relation

$$\|u\|_{L_x^2} \simeq \left(\sum_{\beta \in \mathcal{S}_\rho} \|\sigma_\rho^\beta(x) u(x)\|_{L_x^2}^2 \right)^{\frac{1}{2}}$$

holds uniformly in $\rho \leq 0$.

For each ρ sufficiently small, $\beta \in \mathcal{S}_\rho$, we also introduce a Whitney-type decomposition of the sphere in Fourier space. This consists of functions $\eta_r^{(r,l)}$ cutting off to discs of radius $\sim 2^r$ at distance $\sim 2^r$ from β, made precise in the following proposition. These cut-offs are turned into operators by defining, for example,

$$\eta_r^{(r,l)}(D)\phi(x) := \mathcal{F}^{-1}(\eta_r^{(r,l)}(\xi)\hat{\phi}(\xi))(x)$$

Proposition 2.2.6. *There exist absolute constants $C_1, C_2, C_3, C_4, N > 0$ such that the following holds. For any $\rho \leq -C_1$, $\beta \in \mathcal{S}_\rho$ there is a partition of unity consisting of functions*

$$\eta_\rho^\beta \text{ and } \eta_r^{(r,l)} \ (\rho + C_1 \leq r \leq 0, l = 1, \ \dots, N) \tag{2.2.8}$$

with the following properties:

1. *There are points $\alpha_{r,l} \in \mathbb{S}^2$ and functions $\tilde{\eta}_\rho^\beta$, $\tilde{\eta}_r^{(r,l)}$ of the form $\tilde{\eta}_\rho^\beta(\xi) = \sigma(2^{-(\rho+C_2)}\|\hat{\xi} \times \beta\|)$ and $\tilde{\eta}_r^{(r,l)}(\xi) = \sigma(2^{-(r-C_3)}\|\hat{\xi} \times \alpha_{r,l}\|)$ for σ as before such that*

$$\eta_\rho^\beta = \frac{\tilde{\eta}_\rho^\beta}{\tilde{\eta}_\rho^\beta + \sum \tilde{\eta}_r^{(r,l)}} \text{ and } \eta_r^{(r,l)} = \frac{\tilde{\eta}_r^{(r,l)}}{\tilde{\eta}_\rho^\beta + \sum \tilde{\eta}_r^{(r,l)}}$$

   We allow for a different constant C_3 when $r = 0$.

2. *$\|\alpha_{r,l} \times \beta\| \simeq 2^r$ and $\|\hat{x} \times \hat{\xi}\| \gtrsim 2^r$ whenever $\hat{x} \in supp(\sigma_\rho^\beta)$ and $\hat{\xi} \in supp(\eta_r^{(r,l)})$, for all (r,l).*

3. *$supp(\eta_{r'}^{(r',l')}) \cap supp(\eta_r^{(r,l)}) = \emptyset$ for all $|r - r'| \geq C_4$ and $supp(\eta_\rho^\beta) \cap supp(\eta_r^{(r,l)}) = \emptyset$ for all $r \geq \rho + C_4$.*

Proof (sketch). We may without loss of generality fix $\beta = e_3$, the unit vector in the z-direction. For $r = 0$ we then choose N equally spaced points $(\alpha_{0,l})_{l=0}^N$ on the equator $\{z = 0\}$ and C sufficiently large such that the functions

$$\eta(C^{-1}\|\hat{\xi} \times \alpha_{0,l}\|)$$

cover the set $\{\hat{\xi} \in \mathbb{S}^2 : |\hat{\xi}_3| \le 3/4\}$. It then remains to find a partition of unity on $\{\hat{\xi} \in \mathbb{S}^2 : |\hat{\xi}_3| > 3/4\}$. By diffeomorphism to the unit disc, it suffices to find functions $\tilde{\eta}_r^{(r,l)}$ with the required properties on $B(0,1)$, which is straightforward. $\square$

We then have the following lemma concerning the boundedness of these multipliers. We omit the proof which is similar to that of Lemma 2.5.2.

Lemma 2.2.7. *Let $1 \le q \le \infty$. For any $j \in \mathbb{Z}$, $\rho \le -C_1$, $\beta \in \mathcal{S}_\rho$, $\rho + C_1 \le r \le 0$ and $l = 1, \dots, N$ it holds*

$$\|\eta^{\beta}_{\rho+C_1}(D)P_j\phi\|_{L^q_x},\ \|\eta_r^{(r,l)}(D)P_j\phi\|_{L^q_x} \lesssim_q \|P_j\phi\|_{L^q_x}$$

2.3 Reduction to main proposition

We will work with frequency envelopes to reduce our critical global result to the subcritical local result of Theorem 2.1.3 (proved in Section 2.10). This section is largely based on Section 3 of [Tao01a].

In what follows we fix $\sigma \in (0,1)$ (which will need to be taken sufficiently small), $s \in (3/2, 3/2+\sigma)$ and $0 < \epsilon \ll 1$ which may depend on σ. We also need the following definition from [Tao01a].

Definition 3 (Frequency envelope)**.** *We call $c = (c_k)_{k\in\mathbb{Z}} \in \ell^1$ a frequency envelope if*

$$\|c\|_{\ell^1} \lesssim \epsilon$$

and

$$2^{-\sigma|k-k'|}c_{k'} \lesssim c_k \lesssim 2^{\sigma|k-k'|}c_{k'}$$

We say that (f,g) lies underneath the envelope c if

$$\|P_k(f,g)\|_{\dot{H}^{3/2}\times\dot{H}^{1/2}} \le c_k$$

for all $k \in \mathbb{Z}$.

Our first step in proving Theorem 2.1.2 is to make the following reduction, saying that the frequency profile of the solution stays roughly constant along the evolution.

Proposition 2.3.1 (Main Proposition)**.** *Let $0 < T < \infty$, c be a frequency envelope and ϕ a smooth solution to* (2.1.2) *on $[0,T]\times\mathbb{R}^3$ with initial data $\phi[0]$ satisfying the smallness* *condition*

$$\|\langle\Omega\rangle P_k\phi[0]\|_{\dot H^{3/2}\times\dot H^{1/2}} + \|\langle\Omega\rangle(x\cdot\nabla)P_k\phi[0]\|_{\dot H^{3/2}\times\dot H^{1/2}} \le c_k \tag{2.3.1}$$

for all k. Let ϕ^L be as in (2.2.6)*. Then if ϵ is sufficiently small it holds*

$$\|P_k\phi^L\|_{S_k([0,T])} \le C_0 c_k \tag{2.3.2}$$

for all $k\in\mathbb{Z}$, where $C_0 \gg 1$ is an absolute constant. In particular, $\phi[t]$ lies underneath the frequency envelope $C_0 c$ for all $t\in[0,T]$.

We quickly outline how Theorem 2.1.2 follows from this proposition (and Theorem 2.1.3). Given data $\phi[0]$ as in the statement with ϵ sufficiently small, define a frequency envelope

$$c_k := \sum_{j\in\mathbb{Z}} 2^{-\sigma|j-k|}(\|\langle\Omega\rangle P_j\phi[0]\|_{\dot H^{3/2}\times\dot H^{1/2}} + \|\langle\Omega\rangle(x\cdot\nabla)P_j\phi[0]\|_{\dot H^{3/2}\times\dot H^{1/2}})$$

It is then clear that (2.3.1) holds, so we see from the proposition that the local solution $\phi:[0,T]\times\mathbb{R}^3\to\mathbb{S}^2$ of Theorem 2.1.3 satisfies

$$\|P_k\phi\|_{S_k([0,T])} \le C_0 c_k$$

for all k. It follows that for any $s>3/2$ with $|s-3/2|<\sigma$ we have

$$\begin{aligned}
&\|P_k\phi\|_{L^\infty\dot H^s([0,T]\times\mathbb{R}^3)} + \|P_k\partial_t\phi\|_{L^\infty\dot H^{s-1}([0,T]\times\mathbb{R}^3)}\\
&\quad\lesssim 2^{(s-3/2)k}\|P_k\phi\|_{S_k([0,T])}\\
&\quad\lesssim 2^{(s-3/2)k}C_0c_k\\
&\quad\lesssim C_0\sum_{j\in\mathbb{Z}} 2^{(|s-3/2|-\sigma)|k-j|}(\|P_j\langle\Omega\rangle\phi[0]\|_{\dot H^s\times\dot H^{s-1}} + \|P_j\langle\Omega\rangle(x\cdot\nabla)\phi[0]\|_{\dot H^s\times\dot H^{s-1}})
\end{aligned}$$

for all $k\in\mathbb{Z}$, from which we see that

$$\|\phi[t]\|_{\dot B^s_{2,1}\times\dot B^{s-1}_{2,1}} \lesssim C_0(\|\langle\Omega\rangle\phi[0]\|_{\dot B^s_{2,1}\times\dot B^{s-1}_{2,1}} + \|\langle\Omega\rangle(x\cdot\nabla)\phi[0]\|_{\dot B^s_{2,1}\times\dot B^{s-1}_{2,1}})$$

for all $t\in[0,T]$. The low-frequency portion of $\phi[t]$ is straightforward to bound using energy estimates, however it is something of a distraction at this point so we postpone this to Appendix 2.A.

In summary we obtain uniform bounds on the $B^s_{2,1}\times B^{s-1}_{2,1}$ norm of the solution. Since Proposition 2.3.1 also shows that smallness in the critical space is (almost) conserved, it follows from the local theory that the solution extends globally.

Using the same argument as in [Tao01a], we see that Proposition 2.3.1 can be further reduced to the following statement, to whose proof the bulk of this chapter is dedicated.

Proposition 2.3.2 (Reduced Main Proposition)**.** *Let c be a frequency envelope, $0<T<\infty$ and ϕ be a smooth half-wave map on $[0,T]\times\mathbb{R}^3$ such that $\phi[0]$ satisfies* (2.3.1)*.*

Suppose that

$$\|P_k\phi^L\|_{S_k([0,T])} \leq 2C_0c_k \tag{2.3.3}$$

for all $k \in \mathbb{Z}$*. Then in fact*

$$\|P_k\phi^L\|_{S_k([0,T])} \leq C_0c_k \tag{2.3.4}$$

for all $k \in \mathbb{Z}$ *(assuming that* C_0 *is sufficiently large and* ϵ *is sufficiently small).*

2.4 Discarding some error terms

We now begin the first step in the proof of Proposition 2.3.2, where we will show that some terms in the forcing of equation (2.1.2) can be ignored.

Fix c, T and ϕ satisfying the hypotheses of the proposition. We need to show (2.3.4). By scaling invariance it suffices to prove that

$$\|\psi^L\|_{S_0([0,T])} \leq C_0c_0$$

for $\psi^L := P_0(\phi^L)$. We will use the notation $\psi^L \equiv (\psi_0, \psi_1, \psi_2, \psi_3)$ so that $\psi_n = P_0(L_n\phi)$.

Thanks to the linear estimate, it would be sufficient to show that

$$\|\langle\Omega\rangle\Box\psi^L\|_{L^1_tL^2_x} \lesssim C_0^3c_0\epsilon$$

and take $\epsilon(C_0)$ sufficiently small (the initial data term involving $\Box\phi$ can be bounded straightforwardly, see (2.6.3)). Unfortunately, it will not be possible to show this directly, however after some transformations we will be able to achieve a similar form, as we shall see in the coming sections.

This motivates the definition of an "error" term:

Definition 4 (Error terms)**.** *A function* $F = (F_0, \ldots, F_3)$ *on* $[0,T] \times \mathbb{R}^n$ *is said to be an* acceptable error *if*

$$\|\langle\Omega\rangle F\|_{L^1_tL^2_x} \lesssim C_0^3c_0\epsilon$$

In this case we write $F = error$*. We will also denote by* error *the components of such a vector.*

Applying P_0 to equation (2.1.2) we find

$$\Box\psi^L = P_0L(-\phi\partial_\alpha\phi^T\partial^\alpha\phi + HWM(\phi)) \tag{2.4.1}$$

where

$$HWM(\phi) := \Pi_{\phi^\perp}[((-\Delta)^{1/2}\phi)(\phi\cdot(-\Delta)^{1/2}\phi)] + \phi\times[(-\Delta)^{1/2}(\phi\times(-\Delta)^{1/2}\phi) - (\phi\times(-\Delta)\phi)]$$

Our first step is to remove the most simple frequency interactions from the wave-maps source term. We introduce the following notation for some commonly used Strichartz pairs, with $M = \infty-$ a large constant which η, σ and ϵ will all depend on.

$$\begin{aligned} (2+,\infty-) &:= (\frac{2M}{M-1}, 2M), \quad \|\langle\Omega\rangle P_k\phi^L\|_{2+,\infty-} \lesssim 2^{-(\frac{1}{2}+\frac{1}{M})k}\|P_k\phi^L\|_{S_k} \\ (\infty-,2+) &:= (M, \frac{2M}{M-2}), \quad \|\langle\Omega\rangle P_k\phi^L\|_{\infty-,2+} \lesssim 2^{-(\frac{3}{2}-\frac{2}{M})k}\|P_k\phi^L\|_{S_k} \end{aligned} \tag{2.4.2}$$

Proposition 2.4.1. *We have*

$$P_0(\phi\partial_\alpha\phi^T\partial^\alpha\phi) = 2P_0(\phi_{\leq-10}\partial_\alpha\phi^T_{\leq-10}\partial^\alpha\phi_{>-10}) + P_0(\phi_{\leq-10}\partial_\alpha\phi^T_{>-10}\partial^\alpha\phi_{>-10}) + error \tag{2.4.3}$$

and

$$\begin{aligned} P_0L_n(\phi\partial_\alpha\phi^T\partial^\alpha\phi) = {} & 2P_0((L_n\phi)_{\leq-10}\partial_\alpha\phi^T_{\leq-10}\partial^\alpha\phi_{>-10}) \\ & + 2P_0(\phi_{\leq-10}\partial_\alpha(L_n\phi)^T_{\leq-10}\partial^\alpha\phi_{>-10}) \qquad (2.4.4) \\ & + 2P_0(\phi_{\leq-10}\partial_\alpha\phi^T_{\leq-10}\partial^\alpha(L_n\phi)_{>-10}) \\ & + P_0((L_n\phi)_{\leq-10}\partial_\alpha\phi^T_{>-10}\partial^\alpha\phi_{>-10}) \\ & + P_0(\phi_{\leq-10}\partial_\alpha(L_n\phi)^T_{>-10}\partial^\alpha\phi_{>-10}) \\ & + P_0(\phi_{\leq-10}\partial_\alpha\phi^T_{>-10}\partial^\alpha(L_n\phi)_{>-10}) \\ & + error \qquad (2.4.5) \end{aligned}$$

for $n = 1, 2, 3$.

Proof. We will only show (2.4.5), (2.4.3) being similar. We start with the following observation, using that L_n commutes with the wave operator and satisfies the Leibniz rule:

$$L_n(\partial_\alpha\phi^T\partial^\alpha\phi) = \frac{1}{2}L_n(\Box(\phi^T\phi) - 2\phi^T\Box\phi) = 2\partial_\alpha(L_n\phi)^T\partial^\alpha\phi$$

Applying this property and the Leibniz rule on the whole nonlinearity we have

$$P_0L_n(\phi\partial_\alpha\phi^T\partial^\alpha\phi) = P_0((L_n\phi)\partial_\alpha\phi^T\partial^\alpha\phi) + 2P_0(\phi\partial_\alpha(L_n\phi)^T\partial^\alpha\phi)$$

Henceforth we restrict our attention to the first term, the other term being treated identically. We also drop the subscript on L_n.

Decomposing each factor of ϕ into low and high frequencies, and noting that the term vanishes when all three factors are at low frequency, we write

$$\begin{aligned} P_0((L\phi)\partial_\alpha\phi^T\partial^\alpha\phi) = {} & 2P_0((L\phi)_{>-10}\partial_\alpha\phi^T_{>-10}\partial^\alpha\phi_{\leq-10}) \qquad (2.4.6) \\ & + P_0((L\phi)_{>-10}\partial_\alpha\phi^T_{\leq-10}\partial^\alpha\phi_{\leq-10}) \qquad (2.4.7) \\ & + P_0((L\phi)_{>-10}\partial_\alpha\phi^T_{>-10}\partial^\alpha\phi_{>-10}) \qquad (2.4.8) \end{aligned}$$

$$+ 2P_0((L\phi)_{\leq -10}\partial_\alpha \phi^T_{\leq -10}\partial^\alpha \phi_{>-10}) \tag{2.4.9}$$
$$+ P_0((L\phi)_{\leq -10}\partial_\alpha \phi^T_{>-10}\partial^\alpha \phi_{>-10}) \tag{2.4.10}$$

Of these terms, (2.4.9) and (2.4.10) appear in (2.4.5) so we have to show that (2.4.6) = (2.4.7) = (2.4.8) = *error*.

- (2.4.6), (high)∇(high)∇(low):

  We have

$$\begin{aligned}&\|\langle\Omega\rangle P_0((L\phi)_{>-10}\partial_\alpha \phi^T_{>-10}\partial^\alpha \phi_{\leq -10})\|_{1,2}\\ &\lesssim \|\langle\Omega\rangle(L\phi)_{>-10}\|_{2+,\infty-}\|\langle\Omega\rangle\partial^\alpha\phi_{>-10}\|_{\infty-,2+}\|\langle\Omega\rangle\partial_\alpha\phi_{\leq-10}\|_{2+,\infty-}\end{aligned}$$

  where we used the Leibniz rule to spread the angular derivatives across the 3 terms, followed by the monotonicity of the angular Sobolev spaces. Using the definition of the S-norm and the local constancy of the frequency envelope we therefore see that

$$\begin{aligned}\|\langle\Omega\rangle(2.4.6)\|_{1,2} &\lesssim \left(\sum_{j>-10} 2^{-(\frac{1}{2}+\frac{1}{M})j}\|P_j\phi^L\|_{S_j}\right)\left(\sum_{k>-10} 2^{-(\frac{1}{2}-\frac{2}{M})k}\|P_k\phi\|_{S_k}\right)\\ &\qquad\cdot\left(\sum_{l\leq -10} 2^{(\frac{1}{2}-\frac{1}{M})l}\|P_l\phi\|_{S_l}\right)\\ &\lesssim C_0^3\epsilon^2 c_0\end{aligned}$$

- (2.4.7), (high)∇(low)∇(low):

  We similarly estimate

$$\begin{aligned}&\|\langle\Omega\rangle P_0((L\phi)_{>-10}\partial_\alpha \phi^T_{\leq -10}\partial^\alpha \phi_{\leq -10})\|_{1,2}\\ &\lesssim \|\langle\Omega\rangle(L\phi)_{>-10}\|_{\infty-,2+}\|\langle\Omega\rangle\partial^\alpha\phi_{\leq-10}\|_{2+,\infty-}\|\langle\Omega\rangle\partial_\alpha\phi_{\leq-10}\|_{2+,\infty-}\\ &\lesssim C_0^3\epsilon^2 c_0\end{aligned}$$

- (2.4.8), (high)∇(high)∇(high):

  This term cannot be handled in the standard Strichartz spaces. Observe that when $\langle\Omega\rangle$ spreads over the three terms according to the Leibniz rule, in each case there will be at least one differentiated term which is not hit by an angular derivative. This term can then be placed into a non-standard Strichartz space. For example, when $\langle\Omega\rangle$ falls on the first differentiated factor we have

$$\begin{aligned}&\|P_0((L\phi)_{>-10}\langle\Omega\rangle\partial_\alpha \phi^T_{>-10}\partial^\alpha \phi_{>-10})\|_{1,2}\\ &\lesssim \|(L\phi)_{>-10}\|_{\frac{18}{7},\infty}\|\langle\Omega\rangle\partial_\alpha\phi_{>-10}\|_{9,\frac{10}{3}}\|\partial^\alpha\phi_{>-10}\|_{2,5}\\ &\lesssim \sum_{j,k,l>-10} 2^{-\frac{7}{18}j}\|P_j\phi^L\|_{S_j}\cdot 2^{-\frac{1}{90}k}\|\phi_k\|_{S_k}\cdot 2^{-\frac{1}{10}l}\|\phi_l\|_{S_l}\\ &\lesssim C_0^3 c_0^2\epsilon\end{aligned}$$

□

Thanks to this proposition we can rewrite the frequency-localised equation as

$$\begin{aligned}\Box\psi_0 =& - 2\phi_{\leq -10}\partial_\alpha\phi^T_{\leq -10}\partial^\alpha\psi_0 - 2[P_0(\phi_{\leq -10}\partial_\alpha\phi^T_{\leq -10}\partial^\alpha\phi_{>-10}) - \phi_{\leq -10}\partial_\alpha\phi^T_{\leq -10}\partial^\alpha\psi_0] \\ & - P_0(\phi_{\leq -10}\partial_\alpha\phi^T_{>-10}\partial^\alpha\phi_{>-10}) + P_0(HWM(\phi)) + error \end{aligned} \tag{2.4.11}$$

and

$$\begin{aligned}\Box\psi_n = & -2(L_n\phi)_{\leq -10}\partial_\alpha\phi^T_{\leq -10}\partial^\alpha\psi_0 \\ & \quad - 2\phi_{\leq -10}\partial_\alpha(L_n\phi)^T_{\leq -10}\partial^\alpha\psi_0 \\ & \quad - 2\phi_{\leq -10}\partial_\alpha\phi^T_{\leq -10}\partial^\alpha\psi_n \\ & - 2[P_0((L_n\phi)_{\leq -10}\partial_\alpha\phi^T_{\leq -10}\partial^\alpha\phi_{>-10}) - (L_n\phi)_{\leq -10}\partial_\alpha\phi^T_{\leq -10}\partial^\alpha\psi_0] \\ & \quad - 2[P_0(\phi_{\leq -10}\partial_\alpha(L_n\phi)^T_{\leq -10}\partial^\alpha\phi_{>-10}) - \phi_{\leq -10}\partial_\alpha(L_n\phi)^T_{\leq -10}\partial^\alpha\psi_0] \\ & \quad - 2[P_0(\phi_{\leq -10}\partial_\alpha\phi^T_{\leq -10}\partial^\alpha(L_n\phi)_{>-10}) - \phi_{\leq -10}\partial_\alpha\phi^T_{\leq -10}\partial^\alpha\psi_n] \\ & - P_0((L_n\phi)_{\leq -10}\partial_\alpha\phi^T_{>-10}\partial^\alpha\phi_{>-10}) \\ & \quad - P_0(\phi_{\leq -10}\partial_\alpha(L_n\phi)^T_{>-10}\partial^\alpha\phi_{>-10}) \\ & \quad - P_0(\phi_{\leq -10}\partial_\alpha\phi^T_{>-10}\partial^\alpha(L_n\phi)_{>-10}) \\ & + P_0L_n(HWM(\phi)) + error \end{aligned} \tag{2.4.12}$$

for $n = 1, 2, 3$. We have now clearly identified the troublesome frequency interactions in the wave maps source term. In the next section we will show that the half-wave maps terms are acceptable, and in Section 2.6 we will discard the second and third terms (or groups of terms) via normal transformations. Lastly in Sections 2.7 and 2.8 we will show that the remaining (low)∇(low)∇(high) term can be gauged away using Tao's approximate parallel transport.

2.5 The half-wave maps contributions are negligible

We decompose the half-wave maps forcing into two terms:

$$HWM(\phi) = HWM_1(\phi) + HWM_2(\phi)$$

with

$$HWM_1(\phi) := \Pi_{\phi^\perp}[((-\Delta)^{1/2}\phi)(\phi\cdot(-\Delta)^{1/2}\phi)]$$

and

$$HWM_2(\phi) := \phi\times[(-\Delta)^{1/2}(\phi\times(-\Delta)^{1/2}\phi) - (\phi\times(-\Delta)\phi)]$$

As discussed in the introduction, we are able to discard of these terms entirely due to their geometric structures. We largely use techniques from [KK21], with a novel ingredient for handling the (low)∇(lowest)∇(high) frequency interactions (see Lemma 2.5.3).

Before preceding to the estimates, we present some lemmas which will be used frequently in the sequel. Denote

$$\mathcal{L}_k(u_{k_1}, v_{k_2}) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} m_k(\xi,\eta) e^{ix\cdot(\xi+\eta)} \chi_{k_1}(\xi)\hat{u}(\xi)\chi_{k_2}(\eta)\hat{v}(\eta) d\xi d\eta \tag{2.5.1}$$

for m_k any smooth multiplier satisfying the pointwise bounds

$$|m_k(\xi,\eta)| \lesssim 2^k, \ |(2^{k_1}\nabla_\xi)^i (2^{k_2}\nabla_\eta)^j m_k(\xi,\eta)| \lesssim_{i,j} 2^k$$

on the support of $\chi_{k_1}(\xi)\chi_{k_2}(\eta)$. Note that such a multiplier can be expanded as a Fourier series with rapidly decaying coefficients on the support of $\chi_{k_1}(\xi)\chi_{k_2}(\eta)$:

$$m_k(\xi,\eta) = \sum_{a,b\in\mathbb{Z}^3} c^{(k)}_{a,b} e^{-i(2^{-k_1}a\cdot\xi + 2^{-k_2}b\cdot\eta)} \quad \text{with } |c^{(k)}_{a,b}| \lesssim_N 2^k \langle a\rangle^{-N}\langle b\rangle^{-N} \text{ for any } N\in\mathbb{N}. \tag{2.5.2}$$

We can therefore, at least formally, write

$$\mathcal{L}_k(u_{k_1}, v_{k_2}) = \sum_{a,b\in\mathbb{Z}^3} c^{(k)}_{a,b} u_{k_1}(x - 2^{-k_1}a) v_{k_2}(x - 2^{-k_2}b) \tag{2.5.3}$$

Operators of this form arise in studying cancellations in $HWM_2(\phi)$, and an important property is given by the following

Lemma 2.5.1 (Lemma 3.1, [KS17]). *Let $\mathcal{L}_k$ be as above. Then if $\|\cdot\|_Z$, $\|\cdot\|_X$, $\|\cdot\|_Y$ are translation invariant norms with the property that*

$$\|u\cdot v\|_Z \le \|u\|_X \|v\|_Y$$

it holds

$$\|\mathcal{L}_k(u_{k_1}, v_{k_2})\|_Z \lesssim 2^k \|u_{k_1}\|_X \|v_{k_2}\|_Y$$

In particular this lemma tells us we can (and should) think of $\mathcal{L}_j(\phi_j, \phi_k)$ as $\partial\phi_j \cdot \phi_k$.

Due to the generally nonlocal nature of these operators, they interact non-trivially with the non-translation invariant commuting vector fields. In fact for $k, k_1, k_2 \in \mathbb{Z}$, $n = 1,2,3$, $i,j = 1,2,3$ it holds

$$L_n(\mathcal{L}_k(u_{k_1}, v_{k_2})) = \mathcal{L}_k(L_n u_{k_1}, v_{k_2}) + \mathcal{L}_k(u_{k_1}, L_n v_{k_2}) + \mathcal{L}_{k-k_1}(\partial_t u_{k_1}, u_{k_2}) + \mathcal{L}_{k-k_2}(u_{k_1}, \partial_t u_{k_2}) \tag{2.5.4}$$

and

$$\Omega_{ij}(\mathcal{L}_k(u_{k_1}, v_{k_2})) = \mathcal{L}_k(\Omega_{ij} u_{k_1}, v_{k_2}) + \mathcal{L}_k(u_{k_1}, \Omega_{ij} v_{k_2}) + \mathcal{L}_k(u_{k_1}, v_{k_2}) \tag{2.5.5}$$

The $\mathcal{L}_k$ in these expressions need not all correspond to the same multiplier m_k.

Lastly, we note the following basic facts which follow from the commutation relations between Ω, L and P_k. For any $(p,q) \in \mathcal{Q}$ and $a \in \mathbb{R}$ it holds[9]

$$\|\langle\Omega\rangle^{1-\delta(p,q)} L(u_k(\cdot + a))\|_{p,q} \lesssim \langle 2^k a\rangle^2 2^{-(\frac{1}{p}+\frac{3}{q})k} \|P_{\sim k} u^L\|_{S_{\sim k}} \tag{2.5.6}$$

and if further $q \neq \infty$ we also have[10]

$$\|\langle\Omega\rangle^{1-\delta(p,q)} L(-\Delta)^{1/2} u_k\|_{p,q} \lesssim 2^{(1-\frac{1}{p}-\frac{3}{q})k} \|P_{\sim k} u^L\|_{S_{\sim k}}$$

We now come to the most important lemma of this chapter, which is the key ingredient for handling the (low)∇(lowest)∇(high) terms in 3 dimensions. For notation used in the statement we refer the reader to Section 2.2.3.

Lemma 2.5.2 (Angular Separation Estimate)**.** *Fix $\rho \leq -C_1$. Then for any $\lambda, k \in \mathbb{Z}$, $\rho + C_1 \leq r \leq 0$, $l = 0, \ldots, N$ it holds*

$$\|\varphi_\lambda(x)\sigma_\rho^\beta(x)\eta_r^{(r,l)}(D)\phi_k\|_{L^q_x} \lesssim 2^{-(\lambda+k+2r)}(\|\phi_k\|_{L^q_x} + \max_{i,j} \|\Omega_{ij}\phi_k\|_{L^q_x})$$

for any $1 \leq q \leq \infty$.

The intuition for this estimate was discussed in the introduction, and leads us to expect a preferable loss of $2^{-(\lambda+j+r)}$. Unfortunately, we were not quite able to achieve this. This comes from the Ω_{ij} being non-translation invariant.

Proof. We may assume without loss of generality that $\alpha_{r,l} = e_1$, and by scaling it suffices to consider $k = 0$. We may further assume that β lies in the $x - y$ plane so that $|\hat{x}_1\hat{\xi}_2 - \hat{x}_2\hat{\xi}_1| \gtrsim 2^r$ for any $\hat{x}$, $\hat{\xi} \in \mathbb{S}^2$ in the supports of σ_ρ^β, $\eta_r^{(r,l)}$ respectively.

Write

$$\varphi_\lambda(x)\sigma_\rho^\beta(x)\eta_r^{(r,l)}(D)\phi_0 = \int_\xi e^{ix\cdot\xi} m(x,\xi) \cdot (x_1\xi_2 - x_2\xi_1)\hat{\phi}_0(\xi)d\xi$$

for

$$m(x,\xi) := \varphi_\lambda(x)\sigma_\rho^\beta(x)\tilde{\chi}_0(\xi)\eta_r^{(r,l)}(\xi)(x_1\xi_2 - x_2\xi_1)^{-1}$$

Expand m as a Fourier series in ξ. Since $\alpha_{r,l} = e_1$ we have $\operatorname{supp} m \subset \{\xi_1 \sim 1, |\xi_2|, |\xi_3| \lesssim 2^r\}$ so

$$m(x,\xi) = \varphi_\lambda(x)\sigma_\rho^\beta(x) \sum_{p\in\mathbb{Z}^3} c_p(x) e^{2\pi i(\xi_1 p_1 + 2^{-r}\xi_2 p_2 + 2^{-r}\xi_3 p_3)}$$

[9] This may be interpreted as saying that translation does not affect the norm provided we translate on scales at most comparable to the natural oscillation length of u_k.

[10] The restriction to $q < \infty$ comes from the need to bound the Riesz transform appearing in $[L, (-\Delta)^{1/2}]$. In practice this is not important since there is usually enough flexibility in the estimates to lower q using Bernstein's inequality.

where

$$c_p(x) \simeq 2^{-2r} \int_{\substack{|\xi_1|\sim 1 \\ |\xi_2|,|\xi_3|\lesssim 2^r}} \tilde{\chi}_0(\xi)\eta_r^{(r,l)}(\xi)(x_1\xi_2 - x_2\xi_1)^{-1} e^{-2\pi i(\xi_1 p_1 + 2^{-r}\xi_2 p_2 + 2^{-r}\xi_3 p_3)} d\xi$$

We want to integrate by parts so need bounds on the derivatives of the integrand. A calculation yields

$$|\nabla_\xi^\gamma \eta_r^{(r,l)}(\xi)| \lesssim_\gamma 2^{-(\gamma_2+\gamma_3)r}$$

for all $\gamma \in \mathbb{N}^3$, $\xi \in \operatorname{supp}(\tilde{\chi}_0) \cap \operatorname{supp}(\eta_r^{(r,l)})$.

Furthermore, for x, ξ in the support of m, we have $|x_2| \leq |x| \|\hat{x} \times \alpha_{r,l}\| \lesssim 2^{\lambda+r}$ and so

$$\begin{aligned} |\partial_{\xi_1}^{\gamma_1}\partial_{\xi_2}^{\gamma_2}(x_1\xi_2 - x_2\xi_1)^{-1}| =& |x_1\xi_2 - x_2\xi_1|^{-(\gamma_1+\gamma_2+1)}|x_2|^{\gamma_1}|x_1|^{\gamma_2} \\ \lesssim& 2^{-(\lambda+r)(\gamma_1+\gamma_2+1)}2^{(\lambda+r)\gamma_1}2^{\lambda\gamma_2} = 2^{-(\lambda+r)}2^{-r\gamma_2} \end{aligned}$$

It follows that

$$\begin{aligned} &|\nabla_\xi^\gamma(\tilde{\chi}_0(\xi)\eta_r^{(r,l)}(\xi)(x_1\xi_2 - x_2\xi_1)^{-1})| \\ \lesssim& \sum_{\gamma^{(1)}+\gamma^{(2)}+\gamma^{(3)}=\gamma} |\nabla_\xi^{\gamma^{(1)}}\tilde{\chi}_0(\xi) \cdot \nabla_\xi^{\gamma^{(2)}}\eta_r^{(r,l)}(\xi) \cdot \nabla_\xi^{\gamma^{(3)}}((x_1\xi_2 - x_2\xi_2)^{-1})| \\ \lesssim& \sum_{\substack{\gamma^{(1)}+\gamma^{(2)}+\gamma^{(3)}=\gamma, \\ \gamma_3^{(3)}=0}} 1 \cdot 2^{-(\gamma_2^{(2)}+\gamma_3^{(2)})r} \cdot 2^{-(\lambda+r)}2^{-\gamma_2^{(3)}r} \\ \lesssim& 2^{-(\lambda+r)} \sum_{\substack{\gamma^{(1)}+\gamma^{(2)}+\gamma^{(3)}=\gamma, \\ \gamma_3^{(3)}=0}} 2^{-(\gamma_2^{(2)}+\gamma_2^{(3)}+\gamma_3^{(2)})r} \end{aligned}$$

For $r \leq 0$, the right hand side of this expression is largest when $\gamma_2^{(2)} + \gamma_2^{(3)} = \gamma_2$ and $\gamma_3^{(2)} = \gamma_3$, leading to

$$|\nabla_\xi^\gamma(\tilde{\chi}_0(\xi)\eta_r^{(r,l)}(\xi)(x_1\xi_2 - x_2\xi_1)^{-1})| \lesssim 2^{-(\lambda+r)}2^{-(\gamma_2+\gamma_3)r}$$

Integrating by parts in the expression for $c_p(x)$ we therefore obtain

$$|c_p(x)| \lesssim \frac{2^{-2r}}{p_1^{\gamma_1}(2^{-r}p_2)^{\gamma_2}(2^{-r}p_3)^{\gamma_3}} \int_{\substack{|\xi_1|\sim 1 \\ |\xi_2|,|\xi_3|\lesssim 2^r}} 2^{-(\lambda+r)}2^{-(\gamma_2+\gamma_3)r} d\xi \lesssim \frac{2^{-(\lambda+r)}}{\langle p\rangle^{|\gamma|}}$$

With this bound we calculate, for any $N \in \mathbb{N}$,

$$\|\mu_2(x)\varphi_\lambda(x)\sigma_\rho^\beta(x)\eta_r^{(r,l)}(D)\phi_0\|_{L_x^q}$$

$$\lesssim_N \sum_p \frac{2^{-(\lambda+r)}}{\langle p\rangle^N} \| \int_\xi e^{i((x_1+2\pi p_1)\xi_1+(x_2+2\pi 2^{-r}p_2)\xi_2+(x_3+2\pi 2^{-r}p_3)\xi_3)}(x_1\xi_2 - x_2\xi_1)\hat{\phi}_0(\xi)d\xi\|_{L^q_x}$$
$$\lesssim_N \sum_p \frac{2^{-(\lambda+r)}}{\langle p\rangle^N} \|(x_1\partial_2 - x_2\partial_1)(\phi_0(x_1+2\pi p_1, x_2+2^{-r}2\pi p_2, x_3+2^{-r}2\pi p_3))\|_{L^q_x}$$
$$\lesssim_N \sum_p \frac{2^{-(\lambda+r)}}{\langle p\rangle^N} (\|(x_1\partial_2 - x_2\partial_1)\phi_0\|_{L^q_x} + 2^{-r}|p|\|\nabla\phi_0\|_{L^q_x})$$
$$\lesssim_N \sum_p \frac{2^{-(\lambda+2r)}}{\langle p\rangle^{N-1}} (\|\Omega_{1,2}\phi_0\|_{L^q_x} + \|\phi_0\|_{L^q_x})$$

Choosing N sufficiently large and summing over p gives the desired result. □

As a consequence of this lemma we can bound certain trilinear terms as follows.

Lemma 2.5.3. *Let $m, j, k \in \mathbb{Z}$ and fix M sufficiently large. Then for any (scalar) functions $\phi_j^{(1)}$, $\phi_k^{(2)}$, $\phi_m^{(3)}$ we have the following estimates:*

1. *If $j \lesssim k$, $j \lesssim m$ we have*

$$\begin{aligned}&\|\phi_j^{(1)} \cdot \phi_k^{(2)} \cdot \phi_m^{(3)}\|_{1,2} \\ &\lesssim 2^{-j/M}\|\phi_j^{(1)}\|_{\frac{2M}{M-1},\infty}(2^{3k/2M}\|\langle\Omega\rangle\phi_k^{(2)}\|_{\frac{2M}{M-1},2M} + 2^{3k\frac{M-1}{4M}}\|\phi_k^{(2)}\|_{\frac{2M}{M-1},\frac{4M}{M-1}}) \\ &\qquad \cdot (\|\phi_m^{(3)}\|_{\infty,2} + 2^{j-m}\|L\phi_m^{(3)}\|_{\infty,2} + 2^{-m}\|\partial_t\phi_m^{(3)}\|_{\infty,2})\end{aligned}$$

2. *If $j \lesssim m$ we have*

$$\begin{aligned}&\|\phi_j^{(1)} \cdot \phi_k^{(2)} \cdot \phi_m^{(3)}\|_{1,2} \\ &\lesssim 2^{-j/M}(2^{3j/2M}\|\langle\Omega\rangle\phi_j^{(1)}\|_{\frac{2M}{M-1},2M} + 2^{3j\frac{M-1}{4M}}\|\phi_j^{(1)}\|_{\frac{2M}{M-1},\frac{4M}{M-1}})\|\phi_k^{(2)}\|_{\frac{2M}{M-1},\infty} \\ &\qquad \cdot (\|\phi_m^{(3)}\|_{\infty,2} + 2^{j-m}\|L\phi_m^{(3)}\|_{\infty,2} + 2^{-m}\|\partial_t\phi_m^{(3)}\|_{\infty,2})\end{aligned}$$

3. *If $j \lesssim k$ we have*

$$\begin{aligned}&\|\phi_j^{(1)} \cdot \phi_k^{(2)} \cdot \phi_m^{(3)}\|_{1,2} \\ &\lesssim 2^{-j/M}2^{3k/2M}\|\phi_j^{(1)}\|_{\frac{2M}{M-1},\infty} \\ &\quad \cdot (\|\langle\Omega\rangle\phi_k^{(2)}\|_{\frac{2M}{M-1},2M} + 2^{j-k}\|\langle\Omega\rangle L\phi_k^{(2)}\|_{\frac{2M}{M-1},2M} + 2^{-k}\|\langle\Omega\rangle\partial_t\phi_k^{(2)}\|_{\frac{2M}{M-1},2M})\|\phi_m^{(3)}\|_{\infty,2} \\ &\quad + 2^{-j/M}2^{3k\frac{M-1}{4M}}\|\phi_j^{(1)}\|_{\frac{2M}{M-1},\infty} \\ &\quad \cdot (\|\phi_k^{(2)}\|_{\frac{2M}{M-1},\frac{4M}{M-1}} + 2^{j-k}\|L\phi_k^{(2)}\|_{\frac{2M}{M-1},\frac{4M}{M-1}} + 2^{-k}\|\partial_t\phi_k^{(2)}\|_{\frac{2M}{M-1},\frac{4M}{M-1}})\|\phi_m^{(3)}\|_{\infty,2}\end{aligned}$$

4. *If $j \lesssim k$ we have*

$$\|\phi_j^{(1)} \cdot \phi_k^{(2)} \cdot \phi_m^{(3)}\|_{1,2}$$

$$
\begin{aligned}
&\lesssim 2^{-j/M}2^{3j/2M}\|\langle\Omega\rangle\phi_j^{(1)}\|_{\frac{2M}{M-1},2M} \\
&\cdot(\|\phi_k^{(2)}\|_{\frac{2M}{M-1},\infty}+2^{j-k}\|L\phi_k^{(2)}\|_{\frac{2M}{M-1},\infty}+2^{-k}\|\partial_t\phi_k^{(2)}\|_{\frac{2M}{M-1},\infty})\|\phi_m^{(3)}\|_{\infty,2} \\
&+2^{-j/M}2^{3j\frac{M-1}{4M}}\|\phi_j^{(1)}\|_{\frac{2M}{M-1},\frac{4M}{M-1}} \\
&\cdot(\|\phi_k^{(2)}\|_{\frac{2M}{M-1},\infty}+2^{j-k}\|L\phi_k^{(2)}\|_{\frac{2M}{M-1},\infty}+2^{-k}\|\partial_t\phi_k^{(2)}\|_{\frac{2M}{M-1},\infty})\|\phi_m^{(3)}\|_{\infty,2}
\end{aligned}
$$

We note that since the above holds for arbitrary scalar functions, in our setting the same will hold for vector functions independent of the type of multiplication used (dot product, cross product,...) and the order of the terms.

Before going into the proof, let us shed some light on the relevance of this lemma. It will be applied to terms of the form

$$
\sum_{j\le k\le -10}\|L\langle\Omega\rangle(\phi_k\ \nabla\phi_j\cdot\nabla\phi_{\sim 0})\|_{1,2} \tag{2.5.7}
$$

which are beyond the reach of the standard Strichartz estimates. For example, when L and $\langle\Omega\rangle$ both fall on $\nabla\phi_j$ we use point 1 of the above lemma to find

$$
\begin{aligned}
&\sum_{j\le k\le -10}\|\phi_k\ L\langle\Omega\rangle\nabla\phi_j\cdot\nabla\phi_{\sim 0}\|_{1,2} \\
&\lesssim \sum_{j\le k\le -10}2^{-j/M}\|L\langle\Omega\rangle\nabla\phi_j\|_{\frac{2M}{M-1},\infty}(2^{3k/2M}\|\langle\Omega\rangle\phi_k\|_{\frac{2M}{M-1},2M}+2^{3k\frac{M-1}{4M}}\|\phi_k\|_{\frac{2M}{M-1},\frac{4M}{M-1}}) \\
&\qquad\qquad\qquad\qquad\cdot(\|\nabla\phi_{\sim 0}\|_{\infty,2}+2^j\|L\nabla\phi_{\sim 0}\|_{\infty,2}+\|\partial_t\nabla\phi_{\sim 0}\|_{\infty,2}) \\
&\lesssim \sum_{j\le k\le -10}2^{(\frac12-\frac{1}{2M})(j-k)}C_0^3c_jc_kc_0\lesssim C_0^3\epsilon^2c_0
\end{aligned} \tag{2.5.8}
$$

where we were able to place ϕ_k into S_k in both cases since it only appears in a non-standard Strichartz space when not accompanied by an angular derivative.

The same argument works for any combination in which $\langle\Omega\rangle$ falls on ϕ_j or $\phi_{\sim 0}$, and L on ϕ_k or ϕ_j. If $\langle\Omega\rangle$ falls on ϕ_k, and L still doesn't hit $\phi_{\sim 0}$, we obtain the same result using point 2 of the lemma. Points 3 and 4 are for when L hits $\phi_{\sim 0}$ and $\langle\Omega\rangle$ avoids or hits ϕ_k respectively.

Due to the non-local nature of our equation, frequency interactions of the type discussed will appear in many different guises, which is why we give the lemma in such generality.

Proof of Lemma 2.5.3. We focus on point 1, noting the adaptations needed for the other cases at the end.

Using the notation introduced in Section 2.2 we first split the term over regions where

$|x|$ is large or small compared to the natural oscillations of $\phi_j^{(1)}$:

$$\|\phi_j^{(1)} \cdot \phi_k^{(2)} \cdot \phi_m^{(3)}\|_{1,2} \lesssim \underbrace{\|\varphi_{<-j}(x)(\phi_j^{(1)} \cdot \phi_k^{(2)} \cdot \phi_m^{(3)})\|_{1,2}}_{(A)} + \underbrace{\|\varphi_{\geq -j}(x)(\phi_j^{(1)} \cdot \phi_k^{(2)} \cdot \phi_m^{(3)})\|_{1,2}}_{(B)}$$

Starting with (A), we further split the norm depending on the size of $|t|$:

$$(A) \lesssim \underbrace{\|\varphi_{<-j}(x)\varphi_{<-j}(t)(\phi_j^{(1)} \cdot \phi_k^{(2)} \cdot \phi_m^{(3)})\|_{1,2}}_{(A.I)} + \underbrace{\|\varphi_{<-j}(x)\varphi_{\geq -j}(t)(\phi_j^{(1)} \cdot \phi_k^{(2)} \cdot \phi_m^{(3)})\|_{1,2}}_{(A.II)}$$

To reduce notation, we will often omit the space/time cut-offs. Starting with (A.I), we use Hölder in the time variable to obtain

$$(A.I) \lesssim \|\phi_j^{(1)}\|_{2,\infty}\|\phi_k^{(2)}\|_{2,\infty}\|\phi_m^{(3)}\|_{\infty,2} \lesssim 2^{-j/M}\|\phi_j^{(1)}\|_{\frac{2M}{M-1},\infty}\|\phi_k^{(2)}\|_{\frac{2M}{M-1},\infty}\|\phi_m^{(3)}\|_{\infty,2}$$

Using Bernstein's inequality on $\phi_k^{(2)}$ and the monotonicity of the angular Sobolev spaces we see that this term is as required.

We now study (A.II). To counteract the loss coming from the use of Hölder's inequality in time we use a trick that will come up frequently in the sequel: since $\phi_m^{(3)}$ has not yet been acted on by any Lorentz boost we can write

$$\phi_m^{(3)} = t^{-1}\Delta^{-1}\partial_n L_n\phi_m^{(3)} - \Delta^{-1}\partial_n[(t^{-1}x_n)\partial_t\phi_m^{(3)}] \tag{2.5.9}$$

with the implicit sum over $n = 1, 2, 3$. A simple computation using that the spatial localisation passes through the Fourier multipliers up to exponentially decaying tails (since $m \gtrsim j$) yields

$$\|\varphi_{<-j}(x)\varphi_{k_1}(t)\phi_m^{(3)}\|_{\infty,2} \lesssim 2^{-k_1-m}\|L\phi_m^{(3)}\|_{\infty,2} + 2^{-k_1-j-m}\|\partial_t\phi_m^{(3)}\|_{\infty,2} \tag{2.5.10}$$

Therefore

$$\begin{aligned}(A.II) &\lesssim \sum_{k_1 \geq -j} \|\phi_j^{(1)}\|_{2,\infty}\|\phi_k^{(2)}\|_{2,\infty}\|\varphi_{<-j}(x)\varphi_{k_1}(t)\phi_m^{(3)}\|_{\infty,2} \\ &\lesssim \sum_{k_1 \geq -j} 2^{k_1/M}\|\phi_j^{(1)}\|_{\frac{2M}{M-1},\infty}\|\phi_k^{(2)}\|_{\frac{2M}{M-1},\infty}(2^{-k_1-m}\|L\phi_m^{(3)}\|_{\infty,2} + 2^{-k_1-j-m}\|\partial_t\phi_m^{(3)}\|_{\infty,2}) \\ &\lesssim 2^{-j/M}\|\phi_j^{(1)}\|_{\frac{2M}{M-1},\infty}\|\phi_k^{(2)}\|_{\frac{2M}{M-1},\infty}(2^{j-m}\|L\phi_m^{(3)}\|_{\infty,2} + 2^{-m}\|\partial_t\phi_m^{(3)}\|_{\infty,2})\end{aligned}$$

which is as required.

We now turn to

$$(B) = \|\varphi_{\geq -j}(x)(\phi_j^{(1)} \cdot \phi_k^{(2)} \cdot \phi_m^{(3)})\|_{1,2}$$

It is here that we need to invoke the angular multipliers. Write

$$(B) \leq \sum_{k_1 \geq -j} \sum_{k_2 \in \mathbb{Z}} \underbrace{\|\varphi_{k_1}(x)\varphi_{k_2}(t)(\phi_j^{(1)} \cdot \phi_k^{(2)} \cdot \phi_m^{(3)})\|_{1,2}}_{(B)_{k_1,k_2}}$$

Fix $k_1 \geq -j$. Let $(\sigma^\beta_{-(j+k_1)/3})_{\beta \in \mathcal{S}_{j,k_1}}$ be a partition of unity on $\mathbb{S}^2$ as in (2.2.7) (denoting $\mathcal{S}_{j,k_1} := \mathcal{S}_{-(j+k_1)/3}$). Then for fixed k_1, k_2 we can split $(B)_{k_1,k_2}$ into a square-sum

$$(B)_{k_1,k_2} \lesssim \left\| \left(\sum_{\beta \in \mathcal{S}_{j,k_1}} \|\sigma^\beta_{-(j+k_1)/3}(x)(\phi_j^{(1)} \cdot \phi_k^{(2)} \cdot \phi_m^{(3)})\|^2_{L^2_x(|x|\sim 2^{k_1})} \right)^{\frac{1}{2}} \right\|_{L^1_t(|t|\sim 2^{k_2})}$$

Now, for each $\beta \in \mathcal{S}_{j,k_1}$, use the Fourier multipliers introduced in (2.2.8) to write

$$\phi_k^{(2)} = \eta^\beta_{-(j+k_1)/3}(D)\phi_k^{(2)} + \sum_{l=1}^{N} \sum_{-(j+k_1)/3 \ll r \leq 0} \eta_r^{(r,l)}(D)\phi_k^{(2)} \tag{2.5.11}$$

Let's start with the first term, where the angular localisations in Fourier space and physical space are forced to be close. We find

$$\begin{aligned} &\left\| \left(\sum_{\beta \in \mathcal{S}_{j,k_1}} \|\sigma^\beta_{-(j+k_1)/3}(x)(\phi_j^{(1)} \cdot \eta^\beta_{-(j+k_1)/3}(D)\phi_k^{(2)} \cdot \phi_m^{(3)})\|^2_{L^2_x(|x|\sim 2^{k_1})} \right)^{\frac{1}{2}} \right\|_{L^1_t(|t|\sim 2^{k_2})} \\ &\quad \lesssim \left\| \left(\sum_{\beta \in \mathcal{S}_{j,k_1}} (\|\phi_j^{(1)}\|_\infty \|\eta^\beta_{-(j+k_1)/3}(D)\phi_k^{(2)}\|_\infty \|\sigma^\beta_{-(j+k_1)/3}(x)\phi_m^{(3)}\|_2)^2 \right)^{\frac{1}{2}} \right\|_{L^1_t(|t|\sim 2^{k_2})} \end{aligned} \tag{2.5.12}$$

We then use Bernstein's inequality on the middle term to benefit from the close angular localisation, and thereby bound the above by

$$\begin{aligned} &\|\phi_j^{(1)}\|_{2,\infty} \left\| \left(\sum_{\beta \in \mathcal{S}_{j,k_1}} (2^{(3k-2(j+k_1)/3)(\frac{M-1}{4M})} \|\eta^\beta_{-(j+k_1)/3}(D)\phi_k^{(2)}\|_{\frac{4M}{M-1}} \|\sigma^\beta_{-(j+k_1)/3}(x)\phi_m^{(3)}\|_2)^2 \right)^{\frac{1}{2}} \right\|_2 \\ &\lesssim \|\phi_j^{(1)}\|_{2,\infty} \cdot 2^{(3k-2(j+k_1)/3)(\frac{M-1}{4M})} \|\phi_k^{(2)}\|_{2,\frac{4M}{M-1}} \cdot \|\phi_m^{(3)}\|_{\infty,2} \\ &\lesssim 2^{k_2/M} 2^{(3k-2(j+k_1)/3)(\frac{M-1}{4M})} \|\phi_j^{(1)}\|_{\frac{2M}{M-1},\infty} \|\phi_k^{(2)}\|_{\frac{2M}{M-1},\frac{4M}{M-1}} \|\phi_m^{(3)}\|_{\infty,2} \end{aligned} \tag{2.5.13}$$

where we used that the operator $\eta^\beta_{-(j+k_1)/3}(D)$ is bounded and square summed over the $\sigma^\beta_{-(j+k_1)/3}$.

Now, fixing M sufficiently large this gives an acceptable bound in the range $k_2 \leq k_1$:

$$\sum_{\substack{k_1\geq -j,\\ k_2\leq k_1}}\left\|\left(\sum_{\beta\in\mathcal{S}_{j,k_1}}\|\sigma^\beta_{-(j+k_1)/3}(x)(\phi^{(1)}_j\cdot\eta^\beta_{-(j+k_1)/3}(D)\phi^{(2)}_k\cdot\phi^{(3)}_m)\|^2_{L^2_x(|x|\sim 2^{k_1})}\right)^{\frac{1}{2}}\right\|_{L^1_t(|t|\sim 2^{k_2})}$$
$$\lesssim 2^{-j/M}2^{3k\frac{M-1}{4M}}\|\phi^{(1)}_j\|_{\frac{2M}{M-1},\infty}\|\phi^{(2)}_k\|_{\frac{2M}{M-1},\frac{4M}{M-1}}\|\phi^{(3)}_m\|_{\infty,2}$$

For $k_2>k_1$, we obtain decay in t via an estimate analogous to (2.5.10),

$$\|\varphi_{k_1}(x)\varphi_{k_2}(t)\phi^{(3)}_m\|_{\infty,2}\lesssim 2^{-k_2-m}\|L\phi^{(3)}_m\|_{\infty,2}+2^{k_1-k_2-m}\|\partial_t\phi^{(3)}_m\|_{\infty,2}\tag{2.5.14}$$

and the desired result follows.

We now turn to the second term in (2.5.11), the "far-angle case". We use the angular separation estimate, Lemma 2.5.2, followed by Bernstein's inequality and the Riesz estimate for angular derivatives (which only holds for finite exponents) to bound

$$\|\varphi_{k_1}(x)\sigma^\beta_{-(j+k_1)/3}(x)\eta^{(r,l)}_r(D)\phi^{(2)}_k\|_\infty\lesssim 2^{-(k_1+k+2r)}(\|\phi^{(2)}_k\|_\infty+\max_{i,j}\|\Omega_{ij}\phi^{(2)}_k\|_\infty)$$
$$\lesssim 2^{-(k_1+k+2r)}2^{3k/2M}\|\langle\Omega\rangle\phi^{(2)}_k\|_{2M}$$

Therefore

$$\sum_{l=1}^N\sum_{-(j+k_1)/3\ll r}\|\sigma^\beta_{-(j+k_1)/3}(x)(\phi^{(1)}_j\cdot\eta^{(r,l)}_r(D)\phi^{(2)}_k\cdot\phi^{(3)}_m)\|_{L^2_x(|x|\sim 2^{k_1})}$$
$$\lesssim 2^{-(k_1+k)}2^{2(j+k_1)/3}2^{3k/2M}\|\phi^{(1)}_j\|_\infty\|\langle\Omega\rangle\phi^{(2)}_k\|_{2M}\|\sigma^\beta_{-(j+k_1)/3}(x)\phi^{(3)}_m\|_2\tag{2.5.15}$$

Lastly the $L^1_{(|t|\sim 2^{k_2})}$ norm of the square-sum of (2.5.15) over $\beta\in\mathcal{S}_{j,k_1}$ is bounded by

$$2^{k_2/M}2^{-(k_1+k)}2^{2(j+k_1)/3}2^{3k/2M}\|\phi^{(1)}_j\|_{\frac{2M}{M-1},\infty}\|\langle\Omega\rangle\phi^{(2)}_k\|_{\frac{2M}{M-1},2M}\|\phi^{(3)}_m\|_{\infty,2}\tag{2.5.16}$$

Summed over $k_2\leq k_1$ and $k_1\geq -j$ this gives

$$2^{-j/M}2^{j-k}2^{3k/2M}\|\phi^{(1)}_j\|_{\frac{2M}{M-1},\infty}\|\langle\Omega\rangle\phi^{(2)}_k\|_{\frac{2M}{M-1},2M}\|\phi^{(3)}_m\|_{\infty,2}$$

which is acceptable since $j\lesssim k$. In the case of large t we again apply (2.5.14) before summing over $k_2>k_1$.

To prove point 2 of the Lemma, we perform the same argument but carry out the angular decomposition in Fourier space on $\phi^{(1)}_j$ instead of $\phi^{(2)}_k$. In the far-angle case we no longer gain a factor of 2^{j-k} so the restriction $j\lesssim k$ is not necessary. For point 3, we do not change the angular decomposition but get the gain in $|t|^{-1}$ from $\phi^{(2)}_k$ rather than $\phi^{(3)}_m$, using the estimate

$$\|\varphi_{k_1}(x)\varphi_{k_2}(t)\phi^{(2)}_k\|_{2,\infty}\lesssim 2^{-k_2-k}\|L\phi^{(2)}_k\|_{2,\infty}+2^{k_1-k_2-k}\|\partial_t\phi^{(2)}_k\|_{2,\infty}$$

for $k + k_1 \gtrsim 0$. For point 4 we use both of the adaptations described above. □

Due to some error terms which appear as a result of the commutation relations (2.2.1)-(2.2.3), we will also need the following form of Lemma 2.5.3.

Corollary 2.5.4. *Let K_j be a convolution operator given by a Schwarz kernel $k_j(x) := 2^{3k}k(2^j x)$. Then for $m, r, s \gtrsim j$ it holds*

1. $$\|K_j(\phi_r^{(1)}\phi_s^{(2)})\,\phi_m^{(3)}\|_{1,2} \lesssim 2^{-j/M}(2^{3j\frac{M-1}{4M}}2^{\frac{21}{4M}(r-j)}\|\phi_r^{(1)}\|_{\frac{2M}{M-1},\frac{4M}{M-1}} + 2^{3r/2M}2^{j-r}\|\langle\Omega\rangle\phi_r^{(1)}\|_{\frac{2M}{M-1},2M}) \cdot \|\phi_s^{(2)}\|_{\frac{2M}{M-1},\infty}(\|\phi_m^{(3)}\|_{\infty,2} + 2^{j-m}\|L\phi_m^{(3)}\|_{\infty,2} + 2^{-m}\|\partial_t\phi_m^{(3)}\|_{\infty,2})$$
2. $$\|K_j(\phi_r^{(1)}\phi_s^{(2)})\,\phi_m^{(3)}\|_{1,2} \lesssim 2^{-j/M}(2^{3j\frac{M-1}{4M}}2^{\frac{21}{4M}(r-j)}\|\phi_r^{(1)}\|_{\frac{2M}{M-1},\frac{4M}{M-1}} + 2^{3r/2M}2^{j-r}\|\langle\Omega\rangle\phi_r^{(1)}\|_{\frac{2M}{M-1},2M}) \cdot (\|\phi_s^{(2)}\|_{\frac{2M}{M-1},\infty} + 2^{j-s}\|L\phi_s^{(2)}\|_{\frac{2M}{M-1},\infty} + 2^{-s}\|\partial_t\phi_s^{(2)}\|_{\frac{2M}{M-1},\infty})\|\phi_m^{(3)}\|_{\infty,2}$$

The important thing to note here is the gain in powers of 2^j rather than 2^r (up to a small amount of leakage).

The proof of this corollary relies on the following simple proposition, which says that angular localisation passes through convolution up to exponentially decaying tails.

Proposition 2.5.5. *Let $1 \le q \le p \le \infty$. Let K_j be as in the corollary. Then the following commutator estimates hold for any $N \in \mathbb{N}$.*

1. *Let $l > k_1 + 5$. Then* $\|\varphi_{k_1} \cdot K_j(\varphi_l F)\|_{L^p_x} \lesssim_N 2^{-(l+j)N}2^{3(\frac{1}{q}-\frac{1}{p})j}\|F\|_{L^q_x}$.
2. $\|\varphi_{k_1} \cdot K_j(\varphi_{<k_1-5}F)\|_{L^p_x} \lesssim_N 2^{-(k_1+j)N}2^{3(\frac{1}{q}-\frac{1}{p})j}\|F\|_{L^q_x}$.
3. *Let $r \ge -(j+k_1)/3 + C_1$. Then*
$$\|\sigma^{\beta}_{-\frac{(j+k_1)}{3}}\varphi_{k_1} \cdot K_j(\eta_r^{(r,l)}\widetilde{\varphi}_{k_1}F)\|_{L^p_x} \lesssim_N 2^{-(j+k_1+r)N}2^{3(\frac{1}{q}-\frac{1}{p})j}\|F\|_{L^p_x}.$$

Proof of Proposition 2.5.5. Estimates (1) and (2) are standard, so we focus on (3). Write

$$\left[\sigma^{\beta}_{-\frac{(j+k_1)}{3}}\varphi_{k_1} \cdot K_j(\eta_r^{(r,l)}\widetilde{\varphi}_{k_1}F)\right](x) = \sigma^{\beta}_{-\frac{(j+k_1)}{3}}(x)\varphi_{k_1}(x)\int_y k_j(y)(\eta_r^{(r,l)}\widetilde{\varphi}_{k_1}F)(x-y)dy$$

Observe that from the restrictions

$$|x| \sim 2^{k_1}, \qquad |\hat{x} \times \beta| \lesssim 2^{-(j+k_1)/3}, \qquad |x-y| \sim 2^{k_1}, \qquad |\widehat{(x-y)} \times \beta| \sim 2^r,$$

we find that $|y| \gtrsim 2^{k_1+r}$. Indeed,

$$\begin{aligned} 2^r \simeq |\widehat{(x-y)} \times \beta| \simeq 2^{-k_1}|(x-y) \times \beta| &\lesssim 2^{-k_1}(|x \times \beta| + |y \times \beta|) \\ &\lesssim 2^{-(j+k_1)/3} + 2^{-k_1}|y||\hat{y} \times \beta| \end{aligned}$$

Since $r \gg -(j+k_1)/3$ this implies that $|y| \gtrsim 2^{k_1+r}/|\hat{y} \times \beta| \gtrsim 2^{k_1+r}$ (since $|\hat{y} \times \beta| \lesssim 1$). Therefore, for c such that $1 + \frac{1}{p} = \frac{1}{c} + \frac{1}{q}$, it holds

$$\begin{aligned} &\|\sigma^{\beta}_{-\frac{(j+k_1)}{3}} \varphi_{k_1} \cdot K_j(\eta_r^{(r,l)} \widetilde{\varphi}_{k_1} F)\|_{L^p_x} \\ &\lesssim \left\| \int_{|y| \gtrsim 2^{k_1+r}} k_j(y) \eta_r^{(r,l)}(x-y) \widetilde{\varphi}_{k_1}(x-y) F(x-y) dy \right\|_{L^p_x} \\ &\lesssim \|\mathbb{1}_{|y| \gtrsim 2^{k_1+r}} k_j(y)\|_{L^c_x} \|\eta_r^{(r,l)} \widetilde{\varphi}_{k_1} F\|_{L^q_x} \\ &\lesssim 2^{-(j+k_1+r)N} 2^{3j(\frac{1}{q}-\frac{1}{p})} \|\eta_r^{(r,l)} \widetilde{\varphi}_{k_1} F\|_{L^q_x} \end{aligned}$$

□

Proof of Corollary 2.5.4. We show only point (1), the adaptations for (2) being as in the previous proof. As in the proof of Lemma 2.5.3 we decompose

$$\|K_j(\phi_a^{(1)} \phi_b^{(2)})\, \phi_m^{(3)}\|_{1,2} \lesssim \|\cdot\|_{L^1_t L^2_x(|x| \lesssim 2^{-j})} + \|\cdot\|_{L^1_t L^2_x(|x| \gg 2^{-j})} =: (A) + (B)$$

Here (A) can be treated similarly to in the lemma, so we focus on (B). Further decompose

$$(B) \leq \sum_{k_1 \gg -j} \sum_{k_2 \in \mathbb{Z}} \|\cdot\|_{L^1_t L^2_x(|x| \sim 2^{k_1}, |t| \sim 2^{k_2})}$$

Performing an angular decomposition in the physical variable as in the previous proof and moving the spatial localisation inside the convolution, we have

$$\begin{aligned} &\|\cdot\|_{L^1_t L^2_x(|x| \sim 2^{k_1}, |t| \sim 2^{k_2})} \\ &\lesssim \left\| \left(\sum_{\beta \in \mathcal{S}_{j,k_1}} \left\| \sigma^{\beta}_{-(j+k_1)/3}(x) \varphi_{k_1}(x) \cdot K_j(\phi_a^{(1)} \phi_b^{(2)})\, \phi_m^{(3)} \right\|^2_{L^2_x} \right)^{\frac{1}{2}} \right\|_{L^1_t(|t| \sim 2^{k_2})} \\ &\lesssim \left\| \left(\sum_{\beta} \left\| (\sigma^{\beta}_{-(j+k_1)/3}\, \varphi_{k_1})(x) \cdot K_j(\varphi_{[k_1-5, k_1+5]}\, \phi_a^{(1)} \phi_b^{(2)})\, \phi_m^{(3)} \right\|^2_{L^2_x} \right)^{\frac{1}{2}} \right\|_{L^1_t(|t| \sim 2^{k_2})} \\ &\quad + \left\| \left(\sum_{\beta} \left\| (\sigma^{\beta}_{-(j+k_1)/3}\, \varphi_{k_1})(x) \cdot K_j(\varphi_{<k_1-5}\, \phi_a^{(1)} \phi_b^{(2)})\, \phi_m^{(3)} \right\|^2_{L^2_x} \right)^{\frac{1}{2}} \right\|_{L^1_t(|t| \sim 2^{k_2})} \end{aligned}$$

$$+\left\|\left(\sum_{\beta}\left\|(\sigma^{\beta}_{-(j+k_1)/3}\,\varphi_{k_1})(x)\cdot K_j(\varphi_{>k_1+5}\,\phi^{(1)}_a\phi^{(2)}_b)\,\phi^{(3)}_m\right\|^2_{L^2_x}\right)^{\frac{1}{2}}\right\|_{L^1_t(|t|\sim 2^{k_2})}$$

The second and third terms above are error terms bounded using Proposition 2.5.5. For instance for the third term we can use Point 1 of Proposition 2.5.5 to find

$$\begin{aligned}
&\sum_{l>k_1+5}\left\|\left(\sum_{\beta}\left\|(\sigma^{\beta}_{-(j+k_1)/3}\,\varphi_{k_1})(x)\cdot K_j(\varphi_l\,\phi^{(1)}_a\phi^{(2)}_b)\,\phi^{(3)}_m\right\|^2_{L^2_x}\right)^{\frac{1}{2}}\right\|_{L^1_t(|t|\sim 2^{k_2})}\\
&\lesssim \sum_{l>k_1+5}\left\|\left(\sum_{\beta}\|\sigma^{\beta}_{-(j+k_1)/3}\phi^{(3)}_m\|^2_{L^2_x}\|\varphi_{k_1}K_j(\varphi_l\,\phi^{(1)}_a\phi^{(2)}_b)\|^2_{L^\infty_x}\right)^{\frac{1}{2}}\right\|_{L^1_t(|t|\sim 2^{k_2})}\\
&\lesssim \sum_{l>k_1+5} 2^{k_2/M}2^{-(l+j)N}2^{3j(\frac{M-1}{4M})}\|\phi^{(3)}_m\|_{\infty,2}\|\phi^{(1)}_a\phi^{(2)}_b\|_{\frac{M}{M-1},\frac{4M}{M-1}}\\
&\lesssim 2^{k_2/M}2^{-(k_1+j)N}2^{3j(\frac{M-1}{4M})}\|\phi^{(1)}_a\|_{\frac{2M}{M-1},\frac{4M}{M-1}}\|\phi^{(2)}_b\|_{\frac{2M}{M-1},\infty}\|\phi^{(3)}_m\|_{\infty,2}
\end{aligned}$$

This is as required when summed over $k_2 \le k_1$, $k_1 \gg -j$ (since $a \gtrsim j$). When $k_2 > k_1$, we again use (2.5.14).

For the first term we still have to exchange the angular localisation and the convolution. Denoting $\varphi_{[k_1-5,k_1+5]} =: \widetilde{\varphi}_{k_1}$, we have (leaving the restriction to $|t| \sim 2^{k_2}$ implicit.)

$$\begin{aligned}
&\left\|\left(\sum_{\beta}\left\|(\sigma^{\beta}_{-(j+k_1)/3}\,\varphi_{k_1})(x)\cdot K_j(\widetilde{\varphi}_{k_1}\,\phi^{(1)}_a\phi^{(2)}_b)\,\phi^{(3)}_m\right\|^2_{L^2_x}\right)^{\frac{1}{2}}\right\|_{L^1_t(|t|\sim 2^{k_2})}\\
&\lesssim \left\|\left(\sum_{\beta}\left\|(\sigma^{\beta}_{-(j+k_1)/3}\,\varphi_{k_1})(x)\cdot K_j(\eta^{\beta}_{-(j+k_1)/3}(x)\widetilde{\varphi}_{k_1}(x)\,\phi^{(1)}_a\phi^{(2)}_b)\,\phi^{(3)}_m\right\|^2_{L^2_x}\right)^{\frac{1}{2}}\right\|_{L^1_t}\\
&\quad+\sum_{\substack{r\gg-(j+k_1)/3\\ l=1,\dots,N}}\left\|\left(\sum_{\beta}\left\|(\sigma^{\beta}_{-(j+k_1)/3}\,\varphi_{k_1})(x)\cdot K_j(\eta^{(r,l)}_r(x)\widetilde{\varphi}_{k_1}(x)\,\phi^{(1)}_a\phi^{(2)}_b)\,\phi^{(3)}_m\right\|^2_{L^2_x}\right)^{\frac{1}{2}}\right\|_{L^1_t}
\end{aligned}$$

Note that here the multipliers η are acting on the space variable. Here the second term is an error and again treated using Proposition 2.5.5, while for the main term we use the close/far angle decomposition from the previous proof on $\phi^{(1)}$ to reduce to

$$\left\|\left(\sum_{\beta}\left\|(\sigma^{\beta}_{-(j+k_1)/3}\,\varphi_{k_1})(x)K_j((\eta^{\beta}_{-(j+k_1)/3}\widetilde{\varphi}_{k_1})(x)\cdot\tilde{\eta}^{\beta}_{-(j+k_1)/3}(D)\phi^{(1)}_a\cdot\phi^{(2)}_b)\,\phi^{(3)}_m\right\|^2_{L^2_x}\right)^{\frac{1}{2}}\right\|_{L^1_t} \tag{C-A}$$

$$+\sum_{r,l}\left\|\left(\sum_{\beta}\left\|(\sigma^{\beta}_{-(j+k_1)/3}\,\varphi_{k_1})(x)K_j((\eta^{\beta}_{-(j+k_1)/3}\widetilde{\varphi}_{k_1})(x)\cdot\tilde{\eta}_r^{(r,l)}(D)\phi_a^{(1)}\cdot\phi_b^{(2)})\,\phi_m^{(3)}\right\|^2_{L^2_x}\right)^{\frac{1}{2}}\right\|_{L^1_t} \tag{F-A}$$

where $\tilde{\eta}^{\beta}_{-(j+k_1)/3}+\sum_{r,l}\tilde{\eta}_r^{(r,l)}$ ($r\gg-(j+k_1)/3$, $l=1,\ldots,N$) is an angular decomposition as in Proposition 2.2.6 with $\eta^{\beta}_{-(j+k_1)/3}$ playing the role of $\sigma^{\beta}_{-(j+k_1)/3}$. For the close-angle term we proceed almost as in the proof of Lemma 2.5.3, however we must be careful applying Bernstein's inequality. We first apply Bernstein in the form of Young's inequality on the convolution K_j then directly on the term $\tilde{\eta}^{\beta}_{-(j+k_1)/3}(D)\phi_a^{(1)}$ to find

$$\begin{aligned}
\text{(C-A)}&\lesssim\left\|\left(\sum_{\beta}(\|\sigma^{\beta}_{-(j+k_1)/3}\phi_m^{(3)}\|_2\,2^{3j\frac{M-8}{4M}}\|\tilde{\eta}^{\beta}_{-(j+k_1)/3}(D)\phi_a^{(1)}\cdot\phi_b^{(2)}\|_{\frac{4M}{M-8}})^2\right)^{\frac{1}{2}}\right\|_{L^1_t}\\
&\lesssim 2^{k_2/M}2^{3j\frac{M-8}{4M}}2^{(3a-2(j+k_1)/3)\frac{7}{4M}}\|\phi_a^{(1)}\|_{\frac{2M}{M-1},\frac{4M}{M-1}}\|\phi_b^{(2)}\|_{\frac{2M}{M-1},\infty}\|\phi_m^{(3)}\|_{\infty,2}
\end{aligned}$$

Summing first over $k_2\le k_1$ this may be summed over $k_1\gg-j$ (due to our careful application of Bernstein) yielding

$$2^{-j/M}2^{3j\frac{M-1}{4M}}2^{3(a-j)\frac{7}{4M}}\|\phi_a^{(1)}\|_{\frac{2M}{M-1},\frac{4M}{M-1}}\|\phi_b^{(2)}\|_{\frac{2M}{M-1},\infty}\|\phi_m^{(3)}\|_{\infty,2}$$

as required. We make the usual adaptation involving (2.5.14) in the case $k_2>k_1$.

For the far-angle term, we apply the angular separation lemma, Lemma 2.5.2, to bound

$$\begin{aligned}
\text{(F-A)}&\lesssim\sum_{r,l}\left\|\left(\sum_{\beta}(\|\sigma^{\beta}_{-(j+k_1)/3}\phi_m^{(3)}\|_2\|(\eta^{\beta}_{-(j+k_1)/3}\widetilde{\varphi}_{k_1})(x)\tilde{\eta}_r^{(r,l)}(D)\phi_a^{(1)}\|_\infty\|\phi_b^{(2)}\|_\infty)^2\right)^{\frac{1}{2}}\right\|_{L^1_t}\\
&\lesssim 2^{k_2/M}\sum_{r,l}2^{-(a+k_1+2r)}2^{3a/2M}\|\langle\Omega\rangle\phi_a^{(1)}\|_{\frac{2M}{M-1},2M}\|\phi_b^{(2)}\|_{\frac{2M}{M-1},\infty}\|\phi_m^{(3)}\|_{\infty,2}\\
&\lesssim 2^{k_2/M}2^{-(a+k_1)}2^{2(j+k_1)/3}2^{3a/2M}\|\langle\Omega\rangle\phi_a^{(1)}\|_{\frac{2M}{M-1},2M}\|\phi_b^{(2)}\|_{\frac{2M}{M-1},\infty}\|\phi_m^{(3)}\|_{\infty,2}
\end{aligned}$$

which yields

$$2^{-j/M}2^{j-a}2^{3a/2M}\|\langle\Omega\rangle\phi_a^{(1)}\|_{\frac{2M}{M-1},2M}\|\phi_b^{(2)}\|_{\frac{2M}{M-1},\infty}\|\phi_m^{(3)}\|_{\infty,2}$$

when summed over $k_2\le k_1$, $k_1\gg-j$, and

$$2^{-j/M}2^{j-a}2^{3a/2M}\|\langle\Omega\rangle\phi_a^{(1)}\|_{\frac{2M}{M-1},2M}\|\phi_b^{(2)}\|_{\frac{2M}{M-1},\infty}(2^{j-m}\|L\phi_m^{(3)}\|_{\infty,2}+2^{-m}\|\partial_t\phi_m^{(3)}\|_{\infty,2})$$

for $k_2>k_1$ after applying (2.5.14). $\square$

2.5.1 Showing that $P_0(HWM_1(\phi)) = error$

The goal of this subsection is the following

Proposition 2.5.6. *We have*

$$P_0 L(HWM_1(\phi)) = error$$

First note the following Moser-type estimate:

Lemma 2.5.7. *Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth function with bounded derivatives up to order 4. Then for any $k \in \mathbb{Z}$ it holds*

$$\max_{(p,q)\in\mathcal{Q}} 2^{(\frac{1}{p}+\frac{3}{q})k} \|P_k \langle\Omega\rangle^{1-\delta(p,q)} Lg(\phi)\|_{p,q} \lesssim_{\mathcal{Q}} C(\|\phi^L\|_S)$$

and

$$\max_{(p,q)\in\mathcal{Q}} 2^{(\frac{1}{p}+\frac{3}{q}-1)k} \|P_k \langle\Omega\rangle^{1-\delta(p,q)} \partial_t g(\phi)\|_{p,q} \lesssim_{\mathcal{Q}} C(\|\phi\|_S)$$

for $\mathcal{Q}$ as in the definition of S and $C(\cdot)$ a polynomial.

Proof. In the absence of any vector fields, we have the following standard estimate assuming only bounded derivatives up to second order:

$$\|P_k g(\phi)\|_{p,q} \lesssim_{p,q} 2^{-(\frac{1}{p}+\frac{3}{q})k} \|\phi\|_S^2 (1 + \|\phi\|_S) \tag{2.5.17}$$

This is proved in Appendix 2.B.

To incorporate the vector fields we apply the chain rule (omitting the angular derivative for a radially admissible pair) to find

$$P_k \Omega_{ij} L_n g(\phi) = P_k(\Omega_{ij} L_n \phi \cdot g'(\phi)) + P_k(L_n\phi \cdot \Omega_{ij}\phi \cdot g''(\phi)) \tag{2.5.18}$$

Then since g' satisfies the hypotheses for (2.5.17) we have

$$\begin{aligned}\|P_k(\Omega_{ij} L_n \phi \cdot g'(\phi))\|_{p,q} &\lesssim \|P_{<k-10}\Omega_{ij} L_n\phi\|_{\infty,\infty} \|P_{\sim k} g'(\phi)\|_{p,q} + \|P_{\geq k-10}\Omega_{ij} L_n \phi\|_{p,q} \|g'\|_\infty \\ &\lesssim \|\phi^L\|_S \cdot 2^{-(\frac{1}{p}+\frac{3}{q})k} \|\phi\|_S (1 + \|\phi\|_S) + 2^{-(\frac{1}{p}+\frac{3}{q})k} \|\phi^L\|_S\end{aligned}$$

which is acceptable. A similar argument works for the remaining term in (2.5.18) and the second estimate of the statement can be proved similarly. □

From this lemma we can deduce the following result which effectively allows us to ignore the projection when estimating $P_k(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))$:

Lemma 2.5.8. *Let ϕ such that $\|\phi^L\|_S \leq 1$. Then there exists a constant $C_{\mathcal{Q}} > 0$ such that*

$$\max_{(p,q)\in\mathcal{Q}} 2^{(\frac{1}{p}+\frac{3}{q}-1)k} \|\langle\Omega\rangle^{1-\delta(p,q)} L P_k(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\|_{p,q} \lesssim_{\mathcal{Q}} \sum_{k_1\in\mathbb{Z}} 2^{-C_{\mathcal{Q}}|k-k_1|} \|P_{k_1}\phi^L\|_{S_{k_1}}$$

Under the same conditions we also have

$$\max_{(p,q)\in\mathcal{Q}} 2^{(\frac{1}{p}+\frac{3}{q}-2)k} \|\partial_t \langle\Omega\rangle^{1-\delta(p,q)} P_k(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\|_{p,q} \lesssim_{\mathcal{Q}} \sum_{k_1\in\mathbb{Z}} 2^{-C_{\mathcal{Q}}|k-k_1|} \|\phi_{k_1}\|_{S_{k_1}}$$

Our final preparation for the proof of Proposition 2.5.6 is the following lemma, a variant of Lemma 4.3 [Tao01a], which allows us to apply the geometric identity (GeId) in a more general setting:

Lemma 2.5.9. *Let $r \in \mathbb{Z}$, $p, q \geq 1$ with $p^{-1} = p_1^{-1} + p_2^{-1}$ and $q^{-1} = q_1^{-1} + q_2^{-1}$. It holds*

$$u(x) \cdot P_r v(x) - P_r(u \cdot v)(x) = 2^{-r} \int_y \int_{\theta=0}^1 (\check{\chi}_r(y)(2^r y)^T) \nabla u(x - \theta y) v(x-y) dy d\theta$$

from which

$$\|u \cdot P_r v - P_r(u \cdot v)\|_q \lesssim 2^{-r} \|\nabla u\|_{q_1} \|v\|_{q_2}$$

and more generally

$$\|\langle\Omega\rangle L[u \cdot P_r v - P_r(u\cdot v)]\|_q \lesssim 2^{-r} \|\langle\Omega\rangle \nabla_{t,x} u^L\|_{q_1} \|\langle\Omega\rangle v^L\|_{q_2} + 2^{-2r} \|\langle\Omega\rangle \nabla u\|_{q_1} \|\langle\Omega\rangle \partial_t v\|_{q_2}$$

These statements also hold for $\mathcal{P}_r$ as in Section 2.2.1.

Henceforth we will use the following shorthand, adding to that introduced in (2.4.2):

$$\begin{aligned}
(2+,\infty) &:= (\frac{2M}{M-1}, \infty), & \|\langle\Omega\rangle P_k\phi^L\|_{2+,\infty} &\lesssim 2^{-(\frac{1}{2}-\frac{1}{2M})k} \|P_k\phi^L\|_{S_k} \\
(2+,4+) &:= (\frac{2M}{M-1}, \frac{4M}{M-1}), & \|\langle\Omega\rangle P_k\phi^L\|_{2+,4+} &\lesssim 2^{-(\frac{1}{2}-\frac{1}{2M})k} 2^{-3\frac{M-1}{4M}k} \|P_k\phi^L\|_{S_k}
\end{aligned} \tag{2.5.19}$$

Proof of Proposition 2.5.6. First note that it is sufficient to find some $\delta > \lambda > 0$ such that

$$\|\langle\Omega\rangle P_0 L[P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ (\phi\cdot(-\Delta)^{1/2}\phi)]\|_{1,2} \lesssim C_0^3 \epsilon^2 2^{-\delta|m|} \sum_{k\in\mathbb{Z}} 2^{-\lambda|k-m|} c_k \tag{2.5.20}$$

for every $m \in \mathbb{Z}$, provided we then fix $\sigma < \lambda$.

We start by studying (2.5.20) with $m > -10$. Further decompose

$$\begin{aligned}
&\|\langle\Omega\rangle P_0 L[P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ (\phi\cdot(-\Delta)^{1/2}\phi)]\|_{1,2} \\
&\lesssim \|\langle\Omega\rangle L[P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ \sum_{k\in\mathbb{Z}}(\phi_k\cdot(-\Delta)^{1/2}\phi_{<k+10})]\|_{1,2} \qquad (2.5.21)\\
&\quad + \|\langle\Omega\rangle L[P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ \sum_{j\in\mathbb{Z}}(\phi_{\le j-10}\cdot(-\Delta)^{1/2}\phi_j)]\|_{1,2} \qquad (2.5.22)
\end{aligned}$$

We first study (2.5.21). For the sum over $k \geq -10$ we use Lemma 2.5.8 to see that

$$\begin{aligned}
&\sum_{k\geq -10}\|\langle\Omega\rangle L[P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ (\phi_k\cdot(-\Delta)^{1/2}\phi_{<k+10})]\|_{1,2}\\
&\lesssim \|\langle\Omega\rangle L(P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi)))\|_{\infty-,2+}\sum_{k\geq-10}\|\langle\Omega\rangle L\phi_k\|_{2+,\infty-}\|\langle\Omega\rangle L(-\Delta)^{1/2}\phi_{<k+10}\|_{2+,\infty-}\\
&\lesssim 2^{-(\frac{1}{2}-\frac{2}{M})m}C_0^3\epsilon^2\sum_{k_1}2^{-\lambda|m-k_1|}c_{k_1}
\end{aligned}$$

for some $\lambda > 0$.

The case $k < -10$ is handled by a direct application of Lemma 2.5.3. For example if $\langle\Omega\rangle$ and L both fall on ϕ_k we have by point 2 of said lemma that

$$\begin{aligned}
&\sum_{k<-10}\|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ (\langle\Omega\rangle L\phi_k\cdot(-\Delta)^{1/2}\phi_{<k+10})\|_{1,2}\\
&\lesssim \sum_{k<-10}\sum_{j<k+10}2^{(\frac{1}{2}-\frac{1}{2M})(j-k)}C_0^2c_jc_k(\|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\|_{\infty,2}\\
&\qquad + 2^{j-m}\|LP_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\|_{\infty,2} + 2^{-m}\|\partial_t P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\|_{\infty,2})
\end{aligned}$$

which, again thanks to Lemma 2.5.8, is bounded by

$$C_0^3\epsilon^2 2^{-m/2}\sum_{k_1}2^{-\lambda|m-k_1|}c_{k_1}$$

for some $\lambda > 0$. When $\langle\Omega\rangle$ and L distribute in other combinations, we apply the other parts of Lemma 2.5.3.

We now turn to (2.5.22), in which there are no derivatives falling on the lowest frequency factor. To handle this delicate situation we use the geometric relation (GeId). As an example we consider only the case where L and Ω both fall on the product $\phi_{\le j-10}\cdot(-\Delta)^{1/2}\phi_j$. This case presents the most technical difficulties from interchanging L with the nonlocal derivative $(-\Delta)^{1/2}$ and the frequency projections. We have

$$\|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ \sum_{j\in\mathbb{Z}}\langle\Omega\rangle L(\phi_{\le j-10}\cdot(-\Delta)^{1/2}\phi_j)\|_{1,2}$$

$$\leq \sum_{j\in\mathbb{Z}} \|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ \langle\Omega\rangle L(\phi_{\leq j-10}\cdot(-\Delta)^{1/2}\phi_j - (-\Delta)^{1/2}(\phi_{\leq j-10}\cdot\phi_j))\|_{1,2} \tag{2.5.23}$$

$$+ \|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ \langle\Omega\rangle L(-\Delta)^{1/2}(\phi_{\leq j-10}\cdot\phi_j - P_j(\phi_{\leq j-10}\cdot\phi_{>j-10}))\|_{1,2} \tag{2.5.24}$$

$$+ \|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ \langle\Omega\rangle L(-\Delta)^{1/2}P_j(\phi_{\leq j-10}\cdot\phi_{>j-10})\|_{1,2} \tag{2.5.25}$$

The idea for handling these terms is that the first two effectively see a derivative moved onto the low frequency factor, and for the third we can apply (GeId). We start by rewriting (2.5.23) as

$$\sum_{j\in\mathbb{Z}}\sum_{k\leq j-10} \|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ \langle\Omega\rangle L\mathcal{L}_k(\phi_k,\phi_j)\|_{1,2}$$

which is similar to (2.5.21). Indeed when $j \geq -10$ we can bound this using the same estimates as for (2.5.21) combined with identities (2.5.4)-(2.5.5) and Lemma 2.5.1. In the case $j < -10$ we need a small adaptation before applying Lemma 2.5.3 due to the nonlocal operator $\mathcal{L}_k$. As in (2.5.3) we expand the corresponding multiplier m_k as a Fourier series to write

$$\mathcal{L}_k(\phi_k,\phi_j) = \sum_{a,b} c^{(k)}_{a,b}\phi_k(x-2^{-k}a)\cdot\phi_j(x-2^{-j}b)$$

Since we are applying Lemma 2.5.3 we must then pay attention to where the vector derivatives fall on this expression. For example if both fall on the low frequency term ϕ_k we have to bound

$$\sum_{\substack{j\leq -10\\ k\leq j-10}}\sum_{a,b} |c^{(k)}_{a,b}| \|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ \langle\Omega\rangle L(\phi_k(x-2^{-k}a))\cdot\phi_j(x-2^{-j}b)\|_{1,2} \tag{2.5.26}$$

Then by point 1 of the Lemma we have

$$\begin{aligned}
&\|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ \langle\Omega\rangle L(\phi_k(x-2^{-k}a))\cdot\phi_j(x-2^{-j}b)\|_{1,2}\\
&\lesssim 2^{-k/M}\|\langle\Omega\rangle L(\phi_k(x-2^{-k}a))\|_{2+,\infty}\\
&\qquad\cdot(2^{3j/2M}\|\langle\Omega\rangle(\phi_j(x-2^{-j}b))\|_{2+,\infty-} + 2^{3j\frac{M-1}{4M}}\|\phi_j\|_{2+,4+})\\
&\qquad\cdot(\|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\|_{\infty,2}+\dots)\\
&\lesssim 2^{-k}2^{(\frac{1}{2}-\frac{1}{2M})(k-j)}\langle a\rangle^2\langle b\rangle^2 C_0^3 c_j c_k\cdot 2^{-m/2}\sum_{k_1}2^{-\lambda|m-k_1|}c_{k_1}
\end{aligned}$$

where we used (2.5.6) to achieve the final line. This is acceptable when summed as in (2.5.26).

For (2.5.24) we first consider $j > -10$, using the commutation relations $[\Omega, L] \simeq L$,

$[L, (-\Delta)^{1/2}] \simeq R\partial_t$ and $[\Omega, R] \simeq R$ to find

$$\sum_{j>-10} \|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ \langle\Omega\rangle L(-\Delta)^{1/2}(\phi_{\le j-10}\cdot\phi_j - P_j(\phi_{\le j-10}\cdot\phi_{>j-10}))\|_{1,2}$$
$$\lesssim \sum_{j>-10} \|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ (-\Delta)^{1/2}\langle\Omega\rangle L(\phi_{\le j-10}\cdot\phi_j - P_j(\phi_{\le j-10}\cdot\phi_{>j-10}))\|_{1,2} \tag{2.5.27}$$
$$+ \|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ R\langle\Omega\rangle\partial_t(\phi_{\le j-10}\cdot\phi_j - P_j(\phi_{\le j-10}\cdot\phi_{>j-10}))\|_{1,2} \tag{2.5.28}$$

These terms can both be handled using Lemma 2.5.9. For (2.5.27) we have

$$\begin{aligned}
&(2.5.27)\\
&\lesssim \sum_{j>-10} \|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\|_{\infty-,2+}\\
&\qquad\qquad \cdot \|(-\Delta)^{1/2}\langle\Omega\rangle L(\phi_{\le j-10}\cdot\phi_j - P_j(\phi_{\le j-10}\cdot\phi_{\sim j}))\|_{\frac{M}{M-1},M}\\
&\lesssim \sum_{j>-10} \|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\|_{\infty-,2+}\cdot 2^j(2^{-j}\|\langle\Omega\rangle\nabla_{t,x}L\phi_{\le j-10}\|_{2+,\infty-}\|\langle\Omega\rangle L\phi_{\sim j}\|_{2+,\infty-}\\
&\qquad\qquad + 2^{-2j}\|\langle\Omega\rangle\nabla\phi_{\le j-10}\|_{2+,\infty-}\|\langle\Omega\rangle\partial_t\phi_{\sim j}\|_{2+,\infty-})\\
&\lesssim \sum_{j>-10} 2^{-(\frac{1}{2}-\frac{2}{M})m}\left(\sum_{k_1} 2^{-\lambda|m-k_1|}C_0c_{k_1}\right)\cdot 2^{-2j/M}C_0^2\epsilon^2
\end{aligned}$$

which is acceptable. (2.5.28) is similar.

For (2.5.24) with $j \le -10$ we will use Lemma 2.5.3 but must incorporate the cancellation structure via Lemma 2.5.9. We write, for $\Phi_j(\xi) := |2^{-j}\xi|\tilde{\chi}_0(2^{-j}\xi)$, $\tilde{\chi}_0$ as in Section 2.1.1,

$$\begin{aligned}
&(-\Delta)^{1/2}(\phi_{\le j-10}\cdot\phi_j - P_j(\phi_{\le j-10}\cdot\phi_{>j-10}))\\
&\qquad = \int_{z,y,\theta}\check{\Phi}_j(y)(2^jz\check{\chi}_j(z))\nabla\phi_{\le j-10}(x-y-\theta z)\cdot\phi_{\sim j}(x-y-z)dyd\theta dz
\end{aligned}$$

Thus by Lemma 2.5.3, using the notation $\langle\Omega\rangle_x$ and L_x to emphasise that these fields act with respect to the x variable only, we have

$$\begin{aligned}
&\sum_{j\le-10} \|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ \langle\Omega\rangle L(-\Delta)^{1/2}(\phi_{\le j-10}\cdot\phi_j - P_j(\phi_{\le j-10}\cdot\phi_{>j-10}))\|_{1,2}\\
&\lesssim \sum_{j\le-10}\int_{z,y,\theta} dyd\theta dz|\check{\Phi}_j(y)(2^jz\check{\chi}_j(z))|\\
&\qquad \cdot\|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))(x)\ \langle\Omega\rangle_xL_x[\nabla\phi_{\le j-10}(x-y-\theta z)]\cdot\phi_{\sim j}(x-y-z)\|_{L^1_tL^2_x}\\
&\quad + \text{similar terms}\\
&\lesssim \sum_{\substack{j\le-10\\k\le j-10}}\int_{z,y,\theta} dyd\theta dz|\check{\Phi}_j(y)(2^jz\check{\chi}_j(z))|\cdot 2^{-k/M}\|\langle\Omega\rangle_xL_x(\nabla\phi_k(x-y-\theta z))\|_{2+,\infty}
\end{aligned}$$

$$
\begin{aligned}
&\cdot (2^{3j/2M}\|\langle\Omega\rangle_x(\phi_{\sim j}(x-y-z))\|_{2+,\infty-} + 2^{3j\frac{M-1}{4M}}\|\phi_{\sim j}\|_{2+,4+})\\
&\cdot (\|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\|_{\infty,2} + \ldots)\\
&+ \text{similar terms}
\end{aligned}
$$

Then using (2.5.6) we bound this by

$$
\begin{aligned}
&\sum_{\substack{j\leq -10\\ k\leq j-10}} 2^{-m/2}2^{(\frac{1}{2}-\frac{1}{2M})(k-j)}C_0^3 c_j c_k \left(\sum_{k_1} 2^{-\lambda|m-k_1|}c_{k_1}\right)\\
&\qquad\cdot \int_{z,y,\theta} |\check{\Phi}_j(y)(2^j z\check{\chi}_j(z))|\langle 2^k(y+\theta z)\rangle^2\langle 2^j(y+z)\rangle^2 dyd\theta dz
\end{aligned}
$$

Thanks to the scaling of $\check{\Phi}_j$ and $\check{\chi}_j$, both of which are rapidly decaying, we see that the integral above is $O(1)$ and this term is acceptable.

To complete the case $m > -10$ we need to study (2.5.25). Applying (GeId) and commuting $\langle\Omega\rangle L$ through $(-\Delta)^{1/2}P_j$ we have

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}} \|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ \langle\Omega\rangle L(-\Delta)^{1/2}P_j(\phi_{\leq j-10}\cdot\phi_{>j-10})\|_{1,2}\\
&\lesssim \sum_{j\in\mathbb{Z}} \|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ (-\Delta)^{1/2}P_j\langle\Omega\rangle L(\phi_{>j-10}\cdot\phi_{>j-10})\|_{1,2} &(2.5.29)\\
&\qquad + \|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ RP_j\langle\Omega\rangle\partial_t(\phi_{>j-10}\cdot\phi_{>j-10})\|_{1,2} &(2.5.30)\\
&\qquad + \|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ (-\Delta)^{1/2}(2^{-j}\partial_t\mathcal{P}_j)\langle\Omega\rangle(\phi_{>j-10}\cdot\phi_{>j-10})\|_{1,2} &(2.5.31)
\end{aligned}
$$

This is easiest to handle when $j \geq -10$. For instance the sum of (2.5.29) over $j \geq -10$ is bounded by

$$
\sum_{j\geq -10} \|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\|_{\infty-,2+}\cdot 2^j\|\langle\Omega\rangle L\phi_{>j-10}\|_{2+,\infty-}\|\langle\Omega\rangle L\phi_{>j-10}\|_{2+,\infty-}
$$

which is fine. The commutator terms (2.5.30) and (2.5.31) correspond to (high)∇(high)-∇(high) interactions and can be treated like (2.4.8).

For $j < -10$ we use Corollary 2.5.4 with $K_j = 2^{-j}(-\Delta)^{1/2}P_j$. For (2.5.29) we have

$$
\begin{aligned}
&\sum_{j<-10} \|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ (-\Delta)^{1/2}P_j\langle\Omega\rangle L(\phi_{>j-10}\cdot\phi_{>j-10})\|_{1,2}\\
&\lesssim \sum_{j<-10} \|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ (-\Delta)^{1/2}P_j(L\phi_{>j-10}\cdot\langle\Omega\rangle L\phi_{>j-10})\|_{1,2}\\
&\lesssim \sum_{\substack{j<-10\\ r,s>j-10}} 2^j 2^{-j/M}\left(2^{3j\frac{M-1}{4M}}2^{\frac{21}{4M}(r-j)}\|L\phi_{\sim r}\|_{2+,4+} + 2^{3r/2M}2^{j-r}\|\langle\Omega\rangle L\phi_{\sim r}\|_{2+,\infty-}\right)\\
&\qquad\qquad\cdot \|\langle\Omega\rangle L\phi_{\sim s}\|_{2+,\infty}(\|P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\|_{\infty,2} + \ldots)
\end{aligned}
$$

$$\lesssim C_0^3\epsilon 2^{-m/2}\left(\sum_{\substack{j<-10\\ r>j-10}}(2^{(\frac{1}{2}-\frac{1}{2M}+3\frac{M-1}{4M}-\frac{21}{4M})(j-r)}+2^{(\frac{1}{2}-\frac{1}{2M}+1-\frac{3}{2M})(j-r)})c_r\right)$$
$$\cdot\left(\sum_{k_1}2^{-\lambda|m-k_1|}c_{k_1}\right)$$

which is acceptable. (2.5.30) and (2.5.31) can be treated in the same way, using the additional information that P_j localises the two factors of $\phi_{>j-10}$ to comparable frequencies ($r\sim s$) in order to handle the high frequency time derivative which appears. This completes the case $m>-10$.

The case $m\leq -10$ is actually easier to handle and we do not need to invoke Lemma 2.5.3, since the geometry rules out any (low)∇(low)∇(high) interactions. When the lone factor of ϕ appears at high frequency ($\geq 2^{-10}$), we refer to (2.4.6) and (2.4.7) of Proposition 2.4.1 for the cases when $(-\Delta)^{1/2}\phi$ appears at high or low frequency respectively. It thus remains to study the case when ϕ is at very low frequency. Here we have

$$\|P_0\langle\Omega\rangle L[P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ (\phi_{\leq -10}\cdot(-\Delta)^{1/2}\phi_{>-10})]\|_{1,2} \tag{2.5.32}$$
$$\leq \|P_0\langle\Omega\rangle L[P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ (\phi_{\leq -10}\cdot(-\Delta)^{1/2}\phi_{>-10}-(-\Delta)^{1/2}(\phi_{\leq-10}\cdot\phi_{>-10}))]\|_{1,2}$$
$$+\|P_0\langle\Omega\rangle L[P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ (-\Delta)^{1/2}(\phi_{\leq-10}\cdot\phi_{>-10})]\|_{1,2} \tag{2.5.33}$$

The first term above is of the form

$$\sum_{j\leq-10}\sum_{k>-10}\|P_0\langle\Omega\rangle L[P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ \mathcal{L}_j(\phi_j,\phi_k))]\|_{1,2}$$

which can be handled like (2.4.7) from Proposition 2.4.1 (or directly when $|j-k|\sim 0$).

For the second term in (2.5.33) we use (GeId) to replace the low frequency term with a high one. We may also insert a projection $\tilde{P}_0$ before the $(-\Delta)^{1/2}$ since m is very small and the whole term is restricted to frequency $\sim 2^0$. We thus bound

$$\begin{aligned}&\|P_0\langle\Omega\rangle L[P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ (-\Delta)^{1/2}(\phi_{\leq-10}\cdot\phi_{>-10})]\|_{1,2}\\ &\lesssim\|\langle\Omega\rangle LP_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ (-\Delta)^{1/2}\langle\Omega\rangle L(\phi_{>-10}\cdot\phi_{>-10})\|_{1,2}\\ &\quad+\|\langle\Omega\rangle P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ \tilde{P}_0R(\partial_t\phi_{>-10}\cdot\langle\Omega\rangle\phi_{>-10})]\|_{1,2}\\ &\quad+\|\langle\Omega\rangle P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ \tilde{P}_0R(\phi_{>-10}\cdot\langle\Omega\rangle\partial_t\phi_{>-10})]\|_{1,2}\end{aligned}$$

The first of these lines is straightforwardly bounded by

$$2^{(\frac{1}{2}-\frac{1}{M})m}\left(\sum_{k_1\in\mathbb{Z}}2^{-\lambda|m-k_1|}C_0c_{k_1}\right)\|\langle\Omega\rangle L\phi_{>-10}\|_{2+,\infty-}\|\langle\Omega\rangle L\phi_{>-10}\|_{\infty-,2+}$$

For the second and third we must again invoke Corollary 2.5.4 to see, for example,

$$
\begin{aligned}
&\|\langle\Omega\rangle P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\ \tilde{P}_0 R(\partial_t\phi_{>-10}\cdot\langle\Omega\rangle\phi_{>-10})]\|_{1,2}\\
&\lesssim \sum_{r\sim s>-10}(2^{\frac{21}{4M}r}\|\partial_t\phi_r\|_{2+,4+}+2^{3r/2M}2^{-r}\|\langle\Omega\rangle\partial_t\phi_r\|_{2+,\infty-})\|\langle\Omega\rangle\phi_s\|_{2+,\infty}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot(\|\langle\Omega\rangle P_m(\Pi_{\phi^\perp}((-\Delta)^{1/2}\phi))\|_{\infty,2}+\ldots)\\
&\lesssim C_0^3\epsilon\cdot 2^{-m/2}\left(\sum_{k_1}2^{-\lambda|m-k_1|}c_{k_1}\right)\sum_{r>-10}(2^{(\frac{21}{4M}-3\frac{M-1}{4M}+\frac{1}{M})r}+2^{-(1-\frac{1}{M})r})c_r
\end{aligned}
$$

which is acceptable for M sufficiently large. The third term can be treated identically and this completes the proof. □

2.5.2 Showing that $P_0(HWM_2(\phi)) = error$

In this section we will prove that the remaining nonlocal terms in the forcing are acceptable. We will use the notation

$$X\lesssim_{a,b}Y$$

to mean that $X\leq C_{a,b}Y$ where $C_{a,b}$ grows at most polynomially in a, b. This is specific to both this section and the letters a, b.

Proposition 2.5.10. *We have*

$$P_0L(HWM_2(\phi)) = error$$

Proof. We decompose

$$HWM_2(\phi)=\sum_{k\in\mathbb{Z}}\phi\times[(-\Delta)^{1/2}(\phi_k\times(-\Delta)^{1/2}\phi)-(\phi_k\times(-\Delta)\phi)]$$

and study the regions $k<-10$, $k\in[-10,10]$ and $k>10$ separately.

- $k<-10$: We make the decomposition

$$
\begin{aligned}
&\|P_0\langle\Omega\rangle L[\phi\times[(-\Delta)^{1/2}(\phi_k\times(-\Delta)^{1/2}\phi)-(\phi_k\times(-\Delta)\phi)]]\|_{1,2}\\
&\leq\|P_0\langle\Omega\rangle L[\phi_{<k-10}\times[(-\Delta)^{1/2}(\phi_k\times(-\Delta)^{1/2}\phi)-(\phi_k\times(-\Delta)\phi)]]\|_{1,2} \quad (2.5.34)\\
&\quad+\|P_0\langle\Omega\rangle L[\phi_{[k-10,-15]}\times[(-\Delta)^{1/2}(\phi_k\times(-\Delta)^{1/2}\phi)-(\phi_k\times(-\Delta)\phi)]]\|_{1,2} \quad (2.5.35)\\
&\quad+\|P_0\langle\Omega\rangle L[\phi_{\geq-15}\times[(-\Delta)^{1/2}(\phi_k\times(-\Delta)^{1/2}\phi)-(\phi_k\times(-\Delta)\phi)]]\|_{1,2} \quad (2.5.36)
\end{aligned}
$$

The last term above is the easiest to handle, splitting

$$\begin{aligned}(2.5.36) \lesssim & \|\langle\Omega\rangle L[\phi_{\geq -15} \times [(-\Delta)^{1/2}(\phi_k \times (-\Delta)^{1/2}\phi_{>k+10}) - (\phi_k \times (-\Delta)\phi_{>k+10})]]\|_{1,2} \\ & + \|\langle\Omega\rangle L[\phi_{\geq -15} \times [(-\Delta)^{1/2}(\phi_k \times (-\Delta)^{1/2}\phi_{\leq k+10}) - (\phi_k \times (-\Delta)\phi_{\leq k+10})]]\|_{1,2}\end{aligned}$$

The first term here can be handled like (2.4.6) upon writing

$$(-\Delta)^{1/2}(\phi_k \times (-\Delta)^{1/2}\phi_{>k+10}) - (\phi_k \times (-\Delta)\phi_{>k+10}) = \sum_{j>k+10} \mathcal{L}_{k+j}(\phi_k, \phi_j) \quad (2.5.37)$$

and using (2.5.4)-(2.5.5), while the second term can be handled like (2.4.7).

Next consider (2.5.35). This term is of the form (low)∇(lowest)∇(high), so we will rely heavily on Lemma 2.5.3. Since the outer projection P_0 almost passes through the operators $\langle\Omega\rangle$ and L, the third factor of ϕ is restricted to $\phi_{\sim 0}$. Then using (2.5.2) to write the commutator expression $\mathcal{L}_k(\phi_k, (-\Delta)^{1/2}\phi_{\sim 0}) = \mathcal{L}_k(\phi_k, \phi_{\sim 0})$ as a Fourier series, we find

$$(2.5.35) \lesssim \sum_{j=k-10}^{-15} \sum_{a,b} |c_{a,b}^{(k)}| \|\langle\Omega\rangle L[\phi_j(x) \times [\phi_k(x - 2^{-k}a) \times \phi_{\sim 0}(x-b)]]\|_{1,2}$$

Then for instance if the derivatives $\langle\Omega\rangle L$ both fall on ϕ_j we can apply point 2 of Lemma 2.5.3 and bound

$$\begin{aligned}&\sum_{j=k-10}^{-15} \sum_{a,b} c_{a,b}^{(k)} \|\langle\Omega\rangle L\phi_j(x) \times [\phi_k(x-2^{-k}a) \times \phi_{\sim 0}(x-b)]\|_{1,2} \\ &\lesssim \sum_{j=k-10}^{-15} \sum_{a,b} c_{a,b}^{(k)} 2^{-k/M} \|\langle\Omega\rangle L\phi_j\|_{2+,\infty} \\ &\qquad \cdot (2^{3k/2M}\|\langle\Omega\rangle(\phi_k(x-2^{-k}a))\|_{2+,\infty-} + 2^{3k\frac{M-1}{4M}}\|\phi_k\|_{2+,4+}) \\ &\qquad \cdot (\|\phi_{\sim 0}\|_{\infty,2} + \|L(\phi_{\sim 0}(x-b))\|_{\infty,2} + \|\partial_t\phi_{\sim 0}\|_{\infty,2}) \\ &\lesssim_N \sum_{j=k-10}^{-15} \sum_{a,b} \langle a\rangle^{-N}\langle b\rangle^{-N} 2^{(\frac{1}{2}-\frac{1}{2M})(k-j)} C_0 c_j \cdot \langle a\rangle C_0 c_k \cdot \langle b\rangle C_0 c_0 \\ &\lesssim C_0^3 c_0 \sum_{j=k-10}^{-15} 2^{(\frac{1}{2}-\frac{1}{2M})(k-j)} c_j c_k\end{aligned}$$

which is acceptable when summed over $k < -10$.

To complete the case $k < -10$ it remains to study (2.5.34). Here there are no derivatives falling on the lowest frequency term, so we must use that ϕ lies on the sphere. Observe that the third factor of ϕ is restricted to frequency $\sim 2^0$ by the outer projection and write

$$(-\Delta)^{1/2}(\phi_k \times (-\Delta)^{1/2}\phi_{\sim 0}) - \phi_k \times (-\Delta)\phi_{\sim 0} = \mathcal{L}_k(\phi_k, \phi_{\sim 0})$$

to find

$$(2.5.34) \leq \sum_{a,b} |c_{a,b}^{(k)}| \| \langle \Omega \rangle L[\phi_{<k-10}(x) \times [\phi_k(x - 2^{-k}a) \times \phi_{\sim 0}(x-b)]] \|_{1,2} \quad (2.5.38)$$

We then invoke the vector identity

$$a \times (b \times c) = b(a \cdot c) - c(b \cdot a) \quad (2.5.39)$$

to rewrite

$$\begin{aligned} &\| \langle \Omega \rangle L[\phi_{<k-10}(x) \times [\phi_k(x - 2^{-k}a) \times \phi_{\sim 0}(x-b)]] \|_{1,2} \\ &\leq \| \langle \Omega \rangle L[\phi_k(x - 2^{-k}a)\ \phi_{<k-10}(x) \cdot \phi_{\sim 0}(x-b)] \|_{1,2} \\ &\quad + \| \langle \Omega \rangle L[\phi_{\sim 0}(x-b)\ \phi_{<k-10}(x) \cdot \phi_k(x - 2^{-k}a)] \|_{1,2} \end{aligned} \quad (2.5.40)$$

Let's start with the first term. In order to use (GeId) we need the two terms in the dot product to be evaluated at the same point, so write

$$\phi_{<k-10}(x) = \phi_{<k-10}(x-b) + \int_0^1 b \cdot \nabla \phi_{<k-10}(x - \theta b) d\theta$$

Putting the integral expression into (2.5.40) we get a term of the form (high)∇(low)∇(low) which is easily handled. Indeed, borrowing the factor of 2^k from $c_{a,b}^{(k)}$, we may write

$$\begin{aligned} &2^k \| \langle \Omega \rangle L[\phi_k(x - 2^{-k}a)\ (\int_0^1 b \cdot \nabla \phi_{<k-10}(x - \theta b) d\theta) \cdot \phi_{\sim 0}(x-b)] \|_{1,2} \\ &\lesssim_b 2^k \int_0^1 \| \langle \Omega \rangle L(\phi_k(x - 2^{-k}a)) \|_{2+,\infty-} \| \langle \Omega \rangle L(\nabla \phi_{<k-10}(x - \theta b)) \|_{2+,\infty-} \\ &\qquad \cdot \| \langle \Omega \rangle L(\phi_{\sim 0}(x-b)) \|_{\infty-,2+} d\theta \\ &\lesssim_{a,b} C_0^3 \sum_{j<k-10} \int_0^1 c_j c_k c_0 2^{(\frac{1}{2} - \frac{1}{M})(j+k)} \langle 2^j \theta b \rangle^2 d\theta \end{aligned}$$

which is acceptable when summed as in (2.5.38) and over $k < -10$.

We then come to

$$\begin{aligned} &2^k \| \langle \Omega \rangle L[\phi_k(x - 2^{-k}a)(\phi_{<k-10} \cdot \phi_{\sim 0})(x-b)] \|_{1,2} \\ &\lesssim 2^k \| \langle \Omega \rangle L[\phi_k(x - 2^{-k}a)(\phi_{[k-10,-10]} \cdot \phi_{\sim 0})(x-b)] \|_{1,2} \\ &\quad + 2^k \| \langle \Omega \rangle L[\phi_k(x - 2^{-k}a)(\phi_{\leq -10} \cdot \phi_{\sim 0})(x-b)] \|_{1,2} \end{aligned} \quad (2.5.41)$$

The first of these terms can be handled by a straightforward application of Lemma 2.5.3. For the second we use (GeId) to bound

$$\begin{aligned} &2^k \| \langle \Omega \rangle L[\phi_k(x - 2^{-k}a)(\phi_{\leq -10} \cdot \phi_{\sim 0})(x-b)] \|_{1,2} \\ &\lesssim 2^k \| \langle \Omega \rangle L[\phi_k(x - 2^{-k}a)(\phi_{\leq -10} \cdot \phi_{\sim 0} - P_{\sim 0}(\phi_{\leq -10} \cdot \phi_{\sim 0}))(x-b)] \|_{1,2} \\ &\quad + 2^k \| \langle \Omega \rangle L[\phi_k(x - 2^{-k}a) P_{\sim 0}(\phi_{>-10} \cdot \phi_{>-10})(x-b)] \|_{1,2} \end{aligned}$$

The first line is easy to handle using Lemma 2.5.9 to move a derivative onto the low frequency term, so it remains to consider the second line. From the estimate

$$\begin{aligned}&2^k\|\langle\Omega\rangle L[\phi_k(x-2^{-k}a)P_{\sim 0}(\phi_{>-10}\cdot\phi_{>-10})(x-b)]\|_{1,2}\\&\lesssim 2^k\|\langle\Omega\rangle L(\phi_k(x-2^{-k}a))\|_{2+,\infty-}\|\langle\Omega\rangle L(P_{\sim 0}(\phi_{>-10}\cdot\phi_{>-10})(x-b))\|_{\frac{2M}{M+1},\frac{2M}{M-1}}\end{aligned}$$

it remains to show that

$$\|\langle\Omega\rangle L(P_{\sim 0}(\phi_{>-10}\cdot\phi_{>-10})(x-b))\|_{\frac{2M}{M+1},\frac{2M}{M-1}}\lesssim_b C_0^2\epsilon^2 \tag{2.5.42}$$

First permuting the vector derivatives and the translation by b we find

$$\begin{aligned}\|\langle\Omega\rangle L(P_{\sim 0}(\phi_{>-10}\cdot\phi_{>-10})(x-b))\|_{\frac{2M}{M+1},\frac{2M}{M-1}}\lesssim_b{}&\|\langle\Omega\rangle LP_{\sim 0}(\phi_{>-10}\cdot\phi_{>-10})\|_{\frac{2M}{M+1},\frac{2M}{M-1}}\\&+\|\langle\Omega\rangle P_{\sim 0}(\partial_t\phi_{>-10}\cdot\phi_{>-10})\|_{\frac{2M}{M+1},\frac{2M}{M-1}}\end{aligned} \tag{2.5.43}$$

For the first term we commute $\langle\Omega\rangle L$ and P_0 to see

$$\begin{aligned}\|\langle\Omega\rangle LP_{\sim 0}(\phi_{>-10}\cdot\phi_{>-10})\|_{\frac{2M}{M+1},\frac{2M}{M-1}}&\lesssim\|P_{\sim 0}\langle\Omega\rangle L(\phi_{>-10}\cdot\phi_{>-10})\|_{\frac{2M}{M+1},\frac{2M}{M-1}}\\&\quad+\|\mathcal{P}_{\sim 0}\langle\Omega\rangle\partial_t(\phi_{>-10}\cdot\phi_{>-10})\|_{\frac{2M}{M+1},\frac{2M}{M-1}}\\&\lesssim\|\langle\Omega\rangle L\phi_{>-10}\|_{2+,\infty-}\|\langle\Omega\rangle L\phi_{>-10}\|_{\infty-,2+}\\&\quad+\|\langle\Omega\rangle\partial_t\phi_{>-10}\|_{\infty-,2+}\|\langle\Omega\rangle\phi_{>-10}\|_{2+,\infty-}\end{aligned}$$

which is as required. This calculation also covered the second term in (2.5.43) so (2.5.42) is shown, completing the study of (2.5.41).

It remains to study the second term in (2.5.40). Again we write

$$\phi_{<k-10}(x)=\phi_{<k-10}(x-2^{-k}a)+2^{-k}a\cdot\int_0^1\nabla\phi_{<k-10}(x-2^{-k}a\theta)d\theta \tag{2.5.44}$$

For the term involving the integral we have

$$\begin{aligned}&\|\langle\Omega\rangle L[\phi_{\sim 0}(x-b)(2^{-k}a\cdot\int_0^1\nabla\phi_{<k-10}(x-2^{-k}a\theta)d\theta)\cdot\phi_k(x-2^{-k}a)]\|_{1,2}\\&\lesssim_a 2^{-k}\int_0^1\|\langle\Omega\rangle L[\phi_{\sim 0}(x-b)\nabla\phi_{<k-10}(x-2^{-k}a\theta)\cdot\phi_k(x-2^{-k}a)]\|_{1,2}d\theta\end{aligned}$$

Remembering that we can absorb the 2^{-k} into $c_{a,b}^{(k)}$ from (2.5.38), we see that this can be treated directly using Lemma 2.5.3 after splitting $\langle\Omega\rangle L$ over the three factors.

For the remaining part of $\phi_{<k-10}$ we use the geometry to bound

$$\begin{aligned}&\|\langle\Omega\rangle L[\phi_{\sim 0}(x-b)(\phi_{<k-10}\cdot\phi_k)(x-2^{-k}a)]\|_{1,2}\\&\lesssim\|\langle\Omega\rangle L[\phi_{\sim 0}(x-b)(\phi_{<k-10}\cdot\phi_k-P_k(\phi_{<k-10}\cdot\phi_{\geq k-10}))(x-2^{-k}a)]\|_{1,2}\end{aligned} \tag{2.5.45}$$

$$+ \|\langle\Omega\rangle L[\phi_{\sim 0}(x-b)P_k(\phi_{\geq k-10}\cdot\phi_{\geq k-10})(x-2^{-k}a)]\|_{1,2} \tag{2.5.46}$$

First consider (2.5.46). We consider only the more difficult case where $\langle\Omega\rangle L$ falls on the P_k. By a series of calculations as in (2.5.43), we reduce to studying terms of the form

$$\|\phi_{\sim 0}(x-b)(\mathcal{P}_k\langle\Omega\rangle L(\phi_{\geq k-10}\cdot\phi_{\geq k-10}))(x-2^{-k}a)\|_{1,2}$$

and

$$2^{-k}\|\phi_{\sim 0}(x-b)(\mathcal{P}_k\langle\Omega\rangle\partial_t(\phi_{\geq k-10}\cdot\phi_{\geq k-10}))(x-2^{-k}a)\|_{1,2}$$

We restrict our attention to the more delicate second term, as the first can be treated similarly. Considering for example the case in which the angular and time derivative fall on different factors, we use Corollary 2.5.4 to bound

$$\begin{aligned}
&2^{-k}\|\phi_{\sim 0}(x-b)(\mathcal{P}_k(\partial_t\phi_{\geq k-10}\cdot\langle\Omega\rangle\phi_{\geq k-10}))(x-2^{-k}a)\|_{1,2}\\
&\lesssim 2^{-k}\sum_{r,s\geq k-10}\|\phi_{\sim 0}(x-b+2^{-k}a)\mathcal{P}_k(\partial_t\phi_r\cdot\langle\Omega\rangle\phi_s)(x)]\|_{1,2}\\
&\lesssim 2^{-k}\sum_{r\sim s\geq k-10}2^{-k/M}(2^{3k\frac{M-1}{4M}}2^{\frac{21}{4M}(r-k)}\|\partial_t\phi_r\|_{2+,4+}+2^{k-r}2^{3r/2M}\|\langle\Omega\rangle\partial_t\phi_r\|_{2+,\infty-})\\
&\qquad\cdot\|\langle\Omega\rangle\phi_s\|_{2+,\infty}(\|\phi_{\sim 0}\|_{\infty,2}+2^k\|L(\phi_{\sim 0}(x-b+2^{-k}a))\|_{\infty,2}+\|\partial_t\phi_{\sim 0}\|_{\infty,2})\\
&\lesssim_{a,b} 2^{-k}C_0^3\epsilon c_0\sum_{r\geq k-10}(2^{(3\frac{M-1}{4M}-\frac{21}{4M}-\frac{1}{M})(k-r)}+2^{(1-\frac{1}{M})(k-r)})c_r
\end{aligned}$$

which is acceptable when multiplied by $c_{a,b}^{(k)}$ and summed over a,b and $k<-10$. Note that here the gain of 2^k before the factor $L(\phi_{\sim 0}(x-b+2^{-k}a))$ was necessary in order to cancel out the loss from the translation by $2^k a$.

For (2.5.45) we expand

$$\phi_{<k-10}\cdot\phi_k - P_k(\phi_{<k-10}\cdot\phi_{\geq k-10}) = 2^{-k}\int_0^1\int_y(2^k y\check{\chi}_k(y))\nabla\phi_{<k-10}(x-\theta y)\phi_{\sim k}(x-y)dyd\theta$$

It therefore remains to bound

$$\begin{aligned}
&\sum_{a,b}2^{-k}|c_{a,b}^{(k)}|\int_{y,\theta}dyd\theta\,|2^k y\check{\chi}_k(y)|\\
&\qquad\cdot\|\langle\Omega\rangle L[\phi_{\sim 0}(x-b)\nabla\phi_{<k-10}(x-2^{-k}a-\theta y)\phi_{\sim k}(x-2^{-k}a-y)]\|_{1,2}
\end{aligned}$$

which can be handled using Lemma 2.5.3.

- <u>$k>10$:</u> We study

$$\sum_{j\in\mathbb{Z}}\|P_0\langle\Omega\rangle L[\phi\times[(-\Delta)^{1/2}(\phi_k\times(-\Delta)^{1/2}\phi_j)-(\phi_k\times(-\Delta)\phi_j)]]\|_{1,2}$$

If the third factor of ϕ is restricted to $j < k - 10$, then the first ϕ is restricted to $\phi_{\sim k}$ by the outer projection P_0. The term is therefore of type (high)∇(high)∇(low) and may be treated as (2.4.6) upon writing

$$(-\Delta)^{1/2}(\phi_k \times (-\Delta)^{1/2}\phi_j) - (\phi_k \times (-\Delta)\phi_j) = \mathcal{L}_k(\phi_k, (-\Delta)^{1/2}\phi_j)$$

and appealing to the identities (2.5.4)-(2.5.5).

When $j > k + 10$, again the first factor is restricted to $\phi_{\sim j}$ and this may be treated similarly.

It thus remains to study $j \sim k$. Here there is nothing to be gained by cancellation so we split the term up into its two parts:

$$\begin{aligned} &\|P_0\langle\Omega\rangle L[\phi \times [(-\Delta)^{1/2}(\phi_k \times (-\Delta)^{1/2}\phi_{\sim k}) - (\phi_k \times (-\Delta)\phi_{\sim k})]]\|_{1,2} \\ &\leq \underbrace{\|P_0\langle\Omega\rangle L[\phi \times (-\Delta)^{1/2}(\phi_k \times (-\Delta)^{1/2}\phi_{\sim k})]\|_{1,2}}_{(A)} + \underbrace{\|P_0\langle\Omega\rangle L[\phi \times (\phi_k \times (-\Delta)\phi_{\sim k})]\|_{1,2}}_{(B)} \end{aligned} \tag{2.5.47}$$

We first study (A). This term presents some more complications due to its nonlocal expression, however it also has the advantage that when the remaining ϕ is at low frequency, the outer derivative $(-\Delta)^{1/2}$ is acting at frequency ~ 1. First split ϕ into low and high frequencies:

$$\begin{aligned} (A) \leq\, &\underbrace{\|P_0\langle\Omega\rangle L[\phi_{\leq -10} \times (-\Delta)^{1/2}(\phi_k \times (-\Delta)^{1/2}\phi_{\sim k})]\|_{1,2}}_{(A)_{\leq -10}} \\ &+ \underbrace{\|P_0\langle\Omega\rangle L[\phi_{> -10} \times (-\Delta)^{1/2}(\phi_k \times (-\Delta)^{1/2}\phi_{\sim k})]\|_{1,2}}_{(A)_{> -10}} \end{aligned}$$

Here $(A)_{>-10}$ is of type (high)∇(high)∇(high) so can be handled like (2.4.8) using the radially admissible Strichartz spaces.

$(A)_{\leq -10}$ is of the form (low)∇(high)∇(high) so must be handled using the geometry. In this case it is especially important to keep track of how the vector derivatives are falling as the commutator terms can rapidly cause a build up of derivatives if treated too crudely.

To clarify the calculations we then fix a particular Ω_{ij} and L_n (the inhomogeneous parts of $\langle\Omega\rangle$ and L are easier to handle) and make a very precise decomposition. Note that we are free to switch the order of Ω_{ij} and L_n up to a term of the same form.

Writing $\tilde{P}_0(-\Delta)^{1/2} = \mathcal{P}_0$, a radial operator, and using the Leibniz rule, we have

$$\left.\begin{aligned}
&P_0 L_n \Omega_{ij}[\phi_{\le -10} \times (-\Delta)^{1/2}(\phi_k \times (-\Delta)^{1/2}\phi_{\sim k})] \\
&= P_0[L_n\Omega_{ij}\phi_{\le -10} \times \mathcal{P}_0(\phi_k \times (-\Delta)^{1/2}\phi_{\sim k})] \\
&\quad + P_0[L_n\phi_{\le -10} \times \mathcal{P}_0(\Omega_{ij}\phi_k \times (-\Delta)^{1/2}\phi_{\sim k})] \\
&\quad + P_0[L_n\phi_{\le -10} \times \mathcal{P}_0(\phi_k \times (-\Delta)^{1/2}\Omega_{ij}\phi_{\sim k})]
\end{aligned}\right\} \quad \text{(A1)}$$

$$\left.\begin{aligned}
&+ P_0[\Omega_{ij}\phi_{\le -10} \times \mathcal{P}_0(L_n\phi_k \times (-\Delta)^{1/2}\phi_{\sim k})] \\
&+ P_0[\phi_{\le -10} \times \mathcal{P}_0(L_n\Omega_{ij}\phi_k \times (-\Delta)^{1/2}\phi_{\sim k})] \\
&+ P_0[\phi_{\le -10} \times \mathcal{P}_0(L_n\phi_k \times (-\Delta)^{1/2}\Omega_{ij}\phi_{\sim k})]
\end{aligned}\right\} \quad \text{(A2)}$$

$$\left.\begin{aligned}
&+ P_0[\Omega_{ij}\phi_{\le -10} \times \mathcal{P}_0(\phi_k \times L_n(-\Delta)^{1/2}\phi_{\sim k})] \\
&+ P_0[\phi_{\le -10} \times \mathcal{P}_0(\Omega_{ij}\phi_k \times L_n(-\Delta)^{1/2}\phi_{\sim k})] \\
&+ P_0[\phi_{\le -10} \times \mathcal{P}_0(\phi_k \times L_n\Omega_{ij}(-\Delta)^{1/2}\phi_{\sim k})]
\end{aligned}\right\} \quad \text{(A3)}$$

$$\left.\begin{aligned}
&+ P_0[\Omega_{ij}\phi_{\le -10} \times \mathcal{P}_0'\partial_t(\phi_k \times (-\Delta)^{1/2}\phi_{\sim k})] \\
&+ P_0[\phi_{\le -10} \times \mathcal{P}_0'\partial_t(\Omega_{ij}\phi_k \times (-\Delta)^{1/2}\phi_{\sim k})] \\
&+ P_0[\phi_{\le -10} \times \mathcal{P}_0'\partial_t(\phi_k \times (-\Delta)^{1/2}\Omega_{ij}\phi_{\sim k})]
\end{aligned}\right\} \quad \text{(A4)}$$

In the above $\mathcal{P}_0'$ is another operator of the type described in Section 2.2.1, which may not be radial.

We start by considering the group (A4) which is the most difficult since there is an additional derivative falling on the high frequency terms. Consider first the case where ∂_t falls on ϕ_k. Writing the operator $\mathcal{P}_0'$ as an explicit convolution by some $\mathcal{K}_0$, we have for the first line

$$\begin{aligned}
&\Omega_{ij}\phi_{\le -10} \times \mathcal{P}_0'(\partial_t\phi_k \times (-\Delta)^{1/2}\phi_{\sim k}) \\
&= \Omega_{ij}\phi_{\le -10}(x) \times \int_y \mathcal{K}_0(y)(\partial_t\phi_k(x-y) \times (-\Delta)^{1/2}\phi_{\sim k}(x-y))dy \\
&= \int_y \mathcal{K}_0(y)\partial_t\phi_k(x-y)\ \Omega_{ij}\phi_{\le -10}(x) \cdot (-\Delta)^{1/2}\phi_{\sim k}(x-y)dy && (2.5.48) \\
&\quad - \int_y \mathcal{K}_0(y)(-\Delta)^{1/2}\phi_{\sim k}(x-y)\ \Omega_{ij}\phi_{\le -10}(x) \cdot \partial_t\phi_k(x-y)dy && (2.5.49)
\end{aligned}$$

We must then split

$$\Omega_{ij}\phi_{\le -10}(x) = (\Omega_{ij}\phi_{\le -10})(x-y) + y \cdot \int_{\theta=0}^1 \nabla(\Omega_{ij}\phi_{\le -10})(x-\theta y)d\theta$$

Putting the integral term into (2.5.48) we find

$$\begin{aligned}
&\left\| \int_y \mathcal{K}_0(y)\partial_t\phi_k(x-y)\ (y \cdot \int_{\theta=0}^1 \nabla(\Omega_{ij}\phi_{\le -10})(x-\theta y)d\theta) \cdot (-\Delta)^{1/2}\phi_{\sim k}(x-y)dy \right\|_{1,2} \\
&\lesssim \int_y \int_0^1 |\mathcal{K}_0(y)||y| \|\partial_t\phi_k\|_{9,\frac{10}{3}} \|\nabla\Omega_{ij}\phi_{\le -10}\|_{\frac{18}{7},\infty} \|(-\Delta)^{1/2}\phi_{\sim k}\|_{2,5} d\theta dy
\end{aligned}$$

which is acceptable when summed over $k > 10$ since the factor of $|y|$ is absorbed by the kernel $\mathcal{K}_0$. The same argument works for (2.5.49), and for the corresponding terms in the second and third lines of (A4).

For (2.5.48) it therefore remains to consider

$$\int_y \mathcal{K}_0(y)[\partial_t\phi_k\ \Omega_{ij}\phi_{\leq -10}\cdot(-\Delta)^{1/2}\phi_{\sim k}](x-y)dy$$

The corresponding terms from the second and third lines of (A4) are

$$\int_y \mathcal{K}_0(y)[\Omega_{ij}\partial_t\phi_k\ \phi_{\leq -10}\cdot(-\Delta)^{1/2}\phi_{\sim k}](x-y)dy \tag{2.5.50}$$

and

$$\int_y \mathcal{K}_0(y)[\partial_t\phi_k\ \phi_{\leq -10}\cdot(-\Delta)^{1/2}\Omega_{ij}\phi_{\sim k}](x-y)dy \tag{2.5.51}$$

and we have to bound the sum of these in $L^1_tL^2_x$.

In order to use (GeId) we rewrite $\phi_{\leq -10} = \phi_{\leq k-10} - \phi_{[-10,k-10]}$. Then for the high frequency part we can again use a bound as for (2.4.8) to see e.g.

$$\|\partial_t\phi_k\ \Omega_{ij}\phi_{[-10,k-10]}\cdot(-\Delta)^{1/2}\phi_{\sim k}\|_{1,2} \lesssim \|\partial_t\phi_k\|_{9,\frac{10}{3}}\|\Omega_{ij}\phi_{[-10,k-10]}\|_{\frac{18}{7},\infty}\|(-\Delta)^{1/2}\phi_{\sim k}\|_{2,5}$$

which is acceptable. The same argument works for (2.5.50) and (2.5.51).

For the low frequency part $\phi_{\leq k-10}$ we want to use (GeId). We have

$$\begin{aligned}
&\partial_t\phi_k\ \Omega_{ij}\phi_{\leq k-10}\cdot(-\Delta)^{1/2}\phi_{\sim k}\\
&= \partial_t\phi_k\ (\Omega_{ij}\phi_{\leq k-10}\cdot(-\Delta)^{1/2}\phi_{\sim k} - (-\Delta)^{1/2}(\Omega_{ij}\phi_{\leq k-10}\cdot\phi_{\sim k})) && (2.5.52)\\
&\quad + \partial_t\phi_k\ (-\Delta)^{1/2}(\Omega_{ij}\phi_{\leq k-10}\cdot\phi_{\sim k} - P_{\sim k}(\Omega_{ij}\phi_{\leq k-10}\cdot\phi_{>k-10})) && (2.5.53)\\
&\quad + \partial_t\phi_k\ (-\Delta)^{1/2}P_{\sim k}(\Omega_{ij}\phi_{\leq k-10}\cdot\phi_{>k-10}) && (2.5.54)
\end{aligned}$$

For the first line we use Lemma 2.5.1 to bound

$$\begin{aligned}
\|(2.5.52)\|_{1,2} &\lesssim \sum_{j\leq k-10}\|\partial_t\phi_k\ \mathcal{L}_j(\Omega_{ij}\phi_j,\phi_{\sim k})\|_{1,2}\\
&\lesssim \sum_{j\leq k-10} 2^j\|\partial_t\phi_k\|_{2+,\infty-}\|\Omega_{ij}\phi_j\|_{2+,\infty-}\|\phi_{\sim k}\|_{\infty-,2+}
\end{aligned}$$

and for the second line Lemma 2.5.9 to find

$$\|(2.5.53)\|_{1,2} \lesssim \|\partial_t\phi_k\|_{2+,\infty-}2^k2^{-k}\|\nabla\Omega_{ij}\phi_{\leq k-10}\|_{2+,\infty-}\|\phi_{>k-10}\|_{\infty-,2+}$$

Both of these bounds are acceptable, and we can treat the corresponding parts of (2.5.50) and (2.5.51) in the same way.

We at last come to the interesting part, (2.5.54). We want to use (GeId), but are obstructed by the presence of the Ω_{ij}. The solution is to combine this term with the corresponding part of (2.5.51). We have

$$\begin{aligned}
&\partial_t\phi_k\ (-\Delta)^{1/2}P_{\sim k}(\Omega_{ij}\phi_{\leq k-10}\cdot\phi_{>k-10}) + \partial_t\phi_k\ (-\Delta)^{1/2}P_{\sim k}(\phi_{\leq k-10}\cdot\Omega_{ij}\phi_{>k-10})\\
&= \partial_t\phi_k\ (-\Delta)^{1/2}\Omega_{ij}P_{\sim k}(\phi_{\leq k-10}\cdot\phi_{>k-10})\\
&= -\frac{1}{2}\partial_t\phi_k\ (-\Delta)^{1/2}\Omega_{ij}P_{\sim k}(\phi_{>k-10}\cdot\phi_{>k-10})
\end{aligned}$$

We then bound

$$\begin{aligned}
&\|\partial_t\phi_k\ (-\Delta)^{1/2}\Omega_{ij}P_{\sim k}(\phi_{>k-10}\cdot\phi_{>k-10})\|_{1,2}\\
&\lesssim 2^k\|\partial_t\phi_k\|_{2+,\infty-}\|\Omega_{ij}\phi_{>k-10}\|_{2+,\infty-}\|\phi_{>k-10}\|_{\infty-,2+}
\end{aligned}$$

which is acceptable. The corresponding term in (2.5.50) can be handled similarly on its own.

The term (2.5.49) can be handled in the same way, using the Leibniz rule on the time-derivative in place of Lemma 2.5.1.

To complete the study of (A4) we have to consider the case where the time derivative falls on $(-\Delta)^{1/2}\phi_{\sim k}$ instead of ϕ_k. In this case the argument carries through identically until it comes to handling the term analogous to (2.5.54),

$$\phi_k\ (-\Delta)^{1/2}P_{\sim k}(\Omega_{ij}\phi_{\leq k-10}\cdot\partial_t\phi_{>k-10})$$

with similar contributions

$$\Omega_{ij}\phi_k\ (-\Delta)^{1/2}P_{\sim k}(\phi_{\leq k-10}\cdot\partial_t\phi_{>k-10}) \tag{2.5.55}$$

and

$$\phi_k\ (-\Delta)^{1/2}P_{\sim k}(\phi_{\leq k-10}\cdot\partial_t\Omega_{ij}\phi_{>k-10}) \tag{2.5.56}$$

from the second and third lines of (A4). First note that if the derivative were instead on $\phi_{\leq k-10}$ we would be fine in all three cases, for instance

$$\begin{aligned}
&\|\phi_k\ (-\Delta)^{1/2}P_{\sim k}(\Omega_{ij}\partial_t\phi_{\leq k-10}\cdot\phi_{>k-10})\|_{1,2}\\
&\lesssim \|\phi_k\|_{2+,\infty-}2^k\|\Omega_{ij}\partial_t\phi_{\leq k-10}\|_{2+,\infty-}\|\phi_{>k-10}\|_{\infty-,2+}
\end{aligned}$$

It therefore remains to study

$$\begin{aligned}
&\phi_k\ (-\Delta)^{1/2}P_{\sim k}\partial_t(\Omega_{ij}\phi_{\leq k-10}\cdot\phi_{>k-10})\\
&\Omega_{ij}\phi_k\ (-\Delta)^{1/2}P_{\sim k}\partial_t(\phi_{\leq k-10}\cdot\phi_{>k-10})\\
&\phi_k\ (-\Delta)^{1/2}P_{\sim k}\partial_t(\phi_{\leq k-10}\cdot\Omega_{ij}\phi_{>k-10})
\end{aligned}$$

Combining the first and last terms and using (GeId) we bound

$$\begin{aligned}
&\|\phi_k\ (-\Delta)^{1/2}P_{\sim k}\partial_t(\Omega_{ij}\phi_{\le k-10}\cdot\phi_{>k-10}) + \phi_k\ (-\Delta)^{1/2}P_{\sim k}\partial_t(\phi_{\le k-10}\cdot\Omega_{ij}\phi_{>k-10})\|_{1,2}\\
&= \|\phi_k\ (-\Delta)^{1/2}\partial_t\Omega_{ij}P_{\sim k}(\phi_{>k-10}\cdot\phi_{>k-10})\|_{1,2}\\
&\lesssim 2^k\|\phi_k\|_{2+,\infty-}\|\partial_t\langle\Omega\rangle\phi_{>k-10}\|_{\infty-,2+}\|\langle\Omega\rangle\phi_{>k-10}\|_{2+,\infty-}
\end{aligned}$$

which is acceptable when summed over $k > 10$. The middle term can be dealt with in the same way on its own. This completes the analysis for (A4).

The groups (A1), (A2) and (A3) must be treated simultaneously in order to preserve the structure for (GeId). In all cases, we can work as for (A4) up to the decomposition (2.5.52)-(2.5.54). At this point for (A1) we will be studying

$$\begin{aligned}
&\phi_k\ L_n\Omega_{ij}\phi_{\le k-10}\cdot(-\Delta)^{1/2}\phi_{\sim k}\\
&\Omega_{ij}\phi_k\ L_n\phi_{\le k-10}\cdot(-\Delta)^{1/2}\phi_{\sim k}\\
&\phi_k\ L_n\phi_{\le k-10}\cdot(-\Delta)^{1/2}\Omega_{ij}\phi_{\sim k}
\end{aligned}$$

for the first, second and third lines respectively. For (A2) we will have

$$\begin{aligned}
&L_n\phi_k\ \Omega_{ij}\phi_{\le k-10}\cdot(-\Delta)^{1/2}\phi_{\sim k}\\
&L_n\Omega_{ij}\phi_k\ \phi_{\le k-10}\cdot(-\Delta)^{1/2}\phi_{\sim k}\\
&L_n\phi_k\ \phi_{\le k-10}\cdot(-\Delta)^{1/2}\Omega_{ij}\phi_{\sim k}
\end{aligned}$$

and for (A3)

$$\begin{aligned}
&\phi_k\ \Omega_{ij}\phi_{\le k-10}\cdot L_n(-\Delta)^{1/2}\phi_{\sim k}\\
&\Omega_{ij}\phi_k\ \phi_{\le k-10}\cdot L_n(-\Delta)^{1/2}\phi_{\sim k}\\
&\phi_k\ \phi_{\le k-10}\cdot L_n\Omega_{ij}(-\Delta)^{1/2}\phi_{\sim k}
\end{aligned}$$

(as well as a second set of easier terms from the expansion of the cross product). Adding these nine terms together and reversing the Leibniz rule on Ω_{ij} and L_n this comes to

$$L_n\Omega_{ij}[\phi_k\ \phi_{\le k-10}\cdot(-\Delta)^{1/2}\phi_{\sim k}]$$

which we can split up as

$$\begin{aligned}
&L_n\Omega_{ij}[\phi_k\ (\phi_{\le k-10}\cdot(-\Delta)^{1/2}\phi_{\sim k} - (-\Delta)^{1/2}(\phi_{\le k-10}\cdot\phi_{\sim k}))]\\
&+ L_n\Omega_{ij}[\phi_k\ (-\Delta)^{1/2}(\phi_{\le k-10}\cdot\phi_{\sim k} - P_{\sim k}(\phi_{\le k-10}\cdot\phi_{>k-10}))]\\
&+ L_n\Omega_{ij}[\phi_k\ (-\Delta)^{1/2}P_{\sim k}(\phi_{\le k-10}\cdot\phi_{>k-10})] \qquad (2.5.57)
\end{aligned}$$

The first term is of the form

$$\sum_{j\leq k-10}\Omega_{ij}L_n[\phi_k\mathcal{L}_j(\phi_j,\phi_{\sim k})]$$

which can be treated using (2.5.4)-(2.5.5) and placing ϕ_k and ϕ_j into $L_t^{2+}L_x^{\infty-}$, and $\phi_{\sim k}$ into $L_t^{\infty-}L_x^{2+}$.

For the second term, we use Lemma 2.5.9 to write, for example when $L_n\Omega_{ij}$ falls on the difference term

$$\begin{aligned}
&\|\phi_k\ L_n\Omega_{ij}(-\Delta)^{1/2}(\phi_{\leq k-10}\cdot\phi_{\sim k}-P_{\sim k}(\phi_{\leq k-10}\cdot\phi_{>k-10}))\|_{1,2}\\
&\lesssim\|\phi_k\|_{\frac{2M}{M-1},2M}\\
&\qquad\cdot\left\|2^{-k}L_n\Omega_{ij}(-\Delta)^{1/2}\int_{y,\theta}(2^ky\check{\chi}_k(y))\nabla\phi_{\leq k-10}(x-\theta y)\phi_{\sim k}(x-y)dyd\theta\right\|_{\frac{2M}{M+1},\frac{2M}{M-1}}\\
&\lesssim 2^{-k}2^{-(\frac12+\frac1M)k}C_0c_k\\
&\qquad\cdot\int_{y,\theta}|2^ky\check{\chi}_k(y)|\left\|L_n\Omega_{ij}(-\Delta)^{1/2}[\nabla\phi_{\leq k-10}(x-\theta y)\phi_{\sim k}(x-y)]\right\|_{\frac{2M}{M+1},\frac{2M}{M-1}}dyd\theta
\end{aligned}$$

where

$$\begin{aligned}
&\left\|L_n\Omega_{ij}(-\Delta)^{1/2}[\nabla\phi_{\leq k-10}(x-\theta y)\phi_{\sim k}(x-y)]\right\|_{\frac{2M}{M+1},\frac{2M}{M-1}}\\
&\lesssim 2^k\|L\langle\Omega\rangle(\nabla\phi_{\leq k-10}(x-\theta y))\|_{2+,\infty-}\|L\langle\Omega\rangle(\phi_{\sim k}(x-y))\|_{\infty-,2+}\\
&\quad+\|\langle\Omega\rangle(\partial_t\nabla\phi_{\leq k-10}(x-\theta y))\|_{2+,\infty-}\|\langle\Omega\rangle(\phi_{\sim k}(x-y))\|_{\infty-,2+}\\
&\quad+\|\langle\Omega\rangle(\nabla\phi_{\leq k-10}(x-\theta y))\|_{2+,\infty-}\|\langle\Omega\rangle(\partial_t\phi_{\sim k}(x-y))\|_{\infty-,2+}
\end{aligned}$$

all of which are acceptable using (2.5.6), because $|y|$ behaves like 2^{-k} in the integral.

The third term of (2.5.57) can be treated using (GeId) and the commutation relation between L_n and $(-\Delta)^{1/2}P_{\sim k}$. We place ϕ_k into $L_t^{2+}L_x^{\infty-}$, one of the high frequency factors into $L_t^{2+}L_x^{\infty-}$ and the other into $L_t^{\infty-}L_x^{2+}$, in particular the one accompanied by a ∂_t when this arises from $[L_n,(-\Delta)^{1/2}P_{\sim k}]$.

To conclude the case $k>10$, it remains to consider (B). When ϕ appears at high frequency this term is again easily handled like (2.4.8). In the low frequency case, $\phi_{\leq -10}$, the term can be treated analogously to the group (A4) which also contains two high frequency derivatives, but with significant simplifications.

- $k\in[-10,10]$: This time we consider

$$\|P_0\langle\Omega\rangle L[\phi\times((-\Delta)^{1/2}(\phi_{\sim 0}\times(-\Delta)^{1/2}\phi)-(\phi_{\sim 0}\times(-\Delta)\phi))]\|_{1,2}$$

This term is easiest to handle when the outer factor of ϕ is at high frequency. Indeed we have

$$\|P_0\langle\Omega\rangle L[\phi_{>-10}\times((-\Delta)^{1/2}(\phi_{\sim 0}\times(-\Delta)^{1/2}\phi)-(\phi_{\sim 0}\times(-\Delta)\phi))]\|_{1,2}$$

$$\begin{aligned}&\lesssim \|\langle\Omega\rangle L[\phi_{>-10}\times((-\Delta)^{1/2}(\phi_{\sim 0}\times(-\Delta)^{1/2}\phi_{\leq 20})-(\phi_{\sim 0}\times(-\Delta)\phi_{\leq 20}))]\|_{1,2}\\ &\quad+\|\langle\Omega\rangle L[\phi_{>-10}\times((-\Delta)^{1/2}(\phi_{\sim 0}\times(-\Delta)^{1/2}\phi_{>20})-(\phi_{\sim 0}\times(-\Delta)\phi_{>20}))]\|_{1,2}\end{aligned}$$

Upon carefully commuting $\langle\Omega\rangle L$ through the operators $(-\Delta)^{1/2}$, we can bound the first line above by placing $\phi_{>-10}$ into $L_t^{2+}L_x^{\infty-}$, $\phi_{\sim 0}$ into $L_t^{\infty-}L_x^{2+}$ and $\phi_{<20}$ also into $L_t^{2+}L_x^{\infty-}$, without needing to use the cancellation structure. For the second line we do need the cancellation, since we cannot handle two derivatives falling on a high frequency factor, so bound this by

$$\sum_{j>20}\|\langle\Omega\rangle L[\phi_{>-10}\times\mathcal{L}_0(\phi_{\sim 0},(-\Delta)^{1/2}\phi_j)]\|_{1,2}$$

which can be dealt with by placing $\phi_{>-10}$ and $\phi_{\sim 0}$ into $L_t^{2+}L_x^{\infty-}$ and ϕ_j into $L_t^{\infty-}L_x^{2+}$.

The case $\phi_{\leq -10}$ is more delicate. Note that in this case the final factor of ϕ is also restricted to frequency $\lesssim 1$. First suppose it is at frequency ~ 1. Write

$$\tilde{P}_0((-\Delta)^{1/2}(\phi_{\sim 0}\times(-\Delta)^{1/2}\phi_{\sim 0})-(\phi_{\sim 0}\times(-\Delta)\phi_{\sim 0}))=\mathcal{L}_0(\phi_{\sim 0},\phi_{\sim 0})$$

Then using (2.5.3) we have

$$\begin{aligned}&\|P_0\langle\Omega\rangle L[\phi_{\leq -10}\times((-\Delta)^{1/2}(\phi_{\sim 0}\times(-\Delta)^{1/2}\phi_{\sim 0})-(\phi_{\sim 0}\times(-\Delta)\phi_{\sim 0}))]\|_{1,2}\\ &\lesssim\sum_{a,b}|c^{(0)}_{a,b}|\|\langle\Omega\rangle L[\phi_{\leq -10}(x)\times(\phi_{\sim 0}(x-a)\times\phi_{\sim 0}(x-b))]\|_{1,2}\\ &\lesssim\sum_{a,b}|c^{(0)}_{a,b}|\|\langle\Omega\rangle L[\phi_{\sim 0}(x-a)\ \phi_{\leq -10}(x)\cdot\phi_{\sim 0}(x-b)]\|_{1,2}+\text{similar term}\end{aligned}$$

We can then replace $\phi_{\leq -10}(x)$ with $\phi_{\leq -10}(x-b)$ up to an integral term of the form (high)∇(high)∇(low). Using Lemma 2.5.9 we can then exchange $\phi_{\leq -10}\cdot\phi_{\sim 0}$ for $\tilde{P}_0(\phi_{\leq -10}\cdot\phi_{>-10})\simeq\tilde{P}_0(\phi_{>-10}\cdot\phi_{>-10})$, and bound

$$\begin{aligned}&\|\langle\Omega\rangle L[\phi_{\sim 0}(x-a)\ \tilde{P}_0(\phi_{>-10}\cdot\phi_{>-10})(x-b)]\|_{1,2}\\ &\lesssim_{a,b}\|\langle\Omega\rangle L\phi_{\sim 0}\|_{2+,\infty-}\\ &\qquad\cdot(\|\langle\Omega\rangle L(\phi_{>-10}\cdot\phi_{>-10})\|_{\frac{2M}{M+1},\frac{2M}{M-1}}+\|\langle\Omega\rangle(\partial_t\phi_{>-10}\cdot\phi_{>-10})\|_{\frac{2M}{M+1},\frac{2M}{M-1}})\end{aligned}$$

which can be handled by placing one of the high frequency factors (the differentiated one in the second case) into $L_t^{\infty-}L_x^{2+}$ and the other into the other into $L_t^{2+}L_x^{\infty-}$.

We now consider the case where the third factor of ϕ is at low frequency, say $\leq 2^{-20}$. We start by writing

$$(-\Delta)^{1/2}(\phi_{\sim 0}\times(-\Delta)^{1/2}\phi_j)-(\phi_{\sim 0}\times(-\Delta)\phi_j)=\mathcal{L}_j(\phi_{\sim 0},\phi_j)$$

for $j<-20$. Then we have

$$\|P_0\langle\Omega\rangle L[\phi_{\leq -10}\times((-\Delta)^{1/2}(\phi_{\sim 0}\times(-\Delta)^{1/2}\phi_{<-20})-(\phi_{\sim 0}\times(-\Delta)\phi_{<-20}))]\|_{1,2}$$

$$\lesssim \sum_{j<-20} \sum_{a,b} |c^{(j)}_{a,b}| \|P_0 \langle\Omega\rangle L[\phi_{\sim 0}(x-a)\ \phi_{\leq -10}(x) \cdot \phi_j(x - 2^{-j}b)]\|_{1,2} \tag{2.5.58}$$

$$+ \sum_{j<-20} \sum_{a,b} |c^{(j)}_{a,b}| \|P_0 \langle\Omega\rangle L[\phi_j(x - 2^{-j}b)\ \phi_{\leq -10}(x) \cdot \phi_{\sim 0}(x-a)]\|_{1,2} \tag{2.5.59}$$

We will study the first line above, the second being similar (in fact significantly easier). In order to use (GeId), we split $\phi_{\leq -10}$ into $\phi_{[j-10,-10]} + \phi_{<j-10}$. The first component here is handled by a straightforward application of Lemma 2.5.3, so we are left to study

$$\sum_{j<-20} \sum_{a,b} |c^{(j)}_{a,b}| \|P_0 \langle\Omega\rangle L[\phi_{\sim 0}(x-a)\ \phi_{<j-10}(x) \cdot \phi_j(x - 2^{-j}b)]\|_{1,2}$$

We first replace $\phi_{<j-10}(x)$ with $\phi_{<j-10}(x - 2^{-j}b)$ up to an acceptable integrable term of the form (low)∇(lowest)∇(high). We are then left with

$$\sum_{j<-20} \sum_{a,b} |c^{(j)}_{a,b}| \|P_0 \langle\Omega\rangle L[\phi_{\sim 0}(x-a)\ (\phi_{<j-10} \cdot \phi_j)(x - 2^{-j}b)]\|_{1,2}$$

Similarly to before, we can replace $\phi_{<j-10} \cdot \phi_j$ with $P_j(\phi_{<j-10} \cdot \phi_{\geq j-10})$ up to the term

$$\begin{aligned} 2^{-j} |c^{(j)}_{a,b}| \int_{y,\theta} dy d\theta |(2^j y)^T \check{\chi}_j(y)| \\ \cdot \|\langle\Omega\rangle L[\phi_{\sim 0}(x-a)\ \nabla\phi_{<j-10}(x - 2^{-j}b - \theta y) \cdot \phi_{\sim j}(x - 2^{-j}b - y)]\|_{1,2} \end{aligned}$$

which is also of type (low)∇(lowest)∇(high) and can be handled using Lemma 2.5.3. We can then finally invoke (GeId) to bound

$$\begin{aligned} &\sum_{j<-20} \sum_{a,b} |c^{(j)}_{a,b}| \|P_0 \langle\Omega\rangle L[\phi_{\sim 0}(x-a)\ P_j(\phi_{<j-10} \cdot \phi_{\geq j-10})(x - 2^{-j}b)]\|_{1,2} \\ &\lesssim \sum_{r\sim s\geq j-10} \sum_{j<-20} \sum_{a,b} |c^{(j)}_{a,b}| \|P_0 \langle\Omega\rangle L[\phi_{\sim 0}(x-a)\ P_j(\phi_r \cdot \phi_s)(x - 2^{-j}b)]\|_{1,2} \end{aligned} \tag{2.5.60}$$

This term can be handled using Corollary 2.5.4. For example, when $\langle\Omega\rangle$ and L both fall on P_j, we have (up to some terms which are symmetric in r and s)

$$\begin{aligned} &2^j \|\phi_{\sim 0}(x-a)\ \langle\Omega\rangle L(P_j(\phi_r \cdot \phi_s)(x - 2^{-j}b))\|_{1,2} \\ &\lesssim_b \|\phi_{\sim 0}(x - a + 2^{-j}b)\mathcal{P}_j(\langle\Omega\rangle\partial_t\phi_r \cdot \phi_s)(x)\|_{1,2} \\ &\quad + \|\phi_{\sim 0}(x - a + 2^{-j}b)\mathcal{P}_j(\partial_t\phi_r \cdot \langle\Omega\rangle\phi_s)(x)\|_{1,2} \\ &\quad + 2^j \|\phi_{\sim 0}(x - a + 2^{-j}b)\mathcal{P}_j(\langle\Omega\rangle L\phi_r \cdot \phi_s)(x)\|_{1,2} \\ &\quad + 2^j \|\phi_{\sim 0}(x - a + 2^{-j}b)\mathcal{P}_j(L\phi_r \cdot \langle\Omega\rangle\phi_s)(x)\|_{1,2} \\ &\lesssim 2^{-j/M} 2^{3j\frac{M-1}{4M}} 2^{\frac{21}{4M}(s-j)} \|\phi_s\|_{2+,4+} \|\langle\Omega\rangle\partial_t\phi_r\|_{2+,\infty} C_0 c_0 \\ &\quad + 2^{-j/M} 2^{j-s} 2^{3s/2M} \|\langle\Omega\rangle\phi_s\|_{2+,\infty-} \|\langle\Omega\rangle\partial_t\phi_r\|_{2+,\infty} C_0 c_0 \\ &\quad + 2^{-j/M} 2^{3j\frac{M-1}{4M}} 2^{\frac{21}{4M}(r-j)} \|\partial_t\phi_r\|_{2+,4+} \|\langle\Omega\rangle\phi_s\|_{2+,\infty} C_0 c_0 \end{aligned}$$

$$
\begin{aligned}
&+ 2^{-j/M}2^{j-r}2^{3r/2M}\|\langle\Omega\rangle\partial_t\phi_r\|_{2+,\infty-}\|\langle\Omega\rangle\phi_s\|_{2+,\infty}C_0c_0 \\
&+ 2^j(\text{same terms with } L \text{ instead of } \partial_t) \\
\lesssim\ &(2^{[-\frac{1}{M}+3\frac{M-1}{4M}-\frac{21}{4M}](j-r)} + 2^{(1-\frac{1}{M})(j-r)})C_0^3c_0c_rc_s
\end{aligned}
$$

which is acceptable when summed as in (2.5.60). This concludes the study of (2.5.58).

□

2.6 Normal Forms

The goal of this section is to perform a series of normal transformations to reduce the second and third terms on the right hand side of equation (2.4.11) to *error*.

2.6.1 Low-high-high term

To handle the third term, we make the transformation

$$
\psi^L \mapsto \tilde{\psi}^L := \psi^L + \frac{1}{2}(\Delta_1) \quad \text{for} \quad (\Delta_1) = \begin{pmatrix} \Delta_1^0 \\ \Delta_1^{1,1} + \Delta_1^{1,2} + \Delta_1^{1,3} \\ \vdots \\ \Delta_1^{3,1} + \Delta_1^{3,2} + \Delta_1^{3,3} \end{pmatrix}
$$

with

$$
\begin{aligned}
\Delta_1^0 &:= P_0(\phi_{\leq -10}\phi^T_{>-10}\phi_{>-10}) \\
\Delta_1^{n,1} &:= P_0((L_n\phi)_{\leq -10}\phi^T_{>-10}\phi_{>-10}) \\
\Delta_1^{n,2} &:= P_0(\phi_{\leq -10}(L_n\phi)^T_{>-10}\phi_{>-10}) \\
\Delta_1^{n,3} &:= P_0(\phi_{\leq -10}\phi^T_{>-10}(L_n\phi)_{>-10})
\end{aligned}
$$

We start by showing that this transformation is bounded in the following sense:

Proposition 2.6.1. *For* (Δ_1) *as above, it holds*

$$
\|(\Delta_1)\|_{S_0} \lesssim C_0^2\epsilon c_0
$$

and

$$
\|\langle\Omega\rangle(\Delta_1)[0]\|_{\dot{H}^{3/2}\times\dot{H}^{1/2}} \lesssim c_0
$$

Proof. By Bernstein's inequality we have

$$
\|\Delta_1^0\|_{S_0} \simeq \max_{\mathcal{Q}}\|\langle\Omega\rangle^{1-\delta(p,q)}\nabla_{t,x}P_0(\phi_{\leq -10}\phi^T_{>-10}\phi_{>-10})\|_{p,q}
$$

$$\lesssim \max_{\mathcal{Q}} \|\langle\Omega\rangle\nabla_{t,x}(\phi_{\leq -10}\phi^T_{>-10}\phi_{>-10})\|_{p,2}$$

When the derivative falls on a high frequency term we have, noting that $p \neq 2$ for $(p,q) \in \mathcal{Q}$,

$$\begin{aligned}\|\langle\Omega\rangle(\phi_{\leq -10}\nabla_{t,x}\phi^T_{>-10}\phi_{>-10})\|_{p,2} &\lesssim \|\langle\Omega\rangle\phi_{\leq -10}\|_{\infty,\infty}\|\langle\Omega\rangle\nabla_{t,x}\phi_{>-10}\|_{\infty,2}\|\langle\Omega\rangle\phi_{>-10}\|_{p,\infty}\\ &\lesssim C_0^2\epsilon c_0\end{aligned}$$

and in the same way

$$\begin{aligned}\|\langle\Omega\rangle(\nabla_{t,x}\phi_{\leq -10}\phi^T_{>-10}\phi_{>-10})\|_{p,2} &\lesssim \|\langle\Omega\rangle\nabla_{t,x}\phi_{\leq -10}\|_{\infty,\infty}\|\langle\Omega\rangle\phi_{>-10}\|_{\infty,2}\|\langle\Omega\rangle\phi_{>-10}\|_{p,\infty}\\ &\lesssim C_0^2\epsilon c_0\end{aligned}$$

The argument for the remaining $\Delta_1^{n,i}$ $(n, i = 1,2,3)$ is identical.

We now show the bound on the initial data. Recall the smallness assumption (2.3.1):

$$\|\langle\Omega\rangle P_k\phi[0]\|_{\dot{H}^{3/2}\times\dot{H}^{1/2}} + \|\langle\Omega\rangle(x\cdot\nabla)P_k\phi[0]\|_{\dot{H}^{3/2}\times\dot{H}^{1/2}} \leq c_k \tag{2.6.1}$$

It immediately follows that

$$\|\langle\Omega\rangle(\Delta_1^0)(0)\|_{\dot{H}^{3/2}} \lesssim \|\langle\Omega\rangle\phi_{\leq -10}(0)\|_\infty\|\langle\Omega\rangle\phi_{>-10}(0)\|_\infty\|\langle\Omega\rangle\phi_{>-10}(0)\|_2 \ll c_0$$

with a similar argument for the initial velocity.

The terms involving L are slightly more complicated. The initial bound in $\dot{H}^{3/2}$ presents no particular difficulties, however to study the initial velocity we have to iterate the equation. We consider only $\Delta_1^{n,1}$ as an example, in which case we have

$$\begin{aligned}\|\langle\Omega\rangle\partial_t(\Delta_1^{n,1})(0)\|_{\dot{H}^{1/2}} &\lesssim \|\langle\Omega\rangle\partial_t(L_n\phi)_{\leq -10}(0)\|_\infty\|\langle\Omega\rangle\phi_{>-10}(0)\|_\infty\|\langle\Omega\rangle\phi_{>-10}(0)\|_2\\ &\quad + \|\langle\Omega\rangle(L_n\phi)_{\leq -10}(0)\|_\infty\|\langle\Omega\rangle\partial_t\phi_{>-10}(0)\|_2\|\langle\Omega\rangle\phi_{>-10}(0)\|_\infty\\ &\lesssim \epsilon^2\|\langle\Omega\rangle\partial_t(L_n\phi)_{\leq -10}(0)\|_\infty + \epsilon^2 c_0 \end{aligned}\tag{2.6.2}$$

where we used

$$\begin{aligned}\|\langle\Omega\rangle(L_n\phi)_{\leq -10}(0)\|_\infty &\lesssim \sum_{k\leq -10} 2^{3k/2}\|\langle\Omega\rangle P_k(x_n\partial_t\phi(0))\|_2\\ &\lesssim \sum_{k\leq -10}\|\langle\Omega\rangle\langle x\cdot\nabla\rangle P_k\partial_t\phi(0)\|_{\dot{H}^{1/2}} \lesssim \epsilon\end{aligned}$$

To bound the term involving $\partial_t(L_n\phi)$ we need to refer back to the equation. Indeed, the necessary bound will follow and the proof will be complete given the following claim.

Claim 3. *Let $k \in \mathbb{Z}$. It holds*

$$\|\langle\Omega\rangle P_k(x_n\cdot\Box\phi)(0)\|_{L_x^2} \lesssim 2^{-k/2}\epsilon c_k \tag{2.6.3}$$

for all $n = 1, 2, 3$. *It follows that*

$$\|\langle\Omega\rangle\partial_t P_k(L_n\phi)(0)\|_2 \lesssim \|\langle\Omega\rangle P_k(x_n\partial_t^2 + \partial_{x_n})\phi(0)\|_2 \lesssim 2^{-k/2}c_k$$

Proof of claim 3. By scaling it suffices to consider $k = 0$.

We start with the wave maps source terms, placing high frequency factors are placed into L^2 and others into L^∞. If a high frequency derivative is forced into L^∞, we let it absorb the multiplier x_n which scales like an inverse derivative. Explicitly, we have

$$\begin{aligned}
&\|\langle\Omega\rangle P_0(x_n(\phi\partial^\alpha\phi^T\partial_\alpha\phi)(0))\|_{L^2}\\
&\lesssim \|\langle\Omega\rangle(x_n(\phi\partial^\alpha\phi^T_{\leq -10}\partial_\alpha\phi_{>-10})(0)\|_{L^2}\\
&\quad + \|\langle\Omega\rangle(x_n(\phi\partial^\alpha\phi^T_{>-10}\partial_\alpha\phi_{>-10})(0)\|_{L^2}\\
&\quad + \|\langle\Omega\rangle(x_n(\phi\partial^\alpha\phi^T_{\leq -10}\partial_\alpha\phi_{\leq -10})(0)\|_{L^2}\\
&\lesssim \|\langle\Omega\rangle P_0\phi(0)\|_\infty\|\langle\Omega\rangle\partial^\alpha\phi_{\leq -10}(0)\|_\infty\|\langle\Omega\rangle(x_n\partial_\alpha\phi_{>-10}(0))\|_2\\
&\quad + \|\langle\Omega\rangle P_0\phi(0)\|_\infty\|\langle\Omega\rangle\partial^\alpha\phi_{>-10}(0)\|_2\|\langle\Omega\rangle(x_n\partial_\alpha\phi_{>-10}(0))\|_\infty\\
&\quad + \|\langle\Omega\rangle P_0(x_n\phi_{\sim 0}(0))\|_2\|\langle\Omega\rangle\partial^\alpha\phi_{\leq -10}(0)\|_\infty\|\langle\Omega\rangle\partial_\alpha\phi_{\leq -10}(0)\|_\infty\\
&\lesssim \epsilon c_0
\end{aligned}$$

The first half-wave maps terms, $HWM_1(\phi)$, can be treated similarly. To study HWM_2 we decompose

$$\begin{aligned}
&\|\langle\Omega\rangle P_0(x_n\cdot HWM_2(\phi))(0)\|_{L^2_x}\\
&\lesssim \sum_{j,k\in\mathbb{Z}}\|\langle\Omega\rangle P_0(x_n\cdot[\phi\times((-\Delta)^{1/2}(\phi_k\times(-\Delta)^{1/2}\phi_j)-\phi_k\times\Delta\phi_j)])(0)\|_{L^2_x}
\end{aligned}$$

Note that if $j \gg k$ or $k \gg j$ we can write

$$(-\Delta)^{1/2}(\phi_k\times(-\Delta)^{1/2}\phi_j)-\phi_k\times\Delta\phi_j = \mathcal{L}_{k+j}(\phi_k,\phi_j)$$

In this symmetric form we see that it suffices to consider $j \gg k$. Starting with $k \geq -10$ we use the Fourier expansion (2.5.3) and find

$$\begin{aligned}
&\sum_{\substack{k\geq -10\\ j\gg k}}\|\langle\Omega\rangle P_0(x_n\cdot(\phi\times\mathcal{L}_{j+k}(\phi_k,\phi_j)))(0)\|_{L^2_x}\\
&\lesssim \sum_{\substack{k\geq -10\\ j\gg k}}\sum_{a,b\in\mathbb{Z}^3}|c^{(k+j)}_{a,b}|\|\langle\Omega\rangle(x_n\cdot\phi\times(\phi_k(x-2^{-k}a)\times\phi_j(x-2^{-j}b)))\|_{L^2_x}\\
&\lesssim \sum_{\substack{k\geq -10\\ j\gg k}}\sum_{a,b\in\mathbb{Z}^3}|c^{(k+j)}_{a,b}|\|\langle\Omega\rangle\phi(0)\|_\infty\|\langle\Omega\rangle(\phi_k(x-2^{-k}a))(0)\|_\infty\|\langle\Omega\rangle(x_n\cdot\phi_j(x-2^{-j}b))(0)\|_2\\
&\lesssim \sum_{\substack{k\geq -10\\ j\gg k}} 2^{j+k}\cdot c_k\cdot 2^{-5j/2}c_j \lesssim \epsilon c_0
\end{aligned}$$

The case $k < -10$ can be handled similarly upon further localising the outer factor of ϕ to low and high frequencies.

It remains to study the case $j \simeq k$. When this frequency is low we have

$$\begin{aligned}
&\sum_{k<-10} \|\langle\Omega\rangle P_0(x_n \cdot [\phi \times ((-\Delta)^{1/2}(\phi_k \times (-\Delta)^{1/2}\phi_{\sim k}) - \phi_k \times \Delta\phi_{\sim k})])(0)\|_{L^2_x} \\
&\simeq \sum_{k<-10} \|\langle\Omega\rangle P_0(x_n \cdot [\phi_{\sim 0} \times ((-\Delta)^{1/2}(\phi_k \times (-\Delta)^{1/2}\phi_{\sim k}) - \phi_k \times \Delta\phi_{\sim k})])(0)\|_{L^2_x} \\
&\lesssim \sum_{k<-10} \|\langle\Omega\rangle(x_n \cdot \phi_{\sim 0})(0)\|_2 \cdot 2^{2k}\|\langle\Omega\rangle\phi_k(0)\|_\infty\|\langle\Omega\rangle\phi_{\sim k}(0)\|_\infty \lesssim \epsilon^2 c_0
\end{aligned}$$

and when it is high we have

$$\begin{aligned}
&\sum_{k\geq-10} \|\langle\Omega\rangle P_0(x_n \cdot [\phi \times ((-\Delta)^{1/2}(\phi_k \times (-\Delta)^{1/2}\phi_{\sim k}) - \phi_k \times \Delta\phi_{\sim k})])(0)\|_{L^2_x} \\
&\lesssim \sum_{k\geq-10} \|\langle\Omega\rangle P_0(\phi \times x_n(-\Delta)^{1/2}(\phi_k \times (-\Delta)^{1/2}\phi_{\sim k}))(0)\|_{L^2_x} \\
&\qquad + \|\langle\Omega\rangle P_0(\phi \times ((x_n\phi_k) \times \Delta\phi_{\sim k})(0)\|_{L^2_x}
\end{aligned}$$

Interchanging the x_n and $(-\Delta)^{1/2}$ up to a term involving a Riesz transform this is seen to be acceptable upon placing ϕ and $\phi_{\sim k}$ into L^∞ and the remaining factor into L^2. This completes the proof of the claim. $\square$

$\square$

We now show that this transformation reduces the equations to the form

$$\begin{aligned}
\Box\tilde{\psi}_0 = &-2\phi_{\leq-10}\partial_\alpha\phi^T_{\leq-10}\partial^\alpha\psi_0 \\
&- 2[P_0(\phi_{\leq-10}\partial_\alpha\phi^T_{\leq-10}\partial^\alpha\phi_{>-10}) - \phi_{\leq-10}\partial_\alpha\phi^T_{\leq-10}\partial^\alpha\psi_0] + error
\end{aligned} \tag{2.6.4}$$

and

$$\begin{aligned}
\Box\tilde{\psi}_n = &-2(L_n\phi)_{\leq-10}\partial_\alpha\phi^T_{\leq-10}\partial^\alpha\psi_0 \\
&- 2\phi_{\leq-10}\partial_\alpha(L_n\phi)^T_{\leq-10}\partial^\alpha\psi_0 \\
&- 2\phi_{\leq-10}\partial_\alpha\phi^T_{\leq-10}\partial^\alpha\psi_n \\
&- 2[P_0((L_n\phi)_{\leq-10}\partial_\alpha\phi^T_{\leq-10}\partial^\alpha\phi_{>-10}) - (L_n\phi)_{\leq-10}\partial_\alpha\phi^T_{\leq-10}\partial^\alpha\psi_0] \\
&- 2[P_0(\phi_{\leq-10}\partial_\alpha(L_n\phi)^T_{\leq-10}\partial^\alpha\phi_{>-10}) - \phi_{\leq-10}\partial_\alpha(L_n\phi)^T_{\leq-10}\partial^\alpha\psi_0] \\
&- 2[P_0(\phi_{\leq-10}\partial_\alpha\phi^T_{\leq-10}\partial^\alpha(L_n\phi)_{>-10}) - \phi_{\leq-10}\partial_\alpha\phi^T_{\leq-10}\partial^\alpha\psi_n] \\
&+ error
\end{aligned} \tag{2.6.5}$$

Indeed, clearly

$$\Box\tilde{\psi}_0 = \Box\psi_0 + \frac{1}{2}P_0\Box(\phi_{\leq-10}\phi^T_{>-10}\phi_{>-10})$$

where

$$
\begin{aligned}
P_0\Box(\phi_{\leq -10}\phi^T_{>-10}\phi_{>-10}) =& P_0(\Box\phi_{\leq -10}\ \phi^T_{>-10}\phi_{>-10}) && (2.6.6)\\
&+4P_0(\partial^\alpha\phi_{\leq -10}\ \partial_\alpha\phi^T_{>-10}\phi_{>-10}) && (2.6.7)\\
&+2P_0(\phi_{\leq -10}\ (\Box\phi_{>-10})^T\phi_{>-10}) && (2.6.8)\\
&+2P_0(\phi_{\leq -10}\ \partial^\alpha\phi^T_{>-10}\partial_\alpha\phi_{>-10})) && (2.6.9)
\end{aligned}
$$

We then need to show that (2.6.6) = (2.6.7) = (2.6.8) = *error*, since (2.6.9) cancels the (LHH) term in the equation for ψ_0, (2.4.11). Arguing similarly for $\tilde{\psi}_n$ we see that (2.6.4) and (2.6.5) follow from the following proposition.

Proposition 2.6.2. *Denote*

$$
\begin{aligned}
T_1(\varphi^{(1)},\varphi^{(2)},\varphi^{(3)}) &:= P_0(\Box\varphi^{(1)}_{\leq -10}\ (\varphi^{(2)}_{>-10})^T\varphi^{(3)}_{>-10})\\
T_2(\varphi^{(1)},\varphi^{(2)},\varphi^{(3)}) &:= P_0(\partial^\alpha\varphi^{(1)}_{\leq -10}\ \partial_\alpha(\varphi^{(2)}_{>-10})^T\varphi^{(3)}_{>-10})\\
T_3(\varphi^{(1)},\varphi^{(2)},\varphi^{(3)}) &:= P_0(\varphi^{(1)}_{\leq -10}\ (\Box\varphi^{(2)}_{>-10})^T\varphi^{(3)}_{>-10})
\end{aligned}
$$

Then it holds

$$
T_1(\varphi^{(1)},\varphi^{(2)},\varphi^{(3)}) = T_2(\varphi^{(1)},\varphi^{(2)},\varphi^{(3)}) = T_3(\varphi^{(1)},\varphi^{(2)},\varphi^{(3)}) = error
$$

for any of

$$
(\varphi^{(1)},\varphi^{(2)},\varphi^{(3)}) \in \{(\phi,\phi,\phi),(L_n\phi,\phi,\phi),(\phi,L_n\phi,\phi),(\phi,\phi,L_n\phi)\}
$$

Proof. For simplicity, we will only prove the statement for (ϕ,ϕ,ϕ) and the other cases follow in the same way.[11] Let's start with T_1. We have

$$
\begin{aligned}
&\|\langle\Omega\rangle P_0(\Box\phi_{\leq -10}\phi^T_{>-10}\phi_{>-10})\|_{1,2}\\
&\lesssim \|\langle\Omega\rangle\Box\phi_{\leq -10}\|_{2+,\infty-}\|\langle\Omega\rangle\phi_{>-10}\|_{\infty-,2+}\|\langle\Omega\rangle\phi_{>-10}\|_{2+,\infty-}
\end{aligned}
$$

so it suffices to show the following:

Claim 4. *It holds*

$$
\|\langle\Omega\rangle P_k\Box\phi\|_{\frac{2M}{M-1},2M} \lesssim 2^{(\frac{3}{2}-\frac{1}{M})k}C_0c_k \tag{2.6.10}
$$

Proof of claim. It again suffices to consider $k=0$. By the usual frequency decomposition and Bernstein's inequality we have

$$
\|\langle\Omega\rangle P_0(\phi\partial^\alpha\phi^T\partial_\alpha\phi)\|_{\frac{2M}{M-1},2M} \lesssim \|\langle\Omega\rangle P_0(\phi\partial^\alpha\phi^T_{\leq -10}\partial_\alpha\phi_{>-10})\|_{\frac{2M}{M-1},2} \tag{2.6.11}
$$

[11] The only difference when including the factors of L_n comes in estimating the half-wave maps terms upon iterating the equation. Here one must simply pay a little attention when exchanging L with the operators $(-\Delta)^{1/2}$ and P_j, however this causes no problems thanks to the commutation relations of Section 2.2.1.

$$+ \|\langle\Omega\rangle P_0(\phi\partial^\alpha\phi^T_{>-10}\partial_\alpha\phi_{>-10})\|_{\frac{2M}{M-1},\frac{2M}{M+1}} \tag{2.6.12}$$

$$+ \|\langle\Omega\rangle P_0(\phi_{\sim 0}\partial^\alpha\phi^T_{\leq-10}\partial_\alpha\phi_{\leq-10})\|_{\frac{2M}{M-1},2M} \tag{2.6.13}$$

Always placing the lone ϕ into $L^\infty_{t,x}$ we have

$$(2.6.11) \lesssim \|\langle\Omega\rangle\partial^\alpha\phi_{\leq-10}\|_{\frac{2M}{M-1},\infty}\|\langle\Omega\rangle\partial_\alpha\phi_{>-10}\|_{\infty,2} \lesssim C_0^2c_0^2$$

Likewise

$$(2.6.12) \lesssim \|\langle\Omega\rangle\partial^\alpha\phi_{>-10}\|_{\frac{4M}{M-1},\frac{4M}{M+1}}\|\langle\Omega\rangle\partial_\alpha\phi_{>-10}\|_{\frac{4M}{M-1},\frac{4M}{M+1}} \lesssim C_0^2c_0^2$$

and lastly

$$(2.6.13) \lesssim \|\langle\Omega\rangle\phi_{\sim 0}\|_{\frac{2M}{M-1},2M}\|\langle\Omega\rangle\partial^\alpha\phi_{\leq-10}\|_{\infty,\infty}\|\langle\Omega\rangle\partial_\alpha\phi_{\leq-10}\|_{\infty,\infty} \lesssim C_0^3c_0^3$$

The first half-wave maps terms can be treated in the same way and the remaining such terms can be handled analogously upon incorporating the modifications as in the proof of Claim 3.

□

For T_2 we have

$$\begin{aligned}&\|\langle\Omega\rangle P_0(\partial^\alpha\phi_{\leq-10}\partial_\alpha\phi^T_{>-10}\phi_{>-10})\|_{1,2}\\ &\lesssim \|\langle\Omega\rangle\partial^\alpha\phi_{\leq-10}\|_{2+,\infty-}\|\langle\Omega\rangle\partial_\alpha\phi_{>-10}\|_{\infty-,2+}\|\langle\Omega\rangle\phi_{>-10}\|_{2+,\infty-} \lesssim C_0^3c_0^3\end{aligned}$$

Lastly, for T_3 we must expand the wave operator within the overall expression. Starting with the wave maps source terms we have

$$\|\langle\Omega\rangle P_0(\phi_{\leq-10}P_{>-10}(\phi\partial_\alpha\phi^T\partial^\alpha\phi)^T\phi_{>-10})\|_{1,2}$$

$$\lesssim \sum_{k>-10}\|\langle\Omega\rangle P_0(\phi_{\leq-10}P_k(\phi\partial_\alpha\phi^T_{\leq k-10}\partial^\alpha\phi_{>k-10})^T\phi_{\sim k})\|_{1,2} \tag{2.6.14}$$

$$+\|\langle\Omega\rangle P_0(\phi_{\leq-10}P_k(\phi\partial_\alpha\phi^T_{>k-10}\partial^\alpha\phi_{>k-10})^T\phi_{\sim k})\|_{1,2} \tag{2.6.15}$$

$$+\|\langle\Omega\rangle P_0(\phi_{\leq-10}P_k(\phi_{\sim k}\partial_\alpha\phi^T_{\leq k-10}\partial^\alpha\phi_{\leq k-10})^T\phi_{\sim k})\|_{1,2} \tag{2.6.16}$$

which can be treated like (2.4.6), (2.4.8) and (2.4.7) respectively. The $HWM_1(\phi)$ terms are analogous. For HWM_2 we have

$$\sum_{k>-10}\|\langle\Omega\rangle P_0(\phi_{\leq-10}P_k(HWM_2(\phi))^T\phi_{\sim k})\|_{1,2}$$

$$\lesssim \sum_{k>-10}\sum_{l,j}\|\langle\Omega\rangle(P_k(\phi\times((-\Delta)^{1/2}(\phi_l\times(-\Delta)^{1/2}\phi_j)-\phi_l\times\Delta\phi_j))^T\cdot\phi_{\sim k})\|_{1,2}$$

First consider $j \gg l$, $l \geq k-10$, comparable to (2.6.15). Using the expansion (2.5.5) we have

$$\begin{aligned}
&\sum_{k>-10}\sum_{\substack{l\geq k-10\\ j\gg l}} \|\langle\Omega\rangle(P_k(\phi\times((-\Delta)^{1/2}(\phi_l\times(-\Delta)^{1/2}\phi_j)-\phi_l\times\Delta\phi_j))^T\cdot\phi_{\sim k})\|_{1,2}\\
&\simeq \sum_{k>-10}\sum_{\substack{l\geq k-10\\ j\gg l}} \|\langle\Omega\rangle P_k(\phi\times\mathcal{L}_{l+j}(\phi_l,\phi_j))^T\cdot\langle\Omega\rangle\phi_{\sim k})\|_{1,2}\\
&\lesssim \sum_{k>-10}\sum_{\substack{l\geq k-10\\ j\gg l}} 2^{l+j}\|\langle\Omega\rangle\phi_l\|_{9,10/3}\|\phi_j\|_{2,5}\|\langle\Omega\rangle\phi_{\sim k}\|_{18/7,\infty}\\
&\qquad\qquad + 2^{l+j}\|\phi_l\|_{2,5}\|\langle\Omega\rangle\phi_j\|_{9,10/3}\|\langle\Omega\rangle\phi_{\sim k}\|_{18/7,\infty}\\
&\lesssim C_0^3\epsilon^2 c_0
\end{aligned}$$

as required. When $l<k-10$ the term behaves like (2.6.14) and we have

$$\begin{aligned}
&\sum_{k>-10}\sum_{\substack{l<k-10\\ j\gg l}} \|\langle\Omega\rangle(P_k(\phi\times((-\Delta)^{1/2}(\phi_l\times(-\Delta)^{1/2}\phi_j)-\phi_l\times\Delta\phi_j))^T\cdot\phi_{\sim k})\|_{1,2}\\
&\lesssim \sum_{k>-10}\sum_{\substack{l<k-10\\ j\gg l}} 2^{l+j}\|\langle\Omega\rangle\phi_l\|_{2+,\infty-}\|\langle\Omega\rangle\phi_j\|_{\infty-,2+}\|\langle\Omega\rangle\phi_{\sim k}\|_{2+,\infty-}\lesssim C_0^3\epsilon^2 c_0
\end{aligned}$$

The case $l\gg j$ can be treated in the same way. When $j\simeq l<k-10$ we are in the regimen of (2.6.16) and have

$$\begin{aligned}
&\sum_{k>-10}\sum_{l<k-10} \|\langle\Omega\rangle(P_k(\phi\times((-\Delta)^{1/2}(\phi_l\times(-\Delta)^{1/2}\phi_{\sim l})-\phi_l\times\Delta\phi_{\sim l}))^T\cdot\phi_{\sim k})\|_{1,2}\\
&\lesssim \sum_{k>-10}\sum_{l<k-10} 2^{2l}\|\langle\Omega\rangle\phi_l\|_{2+,\infty-}\|\langle\Omega\rangle\phi_{\sim l}\|_{2+,\infty-}\|\langle\Omega\rangle\phi_{\sim k}\|_{\infty-,2+}\lesssim C_0^3\epsilon^2 c_0
\end{aligned}$$

and finally if $j\simeq l\geq k-10$ we refer to (2.6.15) and find

$$\begin{aligned}
&\sum_{k>-10}\sum_{l\geq k-10} \|\langle\Omega\rangle(P_k(\phi\times((-\Delta)^{1/2}(\phi_l\times(-\Delta)^{1/2}\phi_{\sim l})-\phi_l\times\Delta\phi_{\sim l}))^T\cdot\phi_{\sim k})\|_{1,2}\\
&\lesssim \sum_{k>-10}\sum_{l\geq k-10} 2^{2l}\|\langle\Omega\rangle\phi_{\sim l}\|_{9,10/3}\|\phi_{\sim l}\|_{2,5}\|\langle\Omega\rangle\phi_{\sim k}\|_{18/7,\infty}\lesssim C_0^3\epsilon^2 c_0
\end{aligned}$$

which completes the proof. □

2.6.2 Low-low-high error term

Write

$$P_0(\phi_{\leq -10}\partial_\alpha\phi_{\leq -10}^T\partial^\alpha\phi_{>-10}) - \phi_{\leq -10}\partial_\alpha\phi_{\leq -10}^T\partial^\alpha\psi_0$$

$$= -\tilde{P}_0 \int_0^1 \int_y \check{\chi}_0(y) y^T \nabla_x (\phi_{\leq -10}(x-\theta y) \partial_\alpha \phi_{\leq -10}^T (x-\theta y)) \partial^\alpha \phi_{\sim 0}(x-y) d\theta dy \quad (2.6.17)$$

This splits into two terms by Leibniz's rule: one where the derivative ∇_x falls on the non-differentiated term, and one where it falls on the $\partial_\alpha \phi_{\leq -10}$. The first such term is unproblematic:

$$\left\| \tilde{P}_0 \int_0^1 \int_y \check{\chi}_0(y) y^T \nabla_x \phi_{\leq -10}(x-\theta y\ ) \partial_\alpha \phi_{\leq -10}^T (x-\theta y) \partial^\alpha \phi_{\sim 0}(x-y) d\theta dy \right\|_{1,2}$$
$$\lesssim \int_0^1 \int_y |y \check{\chi}_0(y)| \, \|\nabla_x \phi_{\leq -10}\|_{2+,\infty-} \|\partial_\alpha \phi_{\leq -10}\|_{2+,\infty-} \|\partial^\alpha \phi_{\sim 0}\|_{\infty-,2+} dy d\theta \lesssim C_0^3 \epsilon^2 c_0$$

so that

$$\begin{aligned} \Box \tilde{\psi}_0 &= -2\phi_{\leq -10} \partial_\alpha \phi_{\leq -10}^T \partial^\alpha \psi_0 \\ &+ 2\tilde{P}_0 \int_0^1 \int_y \check{\chi}_0(y) y^T \phi_{\leq -10}(x-\theta y)\ \nabla_x \partial^\alpha \phi_{\leq -10}^T (x-\theta y) \partial_\alpha \phi_{\sim 0}(x-y) d\theta dy + error \end{aligned} \quad (2.6.18)$$

and similar for the $\tilde{\psi}_n$. This motivates our second transformation

$$\tilde{\psi}^L \mapsto \Phi^L := \tilde{\psi}^L - (\Delta_2) \quad \text{for} \quad \Delta_2 = \begin{pmatrix} \Delta_2^0 \\ \Delta_2^{1,1} + \Delta_2^{1,2} + \Delta_2^{1,3} \\ \vdots \\ \Delta_2^{3,1} + \Delta_2^{3,2} + \Delta_2^{3,3} \end{pmatrix}$$

with

$$\begin{aligned} \Delta_2^0 &:= \tilde{P}_0 \int_0^1 \int_y \check{\chi}_0(y) y^T \phi_{\leq -10}(x-\theta y) \nabla_x \phi_{\leq -10}^T (x-\theta y) \phi_{\sim 0}(x-y) d\theta dy \\ \Delta_2^{n,1} &:= \tilde{P}_0 \int_0^1 \int_y \check{\chi}_0(y) y^T (L_n \phi)_{\leq -10}(x-\theta y) \nabla_x \phi_{\leq -10}^T (x-\theta y) \phi_{\sim 0}(x-y) d\theta dy \\ \Delta_2^{n,2} &:= \tilde{P}_0 \int_0^1 \int_y \check{\chi}_0(y) y^T \phi_{\leq -10}(x-\theta y) \nabla_x (L_n \phi)_{\leq -10}^T (x-\theta y) \phi_{\sim 0}(x-y) d\theta dy \\ \Delta_2^{n,3} &:= \tilde{P}_0 \int_0^1 \int_y \check{\chi}_0(y) y^T \phi_{\leq -10}(x-\theta y) \nabla_x \phi_{\leq -10}^T (x-\theta y) (L_n \phi)_{\sim 0}(x-y) d\theta dy \end{aligned}$$

Henceforth we drop the $\tilde{P}_0$ since it does not affect the calculations.

As usual write $\Phi^L = (\Phi_0, \Phi_1, \Phi_2, \Phi_3)$ and start by noting the boundedness of the transformation, the proof of which is very similar to Proposition 2.6.1 and so omitted.

Proposition 2.6.3. *It holds*

$$\|(\Delta_2)\|_{S_0} \lesssim C_0^2 \epsilon c_0$$

and moreover

$$\|\langle \Omega \rangle (\Delta_2)[0]\|_{\dot{H}^{3/2} \times \dot{H}^{1/2}} \lesssim c_0$$

This time we show that the transformation reduces the equations to the form

$$\Box\Phi_0 = -2\phi_{\leq -10}\partial_\alpha\phi^T_{\leq -10}\partial^\alpha\psi_0 + error \tag{2.6.19}$$

and

$$\begin{aligned}\Box\Phi_n = &-2(L_n\phi)_{\leq -10}\partial_\alpha\phi^T_{\leq -10}\partial^\alpha\psi_0\\ &- 2\phi_{\leq -10}\partial_\alpha(L_n\phi)^T_{\leq -10}\partial^\alpha\psi_0\\ &- 2\phi_{\leq -10}\partial_\alpha\phi^T_{\leq -10}\partial^\alpha\psi_n + error\end{aligned} \tag{2.6.20}$$

Observe that

$$\Box(\Delta_2^0) = \Box\left(\int_0^1\int_y \check{\chi}_0(y)y^T\phi_{\leq -10}(x-\theta y)\nabla_x\phi^T_{\leq -10}(x-\theta y)\phi_{\sim 0}(x-y)d\theta dy\right)$$

$$= \int_0^1\int_y \check{\chi}_0(y)y^T\Box\phi_{\leq -10}(x-\theta y)\nabla_x\phi^T_{\leq -10}(x-\theta y)\phi_{\sim 0}(x-y)d\theta dy \tag{2.6.21}$$

$$+ \int_0^1\int_y \check{\chi}_0(y)y^T\phi_{\leq -10}(x-\theta y)\Box\nabla_x\phi^T_{\leq -10}(x-\theta y)\phi_{\sim 0}(x-y)d\theta dy \tag{2.6.22}$$

$$+ \int_0^1\int_y \check{\chi}_0(y)y^T\phi_{\leq -10}(x-\theta y)\nabla_x\phi^T_{\leq -10}(x-\theta y)\Box\phi_{\sim 0}(x-y)d\theta dy \tag{2.6.23}$$

$$+ 2\int_0^1\int_y \check{\chi}_0(y)y^T\partial^\alpha\phi_{\leq -10}(x-\theta y)\partial_\alpha\nabla_x\phi^T_{\leq -10}(x-\theta y)\phi_{\sim 0}(x-y)d\theta dy \tag{2.6.24}$$

$$+ 2\int_0^1\int_y \check{\chi}_0(y)y^T\partial^\alpha\phi_{\leq -10}(x-\theta y)\nabla_x\phi^T_{\leq -10}(x-\theta y)\partial_\alpha\phi_{\sim 0}(x-y)d\theta dy \tag{2.6.25}$$

$$+ 2\int_0^1\int_y \check{\chi}_0(y)y^T\phi_{\leq -10}(x-\theta y)\partial^\alpha\nabla_x\phi^T_{\leq -10}(x-\theta y)\partial_\alpha\phi_{\sim 0}(x-y)d\theta dy \tag{2.6.26}$$

and similar expressions for the $\Delta_2^{n,i}$. The final term (2.6.26) cancels with the integral expression in equation (2.6.18), so we must show the following:

Proposition 2.6.4. *We have*

$$(2.6.21) = \ldots = (2.6.25) = error$$

and the same holds when any one factor of ϕ in the expressions (2.6.21),...,(2.6.25) *is replaced by $L_n\phi$ ($n = 1, 2, 3$).*

Proof. We will neglect the angular and vector derivatives in what follows in order to reduce notation. We also drop the transpose symbols.

Let's start with (2.6.21). Using the estimate (2.6.10) on $\Box\phi$ and the L^1 boundedness of the kernel $\check{\chi}_0(y)y$ we have

$$\|(2.6.21)\|_{1,2} \lesssim \|\Box\phi_{\leq -10}\|_{2+,\infty-}\|\nabla_x\phi_{\leq -10}\|_{2+,\infty-}\|\phi_{\sim 0}\|_{\infty-,2+} \lesssim C_0^3\epsilon c_0$$

which is acceptable. For (2.6.22) we again have to iterate the equation. For the wave maps source terms we have

$$\begin{aligned}
&\left\|\int_0^1\int_y \check{\chi}_0(y)y\,(\phi_{\leq -10}\cdot\nabla_x P_{\leq -10}(\phi\partial_\alpha\phi\partial^\alpha\phi))(x-\theta y)\phi_{\sim 0}(x-y)d\theta dy\right\|_{1,2}\\
&\lesssim \sum_{k\leq -10}\left\|\int_0^1\int_y \check{\chi}_0(y)y\,(\phi_{\leq -10}\cdot\nabla_x P_k(\phi\partial_\alpha\phi_{\leq k-10}\partial^\alpha\phi_{>k-10}))(x-\theta y)\phi_{\sim 0}(x-y)d\theta dy\right\|_{1,2} && (2.6.27)\\
&\quad+\left\|\int_0^1\int_y \check{\chi}_0(y)y\,(\phi_{\leq -10}\cdot\nabla_x P_k(\phi\partial_\alpha\phi_{>k-10}\partial^\alpha\phi_{>k-10}))(x-\theta y)\phi_{\sim 0}(x-y)d\theta dy\right\|_{1,2} && (2.6.28)\\
&\quad+\left\|\int_0^1\int_y \check{\chi}_0(y)y\,(\phi_{\leq -10}\cdot\nabla_x P_k(\phi_{\sim k}\partial_\alpha\phi_{\leq k-10}\partial^\alpha\phi_{\leq k-10}))(x-\theta y)\phi_{\sim 0}(x-y)d\theta dy\right\|_{1,2} && (2.6.29)
\end{aligned}$$

Placing all the low frequency factors of ϕ into $L^\infty_{t,x}$ we bound the above by

$$\begin{aligned}
&\sum_{k\leq -10} 2^k\|\partial_\alpha\phi_{\leq k-10}\|_{2+,\infty-}\|\partial^\alpha\phi_{>k-10}\|_{\infty-,2+}\|\phi_{\sim 0}\|_{2+,\infty-}\\
&\qquad + 2^k\|\partial_\alpha\phi_{>k-10}\|_{2,5}\|\partial^\alpha\phi_{>k-10}\|_{9,10/3}\|\phi_{\sim 0}\|_{18/7,\infty}\\
&\qquad + 2^k\|\phi_{\sim k}\|_{\infty,\infty}\|\partial_\alpha\phi_{\leq k-10}\|_{2+,\infty-}\|\partial^\alpha\phi_{\leq k-10}\|_{2+,\infty-}\|\phi_{\sim 0}\|_{\infty-,2+}\\
&\lesssim C_0^3\epsilon c_0
\end{aligned}$$

The half-wave maps terms are treated analogously using methods as in the proof of Proposition 2.6.2.

We now turn to (2.6.23). Considering the wave maps portion of $\Box\phi$ we have

$$\begin{aligned}
&\left\|\int_0^1\int_y \check{\chi}_0(y)y\,(\phi_{\leq -10}\nabla_x\phi_{\leq -10})(x-\theta y)P_{\sim 0}(\phi\partial_\alpha\phi_{>-10}\partial^\alpha\phi_{>-10})(x-y)d\theta dy\right\|_{1,2}\\
&\quad+\left\|\int_0^1\int_y \check{\chi}_0(y)y\,(\phi_{\leq -10}\nabla_x\phi_{\leq -10})(x-\theta y)P_{\sim 0}(\phi\partial_\alpha\phi_{\leq -10}\partial^\alpha\phi_{>-10})(x-y)d\theta dy\right\|_{1,2}\\
&\quad+\left\|\int_0^1\int_y \check{\chi}_0(y)y\,(\phi_{\leq -10}\nabla_x\phi_{\leq -10})(x-\theta y)P_{\sim 0}(\phi\partial_\alpha\phi_{\leq -10}\partial^\alpha\phi_{\leq -10})(x-y)d\theta dy\right\|_{1,2}\\
&\lesssim \|\nabla_x\phi_{\leq -10}\|_{\frac{18}{7},\infty}\|\partial_\alpha\phi_{>-10}\|_{2,5}\|\partial^\alpha\phi_{>-10}\|_{9,\frac{10}{3}}\\
&\quad+\|\nabla_x\phi_{\leq -10}\|_{2+,\infty-}\|\partial_\alpha\phi_{\leq -10}\|_{2+,\infty-}\|\partial^\alpha\phi_{>-10}\|_{\infty-,2+}\\
&\quad+\|\nabla_x\phi_{\leq -10}\|_{\infty,\infty}\|\phi_{\sim 0}\|_{\infty-,2+}\|\partial_\alpha\phi_{\leq -10}\|_{2+,\infty-}\|\partial^\alpha\phi_{\leq -10}\|_{2+,\infty-}\\
&\lesssim C_0^3\epsilon c_0
\end{aligned}$$

Again half-wave maps terms are analogous.

Lastly,

$$\|(2.6.24)\|_{1,2} \lesssim \|\partial^\alpha \phi_{\leq -10}\|_{2+,\infty-}\|\partial_\alpha \nabla_x \phi_{\leq -10}\|_{2+,\infty-}\|\phi_{\sim 0}\|_{\infty-,2+} \lesssim C_0^3 \epsilon c_0$$

while

$$\|(2.6.25)\|_{1,2} \lesssim \|\partial^\alpha \phi_{\leq -10}\|_{2+,\infty-}\|\nabla_x \phi_{\leq -10}\|_{2+,\infty-}\|\partial_\alpha \phi_{\sim 0}\|_{\infty-,2+} \lesssim C_0^3 \epsilon c_0$$

$\square$

We can now recast the equations in an even more distilled form. Denote

$$A_\alpha := -\phi_{\leq -10}\partial_\alpha \phi_{\leq -10}^T; \qquad A_\alpha^n := -((L_n\phi)_{\leq -10}\partial_\alpha \phi_{\leq -10}^T + \phi_{\leq -10}\partial_\alpha (L_n\phi)_{\leq -10}^T)$$

Then writing A_α^n for the block vector $(A_\alpha^1, A_\alpha^2, A_\alpha^3)$ and $\mathbf{I}_3$ the block 3×3 identity matrix (so a 9×9 matrix), we arrive at

$$\square \Phi^L = 2\begin{pmatrix} A_\alpha & 0 \\ A_\alpha^n & A_\alpha \mathbf{I}_3 \end{pmatrix} \partial^\alpha \psi^L + error \tag{2.6.30}$$

with

$$\|\Phi^L\|_{S_0} \leq \|\psi^L\|_{S_0} + C_0^3 \epsilon c_0 \lesssim C_0 c_0 \tag{2.6.31}$$

Henceforth we will use the notation

$$\mathbf{A}_\alpha^L := \begin{pmatrix} A_\alpha & 0 \\ A_\alpha^n & A_\alpha \mathbf{I}_3 \end{pmatrix}$$

2.7 The gauge transformation

In this section, we perform a nonlinear transformation to cancel out the remaining nontrivial term in the equation above. Our construction is a simplification of that in [Tao01a] (possible due to our working in Besov rather than Sobolev spaces).

Fix a large integer N *depending on* T. Define the matrix field $\mathbf{U}$ by

$$\mathbf{U} := \mathbf{I}_4 + \sum_{-N<k\leq -10} \mathbf{U}_k$$

Here $\mathbf{I}_4$ is the block 4×4 identity matrix and the $\mathbf{U}_k$ are defined inductively by

$$\mathbf{U}_k := \begin{pmatrix} -\phi_{<k}\phi_k^T & 0 \\ -(\phi_{<k}(L_n\phi)_k^T + (L_n\phi)_{<k}\phi_k^T) & -\phi_{<k}\phi_k^T \cdot \mathbf{I}_3 \end{pmatrix} \mathbf{U}_{<k}$$

and

$$\mathbf{U}_{<k} := \mathbf{I}_4 + \sum_{-N<k'<k} \mathbf{U}_{k'}$$

This $\mathbf{U}$ is constructed so as to almost satisfy $\partial_\alpha \mathbf{U} = \mathbf{A}^L_\alpha \mathbf{U}$ and so cancel out the troublesome frequency interactions remaining in our equation. This will be discussed in the next section.

One may verify inductively that $\mathbf{U}_{<k}$ has frequency support on $\{|\xi| \lesssim 2^k\}$, so $\mathbf{U}$ has frequency support on $\{|\xi| \lesssim 1\}$. Moreover $\mathbf{U}$ is of the form

$$\mathbf{U} = \begin{pmatrix} U & 0 \\ U^n & U\mathbf{I}_3 \end{pmatrix}$$

for a 3×3 matrix U and a block vector $U^n = (U^1, U^2, U^3)$ with each U^i a 3×3 matrix. The same lower triangular structure of couse holds for $\mathbf{U}_k$, $\mathbf{U}_{<k}$ and $\mathbf{U}^{-1}$, the existence of which is shown in the following proposition.

Proposition 2.7.1. *For $C_0 \gg 1$ fixed, $\epsilon(C_0)$ sufficiently small, the matrix $\mathbf{U}$ is a perturbation of the identity,*

$$\|\langle\Omega\rangle(\mathbf{U} - \mathbf{I}_4)\|_{\infty,\infty}, \|\langle\Omega\rangle\partial_t(\mathbf{U} - \mathbf{I}_4)\|_{\infty,\infty} \lesssim C_0\epsilon \tag{2.7.1}$$

and is invertible with

$$\|\langle\Omega\rangle\mathbf{U}\|_{\infty,\infty}, \|\langle\Omega\rangle(\mathbf{U}^{-1})\|_{\infty,\infty} \lesssim 1 \tag{2.7.2}$$

Moreover, for any admissible pair $(p,q) \in \mathcal{Q}$ with $1 - \frac{1}{p} - \frac{3}{q} > \sigma > 0$, we have

$$\|\langle\Omega\rangle^{1-\delta(p,q)}\partial_\alpha \mathbf{U}\|_{p,q}, \|\langle\Omega\rangle^{1-\delta(p,q)}\partial_\alpha(\mathbf{U}^{-1})\|_{p,q} \lesssim_{p,q} C_0\epsilon \tag{2.7.3}$$

for each $\alpha = 0, \ldots, 3$. More precisely,

$$\|\langle\Omega\rangle^{1-\delta(p,q)}\partial_\alpha \mathbf{U}_K\|_{p,q} \lesssim_{p,q} 2^{(1-\frac{1}{p}-\frac{3}{q})K} C_0 c_K \tag{2.7.4}$$

for all $-N < K \leq -10$.

Proof. We first show (2.7.2). We will show by induction that

$$\|\langle\Omega\rangle\mathbf{U}_{<K}\|_{\infty,\infty} \leq 2$$

for all $-N \leq K \leq -9$. Since $\mathbf{U} = \mathbf{U}_{<-9}$ this proves the first part of (2.7.2). When $K = -N$ this is clearly true, so suppose it holds for all k below some fixed $K > -N$. Then for any $-N < k < K$ we have, for ϕ^L as in (2.2.6),

$$\|\langle\Omega\rangle\mathbf{U}_k\|_{\infty,\infty} \lesssim \|\langle\Omega\rangle P_{<k}\phi^L\|_{\infty,\infty}\|\langle\Omega\rangle P_k\phi^L\|_{\infty,\infty}\|\langle\Omega\rangle\mathbf{U}_{<k}\|_{\infty,\infty} \lesssim C_0 c_k \tag{2.7.5}$$

It follows that

$$\|\langle\Omega\rangle \mathbf{U}_{<K}\|_{\infty,\infty} \lesssim 1 + \sum_{-N<k<K} C_0 c_k \lesssim 2$$

for $\epsilon(C_0)$ sufficiently small. Note that the first part of (2.7.1) is also a consequence of (2.7.5), and it follows that $\mathbf{U}$ is invertible with

$$\|\mathbf{U}^{-1}\|_{\infty,\infty} = \|\mathbf{I}_4 + (\mathbf{I}_4 - \mathbf{U}) + (\mathbf{I}_4 - \mathbf{U})^2 + \dots\|_{\infty,\infty} \lesssim 2$$

for $\epsilon(C_0)$ sufficiently small as required.

Using the relation $\mathbf{U}^{-1}\mathbf{U} = \mathbf{I}_4$, we can express the angular derivatives of $\mathbf{U}^{-1}$ in terms of those of $\mathbf{U}$:

$$\Omega_{ij}(\mathbf{U}^{-1}) = -\mathbf{U}^{-1}(\Omega_{ij}\mathbf{U})\mathbf{U}^{-1}$$

from which

$$\|\langle\Omega\rangle(\mathbf{U}^{-1})\|_{\infty,\infty} \lesssim \|\langle\Omega\rangle\mathbf{U}\|_{\infty,\infty} \lesssim 1$$

completing the proof of (2.7.2).

We now turn to (2.7.3). Note that this immediately implies the second part of (2.7.1), and also that the second part of (2.7.3) follows from the first thanks to the identity $\partial_\alpha(\mathbf{U}^{-1}) = -\mathbf{U}^{-1} \cdot \partial_\alpha \mathbf{U} \cdot \mathbf{U}^{-1}$.

We will show by induction that

$$\|\langle\Omega\rangle^{1-\delta(p,q)} \partial_\alpha \mathbf{U}_{<K}\|_{p,q} \lesssim 2^{\beta K} C_0 c_K \tag{2.7.6}$$

for all $-N \le K \le -9$. Here $\beta = \beta_{p,q} = 1 - \frac{1}{p} - \frac{3}{q}$.

For $K = -N$, the claim is trivial. Now suppose that $K > -N$ and the claim has been proven for all smaller k. By differentiating the formula defining $\mathbf{U}_{K-1}$ we have (neglecting the factor of $\langle\Omega\rangle^{1-\delta(p,q)}$ which plays no role)

$$\begin{aligned}
\|\partial_\alpha \mathbf{U}_{K-1}\|_{p,q} \lesssim & \|\partial_\alpha \phi^L_{<K-1}\|_{p,q} \|\phi^L_{K-1}\|_{\infty,\infty} \|\mathbf{U}_{<K-1}\|_{\infty,\infty} \\
& + \|\phi^L_{<K-1}\|_{\infty,\infty} \|\partial_\alpha \phi^L_{K-1}\|_{p,q} \|\mathbf{U}_{<K-1}\|_{\infty,\infty} \\
& + \|\phi^L_{<K-1}\|_{\infty,\infty} \|\phi^L_{K-1}\|_{\infty,\infty} \|\partial_\alpha \mathbf{U}_{<K-1}\|_{p,q} \lesssim 2^{\beta K} C_0 c_K
\end{aligned}$$

and the claim follows. □

We can now use this proposition to transform our equation (2.6.30) into a form in which the only non-trivial term in the forcing cancels. We make the transformation $\Phi^L = \mathbf{U} w^L$, so

$$\Box \Phi^L = (\Box \mathbf{U}) w^L + 2\partial_\alpha \mathbf{U} \partial^\alpha w^L + \mathbf{U} \Box w^L$$

Setting this equal to the right hand side of equation (2.6.30) and multiplying on the left by $\mathbf{U}^{-1}$ we obtain

$$\Box w^L = -2\mathbf{U}^{-1}(\partial_\alpha \mathbf{U}\partial^\alpha w^L - \mathbf{A}^L_\alpha \partial^\alpha \psi^L) - \mathbf{U}^{-1}(\Box \mathbf{U})\mathbf{U}^{-1}\Phi^L + \mathbf{U}^{-1}(error)$$

Note that $\mathbf{U}^{-1}(error) = error$ by (2.7.2), so we don't need to worry about the final term above.

In order to make use of the fact that $\partial_\alpha \mathbf{U} \simeq \mathbf{A}^L_\alpha \mathbf{U}$, we go back a step and use that $\Phi^L = \psi^L + \frac{1}{2}(\Delta_1) - (\Delta_2)$ to decompose w^L as

$$w^L = \underbrace{\mathbf{U}^{-1}\psi^L}_{w^L_1} + \underbrace{\frac{1}{2}\mathbf{U}^{-1}(\Delta_1)}_{w^L_2} - \underbrace{\mathbf{U}^{-1}(\Delta_2)}_{w^L_3}$$

In particular, since $\partial^\alpha \psi^L = (\partial^\alpha \mathbf{U})w^L_1 + \mathbf{U}\partial^\alpha w^L_1$, we can write

$$\begin{aligned}\Box w^L = &-2\mathbf{U}^{-1}(\partial_\alpha \mathbf{U} - \mathbf{A}^L_\alpha \mathbf{U})\partial^\alpha w^L_1 + 2\mathbf{U}^{-1}\mathbf{A}^L_\alpha(\partial^\alpha \mathbf{U})w^L_1 - 2\mathbf{U}^{-1}\partial_\alpha \mathbf{U}\partial^\alpha(w^L_2 + w^L_3)\\ &- \mathbf{U}^{-1}(\Box \mathbf{U})\mathbf{U}^{-1}\Phi^L + error\end{aligned}$$

The remainder of this section will be dedicated to showing that the second, third and fourth terms above are all of the form *error*. The remaining term will be studied in Section 2.8.

Proposition 2.7.2.

$$\mathbf{U}^{-1}\mathbf{A}^L_\alpha(\partial^\alpha \mathbf{U})w^L_1 = error \tag{2.7.7}$$

and

$$\mathbf{U}^{-1}\partial_\alpha \mathbf{U}\partial^\alpha(w^L_2 + w^L_3) = error \tag{2.7.8}$$

Proof. Again using (2.7.2) we may neglect the $\mathbf{U}^{-1}$. We will also work entirely in standard Strichartz spaces so neglect the angular derivatives.

To bound (2.7.7) note that by definition of $\mathbf{A}^L_\alpha$ we have

$$\|\mathbf{A}^L_\alpha\|_{\frac{2M}{M-1},2M} \lesssim \|\phi^L_{\leq -10}\|_{\infty,\infty}\|\partial_\alpha \phi^L_{\leq -10}\|_{\frac{2M}{M-1},2M} \lesssim C_0\epsilon$$

Hence by (2.7.3) it holds

$$\|\mathbf{A}^L_\alpha(\partial^\alpha \mathbf{U})\mathbf{U}^{-1}\psi^L\|_{1,2} \lesssim \|\mathbf{A}^L_\alpha\|_{\frac{2M}{M-1},2M}\|\partial^\alpha \mathbf{U}\|_{\frac{2M}{M-1},2M}\|\psi^L\|_{M,\frac{2M}{M-2}} \lesssim C_0^3\epsilon^2 c_0$$

as required.

We now turn to (2.7.8). Let's start with w^L_2. Heuristically, we can write $w^L_2 \simeq \mathbf{U}^{-1}P_0(\phi^L_{\leq -10}(\phi^L_{>-10})^T\phi^L_{>-10})$. Therefore

$$\|\partial_\alpha \mathbf{U}\partial^\alpha w^L_2\|_{1,2} \lesssim \|\partial_\alpha \mathbf{U}\partial^\alpha(\mathbf{U}^{-1})\ P_0(\phi^L_{\leq -10}(\phi^L_{>-10})^T\phi^L_{>-10})\|_{1,2} \tag{2.7.9}$$

$$+ \|\partial_\alpha \mathbf{U}\ \mathbf{U}^{-1} P_0 \partial^\alpha (\phi^L_{\leq -10} (\phi^L_{>-10})^T \phi^L_{>-10})\|_{1,2} \tag{2.7.10}$$

where

$$(2.7.9) \lesssim \|\partial_\alpha \mathbf{U}\|_{\frac{2M}{M-1},2M} \|\partial^\alpha(\mathbf{U}^{-1})\|_{\infty,\infty} \|\phi^L_{\leq -10}\|_{\infty,\infty} \|\phi^L_{>-10}\|_{\frac{2M}{M-1},2M} \|\phi^L_{>-10}\|_{M,\frac{2M}{M-2}} \lesssim C_0^3 \epsilon^2 c_0$$

On the other hand, for (2.7.10) we have

$$\begin{aligned}
(2.7.10) \lesssim& \|\partial_\alpha \mathbf{U}\ \mathbf{U}^{-1} P_0 (\partial^\alpha \phi^L_{\leq -10} (\phi^L_{>-10})^T \phi^L_{>-10})\|_{1,2} \\
&+ \|\partial_\alpha \mathbf{U}\ \mathbf{U}^{-1} P_0 (\phi^L_{\leq -10} (\partial^\alpha \phi^L_{>-10})^T \phi^L_{>-10})\|_{1,2} \\
\lesssim& \|\partial_\alpha \mathbf{U}\|_{\infty,\infty} \|\partial^\alpha \phi^L_{\leq -10}\|_{\frac{2M}{M-1},2M} \|\phi^L_{>-10}\|_{\frac{2M}{M-1},2M} \|\phi^L_{>-10}\|_{M,\frac{2M}{M-2}} \\
&+ \|\partial_\alpha \mathbf{U}\|_{\frac{2M}{M-1},2M} \|\phi^L_{\leq -10}\|_{\infty,\infty} \|\partial^\alpha \phi^L_{>-10}\|_{M,\frac{2M}{M-2}} \|\phi^L_{>-10}\|_{\frac{2M}{M-1},2M} \\
\lesssim& C_0^3 \epsilon^2 c_0
\end{aligned}$$

Next we study

$$w_3^L \simeq -\mathbf{U}^{-1} \int_0^1 \int_y \check{\psi}_0(y) y^T\ \phi^L_{\leq -10}(x-\theta y) \nabla_x \phi^L_{\leq -10}(x-\theta y)^T \phi^L_{\sim 0}(x-y) d\theta dy$$

We have

$$\begin{aligned}
&\left\| \partial^\alpha \mathbf{U} \partial_\alpha(\mathbf{U}^{-1}) \int_0^1 \int_y \check{\psi}_0(y) y^T\ \phi^L_{\leq -10}(x-\theta y) \nabla_x \phi^L_{\leq -10}(x-\theta y)^T \phi^L_{\sim 0}(x-y) d\theta dy \right\|_{1,2} \\
\lesssim& \|\partial^\alpha \mathbf{U}\|_{\frac{2M}{M-1},2M} \|\partial_\alpha(\mathbf{U}^{-1})\|_{\infty,\infty} \|\phi^L_{\leq -10}\|_{\infty,\infty} \|\nabla \phi^L_{\leq -10}\|_{\frac{2M}{M-1},2M} \|\phi^L_{\sim 0}\|_{M,\frac{2M}{M-2}}
\end{aligned}$$

which is acceptable. On the other hand,

$$\begin{aligned}
&\left\| \partial^\alpha \mathbf{U}\ \mathbf{U}^{-1} \partial_\alpha \int_0^1 \int_y \check{\psi}_0(y) y^T\ \phi^L_{\leq -10}(x-\theta y) \nabla_x \phi^L_{\leq -10}(x-\theta y)^T \phi^L_{\sim 0}(x-y) d\theta dy \right\|_{1,2} \\
\lesssim& \|\partial^\alpha \mathbf{U}\|_{\frac{2M}{M-1},2M} \|\partial_\alpha \phi^L_{\leq -10}\|_{\infty,\infty} \|\nabla \phi^L_{\leq -10}\|_{\frac{2M}{M-1},2M} \|\phi^L_{\sim 0}\|_{M,\frac{2M}{M-2}} \\
&+ \|\partial^\alpha \mathbf{U}\|_{\frac{2M}{M-1},2M} \|\phi^L_{\leq -10}\|_{\infty,\infty} \|\partial_\alpha \nabla \phi^L_{\leq -10}\|_{\frac{2M}{M-1},2M} \|\phi^L_{\sim 0}\|_{M,\frac{2M}{M-2}} \\
&+ \|\partial^\alpha \mathbf{U}\|_{\frac{2M}{M-1},2M} \|\phi^L_{\leq -10}\|_{\infty,\infty} \|\nabla \phi^L_{\leq -10}\|_{\frac{2M}{M-1},2M} \|\partial_\alpha \phi^L_{\sim 0}\|_{M,\frac{2M}{M-2}}
\end{aligned}$$

which is also acceptable. □

Proposition 2.7.3.

$$\mathbf{U}^{-1}(\Box \mathbf{U}) \mathbf{U}^{-1} \Phi^L = error$$

Proof. We still ignore the $\mathbf{U}^{-1}$, however in this case we cannot simply neglect the angular derivatives. We will show inductively that

$$\|\langle\Omega\rangle(\Box\mathbf{U}_{<K}\mathbf{U}^{-1}\Phi^L)\|_{1,2} \leq 2^{K/M}C_0^2\epsilon c_0$$

for all $-N \leq K \leq -9$, for M sufficiently large.

The claim is trivial for $K=-N$, so suppose it is true up to $K-1 \geq -N$. Observe that

$$\begin{aligned}\Box\mathbf{U}_{K-1} \simeq &\phi^L_{<K-1}\phi^L_{K-1}\Box\mathbf{U}_{<K-1} + \phi^L_{<K-1}\Box\phi^L_{K-1}\mathbf{U}_{<K-1}\\ &+ \Box\phi^L_{<K-1}\phi^L_{K-1}\mathbf{U}_{<K-1} + \partial_\alpha\phi^L_{<K-1}\partial^\alpha\phi^L_{K-1}\mathbf{U}_{<K-1}\\ &+ \partial_\alpha\phi^L_{<K-1}\phi^L_{K-1}\partial^\alpha\mathbf{U}_{<K-1} + \phi^L_{<K-1}\partial_\alpha\phi^L_{K-1}\partial^\alpha\mathbf{U}_{<K-1}\end{aligned} \tag{2.7.11}$$

The last three terms are the easiest to handle. For instance, using (2.6.31) to bound Φ we have

$$\begin{aligned}&\|\langle\Omega\rangle(\partial_\alpha\phi^L_{<K-1}\partial^\alpha\phi^L_{K-1}\mathbf{U}_{<K-1}\cdot\mathbf{U}^{-1}\Phi^L)\|_{1,2}\\ &\lesssim \|\langle\Omega\rangle\partial_\alpha\phi^L_{<K-1}\|_{\frac{2M}{M-1},2M}\|\langle\Omega\rangle\partial^\alpha\phi^L_{K-1}\|_{\frac{2M}{M-1},2M}\|\langle\Omega\rangle\Phi^L\|_{M,\frac{2M}{M-2}}\\ &\lesssim 2^{(1-\frac{2}{M})K}C_0^3\epsilon^2c_0\end{aligned}$$

which is more than we need. The last 2 terms of (2.7.11) can be bounded in the same way.

We now study the terms of (2.7.11) involving the wave operator. The first one will be bounded using the induction hypothesis so let's start with the second term. We have to bound

$$\|\langle\Omega\rangle(\phi^L_{<K-1}\Box\phi^L_{K-1}\mathbf{U}_{<K-1}\cdot\mathbf{U}^{-1}\Phi^L)\|_{1,2} \lesssim \|\langle\Omega\rangle(\Box\phi^L_{K-1})\|_{\frac{2M}{M+1},\frac{2M}{M-1}}\|\langle\Omega\rangle\Phi^L\|_{\frac{2M}{M-1},2M}$$

We then need the following claim.

Claim 5.

$$\|\langle\Omega\rangle(\Box\phi^L_{K-1})\|_{\frac{2M}{M+1},\frac{2M}{M-1}} \lesssim 2^{k/M}C_0^2\epsilon c_K$$

Proof of claim. By scaling, it suffices to prove the claim for $K=1$, so study $\Box\psi^L$ (see (2.4.1)). First consider the wave maps part of $\Box\psi^L$. The action of L does not play an important role here, so for simplicity we only study $\Box\phi$. Using Bernstein's inequality to lower the exponent for the high frequency interactions, we have

$$\begin{aligned}\|\langle\Omega\rangle P_0(\phi\partial^\alpha\phi^T\partial_\alpha\phi)\|_{\frac{2M}{M+1},\frac{2M}{M-1}} &\lesssim \|\langle\Omega\rangle P_0(\phi\partial^\alpha\phi^T_{>-10}\partial_\alpha\phi_{>-10})\|_{\frac{2M}{M+1},\frac{2M}{M-1}}\\ &\quad+ \|\langle\Omega\rangle P_0(\phi\partial^\alpha\phi^T_{\leq-10}\partial_\alpha\phi_{>-10})\|_{\frac{2M}{M+1},\frac{2M}{M-1}}\\ &\quad+ \|\langle\Omega\rangle P_0(\phi\partial^\alpha\phi^T_{\leq-10}\partial_\alpha\phi_{\leq-10})\|_{\frac{2M}{M+1},\frac{2M}{M-1}}\\ &\lesssim \|\langle\Omega\rangle\partial^\alpha\phi_{>-10}\|_{\frac{4M}{M-1},\frac{4M}{M+1}}\|\partial^\alpha\phi_{>-10}\|_{\frac{4M}{M+3},\frac{4M}{M+1}}\\ &\quad+ \|\langle\Omega\rangle\partial^\alpha\phi_{\leq-10}\|_{\frac{4M}{M+3},\frac{4M}{M-3}}\|\langle\Omega\rangle\partial^\alpha\phi_{>-10}\|_{\frac{4M}{M-1},\frac{4M}{M+1}}\end{aligned}$$

$$+\|\langle\Omega\rangle\partial^\alpha\phi_{\le -10}\|_{\frac{4M}{M+1},\frac{4M}{M-1}}\|\langle\Omega\rangle\partial_\alpha\phi_{\le -10}\|_{\frac{4M}{M+1},\frac{4M}{M-1}}$$
$$\lesssim C_0^2\epsilon c_0$$

provided M is sufficiently large and σ sufficiently small. The half-wave maps terms can be treated similarly. $\square$

We therefore have

$$\|\langle\Omega\rangle(\phi^L_{<K-1}\Box\phi^L_{K-1}\mathbf{U}_{<K-1}\cdot\mathbf{U}^{-1}\Phi^L)\|_{1,2}\lesssim 2^{K/M}C_0^3\epsilon^2c_0$$

Similarly,

$$\|\langle\Omega\rangle(\Box\phi^L_{<K-1}\phi^L_{K-1}\mathbf{U}_{<K-1}\cdot\mathbf{U}^{-1}\Phi^L)\|_{1,2}\lesssim\sum_{J<K-1}2^{J/M}C_0^2\epsilon c_0\lesssim 2^{K/M}C_0^3\epsilon^2c_0$$

and lastly by the induction hypothesis we have

$$\begin{aligned}\|\langle\Omega\rangle(\phi^L_{<K-1}\phi^L_{K-1}\Box\mathbf{U}_{<K-1}\cdot\mathbf{U}^{-1}\Phi^L)\|_{1,2}&\lesssim\|\langle\Omega\rangle\phi^L_{K-1}\|_{\infty,\infty}\|\langle\Omega\rangle(\Box\mathbf{U}_{<K-1}\cdot\mathbf{U}^{-1}\Phi^L)\|_{1,2}\\&\lesssim C_0\epsilon\cdot 2^{K/M}C_0^2\epsilon c_0\lesssim 2^{K/M}C_0^3\epsilon^2c_0\end{aligned}$$

Therefore, letting D denote the sum of the implicit constants above and using (2.7.11), we have

$$\begin{aligned}\|\langle\Omega\rangle(\Box\mathbf{U}_{<K}\cdot\mathbf{U}^{-1}\Phi^L)\|_{1,2}&\le\|\langle\Omega\rangle(\Box\mathbf{U}_{<K-1}\cdot\mathbf{U}^{-1}\Phi^L)\|_{1,2}+D\cdot 2^{K/M}C_0^3\epsilon^2c_0\\&\le 2^{K/M}C_0^2\epsilon c_0(2^{-1/M}+DC_0\epsilon)\end{aligned}$$

Hence choosing $\epsilon\ll(DC_0)^{-1}$ completes the induction. $\square$

Combining these two propositions with the equation previously obtained for w^L, we arrive at

$$\Box w^L=-2\mathbf{U}^{-1}(\partial_\alpha\mathbf{U}-\mathbf{A}^L_\alpha\mathbf{U})\partial^\alpha w^L_1+error$$

2.8 The (very low-low-high) cancellation

In this section, we will show that

$$\mathbf{U}^{-1}(\partial_\alpha\mathbf{U}-\mathbf{A}^L_\alpha\mathbf{U})\partial^\alpha w^L_1=error,\tag{2.8.1}$$

finally doing away with the difficult (lowest)∇(low)∇(high) frequency interactions in the wave maps source term.

As usual we may neglect the $\mathbf{U}^{-1}$. The first step in proving (2.8.1) is to use the telescoping identity

$$\sum_{-N<k\leq -10} (\mathbf{A}^L_{\alpha,\leq k}\mathbf{U}_{\leq k} - \mathbf{A}^L_{\alpha,<k}U_{<k}) = \mathbf{A}^L_\alpha \mathbf{U} - \mathbf{A}^L_{\alpha,\leq -N}$$

where

$$\mathbf{A}^L_{\alpha,\leq k} := \mathbf{A}^L_{\alpha,<k+1} := \begin{pmatrix} -\phi_{\leq k}\partial_\alpha \phi^T_{\leq k} & 0 \\ -(\phi_{\leq k}\partial_\alpha (L_n\phi)^T_{\leq k} + (L_n\phi)_{\leq k}\partial_\alpha \phi^T_{\leq k}) & -\phi_{\leq k}\partial_\alpha\phi^T_{\leq k}\,\mathbf{I}_3 \end{pmatrix}$$

(so $\mathbf{A}^L_\alpha = \mathbf{A}^L_{\alpha,\leq -10}$) to write

$$\partial_\alpha \mathbf{U} - \mathbf{A}^L_\alpha U = \left[\sum_{-N<k\leq -10} \partial_\alpha \mathbf{U}_k - (\mathbf{A}^L_{\alpha,\leq k}\mathbf{U}_{\leq k} - \mathbf{A}^L_{\alpha,<k}\mathbf{U}_{<k})\right] - \mathbf{A}^L_{\alpha,\leq -N} \tag{2.8.2}$$

We first show that the $\mathbf{A}^L_{\alpha,\leq -N}$ part is acceptable:

Lemma 2.8.1.

$$\mathbf{A}^L_{\alpha,\leq -N}\partial^\alpha w^L_1 = error$$

Proof. We have

$$\|\langle\Omega\rangle(\mathbf{A}^L_{\alpha,\leq -N}\partial^\alpha w^L_1)\|_{1,2} \lesssim \|\langle\Omega\rangle\mathbf{A}^L_{\alpha,\leq -N}\|_{1,\infty}\|\langle\Omega\rangle\partial^\alpha w^L_1\|_{\infty,2}$$

Then

$$\|\langle\Omega\rangle\mathbf{A}^L_{\alpha,\leq -N}\|_{1,\infty} \lesssim T\|\langle\Omega\rangle\phi^L_{\leq -N}\|_{\infty,\infty}\|\langle\Omega\rangle\partial_\alpha\phi^L_{\leq -N}\|_{\infty,\infty} \lesssim 2^{-N}TC_0\epsilon$$

and by the identity

$$\partial^\alpha w^L_1 = \mathbf{U}^{-1}\partial^\alpha\psi^L - \mathbf{U}^{-1}(\partial^\alpha\mathbf{U})\mathbf{U}^{-1}\psi^L \tag{2.8.3}$$

also

$$\|\langle\Omega\rangle\partial^\alpha w^L_1\|_{\infty,2} \lesssim C_0c_0 \tag{2.8.4}$$

The result is now immediate upon taking $N(T,C_0)$ sufficiently large. □

Next we study the sum in (2.8.2). It is here that we observe the critical cancellation of the $\phi_{<k}\partial^\alpha\phi^T_k$ terms. Indeed, as in [Tao01a] we may write

$$\begin{aligned}\partial_\alpha\mathbf{U}_k = &\begin{pmatrix} -\partial_\alpha\phi_{<k}\phi^T_k & 0 \\ -(\partial_\alpha\phi_{<k}(L_n\phi)^T_k + \partial_\alpha(L_n\phi)_{<k}\phi^T_k) & -\partial_\alpha\phi_{<k}\phi^T_k\,\mathbf{I}_3\end{pmatrix}\mathbf{U}_{<k} \\ &+ \begin{pmatrix} -\phi_{<k}\partial_\alpha\phi^T_k & 0 \\ -(\phi_{<k}\partial_\alpha(L_n\phi)^T_k + (L_n\phi)_{<k}\partial_\alpha\phi^T_k) & -\phi_{<k}\partial_\alpha\phi^T_k\,\mathbf{I}_3\end{pmatrix}\mathbf{U}_{<k} \\ &+ \begin{pmatrix} -\phi_{<k}\phi^T_k & 0 \\ -(\phi_{<k}(L_n\phi)^T_k + (L_n\phi)_{<k}\phi^T_k) & -\phi_{<k}\phi^T_k\,\mathbf{I}_3\end{pmatrix}\partial_\alpha\mathbf{U}_{<k}\end{aligned} \tag{2.8.5}$$

and

$$\mathbf{A}^L_{\alpha,\leq k}\mathbf{U}_{\leq k} - \mathbf{A}^L_{\alpha,<k}\mathbf{U}_{<k} = \begin{pmatrix} -\phi_{<k}\partial_\alpha\phi_k^T & 0 \\ -(\phi_{<k}\partial_\alpha(L_n\phi)_k^T + (L_n\phi)_{<k}\partial_\alpha\phi_k^T) & -\phi_{<k}\partial_\alpha\phi_k^T\,\mathbf{I}_3 \end{pmatrix}\mathbf{U}_{<k}$$
$$+ \begin{pmatrix} -\phi_k\partial_\alpha\phi_{\leq k}^T & 0 \\ -(\phi_k\partial_\alpha(L_n\phi)_{\leq k}^T + (L_n\phi)_k\partial_\alpha\phi_{\leq k}^T) & -\phi_k\partial_\alpha\phi_{\leq k}^T\,\mathbf{I}_3 \end{pmatrix}\mathbf{U}_{<k}$$
$$+ \begin{pmatrix} -\phi_{\leq k}\partial_\alpha\phi_{\leq k}^T & 0 \\ -(\phi_{\leq k}\partial_\alpha(L_n\phi)_{\leq k}^T + (L_n\phi)_{\leq k}\partial_\alpha\phi_{\leq k}^T) & -\phi_{\leq k}\partial_\alpha\phi_{\leq k}^T\,\mathbf{I}_3 \end{pmatrix}\mathbf{U}_k \tag{2.8.6}$$

Crucially, the second line in (2.8.5) and the first line in (2.8.6) cancel and we are left with

$$\begin{aligned}
&\partial_\alpha\mathbf{U}_k - (\mathbf{A}^L_{\alpha,\leq k}\mathbf{U}_{\leq k} - \mathbf{A}^L_{\alpha,<k}\mathbf{U}_{<k}) \\
&= \begin{pmatrix} -\partial_\alpha\phi_{<k}\phi_k^T & 0 \\ -(\partial_\alpha\phi_{<k}(L_n\phi)_k^T + \partial_\alpha(L_n\phi)_{<k}\phi_k^T) & -\partial_\alpha\phi_{<k}\phi_k^T\,\mathbf{I}_3 \end{pmatrix}\mathbf{U}_{<k} \\
&+ \begin{pmatrix} -\phi_{<k}\phi_k^T & 0 \\ -(\phi_{<k}(L_n\phi)_k^T + (L_n\phi)_{<k}\phi_k^T) & -\phi_{<k}\phi_k^T\,\mathbf{I}_3 \end{pmatrix}\partial_\alpha\mathbf{U}_{<k} \\
&- \begin{pmatrix} -\phi_k\partial_\alpha\phi_{\leq k}^T & 0 \\ -(\phi_k\partial_\alpha(L_n\phi)_{\leq k}^T + (L_n\phi)_k\partial_\alpha\phi_{\leq k}^T) & -\phi_k\partial_\alpha\phi_{\leq k}^T\,\mathbf{I}_3 \end{pmatrix}\mathbf{U}_{<k} \\
&- \begin{pmatrix} -\phi_{\leq k}\partial_\alpha\phi_{\leq k}^T & 0 \\ -(\phi_{\leq k}\partial_\alpha(L_n\phi)_{\leq k}^T + (L_n\phi)_{\leq k}\partial_\alpha\phi_{\leq k}^T) & -\phi_{\leq k}\partial_\alpha\phi_{\leq k}^T\,\mathbf{I}_3 \end{pmatrix}\mathbf{U}_k
\end{aligned}$$

(2.8.1) is therefore implied by the following result.

Lemma 2.8.2.

$$\sum_{-N<k\leq -10}\begin{pmatrix} \partial_\alpha\phi_{<k}\phi_k^T & 0 \\ (\partial_\alpha\phi_{<k}(L_n\phi)_k^T + \partial_\alpha(L_n\phi)_{<k}\phi_k^T) & \partial_\alpha\phi_{<k}\phi_k^T\,\mathbf{I}_3 \end{pmatrix}\mathbf{U}_{<k}\,\partial^\alpha w_1^L = \mathit{error} \tag{2.8.7}$$

$$\sum_{-N<k\leq -10}\begin{pmatrix} \phi_{<k}\phi_k^T & 0 \\ (\phi_{<k}(L_n\phi)_k^T + (L_n\phi)_{<k}\phi_k^T) & \phi_{<k}\phi_k^T\,\mathbf{I}_3 \end{pmatrix}\partial_\alpha\mathbf{U}_{<k}\,\partial^\alpha w_1^L = \mathit{error} \tag{2.8.8}$$

$$\sum_{-N<k\leq -10}\begin{pmatrix} \phi_k\partial_\alpha\phi_{\leq k}^T & 0 \\ (\phi_k\partial_\alpha(L_n\phi)_{\leq k}^T + (L_n\phi)_k\partial_\alpha\phi_{\leq k}^T) & \phi_k\partial_\alpha\phi_{\leq k}^T\,\mathbf{I}_3 \end{pmatrix}\mathbf{U}_{<k}\,\partial^\alpha w_1^L = \mathit{error} \tag{2.8.9}$$

$$\sum_{-N<k\leq -10}\begin{pmatrix} \phi_{\leq k}\partial_\alpha\phi_{\leq k}^T & 0 \\ (\phi_{\leq k}\partial_\alpha(L_n\phi)_{\leq k}^T + (L_n\phi)_{\leq k}\partial_\alpha\phi_{\leq k}^T) & \phi_{\leq k}\partial_\alpha\phi_{\leq k}^T\,\mathbf{I}_3 \end{pmatrix}\mathbf{U}_k\,\partial^\alpha w_1^L = \mathit{error} \tag{2.8.10}$$

Proof.

1. Proof of (2.8.7), (2.8.9). These two inequalities are to all intents and purposes the same, so we consider only (2.8.7). Recall that $\partial^\alpha w_1^L = \mathbf{U}^{-1}\partial^\alpha\psi^L - \mathbf{U}^{-1}(\partial^\alpha\mathbf{U})\mathbf{U}^{-1}\psi^L$

where $\mathbf{U}^{-1}$ has the form

$$\mathbf{U}^{-1} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$$

and $\mathbf{U}_{<k}$ has the same structure. We first study the part of $\partial^\alpha w_1^L$ involving $\partial^\alpha \psi^L$. Write

$$\mathbf{U}_{<k}\mathbf{U}^{-1} = \begin{pmatrix} U_1 & 0 \\ U_2 & U_1 \mathbf{I}_3 \end{pmatrix}$$

Then expanding the matrix product we have

$$\begin{aligned} &\begin{pmatrix} \partial_\alpha \phi_{<k} \phi_k^T & 0 \\ (\partial_\alpha \phi_{<k} (L_n\phi)_k^T + \partial_\alpha (L_n\phi)_{<k} \phi_k^T) & \partial_\alpha \phi_{<k} \phi_k^T \, \mathbf{I}_3 \end{pmatrix} \mathbf{U}_{<k}\mathbf{U}^{-1} \partial^\alpha \psi^L \\ &= \begin{pmatrix} \partial_\alpha \phi_{<k} \phi_k^T U_1 & 0 \\ (\partial_\alpha \phi_{<k} (L_n\phi)_k^T + \partial_\alpha (L_n\phi)_{<k} \phi_k^T) U_1 + \partial_\alpha \phi_{<k} \phi_k^T U_2 & \partial_\alpha \phi_{<k} \phi_k^T U_1 \, \mathbf{I}_3 \end{pmatrix} \begin{pmatrix} \partial^\alpha \psi \\ \partial^\alpha \psi_n^L \end{pmatrix} \end{aligned}$$

Further expanding this product, it remains to study the following:

(a) $\partial_\alpha \phi_{<k} \phi_k^T U_i \partial^\alpha \psi$, $i = 1, 2$.
(b) $\partial_\alpha \phi_{<k} (L_n\phi)_k^T U_1 \partial^\alpha \psi$
(c) $\partial_\alpha (L_n\phi)_{<k} \phi_k^T U_1 \partial^\alpha \psi$
(d) $\partial_\alpha \phi_{<k} \phi_k^T U_1 \partial^\alpha \psi_n^L$

For the rest of this proof we will treat all functions as scalars, even though they are really vector or matrix fields, by working componentwise. This reduction is possible since none of the arguments that follow rely on any geometric structure and in particular Lemma 2.5.3 held for scalar functions (see remark at end of said lemma). In this spirit, since all the $\langle \Omega \rangle U_i$ are bounded in $L^\infty_{t,x}$ (by Proposition 2.7.1) we may ignore these terms in the above expressions. What's left is treated by direct application of Lemma 2.5.3.

Starting with (a), we easily reduce to the following three terms:

$$\begin{aligned} \|\langle \Omega \rangle (\partial_\alpha \phi_{<k} \phi_k \partial^\alpha \psi)\|_{1,2} &\lesssim \|\langle \Omega \rangle \partial_\alpha \phi_{<k} \cdot \phi_k \cdot \partial^\alpha \psi\|_{1,2} && (2.8.11) \\ &+ \|\partial_\alpha \phi_{<k} \cdot \langle \Omega \rangle \phi_k \cdot \partial^\alpha \psi\|_{1,2} && (2.8.12) \\ &+ \|\partial_\alpha \phi_{<k} \cdot \phi_k \cdot \langle \Omega \rangle \partial^\alpha \psi\|_{1,2} && (2.8.13) \end{aligned}$$

For (2.8.11) we apply point 1 of Lemma 2.5.3 as in (2.5.8) (recalling that ψ is at unit frequency) to see

$$\|\langle \Omega \rangle \partial_\alpha \phi_{<k} \cdot \phi_k \cdot \partial^\alpha \psi\|_{1,2} \lesssim \sum_{j<k} 2^{(\frac{1}{2} - \frac{1}{2M})(j-k)} C_0^2 c_j c_k (C_0 c_0 + \|\partial_t \partial^\alpha \psi\|_{\infty,2})$$

and we have to do just a little work to bound $\|\partial_t \partial^\alpha \psi\|_{\infty,2}$ in the case $\alpha = 0$. We use the equation to find

$$\|\partial_t^2 \psi\|_{\infty,2} \leq \|\Delta \psi\|_{\infty,2} + \|\langle \Omega \rangle \Box P_0 \phi\|_{\infty,2}$$

where for example

$$\begin{aligned}\|P_0(\phi\partial^\alpha\phi^T\partial_\alpha\phi)\|_{\infty,2} &\lesssim \|P_0(\phi\partial^\alpha\phi^T_{>-10}\partial_\alpha\phi_{>-10})\|_{\infty,1} + \|P_0(\phi\partial^\alpha\phi^T_{\leq-10}\partial_\alpha\phi_{>-10})\|_{\infty,2}\\ &\quad + \|P_0(\phi\partial^\alpha\phi^T_{\leq-10}\partial_\alpha\phi_{\leq-10})\|_{\infty,2}\\ &\lesssim \|\partial^\alpha\phi_{>-10}\|_{\infty,2}\|\partial^\alpha\phi_{>-10}\|_{\infty,2} + \|\partial^\alpha\phi_{\leq-10}\|_{\infty,\infty}\|\partial^\alpha\phi_{>-10}\|_{\infty,2}\\ &\quad + \|\partial^\alpha\phi_{\leq-10}\|_{\infty,4}\|\partial^\alpha\phi_{\leq-10}\|_{\infty,4}\\ &\ll C_0c_0 \end{aligned} \tag{2.8.14}$$

Bounding the half-wave maps terms similarly we find

$$\|\langle\Omega\rangle\partial_\alpha\phi_{<k}\cdot\phi_k\cdot\partial^\alpha\psi\|_{1,2} \lesssim \sum_{j<k} 2^{(\frac{1}{2}-\frac{1}{2M})(j-k)} C_0^3 c_j c_k c_0$$

which is acceptable when summed over $k \leq -10$.

(2.8.13) can be handled in the same way and for (2.8.12) we use Point 2 of Lemma 2.5.3. (b) and (c) can be treated identically to (a), and (d) is similar upon using points 3 and 4 of Lemma 2.5.3 rather than 1 and 2 respectively.

The remaining part $(\mathbf{U}^{-1}\partial^\alpha\mathbf{U}\mathbf{U}^{-1})\psi^L$ of $\partial^\alpha w_1^L$ can be treated in the same way, since $\mathbf{U}^{-1}\partial^\alpha\mathbf{U}\mathbf{U}^{-1}$ has the same block structure as $\mathbf{U}_{<k}\mathbf{U}^{-1}$ and ψ^L is at unit frequency so behaves like $\partial^\alpha\psi^L$.

2. Proof of (2.8.8). Expanding $\partial^\alpha w_1^L$ as before and restricting to the term $\mathbf{U}^{-1}\partial^\alpha\psi^L$ for simplicity, we have to consider

$$\left\|\langle\Omega\rangle\left[\begin{pmatrix}\phi_{<k}\phi_k^T & 0\\ (\phi_{<k}(L_n\phi)_k^T + (L_n\phi)_{<k}\phi_k^T) & \phi_{<k}\phi_k^T\,\mathbf{I}_3\end{pmatrix}\partial_\alpha\mathbf{U}_{<k}\mathbf{U}^{-1}\partial^\alpha\psi^L\right]\right\|_{1,2}$$

Generally speaking, the argument for this term is similar to the previous one, using Lemma 2.5.3 and the lower triangular structure of the matrices involved to limit the interactions. For the sake of presentation, we will only consider the top left component of the expression above,

$$\|\langle\Omega\rangle(\phi_{<k}\phi_k^T\partial_\alpha U_{<k}U^{-1}\partial^\alpha\psi)\|_{1,2}$$

We will also restrict to the case where the angular derivative falls on ϕ_k, the other cases bring similar. Note that by placing $\phi_{<k}$ and U^{-1} into $L^\infty_{t,x}$ it suffices to consider

$$\|\langle\Omega\rangle\phi_k\cdot\partial_\alpha U_{<k}\cdot\partial^\alpha\psi\|_{1,2}$$

(working componentwise). We proceed by induction. Set

$$R(j) := \|\langle\Omega\rangle\phi_k\cdot\partial_\alpha U_{<j}\cdot\partial^\alpha\psi\|_{1,2}$$

Claim 6. *For all $-N \leq j \leq k$ it holds*

$$R(j) \lesssim 2^{(\frac{1}{2}-\frac{1}{2M})(j-k)} C_0^3 c_j c_k c_0$$

Proof of claim. The claim is trivial for $j = -N$ so suppose it is true up to some fixed $-N < j \le k$. By definition of U_j we have

$$R(j) \le R(j-1) + \|\langle\Omega\rangle\phi_k \cdot \phi_{<j-1}\phi_{j-1}^T \partial_\alpha U_{<j-1} \cdot \partial^\alpha \psi\|_{1,2} \tag{2.8.15}$$
$$+ \|\langle\Omega\rangle\phi_k \cdot \phi_{<j-1}\partial_\alpha\phi_{j-1}^T U_{<j-1} \cdot \partial^\alpha \psi\|_{1,2} \tag{2.8.16}$$
$$+ \|\langle\Omega\rangle\phi_k \cdot \partial_\alpha\phi_{<j-1}\phi_{j-1}^T U_{<j-1} \cdot \partial^\alpha \psi\|_{1,2} \tag{2.8.17}$$

For (2.8.15) we pull out $\|\phi_{<j-1}\phi_{j-1}^T\|_{L^\infty_{t,x}}$ to find

$$(2.8.15) \lesssim C_0 c_{j-1} R(j-1)$$

using the induction hypothesis.

For (2.8.16) we place $U_{<j-1}$ and $\phi_{<j-1}$ into $L^\infty_{t,x}$ and apply part 2 of Lemma 2.5.3 in conjunction with (2.8.14) to bound

$$\begin{aligned}\|\langle\Omega\rangle\phi_k \cdot \phi_{<j-1}\partial_\alpha\phi_{j-1}^T U_{<j-1} \cdot \partial^\alpha \psi\|_{1,2} &\lesssim \|\langle\Omega\rangle\phi_k \cdot \partial_\alpha\phi_{j-1} \cdot \partial^\alpha\psi\|_{1,2} \\ &\lesssim 2^{(\frac{1}{2}-\frac{1}{2M})(j-k)} C_0^3 c_j c_k c_0\end{aligned}$$

Similarly, for (2.8.17) we have

$$\begin{aligned}\|\langle\Omega\rangle\phi_k \cdot \partial_\alpha\phi_{<j-1}\phi_{j-1}^T U_{<j-1} \cdot \partial^\alpha \psi\|_{1,2} &\lesssim C_0 c_j \sum_{l<j-1} \|\langle\Omega\rangle\phi_k \cdot \partial_\alpha\phi_l \cdot \partial^\alpha\psi\|_{1,2} \\ &\lesssim C_0 c_j \sum_{l<j-1} 2^{(\frac{1}{2}-\frac{1}{2M})(l-k)} C_0^3 c_l c_k c_0 \\ &\lesssim 2^{(\frac{1}{2}-\frac{1}{2M})(j-k)} C_0^3 c_j c_k c_0\end{aligned}$$

We deduce that, for some constant $D > 0$,

$$R(j) \le (1 + D \cdot C_0\epsilon) R(j-1) + D \cdot 2^{(\frac{1}{2}-\frac{1}{2M})(j-k)} C_0^3 c_j c_k c_0$$

and the claim follows upon taking $\epsilon(C_0)$ sufficiently small. $\square$

With this claim in hand, we have

$$\sum_{-N<k\le -10} \|\langle\Omega\rangle\phi_k \cdot \partial_\alpha U_{<k} \cdot \partial^\alpha\psi\|_{1,2} \lesssim \sum_{-N<k\le -10} C_0^3 c_k^2 \epsilon$$

which is as required.

3. Proof of (2.8.10). This is another straightforward application of Lemma 2.5.3. We again focus only on the top left component of the term, that is

$$\phi_{\le k}\partial_\alpha\phi_{\le k}^T \cdot U_k \cdot \partial^\alpha w_1$$

Expand

$$U_k := -\phi_{<k}\phi_k^T U_{<k}$$

and place $\phi_{<k}$, $U_{<k}$ and the other $\phi_{\leq k}$ appearing in the term into $L^\infty_{t,x}$ to reduce to bounding

$$\sum_{-N<k\leq -10} \|\langle\Omega\rangle(\partial_\alpha\phi_{\leq k}\cdot\phi_k\cdot\partial^\alpha w_1)\|_{1,2}$$

Upon expanding $\partial^\alpha w_1 = U^{-1}\partial^\alpha\psi + U^{-1}\partial^\alpha U U^{-1}\psi$ as before, one sees that this can be treated via a direct application of Lemma 2.5.3 as in part (1) of this proof.

□

2.9 Putting it all together

We have succeeded in reducing our equation to

$$\Box w^L = error$$

for w^L defined through $\Phi^L = \mathbf{U}w^L$. In order to exploit the linear estimate, we need to check that we still have the correct smallness on the initial data.

Proposition 2.9.1. *Let* $\phi[0]$ *satisfy assumption* (2.3.1)*. Then*

$$\|\langle\Omega\rangle\tilde{P}_0 w^L[0]\|_{\dot{H}^{3/2}\times\dot{H}^{1/2}} \lesssim c_0$$

Proof. By (2.7.2) and (2.7.3) it suffices to show the corresponding bound on Φ^L, which by Propositions 2.6.1 and 2.6.3 further reduces to

$$\|\langle\Omega\rangle\psi^L[0]\|_{\dot{H}^{3/2}\times\dot{H}^{1/2}} \lesssim c_0$$

In the absence of L the bound is immediate. Then for $n\in\{1,2,3\}$ we have

$$\begin{aligned}
&\|\langle\Omega\rangle P_0(L_n\phi)[0]\|_{\dot{H}^{3/2}\times\dot{H}^{1/2}}\\
&\lesssim \|\langle\Omega\rangle P_0(x_n\partial_t\phi(0))\|_{L^2} + \|\langle\Omega\rangle P_0(x_n\partial_t^2\phi(0))\|_{L^2} + \|\langle\Omega\rangle P_0(\partial_{x_n}\phi(0))\|_{L^2}\\
&\lesssim c_0 + \|\langle\Omega\rangle P_0(x_n\Box\phi(0))\|_{L^2}
\end{aligned}$$

which is acceptable thanks to (2.6.3). □

Remember that our actual goal is to bound ψ^L. By Propositions 2.6.1 and 2.6.3 we see that

$$\|\psi^L\|_{S_0} \lesssim \|\Phi^L\|_{S_0} + \|(\Delta_1)\|_{S_0} + \|(\Delta_2)\|_{S_0} \lesssim \|\mathbf{U}w^L\|_{S_0} + C_0^2\epsilon c_0 \tag{2.9.1}$$

where by (2.7.2) and (2.7.3),

$$\|\mathbf{U}w^L\|_{S_0} \lesssim \|w^L\|_{S_0} + \max_{\mathcal{Q}}\|\langle\Omega\rangle^{1-\delta(p,q)}w^L\|_{p,q} =: \|w^L\|_{\tilde{S}_0}$$

Using the linear estimate, Theorem 2.2.5, we are now almost done, modulo the fact that w^L is not quite at unit frequency. To get around this, use that $w^L \simeq \Phi^L$ by writing

$$w^L = \Phi^L - (\mathbf{U} - \mathbf{I}_4)w^L$$

where $\Phi^L = \tilde{P}_0(\Phi^L)$. Then since $\tilde{S}_0$ is equivalent to S_0 at unit frequency, we can use Theorem 2.2.5 to bound

$$\|\tilde{P}_0 w^L\|_{\tilde{S}_0} \lesssim \|\langle\Omega\rangle w^L[0]\|_{\dot{H}^{3/2}\times\dot{H}^{1/2}} + C_0^3 c_0 \epsilon \lesssim c_0$$

upon taking $\epsilon(C_0)$ sufficiently small. On the other hand by Proposition 2.7.1 we have

$$\|(1-\tilde{P}_0)w^L\|_{\tilde{S}_0} \lesssim \|(\mathbf{U}-\mathbf{I}_4)w^L\|_{\tilde{S}_0} \lesssim C_0\epsilon\|w^L\|_{\tilde{S}_0}$$

We have thus found $\|w^L\|_{\tilde{S}_0} \lesssim c_0 + C_0\epsilon\|w^L\|_{\tilde{S}_0}$ and taking $\epsilon(C_0)$ sufficiently small deduce that

$$\|w^L\|_{\tilde{S}_0} \lesssim c_0$$

Plugging this into (2.9.1) completes the proof of Proposition 2.3.2, and hence of the global existence of ϕ.

2.10 Proof of local wellposedness

This section is devoted to the proof of Theorem 2.1.3. The argument is a combination of the scheme from [KS17] with standard methods for studying subcritical wave maps (see for instance [KS97, KM96a, Sel99, GG16]), however we run into various technical issues which lengthen the presentation. In the first subsection, we prove the local wellposedness of the differentiated half-wave maps equation (2.1.2), and in the second subsection we prove that this solution indeed solves the original half-wave maps equation for compatible initial data.

Throughout this section, $p \in \mathbb{S}^2$ is fixed.

2.10.1 Local Wellposedness of the Differentiated Equation (2.1.2).

We start by outlining the argument. We will work in the subcritical function space $X_1^{s,\theta}$ defined by the norm[12]

$$\|\phi\|_{X_1^{s,\theta}} := \sum_{k\geq 0}\|\phi_k\|_{X^{s,\theta}} := \sum_{k\geq 0}\|\langle|\tau|+|\xi|\rangle^s\langle||\tau|-|\xi||\rangle^\theta \tilde{\mathcal{F}}(\phi_k)(\tau,\xi)\|_{L^2_{\tau,\xi}}$$

[12]As before we say ϕ in $X_1^{s,\theta}$ when $\phi \in p + X_1^{s,\theta}$ and write $\|\phi\|_{X_1^{s,\theta}}$ to meant $\|\phi - p\|_{X_1^{s,\theta}}$. We have a similar statement for $B^s_{2,1}$.

for $3/2+\nu > s > 3/2$, $\theta > 1/2$ and $s - 3/2 > \theta - 1/2$. Here $\tilde{\mathcal{F}}$ denotes the spacetime Fourier transform, and henceforth we denote $\phi_0 := \mathcal{F}^{-1}(\chi(\xi)\hat{\phi}(\xi))$ the low frequency portion of ϕ, for χ as in Section 2.1.1. Note that $X_1^{s,\theta}$ controls the Besov norm (see, for example, Proposition 2.7, [GG16]):

$$\|\phi\|_{L_t^\infty B^s_{2,1}} + \|\partial_t \phi\|_{L_t^\infty B^{s-1}_{2,1}} \lesssim \|\phi\|_{X_1^{s,\theta}}$$

The iteration argument, inspired by the scheme of [KS17], is then as follows.

1. Set $\phi^{(0)} = p$, the limit of the initial data at infinity.
2. Construct $\phi^{(1)}$ as the local solution to the wave maps equation by iteration in the space $X_1^{s,\theta}$. This solution lies on the sphere, satisfies $\|\phi^{(1)}\|_{X_1^{s,\theta}} \leq 2C\|\phi[0]\|_{B^s_{2,1}\times B^{s-1}_{2,1}}$ (which may be large), and has minimal time of existence depending only on $\|\phi[0]\|_{B^s_{2,1}\times B^{s-1}_{2,1}}$. By standard persistence of regularity arguments, it can be seen that $\phi^{(1)}$ is smooth.
3. Suppose that for all $1 \leq j < J$ we have found smooth $\phi^{(j)} \in X_1^{s,\theta}$ solving
$$\begin{cases} \Box\phi^{(j)} = -\phi^{(j)}\partial^\alpha\phi^{(j)}\partial_\alpha\phi^{(j)} + \Pi_{\phi^{(j)}_\perp}(HWM(\phi^{(j-1)})) \\ \phi^{(j)}[0] = \phi[0] \end{cases}$$
on some interval $[0, T(\|\phi[0]\|_{B^s_{2,1}\times B^{s-1}_{2,1}})]$ which lies on the sphere and satisfies $\|\phi^{(j)}\|_{X_1^{s,\theta}} \leq 2C\|\phi[0]\|_{B^s_{2,1}\times B^{s-1}_{2,1}}$ ($j = 1$ is done).
4. Construct $\phi^{(J)}$ as the local solution to
$$\begin{cases} \Box\phi^{(J)} = -\phi^{(J)}\partial^\alpha\phi^{(J)}\partial_\alpha\phi^{(J)} + \tilde{\Pi}_{\phi^{(J)}_\perp}(HWM(\phi^{(J-1)})) \\ \phi^{(J)}[0] = \phi[0] \end{cases} \tag{2.10.1}$$
on the same time interval with $\|\phi^{(J)}\|_{X_1^{s,\theta}} \leq 2C\|\phi[0]\|_{B^s_{2,1}\times B^{s-1}_{2,1}}$. Here
$$\tilde{\Pi}_{\tilde{\phi}_\perp}(\phi) := \phi - (\phi\cdot g(\tilde{\phi}))g(\tilde{\phi}) \tag{2.10.2}$$
for g a smooth, compactly supported version of $\tilde{\phi}/\|\tilde{\phi}\|$ equal to that function for $\|\tilde{\phi}\| \simeq 1$ but vanishing in a neighbourhood of the origin. We make this modification since the subcritical argument assumes no smallness on $\phi^{(J)}$ and there is nothing to stop it crossing the origin, at which point the projection operator is non-smooth. Note that we must evaluate the half-wave maps terms in $\phi^{(J-1)}$ rather than $\phi^{(J)}$ in order to control this part of the forcing, since we do not yet know that $\phi^{(J)}$ lies on the sphere.
5. To close the iteration we must show that $\phi^{(J)}$ lies on the sphere. We can only show this for the true projection $\Pi_{\phi^{(J)}_\perp}$ (by the same argument as in [KS17]), as opposed

to the modified version above. We therefore need $\|\phi^{(J)} - p\|_\infty \ll 1$ so that the two projections coincide, and this follows from the assumed smallness of the data in the critical space. In particular, after constructing each iterate $\phi^{(J)}$ we use the main argument of this chapter (Sections 2.3-2.8) to show that the iterate remains small in the critical space, and so in $p + L^\infty_{t,x}$ as required.[13]

6. Having constructed the sequence $\phi^{(J)}$, we take the limit $J \to \infty$ in $X_1^{s,\theta}$ to obtain a solution $\phi \in X_1^{s,\theta}$ solving the half-wave maps equation on the interval $[0, T(\|\phi[0]\|_{B^s_{2,1}\times B^{s-1}_{2,1}})]$. The higher regularity of the solution follows by standard arguments.

To carry out the above argument, we must establish the necessary estimates for the iteration steps (2), (4) and (6). The following linear estimate shows that we must control the forcing in the space $X_1^{s-1,\theta-1}$.

Lemma 2.10.1 (Linear Estimate, see e.g. Theorems 2.9 and 2.10, [GG16])**.** *Fix $\phi[0] = (\phi_0, \phi_1) \in B^s_{2,1} \times B^{s-1}_{2,1}$ and define the solution operator*

$$\Phi(F) := p + \eta(t)\left(S_{-p}(\phi[0]) - \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(s)ds\right)$$

for

$$S_{-p}(\phi[0])(t) := \cos(t\sqrt{-\Delta})(\phi_0 - p) + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\phi_1$$

and $\eta : \mathbb{R} \to [0,1]$ a smooth, compactly supported cut-off function equal to 1 on $[-1,1]$. Hence $\Phi(F)$ solves the wave equation with data $\phi[0]$ and forcing F on the interval $[-T,T]$. It holds

$$\|\Phi(F)\|_{X_1^{s,\theta}} \lesssim \|\phi[0]\|_{B^s_{2,1}\times B^{s-1}_{2,1}} + \|F\|_{X_1^{s-1,\theta-1}}$$

Subcritical Multilinear Estimates

The following proposition contains the multilinear estimates needed for the iteration argument. We note that the result for the wave maps source terms is considered standard, however we were unable to find a proof in the literature including the necessary gain in T^ϵ, so we provide a proof for completeness.

Using this proposition in conjunction with the linear estimate Lemma 2.10.1 in the scheme outlined at the beginning of this section, the proof of local wellposedness is complete. Henceforth denote $s = 3/2 + s'$, $\theta = 1/2 + \theta'$ for $0 < \theta' < s' < \nu$.

[13] Two very minor adaptations are required in Sections 2.3-2.8 to handle the current situation. Firstly, (2.10.1) involves both $\phi^{(J)}$ and $\phi^{(J-1)}$, however this causes no issues since the wave maps and half-wave maps source terms are treated wholly independently in the main argument, and we may assume iteratively that (2.3.2) already holds for $\phi^{(J-1)}$. The second adaptation is that the terms HWM_2 are now accompanied by a projection which must be taken into account in the steps where one iterates the equation (e.g. in Section 2.6), however this is easily seen to be unproblematic using Moser estimates.

Proposition 2.10.2. *Fix $0 < T < 1$ and set $\eta_T(t) := \eta(T^{-1}t)$ for η as in Lemma 2.10.1. For ϕ, $\tilde\phi \in X_1^{s,\theta}$, define (suppressing the dependence on $\tilde\phi$),*

$$WM(\phi) = -\phi\partial^\alpha\phi\partial_\alpha\phi \tag{2.10.3}$$

$$HWM_1(\phi) = \widetilde{\Pi}_{\tilde\phi^\perp}((-\Delta)^{1/2}\phi)\,(\phi\cdot(-\Delta)^{1/2}\phi) \tag{2.10.4}$$

$$HWM_2(\phi) = \widetilde{\Pi}_{\tilde\phi^\perp}[\phi\times((-\Delta)^{1/2}(\phi\times(-\Delta)^{1/2}\phi) - \phi\times(-\Delta)\phi)] \tag{2.10.5}$$

with $\widetilde{\Pi}_{\tilde\phi_\perp}(\phi)$ as in (2.10.2)*. Then there exists $\epsilon(s',\theta') > 0$ and a function $C(\|\tilde\phi\|_{X_1^{s,\theta}})$ growing polynomially in $\|\tilde\phi\|_{X_1^{s,\theta}}$ such that for $\mathcal{T}$ any of the trilinear terms* (2.10.3)*-*(2.10.5) *it holds*

$$\|\eta_T\cdot\mathcal{T}(\phi)\|_{X_1^{s-1,\theta-1}} \lesssim C(\|\tilde\phi\|_{X_1^{s,\theta}})\,T^\epsilon(1+\|\phi\|_{X_1^{s,\theta}})\|\phi\|^2_{X_1^{s,\theta}} \tag{2.10.6}$$

We also have the difference estimates

$$\begin{aligned}
&\|\eta_T(\mathcal{T}(\phi^{(1)}) - \mathcal{T}(\phi^{(2)}))\|_{X_1^{s-1,\theta-1}}\\
&\quad\lesssim C(\|\tilde\phi^{(1)}\|_{X_1^{s,\theta}}, \|\tilde\phi^{(2)}\|_{X_1^{s,\theta}})T^\epsilon(\|\phi^{(1)}-\phi^{(2)}\|_{X_1^{s,\theta}}(1+\max_j\|\phi^{(j)}\|_{X_1^{s,\theta}})\max_j\|\phi^{(j)}\|_{X_1^{s,\theta}}\\
&\qquad\qquad + \|\tilde\phi^{(1)}-\tilde\phi^{(2)}\|_{X_1^{s,\theta}}(1+\max_j\|\phi^{(j)}\|_{X_1^{s,\theta}})\max_j\|\phi^{(j)}\|^2_{X_1^{s,\theta}})
\end{aligned}$$

for all i and a similar function C.

We restrict to proving the multilinear estimates (2.10.6), the difference estimates being similar. We will constantly use the following well-known transferred Strichartz estimate, see for example Proposition 26 [Bur20] for a proof.

Lemma 2.10.3 (Strichartz embedding)**.** *Let $p, q \geq 2$, $\frac{1}{p}+\frac{1}{q} \leq \frac{1}{2}$, $(p,q) \neq (2,\infty)$. Then for any $\theta > 1/2$, $s = 3/2 + s' > 3/2$ it holds*

$$\|P_k\phi\|_{p,q} \lesssim_{p,q} 2^{-(\frac{1}{p}+\frac{3}{q}+s')k}\|P_k\phi\|_{X^{s,\theta}} \tag{2.10.7}$$

for all $k > 0$.

It follows that the norm S^s defined as the $\ell^1_{k\geq 0}$ sum over

$$\|\phi_k\|_{S^s_k} := \max_{(p,q)} 2^{(\frac{1}{p}+\frac{3}{q}-1+s')k}\|\nabla_{t,x}\phi_k\|_{p,q}, \qquad \|\phi_0\|_{S^s_0} := \max_{\substack{(p,q),\\ \frac{1}{p}+\frac{3}{q}<1-s'}} \|\nabla_{t,x}\phi_0\|_{p,q} + \|\phi_0\|_{\infty,\infty}$$

is controlled by the $X_1^{s,\theta}$ norm whenever the maxima are taken over a finite number of standard Strichartz pairs (taking slight care including the ∇_t at high modulations):

$$\|\phi_k\|_{S^s_k} \lesssim \|\phi_k\|_{X^{s,\theta}}, \qquad \|\phi_0\|_{S^s_0} \leq 1 + \|\phi_0\|_{X^{s,\theta}}$$

The restriction on (p,q) in the low frequency case results from the fact that (2.10.7) holds only at frequencies localised away from the origin, while the 1 appears from the constant p which is implicit in the $X^{s,\theta}$ norm.

We start by recording the following key bilinear estimates. Such estimates first appeared in [KS97], however the reader may consult Theorems 2.11 and 2.12 of [GG16] for a textbook proof of (1)-(2). For the third estimate see Lemma 2.11, [Tao06].

Lemma 2.10.4 (Bilinear estimates)**.** *Let $s' > \theta' > 0$. Then the following hold [KS97]:*

1. $\|\varphi \cdot \phi\|_{X^{s,\theta}} \lesssim \|\varphi\|_{X^{s,\theta}} \|\phi\|_{X^{s,\theta}}$
2. $\|\varphi \cdot F\|_{X^{s-1,\theta-1}} \lesssim \|\varphi\|_{X^{s,\theta}} \|F\|_{X^{s-1,\theta-1}}$

Moreover, for $\tilde{s} \in \mathbb{R}$, $-\frac{1}{2} < \tilde{\theta} < \frac{1}{2}$, it holds

$$\|\eta_T \varphi\|_{X^{\tilde{s},\tilde{\theta}}} \lesssim_\eta \|\varphi\|_{X^{\tilde{s},\tilde{\theta}}}$$

uniformly in $T \in (0,1)$.

We also need estimates to control the projection from the half-wave maps terms in the $X^{s,\theta}$ spaces. We introduce the notation

$$Q_k F := \tilde{\mathcal{F}}^{-1}(\chi_k(||\tau| - |\xi||)\tilde{\mathcal{F}}(F)(\tau,\xi))$$

which decomposes the modulation of a function on dyadic scales. We again use $Q_0 F := \tilde{\mathcal{F}}^{-1}(\chi(||\tau| - |\xi||)\tilde{\mathcal{F}}(F)(\tau,\xi))$ to cover the low modulations. Observe the following modulation Bernstein-type estimate:

$$\|Q_j P_k \varphi\|_{p,q} \lesssim 2^{3(\frac{1}{2}-\frac{1}{q})k} 2^{(\frac{1}{2}-\frac{1}{p})j} \|P_k Q_j \varphi\|_{2,2} \qquad (p, q \geq 2) \tag{MB}$$

Lemma 2.10.5 (Subcritical projection estimate)**.** *Fix $0 < s' < 1/4$ and let ϕ, $\tilde{\phi} \in S^s$. Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be smooth with bounded derivatives and (p,q) an admissible pair. There exists a constant $C(\|\phi\|_{S^s})$ growing polynomially in $\|\phi\|_{S^s}$ such that for any $k > 0$ we have*

$$\|P_k g(\tilde{\phi})\|_{p,q} + 2^{-k}\|P_k \partial_t(g(\tilde{\phi}))\|_{p,q} \lesssim C(\|\tilde{\phi}\|_{S^s}) 2^{-(\frac{1}{p}+\frac{3}{q}+s')k} \tag{2.10.8}$$

and

$$\|P_k(\widetilde{\Pi}_{\tilde{\phi}^\perp}((-\Delta)^{1/2}\phi))\|_{p,q} \lesssim C(\|\tilde{\phi}\|_{S^s}) 2^{(1-\frac{1}{p}-\frac{3}{q}-s')k} \sum_{k_1 \geq 0} 2^{-\sigma|k-k_1|} \|P_{k_1}\phi\|_{S^s_{k_1}} \tag{2.10.9}$$

for some $\sigma(s',p,q) > 0$. The second estimate (2.10.9) *also holds for $k = 0$.*

We omit the proof of this Lemma which is analogous to that in the critical case, see Lemmas 2.5.7 and 2.5.8. From these estimates we can deduce similar bounds in the $X^{s,\theta}$ spaces, discussed further in Appendix 2.B.

Lemma 2.10.6. *Let $0 < s' < 1/4$, ϕ, $\tilde{\phi} \in X_1^{s,\theta}$. Then for $(j,k) \neq (0,0)$ we have the Moser inequality*

$$\|P_k Q_j g(\tilde{\phi})\|_{X^{s,\theta}} \lesssim C(\|\tilde{\phi}\|_{X_1^{s,\theta}}) \tag{2.10.10}$$

and the projection estimate

$$\begin{aligned} &\|P_k Q_j(\widetilde{\Pi}_{\tilde{\phi}^\perp}(-\Delta)^{1/2}\phi)\|_{X^{s,\theta}} \\ &\lesssim C(\|\tilde{\phi}\|_{X_1^{s,\theta}}) \left(2^k \sum_{k'\geq 0} 2^{-\sigma|k-k'|}\|\phi_{k'}\|_{X^{s,\theta}} + \delta_{j\gg k}\cdot 2^j \sum_{k'\gtrsim j} 2^{-\sigma|j-k'|}\|\phi_{k'}\|_{X^{s,\theta}} \right) \end{aligned} \tag{2.10.11}$$

which also holds for $(j,k) = (0,0)$. Here $\delta_{j\gg k} = 1$ if $j \geq k + 20$ and 0 otherwise, and $C(\|\tilde{\phi}\|_{X_1^{s,\theta}})$ is a constant which grows polynomially in $\|\tilde{\phi}\|_{X_1^{s,\theta}}$.

Furthermore, the projections are continuous on $X_1^{s-1,\theta-1}$: for j, $k \geq 0$ it holds

$$\|P_k Q_j \widetilde{\Pi}_{\tilde{\phi}^\perp} F\|_{X^{s-1,\theta-1}} \lesssim C(\|\tilde{\phi}\|_{X_1^{s,\theta}}) \sum_{r\geq 0}\sum_{l\geq 0} 2^{-\sigma(|r-k|+|l-j|)}\|P_r Q_l F\|_{X^{s-1,\theta-1}} \tag{2.10.12}$$

Remark 2.10.7. The continuity property (2.10.12) allows us to neglect the outer projection $\widetilde{\Pi}_{\tilde{\phi}^\perp}$ appearing in HWM_2 when proving (2.10.6). Henceforth, we therefore redefine

$$HWM_2(\phi) := \phi \times ((-\Delta)^{1/2}(\phi \times (-\Delta)^{1/2}\phi) - \phi \times (-\Delta)\phi)$$

To prove Proposition 2.10.2 we will use the frequency decompositions

$$\eta_T \cdot WM(\phi) = -\sum_{k_1,k_2,k_3\geq 0} \eta_T \underbrace{(\phi_{k_1}\partial^\alpha\phi_{k_2}\partial_\alpha\phi_{k_3})}_{WM_{k_1,k_2,k_3}(\phi)},$$

$$\eta_T \cdot HWM_1(\phi) = \sum_{k_1,k_2,k_3\geq 0} \eta_T \underbrace{(P_{k_2}(\widetilde{\Pi}_{\tilde{\phi}^\perp}(-\Delta)^{1/2}\phi)\,(\phi_{k_1}\cdot(-\Delta)^{1/2}\phi_{k_3}))}_{HWM_{1;k_1,k_2,k_3}(\phi)},$$

and

$$\eta_T \cdot HWM_2(\phi) = \sum_{k_1,k_2,k_3\geq 0} \eta_T \underbrace{(\phi_{k_1} \times [(-\Delta)^{1/2}(\phi_{k_2}\times(-\Delta)^{1/2}\phi_{k_3}) - \phi_{k_2}\times(-\Delta)\phi_{k_3}])}_{HWM_{2;k_1,k_2,k_3}(\phi)}$$

We will first deal with all but the (low)∇(high)∇(high) interactions, for which we can get by using Strichartz estimates, the gain in T coming from Hölder's inequality. For this we need the following straightforward bound which tells us that multiplication by a time cut-off does not affect the geometry of the interactions in a serious way.

Lemma 2.10.8. *Let $2 \leq p \leq \infty$, $l \geq 0$. Then it holds*

$$\|P_l^{(t)}\eta_T\|_{L_t^p} \lesssim_{\eta,N} T^{1/p}(2^l T)^{-N}$$

for any $N > 0$. Here $P_l^{(t)}$ is the projection to temporal frequency $\sim 2^l$.

Most of the frequency interactions are then handled in the following proposition.

Proposition 2.10.9. *Fix $T > 0$. Set*

$$\mathcal{S}_* := \{(k_1, k_2, k_3) \in \mathbb{N}^3_{\geq 0} : 2^{k_2}T > 1,\ 2^{k_3}T > 1,\ k_1 < \max\{k_2, k_3\} - 10\}$$

Then there exist $\epsilon > 0$ such that for any ϕ, $\tilde{\phi} \in X_1^{s,\theta}$ it holds

$$\|\sum_{(k_1,k_2,k_3)\notin\mathcal{S}_*} \eta_T \cdot \mathcal{T}_{k_1,k_2,k_3}(\phi)\|_{X_1^{s-1,\theta-1}} \lesssim C(\|\tilde{\phi}\|_{X_1^{s,\theta}})\, T^\epsilon (1 + \|\phi\|_{X_1^{s,\theta}})\|\phi\|^2_{X_1^{s,\theta}}$$

for $\mathcal{T} \in \{WM, HWM_1, HWM_2\}$ and $C(\|\tilde{\phi}\|_{X_1^{s,\theta}})$ as in Lemma 2.10.11.

Proof. Throughout this proof any implicit constants may depend polynomially on $\|\tilde{\phi}\|_{X_1^{s,\theta}}$. We will only prove the estimates for the wave maps terms, WM, the other terms being entirely analogous using Lemma 2.10.6 and the fact that

$$\|(-\Delta)^{1/2}(\phi_{k_2} \times (-\Delta)^{1/2}\phi_{k_3}) - \phi_{k_2} \times (-\Delta)\phi_{k_3}\|_{p,q} \lesssim 2^{k_2+k_3}\|\phi_{k_2}\|_{p_1,q_1}\|\phi_{k_3}\|_{p_2,q_2}$$

for all k_2, $k_3 \geq 0$ (see Lemma 2.5.1) and conjugate triples $p^{-1} = p_1^{-1} + p_2^{-1}$, $q^{-1} = q_1^{-1} + q_2^{-1}$.

Fix $k \geq 0$ and consider

$$\sum_{(k_1,k_2,k_3)\notin\mathcal{S}_*} \|P_k(\eta_T \cdot WM_{k_1,k_2,k_3}(\phi))\|_{X^{s-1,\theta-1}}$$

We start with the case $k_1 \geq \max\{k_2, k_3\} - 10$. Note that in this case the whole term is restricted to frequency $P_{\lesssim k_1}$ so we must have $k_1 \gtrsim k$. We then consider different cases for the modulation.

- $P_{\lesssim k_1}Q_{\lesssim k_1}$: Using that $k \lesssim k_1$ followed by the modulation-Bernstein estimate and Hölder's inequality we have

$$\begin{aligned}
&\|P_kQ_{\lesssim k_1}(\eta_T\phi_{k_1}\partial^\alpha\phi_{k_2}\partial_\alpha\phi_{k_3})\|_{X^{s-1,\theta-1}}\\
&\lesssim 2^{(s-1)k_1}\sum_{j\lesssim k_1} 2^{(\theta-1)j}2^{j/2}\|\eta_T\|_M\|\phi_{k_1}\|_{\infty,2}\|\partial^\alpha\phi_{k_2}\|_{\frac{2M}{M-1},\infty}\|\partial_\alpha\phi_{k_3}\|_{\frac{2M}{M-1},\infty}\\
&\lesssim T^{\frac{1}{M}}2^{(s-1)k_1}2^{\theta' k_1}\cdot 2^{-sk_1}\|\phi_{k_1}\|_{X^{s,\theta}}\cdot 2^{(\frac{1}{2}+\frac{1}{2M}-s')(k_2+k_3)}\|\phi_{k_2}\|_{X^{s,\theta}}\|\phi_{k_3}\|_{X^{s,\theta}}\\
&\lesssim T^{\frac{1}{M}}2^{-(2s'-\theta'-\frac{1}{M})k_1}\|\phi_{k_1}\|_{X^{s,\theta}}\|\phi_{k_2}\|_{X^{s,\theta}}\|\phi_{k_3}\|_{X^{s,\theta}}
\end{aligned}$$

  We may then take, for example, $\frac{1}{M} = \theta'$, $s' > \theta'$ so bound this by

$$T^{\theta'}2^{-2(s'-\theta')(k_1-k)}2^{-2(s'-\theta')k}\|\phi_{k_1}\|_{X^{s,\theta}}\|\phi_{k_2}\|_{X^{s,\theta}}\|\phi_{k_3}\|_{X^{s,\theta}}$$

  which is summable over $k_2\,k_3 \leq k_1 - 10$, $k_1 \gtrsim k$, $k \geq 0$ as required.

- $\sum_{l\gg k_1} P_{\lesssim k_1}Q_l$: In this case one of the four factors must be at modulation (or frequency in the case of η_T) at least comparable to 2^l. We study each option separately.

  1. $(P_{\gtrsim l}\eta_T)\,\phi_{k_1}\partial^\alpha\phi_{k_2}\partial_\alpha\phi_{k_3}$: Here we use Lemma 2.10.8 to see that

$$\begin{aligned}
&\sum_{l\gg k_1}\|P_kQ_l((P^{(t)}_{\gtrsim l}\eta_T)\phi_{k_1}\partial^\alpha\phi_{k_2}\partial_\alpha\phi_{k_3})\|_{X^{s-1,\theta-1}}\\
&\lesssim \sum_{l\gg k_1} 2^{(s'+\theta')l}\|P^{(t)}_{\gtrsim l}\eta_T\|_M\|\phi_{k_1}\|_{\infty,2}\|\partial^\alpha\phi_{k_2}\|_{\frac{2M}{M-2},\infty}\|\partial^\alpha\phi_{k_3}\|_{\infty,\infty}\\
&\lesssim \sum_{l\gg k_1} 2^{(s'+\theta')l}\cdot T^{1/M}(2^lT)^{-N}\cdot 2^{-sk_1}\|\phi_{k_1}\|_{X^{s,\theta}}\cdot 2^{(\frac12+\frac1M-s')k_2}\|\phi_{k_2}\|_{X^{s,\theta}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\cdot 2^{(1-s')k_3}\|\phi_{k_3}\|_{X^{s,\theta}}
\end{aligned}$$

  We may then take, e.g., $N=s'+\theta'+\frac{1}{2M}$ and $1/2M=s'+2\theta'$ to bound this by

$$\begin{aligned}
&T^{\frac{1}{2M}-s'-\theta'}2^{-k_1/2M}2^{-sk_1}2^{(\frac12+\frac1M-s')k_1}2^{(1-s')k_1}\|\phi_{k_1}\|_{X^{s,\theta}}\|\phi_{k_2}\|_{X^{s,\theta}}\|\phi_{k_3}\|_{X^{s,\theta}}\\
&\lesssim T^{\theta'}2^{-2(s'-\theta')(k_1-k)}2^{-2(s'-\theta')k}\|\phi_{k_1}\|_{X^{s,\theta}}\|\phi_{k_2}\|_{X^{s,\theta}}\|\phi_{k_3}\|_{X^{s,\theta}}
\end{aligned}$$

  which is acceptable.

  2. $\eta_T\,(Q_{\gtrsim l}\phi_{k_1})\partial^\alpha\phi_{k_2}\partial_\alpha\phi_{k_3}$: This time we place $Q_{\gtrsim l}\phi_{k_1}$ directly into $X^{s,\theta}$ to find

$$\begin{aligned}
&\sum_{l\gg k_1}\|P_kQ_l(\eta_T(Q_{\gtrsim l}\phi_{k_1})\partial^\alpha\phi_{k_2}\partial_\alpha\phi_{k_3})\|_{X^{s-1,\theta-1}}\\
&\lesssim \sum_{l\gg k_1} 2^{(s'+\theta')l}\|\eta_T\|_M\|Q_{\gtrsim l}\phi_{k_1}\|_{\frac{2M}{M-2},2}\|\partial^\alpha\phi_{k_2}\|_{\infty,\infty}\|\partial^\alpha\phi_{k_3}\|_{\infty,\infty}\\
&\lesssim T^{1/M}\sum_{l\gg k_1} 2^{(s'+\theta')l}2^{l/M}2^{-\theta l}2^{-sl}\|\phi_{k_1}\|_{X^{s,\theta}}\cdot 2^{(1-s')k_2}\|\phi_{k_2}\|_{X^{s,\theta}}\cdot 2^{(1-s')k_3}\|\phi_{k_3}\|_{X^{s,\theta}}\\
&\lesssim T^{1/M}2^{(\frac1M-2s')k_1}\|\phi_{k_1}\|_{X^{s,\theta}}\|\phi_{k_2}\|_{X^{s,\theta}}\|\phi_{k_3}\|_{X^{s,\theta}}
\end{aligned}$$

  which is acceptable choosing $1/M=2\theta'$.

  3. $\eta_T\,\phi_{k_1}(Q_{\gtrsim l}\partial^\alpha\phi_{k_2})\partial_\alpha\phi_{k_3}$:

$$\begin{aligned}
&\sum_{l\gg k_1}\|P_kQ_l(\eta_T\,\phi_{k_1}(Q_{\gtrsim l}\partial^\alpha\phi_{k_2})\partial_\alpha\phi_{k_3})\|_{X^{s-1,\theta-1}}\\
&\lesssim \sum_{l\gg k_1} 2^{(s'+\theta')l}\|\eta_T\|_M\|\phi_{k_1}\|_{\infty,\infty}\|Q_{\gtrsim l}\partial^\alpha\phi_{k_2}\|_{\frac{2M}{M-2},2}\|\partial^\alpha\phi_{k_3}\|_{\infty,\infty}\\
&\lesssim T^{1/M}\sum_{l\gg k_1} 2^{(s'+\theta')l}(\delta_{k_1,0}+\|\phi_{k_1}\|_{X^{s,\theta}})\cdot 2^{(\frac1M-\theta+1-s)l}\|\phi_{k_2}\|_{X^{s,\theta}}\cdot 2^{(1-s')k_3}\|\phi_{k_3}\|_{X^{s,\theta}}\\
&\lesssim T^{1/M}2^{(\frac1M-s')k_1}(\delta_{k_1,0}+\|\phi\|_{X^{s,\theta}})\|\phi_{k_2}\|_{X^{s,\theta}}\|\phi_{k_3}\|_{X^{s,\theta}}
\end{aligned}\tag{2.10.13}$$

  which is acceptable choosing e.g. $1/M=\theta'$.

4. $\eta_T \phi_{k_1} \partial^\alpha \phi_{k_2} (Q_{\gtrsim l} \partial_\alpha \phi_{k_3})$: as above.

This completes the study of the case $k_1 \geq \max\{k_2, k_3\} - 10$.

We next turn to the case $k_1 < \max\{k_2, k_3\} - 10$. WLOG $k_2 \geq k_3$. This time the whole term is at frequency $\lesssim 2^{k_2}$ and we must have $k \lesssim k_2$. We first study the case where the whole term has large modulation,

$$\sum_{l \gg k_2} \| P_{\lesssim k_2} Q_l (\eta_T \phi_{k_1} \partial^\alpha \phi_{k_2} \partial_\alpha \phi_{k_3}) \|_{X^{s-1,\theta-1}}$$

Again, one of the factors must have modulation of order at least 2^l, so we have four cases to consider.

- $\sum_{l \gg k_2} P_{\lesssim k_2} Q_l$:

1. $P^{(t)}_{\gtrsim l} \eta_T$: In this case we again use Lemma 2.10.8 to see that

$$\begin{aligned}
&\sum_{l \gg k_2} \| P_k Q_l ((P^{(t)}_{\gtrsim l} \eta_T) \phi_{k_1} \partial^\alpha \phi_{k_2} \partial_\alpha \phi_{k_3}) \|_{X^{s-1,\theta-1}} \\
&\lesssim \sum_{l \gg k_2} 2^{(s'+\theta')l} \| P^{(t)}_{\gtrsim l} \eta_T \|_M \| \phi_{k_1} \|_{\infty,\infty} \| \partial^\alpha \phi_{k_2} \|_{\infty,2} \| \partial^\alpha \phi_{k_3} \|_{\frac{2M}{M-2},\infty} \\
&\lesssim \sum_{l \gg k_2} 2^{(s'+\theta')l} T^{1/M} (2^l T)^{-N} 2^{(1-s)k_2} 2^{(\frac{1}{2}+\frac{1}{M}-s')k_3} (\delta_{k_1,0} + \| \phi_{k_1} \|_{X^{s,\theta}}) \| \phi_{k_2} \|_{X^{s,\theta}} \\
&\qquad \cdot \| \phi_{k_3} \|_{X^{s,\theta}} \\
&\lesssim T^{\frac{1}{2M}-s'-\theta'} 2^{(\frac{1}{2M}-2s')k_2} \| \phi_{k_1} \|_{X^{s,\theta}} \| \phi_{k_2} \|_{X^{s,\theta}} \| \phi_{k_3} \|_{X^{s,\theta}}
\end{aligned}$$

where we again used $N = \frac{1}{2M} + s' + \theta'$ and $k_3 \leq k_2$. Choosing M such that $s' + \theta' < \frac{1}{2M} < 2s'$ we obtain the result.

2. $Q_{\gtrsim l} \phi_{k_1}$: This is a direct application of Hölder's inequality. Placing all three factors of ϕ into Strichartz spaces we have

$$\begin{aligned}
&\sum_{l \gg k_2} \| P_k Q_l (\eta_T (Q_{\gtrsim l} \phi_{k_1}) \partial^\alpha \phi_{k_2} \partial_\alpha \phi_{k_3}) \|_{X^{s-1,\theta-1}} \\
&\lesssim \sum_{l \gg k_2} 2^{(s'+\theta')l} \| \eta_T \|_M \| Q_{\gtrsim l} \phi_{k_1} \|_{\frac{2M}{M-2},\infty} \| \partial^\alpha \phi_{k_2} \|_{\infty,2} \| \partial^\alpha \phi_{k_3} \|_{\infty,\infty} \\
&\lesssim T^{1/M} \sum_{l \gg k_2} 2^{(s'+\theta')l} \cdot 2^{3k_1/2} 2^{(\frac{1}{M}-\theta)l} 2^{-sl} \| \phi_{k_1} \|_{X^{s,\theta}} \cdot 2^{(1-s)k_2} \| \phi_{k_2} \|_{X^{s,\theta}} \\
&\qquad \cdot 2^{(1-s')k_3} \| \phi_{k_3} \|_{X^{s,\theta}} \\
&\lesssim T^{1/M} 2^{(\frac{1}{M}-2s')k_2} \| \phi_{k_1} \|_{X^{s,\theta}} \| \phi_{k_2} \|_{X^{s,\theta}} \| \phi_{k_3} \|_{X^{s,\theta}}
\end{aligned}$$

which is acceptable for e.g. $1/M = 2\theta'$.

3. $Q_{\gtrsim l}\partial^\alpha \phi_{k_2}$:

$$
\begin{aligned}
&\sum_{l \gg k_2} \|P_k Q_l(\eta_T \phi_{k_1}(Q_{\gtrsim l}\partial^\alpha \phi_{k_2})\partial_\alpha \phi_{k_3})\|_{X^{s-1,\theta-1}} \\
&\lesssim \sum_{l \gg k_2} 2^{(s'+\theta')l}\|\eta_T\|_M \|\phi_{k_1}\|_{\infty,\infty}\|Q_{\gtrsim l}\partial^\alpha \phi_{k_2}\|_{\frac{2M}{M-2},2}\|\partial^\alpha \phi_{k_3}\|_{\infty,\infty} \\
&\lesssim T^{1/M}\sum_{l \gg k_2} 2^{(s'+\theta')l}\cdot(\delta_{k_1,0}+\|\phi_{k_1}\|_{X^{s,\theta}})\cdot 2^{(\frac{1}{M}-\theta)l}2^{(1-s)l}\|\phi_{k_2}\|_{X^{s,\theta}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot 2^{(1-s')k_3}\|\phi_{k_3}\|_{X^{s,\theta}} \\
&\lesssim T^{1/M}2^{(\frac{1}{M}-s')k_2}(\delta_{k_1,0}+\|\phi_{k_1}\|_{X^{s,\theta}})\|\phi_{k_2}\|_{X^{s,\theta}}\|\phi_{k_3}\|_{X^{s,\theta}}
\end{aligned}
$$

which is acceptable for $\frac{1}{M} < s'$.

4. $Q_{\gtrsim l}\partial_\alpha \phi_{k_3}$:

$$
\begin{aligned}
&\sum_{l \gg k_2} \|P_k Q_l(\eta_T \phi_{k_1}\partial^\alpha \phi_{k_2}(Q_{\gtrsim l}\partial_\alpha \phi_{k_3}))\|_{X^{s-1,\theta-1}} \\
&\lesssim \sum_{l \gg k_2} 2^{(s'+\theta')l}\|\eta_T\|_M \|\phi_{k_1}\|_{\infty,\infty}\|\partial^\alpha \phi_{k_2}\|_{\infty,2}\|Q_{\gtrsim l}\partial^\alpha \phi_{k_3}\|_{\frac{2M}{M-2},\infty} \\
&\lesssim T^{1/M}\sum_{l \gg k_2} 2^{(s'+\theta')l}\cdot(\delta_{k_1,0}+\|\phi_{k_1}\|_{X^{s,\theta}})\cdot 2^{(1-s)k_2}\|\phi_{k_2}\|_{X^{s,\theta}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot 2^{3k_3/2}2^{(\frac{1}{M}-\theta)l}2^{(1-s)l}\|\phi_{k_3}\|_{X^{s,\theta}} \\
&\lesssim T^{1/M}2^{(\frac{1}{M}-s')k_2}(\delta_{k_1,0}+\|\phi_{k_1}\|_{X^{s,\theta}})\|\phi_{k_2}\|_{X^{s,\theta}}\|\phi_{k_3}\|_{X^{s,\theta}}
\end{aligned}
$$

which is acceptable for $\frac{1}{M} < s'$.

- $P_{\lesssim k_2}Q_{\lesssim k_2}$: It remains to study the term with overall modulation restricted to $\lesssim 2^{k_2}$. We consider the cases $2^{k_2}T \le 1$ and $2^{k_2}T > 1$ separately. The latter case we further split into $2^{k_3}T \le 1$ and $2^{k_3}T > 1$.

  1. $2^{k_2}T \le 1$: Here we use the modulation Bernstein estimate followed by Bernstein's inequality to bound (for $k \lesssim k_2$)

$$
\begin{aligned}
&\|P_k Q_{\lesssim k_2}(\eta_T \phi_{k_1}\partial^\alpha \phi_{k_2}\partial_\alpha \phi_{k_3})\|_{X^{s-1,\theta-1}} \\
&\lesssim \sum_{l \lesssim k_2} 2^{(s-1)k_2}2^{(\theta-1)l}2^{l/2}\|\eta_T\|_1\|\phi_{k_1}\|_{\infty,\infty}\|\partial^\alpha \phi_{k_2}\|_{\infty,2}\|\partial_\alpha \phi_{k_3}\|_{\infty,\infty} \\
&\lesssim T\sum_{l \lesssim k_2} 2^{(s-1)k_2}2^{\theta' l}2^{(1-s)k_2}2^{(1-s')k_3}(\delta_{k_1,0}+\|\phi_{k_1}\|_{X^{s,\theta}})\|\phi_{k_2}\|_{X^{s,\theta}}\|\phi_{k_3}\|_{X^{s,\theta}} \\
&\lesssim T2^{(1+\theta'-s')k_2}(\delta_{k_1,0}+\|\phi_{k_1}\|_{X^{s,\theta}})\|\phi_{k_2}\|_{X^{s,\theta}}\|\phi_{k_3}\|_{X^{s,\theta}}
\end{aligned}
$$

  We then use that s', θ' are very small and separate $2^{(1+\theta'-s')k_2}$ into $2^{(1+\frac{1}{2}(\theta'-s'))k_2}2^{\frac{1}{2}(\theta'-s')k_2}$. Since $2^{k_2} \le T^{-1}$ this allows us to bound the previous line by

$$
\lesssim T^{\frac{1}{2}(s'-\theta')}2^{\frac{1}{2}(\theta'-s')k_2}(\delta_{k_1,0}+\|\phi_{k_1}\|_{X^{s,\theta}})\|\phi_{k_2}\|_{X^{s,\theta}}\|\phi_{k_3}\|_{X^{s,\theta}}
$$

which is acceptable since $s' > \theta'$.

2. $2^{k_2}T > 1$, $2^{k_3}T \leq 1$: In this case we start by using (MB) to lower the time exponent from 2 to 1+, then place all three factors of ϕ into Strichartz spaces:

$$\begin{aligned}
&\|P_k Q_{\lesssim k_2}(\eta_T \phi_{k_1} \partial^\alpha \phi_{k_2} \partial_\alpha \phi_{k_3})\|_{X^{s-1,\theta-1}} \\
&\lesssim 2^{(s-1)k_2} \sum_{l \lesssim k_2} 2^{(\theta-1)l} 2^{(\frac{1}{2}-\frac{1}{M})l} \|\eta_T\|_{\frac{2M}{M-1}} \|\phi_{k_1}\|_{\infty,\infty} \|\partial^\alpha \phi_{k_2}\|_{\infty,2} \|\partial_\alpha \phi_{k_3}\|_{\frac{2M}{M-1},\infty} \\
&\lesssim T^{\frac{1}{2}-\frac{1}{2M}} 2^{(\frac{1}{2}+\frac{1}{2M}-s')k_3} (\delta_{k_1,0} + \|\phi_{k_1}\|_{X^{s,\theta}}) \|\phi_{k_2}\|_{X^{s,\theta}} \|\phi_{k_3}\|_{X^{s,\theta}}
\end{aligned}$$

where we chose $1/M > \theta'$ and summed over $l \geq 0$. In order to gain some decay in k we need to split into two further sub-cases. Henceforth assume $s' + \theta' > 2/M > 2\theta'$.

(a) $k_2 \simeq k$: In this case, we simply bound $2^{k_3} \leq T^{-1}$ to find

$$\begin{aligned}
&\|P_k Q_{\lesssim k_2}(\eta_T \phi_{k_1} \partial^\alpha \phi_{k_2} \partial_\alpha \phi_{k_3})\|_{X^{s-1,\theta-1}} \\
&\lesssim T^{s'-\frac{1}{M}} (\delta_{k_1,0} + \|\phi_{k_1}\|_{X^{s,\theta}}) \|\phi_{k_2}\|_{X^{s,\theta}} \|\phi_{k_3}\|_{X^{s,\theta}}
\end{aligned}$$

which is acceptable.

(b) $k_2 \gg k$. Since $k_1 \leq k_2 - 10$, we must in this case have $k_3 \simeq k_2$. We find

$$\begin{aligned}
&\|P_k Q_{\lesssim k_2}(\eta_T \phi_{k_1} \partial^\alpha \phi_{k_2} \partial_\alpha \phi_{k_3})\|_{X^{s-1,\theta-1}} \\
&\lesssim T^{\frac{1}{2}-\frac{1}{2M}} 2^{(\frac{1}{2}+\frac{1}{2M}-\frac{1}{2}(s'+\theta'))k_3} 2^{-\frac{1}{2}(s'-\theta')k_3} (\delta_{k_1,0} + \|\phi_{k_1}\|_{X^{s,\theta}}) \|\phi_{k_2}\|_{X^{s,\theta}} \|\phi_{k_3}\|_{X^{s,\theta}} \\
&\lesssim T^{\frac{1}{2}(s'+\theta')-\frac{1}{M}} 2^{-\frac{1}{2}(s'-\theta')k_2} (\delta_{k_1,0} + \|\phi_{k_1}\|_{X^{s,\theta}}) \|\phi_{k_2}\|_{X^{s,\theta}} \|\phi_{k_3}\|_{X^{s,\theta}}
\end{aligned}$$

where we used that $2^{-\frac{1}{2}(s'-\theta')k_3} \simeq^{-\frac{1}{2}(s'-\theta')k_2}$ for the final inequality.

The remaining case $k_1 < \max\{k_2, k_3\} - 10$, $2^{k_2}T$, $2^{k_3}T >$, corresponds to a triple in $\mathcal{S}_*$ so the proof for WM is complete. $\square$

To handle the remaining (low)∇(high)∇(high) interactions we must incorporate the structures in the different terms. For the wave maps source terms we will use the following lemma, proved in Appendix 2.C.

Lemma 2.10.10. *Set $s = 3/2 + s'$, $\theta = 1/2 + \theta'$ for $\nu > s' > \theta' > 0$. Let k_2, $k_3 \geq 0$. It holds*

$$\|\varphi_{k_2} \cdot F_{k_3}\|_{X^{s-1,\theta-1}} \lesssim 2^{-s' \min\{k_2,k_3\}} \|\varphi_{k_2}\|_{X^{s,\theta}} \|F_{k_3}\|_{X^{s-1,\theta-1}} \tag{2.10.14}$$

and

$$\|\varphi^{(2)}_{k_2} \cdot \varphi^{(3)}_{k_3}\|_{X^{s,\theta}} \lesssim 2^{-s' \min\{k_2,k_3\}} \|\varphi^{(2)}_{k_2}\|_{X^{s,\theta}} \|\varphi^{(3)}_{k_3}\|_{X^{s,\theta}} \tag{2.10.15}$$

The remaining interactions are then handled in the following proposition.

Proposition 2.10.11. *Let $(k_1, k_2, k_3) \in \mathcal{S}_*$. Then for any ϕ, $\tilde{\phi} \in X_1^{s,\theta}$ it holds*

$$\| \sum_{(k_1,k_2,k_3)\in\mathcal{S}_*} \eta_T \cdot \mathcal{T}_{k_1,k_2,k_3}(\phi)\|_{X_1^{s-1,\theta-1}} \lesssim C(\|\tilde{\phi}\|_{X_1^{s,\theta}})T^{\epsilon}\|\phi\|^3_{X_1^{s,\theta}}$$

for $\mathcal{T} \in \{WM, HWM_1, HWM_2\}$ and $C(\|\tilde{\phi}\|_{X_1^{s,\theta}})$ a constant as in the previous proposition.

Proof. We start with WM, again taking $k_2 \geq k_3$ without loss of generality, so that $k_1 < k_2 - 10$ and $2^{k_2}T$, $2^{k_3}T > 1$. Use the null structure to write

$$\partial^{\alpha}\phi_{k_2} \cdot \partial_{\alpha}\phi_{k_3} = \frac{1}{2}[\Box(\phi_{k_2} \cdot \phi_{k_3}) - \phi_{k_2} \cdot \Box\phi_{k_3} - \Box\phi_{k_2} \cdot \phi_{k_3}]$$

First consider $\|\eta_T(\phi_{k_1}\Box(\phi_{k_2} \cdot \phi_{k_3}))\|_{X_1^{s-1,\theta-1}}$. Note that we may neglect the cut-off η_T thanks to Lemma 2.10.4. By point 2 of Lemma 2.10.4 followed by the definition of the $X^{s,\theta}$ space and Lemma 2.10.10 we have

$$\begin{aligned}\|P_k(\phi_{k_1}\Box(\phi_{k_2} \cdot \phi_{k_3}))\|_{X^{s-1,\theta-1}} &\lesssim \|\phi_{k_1}\|_{X^{s,\theta}}\|\Box(\phi_{k_2} \cdot \phi_{k_3})\|_{X^{s-1,\theta-1}} \\ &\lesssim \|\phi_{k_1}\|_{X^{s,\theta}}\|\phi_{k_2} \cdot \phi_{k_3}\|_{X^{s,\theta}} \\ &\lesssim 2^{-s'k_3}\|\phi_{k_1}\|_{X^{s,\theta}}\|\phi_{k_2}\|_{X^{s,\theta}}\|\phi_{k_3}\|_{X^{s,\theta}}\end{aligned}$$

If $k_2 \simeq k$ we bound this by $T^{s'}\|\phi_{k_1}\|_{X^{s,\theta}}\|\phi_{k_2}\|_{X^{s,\theta}}\|\phi_{k_3}\|_{X^{s,\theta}}$ which is acceptable, and if $k_2 \gg k$, we know (since $k_1 \ll k_2$) that $k_3 \gtrsim k_2$ so we can bound this by

$$T^{s'/2}2^{-s'(k_3-k)/2}\|\phi_{k_1}\|_{X^{s,\theta}}\|\phi_{k_2}\|_{X^{s,\theta}}\|\phi_{k_3}\|_{X^{s,\theta}}$$

which is also fine.

Similarly for $\|\phi_{k_1}\phi_{k_2} \cdot \Box\phi_{k_3}\|_{X^{s-1,\theta-1}}$ we use Lemma 2.10.10 to bound

$$\|P_k(\phi_{k_1}\phi_{k_2} \cdot \Box\phi_{k_3})\|_{X^{s-1,\theta-1}} \lesssim 2^{-s'k_3}\|\phi_{k_1}\|_{X^{s,\theta}}\|\phi_{k_2}\|_{X^{s,\theta}}\|\phi_{k_3}\|_{X^{s,\theta}}$$

which is acceptable for the same reasons. The remaining term is similar.

For HWM_1 and HWM_2 we don't actually need to use that $2^{k_2}T$, $2^{k_3}T > 1$ and we can get the gain we need from Hölder's inequality. Let's start with HWM_1 in the case $k_3 \geq k_2$, so $k_1 < k_3 - 10$. Note that the high modulation case $||\tau| - |\xi|| \gg k_3$ was handled in the previous proof, so we only have to consider low modulations. Since k_3 is the highest frequency the output is restricted to $2^k \lesssim 2^{k_3}$, so for fixed k_2, k_3 we have

$$\begin{aligned}&\| \sum_{k_1<k_3-10} Q_{\lesssim k_3}P_k(\eta_T \cdot HWM_{1;k_1,k_2,k_3}(\phi))\|_{X^{s-1,\theta-1}} \\ &\lesssim \sum_{l\lesssim k_3} 2^{(\theta-1)l}2^{(s-1)k_3}\|Q_lP_k(\eta_TP_{k_2}(\widetilde{\Pi}_{\tilde{\phi}^{\perp}}(-\Delta)^{1/2}\phi)(\phi_{<k_3-10} \cdot (-\Delta)^{1/2}\phi_{k_3}))\|_{2,2}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{l \lesssim k_3} 2^{(\theta-1)l} 2^{(s-1)k_3} \| Q_l P_k (\eta_T P_{k_2} (\widetilde{\Pi}_{\tilde{\phi}^\perp} (-\Delta)^{1/2} \phi) (\phi_{<k_3-10} \cdot (-\Delta)^{1/2} \phi_{k_3} \\
&\qquad\qquad - (-\Delta)^{1/2} P_{k_3} (\phi_{<k_3-10} \cdot \phi_{\geq k_3-10})) \|_{2,2} \\
&\quad + \sum_{l \lesssim k_3} 2^{(\theta-1)l} 2^{(s-1)k_3} \| Q_l P_k (\eta_T P_{k_2} (\widetilde{\Pi}_{\tilde{\phi}^\perp} (-\Delta)^{1/2} \phi) (-\Delta)^{1/2} P_{k_3} (\phi_{<k_3-10} \cdot \phi_{\geq k_3-10})) \|_{2,2}
\end{aligned}
$$

The first term above sees a derivative moved onto the low frequency factor $\phi_{<k_3-10}$ (see Lemmas 2.5.1 and 2.5.9), so is easier to handle. For the third line we use the geometric identity (GeId) to swap the low frequency factor for a high one and find

$$
\begin{aligned}
&\sum_{l \lesssim k_3} 2^{(\theta-1)l} 2^{(s-1)k_3} \| Q_l P_k (\eta_T P_{k_2} (\widetilde{\Pi}_{\tilde{\phi}^\perp} (-\Delta)^{1/2} \phi) (-\Delta)^{1/2} P_{k_3} (\phi_{<k_3-10} \cdot \phi_{\geq k_3-10})) \|_{2,2} \\
&\lesssim \sum_{l \lesssim k_3} 2^{(\theta-1)l} 2^{(s-1)k_3} 2^{l(\frac{1}{2}-\frac{1}{M})} \|\eta_T\|_M \| P_{k_2} (\widetilde{\Pi}_{\tilde{\phi}^\perp} (-\Delta)^{1/2} \phi) \|_{\frac{2M}{M-4},\infty} \\
&\qquad\qquad \cdot 2^{k_3} \|\phi_{\geq k_3-10}\|_{2M,\frac{2M}{M-1}} \|\phi_{\geq k_3-10}\|_{\frac{2M}{M-1},2M} \\
&\lesssim T^{1/M} 2^{(\frac{1}{2}+\frac{2}{M}-s')(k_2-k_3)} 2^{2(\frac{1}{M}-s')(k_3-k)} 2^{2(\frac{1}{M}-s')k} \|\phi\|^2_{X_1^{s,\theta}} \sum_{k' \geq 0} 2^{-\sigma|k_2-k'|} \|\phi_{k'}\|_{X^{s,\theta}}
\end{aligned}
$$

where we chose M such that $\theta' < M^{-1} < s'$. This can be summed over $k_2 \leq k_3$, $k_3 \gtrsim k$ and $k \geq 0$ as required.

The case $k_2 > k_3$ is similar, with the exception that we must separately study $k_1 < k_3 - 10$ and $k_1 \in [k_3 - 10, k_2 - 10]$ in order to apply (GeId).

Finally, we turn to HWM_2, again restricting to modulation $\lesssim 2^{\max\{k_2,k_3\}}$. We first consider $k_3 \geq k_2 + 10$, in which case we must have output frequency $k \sim k_3$ and can write $HWM_{2;k_1,k_2,k_3}(\phi) = \phi_{k_1} \times \mathcal{L}_{k_2+k_3}(\phi_{k_2}, \phi_{k_3})$ for $\mathcal{L}$ as in (2.5.1). We therefore have

$$
\begin{aligned}
&\| \sum_{k_1 < k_3-10} Q_{\lesssim k_3} P_k (\eta_T \cdot HWM_{2;k_1,k_2,k_3}(\phi)) \|_{X^{s-1,\theta-1}} \\
&\lesssim \sum_{l \lesssim k_3} 2^{(\theta-1)l} 2^{(s-1)k_3} 2^{(\frac{1}{2}-\frac{1}{M})l} \| \eta_T \, \phi_{<k_3-10} \times \mathcal{L}_{k_2+k_3}(\phi_{k_2}, \phi_{k_3}) \|_{\frac{M}{M-1},2} \\
&\lesssim T^{1/M} \sum_{l \lesssim k_3} \sum_{a,b} 2^{(\theta'-\frac{1}{M})l} 2^{(s-1)k_3} c_{a,b}^{(k_2+k_3)} (\| \phi_{k_2}(x + 2^{-k_2}a) \, \phi_{<k_3-10}(x) \cdot \phi_{k_3}(x + 2^{-k_3}b) \|_{\frac{M}{M-2},2} \\
&\qquad\qquad + \| \phi_{k_3}(x + 2^{-k_3}b) \, \phi_{<k_3-10}(x) \cdot \phi_{k_2}(x + 2^{-k_2}a) \|_{\frac{M}{M-2},2})
\end{aligned}
$$

where we used (2.5.2) and (2.5.39) in the second inequality. We then write

$$
\phi_{<k_3-10}(x) = \phi_{<k_3-10}(x + 2^{-k_3}b) - 2^{-k_3} b \int_0^1 \nabla \phi_{<k_3-10}(x + 2^{-k_3} b\theta) d\theta
$$

and use (GeId) to bound

$$
\| \phi_{k_2}(x + 2^{-k_2}a) \, \phi_{<k_3-10}(x) \cdot \phi_{k_3}(x + 2^{-k_3}b) \|_{\frac{M}{M-2},2}
$$

$$\lesssim \langle b\rangle 2^{-(\frac{1}{2}-\frac{2}{M}+s')k_2} 2^{-(2+2s')k_3} \|\phi\|^2_{X_1^{s,\theta}} \left(\sum_{k' \gtrsim k_3} 2^{-(\frac{1}{2}+\frac{1}{M}+s')(k'-k_3)} \|\phi_{k'}\|_{X^{s,\theta}} \right)$$

and similarly

$$\begin{aligned} &\|\phi_{k_3}(x+2^{-k_3}b)\ \phi_{<k_3-10}(x)\cdot\phi_{k_2}(x+2^{-k_2}a)\|_{\frac{M}{M-2},2} \\ &\lesssim \langle a\rangle 2^{-sk_3} 2^{-(1-\frac{2}{M}+2s')k_2} \|\phi_{k_3}\|_{X_1^{s,\theta}} \|\phi\|^2_{X_1^{s,\theta}} \end{aligned}$$

It follows that, choosing $\theta' < M^{-1} < s'$,

$$\begin{aligned} &\| \sum_{k_1<k_3-10} Q_{\lesssim k_3} P_k(\eta_T \cdot HWM_{2;k_1,k_2,k_3}(\phi)) \|_{X^{s-1,\theta-1}} \\ &\lesssim T^{1/M} \sum_{l\lesssim k_3} \sum_{a,b} 2^{(\theta'-\frac{1}{M})l} 2^{(s-1)k_3} c^{k_2+k_3}_{a,b} \langle a\rangle\langle b\rangle 2^{-sk_3} 2^{-(1-\frac{2}{M}+2s')k_2} \|\phi\|^2_{X_1^{s,\theta}} \\ &\qquad\qquad \cdot \left(\sum_{k'\gtrsim k_3} 2^{-\sigma|k'-k_3|} \|\phi_{k'}\|_{X^{s,\theta}} \right) \\ &\lesssim T^{1/M} 2^{2(\frac{1}{M}-s')k_2} \|\phi\|^2_{X_1^{s,\theta}} \left(\sum_{k'\gtrsim k_3} 2^{-\sigma|k'-k_3|} \|\phi_{k'}\|_{X^{s,\theta}} \right) \end{aligned}$$

which is acceptable when summed over $k_2 \geq 0$, $k_3 \sim k$, $k \geq 0$. The case $k_2 \geq k_3 + 10$ can be treated identically.

In the remaining case $k_2 \simeq k_3$, we again call upon the identity (GeId), however this time there is nothing to be gained by cancellation and so one must instead split HWM_2 into its two components and treat each separately. The term involving

$$\phi_{k_2} \times (-\Delta)\phi_{k_3}$$

is easier to handle as there are no nonlocal operators acting so one can directly apply the vector product identity. For the term involving

$$(-\Delta)^{1/2}(\phi_{k_2} \times (-\Delta)^{1/2}\phi_{k_3}),$$

we only need to use (GeId) when the frequency of this output output is comparable to 2^k. The details are left to the reader.

□

In combination with Remark 2.10.7, the previous two propositions complete the proof of Proposition 2.10.2.

2.10.2 Local Wellposedness of the Half-Wave Maps Equation (2.1.1).

It remains to show that the local solution to the differentiated equation in fact solves the original problem (2.1.1) under the compatibility assumption $\phi_1 = \phi_0 \times (-\Delta)^{1/2}\phi_0$. We use an energy argument as in [KS17].

Let ϕ be a smooth local solution to equation (2.1.2) with data (ϕ_0, ϕ_1) as above. Set

$$X := \phi_t - \phi \times (-\Delta)^{1/2}\phi$$

Our goal is to show $X \equiv 0$. To this end, consider the energy type functional

$$\tilde{E}(t) = \frac{1}{2}\int_{\mathbb{R}^3} |X(t,x)|^2 dx$$

A calculation as in [KS17] shows

$$\partial_t X = -\phi(X \cdot (\phi \times (-\Delta)^{1/2}\phi + \phi_t)) - X \times (-\Delta)^{1/2}\phi - \phi \times (-\Delta)^{1/2} X$$

from which

$$\frac{d}{dt}\tilde{E}(t) = -\int_{\mathbb{R}^3} (\phi(X \cdot (\phi \times (-\Delta)^{1/2}\phi + \phi_t))) \cdot X dx - \int_{\mathbb{R}^3} (\phi \times (-\Delta)^{1/2} X) \cdot X dx \tag{2.10.16}$$

We immediately see that

$$\left| \int_{\mathbb{R}^3} (\phi(X \cdot (\phi \times (-\Delta)^{1/2}\phi + \phi_t))) \cdot X \, dx \right| \lesssim \|\phi\|_\infty \|X\|_2^2 \|\phi \times (-\Delta)^{1/2}\phi + \phi_t\|_\infty \lesssim_\phi \|X\|_2^2$$

since ϕ, $\nabla_{t,x}\phi \in X^{s,\theta} \hookrightarrow L^\infty_{t,x}$. For the second term, we subtract a term which is zero (by Plancherel):

$$\int_{\mathbb{R}^3} (\phi \times (-\Delta)^{\frac{1}{2}} X) \cdot X dx = \int_{\mathbb{R}^3} [(\phi \times (-\Delta)^{1/2} X) - (-\Delta)^{\frac{1}{4}}(\phi \times (-\Delta)^{\frac{1}{4}} X)] \cdot X dx$$

then bound

$$\begin{aligned}
&\|(\phi \times (-\Delta)^{1/2} X) - (-\Delta)^{\frac{1}{4}}(\phi \times (-\Delta)^{\frac{1}{4}} X)\|_2 \\
&\lesssim \sum_{k_1 \geq 0} \|\mathcal{L}(\phi_{k_1}, X_{<k_1+10})\|_2 + \left(\sum_{k_2 \geq 0} \|\mathcal{L}(\phi_{<k_2-10}, X_{k_2})\|_2^2 \right)^{\frac{1}{2}}
\end{aligned}$$

with

$$\mathcal{L}(\phi_{k_1}, X_{k_2}) = \int_{\xi,\eta} e^{ix\cdot(\xi+\eta)} |\eta|^{\frac{1}{2}} (|\eta|^{\frac{1}{2}} - |\xi+\eta|^{\frac{1}{2}}) \chi_{k_1}(\xi)\hat{\phi}(\xi)\chi_{k_2}(\eta)\hat{X}(\eta) d\xi d\eta$$

It is then straightforward that

$$\sum_{k_1 \geq 0} \|\mathcal{L}(\phi_{k_1}, X_{<k_1+10})\|_2 \lesssim \sum_{k_1 \geq 0} 2^{k_1} \|\phi_{k_1}\|_\infty \|X\|_2 \lesssim_\phi \|X\|_2$$

and applying Lemma 2.5.1 (using $\left| |\eta|^{\frac{1}{2}} (|\eta|^{\frac{1}{2}} - |\xi+\eta|^{\frac{1}{2}}) \right| \lesssim |\xi|$), we also have

$$\left(\sum_{k_2 \geq 0} \|\mathcal{L}(\phi_{<k_2-10}, X_{k_2})\|_2^2 \right)^{\frac{1}{2}} \lesssim \left(\sum_{k_2 \geq 0} \|\nabla \phi\|_\infty^2 \|X_{k_2}\|_2^2 \right)^{\frac{1}{2}} \lesssim_\phi \|X\|_2$$

We have therefore shown that

$$\frac{d}{dt} \tilde{E}(t) \lesssim_\phi \tilde{E}(t)$$

and since the initial conditions imply that $\tilde{E}(0) = 0$, we conclude that $\tilde{E} \equiv 0$ for all time. This completes the proof.

Appendix

2.A Control of the Low Frequencies

In this short appendix we show that the low frequency portion of ϕ cannot blow up. It is recommended that the reader ignores this appendix until the end of the proof, since some of the methods will by then be familiar.

By the energy estimate for the wave equation we find

$$\|P_{\leq 0}\partial_t\phi\|_{L_t^\infty L_x^2([0,T])} \lesssim \|P_{\leq 0}\phi[0]\|_{\dot{H}^1\times L^2} + \|P_{\leq 0}\Box\phi\|_{L_t^1 L_x^2([0,T])}$$

By our assumptions on the initial data certainly $\|P_{\leq 0}\phi[0]\|_{\dot{H}^1\times L^2} < \infty$. For the nonhomogeneous term we use Hölder's inequality in time and Bernstein in space to find, for instance,

$$\begin{aligned}
\|P_{\leq 0}(\phi\partial_\alpha\phi\partial^\alpha\phi)\|_{L_t^1 L_x^2([0,T])} &\lesssim T(\|P_{\leq 0}(\phi\partial_\alpha\phi_{>10}\partial^\alpha\phi_{>10})\|_{L_t^\infty L_x^1([0,T])} \\
&\quad + \|P_{\leq 0}(\phi\partial_\alpha\phi_{\leq 10}\partial^\alpha\phi_{>10})\|_{L_t^\infty L_x^2([0,T])} \\
&\quad + \|P_{\leq 0}(\phi\partial_\alpha\phi_{\leq 10}\partial^\alpha\phi_{\leq 10})\|_{L_t^\infty L_x^2([0,T])}) \\
&\lesssim T(\|\phi\|_{\infty,\infty}\|\partial_\alpha\phi_{>10}\|_{\infty,2}\|\partial^\alpha\phi_{>10}\|_{\infty,2} \\
&\quad + \|\phi\|_{\infty,\infty}\|\partial_\alpha\phi_{\leq 10}\|_{\infty,\infty}\|\partial^\alpha\phi_{>10}\|_{\infty,2} \\
&\quad + \|\phi\|_{\infty,\infty}\|\partial_\alpha\phi_{\leq 10}\|_{\infty,4}\|\partial^\alpha\phi_{\leq 10}\|_{\infty,4})
\end{aligned}$$

All of these terms are bounded by $T\epsilon^2$ using the definition of S and the local constancy of the frequency envelope. The half-wave maps source terms can be treated similarly using arguments as in Section 2.6 (see for example Claim 3, Proposition 2.6.1).

This shows that the low frequency portion of $\partial_t\phi$ remains bounded for all time (even if this bound is growing in T). For the L^2 norm of the solution itself we can then use that the data is certainly in L^2 (upon subtracting the constant p) and calculate the derivative

$$\begin{aligned}
\frac{d}{dt}\|P_{\leq 0}\phi(t)\|_{L_x^2}^2 &= 2\int_{\mathbb{R}^3} P_{\leq 0}\phi \cdot P_{\leq 0}\partial_t\phi dx \\
&\leq \epsilon^2\|P_{\leq 0}\phi\|_{L_t^\infty L_x^2([0,T])}^2 + \epsilon^{-2}\|P_{\leq 0}\partial_t\phi\|_{L_t^\infty L_x^2([0,T])}^2
\end{aligned}$$

Choosing $\epsilon = (2T)^{-1/2}$ and using the fundamental theorem of calculus this yields

$$\|P_{\leq 0}\phi\|^2_{L^\infty_t L^2_x([0,T])} \leq 2\|\phi_0\|^2_{L^2_x} + 4T^2\|P_{\leq 0}\partial_t\phi\|^2_{L^\infty_t L^2_x([0,T])}$$

In combination with the bound already shown for $\|P_{\leq 0}\partial_t\phi\|_{L^\infty_t L^2_x([0,T])}$ this shows that $\|P_{\leq 0}\phi\|_{L^\infty_t L^2_x([0,T])}$ also remains bounded on the interval $[0,T]$.

We remark that the control on $\|P_{\leq 0}\phi\|_{L^2_x}$ could also be obtained from the conserved mass

$$M(t) := \int_{\mathbb{R}^n} |\phi - p|^2 dx \tag{2.A.1}$$

of the half-wave maps equation, although the approach above is of course more general.

2.B Proof of Moser Estimates

In this appendix we prove the nonlinear Moser estimates which played a crucial in the analysis of this chapter. We first prove the most straightforward such estimate, (2.5.17), involving only Strichartz norms.

Lemma 2.B.1. *Let $g : \mathbb{R}^3 \to \mathbb{R}$ have bounded derivatives up to second order, and (p,q) be a standard Strichartz pair. Then it holds*

$$\|P_k g(\phi)\|_{p,q} \lesssim_{p,q} 2^{-(\frac{1}{p}+\frac{3}{q})k}\|\phi\|^2_S(1 + \|\phi\|_S)$$

Here S is the critical norm of Section 2.2.2.

Proof. First assume that $1 - \frac{1}{p} - \frac{3}{q} \leq 0$. Differentiating at frequency 2^k and using the chain rule, we have

$$\begin{aligned}\|P_k g(\phi)\|_{p,q} &\lesssim 2^{-k}\|P_k(\nabla\phi g'(\phi))\|_{p,q} \\ &\lesssim 2^{-k}\|P_k(\nabla\phi_{<k-10}g'(\phi))\|_{p,q} + 2^{-k}\|P_k(\nabla\phi_{>k-10}g'(\phi))\|_{p,q}\end{aligned} \tag{2.B.1}$$

where

$$2^{-k}\|P_k(\nabla\phi_{>k-10}g'(\phi))\|_{p,q} \lesssim 2^{-k}\|\nabla\phi_{>k-10}\|_{p,q}\|g'\|_\infty \lesssim 2^{-k}2^{(1-\frac{1}{p}+\frac{3}{q})k}\|\phi\|_S$$

is as required. For the low frequency term, we differentiate a second time to find

$$\begin{aligned}2^{-k}\|P_k(\nabla\phi_{<k-10}g'(\phi))\|_{p,q} &\lesssim 2^{-k}\|P_k(\nabla\phi_{<k-10}\nabla^{-1}\cdot\nabla P_{\sim k}g'(\phi))\|_{p,q} \\ &\lesssim \sum_{j<k-10} 2^{-k}\|P_k(\nabla\phi_j\nabla^{-1}\cdot P_{\sim k}(\nabla\phi_{<j}g''(\phi)))\|_{p,q} \\ &\qquad + 2^{-k}\|P_k(\nabla\phi_j\nabla^{-1}\cdot P_{\sim k}(\nabla\phi_{\geq j}g''(\phi)))\|_{p,q} \\ &\lesssim \sum_{j<k-10} 2^{-k}\|\nabla\phi_j\|_{p,q}\cdot 2^{-k}\|\nabla\phi_{<j}\|_{\infty,\infty}\end{aligned}$$

$$
\begin{aligned}
&+ 2^{-k}\|\nabla\phi_j\|_{\infty,\infty} \cdot 2^{-k}\|\nabla\phi_{\geq j}\|_{p,q} \\
\lesssim\ & 2^{-2k} \sum_{j<k-10} 2^{(2-\frac{1}{p}-\frac{3}{q})j}\|\phi\|_S^2
\end{aligned}
$$

which is as required since $p, q \geq 2$ implies $2 - \frac{1}{p} - \frac{3}{q} \geq 0$.

If $1 - \frac{1}{p} - \frac{3}{q} > 0$, we can start from (2.B.1) and see that the low frequency term can now be estimated directly. For the high frequency part we split $\nabla\phi_{>k-10}$ into intermediate and high frequencies then find $1 \leq r \leq q$ such that $\frac{1}{r} = \frac{1}{q} + \frac{1}{2}$ and apply Bernstein's inequality:

$$
\begin{aligned}
2^{-k}\|P_k(\nabla\phi_{>k-10}g'(\phi))\|_{p,q} \lesssim\ & 2^{-k}\|P_k(\nabla\phi_{[k-10,k+10]}g'(\phi))\|_{p,q} \\
&+ 2^{-k}2^{3k/2} \sum_{r>k+10} \|P_k(\nabla\phi_r g'(\phi))\|_{p,r}
\end{aligned}
$$

The first term can then be bounded upon placing $\nabla\phi_{[k-10,k+10]}$ directly into $L_t^p L_x^q$, while for the second term we use that $g(\phi)$ is now also restricted to frequency $\sim 2^r$ and use the bound from the case $1 - \frac{1}{p} - \frac{3}{q} \leq 0$:

$$
\begin{aligned}
2^{-k}2^{3k/2} \sum_{r>k+10} \|P_k(\nabla\phi_r P_{\sim r}g'(\phi))\|_{p,r} &\lesssim 2^{k/2} \sum_{r>k+10} \|\nabla\phi_r\|_{p,q}\|P_{\sim r}g(\phi)\|_{\infty,2} \\
&\lesssim 2^{k/2} \sum_{r>k+10} 2^{(1-\frac{1}{p}-\frac{3}{q})r}\|\phi\|_S \cdot 2^{-3r/2}\|\phi\|_S(1 + \|\phi\|_S)
\end{aligned}
$$

which is as required once summed over r. □

We next turn to Lemma 2.10.6, focusing only on the first and second points, (2.10.10) and (2.10.11), the remaining estimate being similar. The proofs are similar in flavour to that above, however it is often more suitable to differentiate in time rather than space.

Proof of (2.10.10). We study the different regimens of (j, k) separately. In this proof all implicit constants may depend polynomially on $\|\tilde{\phi}\|_{X_1^{s,\theta}}$.

- $\underline{k \ll j}$: We have to show that

$$
\|P_k Q_j g(\tilde{\phi})\|_{2,2} \lesssim_{\|\tilde{\phi}\|_{X_1^{s,\theta}}} 2^{-(s+\theta)j}
$$

  Note that since $j \gg k$ we have $P_k Q_j = P_k Q_j P_{\sim j}^{(t)}$ and $j > 0$ so

$$
\|P_k Q_j g(\tilde{\phi})\|_{2,2} \lesssim \underbrace{2^{-j}\|P_k Q_j[Q_{\gtrsim j}\partial_t\tilde{\phi} \cdot g'(\tilde{\phi})]\|_{2,2}}_{(A)} + \underbrace{2^{-j}\|P_k Q_j[Q_{\ll j}\partial_t\tilde{\phi} \cdot g'(\tilde{\phi})]\|_{2,2}}_{(B)}
$$

Here

$$(A) \lesssim 2^{-j}\|Q_{\gtrsim j}\partial_t\tilde{\phi}\|_{2,2} \lesssim 2^{-j}2^{-(s+\theta-1)j}\|\partial_t\tilde{\phi}\|_{X^{0,s-1+\theta}} \lesssim 2^{-(s+\theta)j}\|\tilde{\phi}\|_{X^{s,\theta}}$$

For (B) we differentiate in t a second time and find

$$(B) \lesssim \underbrace{2^{-2j}\|P_kQ_j[Q_{\ll j}\partial_t^2\tilde{\phi}\cdot g'(\tilde{\phi})]\|_{2,2}}_{(B1)} + \underbrace{2^{-2j}\|P_kQ_j[Q_{\ll j}\partial_t\tilde{\phi}\cdot\partial_t\tilde{\phi}\cdot g''(\tilde{\phi})]\|_{2,2}}_{(B2)}$$

We start with

$$(B1) \lesssim \underbrace{2^{-2j}\|P_kQ_j[Q_{<k-10}\partial_t^2\tilde{\phi}\cdot g'(\tilde{\phi})]\|_{2,2}}_{(B1)_{\ll k}} + \underbrace{2^{-2j}\|P_kQ_j[Q_{[k-10,j-10]}\partial_t^2\tilde{\phi}\cdot g'(\tilde{\phi})]\|_{2,2}}_{(B1)_{\gtrsim k}}$$

For the lowest modulation case we have

$$(B1)_{\ll k} \lesssim \sum_{l<k-10}(2^{-2j}\|P_kQ_j[P_{<l-10}Q_l\partial_t^2\tilde{\phi}\cdot g'(\tilde{\phi})]\|_{2,2} \qquad (B1a)_{\ll k}$$
$$+ 2^{-2j}\|P_kQ_j[P_{\geq l-10}Q_l\partial_t^2\tilde{\phi}\cdot g'(\tilde{\phi})]\|_{2,2}) \qquad (B1b)_{\ll k}$$

Now the real calculations begin. For the first of these terms we use that l is far smaller than the scales k or j so the factor of g' must also be localised to $P_{\sim k}Q_{\sim j}$. It follows that

$$\begin{aligned}(B1a)_{\ll k} &\lesssim 2^{-2j}\sum_{l<k-10}\|P_kQ_j[P_{<l-10}Q_l\partial_t^2\tilde{\phi}\cdot\partial_t^{-1}P_{\sim k}Q_{\sim j}[\partial_t\tilde{\phi}\cdot g''(\tilde{\phi})]]\|_{2,2}\\ &\lesssim 2^{-2j}\sum_{l<k-10}2^{-j}\|P_{<l-10}Q_l\partial_t^2\tilde{\phi}\|_{\infty,\infty}\|Q_{\gtrsim l}\partial_t\tilde{\phi}\|_{2,2}\|g''(\tilde{\phi})\|_{\infty,\infty}\\ &\quad + 2^{-2j}\sum_{l<k-10}\|P_kQ_j[P_{<l-10}Q_l\partial_t^2\tilde{\phi}\cdot\partial_t^{-1}P_{\sim k}Q_{\sim j}[Q_{\ll l}\partial_t\tilde{\phi}\cdot g''(\tilde{\phi})]]\|_{2,2}\end{aligned}$$

The first line above can be bounded by

$$2^{-3j}\sum_{l<k-10}2^{[2-(s'+\theta')]l}\|\tilde{\phi}\|_{X^{s,\theta}}2^{-(s-1+\theta)l}\|Q_{\gtrsim l}\partial_t\tilde{\phi}\|_{X^{0,s-1+\theta}} \lesssim 2^{[3-(s+\theta)-(s'+\theta')](k-j)}2^{-(s+\theta)j}$$

as required. For the second line we further split $Q_{\ll l}\partial_t\tilde{\phi}$ into low and high frequencies to find

$$\begin{aligned}&2^{-2j}\sum_{l<k-10}\|P_kQ_j[P_{<l-10}Q_l\partial_t^2\tilde{\phi}\cdot\partial_t^{-1}P_{\sim k}Q_{\sim j}[Q_{\ll l}\partial_t\tilde{\phi}\cdot g''(\tilde{\phi})]]\|_{2,2}\\ &\lesssim 2^{-3j}\sum_{l<k-10}\|P_{<l-10}Q_l\partial_t^2\tilde{\phi}\|_{2,2}\|P_{\lesssim l}Q_{\ll l}\partial_t\tilde{\phi}\|_{\infty,\infty} + 2^{-3j}\sum_{l<k-10}\|P_{<l-10}Q_l\partial_t^2\tilde{\phi}\|_{2,\infty}\|P_{\gg l}Q_{\ll l}\partial_t\tilde{\phi}\|_{\infty,2}\\ &\lesssim 2^{-3j}\sum_{l<k-10}\left(2^{[2-(s+\theta)l]}2^{(1-s')l} + 2^{3l/2}2^{[2-(s+\theta)]l}2^{(1-s)l}\right)\\ &\lesssim 2^{[3-(s+\theta)](k-j)}2^{-(s+\theta)j}\end{aligned}$$

This completes the study of $(B1a)_{\ll k}$. We now turn to $(B1b)_{\ll k}$. Write

$$(B1b)_{\ll k} \le 2^{-2j} \sum_{l<k-10} \|P_k Q_j [P_{>k+10} Q_l \partial_t^2 \tilde{\phi} \cdot g'(\tilde{\phi})]\|_{2,2} \tag{2.B.2}$$

$$+ 2^{-2j} \sum_{l<k-10} \|P_k Q_j [P_{[l-10,k+10]} Q_l \partial_t^2 \tilde{\phi} \cdot g'(\tilde{\phi})]\|_{2,2} \tag{2.B.3}$$

The easier of these terms is (2.B.2), which we write as follows:

$$\begin{aligned}(2.B.2) &\lesssim 2^{-2j} \sum_{l\ll k} \sum_{r \gg k} \|P_k Q_j [P_r Q_l \partial_t^2 \tilde{\phi} \cdot P_{\sim r} g'(\tilde{\phi})]\|_{2,2} \\ &\lesssim 2^{-2j} \sum_{\substack{l\ll k \\ r\gg k}} \|P_k Q_j [P_r Q_l \partial_t^2 \tilde{\phi} \cdot \nabla^{-1} P_{\sim r} [\nabla \tilde{\phi}_{\ll r} \cdot g''(\tilde{\phi})]]\|_{2,2} &(2.B.4) \\ &\quad + 2^{-2j} \sum_{\substack{l\ll k \\ r\gg k}} \|P_k Q_j [P_r Q_l \partial_t^2 \tilde{\phi} \cdot \nabla^{-1} P_{\sim r} [\nabla \tilde{\phi}_{\gtrsim r} \cdot g''(\tilde{\phi})]]\|_{2,2} &(2.B.5)\end{aligned}$$

To study (2.B.4), we differentiate in t a further time and obtain

$$\begin{aligned}(2.B.4) &\lesssim 2^{-3j} \sum_{\substack{l\ll k \\ r\gg k}} \|P_k Q_j [P_r Q_l \partial_t^3 \tilde{\phi} \cdot \nabla^{-1} P_{\sim r} [\nabla \tilde{\phi}_{\ll r} \cdot P_{\sim r} g''(\tilde{\phi})]]\|_{2,2} &(2.B.6) \\ &\quad + 2^{-3j} \sum_{\substack{l\ll k \\ r\gg k}} \|P_k Q_j [P_r Q_l \partial_t^2 \tilde{\phi} \cdot \nabla^{-1} P_{\sim r} [\nabla \partial_t \tilde{\phi}_{\ll r} \cdot P_{\sim r} g''(\tilde{\phi})]]\|_{2,2} &(2.B.7) \\ &\quad + 2^{-3j} \sum_{\substack{l\ll k \\ r\gg k}} \|P_k Q_j [P_r Q_l \partial_t^2 \tilde{\phi} \cdot \nabla^{-1} P_{\sim r} [\nabla \tilde{\phi}_{\ll r} \cdot \partial_t P_{\sim r} g''(\tilde{\phi})]]\|_{2,2} &(2.B.8)\end{aligned}$$

For (2.B.6) we use Bernstein at frequency 2^k to see

$$\begin{aligned}(2.B.6) &\lesssim 2^{-3j} \sum_{\substack{l\ll k \\ r\gg k}} 2^{3k/2} \|P_r Q_l \partial_t^3 \tilde{\phi}\|_{\infty,2} \cdot 2^{-r} \|\nabla \tilde{\phi}_{\ll r}\|_{\frac{2M}{M-1},2M} \|P_{\sim r} g''(\tilde{\phi})\|_{2M,\frac{2M}{M-1}} \\ &\lesssim 2^{-3j} \sum_{\substack{l\ll k \\ r\gg k}} 2^{3k/2} 2^{-\theta' l} 2^{(3-s)r} 2^{-r} 2^{(\frac{1}{2}-\frac{1}{M}-s')r} 2^{-(\frac{3}{2}-\frac{1}{M}+s')r} \\ &\lesssim 2^{(1-3s')(k-j)} 2^{-(2+3s')j}\end{aligned}$$

which is acceptable. The second term (2.B.7) can be treated in the same way. For (2.B.8) we use Lemma 2.10.5 to bound

$$\begin{aligned}(2.B.8) &\lesssim 2^{-3j} \sum_{\substack{l\ll k \\ r\gg k}} \|P_k Q_j [P_r Q_l \partial_t^2 \tilde{\phi} \cdot \nabla^{-1} P_{\sim r} [\nabla \tilde{\phi}_{\ll r} \cdot \partial_t P_{\sim r} g''(\tilde{\phi})]]\|_{2,2} \\ &\lesssim 2^{-3j} \sum_{\substack{l\ll k \\ r\gg k}} 2^{3k/2} \|P_r Q_l \partial_t^2 \tilde{\phi}\|_{\infty,2} \cdot 2^{-r} \|\nabla \tilde{\phi}_{\ll r}\|_{\frac{2M}{M-1},2M} \|\partial_t P_{\sim r} g''(\tilde{\phi})\|_{2M,\frac{2M}{M-1}} \\ &\lesssim 2^{-3j} \sum_{\substack{l\ll k \\ r\gg k}} 2^{3k/2} 2^{-\theta' l} 2^{(2-s)r} 2^{-r} 2^{(\frac{1}{2}-\frac{1}{M}-s')r} 2^{-(\frac{1}{2}-\frac{1}{M}+s')r}\end{aligned}$$

$$\lesssim 2^{-3j}2^{(1-3s')k}$$

which is as required. We now turn to (2.B.5). If we restrict the sum to $r \gtrsim j$ the term is easily handled:

$$\begin{aligned}
&2^{-2j}\sum_{\substack{l\ll k\\ r\gtrsim j}}\|P_kQ_j[P_rQ_l\partial_t^2\tilde{\phi}\cdot\nabla^{-1}P_{\sim r}[\nabla\tilde{\phi}_{\gtrsim r}\cdot g''(\tilde{\phi})]]\|_{2,2}\\
&\lesssim 2^{-2j}\sum_{\substack{l\ll k\\ r\gtrsim j}}\|P_rQ_l\partial_t^2\tilde{\phi}\|_{2M,\frac{2M}{M-1}}2^{-r}\|P_{\sim r}(\nabla\tilde{\phi}_{\gtrsim r}\cdot g''(\tilde{\phi}))\|_{\frac{2M}{M-1},2M}\\
&\lesssim 2^{-2j}\sum_{\substack{l\ll k\\ r\gtrsim j}}2^{(\frac{1}{2}-\frac{1}{2M}-\theta)l}2^{3r/2M}2^{(2-s)r}\cdot 2^{-r}2^{(\frac{1}{2}-\frac{1}{M}-s')r}\\
&\lesssim 2^{-(2+2s'-\frac{1}{2M})j}
\end{aligned}$$

Here we used the bound

$$\|P_{\sim r}(\nabla\tilde{\phi}_{\gtrsim r}\cdot g''(\tilde{\phi}))\|_{\frac{2M}{M-1},2M}\lesssim\|\nabla\tilde{\phi}_{\sim r}\|_{\frac{2M}{M-1},2M}+\sum_{m\gg r}2^{3r/2}\|\nabla\tilde{\phi}_m\|_{\frac{2M}{M-1},2M}\|P_{\sim m}g''(\tilde{\phi})\|_{\infty,2}$$

to go from the second to the third line. Such decompositions will be used frequently without comment in the sequel. Choosing M such that $s'-\frac{1}{2M}\geq\theta'$, we see that the sum over $r\gtrsim j$ is acceptable.

For the sum over $r\in[k+10,j-10]$ we differentiate again and have

$$\begin{aligned}
&2^{-2j}\sum_{\substack{l\ll k\\ r\in[k+10,j-10]}}\|P_kQ_j[P_rQ_l\partial_t^2\tilde{\phi}\cdot\nabla^{-1}P_{\sim r}[\nabla\tilde{\phi}_{\gtrsim r}\cdot g''(\tilde{\phi})]]\|_{2,2}\\
&\lesssim 2^{-3j}\sum_{\substack{l\ll k\\ r\in[k+10,j-10]}}\|P_kQ_j[P_rQ_l\partial_t^3\tilde{\phi}\cdot\nabla^{-1}P_{\sim r}[\nabla\tilde{\phi}_{\gtrsim r}\cdot g''(\tilde{\phi})]]\|_{2,2} && \text{(I)}\\
&\quad+2^{-3j}\sum_{\substack{l\ll k\\ r\in[k+10,j-10]}}\|P_kQ_j[P_rQ_l\partial_t^2\tilde{\phi}\cdot\nabla^{-1}P_{\sim r}[\nabla\partial_t\tilde{\phi}_{\gtrsim r}\cdot g''(\tilde{\phi})]]\|_{2,2} && \text{(II)}\\
&\quad+2^{-3j}\sum_{\substack{l\ll k\\ r\in[k+10,j-10]}}\|P_kQ_j[P_rQ_l\partial_t^2\tilde{\phi}\cdot\nabla^{-1}P_{\sim r}[\nabla\tilde{\phi}_{\gtrsim r}\cdot\partial_t g''(\tilde{\phi})]]\|_{2,2} && \text{(III)}
\end{aligned}$$

where

$$\text{(I)}\lesssim 2^{-3j}\sum_{\substack{l\ll k\\ r\in[k+10,j-10]}}\|P_rQ_l\partial_t^3\tilde{\phi}\|_{2M,\frac{2M}{M-1}}\cdot 2^{-r}\|P_{\sim r}(\nabla\tilde{\phi}_{\gtrsim r}\cdot g''(\tilde{\phi}))\|_{\frac{2M}{M-1},2M}\lesssim 2^{-(2+2s'-\frac{1}{2M})j}$$

(summing over $l\geq 0$, $r\ll j$) which is again acceptable for $s'-\frac{1}{2M}\geq\theta'$. The bound for (II) is similar. For (III) the bound is straightforward upon placing $P_{\sim r}[\nabla\tilde{\phi}_{\sim r}\cdot\partial_t g''(\tilde{\phi})]$ into $L_t^{2+}L_x^2$ and separately considering the cases where the frequency of $\partial_t g''(\tilde{\phi})$ is comparable to or much smaller than that of $\nabla\tilde{\phi}$.

This completes the work on (2.B.2) so we now turn to (2.B.3):

$$(2.B.3) \lesssim 2^{-2j} \sum_{l<k-10} \|P_k Q_j [P_{[l-10,k-10]} Q_l \partial_t^2 \tilde{\phi} \cdot g'(\tilde{\phi})]\|_{2,2} \tag{2.B.9}$$

$$+ 2^{-2j} \sum_{l<k-10} \|P_k Q_j [P_{[k-10,k+10]} Q_l \partial_t^2 \tilde{\phi} \cdot g'(\tilde{\phi})]\|_{2,2} \tag{2.B.10}$$

For the first line we use that $g'(\tilde{\phi})$ must be restricted to frequency $\sim 2^k$ and modulation $\sim 2^j$, which allows us to swap a 2^j for a 2^k by Moser's inequality (2.10.8):

$$\begin{aligned}
(2.B.9) &\lesssim 2^{-2j} \sum_{l \ll k} \|P_k Q_j [P_{[l-10,k-10]} Q_l \partial_t^2 \tilde{\phi} \cdot P_{\sim k} Q_{\sim j} g'(\tilde{\phi})]\|_{2,2} \\
&\lesssim 2^{-2j} \sum_{l \ll k} \|P_{[l-10,k-10]} Q_l \partial_t^2 \tilde{\phi}\|_{2M, \frac{2M}{M-1}} \cdot 2^{-j} \|\partial_t P_{\sim k} g'(\tilde{\phi})\|_{\frac{2M}{M-1}, 2M} \\
&\lesssim 2^{-3j} \sum_{l \ll k} \sum_{\lambda = l-10}^{k-10} 2^{(\frac{1}{2} - \frac{1}{2M} - \theta) l} 2^{3\lambda/2M} 2^{(2-s)\lambda} 2^{(\frac{1}{2} - \frac{1}{M} - s')k} \\
&\lesssim 2^{(1 + \frac{1}{2M} - 2s')(k-j)} 2^{-(2+2s' - \frac{1}{2M}) j}
\end{aligned}$$

which is acceptable for $s' - \frac{1}{2M} \geq \theta'$.

To complete the work on $(B1b)_{\ll k}$ it remains to study (2.B.10). We use that $k \ll j$ to see that $g'(\tilde{\phi})$ must be at modulation $\sim 2^j$ and so

$$\begin{aligned}
(2.B.10) &\lesssim 2^{-2j} \sum_{l \ll k} \|P_k Q_j [P_{\sim k} Q_l \partial_t^2 \tilde{\phi} \cdot \partial_t^{-1} P_{\lesssim k} Q_{\sim j} \partial_t g'(\tilde{\phi})]\|_{2,2} \\
&\lesssim 2^{-2j} \sum_{\substack{l \ll k \\ \lambda \lesssim k}} \|P_{\sim k} Q_l \partial_t^2 \tilde{\phi}\|_{2M, \frac{2M}{M-1}} 2^{-j} \|P_\lambda \partial_t g'(\tilde{\phi})\|_{\frac{2M}{M-1}, 2M} \\
&\lesssim 2^{-3j} \sum_{\substack{l \ll k \\ \lambda \lesssim k}} 2^{(\frac{1}{2} - \frac{1}{2M} - \theta) l} 2^{3k/2M} 2^{(2-s)k} 2^{(\frac{1}{2} - \frac{1}{M} - s')\lambda} \\
&\lesssim 2^{(1 - 2s' + \frac{1}{2M})(k-j)} 2^{-(2+2s' - \frac{1}{2M}) j}
\end{aligned}$$

which is acceptable for $s' - \frac{1}{2M} > \theta'$. This completes the study of $(B1b)_{\ll k}$, and so of $(B1)_{\ll k}$.

To finish the work on (B1), we therefore now have to study

$$(B1)_{\gtrsim k} \lesssim 2^{-2j} \sum_{l=k-10}^{j-10} \|P_k Q_j [P_{<k-10} Q_l \partial_t^2 \tilde{\phi} \cdot g'(\tilde{\phi})]\|_{2,2} \tag{$(B1a)_{\gtrsim k}$}$$

$$+ 2^{-2j} \sum_{l=k-10}^{j-10} \|P_k Q_j [P_{\geq k-10} Q_l \partial_t^2 \tilde{\phi} \cdot g'(\tilde{\phi})]\|_{2,2} \tag{$(B1b)_{\gtrsim k}$}$$

For $(B1a)_{\gtrsim k}$ we note that $g'(\tilde{\phi})$ must be at frequency $\sim 2^k$ and modulation $\sim 2^j$ and decompose

$$
\begin{aligned}
(B1a)_{\gtrsim k} &\lesssim 2^{-2j} \sum_{l=k-10}^{j-10} \|P_k Q_j[P_{<k-10} Q_l \partial_t^2 \tilde{\phi} \cdot \partial_t^{-1} P_{\sim k} Q_{\sim j} \partial_t P_{\lesssim k} g'(\tilde{\phi})]\|_{2,2} \\
&\lesssim 2^{-2j} \sum_{l \gtrsim k} \|P_{\ll k} Q_l \partial_t^2 \tilde{\phi}\|_{2,2} \cdot 2^{-j} \|P_{\sim k} \partial_t g'(\tilde{\phi})\|_{\infty,\infty} \\
&\lesssim 2^{(1-s'-\theta')(k-j)} 2^{-(2+s'+\theta')j}
\end{aligned}
$$

Then for $(B1b)_{\gtrsim k}$ we note that if $l \gg k$ we can write

$$
\begin{aligned}
2^{-2j} \sum_{l=k+10}^{j-10} \|P_k Q_j[P_{\geq k-10} Q_l \partial_t^2 \tilde{\phi} \cdot g'(\tilde{\phi})]\|_{2,2} &\lesssim 2^{-2j} \sum_{l=k+10}^{j-10} \|P_k Q_j[P_{>k+10} Q_l \partial_t^2 \tilde{\phi} \cdot g'(\tilde{\phi})]\|_{2,2} && \text{(2.B.11)} \\
&+ 2^{-2j} \sum_{l=k+10}^{j-10} \|P_k Q_j[P_{\sim k} Q_l \partial_t^2 \tilde{\phi} \cdot g'(\tilde{\phi})]\|_{2,2} && \text{(2.B.12)}
\end{aligned}
$$

where (2.B.11) can be treated in exactly the same way as (2.B.2), and for (2.B.12) we observe that $g'(\tilde{\phi})$ is restricted to $P_{\lesssim k} Q_{\sim j} g'(\tilde{\phi}) = P^{(t)}_{\sim j} P_{\lesssim k} Q_{\sim j} g'(\tilde{\phi})$ and so

$$
\begin{aligned}
\text{(2.B.12)} &\lesssim 2^{-2j} \sum_{l=k+10}^{j-10} \|P_k Q_j[P_{\sim k} Q_l \partial_t^2 \tilde{\phi} \cdot \partial_t^{-1} P_{\lesssim k} Q_{\sim j} \partial_t g(\tilde{\phi})]\|_{2,2} \\
&\lesssim 2^{-3j} \sum_{l \gg k} \|P_{\sim k} Q_l \partial_t^2 \tilde{\phi}\|_{2,2} \|P_{\lesssim k} \partial_t g(\tilde{\phi})\|_{\infty,\infty} \\
&\lesssim_{\|\tilde{\phi}\|_{X_1^{s,\theta}}} 2^{(1-s'-\theta')(k-j)} 2^{-(2+s'+\theta')j}
\end{aligned}
$$

For the remaining part of $(B1b)_{\gtrsim k}$ with $l \sim k$ we have

$$
\begin{aligned}
2^{-2j} \|P_k Q_j[P_{\gtrsim k} Q_{\sim k} \partial_t^2 \tilde{\phi} \cdot g'(\tilde{\phi})]\|_{2,2} &\lesssim 2^{-2j} \|P_k Q_j[P_{\sim k} Q_{\sim k} \partial_t^2 \tilde{\phi} \cdot P_{\lesssim k} Q_{\sim j} g'(\tilde{\phi})]\|_{2,2} && \text{(2.B.13)} \\
&+ 2^{-2j} \sum_{r \gg k} \|P_k Q_j[P_r Q_{\sim k} \partial_t^2 \tilde{\phi} \cdot P_{\sim r} g'(\tilde{\phi})]\|_{2,2} && \text{(2.B.14)}
\end{aligned}
$$

where

$$
\begin{aligned}
\text{(2.B.13)} &\lesssim 2^{-2j} \|P_k Q_j[P_{\sim k} Q_{\sim k} \partial_t^2 \tilde{\phi} \cdot \partial_t^{-1} P_{\lesssim k} Q_{\sim j} \partial_t g'(\tilde{\phi})]\|_{2,2} \\
&\lesssim 2^{-2j} \|P_{\sim k} Q_{\sim k} \partial_t^2 \tilde{\phi}\|_{2,2} \cdot 2^{-j} \|P_{\lesssim k} \partial_t g'(\tilde{\phi})\|_{\infty,\infty} \\
&\lesssim 2^{(3-s-\theta-s')(k-j)} 2^{-(s+\theta)j} 2^{-s'j}
\end{aligned}
$$

as required. For (2.B.14) we have

$$\begin{aligned}
(2.B.14) &\lesssim 2^{-2j} \sum_{r \geq j-10} \|P_k Q_j [P_r Q_{\sim k} \partial_t^2 \tilde{\phi} \cdot P_{\sim r} g'(\tilde{\phi})]\|_{2,2} \\
&\quad + 2^{-2j} \sum_{r=k+10}^{j-10} \|P_k Q_j [P_r Q_{\sim k} \partial_t^2 \tilde{\phi} \cdot \partial_t^{-1} P_{\sim r} Q_{\sim j} \partial_t g'(\tilde{\phi})]\|_{2,2} \\
&\lesssim 2^{-2j} \sum_{r \gtrsim j} 2^{3k/2} \|P_r Q_{\sim k} \partial_t^2 \tilde{\phi}\|_{2,2} \|P_{\sim r} g'(\tilde{\phi})\|_{\infty,2} \\
&\quad + 2^{-2j} \sum_{r \gg k} 2^{3k/2} \|P_r Q_{\sim k} \partial_t^2 \tilde{\phi}\|_{2,2} \cdot 2^{-j} \|\partial_t P_{\sim r} g'(\tilde{\phi})\|_{\infty,2} \\
&\lesssim_{\|\tilde{\phi}\|_{X_1^{s,\theta}}} 2^{-2j} 2^{3k/2} \sum_{r \gtrsim j} 2^{-\theta k} 2^{(2-s)r} 2^{-sr} + 2^{-3j} 2^{3k/2} \sum_{r \gg k} 2^{-\theta k} 2^{(2-s)r} 2^{(1-s)r} \\
&\lesssim_{\|\tilde{\phi}\|_{X_1^{s,\theta}}} 2^{(s-\theta)(k-j)} 2^{-(s+\theta)j} + 2^{(3-s-\theta)(k-j)} 2^{-(s+\theta)j}
\end{aligned}$$

which is acceptable. This completes the work on (B1).

(B2) can be treated similarly and this complete the study of $j \gg k$.

- $\underline{j \simeq k}$: This time we have to show

$$\|P_k Q_j g(\tilde{\phi})\|_{2,2} \lesssim 2^{-(s+\theta)k}$$

We have

$$\begin{aligned}
\|P_k Q_{\sim k} g(\tilde{\phi})\|_{2,2} &\lesssim 2^{-k} \|P_k Q_{\sim k} (\nabla \tilde{\phi}_{\gtrsim k} \cdot g'(\tilde{\phi}))\|_{2,2} + 2^{-k} \|P_k Q_{\sim k} (\nabla \tilde{\phi}_{\ll k} \cdot P_{\sim k} g'(\tilde{\phi}))\|_{2,2} \\
&\lesssim 2^{-k} \|Q_{\gtrsim k} \nabla \tilde{\phi}_{\gtrsim k}\|_{2,2} + 2^{-k} \|P_k Q_{\sim k} (Q_{\ll k} \nabla \tilde{\phi}_{\gtrsim k} \cdot g'(\tilde{\phi}))\|_{2,2} \\
&\quad + 2^{-k} \|\nabla \tilde{\phi}_{\ll k}\|_{\frac{2M}{M-1}, 2M} \|P_{\sim k} g'(\tilde{\phi})\|_{2M, \frac{2M}{M-1}} \\
&\lesssim 2^{-k} 2^{(1-s-\theta)k} + 2^{-k} \|P_k Q_{\sim k} (Q_{\ll k} \nabla \tilde{\phi}_{\gtrsim k} \cdot g'(\tilde{\phi}))\|_{2,2} \\
&\quad + 2^{-k} 2^{(\frac{1}{2} - \frac{1}{M} - s')k} 2^{-(\frac{3}{2} - \frac{1}{M} + s')k}
\end{aligned}$$

The first and third terms here are as required, so it remains to study

$$\begin{aligned}
2^{-k} \|P_k Q_{\sim k} (Q_{\ll k} \nabla \tilde{\phi}_{\gtrsim k} \cdot g'(\tilde{\phi}))\|_{2,2} &\lesssim 2^{-k} \|P_k Q_{\sim k} (Q_{\ll k} \nabla \tilde{\phi}_{\sim k} \cdot P_{\lesssim k} g'(\tilde{\phi}))\|_{2,2} \quad (2.B.15) \\
&\quad + \sum_{r \gg k} 2^{-k} \|P_k Q_{\sim k} (Q_{\ll k} \nabla \tilde{\phi}_r \cdot P_{\sim r} g'(\tilde{\phi}))\|_{2,2} \quad (2.B.16)
\end{aligned}$$

where

$$(2.B.16) \lesssim 2^{-k} \sum_{r \gg k} \|\nabla \tilde{\phi}_r\|_{\frac{2M}{M-1}, 2M} \|P_{\sim r} g'(\tilde{\phi})\|_{2M, \frac{2M}{M-1}} \lesssim 2^{-k} 2^{(\frac{1}{2} - \frac{1}{M} - s')k} 2^{-(\frac{3}{2} - \frac{1}{M} + s')k}$$

is fine, and

$$
\begin{aligned}
(2.B.15) &\lesssim 2^{-k}\|P_kQ_{\sim k}[Q_{\ll k}\nabla\tilde{\phi}_{\sim k}\cdot P_{\ll k}Q_{\sim k}g'(\tilde{\phi})]\|_{2,2} \\
&\quad + 2^{-k}\|P_kQ_{\sim k}[Q_{\ll k}\nabla\tilde{\phi}_{\sim k}\cdot P_{\sim k}g'(\tilde{\phi})]\|_{2,2} \\
&\lesssim 2^{-k}\|Q_{\ll k}\nabla\tilde{\phi}_{\sim k}\cdot\partial_t^{-1}P_{\ll k}Q_{\sim k}(\partial_t\tilde{\phi}\ g''(\tilde{\phi}))\|_{2,2} \\
&\quad + 2^{-k}\|Q_{\ll k}\nabla\tilde{\phi}_{\sim k}\cdot\nabla^{-1}P_{\sim k}Q_{\sim k}(\nabla\tilde{\phi}\ g''(\tilde{\phi}))\|_{2,2} \\
&\lesssim 2^{-2k}(\|\nabla\tilde{\phi}_{\sim k}\|_{2M,\frac{2M}{M-1}}\|\partial_t\tilde{\phi}_{\lesssim k}\|_{\frac{2M}{M-1},2M} + \|\nabla\tilde{\phi}_{\sim k}\|_{\frac{2M}{M-1},2M}\|\partial_t\tilde{\phi}_{\gg k}\|_{2M,\frac{2M}{M-1}}) \\
&\quad + 2^{-2k}(\|\nabla\tilde{\phi}_{\sim k}\|_{2M,\frac{2M}{M-1}}\|\nabla\tilde{\phi}_{\lesssim k}\|_{\frac{2M}{M-1},2M} + \|\nabla\tilde{\phi}_{\sim k}\|_{\frac{2M}{M-1},2M}\|\nabla\tilde{\phi}_{\gg k}\|_{2M,\frac{2M}{M-1}}) \\
&\lesssim 2^{-(2+2s')k}
\end{aligned}
$$

- $\underline{j \ll k}$: This time our goal is

$$\|P_kQ_jg(\tilde{\phi})\|_{2,2} \lesssim 2^{-sk-\theta j}$$

We have

$$
\begin{aligned}
\|P_kQ_jg(\tilde{\phi})\|_{2,2} &\lesssim 2^{-k}\|P_kQ_j(\nabla\tilde{\phi}_{\ll k}\cdot P_{\sim k}g'(\tilde{\phi}))\|_{2,2} &\quad (2.B.17)\\
&\quad + 2^{-k}\|P_kQ_j(\nabla\tilde{\phi}_{\sim k}\cdot g'(\tilde{\phi}))\|_{2,2} &\quad (2.B.18)\\
&\quad + 2^{-k}\sum_{r\gg k}\|P_kQ_j(\nabla\tilde{\phi}_r\cdot P_{\sim r}g'(\tilde{\phi}))\|_{2,2} &\quad (2.B.19)
\end{aligned}
$$

Here (2.B.17) and (2.B.19) can be handled as in the case $j \simeq k$. For (2.B.18) we separate

$$(2.B.18) \lesssim 2^{-k}\|P_kQ_j(Q_{\ll j}\nabla\tilde{\phi}_{\sim k}\cdot g'(\tilde{\phi}))\|_{2,2} + 2^{-k}\|P_kQ_j(Q_{\gtrsim j}\nabla\tilde{\phi}_{\sim k}\cdot g'(\tilde{\phi}))\|_{2,2}$$

The second line here is straightforward to handle by placing $\nabla\tilde{\phi}$ into $L_t^2L_x^2$, so we consider only the first term. Referring to the result for $j \gg k$ to handle g' we find

$$
\begin{aligned}
2^{-k}\|P_kQ_j(Q_{\ll j}\nabla\tilde{\phi}_{\sim k}\cdot g'(\tilde{\phi}))\|_{2,2} &\lesssim 2^{-k}\|P_kQ_j(Q_{\ll j}\nabla\tilde{\phi}_{\sim k}\cdot Q_{\gtrsim j}P_{\ll j}g'(\tilde{\phi}))\|_{2,2} \\
&\quad + 2^{-k}\|P_kQ_j(Q_{\ll j}\nabla\tilde{\phi}_{\sim k}\cdot P_{\gtrsim j}g'(\tilde{\phi}))\|_{2,2} \\
&\lesssim 2^{-k}\|Q_{\ll j}\nabla\tilde{\phi}_{\sim k}\|_{\infty,2}\sum_{\substack{l\gtrsim j\\ r\ll j}}\|Q_lP_rg'(\tilde{\phi}))\|_{2,\infty} \\
&\quad + 2^{-k}\|Q_{\ll j}\nabla\tilde{\phi}_{\sim k}\|_{2M,2}\sum_{r\gtrsim j}2^{-r}\|P_r(\nabla\tilde{\phi}\ g''(\tilde{\phi}))\|_{\frac{2M}{M-1},\infty} \\
&\lesssim 2^{-sk-\theta j}
\end{aligned}
$$

for $s' - \frac{1}{2M} \geq \theta'$. This completes the proof of the Moser estimate.

$\square$

We are now in a position to prove the projection estimate.

Proof of (2.10.11). As before, we consider the different regimens of (j,k) separately.

- $\underline{k \ll j}$: Here

$$\|P_k Q_j \widetilde{\Pi}_{\tilde{\phi}^\perp}((-\Delta)^{1/2}\phi)\|_{X^{s,\theta}} \lesssim 2^k \|\phi_k\|_{X^{s,\theta}} + 2^{(s+\theta)j}\|P_k Q_j((-\Delta)^{1/2}\phi \cdot g(\tilde{\phi})\ g(\tilde{\phi}))\|_{2,2}$$

Then setting $G(\tilde{\phi}) := g(\tilde{\phi}) \cdot g(\tilde{\phi})$ we have

$$\begin{aligned} &2^{(s+\theta)j}\|P_k Q_j((-\Delta)^{1/2}\phi \cdot G(\tilde{\phi}))\|_{2,2} \\ &\lesssim 2^{(s+\theta)j} \sum_{r \gg k} \|P_k Q_j(P_r(-\Delta)^{1/2}\phi \cdot P_{\sim r}(G(\tilde{\phi}))\|_{2,2} && \text{(A)} \\ &\quad + 2^{(s+\theta)j}\|P_k Q_j(Q_{\gtrsim j} P_{\lesssim k}(-\Delta)^{1/2}\phi \cdot G(\tilde{\phi}))\|_{2,2} && \text{(B)} \\ &\quad + 2^{(s+\theta)j}\|P_k Q_j(Q_{\ll j} P_{\lesssim k}(-\Delta)^{1/2}\phi \cdot G(\tilde{\phi}))\|_{2,2} && \text{(C)} \end{aligned}$$

The easiest of these terms is (B):

$$\text{(B)} \lesssim 2^{(s+\theta)j} \sum_{l \gtrsim j} \sum_{r \lesssim k} \|Q_l P_r(-\Delta)^{1/2}\phi\|_{2,2} \lesssim 2^k \sum_{r \lesssim k} 2^{r-k}\|\phi_r\|_{X^{s,\theta}}$$

For (C) we use the Moser estimate just proved to see that

$$\begin{aligned} \text{(C)} &\lesssim 2^{(s+\theta)j}\|P_k Q_j(Q_{\ll j} P_{\lesssim k}(-\Delta)^{1/2}\phi \cdot P_{\sim k} Q_{\sim j} G(\tilde{\phi}))\|_{2,2} \\ &\lesssim 2^{(s+\theta)j}\|Q_{\ll j} P_{\lesssim k}(-\Delta)^{1/2}\phi\|_{\infty,\infty}\|P_{\sim k} Q_{\sim j} G(\tilde{\phi})\|_{2,2} \\ &\lesssim 2^{(1-s')k} \sum_{r \lesssim k} 2^{(1-s')(r-k)}\|\phi_r\|_{X^{s,\theta}} \end{aligned}$$

Lastly, we turn to (A):

$$\begin{aligned} \text{(A)} &\lesssim 2^{(s+\theta)j} \sum_{r \gtrsim j} \|P_k Q_j(P_r(-\Delta)^{1/2}\phi \cdot P_{\sim r} G(\tilde{\phi}))\|_{2,2} && \text{(A.1)} \\ &\quad + 2^{(s+\theta)j} \sum_{r=k+10}^{j-10} \|P_k Q_j(P_r(-\Delta)^{1/2}\phi \cdot P_{\sim r} G(\tilde{\phi}))\|_{2,2} && \text{(A.2)} \end{aligned}$$

For (A.1) we have

$$\begin{aligned} \text{(A.1)} &\lesssim 2^{(s+\theta)j} \sum_{r \gtrsim j} \|P_r(-\Delta)^{1/2}\phi\|_{\frac{2M}{M-1},2M}\|P_{\sim r} G(\tilde{\phi})\|_{2M,\frac{2M}{M-1}} \\ &\lesssim 2^{(s+\theta)j} \sum_{r \gtrsim j} 2^{(\frac{1}{2}-\frac{1}{M}-s')r} 2^{-(\frac{3}{2}-\frac{1}{M}+s')r}\|\phi_r\|_{X^{s,\theta}} \\ &\lesssim 2^j \sum_{r \gtrsim j} 2^{(1-\theta-s)(r-j)}\|\phi_r\|_{X^{s,\theta}} \end{aligned}$$

which is acceptable. For (A.2) we separate the $(-\Delta)^{1/2}\phi$ into low and high modulations. The high modulation part is easy:

$$
\begin{aligned}
&2^{(s+\theta)j}\sum_{r=k+10}^{j-10}\|P_kQ_j(Q_{\gtrsim j}P_r(-\Delta)^{1/2}\phi\cdot P_{\sim r}G(\tilde{\phi}))\|_{2,2}\\
&\lesssim 2^{(s+\theta)j}\sum_{r=k+10}^{j-10}2^{3k/2}\|Q_{\gtrsim j}P_r(-\Delta)^{1/2}\phi\|_{2,2}\|P_{\sim r}G(\tilde{\phi})\|_{\infty,2}\\
&\lesssim 2^{(1-s')k}\sum_{r\gg k}2^{(1-s)(r-k)}\|\phi_r\|_{X^{s,\theta}}
\end{aligned}
$$

For the low modulation part we observe that the factor $G(\tilde{\phi})$ must have modulation $\sim 2^j$, so we have

$$
\begin{aligned}
&2^{(s+\theta)j}\sum_{r=k+10}^{j-10}\|P_kQ_j(Q_{\ll j}P_r(-\Delta)^{1/2}\phi\cdot P_{\sim r}G(\tilde{\phi}))\|_{2,2}\\
&\lesssim 2^{(s+\theta)j}\sum_{r=k+10}^{j-10}2^{3k/2}\|Q_{\ll j}P_r(-\Delta)^{1/2}\phi\|_{\infty,2}\|P_{\sim r}Q_{\sim j}G(\tilde{\phi})\|_{2,2}\\
&\lesssim 2^{(1-s')k}\sum_{r\gg k}2^{(1-s)(r-k)}\|\phi_r\|_{X^{s,\theta}}
\end{aligned}
$$

- $\underline{k\gtrsim j}$: In this case we have

$$
\begin{aligned}
&2^{sk+\theta j}\|P_kQ_j[(-\Delta)^{1/2}\phi\cdot G(\tilde{\phi})]\|_{2,2}\\
&\lesssim 2^{sk+\theta j}\|P_kQ_j[Q_{\gtrsim j}P_{\gtrsim k}(-\Delta)^{1/2}\phi\cdot G(\tilde{\phi})]\|_{2,2}\\
&\quad+2^{sk+\theta j}\|P_kQ_j[Q_{\ll j}P_{\gtrsim k}(-\Delta)^{1/2}\phi\cdot G(\tilde{\phi})]\|_{2,2}\\
&\quad+2^{sk+\theta j}\|P_kQ_j[P_{\ll k}(-\Delta)^{1/2}\phi\cdot G(\tilde{\phi})]\|_{2,2}\\
&\lesssim 2^{sk+\theta j}\sum_{l\gtrsim j}\sum_{r\gtrsim k}2^{-\theta l}2^{(1-s)r}\|\phi_r\|_{X^{s,\theta}}\\
&\quad+2^{sk+\theta j}\|P_{\ll k}(-\Delta)^{1/2}\phi\|_{\frac{2M}{M-1},2M}\|P_{\gtrsim k}G(\tilde{\phi})\|_{2M,\frac{2M}{M-1}}\\
&\quad+2^{sk+\theta j}\|P_kQ_j[Q_{\ll j}P_{\gtrsim k}(-\Delta)^{1/2}\phi\cdot G(\tilde{\phi})]\|_{2,2}\\
&\lesssim 2^k\sum_{r\gtrsim k}2^{(1-s)(r-k)}\|\phi_r\|_{X^{s,\theta}}+2^{sk+\theta j}2^{-(\frac{3}{2}-\frac{1}{M}+s')k}\sum_{r\ll k}2^{(\frac{1}{2}-\frac{1}{M}-s')r}\|\phi_r\|_{X^{s,\theta}}\\
&\quad+2^{sk+\theta j}\|P_kQ_j[Q_{\ll j}P_{\gtrsim k}(-\Delta)^{1/2}\phi\cdot G(\tilde{\phi})]\|_{2,2}
\end{aligned}
$$

The first two terms above are acceptable so it remains to study the third. We first consider the case where $(-\Delta)^{1/2}\phi$ is at frequency $\gg 2^k$. Here we have

$$
\begin{aligned}
&2^{sk+\theta j}\|P_kQ_j[Q_{\ll j}P_{\gg k}(-\Delta)^{1/2}\phi\cdot G(\tilde{\phi})]\|_{2,2}\\
&\lesssim 2^{sk+\theta j}\sum_{r\gg k}\|P_kQ_j[Q_{\ll j}P_r(-\Delta)^{1/2}\phi\cdot P_{\sim r}G(\tilde{\phi})]\|_{2,2}
\end{aligned}
$$

$$\lesssim 2^{sk+\theta j} \sum_{r \gg k} \|Q_{\ll j} P_r (-\Delta)^{1/2} \phi\|_{\frac{2M}{M-1}, 2M} \|P_{\sim r} G(\tilde{\phi})\|_{2M, \frac{2M}{M-1}}$$

which is acceptable.

For the intermediate frequency case we have

$$\begin{aligned}
&2^{sk+\theta j} \|P_k Q_j [Q_{\ll j} P_{\sim k} (-\Delta)^{1/2} \phi \cdot G(\tilde{\phi})]\|_{2,2} \\
&\lesssim 2^{sk+\theta j} \|P_k Q_j [Q_{\ll j} P_{\sim k} (-\Delta)^{1/2} \phi \cdot P_{\sim k} G(\tilde{\phi})]\|_{2,2} \\
&\quad + 2^{sk+\theta j} \|P_k Q_j [Q_{\ll j} P_{\sim k} (-\Delta)^{1/2} \phi \cdot P_{\ll k} G(\tilde{\phi})]\|_{2,2} \\
&\lesssim 2^{sk+\theta j} \|Q_{\ll j} P_{\sim k} (-\Delta)^{1/2} \phi\|_{\frac{2M}{M-1}, 2M} \|P_{\sim k} G(\tilde{\phi})\|_{2M, \frac{2M}{M-1}} \\
&\quad + 2^{sk+\theta j} \|P_k Q_j [Q_{\ll j} P_{\sim k} (-\Delta)^{1/2} \phi \cdot P_{\ll k} G(\tilde{\phi})]\|_{2,2}
\end{aligned}$$

The first term here is acceptable, however for the second we need to study $j \simeq k$ and $j \gg k$ separately. If $j \simeq k$ we note that $G(\tilde{\phi})$ must have modulation at least $\sim 2^k$ and so

$$\begin{aligned}
&2^{sk+\theta j} \|P_k Q_j [Q_{\ll j} P_{\sim k} (-\Delta)^{1/2} \phi \cdot P_{\ll k} G(\tilde{\phi})]\|_{2,2} \\
&\lesssim 2^{(s+\theta)k} \|Q_{\ll k} P_{\sim k} (-\Delta)^{1/2} \phi\|_{\infty,\infty} \|Q_{\gtrsim k} P_{\ll k} G(\tilde{\phi})\|_{2,2}
\end{aligned}$$

which is acceptable.

In the case $k \gg j$ we note that if $G(\tilde{\phi})$ in fact has frequency $\gtrsim 2^j$ we are fine:

$$\begin{aligned}
&2^{sk+\theta j} \|P_k Q_j [Q_{\ll j} P_{\sim k} (-\Delta)^{1/2} \phi \cdot P_{[j-10,k-10]} G(\tilde{\phi})]\|_{2,2} \\
&\lesssim 2^{sk+\theta j} \|Q_{\ll j} P_{\sim k} (-\Delta)^{1/2} \phi\|_{2M, \frac{2M}{M-1}} \|P_{[j-10,k-10]} G(\tilde{\phi})\|_{\frac{2M}{M-1}, 2M} \\
&\lesssim 2^k \|\phi_k\|_{X^{s,\theta}}
\end{aligned}$$

while if it has extremely low frequency ($\ll 2^j$), then it must have modulation at least comparable to 2^j:

$$\begin{aligned}
&2^{sk+\theta j} \|P_k Q_j [Q_{\ll j} P_{\sim k} (-\Delta)^{1/2} \phi \cdot Q_{\gtrsim j} P_{\ll j} G(\tilde{\phi})]\|_{2,2} \\
&\lesssim 2^{sk+\theta j} \|Q_{\ll j} P_{\sim k} (-\Delta)^{1/2} \phi\|_{\infty,2} \|Q_{\gtrsim j} P_{\ll j} G(\tilde{\phi})\|_{2,\infty} \\
&\lesssim 2^k \|\phi_k\|_{X^{s,\theta}}
\end{aligned}$$

This completes the proof of the nonlinear projection estimate.

□

2.C Proof of Lemma 2.10.10.

In this appendix we will constantly use the Strichartz estimate Lemma 2.10.3 and the modulation-Bernstein estimate (MB). We start with the first statement in Lemma

2.10.10. As usual, $M = \infty-$ is taken to be a very large constant.

Proof of (2.10.14). First suppose $|k_2 - k_3| \le 10$, so $\varphi_{k_2} \cdot F_{k_3} = P_{\lesssim k_2}(\varphi_{k_2} \cdot F_{k_3})$. We first consider the case when the whole term is at low modulation $Q_{\lesssim k_2}$ and consider the different possibilities for the modulation of F_{k_3}.

- $P_{\lesssim k_2} Q_{\lesssim k_2}(\varphi_{k_2} \cdot Q_{\lesssim k_2} F_{k_3})$: Using (MB) we have

$$\begin{aligned}
&\|P_{\lesssim k_2} Q_{\lesssim k_2}(\varphi_{k_2} \cdot Q_{\lesssim k_2} F_{k_3})\|_{X^{s-1,\theta-1}} \\
&\lesssim \sum_{l \lesssim k_2} 2^{(s-1)k_2} 2^{(\theta-1)l} 2^{(\frac{1}{2}-\frac{1}{M})l} \|P_{\lesssim k_2} Q_l(\varphi_{k_2} \cdot Q_{\lesssim k_2} F_{k_3})\|_{\frac{M}{M-1},2} \\
&\lesssim \sum_{l \lesssim k_2} 2^{(s-1)k_2} 2^{(\theta-1)l} 2^{(\frac{1}{2}-\frac{1}{M})l} \|\varphi_{k_2}\|_{\frac{2M}{M-2},\infty} \|Q_{\lesssim k_2} F_{k_3}\|_{2,2} \\
&\lesssim 2^{-(\theta'+s'-\frac{1}{M})k_2} \sum_{l \lesssim k_2} 2^{(\theta'-\frac{1}{M})l} \|\varphi\|_{X^{s,\theta}} \|F\|_{X^{s-1,\theta-1}} \\
&\lesssim 2^{-s'k_2} \|\varphi\|_{X^{s,\theta}} \|F\|_{X^{s-1,\theta-1}}
\end{aligned}$$

  which is acceptable provided we choose $\frac{1}{M} < \theta'$.

- $P_{\lesssim k_2} Q_{\lesssim k_2}(\varphi_{k_2} \cdot Q_{\gg k_2} F_{k_3})$: Here, since $k_2 \simeq k_3$, we note that for the output modulation to be at most $\sim 2^{k_2}$, the modulation of the two inner factors must be comparable. We therefore have

$$\begin{aligned}
&\|P_{\lesssim k_2} Q_{\lesssim k_2}(\varphi_{k_2} \cdot Q_{\gg k_2} F_{k_3})\|_{X^{s-1,\theta-1}} \\
&\lesssim \sum_{l \lesssim k_2} \sum_{r \gg k_2} 2^{(s-1)k_2} 2^{(\theta-1)l} 2^{(\frac{1}{2}-\frac{1}{M})l} \|P_{\lesssim k_2} Q_l(Q_{\sim r}\varphi_{k_2} \cdot Q_r F_{k_3})\|_{\frac{M}{M-1},2} \\
&\lesssim \sum_{l \lesssim k_2} \sum_{r \gg k_2} 2^{(s-1)k_2} 2^{(\theta'-\frac{1}{M})l} \|Q_{\sim r}\varphi_{k_2}\|_{\frac{2M}{M-2},\infty} \|Q_r F_{k_3}\|_{2,2} \\
&\lesssim \sum_{l \lesssim k_2} \sum_{r \gg k_2} 2^{(s-1)k_2} 2^{(\theta'-\frac{1}{M})l} \cdot 2^{(\frac{1}{M}-\theta)r} 2^{(\frac{3}{2}-s)k_2} \|\varphi_{k_2}\|_{X^{s,\theta}} \cdot 2^{(1-\theta)r} 2^{(1-s)k_3} \|F_{k_3}\|_{X^{s-1,\theta-1}} \\
&\lesssim 2^{-(s'+\theta')k_2} \|\varphi_{k_2}\|_{X^{s,\theta}} \|F_{k_3}\|_{X^{s-1,\theta-1}}
\end{aligned}$$

  provided again $\frac{1}{M} < \theta'$.

We next study the case in which the whole term is at high modulation.

- $\sum_{l \gg k_2} P_{\lesssim k_2} Q_l(Q_{\gtrsim l}\varphi_{k_2} \cdot F_{k_3})$: In this case the modulation of F_{k_3} can be at most comparable to that of φ_{k_2}, i.e.

$$\sum_{l \gg k_2} \|P_{\lesssim k_2} Q_l(Q_{\gtrsim l}\varphi_{k_2} \cdot F_{k_3})\|_{X^{s-1,\theta-1}} \lesssim \sum_{l \gg k_2} \sum_{r \gtrsim l} \|P_{\lesssim k_2} Q_l(Q_r\varphi_{k_2} \cdot Q_{\lesssim r} F_{k_3})\|_{X^{s-1,\theta-1}}$$

Using (MB) to place φ_{k_2} into $L^2_t L^2_x$ and F_{k_3} into $L^2_t L^\infty_x$ then applying Lemma 2.10.3 we bound this by

$$\begin{aligned}
&\sum_{l\gg k_2}\sum_{r\gtrsim l} 2^{(s-1)l}2^{(\theta-1)l}2^{l/2}\|Q_r\varphi_{k_2}\|_{2,2}\|Q_{\lesssim r}F_{k_3}\|_{2,\infty}\\
&\lesssim \sum_{l\gg k_2}\sum_{r\gtrsim l} 2^{(s-1)l}2^{(\theta-1)l}2^{l/2}2^{-\theta r}2^{-sr}\|\varphi_{k_2}\|_{X^{s,\theta}}2^{(1-\theta)r}2^{(1-s)k_3}2^{3k_3/2}\|F_{k_3}\|_{X^{s-1,\theta-1}}\\
&\lesssim 2^{-(s'+\theta')k_2}\|\varphi_{k_2}\|_{X^{s,\theta}}\|F_{k_3}\|_{X^{s-1,\theta-1}}
\end{aligned}$$

- $\sum_{l\gg k_2} P_{\lesssim k_2}Q_l(Q_{<l-10}\varphi_{k_2}\cdot F_{k_3})$: This time we use the L^2 structure of $X^{s-1,\theta-1}$ to square-sum over l, and use that F_{k_3} must be at modulation comparable to l to find

$$\begin{aligned}
&\|\sum_{l\gg k_2} P_{\lesssim k_2}Q_l(Q_{<l-10}\varphi_{k_2}\cdot F_{k_3})\|_{X^{s-1,\theta-1}}\\
&\lesssim (\sum_{l\gg k_2}\|P_{\lesssim k_2}Q_l(Q_{<l-10}\varphi_{k_2}\cdot Q_{\sim l}F_{k_3})\|^2_{X^{s-1,\theta-1}})^{\frac12}\\
&\lesssim (\sum_{l\gg k_2}(2^{(s+\theta-2)l}\|Q_{<l-10}\varphi_{k_2}\|_{\infty,\infty}\|Q_{\sim l}F_{k_3}\|_{2,2})^2)^{\frac12}\\
&\lesssim (\sum_{l\gg k_2}(2^{(s+\theta-2)l}2^{-s'k_2}\|\varphi_{k_2}\|_{X^{s,\theta}}2^{(1-\theta)l}2^{(1-s)l}\|Q_{\sim l}F_{k_3}\|_{X^{s-1,\theta-1}})^2)^{\frac12}
\end{aligned}$$

which is acceptable.

We now come to the case $k_2 \geq k_3+10$, $\varphi_{k_2}\cdot F_{k_3} = P_{\sim k_2}(\varphi_{k_2}\cdot F_{k_3})$. Again we split into different cases depending on the whether the term and its factors are at high or low modulation.

- $P_{\sim k_2}Q_{\lesssim k_2}(\varphi_{k_2}\cdot Q_{\lesssim k_3}F_{k_3})$: This time we use (MB) to see

$$\begin{aligned}
&\|P_{\sim k_2}Q_{\lesssim k_2}(\varphi_{k_2}\cdot Q_{\lesssim k_3}F_{k_3})\|_{X^{s-1,\theta-1}}\\
&\lesssim \sum_{l\lesssim k_2} 2^{(s-1)k_2}2^{(\theta-1+\frac12-\frac1M)l}\|\varphi_{k_2}\|_{\frac{2M}{M-2},M}\|Q_{\lesssim k_3}F_{k_3}\|_{2,\frac{2M}{M-2}}\\
&\lesssim 2^{(s-1)k_2}\sum_{l\lesssim k_2}2^{(\theta'-\frac1M)l}\cdot 2^{-(\frac12+\frac2M+s')k_2}\|\varphi_{k_2}\|_{X^{s,\theta}}\cdot 2^{(1-\theta)k_3}2^{(1-s)k_3}2^{3k_3/M}\|F_{k_3}\|_{X^{s-1,\theta-1}}\\
&\lesssim 2^{(\theta'-\frac3M)k_2}2^{(\frac3M-s'-\theta')k_3}\|\varphi_{k_2}\|_{X^{s,\theta}}\|F_{k_3}\|_{X^{s-1,\theta-1}}
\end{aligned}$$

which is acceptable provided we choose $\frac{\theta'}{3}<\frac1M<\theta'$ and $\frac3M<s'+\theta'$.

- $P_{\sim k_2}Q_{\lesssim k_2}(\varphi_{k_2}\cdot Q_{\gg k_3}F_{k_3})$: This time we have

$$\|P_{\sim k_2}Q_{\lesssim k_2}(\varphi_{k_2}\cdot Q_{\gg k_3}F_{k_3})\|_{X^{s-1,\theta-1}}$$

$$
\begin{aligned}
&\lesssim \sum_{l\lesssim k_2}\sum_{j\gg k_3} 2^{(s-1)k_2}2^{(\theta'-\frac{1}{M})l}\|\varphi_{k_2}\|_{\frac{2M}{M-2},M}\|Q_jF_{k_3}\|_{2,\frac{2M}{M-2}}\\
&\lesssim \sum_{l\lesssim k_2}\sum_{j\gg k_3} 2^{(s-1)k_2}2^{(\theta'-\frac{1}{M})l}2^{-(\frac{1}{2}+\frac{2}{M}+s')k_2}\|\varphi_{k_2}\|_{X^{s,\theta}}\cdot 2^{3k_3/M}2^{(1-\theta)j}2^{(1-s)j}\|F_{k_3}\|_{X^{s-1,\theta-1}}\\
&\lesssim 2^{(\theta'-\frac{3}{M})k_2}2^{(\frac{3}{M}-s'-\theta')k_3}\|\varphi_{k_2}\|_{X^{s,\theta}}\|F_{k_3}\|_{X^{s-1,\theta-1}}
\end{aligned}
$$

with M as in the previous case.

- $\sum_{j\gg k_2} P_{\sim k_2}Q_j(\varphi_{k_2}\cdot Q_{\ll j}F_{k_3})$: Since the outer modulation far exceeds any of the other scales involved in this expression, we see that φ_{k_2} must be restricted to modulation at least comparable to 2^j. We then find

$$
\begin{aligned}
&\sum_{j\gg k_2}\|P_{\sim k_2}Q_j(\varphi_{k_2}\cdot Q_{\ll j}F_{k_3})\|_{X^{s-1,\theta-1}}\\
&\lesssim \sum_{j\gg k_2} 2^{(s+\theta-2)j}\|Q_{\gtrsim j}\varphi_{k_2}\|_{\infty,2}\|Q_{\ll j}F_{k_3}\|_{2,\infty}\\
&\lesssim 2^{-(s'+\theta')k_2}\|\varphi_{k_2}\|_{X^{s,\theta}}\|F_{k_3}\|_{X^{s-1,\theta-1}}
\end{aligned}
$$

where we used that $k_3<k_2$.

- $\sum_{j\gg k_2} P_{\sim k_2}Q_j(\varphi_{k_2}\cdot Q_{\sim j}F_{k_3})$: This case can be handled straightforwardly by again square-summing over j and placing φ_{k_2} into $L_t^\infty L_x^2$ and F_{k_3} into $L_t^2L_x^\infty$.

- $\sum_{j\gg k_2} P_{\sim k_2}Q_j(\varphi_{k_2}\cdot Q_{\gg j}F_{k_3})$: Here we observe that φ_{k_2} is restricted to modulation comparable to that of F_{k_3}, and find

$$
\begin{aligned}
&\sum_{j\gg k_2}\|P_{\sim k_2}Q_j(\varphi_{k_2}\cdot Q_{\gg j}F_{k_3})\|_{X^{s-1,\theta-1}}\\
&\lesssim \sum_{j\gg k_2}\sum_{r\gg j} 2^{(s+\theta-2)j}\|Q_{\sim r}\varphi_{k_2}\|_{\infty,2}\|Q_rF_{k_3}\|_{2,\infty}\\
&\lesssim \sum_{j\gg k_2}\sum_{r\gg j} 2^{(s+\theta-2)j}2^{-\theta'r}2^{-sr}\|\varphi_{k_2}\|_{X^{s,\theta}}2^{(1-\theta)r}2^{(1-s)r}2^{3k_3/2}\|F_{k_3}\|_{X^{s-1,\theta-1}}
\end{aligned}
$$

which is acceptable. This concludes the case $k_2\geq k_3+10$.

Lastly, we consider $k_3\geq k_2+10$, so that $\varphi_{k_2}\cdot F_{k_3}=P_{\sim k_3}(\varphi_{k_2}\cdot F_{k_3})$. In this case we have to consider three cases for the outer modulation, depending on both k_2 and k_3.

- $\sum_{k_2\ll l\lesssim k_3} P_{\sim k_3}Q_l(\varphi_{k_2}\cdot Q_{\gg l}F_{k_3})$: Using observations on the modulation restrictions as before we have

$$
\sum_{k_2\ll l\lesssim k_3}\sum_{j\gg l}\|P_{\sim k_3}Q_l(\varphi_{k_2}\cdot Q_jF_{k_3})\|_{X^{s-1,\theta-1}}
$$

$$\lesssim \sum_{k_2 \ll l \lesssim k_3} \sum_{j \gg l} 2^{(s-1)k_3} 2^{(\theta-1)l} 2^{l/2} \|Q_{\gtrsim j}\varphi_{k_2}\|_{2,\infty} \|Q_j F_{k_3}\|_{2,2}$$
$$\lesssim \sum_{k_2 \ll l \lesssim k_3} \sum_{j \gg l} 2^{(s-1)k_3} 2^{(\theta-1)l} 2^{l/2} 2^{3k_2/2} 2^{-\theta j} 2^{-sj} \|\varphi_{k_2}\|_{X^{s,\theta}} 2^{(1-\theta)j} 2^{(1-s)k_3} \|Q_j F_{k_3}\|_{X^{s-1,\theta-1}}$$
$$\lesssim 2^{-(s'+\theta')k_2} \|\varphi_{k_2}\|_{X^{s,\theta}} \|F_{k_3}\|_{X^{s-1,\theta-1}}$$

- $\sum_{k_2 \ll l \lesssim k_3} P_{\sim k_3} Q_l(\varphi_{k_2} \cdot Q_{\lesssim l} F_{k_3})$: Again square-summing we have

$$\| \sum_{k_2 \ll l \lesssim k_3} \sum_{j \lesssim l} P_{\sim k_3} Q_l(\varphi_{k_2} \cdot Q_j F_{k_3}) \|_{X^{s-1,\theta-1}}$$
$$\lesssim \Big(\sum_{k_2 \ll l \lesssim k_3} \Big(\sum_{j \lesssim l} 2^{(s-1)k_3} 2^{(\theta-1)l} \|\varphi_{k_2}\|_{\infty,\infty} \|Q_j F_{k_3}\|_{2,2} \Big)^2 \Big)^{\frac{1}{2}}$$

which is readily seen to be acceptable.

- $P_{\sim k_3} Q_{\lesssim k_2}(\varphi_{k_2} \cdot Q_{\lesssim k_2} F_{k_3})$: In this case we place φ_{k_2} into $L_t^{\frac{2M}{M-2}} L_x^\infty$ and F_{k_3} directly into $L_t^2 L_x^2$ to bound

$$\|P_{\sim k_3} Q_{\lesssim k_2}(\varphi_{k_2} \cdot Q_{\gg k_2} F_{k_3})\|_{X^{s-1,\theta-1}}$$
$$\lesssim \sum_{l \lesssim k_2} 2^{(s-1)k_3} 2^{(\theta-1)l} 2^{(\frac{1}{2}-\frac{1}{M})l} 2^{-(\frac{1}{2}-\frac{1}{M}+s')k_2} \|\varphi_{k_2}\|_{X^{s,\theta}} 2^{(1-\theta)k_2} 2^{(1-s)k_3} \|F_{k_3}\|_{X^{s-1,\theta-1}}$$
$$\lesssim 2^{-s'k_2} \|\varphi_{k_2}\|_{X^{s,\theta}} \|F_{k_3}\|_{X^{s-1,\theta-1}}$$

choosing $\frac{1}{M} < \theta'$.

- $P_{\sim k_3} Q_{\lesssim k_2}(\varphi_{k_2} \cdot Q_{\gg k_2} F_{k_3})$: This time we observe that φ_{k_2} must be at modulation at least of order 2^j and bound

$$\sum_{j \gg k_2} \|P_{\sim k_3} Q_{\lesssim k_2}(\varphi_{k_2} \cdot Q_j F_{k_3})\|_{X^{s-1,\theta-1}}$$
$$\lesssim \sum_{j \gg k_2} \sum_{l \lesssim k_2} 2^{(s-1)k_3} 2^{(\theta-1)l} 2^{l/2} \|Q_{\gtrsim j}\varphi_{k_2}\|_{2,\infty} \|Q_j F_{k_3}\|_{2,2}$$
$$\lesssim 2^{-(\theta'+s')k_2} \|\varphi_{k_2}\|_{X^{s,\theta}} \|F_{k_3}\|_{X^{s-1,\theta-1}}$$

- $\sum_{l \gg k_3} P_{\sim k_3} Q_l(\varphi_{k_2} \cdot Q_{\gtrsim l} F_{k_3})$: This time we square-sum over l and place F_{k_3} directly into $L_t^2 L_x^2$ to find

$$\| \sum_{l \gg k_3} P_{\sim k_3} Q_l(\varphi_{k_2} \cdot Q_{\gtrsim l} F_{k_3}) \|_{X^{s-1,\theta-1}}$$

$$\lesssim \Big(\sum_{l\gg k_3} 2^{(s+\theta-2)l}\big(\sum_{j\gtrsim l} 2^{-s'k_2}\|\varphi_{k_2}\|_{X^{s,\theta}}2^{(2-s-\theta)j}\|Q_jF_{k_3}\|_{X^{s-1,\theta-1}}\big)^2\Big)^{\frac{1}{2}}$$

which is acceptable upon applying the Cauchy-Schwarz inequality in j.

- $\sum_{l\gg k_3} P_{\sim k_3}Q_l(\varphi_{k_2}\cdot Q_{\ll l}F_{k_3})$: For this final case we note that the entire term vanishes unless φ_{k_2} is at modulation at least $\sim 2^l$, and find

$$\begin{aligned}
&\sum_{l\gg k_3}\|P_{\sim k_3}Q_l(\varphi_{k_2}\cdot Q_{\ll l}F_{k_3})\|_{X^{s-1,\theta-1}}\\
&\lesssim \sum_{l\gg k_3} 2^{(s+\theta-2)l}\|Q_{\gtrsim l}\varphi_{k_2}\|_{2,2}\|Q_{\ll l}F_{k_3}\|_{\infty,\infty}\\
&\lesssim \sum_{l\gg k_3} 2^{(s+\theta-2)l}2^{-\theta l}2^{-sl}\|\varphi_{k_2}\|_{X^{s,\theta}}2^{(\frac{3}{2}-\theta)l}2^{(1-s')k_3}\|F_{k_3}\|_{X^{s-1,\theta-1}}\\
&\lesssim 2^{-(s'+\theta')k_3}\|\varphi_{k_2}\|_{X^{s,\theta}}\|F_{k_3}\|_{X^{s-1,\theta-1}}
\end{aligned}$$

This completes the proof of (2.10.14). □

We are now ready to prove the second statement of the lemma, which is similar to the first, however somewhat simpler due to symmetry reductions.

Proof of (2.10.15). Assume without loss of generality $k_2\ge k_3$. First suppose $k_2\ge k_3+10$ so the whole term is at frequency $\sim 2^{k_2}$. We split into the following cases:

- $\sum_{l\lesssim k_3} P_{\sim k_2}Q_l(Q_{\gtrsim l}\varphi^{(2)}_{k_2}\cdot\varphi^{(3)}_{k_3})$: Square summing over l we have

$$\begin{aligned}
\|\sum_{l\lesssim k_3} P_{\sim k_2}Q_l(Q_{\gtrsim l}\varphi^{(2)}_{k_2}\cdot\varphi^{(3)}_{k_3})\|_{X^{s,\theta}} &\lesssim \left(\sum_{l\lesssim k_3}(2^{sk_2}2^{\theta l}\|Q_{\gtrsim l}\varphi^{(2)}_{k_2}\|_{2,2}\|\varphi^{(3)}_{k_3}\|_{\infty,\infty})^2\right)^{\frac{1}{2}}\\
&\lesssim \left(\sum_{l\lesssim k_3}\big(\sum_{j\gtrsim l}2^{\theta(l-j)}\|Q_j\varphi^{(2)}_{k_2}\|_{X^{s,\theta}}\cdot 2^{-s'k_3}\|\varphi^{(3)}_{k_3}\|_{X^{s,\theta}}\big)^2\right)^{\frac{1}{2}}\\
&\lesssim 2^{-s'k_3}\|\varphi^{(2)}_{k_2}\|_{X^{s,\theta}}\|\varphi^{(3)}_{k_3}\|_{X^{s,\theta}}
\end{aligned}$$

where we used Cauchy-Schwarz for the final inequality.

- $\sum_{l\lesssim k_3} P_{\sim k_2}Q_l(Q_{\ll l}\varphi^{(2)}_{k_2}\cdot\varphi^{(3)}_{k_3})$: Using (MB) on $\varphi^{(2)}$ this term can be bounded in $X^{s,\theta}$ by

$$\begin{aligned}
&\sum_{l\lesssim k_3}\sum_{j\ll l}2^{sk_2}2^{\theta l}\|Q_j\varphi^{(2)}_{k_2}\|_{M,2}\|\varphi^{(3)}_{k_3}\|_{\frac{2M}{M-2},\infty}\\
&\lesssim \sum_{l\lesssim k_3}\sum_{j\ll l}2^{\theta l}2^{(\frac{1}{2}-\frac{1}{M})j}2^{-\theta j}\|\varphi^{(2)}_{k_2}\|_{X^{s,\theta}}2^{-(\frac{1}{2}-\frac{1}{M}+s')k_3}\|\varphi^{(3)}_{k_3}\|_{X^{s,\theta}}
\end{aligned}$$

$$\lesssim 2^{(\theta'+\frac{1}{M}-s')k_3}\|\varphi^{(2)}_{k_2}\|_{X^{s,\theta}}\|\varphi^{(3)}_{k_3}\|_{X^{s,\theta}}$$

which is acceptable upon choosing $1/M < s' - \theta'$.

When the outer modulation is $\gg 2^{k_3}$, one of the inner terms must be of at least comparable modulation or the interaction is null.

- $\sum_{k_3 \ll l \lesssim k_2} P_{\sim k_2} Q_l(Q_{>l-10}\varphi^{(2)}_{k_2} \cdot \varphi^{(3)}_{k_3})$: In this case we use the bound

$$\begin{aligned}\|P_{\sim k_2} Q_l(Q_{>l-10}\varphi^{(2)}_{k_2} \cdot \varphi^{(3)}_{k_3})\|_{X^{s,\theta}} &\lesssim \sum_{j\gtrsim l} 2^{sk_2}2^{\theta l}\|Q_j\varphi^{(2)}_{k_2}\|_{2,2}\|\varphi^{(3)}_{k_3}\|_{\infty,\infty}\\ &\lesssim 2^{-s'k_3}\sum_{j\gtrsim l} 2^{\theta(l-j)}\|Q_j\varphi^{(2)}_{k_2}\|_{X^{s,\theta}}\|\varphi^{(3)}_{k_3}\|_{X^{s,\theta}}\end{aligned}$$

  which is again acceptable when square-summed in l.

- $\sum_{k_3 \ll l \lesssim k_2} P_{\sim k_2} Q_l(Q_{\ll l}\varphi^{(2)}_{k_2} \cdot \varphi^{(3)}_{k_3})$: This time we bound

$$\begin{aligned}\|P_{\sim k_2} Q_l(Q_{\ll l}\varphi^{(2)}_{k_2} \cdot \varphi^{(3)}_{k_3})\|_{X^{s,\theta}} &\lesssim 2^{sk_2}2^{\theta l}\|Q_{\ll l}\varphi^{(2)}_{k_2}\|_{\infty,2}\|Q_{\gtrsim l}\varphi^{(3)}_{k_3}\|_{2,\infty}\\ &\lesssim \|\varphi^{(2)}_{k_2}\|_{X^{s,\theta}}\sum_{j\gtrsim l} 2^{-s'k_3}2^{\theta(l-j)}\|Q_j\varphi^{(3)}_{k_3}\|_{X^{s,\theta}}\end{aligned}$$

  which is again acceptable.

When the outer modulation is very large $\gg 2^{k_2}$, we have a similar situation:

- $\sum_{l \gg k_2} P_{\sim k_2} Q_l(Q_{>l-10}\varphi^{(2)}_{k_2} \cdot \varphi^{(3)}_{k_3})$:

$$\begin{aligned}\|P_{\sim k_2} Q_l(Q_{>l-10}\varphi^{(2)}_{k_2} \cdot \varphi^{(3)}_{k_3})\|_{X^{s,\theta}} &\lesssim \sum_{j\gtrsim l} 2^{sl}2^{\theta l}\|Q_j\varphi^{(2)}_{k_2}\|_{2,2}\|\varphi^{(3)}_{k_3}\|_{\infty,\infty}\\ &\lesssim 2^{-s'k_3}\sum_{j\gtrsim l} 2^{(s+\theta)(l-j)}\|Q_j\varphi^{(2)}_{k_2}\|_{X^{s,\theta}}\|\varphi^{(3)}_{k_3}\|_{X^{s,\theta}}\end{aligned}$$

  which is acceptable.

- $\sum_{l \gg k_2} P_{\sim k_2} Q_l(Q_{\ll l}\varphi^{(2)}_{k_2} \cdot \varphi^{(3)}_{k_3})$: Here

$$\begin{aligned}\|P_{\sim k_2} Q_l(Q_{\ll l}\varphi^{(2)}_{k_2} \cdot \varphi^{(3)}_{k_3})\|_{X^{s,\theta}} &\lesssim 2^{sl}2^{\theta l}\|Q_{\ll l}\varphi^{(2)}_{k_2}\|_{\infty,\infty}\|Q_{\gtrsim l}\varphi^{(3)}_{k_3}\|_{2,2}\\ &\lesssim 2^{-s'k_2}\sum_{j\gtrsim l} 2^{(s+\theta)(l-j)}\|\varphi^{(2)}_{k_2}\|_{X^{s,\theta}}\|Q_j\varphi^{(3)}_{k_3}\|_{X^{s,\theta}}\end{aligned}$$

It remains to consider the case $|k_2 - k_3| \leq 10$. We first suppose that the outer modulation is restricted to $\lesssim 2^{k_2}$:

$$\|P_{\lesssim k_2} Q_{\lesssim k_2}(\varphi^{(2)}_{k_2} \cdot \varphi^{(3)}_{k_3})\|_{X^{s,\theta}} \lesssim 2^{(s+\theta)k_2}\|\varphi^{(2)}_{k_2}\|_{\frac{2M}{M-2},M}\|\varphi^{(3)}_{k_3}\|_{M,\frac{2M}{M-2}}$$

$$\lesssim 2^{(s+\theta)k_2} 2^{-(\frac{1}{2}+\frac{2}{M}+s')k_2} \|\varphi_{k_2}^{(2)}\|_{X^{s,\theta}} \cdot 2^{-(\frac{3}{2}-\frac{2}{M}+s')k_3} \|\varphi_{k_3}^{(3)}\|_{X^{s,\theta}}$$

which is acceptable for $s' > \theta'$.

When the outer modulation is at $2^l \gg 2^{k_2}$, at least one of the terms must also be at modulation at least $\sim 2^l$ or else the term is null. Since we are considering $k_2 \sim k_3$ we may assume WLOG it is the factor $\varphi_{k_2}^{(2)}$:

$$\begin{aligned}\sum_{l \gg k_2} \|P_{\lesssim k_2} Q_l (Q_{\gtrsim l}\varphi_{k_2}^{(2)} \cdot \varphi_{k_3}^{(3)})\|_{X^{s,\theta}} &\lesssim \sum_{l \gg k_2} 2^{(s+\theta)l} \|Q_{\gtrsim l}\varphi_{k_2}^{(2)}\|_{2,2} \|\varphi_{k_3}^{(3)}\|_{\infty,\infty} \\ &\lesssim 2^{-s'k_3} \sum_{l \gg k_2} \sum_{j \gtrsim l} 2^{(s+\theta)(l-j)} \|Q_j \varphi_{k_2}^{(2)}\|_{X^{s,\theta}} \|\varphi_{k_3}^{(3)}\|_{X^{s,\theta}}\end{aligned}$$

This was the final case and the proof is complete. $\square$

Bibliography

[AC09] Thomas Alazard and Rémi Carles. Loss of regularity for supercritical nonlinear Schrödinger equations. *Mathematische Annalen*, 343:397–420, 2009.

[BGT05] Nicolas Burq, Patrick Gérard, and Nikolay Tzvetkov. Multilinear eigenfunction estimates and global existence for the three dimensional nonlinear Schrödinger equations. In *Annales scientifiques de l'Ecole normale supérieure*, volume 38, pages 255–301. Elsevier, 2005.

[BK21] Nicolas Burq and Joachim Krieger. Randomization improved Strichartz estimates and global well-posedness for supercritical data. *Annales de l'Institut Fourier*, 71(5):1929–1961, 2021.

[BKL20] Bjorn Berntson, Rob Klabbers, and Edwin Langmann. Multi-solitons of the half-wave maps equation and Calogero–Moser spin–pole dynamics. *Journal of Physics A: Mathematical and Theoretical*, 53(50):505702, 2020.

[BL13] Nicolas Burq and Gilles Lebeau. Injections de Sobolev probabilistes et applications. In *Annales Scientifiques de l'École Normale Supérieure*, volume 46, pages 917–962, 2013.

[BL14] Nicolas Burq and Gilles Lebeau. Probabilistic Sobolev embeddings, applications to eigenfunctions estimates. *Geometric and Spectral Analysis*, 630:307–318, 2014.

[BOP14] Árpád Bényi, Tadahiro Oh, and Oana Pocovnicu. Wiener randomization on unbounded domains and an application to almost sure well-posedness of NLS. *arXiv preprint arXiv:1405.7326*, 2014.

[BOP15a] Árpád Bényi, Tadahiro Oh, and Oana Pocovnicu. On the probabilistic Cauchy theory of the cubic nonlinear Schrödinger equation on $\mathbb{R}^d$, $d \geq 3$. *Transactions of the American Mathematical Society, Series B*, 2(1):1–50, 2015.

[BOP15b] Árpád Bényi, Tadahiro Oh, and Oana Pocovnicu. Wiener randomization on unbounded domains and an application to almost sure well-posedness of NLS. In *Excursions in Harmonic Analysis, Volume 4*, pages 3–25. Springer, 2015.

[Bou93] Jean Bourgain. Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations: Part II: The KdV-equation. *Geometric & Functional Analysis GAFA*, 3(3):209–262, 1993.

[Bou94] Jean Bourgain. Periodic nonlinear Schrödinger equation and invariant measures. *Communications in Mathematical Physics*, 166:1–26, 1994.

[Bou96] Jean Bourgain. Invariant measures for the 2D-defocusing nonlinear Schrödinger equation. *Communications in mathematical physics*, 176(2):421–445, 1996.

[Bou98] Jean Bourgain. Refinements of Strichartz'inequality and applications to $2d$-NLS with critical nonlinearity. *International Mathematics Research Notices*, 1998(5):253–283, 1998.

[Bou99] Jean Bourgain. Global wellposedness of defocusing critical nonlinear schrödinger equation in the radial case. *Journal of the American Mathematical Society*, 12(1):145–171, 1999.

[BT07] Nicolas Burq and Nikolay Tzvetkov. Invariant measure for a three dimensional nonlinear wave equation. *International Mathematics Research Notices*, 2007(9):rnm108–rnm108, 2007.

[BT08] Nicolas Burq and Nikolay Tzvetkov. Random data Cauchy theory for supercritical wave equations I: local theory. *Inventiones mathematicae*, 173(3):449–475, 2008.

[BT13] Nicolas Burq and Nikolay Tzvetkov. Probabilistic well-posedness for the cubic wave equation. *Journal of the European Mathematical Society*, 16(1):1–30, 2013.

[BTT13] Nicolas Burq, Laurent Thomann, and Nikolay Tzvetkov. Long time dynamics for the one dimensional non linear Schrödinger equation. In *Annales de l'Institut Fourier*, volume 63, pages 2137–2198, 2013.

[Bur20] Stefano Burzio. *On long time behaviour of solutions to nonlinear dispersive equations*. PhD thesis, École Polytechnique Fédérale de Lausanne, 2020.

[Caz03] Thierry Cazenave. *Semilinear Schrödinger Equations*, volume 10. American Mathematical Soc., 2003.

[CCT03] Michael Christ, James Colliander, and Terence Tao. Ill-posedness for nonlinear Schrödinger and wave equations. *arXiv preprint math/0311048*, 2003.

[CdS15] Federico Cacciafesta and Anne-Sophie de Suzzoni. Invariant measure for the Schrödinger equation on the real line. *Journal of Functional Analysis*, 269(1):271–324, 2015.

[CFH11] Thierry Cazenave, Daoyuan Fang, and Zheng Han. Continuous dependence for NLS in fractional order spaces. In *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, volume 28, pages 135–147. Elsevier, 2011.

[CKS+08] James Colliander, Markus Keel, Gigliola Staffilani, Hideo Takaoka, and Terence Tao. Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$. *Annals of Mathematics*, pages 767–865, 2008.

[CO12] James Colliander and Tadahiro Oh. Almost sure well-posedness of the cubic nonlinear Schrödinger equation below $L^2(\mathbb{T})$. 2012.

[CW90] Thierry Cazenave and Fred Weissler. The Cauchy problem for the critical nonlinear Schrödinger equation in H^s. *Nonlinear Analysis: Theory, Methods & Applications*, 14(10):807–836, 1990.

[Den10] Yu Deng. Two dimensional NLS equation with random radial data. *arXiv preprint arXiv:1008.2657*, 2010.

[DG04] Piero D'Ancona and Vladimir Georgiev. On the continuity of the solution operator to the wave map system. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 57(3):357–383, 2004.

[DLM19] Benjamin Dodson, Jonas Lührmann, and Dana Mendelson. Almost sure local well-posedness and scattering for the 4D cubic nonlinear Schrödinger equation. *Advances in Mathematics*, 347:619–676, 2019.

[DLM20] Benjamin Dodson, Jonas Lührmann, and Dana Mendelson. Almost sure scattering for the 4d energy-critical defocusing nonlinear wave equation with radial data. *American Journal of Mathematics*, 142(2):475–504, 2020.

[dS13] Anne-Sophie de Suzzoni. Large data low regularity scattering results for the wave equation on the Euclidean space. *Communications in Partial Differential Equations*, 38(1):1–49, 2013.

[dS14] Anne-Sophie de Suzzoni. Invariant measure for the Klein-Gordon equation in a non periodic setting. *arXiv preprint arXiv:1403.2274*, 2014.

[DX13] Feng Dai and Yuan Xu. *Approximation theory and harmonic analysis on spheres and balls*. Springer, 2013.

[EFS22] Eugene Eyeson, Silvino Reyes Farina, and Armin Schikorra. On uniqueness for half-wave maps in dimension $d \geq 3$. *arXiv preprint arXiv:2211.05652*, 2022.

[GG16] Dan-Andrei Geba and Manoussos G Grillakis. *An introduction to the theory of wave maps and related geometric problems*. World Scientific Publishing Company, 2016.

[GL18] Patrick Gérard and Enno Lenzmann. A Lax pair structure for the half-wave maps equation. *Letters in Mathematical Physics*, 108:1635–1648, 2018.

[Gla77] Robert T Glassey. On the blowing up of solutions to the cauchy problem for nonlinear schrödinger equations. *Journal of Mathematical Physics*, 18(9):1794–1797, 1977.

[Gri00] Manoussos G Grillakis. On nonlinear schrödinger equations: Nonlinear schrödinger equations. *Communications in Partial Differential Equations*, 25(9-10):1827–1844, 2000.

[Guo16] Zihua Guo. Sharp spherically averaged Strichartz estimates for the Schrödinger equation. *Nonlinearity*, 29(5):1668, 2016.

[HKO24] Hiroyuki Hirayama, Shinya Kinoshita, and Mamoru Okamoto. Sharp well-posedness for the cauchy problem of the two dimensional quadratic nonlinear schrödinger equation with angular regularity. *Journal of Differential Equations*, 395:181–222, 2024.

[Hon22] Seokchang Hong. Improved multilinear estimates and global regularity for general nonlinear wave equations in $(1+3)$ dimensions. *arXiv preprint arXiv:2207.02412*, 2022.

[KK21] Anna Kiesenhofer and Joachim Krieger. Small data global regularity for half-wave maps in $n = 4$ dimensions. *Communications in Partial Differential Equations*, 46(12):2305–2324, 2021.

[Kla85] Sergiu Klainerman. Uniform decay estimates and the Lorentz invariance of the classical wave equation. *Communications in Pure Applied Mathematics*, 38:321–332, 1985.

[Kla86] Sergiu Klainerman. The null condition and global existence to nonlinear wave equations. *Nonlinear systems of partial differential equations in applied mathematics, Part*, 1:293–326, 1986.

[KM96a] Sergeiu Klainerman and Matei Machedon. Estimates for null forms and the spaces $H^{s,\delta}$. *International Mathematics Research Notices*, 1996(17):853–865, 1996.

[KM96b] Sergiu Klainerman and Matei Machedon. Smoothing estimates for null forms and applications. *Duke Math. J*, 81(1):99–133, 1996.

[KMV19] Rowan Killip, Jason Murphy, and Monica Visan. Almost sure scattering for the energy-critical NLS with radial data below $H^1(\mathbb{R}^4)$. *Communications in Partial Differential Equations*, 44(1):51–71, 2019.

[KPV96] Carlos Kenig, Gustavo Ponce, and Luis Vega. A bilinear estimate with applications to the KdV equation. *Journal of the American Mathematical Society*, 9(2):573–603, 1996.

[KPV01] Carlos E Kenig, Gustavo Ponce, and Luis Vega. On the ill-posedness of some canonical dispersive equations. 2001.

[KS97] Sergiu Klainerman and Sigmund Selberg. Remark on the optimal regularity for equations of wave maps type. *Communications in Partial Differential Equations*, 22(5-6):99–133, 1997.

[KS17] Joachim Krieger and Yannick Sire. Small data global regularity for half-wave maps. *Analysis & PDE*, 11(3):661–682, 2017.

[Lin93] Hans Lindblad. A sharp counterexample to the local existence of low-regularity solutions to nonlinear wave equations. 1993.

[Liu21] Yang Liu. Global well-posedness for half-wave maps with $\mathbb{S}^2$ and $\mathbb{H}^2$ targets for small smooth initial data. *arXiv preprint arXiv:2109.13657*, 2021.

[Liu23] Yang Liu. Global weak solutions for the half-wave maps equation in $\mathbb{R}$. *arXiv preprint arXiv:2308.06836*, 2023.

[LM14] Jonas Lührmann and Dana Mendelson. Random data Cauchy theory for nonlinear wave equations of power-type on $\mathbb{R}^3$. *Communications in Partial Differential Equations*, 39(12):2262–2283, 2014.

[LM16] Jonas Lührmann and Dana Mendelson. On the almost sure global well-posedness of energy sub-critical nonlinear wave equations on $\mathbb{R}^3$. *New York J. Math*, 22:209–227, 2016.

[LP31] John E Littlewood and Raymond EAC Paley. Theorems on Fourier series and power series. *Journal of the London Mathematical Society*, 1(3):230–233, 1931.

[LRS88] Joel L Lebowitz, Harvey A Rose, and Eugene R Speer. Statistical mechanics of the nonlinear Schrödinger equation. *Journal of statistical physics*, 50(3):657–687, 1988.

[LS95] Hans Lindblad and Christopher D Sogge. On existence and scattering with minimal regularity for semilinear wave equations. *Journal of Functional Analysis*, 130(2):357–426, 1995.

[LS18] Enno Lenzmann and Armin Schikorra. On energy-critical half-wave maps into $\mathbb{S}^2$. *Inventiones mathematicae*, 213:1–82, 2018.

[LS20] Enno Lenzmann and Jérémy Sok. Derivation of the half-wave maps equation from Calogero–Moser spin systems. *arXiv preprint arXiv:2007.15323*, 2020.

[LZ11] Dong Li and XiaoYi Zhang. Stability of solutions for nonlinear Schrödinger equations in critical spaces. *Science China Mathematics*, 54(5):973–986, 2011.

[Mar23] Katie Marsden. Almost sure scattering of the energy-critical NLS in $d >$. *Communications on Pure and Applied Analysis*, 22(10):3165–3199, 2023.

[Mar24] Katie Marsden. Global solutions to the 3D half-wave maps equation with angular regularity. *arXiv preprint arXiv:2403.14567*, 2024.

[Mat22] Yoshimasa Matsuno. Integrability, conservation laws and solitons of a many-body dynamical system associated with the half-wave maps equation. *Physica D: Nonlinear Phenomena*, 430:133080, 2022.

[MNT10] Shuji Machihara, Kenji Nakanishi, and Kotaro Tsugawa. Well-posedness for nonlinear Dirac equations in one dimension. 2010.

[MRRS22] Frank Merle, Pierre Raphaël, Igor Rodnianski, and Jeremie Szeftel. On blow up for the energy super critical defocusing nonlinear Schrödinger equations. *Inventiones mathematicae*, pages 1–167, 2022.

[Mur19] Jason Murphy. Random data final-state problem for the mass-subcritical NLS in L^2. *Proceedings of the American Mathematical Society*, 147(1):339–350, 2019.

[NY18] Kenji Nakanishi and Takuto Yamamoto. Randomized final-data problem for systems of nonlinear Schrödinger equations and the Gross-Pitaevskii equation. *arXiv preprint arXiv:1805.05589*, 2018.

[Ohl23] Gaspard Ohlmann. Analytic proofs of well-posedness and non-turbulence results for the half-wave maps equation. *arXiv preprint arXiv:2310.13442*, 2023.

[OOP17] Tadahiro Oh, Mamoru Okamoto, and Oana Pocovnicu. On the probabilistic well-posedness of the nonlinear schr\"{o} dinger equations with non-algebraic nonlinearities. *arXiv preprint arXiv:1708.01568*, 2017.

[OP16] Tadahiro Oh and Oana Pocovnicu. Probabilistic global well-posedness of the energy-critical defocusing quintic nonlinear wave equation on $\mathbb{R}^3$. *Journal de Mathématiques Pures et Appliquées*, 105(3):342–366, 2016.

[OP17] Tadahiro Oh and Oana Pocovnicu. A remark on almost sure global well-posedness of the energy-critical defocusing nonlinear wave equations in the periodic setting. *Tohoku Mathematical Journal*, 69(3):455–481, 2017.

[Poc17] Oana Pocovnicu. Almost sure global well-posedness for the energy-critical defocusing nonlinear wave equation on $\mathbb{R}^d$, $d = 4$ and 5. *Journal of the European Mathematical Society*, 19(8):2521–2575, 2017.

[PRT14] Aurélien Poiret, Didier Robert, and Laurent Thomann. Probabilistic global well-posedness for the supercritical nonlinear harmonic oscillator. *Analysis & PDE*, 7(4):997–1026, 2014.

[RV07] Eric Ryckman and Monica Visan. Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$. *American journal of mathematics*, 129(1):1–60, 2007.

[Sel99] Sigmund Selberg. *Multilinear space-time estimates and applications to local existence theory for nonlinear wave equations.* PhD thesis, Princeton University, 1999.

[Sog08] Christopher Donald Sogge. *Lectures on non-linear wave equations.* International Press Boston, 2008.

[Spi21] Martin Spitz. Almost sure local wellposedness and scattering for the energy-critical cubic nonlinear Schrödinger equation with supercritical data. *arXiv preprint arXiv:2110.11051*, 2021.

[Ste70a] Elias M Stein. *Singular integrals and differentiability properties of functions.* Princeton university press, 1970.

[Ste70b] Elias M Stein. *Topics in harmonic analysis, related to the Littlewood-Paley theory.* Princeton University Press, 1970.

[Ste05] Jacob Sterbenz. Angular regularity and Strichartz estimates for the wave equation. *International Mathematics Research Notices*, 2005(4):187–231, 2005.

[Ste07] Jacob Sterbenz. Global regularity and scattering for general non-linear wave equations II. $(4+1)$ dimensional Yang-Mills equations in the Lorentz gauge. *American journal of mathematics*, 129(3):611–664, 2007.

[Str72] Robert Strichartz. Multipliers for spherical harmonic expansions. *Transactions of the American Mathematical Society*, 167:115–124, 1972.

[SW16] Elias Stein and Guido Weiss. Introduction to Fourier analysis on euclidean spaces, volume 32. Princeton university press, 2016.

[SX16] Chenmin Sun and Bo Xia. Probabilistic well-posedness for supercritical wave equations with periodic boundary condition on dimension three. *Illinois Journal of Mathematics*, 60(2):481–503, 2016.

[SY20] Mouhamadou Sy and Xueying Yu. Almost sure global well-posedness for the energy supercritical NLS on the unit ball of $\mathbb{R}^3$. *arXiv preprint arXiv:2007.00766*, 2020.

[Sy21] Mouhamadou Sy. Almost sure global well-posedness for the energy supercritical Schrödinger equations. *Journal de Mathématiques Pures et Appliquées*, 154:108–145, 2021.

[Tao98] Terence Tao. Counterexamples to the $n = 3$ endpoint strichartz estimate for the wave equation. *preprint*, 1998.

[Tao00] Terence Tao. Ill-posedness for one-dimensional wave maps at the critical regularity. *American Journal of Mathematics*, 122(3):451–463, 2000.

[Tao01a] Terence Tao. Global regularity of wave maps, I: small critical sobolev norm in high dimension. *International Mathematics Research Notices*, 2001(6):299–328, 2001.

[Tao01b] Terence Tao. Global regularity of wave maps II. small energy in two dimensions. *Communications in Mathematical Physics*, 224(2):443–544, 2001.

[Tao06] Terence Tao. *Nonlinear dispersive equations: local and global analysis*. Number 106. American Mathematical Soc., 2006.

[Tat98] Daniel Tataru. Local and global results for wave maps I. *Communications in Partial Differential Equations*, 23(9-10):1781–1793, 1998.

[Tat01] Daniel Tataru. On global existence and scattering for the wave maps equation. *American Journal of Mathematics*, 123(1):37–77, 2001.

[Tat05] Daniel Tataru. Rough solutions for the wave maps equation. *American Journal of Mathematics*, 127(2):293–377, 2005.

[TV05] Terence Tao and Monica Visan. Stability of energy-critical nonlinear Schrödinger equations in high dimensions. *arXiv preprint math/0507005*, 2005.

[Tzv06] Nikolay Tzvetkov. Invariant measures for the nonlinear Schrodinger equation on the disc. *arXiv preprint math/0603112*, 2006.

[Tzv07] N Tzvetkov. Invariant measures for the defocusing NLS. *arXiv preprint math/0701287*, 2007.

[Vis07] Monica Visan. The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions. *Duke Mathematical Journal*, 138(2):281–374, 2007.

[ZF12] Ting Zhang and Daoyuan Fang. Random data Cauchy theory for the generalized incompressible Navier–Stokes equations. *Journal of Mathematical Fluid Mechanics*, 14(2):311–324, 2012.

[ZS15] Tianci Zhou and Michael Stone. Solitons in a continuous classical Haldane–Shastry spin chain. *Physics Letters A*, 379(43-44):2817–2825, 2015.

Katie Marsden

Curriculum Vitae

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Education

PhD candidate, École Polytechnique Fédérale de Lausanne (EPFL). 2020-present
Advisor: Joachim Krieger.

Mathematics MSc, University of Cambridge. 2016-2020

Publications

Almost Sure Scattering of the Energy-Critical Nonlinear Schrödinger Equation in $d > 6$, Marsden, K. *Communications on Pure and Applied Analysis*, (2023). DOI: 10.3934/cpaa.2023106.

Global Solutions to the 3D Half-Wave Maps Equation with Angular Regularity, Marsden, K. *Preprint, submitted to Ars Inveniendi Analytica (2024).* DOI: arXiv:2403.14567.

Research Stays

Visit to Laboratoire Jacques-Louis Lions, Sorbonne University. (1 month, November 2023. Return visit planned for June 2024).

Talks

1. *Almost Sure Scattering of the Energy-Critical Nonlinear Schrödinger Equation in* $d > 6$. Talks given at
   - Harmonic Analysis and Differential Equations seminar, UC Berkeley, January 2023.
   - Geometric Dispersive PDEs Summer School, Obergurgl, September 2022.
   - Women in Nonlinear Dispersive PDEs conference, Banff International Research Station, February 2023 (poster presentation).
2. *Global Solutions for the Half-Wave Maps Equation at Critical Regularity.* Talks given at
   - Analysis & PDE Seminar, UCLA, January 2024.
   - Analysis PhD Seminar, Orsay, April 2024.
3. *The Aharonov-Bohm Effect: A Topological Effect in Physics*, EPFL student seminar, March 2022.
4. Talks in reading group on the paper *Instability and nonuniqueness for the 2D Euler equations in Vorticity Form, after M. Vishik* (Albritton et al.), ongoing.

Teaching Experience

Teaching assistant for the following courses at EPFL: 2020 – Present

- EDPs d'Évolution, Undergraduate course.
- Dispersive PDEs, Masters course.
- Géométrie Différentielle, Undergraduate course.
- Functional Analysis II, Masters course.
- Topics in Complex Analysis, Undergraduate course.
- Dynamics and Bifurcations, Undergraduate course.
- Structures Algébriques, Undergraduate course.

Voluntary teaching experience:

- Online maths tutor for Tutorfair (a UK-based educational charity). Students aged 14-18. 2020-present
- Maths classroom assistant at The North Cambridge Academy, a local high school. 2019-2020
- Maths and reading tutor in local elementary school for the university program STIMULUS. 2019

Service and Outreach

PhD representative of the EPFL maths department. 2021-present

- One of three representatives for the maths doctoral school. Participated in departmental meetings and organised events such as a careers evening with alumnae and student lunchtime talks.

Undergraduate outreach activities: 2016-2020

- Involved in numerous outreach activities throughout my undergraduate studies: volunteered at support sessions for the university admissions exam, twice hosted high school students from under-represented backgrounds for the university Shadowing Scheme, wrote an article for the She Talks Science blog, volunteered with ChaOS Science Roadshow.

Other

Languages: English (fluent), French (C1), German (A1).
Programming skills: Experience in Python, MATLAB.