# RANDOMIZATION IN THE TWO-PARAMETER OPTIMAL STOPPING PROBLEM

## COMBINATORIAL, PROBABILISTIC AND INFINITESIMAL METHODS

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# RESUME

L'ensemble des temps d'arrêt flous constitue un outil important dans l'étude du problème d'arrêt optimal pour processus à deux paramètres. Cet ensemble est convexe, faiblement compact et contient l'ensemble des temps d'arrêt.

Nous avons montré qu'un temps d'arrêt flou optimal existe lorsque les trajectoires du processus de gain sont semi-continues supérieurement. L'existence d'un temps d'arrêt optimal pourrait se déduire sans difficulté de ce résultat, à condition que la propriété suivante soit vérifiée :

(P) *tout élément extrémal de l'ensemble des temps d'arrêt flous est un temps d'arrêt.*

Pour les processus à un paramètre, cette propriété vaut toujours, mais à deux paramètres ce n'est déjà plus le cas. En fait, nous avons montré que la propriété (P) est intimement liée à la structure de la filtration sous-jacente au problème d'arrêt.

Sur un espace de probabilité fini, nous montrons que les filtrations à deux paramètres possèdent déjà la structure combinatoire la plus générale. En établissant, dans le cas fini, que la propriété (P) vaut sous une faible hypothèse d'indépendance conditionnelle qualitative (ICQ) introduite par U. Krengel et L. Sucheston, nous exhibons une nouvelle classe de graphes parfaitement ordonnables.

Une généralisation partielle à la dimension infinie est obtenue à l'aide de la notion de graphe parfait infini. Mais pour montrer que l'hypothèse ICQ entraîne la propriété (P) à temps discret et sur un espace mesurable quelconque, nous avons développé une méthode constructive. Notre construction fait appel à un opérateur non-linéaire, le supremum conditionnel, qui possède des propriétés semblables à celles de l'espérance conditionnelle et qui est invariant par changement de mesure équivalente.

L'extension au temps continu de cette construction nécessiterait la résolution d'une équation différentielle aléatoire sans condition de régularité. Il nous a donc semblé naturel d'aborder ce problème par les méthodes de l'analyse non-standard. Nous avons introduit un espace filtré hyperfini canonique, possédant la propriété F4 de R. Cairoli et J.B. Walsh, et avons montré que (P) vaut sur cet espace.

Deux autres questions liées au problème d'arrêt optimal ont retenu notre attention. Tout d'abord, nous avons étudié la relation entre l'hypothèse F4 et l'hypothèse ICQ. En particulier, en temps discret, nous avons donné une condition nécessaire et suffisante pour l'existence d'une mesure équivalente donnant lieu à la propriété F4. Enfin, nous avons montré que la condition ICQ en temps continu est nécessaire et suffisante pour que tout point d'arrêt appartienne à un chemin croissant optionnel et nous avons exhibé des exemples de points d'arrêt par lesquels il passe un chemin unique.

# FOREWORD

The motivation to this research is the study of the optimal stopping problem for multiparameter processes : given a random process depending on finitely many parameters $t = (t_1,\ldots,t_n)$, which if stopped at point $t$ in the parameter set produces a reward $X_t$, what random stopping point $T = (T_1,\ldots,T_n)$ maximizes the expected reward ?

This is a natural model for a gambler who has the choice of playing several different games and of stopping at the most advantageous time. Of course, the decision to stop at a given point should only depend on the information available at that point, so the maximum is taken over the set $\underline{\underline{T}}$ of non-anticipative strategies.

The case $n = 1$ has been extensively studied over the last twenty years or so, but when $n > 1$, the problem is quite different and much less is known so far. Since most difficulties are already present when $n = 2$, our study is centered on this case.

The first question relative to the optimal stopping problem that needs to be addressed is whether optimal stopping rules exist under suitable regularity assumptions on the reward process. This reduces to showing that the function $T \longmapsto E(X_T)$ attains its maximum at some point $T_o \in \underline{\underline{T}}$. It is thus natural to embed $\underline{\underline{T}}$ into some larger ("randomized") set $\underline{\underline{U}}$ with certain convexity and compactness properties, in such a way that $T \longmapsto E(X_T)$ can be extended to a function with sufficient regularity that a maximum over $\underline{\underline{U}}$ will exist. The choice of randomization should be such that one can then recover a maximum over $\underline{\underline{T}}$.

The regularity question can often be solved by an approximation of upper semi-continuous multiparameter processes by continuous ones (see Chapter 9). This technique will enable us to show that upper semi-continuity of the sample paths of the reward process guarantees regularity of the extension of the map $T \longmapsto E(X_T)$.

As for the randomization question, a natural extension of a method developed for single-parameter processes leads to the convex set $\underline{U}$ of randomized stopping points. Contrary to the single-parameter case, the set of extremal elements of $\underline{U}$ may properly contain $\underline{T}$. An important object of study is to determine sufficient conditions on the underlying information structure that ensure equality of these two sets. This has led to several developments in graph theory, convexity theory and stochastic processes.

Insight about the set of randomized stopping points can be obtained by examining this set when it is finite-dimensional (see Chapter 4). In this case, $\underline{U}$ is a bounded polyhedron $\underline{P}$ defined by a (0,1)-matrix. An interesting analogy appears : maximizing over $\underline{T}$ is similar to solving an integer program, whereas maximizing over $\underline{U}$ is similar to solving a linear program. This observation is of importance for the practical numerical solution of an optimal stopping problem.

These two problems are equivalent when the polytope $\underline{P}$ is perfect, and conditions for this can be expressed in terms of perfect graphs. Relating properties of the information structure in the two-parameter probabilistic model to properties of the associated graph leads to a new class of perfectly orderable graphs. These combinatorial techniques give considerable insight into the increase in complexity of the information structure when passing from one to two parameters : the single-parameter case is quite restricted, whereas the two-parameter case is in a certain sense already the most general (see Chapter 5). Furthermore, these methods enable us to show, in a finite-dimensional setting, that under a classical hypothesis on the two-parameter probabilistic model, all extremal elements of $\underline{U}$ are stopping points.

Structural properties of the two-parameter information structure on arbitrary probability spaces are also examined. Most results on two-parameter processes have been obtained under Hypothesis F4 of Cairoli and Walsh. However, this hypothesis is not invariant under a change of equivalent measure, whereas a weaker hypothesis CQI, introduced by Krengel and Sucheston, is. Though one could hope that CQI would imply the existence of an equivalent measure such that F4 holds, we show through an example that this is not the case. Furthermore, in discrete time, we give necessary and sufficient conditions for the existence of an equivalent measure such that F4 holds (see Chapter 2).

Since the initial optimal stopping problem will usually lead to an infinite-dimensional set $\underline{U}$, extensions of results on finite perfect graphs and polytopes to such settings becomes of interest. Results in this direction are presented in Chapter 6. But the extension to discrete time and arbitrary probability spaces is achieved using a purely probabilistic decomposition of a non-extremal randomized stopping point on $\mathbb{N}^2$ into a convex combination of two others.

The intuition behind this construction comes from a detailed study of "two-parameter graphs", and the probabilistic tool is a "conditional supremum" operator (see Chapter 7), which has many properties of conditional expectation, but is invariant under a change of equivalent measure. In fact, a conditional qualitative independence hypothesis on the two-parameter filtration becomes a commutation property of this operator.

Extensions of this result to a continuous parameter setting are examined. However, extending the decomposition of a randomized stopping point would require the construction of a continuous time optional increasing path. This kind of construction can be done, in the presence of some regularity hypothesis, by solving certain random differential equations (see Chapter 3). But no regularity is present in the continuous parameter setting.

This and other difficulties motivate the use of techniques from infinitesimal stochastic calculus for the study of this extension (see Chapter 8). By constructing a canonical hyperfinite probability space for two-parameter processes, which satisfies Hypothesis F4 of Cairoli and Walsh, we can "lift" two-parameter randomized stopping points to hyperfinite weight processes. This lifting enables, via the Transfer Principle, a continuous time solution by the methods developed in the discrete case. Certain difficulties related to the two-parameters, which do not appear in the hyperfinite theory for single-parameter processes, are solved using the "conditional supremum" operator described above.

With this result, we obtain the existence of optimal stopping points for upper semicontinuous two-parameter processes defined on this canonical hyperfinite probability space. Though this result does not say how a player can achieve an optimal reward, no simple solution to such a question is to be expected. Indeed, some stopping points lie on a unique optional increasing path (see Chapter 3), and so any mistake by the player, even early in the game, may prevent him from attaining the optimal stopping point.

Many questions pertaining to the optimal stopping problem remain to be solved, but it seems reasonable to hope that blending Stochastic Processes and Discrete Mathematics, two areas generally considered disparate, together with the hyperfinite techniques of Non-Standard Analysis, should lead to many further interesting developments.

# TABLE OF CONTENTS


RESUME (en français) i

FOREWORD iii

TABLE OF CONTENTS vii

CHAPTER 1. INTRODUCTION, NOTATIONS AND TERMINOLOGY 1

1.1 Parameter sets 1

1.2 Probability spaces and σ-algebras 3

1.3 Filtrations and stopping points 5

1.4 Classical conditions on two-parameter filtrations 6

1.5 Stochastic processes 8

1.6 The optimal stopping problem 9

1.7 Randomized stopping points 10

1.8 Extremal randomized stopping points 13

CHAPTER 2. THE RELATIONSHIP BETWEEN CONDITIONAL QUALITATIVE INDEPENDENCE AND THE COMMUTATION PROPERTY. 17

2.1 A necessary condition for conditional independence 17

2.2 Sufficient conditions for conditional independence 21

2.3 Countable conditional qualitative independence 30

2.4 The case of finite algebras 31

2.5 Example of a filtration with CQI but without F4 for any equivalent measure 32

2.6 Necessary and sufficient conditions for the existence of an equivalent measure such that F4 holds (discrete case) 33

CHAPTER 3. EXISTENCE OF OPTIONAL INCREASING PATHS PASSING THROUGH A STOPPING POINT 39

3.1 Optional increasing paths 40

3.2 Approximation of optional increasing paths by step optional increasing paths 43

3.3 Cluster points of a sequence of optional increasing paths 45

3.4 Necessary and sufficient conditions for every stopping point to lie on an optional increasing path 47
3.5 Examples of stopping points lying on a unique optional increasing path : discrete case 48
3.6 Optional increasing paths as solutions of random differential equations 52
3.7 Examples of stopping points lying on a unique optional increasing path : continuous case 54

CHAPTER 4. COMBINATORIAL ASPECTS OF TWO-PARAMETER OPTIMAL STOPPING 63
4.1 The fundamental matrix of a two-parameter filtration 63
4.2 Necessary and sufficient conditions for all extremal randomized stopping points to be stopping points (finite case) 67
4.3 A new class of perfect graphs and sufficiency of Hypothesis CQI 70
4.4 Counterexamples for filtrations with the Intersection Property 76
4.5 A counterexample for three-parameter filtrations 77
4.6 Product filtrations, normal products of graphs and canonical two-parameter filtrations 78
4.7 A constructive algorithm for a class of perfect polytopes 80

CHAPTER 5. COMBINATORIAL STRUCTURE OF SINGLE AND TWO-PARAMETER FILTRATIONS 87
5.1 Objectives 87
5.2 Intersection graphs of single-parameter filtrations 88
5.3 Intersection graphs of general two-parameter filtrations 90
5.4 Minimal forbidden induced subgraphs of two-parameter graphs 91
5.5 Minimal forbidden induced subgraphs for CQI-graphs 98
5.6 A characterization of CQI-graphs 100

CHAPTER 6. INFINITE SYSTEMS OF LINEAR INEQUALITIES AND PERFECT GRAPHS 105
6.1 Infinite dimensional polytopes and perfect graphs 106
6.2 Application. Randomized stopping points on a discrete probability space 110

6.3 Application. Infinite doubly stochastic matrices and Dilworth's theorem 112
6.4 Application. Non-measurable randomized stopping points 113

CHAPTER 7. THE CONDITIONAL SUPREMUM OPERATOR AND RANDOMIZED STOPPING POINTS ON $\mathbb{N}^2$ 117
7.1 Some elementary lemmas 117
7.2 The conditional supremum operator 120
7.3 Some equivalent formulations of conditional qualitative independence 124
7.4 Hypothesis CQI is a commutation property of conditional suprema 127
7.5 A constructive algorithm for randomized stopping points on $\mathbb{N}^2$ 129
7.6 Solution to the extremal question on $\mathbb{N}^2$ 137
7.7 Application : an optimal stopping problem for processes with values in a Banach space 138

CHAPTER 8. INFINITESIMAL METHODS IN THE THEORY OF TWO-PARAMETER PROCESSES 143
8.1 Preliminaries on non-standard analysis 144
8.2 A hyperfinite probability space for two-parameter processes 147
8.3 Basic lifting theorems and the commutation property 150
8.4 The Simultaneous Lifting Theorem 155
8.5 A lifting theorem for randomized stopping points 161
8.6 Extension of the discrete construction of Chapter 7.5 to the continuous case 165
8.7 Extremal randomized stopping points on the canonical hyperfinite probability space for two-parameter processes 169

CHAPTER 9. FUNCTIONALS OF RANDOMIZED STOPPING POINTS 173
9.1 Continuous and upper-semicontinuous functions 173
9.2 Approximation of upper-semicontinuous processes 176
9.3 Upper-semicontinuous functionals of randomized stopping points 178

9.4 Application : existence of optimal stopping points 180

REFERENCES 183

INDEX 189

ACKNOWLEDGMENTS 193

CURRICULUM VITAE 195

CHAPTER 1

# INTRODUCTION, NOTATIONS AND TERMINOLOGY

## 1.1 Parameter sets.

If an observer is watching some random phenomenon, it is necessary to describe the state of the observer. For instance, for how long has he been observing the process and what decisions has he already made. This is the role of the parameter set.

When it is only necessary to record the duration of the observation, there is a single parameter, time, and so the parameter set is either $\mathbb{N}$ or $\mathbb{R}_+$. If, on the other hand, several parameters need to be recorded, this will be reflected in the choice of the parameter set. When there are two or several parameters, we speak of two-dimensional time (respectively multidimensional time), or just time, for short.

Throughout this thesis, we will mainly be concerned with stochastic processes indexed by one of the following sets :

$\mathbb{N}$ , $\mathbb{D}_n$ or $\mathbb{R}_+$ in the single-parameter case,

or

$\mathbb{N}^2$ , $\mathbb{D}_n^2$ or $\mathbb{R}_+^2$ in the two-parameter case,

where $\mathbb{D}_n$ denotes the set of dyadic real numbers of order n. The letter I will be used to denote a single-parameter index set, whereas $I^2$ will denote a two-parameter index set. We will also consider subsets of these index sets, usually intervals in I or rectangles in $I^2$.

The set I is equipped with the usual total order, denoted $\leq$, whereas on the sets $I^2$ it is natural to consider the two orders $\leq$ and $\underline{\wedge}$ defined by

$$s = (s_1,s_2) \leq t = (t_1,t_2) \iff s_1 \leq t_1 \text{ and } s_2 \leq t_2 ,$$

$$s = (s_1,s_2) \underline{\wedge} t = (t_1,t_2) \iff s_1 \leq t_1 \text{ and } s_2 \geq t_2 .$$

We will also use the following notations :

$$s < t \iff s \leq t , \quad s \neq t ,$$

$$s \wedge t \iff s \underline{\wedge} t , \quad s \neq t ,$$

and

$$s = (s_1,s_2) \ll t = (t_1,t_2) \iff s_1 < t_1, s_2 < t_2$$

$$s = (s_1,s_2) \hat{\wedge} t = (t_1,t_2) \iff s_1 < t_1, s_2 > t_2 .$$

We also define, coordinatewise, the minimum and maximum (for $\leq$) of two elements of $I^2$ :

$$\min (s,t) = (\min(s_1,t_1), \min(s_2,t_2)) ,$$

$$\max (s,t) = (\max(s_1,t_1), \max(s_2,t_2)) ,$$

and will use the following standard notation for intervals (with respect to $\leq$) :

$$[s,t] = \{u \in I^2 : s \leq u \leq t\} ,$$

$$]s,t] = \{u \in I^2 : s \ll u \leq t\} ,$$

and so forth. In order to avoid introducing special symbols, we will set $]s,t] = \{u \in I^2 : s < u \leq t\}$, $]s,t[ = \{u \in I^2 : s < u < t\}$ and $]s,t[ = \{u \in I^2 : s < u < t\}$, when $s \leq t$ but $s_1 = t_1$ or $s_2 = t_2$.

In several instances we will make use of the lexicographic (total) order on $I^2$, denoted $\underline{\boldsymbol{<}}$:

$$s \preceq t \iff (s_1 < t_1 \text{ or } (s_1 = t_1 \text{ and } s_2 \leq t_2)).$$

The notation $s \prec t$ will mean $s \preceq t$ and $s \neq t$.

Unless otherwise mentioned, $I^2$ is equipped with its usual norm-induced topology. The following norm is the most convenient for our purpose :

$$|t| = |t_1| + |t_2| \ , \quad t = (t_1, t_2).$$

We will often add to $I$ or $I^2$ an extra element, denoted in both cases $\infty$, and will set $\bar{I} = I \cup \{\infty\}$ , $\overline{I^2} = I^2 \cup \{\infty\}$ . These sets will be equipped with their usual metric topologies, making them compact. We will also suppose that $t \leq \infty$ , for all $t$, in either $I$ or $I^2$.

The notation $\underline{\underline{B}}(I)$, $\underline{\underline{B}}(\bar{I})$, $\underline{\underline{B}}(I^2)$, $\underline{\underline{B}}(\overline{I^2})$ will denote in each case the Borel $\sigma$-algebra of the parameter set.

## 1.2 Probability spaces and $\sigma$-algebras.

Except where otherwise specified, we will suppose that all random variables are defined on a common probability space $(\Omega, \underline{\underline{F}}, P)$, such that $\underline{\underline{F}}$ is complete (i.e. contains all P-null sets). We will make a distinction between measurable maps from $(\Omega, \underline{\underline{F}})$ into some measure space $(M, \underline{\underline{M}})$, and random variables on $\Omega$, which we identify with an equivalence class of measurable maps which differ only on P-null sets. The notation $X \in \underline{\underline{F}}$, where $X$ is a random variable, will indicate that $X$ is $\underline{\underline{F}}$-measurable.

If $P^1$ and $P^2$ are two probabilities defined on $\underline{\underline{F}}$, we say that $P^1$ and $P^2$ are equivalent provided they have the same null sets. In Chapter 2, we shall study the existence of equivalent probabilities with certain properties.

Given three sub $\sigma$-algebras $\underline{\underline{F}}^1$, $\underline{\underline{F}}^2$ and $\underline{\underline{G}}$ of $\underline{\underline{F}}$, it will be said that $\underline{\underline{F}}^1$ and $\underline{\underline{F}}^2$ are conditionally independent given $\underline{\underline{G}}$ with respect to $P$ when

(1) $\quad F^1 \in \underline{\underline{F}}^1 \ , \ F^2 \in \underline{\underline{F}}^2 \implies P(F^1 \cap F^2 | \underline{\underline{G}}) = P(F^1 | \underline{\underline{G}}) \ P(F^2 | \underline{\underline{G}})$ .

This property is denoted

$$\underline{\underline{F}}^1 \perp_P \underline{\underline{F}}^2 | \underline{\underline{G}} \, .$$

When the probability is clearly determined by the context, P will be omitted in the above notation. If $\underline{\underline{G}}$ is the σ-algebra generated by P-null sets, conditional independence of $\underline{\underline{F}}^1$ and $\underline{\underline{F}}^2$ given $\underline{\underline{G}}$ becomes independence of $\underline{\underline{F}}^1$ and $\underline{\underline{F}}^2$.

It is to be observed that conditional independence and independence are not merely a structural property of the three σ-algebras $\underline{\underline{F}}^1$, $\underline{\underline{F}}^2$ and $\underline{\underline{G}}$, but depend explicitly on the probability measure P. In particular, this property is not invariant under a change of equivalent measure. The following notion, introduced by Krengel and Sucheston (see [KS]), is a structural property of σ-algebras :

> $\underline{\underline{F}}^1$ and $\underline{\underline{F}}^2$ are conditionally qualitatively independent given $\underline{\underline{G}}$ with respect to P if

(2) $$F^1 \in \underline{\underline{F}}^1 \, , \, F^2 \in \underline{\underline{F}}^2 \implies \{P(F^1 \cap F^2 | \underline{\underline{G}}) > 0\} = \{P(F^1 | \underline{\underline{G}}) > 0\} \cap \{P(F^2 | \underline{\underline{G}}) > 0\}.$$

This property is denoted CQI($\underline{\underline{F}}^1$, $\underline{\underline{F}}^2$, $\underline{\underline{G}}$, P), but if P is determined by the context, it will be omitted and we shall write CQI($\underline{\underline{F}}^1$, $\underline{\underline{F}}^2$, $\underline{\underline{G}}$). Observe that the inclusion "$\subset$" always holds in the place of "=" in (2), but "$\supset$" generally does not. It is clear that (1) implies (2), and it is not difficult to see that (2) is invariant under an equivalent change of measure. However, as we will see in Chapter 2, (2) generally does not imply the existence of a probability measure that is equivalent to P and such that (1) holds.

Both (1) and (2) imply the following property :

(3) $$\underline{\underline{F}}^1 \cap \underline{\underline{F}}^2 \subset \underline{\underline{G}}$$

(see Lemma 7.4.1).

## 1.3 Filtrations and stopping points.

Filtrations are used to model situations where an observer must make decisions in the absence of complete information on the process he is observing. It is usually supposed that the information depends on the time of observation (corresponds to a point in the parameter set) and that the observer gathers more and more information as time passes. Formally, this leads to the following definition.

A single-parameter (resp. two-parameter) filtration is a family $(\underline{\underline{F}}_t)_{t\in I}$ (resp. $(\underline{\underline{F}}_t)_{t\in I^2}$) of sub $\sigma$-algebras of $\underline{\underline{F}}$ with the following properties :

(F1) $\underline{\underline{F}}_0$ contains all P-null sets ;

(F2) $s \leq t \Rightarrow \underline{\underline{F}}_s \subset \underline{\underline{F}}_t$ ;

(F3) when $I = \mathbb{R}_+$ ,

$$\underline{\underline{F}}_s = \bigcap_{t>s} \underline{\underline{F}}_t \ , \quad \forall s \in \mathbb{R}_+ \ \text{(resp.}\ \underline{\underline{F}}_s = \bigcap_{t \gg s} \underline{\underline{F}}_t \ , \quad \forall s \in I^2\text{)}.$$

These properties are often termed the "usual conditions" (see [DM], IV.48).

For two-parameter filtrations, we set

$$\underline{\underline{F}}^1_{t_1} = \underline{\underline{F}}_{t_1,\infty} = \bigvee_{t_2 \in I} \underline{\underline{F}}_{t_1,t_2} \quad \text{and} \quad \underline{\underline{F}}_{t_2} = \underline{\underline{F}}_{\infty,t_2} = \bigvee_{t_1 \in I} \underline{\underline{F}}_{t_1,t_2} \ .$$

Associated with a two-parameter filtration is a set $\underline{\underline{T}}$ of stopping points. A stopping point is the two-parameter analogue of a stopping time (see [DM], IV. 49). A random variable $T : \Omega \to \overline{I^2}$ is a stopping point provided

$$\{T \leq t\} \in \underline{\underline{F}}_t \ , \quad \forall t \in I^2 \ .$$

In this case, $\underline{\underline{F}}_T$ denotes the $\sigma$-algebra

$$\{F \in \underline{\underline{F}} : F \cap \{T \leq t\} \in \underline{\underline{F}}_t \ , \quad \forall t \in I^2\} \ ,$$

which represents the events observable before T.

An apparently innocuous but important difference between stopping times and points is that

$$\{T > t\} \in \underline{F}_t$$

in the first case but not the second, since

$$\{T \leq t\}^c = \{T \wedge t\} \cup \{t < T\} \cup \{t \wedge T\}.$$

This distinction (and others) prevents stopping points from being as universal a tool as stopping times (see [Me2]).

## 1.4 Classical conditions on two-parameter filtrations.

Early in the theory of two-parameter processes, it was realized that for these processes to have properties similar to those of single-parameter processes, further conditions on the two-parameter filtration are necessary. Much of this thesis is devoted to structural properties of two-parameter filtrations that satisfy one or more of the conditions below. It will for instance be shown that for certain questions, two-parameter filtrations that satisfy only (F1), (F2) and (F3) have as general a structure as filtrations depending on three or more parameters. For other questions, two-parameter filtrations have unique properties among n-parameter filtrations.

The most widely used condition on two-parameter filtrations is usually refered to as Hypothesis F4 of Cairoli and Walsh [CW].

1.4.1 Hypothesis F4. We say that a two-parameter filtration $(\underline{F}_t)_{t\in I^2}$ satisfies Hypothesis F4 with respect to P (or that $F4((\underline{F}_t)_{t\in I^2}, P)$ holds) if

$$\underline{F}_{t_1,\infty} \perp_P \underline{F}_{\infty,t_2} \,|\, \underline{F}_{t_1,t_2} \quad , \quad \forall t \in I^2 .$$

When P is determined by the context, we say that $(\underline{\underline{F}}_t)_{t\in I^2}$ satisfies Hypothesis F4 or that $F4((\underline{\underline{F}}_t)_{t\in I^2})$ holds.

An important observation is that this condition is equivalent to the following commutation property.

1.4.2 <u>The commutation property.</u> We say that $(\underline{\underline{F}}_t)_{t\in I^2}$ satisfies the commutation property provided

$$\underline{\underline{F}}_{t_1,t_2} = \underline{\underline{F}}_{t_1,\infty} \cap \underline{\underline{F}}_{\infty,t_2}$$

and the operators

$$E(\cdot \mid \underline{\underline{F}}_{t_1,\infty}) \quad \text{and} \quad E(\cdot \mid \underline{\underline{F}}_{\infty,t_2})$$

commute.

A proof of the equivalence of 1.4.1 and 1.4.2 can be found in [CW] or [Me2].

As is the case for conditional independence, Hypothesis F4 is not invariant under a change of equivalent measure. A natural weakening is that of Krengel and Sucheston :

1.4.3 <u>Hypothesis CQI.</u> We say that $(\underline{\underline{F}}_t)_{t\in I^2}$ satisfies Hypothesis CQI under P (or that $CQI((\underline{\underline{F}}_t)_{t\in I^2}, P)$ holds), if

$$s \curlywedge t,\ u = \min(s,t) \implies CQI(\underline{\underline{F}}_s, \underline{\underline{F}}_t, \underline{\underline{F}}_u, P).$$

As before, when the probability is determined by the context it will be omitted.

It is clear from the definition that Hypothesis F4 implies Hypothesis CQI.

The weakest condition, which has been used by some authors ([Me2], [MS1], [MM]) is the following.

1.4.4 Intersection property. The filtration $(\underline{\underline{F}}_t)_{t\in I^2}$ satisfies the intersection property provided

$$\underline{\underline{F}}_{t_1,t_2} = \underline{\underline{F}}_{t_1,\infty} \cap \underline{\underline{F}}_{t_2,\infty} , \qquad \forall (t_1,t_2) \in I^2 .$$

This property does not depend on the probability P.

## 1.5 Stochastic processes.

A single- (resp. two-) parameter process is a family $(X_t)_{t\in\bar{I}}$ (resp. $(X_t)_{t\in\bar{I}^2}$) of measurable maps from $\Omega$ into some measure space $(M, \underline{\underline{M}})$, usually the real line. Two processes $(X_t)$ and $(Y_t)$ are indistinguishable provided there is a null set N such that

$$\omega \notin N \Rightarrow X_t(\omega) = Y_t(\omega) \quad , \quad \forall t.$$

Indistinguishable processes will be identified.

A process $(X_t)$ is adapted to the filtration $(\underline{\underline{F}}_t)$ provided $X_t$ is $\underline{\underline{F}}_t$-measurable, for all t. In this case, an observer at time t will "know" the values of $X_s(\omega)$, for $s \leqslant t$.

Certain classes of processes will be of special interest for us. These are

- the set $\underline{\underline{C}}$ of continuous real-valued processes $(X_t)_{t\in I^2}$ such that $E(\sup_t |X_t|) < +\infty$ . $\underline{\underline{C}}$ equipped with the norm $\|X\| = E(\sup_t |X_t|)$ is a Banach space, which is in fact the space $L^1(\Omega, C(\overline{I^2}))$ of Bochner P-integrable functions with values in $C(\overline{I^2})$, the set of continuous functions from $\overline{I^2}$ into $\mathbb{R}$.

- the set $\underline{\underline{A}}$ of non-negative right-continuous processes $A = (A_t)_{t\in\overline{I^2}}$ with non-negative planar increments. The (planar) increment of A on the interval $]s,t]$ is

$$\Delta_{]s,t]} A = A_t - A_{s_1,t_2} - A_{t_1,s_2} + A_s \qquad \text{if } s \ll t ,$$

$\Delta_{]s,t]} A = A_t - A_s$ if $s \leq t$ and ($s_1 = t_1$ or $s_2 = t_2$).

A process $(X_t)_{t \in \overline{I^2}}$ is <u>measurable</u> provided the map

$(\omega, t) \longmapsto X_t(\omega)$

is measurable for the product $\sigma$-algebra $\underline{F} \times \underline{B}(\overline{I^2})$. Any right-continuous process is measurable.

## 1.6 The optimal stopping problem.

Given a measurable process $(X_t)_{t \in \overline{I^2}}$ and a stopping point T, $X_T$ denotes the measurable map

$\omega \longmapsto X_{T(\omega)}(\omega)$.

When the quantity $\sup_{T \in \underline{T}} E(X_T)$ is defined and is finite, one would like to know whether or not there is a stopping point $T_o$ such that

$$E(X_{T_o}) = \sup_{T \in \underline{T}} E(X_T).$$

This is known as the optimal stopping problem for two-parameter processes. This problem and variations on it have been studied by several authors (see for instance [MV], [KS], [MS], [LV]).

In the discrete case, relatively constructive solutions are possible (see [KS], [MV], [MS]), but in the continuous case it is not clear whether an optimal stopping point always exists, even when the reward process $(X_t)_{t \in \overline{I^2}}$ is quite regular.

This problem was the initial motivation to this thesis, and we will show in Chapter 9 that optimal stopping points do exist under natural regularity conditions on the reward process when the probability space is hyperfinite.

The problem of existence of optimal stopping points reduces to the following : show that the reward function

$$T \longmapsto E(X_T) \quad \text{from } \underline{T} \text{ into } \mathbb{R}$$

attains its maximum at some $T_0 \in \underline{T}$. Thus it is natural to embed $\underline{T}$ in some larger ("randomized") set $\underline{U}$ with certain convexity and compactness properties and on which $T \longmapsto E(X_T)$ can be extended to a function with sufficient regularity that a maximum over $\underline{U}$ will exist. The choice of randomization should be such that one can then recover a maximum over $\underline{T}$.

The regularity question for upper semi-continuous processes can often be solved using the result of [D], which will be presented in Chapter 9. A natural randomization of $\underline{T}$ is presented in the next section.

## 1.7 Randomized stopping points.

The notion of randomized stopping point presented here is the natural two-parameter extension of that introduced by Baxter and Chacon [BC] in the single-parameter setting. This notion was refined by Meyer [Me] and Ghoussoub [G].

For each $T \in \underline{T}$ ,

$$(X_t)_{t \in \overline{I^2}} \longmapsto E(X_T)$$

defines a linear continuous map from $\underline{C}$ into $\mathbb{R}$ , and so each element of $\underline{T}$ can be identified with an element f of the topological dual $\underline{C}^*$ of $\underline{C}$ such that

(4) - $f(X) \geq 0$ for each non-negative process X ,

(5) - $\|f\| = 1$ and $f(\underline{1}) = 1$.

(here $\underline{1}$ denotes the process which is identically one).

According to Meyer [Me], a <u>randomized point</u> $\mu$ is a non-negative $\sigma$-additive probability measure on $\Omega\times\overline{I^2}$ such that $\mu(F\times\overline{I^2}) = P(F)$, $\forall F\in\underline{F}$. A classical result (see [DM], VII.4, or [DU], IV. 1) states that for each element f of $\underline{C}^*$ satisfying (4) and (5), there is a randomized point $\mu$ such that

$$f(X) = \langle X,\mu\rangle = \int_{\Omega\times\overline{I^2}} X_t(\omega)\ d\mu(\omega,t) \quad , \qquad \forall X\in\underline{C}\ .$$

Clearly the randomized point $\mu$ associated as above with a stopping point T satisfies

$$\mu(F\times B) = E(I_F(\cdot)\delta_{T(\cdot)}(B)) \quad , \quad F\in\underline{F} \quad , \quad B\in\underline{B}(\overline{I^2})$$

where $\delta_t(B) = 1$ if $t\in B$ and $\delta_t(B) = 0$ otherwise.

Using the theorem of disintegration of measures, we see that there is an affine one-to-one correspondence between randomized points and right-continuous processes with non-negative planar increments (see [G] , Lemma I.1 and II.a). If $\mu$ corresponds to $A\in\underline{A}$, then for any $\mu$-integrable random variable X on $\Omega\times\overline{I^2}$,

$$\int_{\Omega\times\overline{I^2}} X_t(\omega)\ d\mu(\omega,t) = E(\int_{\overline{I^2}} X_t(\cdot)\ d_t\ A_t(\cdot)).$$

The element A of $\underline{A}$ associated with a stopping point T clearly satisfies

(6) $\quad A_t(\omega) = I_{[0,t]}\ (T(\omega)) = I_{\{T\leq t\}}\ (\omega)$ ,

and thus has the following properties :

(7) $\quad - A_t \in \{0,1\}$ a.s. ;

(8) $\quad - A_t$ is $\underline{F}_t$ -measurable ;

(9) $\quad - A_\infty \equiv 1$ a.s.

Conversely, it is not difficult to see that each element $A\in\underline{A}$ with these three properties is associated with some stopping point T.

Since we want the set of randomized stopping points to contain the set of stopping points, and to be convex, it is natural to replace (7) by the condition $0 \leq A_t \leq 1$ a.s. Formally, we set :

1.7.1 Definition. The set $\underline{\underline{U}}$ of randomized stopping points is

$$\underline{\underline{U}} = \{(A_t)_{t \in \overline{I^2}} \in \underline{\underline{A}} : \quad 0 \leq A_t \leq 1 \text{ a.s., } A_t \text{ is } \underline{\underline{F}}_t\text{-measurable, and } A_\infty \equiv 1 \text{ a.s.}\}$$

As $\underline{\underline{U}}$ can be identified with a subset of $\underline{\underline{C}}^*$, we can define the weak topology on $\underline{\underline{U}}$ to be the topology induced on $\underline{\underline{U}}$ by the weak topology $\sigma(\underline{\underline{C}}^*, \underline{\underline{C}})$ on $\underline{\underline{C}}^*$. This is the smallest topology that makes the maps

$$(A_t)_{t \in \overline{I^2}} \longmapsto E\left(\int_{\overline{I^2}} X_t(\cdot)\, d_t A_t(\cdot)\right),$$

from $\underline{\underline{U}}$ into $\mathbb{R}$, continuous for all $X \in \underline{\underline{C}}$.

1.7.2 Theorem. $\underline{\underline{U}}$ is compact in the weak topology.

Proof. As has been observed above, an element f of $\underline{\underline{C}}^*$ corresponds to an element of $\underline{\underline{A}}$ if and only if (4) and (5) hold. This element further belongs to $\underline{\underline{U}}$ provided the following condition is satisfied (see [BC], §2) :

- If $\psi : \overline{I^2} \longrightarrow \mathbb{R}$ is continuous with support contained in $[0,t]$, and X is a random variable on $\Omega$, then

$$f(\psi X) = f(\psi E(X \mid \underline{\underline{F}}_t)).$$

Thus $\underline{\underline{U}}$ is a closed subset of $\underline{\underline{C}}^*$. Since $\underline{\underline{U}}$ is contained in the unit ball of $\underline{\underline{C}}^*$ by (5), and since this ball is compact in the $\sigma(\underline{\underline{C}}^*, \underline{\underline{C}})$-topology by the Banach-Alaoglu Theorem (see [DS], V. 4.2), the set $\underline{\underline{U}}$ is compact in the weak topology. □

Given $X \in \underline{C}$, the correspondence in (6) implies that the map

$$(A_t)_{t \in \overline{I^2}} \longmapsto E\left(\int_{\overline{I^2}} X_t \, d_t \, A_t\right)$$

from $\underline{U}$ into $\mathbb{R}$ is an extension of the map $T \longmapsto E(X_T)$ from $\underline{T}$ into $\mathbb{R}$. By definition, the map on $\underline{U}$ is continuous for the weak topology on $\underline{U}$, and since it is affine, it attains its maximum at an extreme point of $\underline{U}$ (see [B], II. 58, Prop. 1).

Thus, at least when $X \in \underline{C}$, the existence of an optimal stopping point will be straightforward provided each extremal element of $\underline{U}$ corresponds to a stopping point. We examine this question more closely in the next section.

## 1.8 Extremal randomized stopping points.

First of all, observe that property (7) implies that any stopping point defines an extremal element of the convex set $\underline{U}$. Indeed, if $A = (A_t)_{t \in \overline{I^2}}$ satisfies (7), and if

$$A = \frac{1}{2} A^1 + \frac{1}{2} A^2 , \qquad A^1, A^2 \in \underline{A} ,$$

then

$$A_t(\omega) = 0 \implies A^1_t(\omega) = A^2_t(\omega) = 0 ,$$

$$A_t(\omega) = 1 \implies A^1_t(\omega) = A^2_t(\omega) = 1 ,$$

and so $A^1 = A = A^2$.

The converse statement was known to hold in the following trivial case.

1.8.1 Proposition. Suppose $\underline{F}_t = \underline{F}$ , $\forall t \in I^2$. Then every extremal element of $\underline{U}$ corresponds to a stopping point.

A simple counterexample when $\underline{F}_t \neq \underline{F}$ was given in [MM1]. However, answering the following question is one of the primary objects of this thesis.

1.8.2 Question. Given a two-parameter filtration $(\underline{F}_t)_{t\in I^2}$, is every extremal element of the set of randomized stopping points a stopping point ?

Proposition 1.8.2 was well-known (see Ghoussoub [G], Prop. 1.2, who refers to a result of Yor ([Y], Prop. 1.6)), but prior to this thesis, as stated in [MM], Proposition 1.8.1 gives the only case where Question 1.8.2 was known to have an affirmative answer. This is rather surprising since the (affirmative) answer to Question 1.8.2 was well-known in the single-parameter case (see [DM], VI. 66), and its proof is almost trivial (see Lemma 1.8.3 below). Several chapters in this thesis are devoted to the study of Question 1.8.2 and to the influence of the structure of $(\underline{F}_t)_{t\in I^2}$ on the nature of the extremal elements of $\underline{U}$.

1.8.3 Lemma. Let $(\underline{F}_t)_{t\in I}$ be a filtration, and let $\underline{U}_1$ be the set of right-continuous single-parameter non-decreasing processes which are adapted to this filtration and take values in [0,1]. Then every extremal element of $\underline{U}_1$ takes only values 0 and 1, and thus corresponds to a stopping point.

Proof. This proof is well-known. Let $A = (A_t)_{t\in I}$ belong to $\underline{U}_1$. Define two elements $A^1 = (A^1_t)_{t\in I}$ and $A^2 = (A^2_t)_{t\in I}$ of $\underline{U}_1$ by setting

$$A^1_t = \min(2A_t, 1) \quad \text{and} \quad A^2_t = \max(2A_t - 1, 0) \ ,$$

and observe that

$$A_t = \frac{1}{2} A^1_t + \frac{1}{2} A^2_t \ , \quad \forall t \in I \ .$$

Furthermore,

$$P\{\omega\in\Omega : \exists t \in I \text{ with } 0 < A_t(\omega) < 1\} > 0 \ \Rightarrow \ A^1 \neq A \neq A^2 \ .$$

Thus every extremal element of $\underline{U}_1$ must take only values 0 or 1. □

We close this section with some remarks on different ways of considering randomized stopping points.

When I is discrete, say $I = \mathbb{N}$, it is often more practical to replace the process $(A_t)_{t\in\overline{\mathbb{N}^2}}$ with the individual weights of elements of $\mathbb{N}^2$. That is, we replace $(A_t)_{t\in\overline{\mathbb{N}^2}}$ with the (unique) weight process $(a_t)_{t\in\mathbb{N}^2}$ such that

$$A_t = \sum_{s\leqslant t} a_s , \quad \forall t \in I^2 .$$

The process $(a_t)_{t\in\mathbb{N}^2}$ satisfies

- $a_t \geqslant 0$ a.s. , $\forall t \in \mathbb{N}^2$,

- $a_t$ is $\underline{\underline{F}}_t$-measurable, $\forall t \in \mathbb{N}^2$,

- $\sum_{t\in\mathbb{N}^2} a_t \leqslant 1$ a.s.

(the weight of $\{\infty\}$ is $a_\infty = 1 - \sum_{t\in\mathbb{N}^2} a_t$).

In view of the above, we will also say that $(a_t)_{t\in\mathbb{N}^2}$ is a randomized stopping point.

When $I = \mathbb{R}_+$, it will often be useful to associate with each element A of $\underline{\underline{A}}$ a (random) measure $\mu_\omega$ (B), $\omega \in \Omega$, $B \in \underline{\underline{B}}(\overline{I^2})$, that satisfies

$$\mu_\omega([0,t]) = A_t(\omega) , \quad \forall t \in \overline{I^2} ,$$

for almost all $\omega\in\Omega$ . In this case, A is the (random) distribution function of $\mu$ .

Observe that if $A = \frac{1}{2} A^1 + \frac{1}{2} A^2$, where $A^1$ and $A^2$ belong to $\underline{\underline{A}}$, then $\text{supp}\ \mu^i_. \subset \text{supp}\ \mu_.$ a.s., where $\mu^i$ is the measure associated with $A^i$, i = 1, 2, and $\text{supp}\ \mu_\omega$ denotes the support of the measure $\mu_\omega$ .

The above associations will be useful in the following chapters.

CHAPTER 2

# THE RELATIONSHIP BETWEEN CONDITIONAL QUALITATIVE INDEPENDENCE AND THE COMMUTATION PROPERTY

In this chapter, we examine the relationship between the conditions on two-parameter filtrations defined in Chapter 1.4. We have already noted that both Hypothesis F4 and Hypothesis CQI imply the Intersection Property 1.4.4, and that Hypothesis F4 implies Hypothesis CQI. Since CQI is invariant under an equivalent change of measure, but F4 is not, one might hope that CQI would imply the existence of an equivalent probability measure such that F4 holds. Unfortunately this is not the case.

The object of this chapter is to study this problem and to give necessary and sufficient conditions on a two-parameter filtration $(\underline{F}_t)_{t\in\mathbb{N}^2}$ that ensure the existence of an equivalent probability measure P such that F4 is satisfied. We will require P to be equivalent to the original measure on each $\underline{F}_t$ (in general, these two measures will not be equivalent on $\underline{F} = \bigvee_t \underline{F}_t$ ).

Though the statement of the problem is quite simple, it turns out that the situation is more complex than might be expected. We shall illustrate this with some examples. The results presented here are related to the work of Marczewski [M] and Kallianpur and Ramachandran [KR] concerning qualitative independence and the splicing of measures.

## 2.1 A necessary condition for conditional independence.

In this section we give necessary conditions for the following problem to have a solution.

2.1.1 Problem. Given a set $\Omega$, two $\sigma$-algebras $\underline{\underline{F}}^1$ and $\underline{\underline{F}}^2$ on $\Omega$, and a probability P on $\underline{\underline{F}}^1 \vee \underline{\underline{F}}^2$, find a probability Q equivalent to P on $\underline{\underline{F}}^1 \vee \underline{\underline{F}}^2$ such that $\underline{\underline{F}}^1$ and $\underline{\underline{F}}^2$ are conditionally independent given $\underline{\underline{F}}^1 \cap \underline{\underline{F}}^2$ with respect to Q.

2.1.2 Theorem. Suppose Problem 2.1.1 has a solution. Then it has a unique solution Q, called fundamental solution, such that

(1) $$Q(A^1 \cap A^2) = E_P\,(P(A^1|\underline{\underline{G}})\; P(A^2|\underline{\underline{G}})),\ \forall A^1 \in \underline{\underline{F}}^1,\ A^2 \in \underline{\underline{F}}^2,$$

where $\underline{\underline{G}} = \underline{\underline{F}}^1 \cap \underline{\underline{F}}^2$. Furthermore, if $\tilde{Q}$ is any solution to Problem 2.1.1, then

$$\tilde{Q} = \frac{f}{f^1 f^2}\, Q\ ,$$

where f (respectively $f^i$) is the Radon-Nikodym derivative of the restriction $P|_{\underline{\underline{G}}}$ with respect to $\tilde{Q}|_{\underline{\underline{G}}}$ (respectively $P|_{\underline{\underline{F}}^i}$ with respect to $\tilde{Q}|_{\underline{\underline{F}}^i}$, $i = 1,2$).

Proof. Suppose $\tilde{Q}$ is a solution to Problem 2.1.1, and set

$$Q = \frac{f^1 f^2}{f}\, \tilde{Q}\ .$$

We shall show that Q is the fundamental solution to Problem 2.1.1. Since $\tilde{Q}$ is equivalent to Q, $f > 0$ and $f^i > 0$ Q - a.s., i = 1,2. Thus Q is equivalent to $\tilde{Q}$, hence to P on $\underline{\underline{F}}^1 \vee \underline{\underline{F}}^2$.

Furthermore, observe that $E_{\tilde{Q}}(f^i|\underline{\underline{G}}) = f$, since f is $\underline{\underline{G}}$-measurable and if $G \in \underline{\underline{G}}$,

$$\int_G f\, d\tilde{Q} = P(G) = \int_G f^i\, d\tilde{Q}\ .$$

Now if $A^i \in \underline{\underline{F}}^i$, i = 1,2, and $G \in \underline{\underline{G}}$, then

$$E_{\tilde{Q}}(I_{A^i}\, f^i\, I_G) = E_P(I_{A^i}\, I_G) = E_P(P(A^i|\underline{\underline{G}})\, I_G)$$

$$= E_{\tilde{Q}}(P(A^i|\underline{\underline{G}}) I_G\, f)\ ,$$

so $E_{\tilde{Q}}(I_{A^i} f^i | G) = P(A^i | G) f$. Thus

$$Q(A^1 \cap A^2) = E_{\tilde{Q}}(I_{A^1 \cap A^2} \frac{f^1 f^2}{f}) = E_{\tilde{Q}}(\frac{1}{f} E_{\tilde{Q}}(I_{A^1} f^1 I_{A^2} f^2 | \underline{G}))$$

$$= E_{\tilde{Q}}(\frac{1}{f} E_{\tilde{Q}}(I_{A^1} f^1 | \underline{G}) E_{\tilde{Q}}(I_{A^2} f^2 | \underline{G}))$$

$$= E_{\tilde{Q}}(f\ P(A^1 | \underline{G})\ P(A^2 | \underline{G}))$$

$$= E_p(P(A^1 | \underline{G})\ P(A^2 | \underline{G})) .$$

The third equality holds since $\underline{F}^1 \perp_{\tilde{Q}} \underline{F}^2 | \underline{G}$. This shows that Q satisfies (1).

Observe that Q is a probability measure, since

$$Q(\Omega) = E_{\tilde{Q}} (\frac{f^1 f^2}{f}) = E_{\tilde{Q}} (\frac{1}{f} E_{\tilde{Q}}(f^1 | \underline{G}) E_{\tilde{Q}}(f^2 | \underline{G}))$$

$$= E_{\tilde{Q}}(f) = P(\Omega) = 1 .$$

In order to show that $\underline{F}^1 \perp_Q f^2 | \underline{G}$, we first observe that if $Z \in \underline{G}$, then

(2) $$E_Q(Z) = E_{\tilde{Q}}(Z \frac{f^1 f^2}{f}) = E_{\tilde{Q}}(\frac{Z}{f} E_{\tilde{Q}}(f^1 | \underline{G}) E_{\tilde{Q}}(f^2 | \underline{G}))$$

$$= E_{\tilde{Q}}(Z\ f)$$

Thus

(3) $$Q(A^i | \underline{G}) = P(A^i | \underline{G}) ,$$

since by (1) and (2),

$$Q(A^i \cap G) = E_p(P(A^i | \underline{G})\ I_G) = E_{\tilde{Q}}(f\ P(A^i | \underline{G}) I_G)$$

$$= E_Q(P(A^i | \underline{G}) I_G) .$$

Now again by (1) and (2),

$$Q(A^1 \cap A^2 \cap G) = E_P(P(A^1|\underline{G})P(A^2|G)I_G)$$

$$= E_{\tilde{Q}}(f\ P(A^1|\underline{G})\ P(A^2|\underline{G})I_G)$$

$$= E_Q(P(A^1|\underline{G})\ P(A^2|\underline{G})I_G)\ .$$

Together with (3), this shows that Q is a solution to Problem 2.1.1 and completes the proof of the theorem. □

The following problem is a special case of Problem 2.1.1, which arises when $\underline{F}^1 \cap \underline{F}^2$ is the σ-algebra generated by P-null sets.

2.1.3 Problem. Given a set Ω, two σ-algebras $\underline{F}^1$ and $\underline{F}^2$ on Ω, and a probability P on $\underline{F}^1 \vee \underline{F}^2$, find a probability Q equivalent to P such that $\underline{F}^1$ and $\underline{F}^2$ are independent with respect to Q.

An obvious necessary condition for Problem 2.1.3 to have a solution is the following :

2.1.4 Condition. $A^i \in \underline{F}^i$ , $P(A^i) > 0$, $i = 1,2 \Rightarrow P(A^1 \cap A^2) > 0$.

It is not difficult to see that Condition 2.1.4 is equivalent to Hypothesis CQI for the filtration $(\underline{F}_t)_{t\in\{0,1\}^2}$ defined by $\underline{F}_{0,0} = \{\emptyset,\Omega\}$, $\underline{F}_{1,0} = \underline{F}^1$, $\underline{F}_{0,1} = \underline{F}^2$ , and $\underline{F}_{1,1} = \underline{F}^1 \vee \underline{F}^2$ .

The following example shows that Condition 2.1.4 is not sufficient for the existence of a solution to Problem 2.1.3, and also that variations on Problem 2.1.3 are possible. For example, we could require Q to be equivalent to P on both $\underline{F}^1$ and $\underline{F}^2$, instead of on $\underline{F}^1 \vee \underline{F}^2$.

2.1.5 Example. Set $\Omega = [0,1]^2$ and let $\underline{F}^1$ (respectively $\underline{F}^2$) be the σ-algebra of all vertical (respectively horizontal) Borel cylinders. Then $\underline{F} = \underline{F}^1 \vee \underline{F}^2$ is the Borel $\sigma$-algebra on $\Omega$. Let P be the probability measure defined by

$$P = \frac{1}{2} P_1 + \frac{1}{2\sqrt{2}} P_2 ,$$

where $P_1$ denotes Lebesgue measure on $\{0,1\}^2$, and $P_2$ is linear Lebesgue measure on the diagonal $\{(s_1,s_2) : s_1 = s_2\}$.

Clearly, Condition 2.1.4 holds, and Lebesgue measure is a probability on $\Omega$ that coincides with P on both $\underline{F}^1$ and $\underline{F}^2$, and $\underline{F}^1$ and $\underline{F}^2$ are independent for Lebesgue measure.

However, there is no solution to Problem 2.1.3 since by Theorem 2.1.2, the fundamental solution would be Lebesgue measure, which is not equivalent to P on $\underline{F}$. □

## 2.2 Sufficient conditions for conditional independence.

Though Example 2.1.5 shows that CQI($\underline{F}^1$, $\underline{F}^2$, $\underline{F}^1 \cap \underline{F}^2$) does not guaranty the existence of an equivalent measure such that $\underline{F}^1$ and $\underline{F}^2$ are conditionally independent given $\underline{F}^1 \cap \underline{F}^2$, this condition does imply the existence of a finitely additive measure satisfying (1).

2.2.1 Proposition. Suppose CQI($\underline{F}^1$, $\underline{F}^2$, $\underline{G} = \underline{F}^1 \cap \underline{F}^2$). Then there is a finitely additive probability Q defined on the smallest algebra $\underline{H}$ containing $\underline{F}^1$, such that :

$$Q(A^1 \cap A^2) = E_P(P(A^1|\underline{G})\ P(A^2|\underline{G})) , \qquad \forall A^1 \in \underline{F}^1 ,\ A^2 \in \underline{F}^2 .$$

Proof. The proof is a straightforward generalization of that of Marczewski [M], Lemma 6.

It is not difficult to see (and a proof can be found in [M], Lemmas 1 and 2) that each element $C \in \underline{\underline{H}}$ can be written in the following manner :

$$(4) \qquad C = \bigcup_{j=1}^{n} (A_j^1 \cap A_j^2) ,$$

where $A_j^i \in F^i$, $i = 1,2$, and the $A_j^1 \cap A_j^2$ are disjoint, $1 \leq j \leq n$.

Suppose

$$C = \bigcup_{j=1}^{m} (B_j^1 \cap B_j^2) , \quad B_j^i \in F^i , \quad i = 1, 2,$$

where the $B_j^1 \cap B_j^2$ are disjoint, is another decomposition of C as in (4). We shall show that

$$(5) \qquad \sum_{j=1}^{n} E_P(P(A_j^1|\underline{\underline{G}})\ P(A_j^2|\underline{\underline{G}})) = \sum_{j=1}^{m} E_P(P(B_j^1|\underline{\underline{G}})\ P(B_j^2|\underline{\underline{G}})).$$

Let $(G_k^i)$ be a finite sequence of disjoint elements of $\underline{\underline{F}}^i$ such that each $A_j^i$ and $B_j^i$ is a disjoint union of some of the $G_k^i$ :

$$A_j^i = \bigcup_{k \in U_j^i} G_k^i , \quad B_j^i = \bigcup_{k \in V_j^i} G_k^i , \quad i = 1, 2,$$

where the $U_j^i$ and $V_j^i$ are finite sets. Set

$$R = \bigcup_{j=1}^{n} (U_j^1 \times U_j^2) , \quad S = \bigcup_{j=1}^{m} (V_j^1 \times V_j^2).$$

Then

$$C = \bigcup_{(k,\ell) \in R} (G_k^1 \cap G_\ell^2) = \bigcup_{(k,\ell) \in S} (G_k^1 \cap G_\ell^2)$$

Since $(G_k^1 \cap G_\ell^2) \cap (G_{k'}^1 \cap G_{\ell'}^2) = \emptyset$ when $(k,\ell) \neq (k',\ell')$ belong to $R \cup S$, we see that $G_k^1 \cap G_\ell^2 = \emptyset$ when $(k,\ell) \in R \setminus S$. Thus

$$(6)\qquad E_P(P(G_k^1|\underline{G})\ P(G_\ell^2\ \underline{G})) = E_P(P(G_k^1|\underline{G})\ P(G_\ell^2|\underline{G})\ I_{\{P(G_k^1|\underline{G})>0\}\cap\{P(G_\ell^2|\underline{G})>0\}})$$

$$= E_P(P(G_k^1|\underline{G})\ P(G_\ell^2|\underline{G})\ I_{\{P(G_k^1\cap G_\ell^2|\underline{G})>0\}})$$

$$= 0\ ,$$

for $(k,\ell)\in R\setminus S$. (the second equality follows from $CQI(\underline{F}^1, \underline{F}^2, \underline{G})$). It follows that

$$(7)\qquad \sum_{(k,\ell)\in R} E_P(P(G_k^1|\underline{G})\ P(G_\ell^2|\underline{G})) = \sum_{(k,\ell)\in S} E_P(P(G_k^1|\underline{G})\ P(G_\ell^2|\underline{G})).$$

Since the $G_k^i$ are disjoint,

$$P(A_j^i|\underline{G}) = \sum_{k\in P_j^i} P(G_k^i|\underline{G})\ ,$$

and thus

$$(8)\qquad \sum_{j=1}^{n} E_P(P(A_j^1|\underline{G})\ P(A_j^2|\underline{G})) = \sum_{j=1}^{n} \sum_{k\in P_j^1} \sum_{\ell\in P_j^2} E_P(P(G_k^1|\underline{G})\ P(G_\ell^2|\underline{G})).$$

We shall show that this expression is equal to the left-hand side of (7). Observe that if some pair $(k,\ell)$ appears twice in (8), i.e., there are $k$, $\ell$, $j$ and $j'$, $j\neq j'$, such that $k\in P_j^1\cap P_{j'}^1$, $\ell\in P_j^2\cap P_{j'}^2$, then

$$G_k^1\cap G_\ell^2 \subset (A_j^1\cap A_j^2)\cap(A_{j'}^1\cap A_{j'}^2) = \emptyset\ .$$

As in (6), this implies that $E_P(P(G_k^1|\underline{G})\ P(G_k^2|\underline{G})) = 0$, and so couples $(k,\ell)$ that appear twice in (8) do not contribute to the sum. Thus

$$\sum_{j=1}^{n} E_P(P(A_j^1|\underline{G})\ P(A_j^2|\underline{G})) = \sum_{(k,\ell)\in R} E_P(P(G_k^1|\underline{G})\ P(G_k^2|\underline{G}))$$

and similarly,

$$\sum_{j=1}^{n} E_P(P(B_j^1|\underline{G})\ P(B_j^2|\underline{G})) = \sum_{(k,\ell)\in S} E_P(P(G_k^1|\underline{G})\ P(G_k^2|\underline{G})).$$

By (7), this proves (5).

Thus we can define

$$Q(C) = \sum_{j=1}^{m} E_P(P(A_j^1|\underline{G})\ P(A_j^2|\underline{G})),$$

for C, $A_j^i$ as in (1). Q is then a finitely additive probability with the desired property. □

By Theorem 2.1.2, Problem 2.1.1 will have a solution if and only if the measure of Proposition 2.2.1 is countably additive and equivalent to P on $\underline{F}^1 \vee \underline{F}^2$ . Let us determine necessary conditions for countable additivity of this measure.

2.2.2 <u>Condition.</u> If $\Omega = \bigcup_{j=1}^{\infty} (A_j^1 \cap A_j^2)$ , $A_j^i \in \underline{F}^i$ for $j \in \mathbb{N}$, $i = 1, 2$, then

$$\sum_{j=1}^{\infty} P(A_j^1|\underline{F}^1 \cap \underline{F}^2)\ P(A_j^2|\underline{F}^1 \cap F^2) \geq 1 \quad \text{a.s.}$$

An important observation is that this condition is stronger than $CQI(\underline{F}^1, \underline{F}^2, \underline{F}^1 \cap \underline{F}^2)$.

2.2.3 <u>Proposition.</u> Condition 2.2.2 implies $CQI(\underline{F}^1, \underline{F}^2, \underline{F}^1 \cap \underline{F}^2)$.

<u>Proof.</u> Fix $A_i \in \underline{F}^i$, $i = 1, 2$, such that $A_1 \cap A_2 = \emptyset$. Then

$$\Omega = (A_1^c \cap A_2) \cup (A_1 \cap A_2^c) \cup (A_1^c \cap A_2^c) .$$

So by Condition 2.2.2 setting $\underline{G} = \underline{F}^1 \cap \underline{F}^2$, we have

$$(1 - P(A_1|\underline{G}))P(A_2|\underline{G}) + P(A_1|\underline{G})(1 - P(A_2|\underline{G})) + (1 - P(A_1|\underline{G}))(1 - P(A_2|\underline{G})) \geq 1 .$$

Rearranging this expression, we get

$$P(A_1|\underline{G}) \;\; P(A_2|\underline{G}) \leq 0 ,$$

which is equivalent to

$$\{P(A_1|\underline{G}) > 0\} \cap \{P(A_2|\underline{G}) > 0\} = \emptyset . \tag{9}$$

To see that $CQI(\underline{F}^1, \underline{F}^2, \underline{G})$ holds, let $B^i$ be an arbitrary element of $\underline{F}^i$ , $i = 1, 2$, and set

$$A_i = B^i \cap \{P(B^1 \cap B^2|\underline{G}) = 0\}$$

Applying (9), we get

$$\{P(B^1|\underline{G}) > 0\} \cap \{P(B^2|\underline{G}) > 0\} \cap \{P(B^1 \cap B^2|\underline{G}) = 0\} = \emptyset ,$$

so

$$\{P(B^1|\underline{G}) > 0\} \cap \{P(B^2|\underline{G}) > 0\} \subset \{P(B^1 \cap B^2|\underline{G}) > 0\}.$$

Since the converse inclusion is trivial, the conclusion follows. □

2.2.4 Theorem.

(a) If Problem 2.1.1 has a solution, then Condition 2.2.2 holds.

(b) Suppose Condition 2.2.2 holds. Then the finitely additive probability Q, whose existence is affirmed in Proposition 2.2.1, extends to a probability on $\underline{F}^1 \vee \underline{F}^2$. Furthermore, $\underline{F}^1 \perp_Q \underline{F}^2 \mid \underline{G} = \underline{F}^1 \cap \underline{F}^2$.

Proof.

(a) Let $A_j^i$ be as in Condition 2.2.2, and let Q be the fundamental solution of Problem 2.1.1. Then

$$(10) \qquad 1 = Q(\Omega) \leq \sum_{j=1}^{\infty} Q(A_j^1 \cap A_j^2) = \sum_{j=1}^{\infty} E_P(P(A_j^1|\underline{\underline{G}})\, P(A_j^2|\underline{\underline{G}})).$$

Let $\quad D = \left\{ \sum_{j=1}^{\infty} E_P(P(A_j^1|\underline{\underline{G}})\, P(A_j^2|\underline{\underline{G}}) < 1 \right\}$. Then $D \in \underline{\underline{G}}$ and

$$\Omega = (\bigcup_{n=1}^{\infty} ((A_n^1 \cap D) \cap A_n^2)) \cup (D^c \cap D^c) ,$$

so by (10),

$$1 \leq E_P( \sum_{j=1}^{\infty} I_D\, P(A_j^1|\underline{\underline{G}})\, P(A_j^2|\underline{\underline{G}}) + (1 - I_P)^2) ,$$

and thus

$$E_P(I_P( \sum_{j=1}^{\infty} P(A_j^1|\underline{\underline{G}})\, P(A_j^2|\underline{\underline{G}}) - 1)) \geq 0 .$$

This implies that P(D) = 0, and so Condition 2.2.2 holds.

(b) This part of the proof follows that of [KR], Proposition 1. Since Q is additive, Q is countably superadditive, and so we establish its countable subadditivity.

Suppose

$$F = \bigcup_{j=1}^{m} (A_j \cap B_j) , \qquad F_N = \bigcup_{j=1}^{m_N} (A_j^N \cap B_j^N)$$

are such that

$$F = \bigcup_{N=1}^{\infty} F_N , \quad A_j, A_j^N \in \underline{\underline{F}}^1, B_j, B_j^N \in \underline{\underline{F}}^2,$$

the $A_j \cap B_j$ are disjoint, as are the $A_j^N \cap B_j^N$ for each N. Then for each j,

$$A_j \cap B_j = \bigcup_{N=1}^{\infty} ((A_j \cap B_j) \cap F_N) = \bigcup_{N=1}^{\infty} \bigcup_{k=1}^{m_N} (A_j \cap A_k^N \cap B_j \cap B_k^N) ,$$

and so

$$\Omega = \left[ \bigcup_{N=1}^{\infty} \bigcup_{k=1}^{m_N} (A_j \cap A_k^N) \cap (B_j \cap B_k^N) \right] \cup (A_j^c \cap B_j) \cup (A_j \cap B_j^c) \cup (A_j^c \cap B_j^c) .$$

Thus by Condition 2.2.2,

$$\sum_{N=1}^{\infty} \sum_{k=1}^{m_N} E_P(P(A_j \cap A_k^N | \underline{G}) \; P(B_j \cap B_k^N | \underline{G})) +$$

$$E_P(P(A_j^c | \underline{G}) \; P(B_j | \underline{G})) + E_P(P(A_j | \underline{G}) \; P(B_j^c | \underline{G})) + E_P(P(A_j^c | \underline{G}) \; P(B_j^c | \underline{G})) \geq 1 .$$

Rearranging this expression gives

$$E_P(P(A_j | \underline{G}) \; P(B_j | \underline{G})) \leq \sum_{N=1}^{\infty} \sum_{k=1}^{m_N} E_P(P(A_j \cap A_k^N | \underline{G}) \; P(B_j \cap B_k^N | \underline{G})) ,$$

and thus

$$\begin{aligned} Q(F) &= \sum_{j=1}^{m} E_P(P(A_j | \underline{G}) \; P(B_j | \underline{G})) \\ &\leq \sum_{j=1}^{m} \sum_{N=1}^{\infty} \sum_{k=1}^{m_N} E_P(P(A_j \cap A_k^N | \underline{G}) \; P(B_j \cap B_k^N | \underline{G})) \\ &= \sum_{N=1}^{\infty} \sum_{k=1}^{m_N} \sum_{j=1}^{m} Q(A_j \cap A_k^N \cap B_j \cap B_k^N) \\ &= \sum_{N=1}^{\infty} Q(F_N) . \end{aligned}$$

This proves that Q is countably additive, and by the Carathéodory extension theorem, Q has a unique extension to $\underline{F}^1 \vee \underline{F}^2$.

To see that $\underline{F}^1$ and $\underline{F}^2$ are conditionally independent given $\underline{G}$ with respect to Q, observe that $Q(G) = P(G)$, $\forall G \in \underline{\underline{G}}$. Thus if $A^i \in \underline{\underline{F}}^i$ and $G \in \underline{\underline{G}}$,

$$Q(A^i \cap G) = E_P(P(A^i|\underline{\underline{G}})I_G) = E_Q(P(A^i|\underline{\underline{G}})I_G) ,$$

and so $Q(A^i|\underline{\underline{G}}) = P(A^i|\underline{\underline{G}})$. Thus

$$Q(A^1 \cap A^2 \cap G) = E_P(P(A^1|\underline{\underline{G}})\ P(A^2|\underline{\underline{G}})I_G)$$

$$= E_Q(Q(A^1|\underline{\underline{G}})\ Q(A^2|\underline{\underline{G}})I_G) ,$$

proving $Q(A^1 \cap A^2|\underline{\underline{G}}) = Q(A^1|\underline{\underline{G}})\ Q(A^2|\underline{\underline{G}})$. □

It is not immediately apparent that Condition 2.2.2 is invariant under a change of equivalent measure. This is however the case. In fact, we have

2.2.5 Proposition.

(a) Suppose $\tilde{P}$ is a probability on $\underline{\underline{F}}^1 \vee \underline{\underline{F}}^2$ such that $\tilde{P}|_{\underline{\underline{F}}^i}$ is equivalent to $P|_{\underline{\underline{F}}^i}$, $i = 1, 2$, and P satisfies Condition 2.2.2. Then $\tilde{P}$ also satisfies Condition 2.2.2.

(b) Suppose in addition that P is equivalent to $\tilde{P}$ on $\underline{\underline{F}}^1 \vee \underline{\underline{F}}^2$. Let Q (respectively $\tilde{Q}$) be the unique probability such that

$$Q(A^1 \cap A^2) = E_P(P(A^1|\underline{\underline{F}}^1 \cap \underline{\underline{F}}^2)\ P(A^2|\underline{\underline{F}}^1 \cap \underline{\underline{F}}^2))$$

(respectively

$$\tilde{Q}(A^1 \cap A^2) = E_{\tilde{P}}(\tilde{P}(A^1|\underline{\underline{F}}^1 \cap \underline{\underline{F}}^2)\ \tilde{P}(A^2|\underline{\underline{F}}^1 \cap \underline{\underline{F}}^2))\Big),\ A^1 \in F^1,\ A^2 \in F^2 .$$

Then Q is equivalent to P on $\underline{\underline{F}}^1 \vee \underline{\underline{F}}^2$ if and only if $\tilde{Q}$ is equivalent to $\tilde{P}$ on $\underline{\underline{F}}^1 \vee \underline{\underline{F}}^2$.

Proof.

(a) Let $f_i$ be the Radon-Nikodym derivative of $\tilde{P}|_{F^i}$ with respect to $P|_{F^i}$, and suppose $\Omega = \bigcup_{j=1}^{\infty} (A_j^1 \cap A_j^2)$, $A_j^i \in \underline{F}^i$, $i = 1, 2$. Then, using the equality

$$\tilde{P}(A_j^i | \underline{G}) = \frac{E_P(f_i I_{A^i} | \underline{G})}{E_P(f_i | \underline{G})},$$

which can be checked directly from the definition of conditional expectation, we get

$$\sum_{j=1}^{\infty} E_{\tilde{P}}(\tilde{P}(A_j^1 | \underline{G})\ \tilde{P}(A_j^2 | \underline{G})) = \sum_{j=1}^{\infty} E_P(E_P(f_1 | \underline{G})\ \tilde{P}(A_j^1 | \underline{G})\ \tilde{P}(A_j^2 | \underline{G}))$$

$$= \sum_{j=1}^{\infty} E_P\left(E_P(f_1 | \underline{G}) \frac{E_P(f_1 I_{A_j^1} | \underline{G})}{E_P(f_1 | \underline{G})} \frac{E_P(f_2 I_{A_j^2} | \underline{G})}{E_P(f_2 | \underline{G})}\right)$$

$$= \sum_{j=1}^{\infty} E_P\left(\frac{E_Q(f_1 I_{A_j^1} | \underline{G})\ E_Q(f_2 I_{A_j^2} | \underline{G})}{E_P(f_2 | \underline{G})}\right)$$

$$= \sum_{j=1}^{\infty} E_P\left(E_Q(f_1 f_2 I_{A_j^1 \cap A_j^2} | \underline{G}) \frac{1}{E_P(f_2 | \underline{G})}\right)$$

$$\geq E_P\left(\frac{E_Q(f_1 f_2 | \underline{G})}{E_P(f_2 | \underline{G})}\right) = E_P\left(\frac{E_Q(f_1 | \underline{G})\ E_Q(f_2 | \underline{G})}{E_P(f_2 | \underline{G})}\right)$$

$$= E_P(E_P(f_1 | \underline{G})) = E_P(f_1) = \tilde{P}(\Omega) = 1 .$$

In the above calculation, Q is the fundamental solution to Problem 2.1.1 for P.

(b) Suppose Q is equivalent to P on $\underline{F}^1 \vee \underline{F}^2$. Then Q is equivalent to $\tilde{P}$ on $\underline{F}^1 \vee \underline{F}^2$ and so by Theorem 2.1.2, $\tilde{Q}$ is equivalent to $\tilde{P}$ on $\underline{F}^1 \vee \underline{F}^2$. □

## 2.3 Countable conditional qualitative independence.

Given the results of the preceeding section, it is natural to introduce the following definition.

2.3.1 Definition. We will say that $\underline{F}^1$ and $\underline{F}^2$ are countably conditionally qualitatively independent given $\underline{G}$ for P (denoted CCQI($\underline{F}^1$, $\underline{F}^2$, $\underline{G}$, P) provided

(a) Condition 2.2.2 holds,

and

(b) the unique probability Q that satisfies (1) is equivalent to P on $\underline{F}^1 \vee \underline{F}^2$.

2.3.2 Remark. Condition 2.2.2 does not imply (b) above, as can be seen from Example 2.1.5. However, Condition 2.2.2 does imply that there is a probability Q that satisfies (1) (see Theorem 2.2.4 (b)).

By Proposition 2.2.5, condition CCQI is invariant under a change of equivalent measure. The following theorem summarizes the results of Section 2.2.

2.3.3 Theorem. Problem 2.1.1 has a solution if and only if CCQI($\underline{F}^1$, $\underline{F}^2$, $\underline{F}^1 \cap \underline{F}^2$, P) holds.

Proof. If Problem 2.1.1 has a solution, Condition 2.2.2 holds by Theorem 2.2.4 (a), and (b) in Definition 2.3.1 holds by Theorem 2.1.2.

If CCQI($\underline{F}^1$, $\underline{F}^2$, $\underline{F}^1 \cap \underline{F}^2$, P) holds, then we can use Theorem 2.2.4 (b) to obtain a probability Q equivalent to P on $\underline{F}^1 \vee \underline{F}^2$ such that $\underline{F}^1 \perp_Q \underline{F}^2 \mid \underline{F}^1 \cap \underline{F}^2$. □

2.3.4 Remark. One could be interested in a generalization of Problem 2.1.1 where the conditioning is on some σ-algebra $\underline{G} \subset \underline{F}^1 \vee \underline{F}^2$ not necessarily equal to $\underline{F}^1 \cap \underline{F}^2$. However, this follows easily from Theorem 2.3.3, since

$$\underline{F}^1 \perp_Q \underline{F}^2 | \underline{G} \iff \underline{F}^1 \vee \underline{G} \perp_Q \underline{F}^2 \vee \underline{G} | \underline{G} \quad ,$$

and as can be observed by examining the proofs, the conditions in Theorem 2.2.3 are valid since $\underline{G} \subset (\underline{F}^1 \vee \underline{G}) \cap (\underline{F}^2 \vee \underline{G})$. Of course, if these conditions hold, $\underline{G} \supset \underline{F}^1 \cap \underline{F}^2$. □

## 2.4 The case of finite algebras.

In this section, we examine the case when $\underline{F}^1$ and $\underline{F}^2$ are finite algebras, which would often be the case in practical applications. In this case, countable additivity plays no role, and a much stronger result is of course to be expected.

2.4.1 Theorem. If $\underline{F}^1$ and $\underline{F}^2$ are finite and conditionally qualitatively independent given $\underline{F}^1 \cap \underline{F}^2$, then the probability Q whose existence is affirmed in Proposition 2.2.1 is a solution to Problem 2.1.1.

Proof. Since $\underline{F}^1 \vee \underline{F}^2$ is also finite, Q is a probability measure on $\underline{F}^1 \vee \underline{F}^2$. Conditional independence of $\underline{F}^1$ and $\underline{F}^2$ given $\underline{F}^1 \cap \underline{F}^2$ can be shown in the same manner as in the second part of the proof of Theorem 2.2.4 (b). It remains to be checked that P and Q are equivalent on $\underline{F}^1 \vee \underline{F}^2$.

Since $\underline{F}^1$ and $\underline{F}^2$ are finite, we need only show that if $A^i \in \underline{F}^i$, $i = 1, 2$, then

$$P(A^1 \cap A^2) > 0 \iff Q(A^1 \cap A^2) > 0 \ .$$

Set $\underline{G} = \underline{F}^1 \cap \underline{F}^2$ and observe that

$$Q(A^1 \cap A^2) = E_P(P(A^1 | \underline{G}) \ P(A^2 | \underline{G})).$$

Furthermore,

$$P\{P(A^1|\underline{G})\ P(A^2|\underline{G}) > 0\} = P(\{P(A^1|\underline{G}) > 0\} \cap \{P(A^2|\underline{G}) > 0\})$$

$$= P\{P(A^1 \cap A^2|\underline{G}) > 0\}$$

by Hypothesis CQI. Thus

$$Q(A^1 \cap A^2) > 0 \iff P\{P(A^1 \cap A^2|\underline{G}) > 0\} > 0 \iff P(A^1 \cap A^2) > 0 . \quad \square$$

2.5 Example of a filtration with CQI but without F4 for any equivalent measure.

We are interested in applying the results of the preceeding sections to establish necessary and sufficient conditions for the existence of an equivalent measure such that F4 is satisfied. In order to illustrate what kind of result is of interest, let us look at the following example.

2.5.1 Example. For i = 1, 2, set $\Omega^i = (0,1]$, let $\underline{F}^i$ be the Borel field on $\Omega^i$, and let $\underline{F}^i_n$ be the (finite) algebra generated by the $n^{th}$ dyadic partition of $\Omega^i$. Set

$$\Omega = \Omega^1 \times \Omega^2 , \quad \underline{F} = \underline{F}^1 \times \underline{F}^2, \quad \underline{F}_{(t_1,t_2)} = \underline{F}^1_{t_1} \times \underline{F}^2_{t_2} ,$$

for $t_1, t_2 \in \mathbb{N}$, and

$$P = \frac{1}{2} P_1 + \frac{1}{2\sqrt{2}} P_2 ,$$

where $P_1$ denotes Lebesgue measure on $(0,1]^2$ and $P_2$ is linear Lebesgue measure on the diagonal $\{(s_1,s_2) : s_1 = s_2\}$.

Observe that $\underline{F}_{t_1,t_1}$ has $2^{2t_1}$ atoms, which have probability $2^{-(2t_1+1)}$

if they do not meet the diagonal and $2^{-(2t_1+1)} + 2^{-(t_1+1)}$ if they do. Hence $P|_{\underline{F}_t}$ is equivalent to the restriction of Lebesgue measure to $\underline{F}_t$, $\forall t \in \mathbb{N}^2$, and F4 is clearly satisfied for Lebesgue measure. Of course, Lebesgue measure is not equivalent to P on $\underline{F}$, and furthermore we have the following result.

2.5.2 Proposition. In the setting of Example 2.5.1, there is no probability measure Q equivalent to P on $\underline{F}$ and such that F4 holds with respect to Q.

Proof. If there were such a probability Q, then $\underline{F}_{\infty,0}$ and $\underline{F}_{0,\infty}$ would be conditionally independent given $\underline{F}_{0,0}$ under Q. But $\underline{F}_{\infty,0}$ is $\underline{F}^1$ of Example 2.1.5, $\underline{F}_{0,\infty}$ is $\underline{F}^2$ and $\underline{F}_{0,0}$ is $\underline{F}^1 \cap \underline{F}^2$. So the statement follows from the fact that Problem 2.1.3 had no solution in Example 2.1.5. □

In Example 2.5.1, the filtration $(\underline{F}_t)_{t \in \mathbb{N}^2}$ is a product filtration, which is a typical structure for Hypothesis F4. The probability measure P is quite reasonable, and yet there is no measure equivalent to P on $\underline{F}$ such that F4 holds. However, Lebesgue measure is equivalent to P on each $\underline{F}_t$, $t \in \mathbb{N}^2$, and F4 holds for this measure.

In view of this example, it seems most interesting to determine under what conditions there is a measure which is equivalent to the original one on each $\underline{F}_t$ and for which F4 holds. This is the object of the next section.

## 2.6 Necessary and sufficient conditions for the existence of an equivalent measure such that F4 holds (discrete case).

We begin by proving the following lemma.

2.6.1 Lemma. Suppose $(\underline{F}_t)_{t \in \mathbb{N}^2}$ is a two-parameter filtration, and $s_1 \leq t_1$, $s_2 \geq t_2 \geq u_2$ (see Figure 1).

(a) Suppose

(a1) $\underline{\underline{F}}_{s_1,s_2} \perp_P \underline{\underline{F}}_{t_1,t_2} \mid \underline{\underline{F}}_{s_1,t_2}$

(a2) $\underline{\underline{F}}_{s_1,t_2} \perp_P \underline{\underline{F}}_{t_1,u_2} \mid \underline{\underline{F}}_{s_1,u_2}$

Then $\underline{\underline{F}}_{s_1,s_2} \perp_P \underline{\underline{F}}_{t_1,u_2} \mid \underline{\underline{F}}_{s_1,u_2}$ .

(b) Suppose

(b1) $\mathrm{CCQI}(\underline{\underline{F}}_{s_1,s_2}, \underline{\underline{F}}_{t_1,t_2}, \underline{\underline{F}}_{s_1,t_2}, P)$

(b2) $\mathrm{CCQI}(\underline{\underline{F}}_{s_1,t_2}, \underline{\underline{F}}_{t_1,u_2}, \underline{\underline{F}}_{s_1,u_2}, P)$.

Then $\mathrm{CCQI}(\underline{\underline{F}}_{s_1}, s_2, \underline{\underline{F}}_{t_1,u_2}, \underline{\underline{F}}_{s_1,u_2}, P)$.

(c) Suppose $\underline{\underline{F}}_{t_1+1,t_2} \perp_P \underline{\underline{F}}_{t_1,t_2+1} \mid \underline{\underline{F}}_{t_1,t_2}, \quad \forall (t_1,t_2) \in \mathbb{N}^2$ .

Then $(\underline{\underline{F}}_t)_{t\in\mathbb{N}^2}$ satisfies Hypothesis F4.

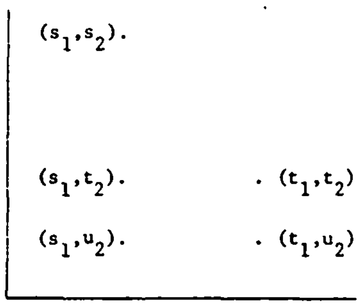



Figure 1.

Proof.

(a) Fix $X \in \mathcal{F}_{t_1,u_2}$. By (a1), $E(X|\mathcal{F}_{s_1,s_2}) = E(X|\mathcal{F}_{s_1,t_2})$. By (a2), this equals $E(X|\mathcal{F}_{s_1,u_2})$, and so (a) is proven.

(b) By Theorem 2.3.3 and (b2), there is $Q^1$ equivalent to $P$ on $\mathcal{F}^1 \vee \mathcal{F}^2$ such that $\mathcal{F}^1 \perp_{Q^1} \mathcal{F}^2 | \mathcal{F}_{s_1,u_2}$, where $\mathcal{F}^1 = \mathcal{F}_{s_1,t_2}$, $\mathcal{F}^2 = \mathcal{F}_{t_1,u_2}$. Let $f$ be the Radon-Nikodym derivative of $Q^1|_{\mathcal{F}^1 \vee \mathcal{F}^2}$ with respect to $P|_{\mathcal{F}^1 \vee \mathcal{F}^2}$, and set $P^1 = f \cdot P$.

Then $P^1$ is equivalent to $P$, so $CCQI(\mathcal{F}_{s_1,s_2}, \mathcal{F}_{t_1,t_2}, \mathcal{F}_{s_1,t_2}, P^1)$ holds by Proposition 2.2.5 (a). By (b1), Theorem 2.3.3 and Theorem 2.1.2, there is $Q^2$ equivalent to $P$ on $\mathcal{F}_{s_1,s_2} \vee \mathcal{F}_{t_1,t_2}$ such that $\mathcal{F}_{s_1,s_2} \perp \mathcal{F}_{t_1,t_2} | \mathcal{F}_{s_1,t_2}$, and $Q^2$ coincides with $P^1$ on $\mathcal{F}_{s_1,s_2} \cup \mathcal{F}_{t_1,t_2}$. In particular, $\mathcal{F}_{s_1,t_2} \perp_{Q^2} \mathcal{F}_{t_1,u_2} | \mathcal{F}_{s_1,u_2}$, and so the conclusion follows from (a).

(c) Using (a), we see by induction that $\mathcal{F}_{t_1,t_2+s_2} \perp \mathcal{F}_{t_1+1,t_2} | \mathcal{F}_{t_1,t_2}$, $\forall\, s_2 \geq 1$. By a statement symetric to (a), we then see by induction that $\mathcal{F}_{t_1,t_2+s_2} \perp_P \mathcal{F}_{t_1+s_1,t_2} | \mathcal{F}_{t_1,t_2}$, $\forall (s_1,s_2) \in \mathbb{N}^2$. This implies that $\mathcal{F}_{t_1,\infty} \perp \mathcal{F}_{\infty,t_2} | \mathcal{F}_{t_1,t_2}$, and thus Hypothesis F4 is satisfied. □

Let us show how Lemma 2.6.1 can be used to obtain a measure such that $(\mathcal{F}_t)_{t \leq (n,n)}$ satisfies F4, for some fixed $n \in \mathbb{N}$. The extension to $(\mathcal{F}_t)_{t \in \mathbb{N}^2}$ requires considerations related to projective limits, and will be dealt with later on.

2.6.2 Theorem. Fix $n \in \mathbb{N}$. Let $(\mathcal{F}_t)_{t \leq (n,n)}$ be a two-parameter filtration. Then the following two conditions are equivalent :

(a) $\mathrm{CCQI}(\underline{F}_{t_1+1,t_2}, \underline{F}_{t_1,t_2+1}, \underline{F}_{t_1,t_2}, P)$ holds, $\forall t \ll (n,n)$;

(b) There is a probability Q equivalent to P on $\underline{F}$ such that F4 $((\underline{F}_t)_{t\leq(n,n)}, Q)$ holds.

Proof. That (b) implies (a) follows from Theorem 2.3.3. So we prove that (a) implies (b).

Suppose $\underline{F}^1$, $\underline{F}^2$ and P are such that $\mathrm{CCQI}(\underline{F}^1, \underline{F}^2, \underline{F}^1 \cap \underline{F}^2, P)$ holds. Then Theorem 2.3.3 affirms the existence of a probability Q on $\underline{F}^1 \vee \underline{F}^2$ such that $\underline{F}^1 \perp_Q \underline{F}^2 \mid \underline{F}^1 \cap \underline{F}^2$, and Q is equivalent to P on $F^1 \vee F^2$. Set $f = \frac{dQ}{dP}$ (calculated on $\underline{F}^1 \vee \underline{F}^2$). We denote the probability f.P by $\varphi(\underline{F}^1, \underline{F}^2, P)$.

Suppose n = 2. Then $Q_2 = \varphi(\underline{F}_{1,0}, \underline{F}_{0,1}, P)$ satisfies (b) for n = 2. We now proceed by induction.

Suppose $Q_n$ is equivalent to P on $\underline{F}_{n,n}$ and $F4((\underline{F}_t)_{t\leq(n,n)}, Q_n)$ holds. Set $Q_n^{(0)} = Q_n$, and define inductively

$$Q_n^{(i+1)} = \varphi(\underline{F}_{n,n}, \underline{F}_{n+1,i}, Q_n^{(i)}) .$$

Then

$$Q_n^{(i)} \text{ is equivalent to P on } \underline{F} ,$$

$$Q_n^{(i+1)}\big|_{\underline{F}_{n,n}} = Q_n^{(i)}\big|_{\underline{F}_{n,n}} , \quad Q_n^{(i+1)}\big|_{\underline{F}_{n+1,i}} = Q_n^{(i)}\big|_{\underline{F}_{n+1,i}} ,$$

$$\underline{F}_{n,n} \perp_{Q_n^{(i)}} \underline{F}_{n+1,j} \,\big|\, \underline{F}_{n,j} , \quad \forall j < i .$$

Using Lemma 2.6.1 (c) we see that $F4((\underline{F}_t)_{t\leq(n+1,n)}, Q_n^{(n)})$ holds. By an analogous procedure, we obtain a probability $Q_{n+1}$ equivalent to P such that $F4((\underline{F}_t)_{t\leq(n+1,n+1)}, Q_{n+1})$ holds. □

2.6.3 Remark. Observe that the $Q_n$ in the proof of Theorem 4 satisfy $Q_{n+1}|_{\underline{F}_{n,n}} = Q_n|_{F_{n,n}}$ . □

In order to extend Theorem 2.6.2 to filtrations indexed by $\mathbb{N}^2$ , we shall put certain restrictions on the filtered probability space. The conditions we give, though not the most general, are sufficient for most applications to stochastic processes. We introduce a few definitions.

2.6.4 Definition. (see [DM], III.16)

(a) A mesure space $(\Omega, \underline{F})$ is termed Lusin if it is isomorphic to some mesure space $(H, \underline{B}(H))$, where H is a Borel subset of some compact metrisable space, and $\underline{B}(H)$ is the Borel $\sigma$-algebra on H.

(b) If $(\Omega, \underline{F})$ is Lusin, isomorphic to $(H, \underline{B}(H))$ and P is a probability on $(\Omega, \underline{F})$, then P is regular if the probability $\tilde{P}$ on $(H, \underline{B}(H))$, image of P under the isomorphism, is regular (meaning $\tilde{P}(A) = \sup \{P(K) : K \subset A, K \text{ compact}\}$ , $\forall A \in \underline{B}(H)$).

(c) $(\Omega, \underline{F}, P, (\underline{F}_t)_{t\in\mathbb{N}^2})$ is a regular filtered two-parameter probability space provided $(\Omega, \underline{F}_t)$ is Lusin, $P|_{\underline{F}_t}$ is regular, $\forall t \in \mathbb{N}^2$ , and $\underline{F} = \bigvee_{t\in\mathbb{N}^2} \underline{F}_t$ .

2.6.5 Remark.

(a) Given $(\Omega, \underline{F}, P, (\underline{F}_t)_{t\in\mathbb{N}^2})$ as in Definition 2.6.4, it is not difficult to see that $(\Omega, \underline{F})$ is isomorphic to the projective limit of $(\Omega,\underline{F}_t)_{t\in\mathbb{N}^2}$ .

(b) The canonical probability space for two-parameter processes satisfies the condition of Definition 2.6.4 (c).

2.6.6 Lemma. Let $(\Omega, \underline{F}, P, (\underline{F}_t)_{t\in\mathbb{N}^2})$ be a regular filtered two-parameter probability space. Suppose that for each $t \in \mathbb{N}^2$, $Q_t$ is a probability on $\underline{F}_t$ such that

(a) $Q_t|_{F_t}$ is equivalent to $P|_{\underline{F}_t}$ ;

(b) $s \leq t \Rightarrow Q_t|_{\underline{F}_s} = Q_s$ .

Then there is a probability $Q$ on $(\Omega, \underline{F})$ such that $Q|_{\underline{F}_t} = Q_t$ , $\forall t \in \mathbb{N}^2$.

Proof. Set $f = \frac{dQ_f}{dP}$ (calculated on $\underline{F}_t$). Then f is integrable with respect to P, and so $Q_t$ is also regular. The conclusion then follows from the definition of a Lusin space and Theorem III.53 of [DM]. □

2.6.7 Theorem. Let $(\Omega, \underline{F}, P, (\underline{F}_t)_{t\in\mathbb{N}^2})$ be a regular filtered two-parameter probability space. Then the following two conditions are equivalent :

(a) $CCQI(\underline{F}_{t_1+1,t_2}, \underline{F}_{t_1,t_2+1}, \underline{F}_{t_1,t_2}, P)$ holds, $\forall t \in \mathbb{N}^2$.

(b) There is a probability Q on $\underline{F}$ such that $P|_{\underline{F}_t}$ is equivalent to $Q|_{\underline{F}_t}$, $\forall t \in \mathbb{N}^2$, and $F4((\underline{F}_t)_{t\in\mathbb{N}^2}, Q)$ holds.

Proof. That (b) implies (a) follows as before from Theorem 2.3.3. To see that (a) implies (b), use Theorem 2.6.2 to obtain a probability $Q_t$ equivalent to P on $\underline{F}_t$ such that $F4((\underline{F}_s)_{s\leq t}, Q_t)$ holds. By Remark 2.6.3, the $Q_t$ satisfy condition (b) in Lemma 2.6.6. So the probability Q of Lemma 2.6.6 satisfies (b). □

2.6.8 Remark. If the filtration $(\underline{F}_t)_{t\in\mathbb{N}^2}$ is complete, whenever the probability Q in Theorem 2.6.7 exists, it is absolutely continuous with respect to P.

CHAPTER 3

# EXISTENCE OF OPTIONAL INCREASING PATHS PASSING TRHOUGH A STOPPING POINT

One of the most important control problems in the theory of two-parameter processes is the following. Given a stopping point, find an optional increasing path that starts at the origin and passes through the stopping point with probability one.

Rather unintuitively, this is not always possible. In fact, when the index set is $\mathbb{N}^2$ , Mandelbaum and Vanderbei [MV] showed that if the filtration satisfies Hypothesis F4, then every stopping point lies on an optional increasing path. Krengel and Sucheston [KS] showed that every stopping point lies on an optional increasing path if and only if the filtration satisfies Hypothesis CQI. In the continuous case, Walsh claimed that every stopping point lies on an optional increasing path when the two-parameter filtration satisfies Hypothesis F4. However, as observed by Mandelbaum [Ma], his proof was incorrect. Nevertheless, Fouque [F], using techniques from the theory of "amarts", proved under Hypothesis F4 the existence of a "minimal" and "maximal" optional increasing path passing through a given stopping point.

The objective of this chapter is twofold. First of all, we extend the result of Krengel and Sucheston, by showing that Hypothesis CQI is again necessary and sufficient in continuous time. Our proof of sufficiency is by pointwise discrete approximation (instead of uniform discrete approximation, as outlined by Walsh [W; Proposition 2.1]), and uses the notion of measurable cluster points.

Our second objective is to show through some examples that certain stopping points lie on a unique optional increasing path. This is the case both in discrete and continuous time, for such standard two-parameter filtrations as those generated by processes with independent increments.

## 3.1 Optional increasing paths.

In the following, we let I denote $\mathbb{N}$, $\mathbb{D}_n$ (the dyadics of order n) or $\mathbb{R}_+$.

**3.1.1 Definition.** (see [CG], [W])

(a) An increasing path is a map $u \mapsto z_u = (z_u^1, z_u^2)$ from $\bar{I}$ into $\overline{I^2}$ which is increasing for $\leq$ and such that

$$|z_u| = z_u^1 + z_u^2 = u , \quad \forall u \in I.$$

The set of increasing paths is denoted $\underline{\underline{z}}$ .

(b) An optional increasing path (o.i.p.) is a map $(\omega,u) \longmapsto Z_u(\omega)$ from $\Omega \times \bar{I}$ into $\overline{I^2}$ such that $Z_u(\cdot)$ is a stopping point, $\forall u \in I$, and $Z_\cdot(\omega) \in \underline{\underline{z}}$, for $\omega \in \Omega$ . The set of o.i.p's is denoted $\underline{\underline{Z}}$. In the sequel, we use the notation $Z = (Z^1, Z^2)$. When $I = \mathbb{D}_n$ , an o.i.p. Z is predictable provided $Z_{(k+1)2^{-n}}$ is $\underline{\underline{F}}_{Z_{k2^{-n}}}$-measurable, $\forall k \in \mathbb{N}$ .

Observe that an increasing path defines a continuous map from $\bar{I}$ to $\overline{I^2}$ when these sets are equipped with the usual metric topology making them compact. Furthermore, the set $\underline{\underline{z}}$ is compact for the topology of uniform convergence. Let us state some elementary properties of these paths.

**3.1.2 Proposition.**

(a) If $z \in \underline{\underline{z}}$ , then $|z_u - z_v| = |u - v|$ , $\quad \forall u, v \in I$ .

(b) If $z, \tilde{z} \in \underline{\underline{z}}$ , $u \in I$ and $|z_u - \tilde{z}_u| < \varepsilon$ , then

$$|z_v - \tilde{z}_v| < \varepsilon + 2\ |v-u|, \quad \forall v \in I .$$

Proof.

(a) If $u \leq v$, then

$$|z_u - z_v| = (z_v^2 - z_u^2) + (z_v^1 - z_u^1) = v - u.$$

(b) The statement follows from (a) since

$$|z_v - \tilde{z}_v| \leq |z_v - z_u| + |z_u - \tilde{z}_u| + |\tilde{z}_u - \tilde{z}_v| . \quad \square$$

The following property in the case $I = \mathbb{N}$ was observed by Mandelbaum [Ma] under Hypothesis F4. It was proved under the weaker hypothesis (1) below by Mazziotto and Millet [MM]. A similar statement clearly holds when $I = \mathbb{D}_n$.

3.1.3 Proposition. Suppose $(\underline{F}_t)_{t \in \mathbb{N}^2}$ is such that

(1) $\quad \underline{F}_{t_1,t_2} = \underline{F}_{t_1+1,t_2} \cap \underline{F}_{t_1,t_2+1}, \quad \forall t \in \mathbb{N}^2.$

Then each o.i.p. $Z$ is predictable, i.e. $Z_{n+1}$ is $\underline{F}_{Z_n}$-measurable, $\forall n \in \mathbb{N}$.

Proof. (see [MM]). On the set $\{Z_n = t\}$, we have $Z_{n+1} = (t_1 + 1, t_2)$ or $Z_{n+1} = (t_1, t_2 + 1)$. Now

$$\{Z_n = t\} \cap \{Z_{n+1} = (t_1 + 1, t_2)\} \in \underline{F}_{t_1+1,t_2},$$

and

$$\{Z_n = t\} \cap \{Z_{n+1} = (t_1 + 1, t_2)\} = \{Z_n = t\} \cap \{Z_{n+1} \leq (t_1, t_2 + 1)\} \in \underline{F}_{t_1,t_2+1}.$$

Thus

$$\{Z_n = t\} \cap \{Z_{n+1} = (t_1 + 1, t_2)\} \in \underline{F}_{t_1+1,t_2} \cap \underline{F}_{t_1,t_2+1} = \underline{F}_t,$$

and similarly,

$$\{Z_n = t\} \cap \{Z_{n+1} = (t_1, t_2 + 1)\} \in \underline{\underline{F}}_t \,.$$

By the definition of $\underline{\underline{F}}_{Z_n}$, this proves the proposition. □

3.1.4 Definition. When $I = \mathbb{R}_+$, $\underline{\underline{Z}}_n$ will denote the set of o.i.p's, called step o.i.p.'s, such that

$$Z_t \in \mathbb{D}_n^2 \,, \quad \forall t \in \mathbb{D}_n \,.$$

3.1.5 Remark. In view of Proposition 3.1.3, whenever the two-parameter filtration satisfies the Intersection Property 1.4.4, each o.i.p. $Z : \Omega \times \overline{\mathbb{D}}_n \longrightarrow \overline{\mathbb{D}_n^2}$ can be identified with a step o.i.p. $\tilde{Z} : \Omega \times \overline{\mathbb{R}_+} \longrightarrow \overline{\mathbb{R}_+^2}$ by setting

$$\tilde{Z}_u = Z_{k2^{-n}} + (2^n u - k)(Z_{(k+1)2^{-n}} - Z_{k2^{-n}}), \; k2^{-n} \leq u \leq (k+1)2^{-n} \,.$$

3.1.6 Definition. We say that an o.i.p. $Z$ passes through a random point $X : \Omega \longrightarrow \overline{I^2}$ if $Z_{|X|} = X$ a.s. We also say that $X$ lies on $Z$.

The following fundamental result is due to Krengel and Sucheston [KS].

3.1.7 Theorem. Given a two-parameter filtration $(\underline{\underline{F}}_t)_{t \in \mathbb{N}^2}$, every stopping point lies on some predictable o.i.p. *if and only if* $(\underline{\underline{F}}_t)_{t \in \mathbb{N}^2}$ satisfies Hypothesis CQI.

For a proof of this result, the reader is referred to [KS], Theorems 2.1 and 2.2. A similar statement clearly holds for $I = \mathbb{D}_n$. However, it is not obvious that Hypothesis CQI is necessary when $I = \mathbb{R}_+$, since stopping points with values in $\mathbb{D}_n$ might lie on an o.i.p. which does not belong to $\underline{\underline{Z}}_n$.

Nevertheless, it will be proved below that Theorem 3.1.7 remains valid for $I = \mathbb{R}_+$.

## 3.2 Approximation of optional increasing paths by step optional increasing paths.

It is useful in certain situations to approximate an o.i.p. by a step o.i.p. Mandelbaum ([Ma], Theorem 7) has given a simple construction of a step o.i.p. in $\underline{\underline{Z}}_n$ which is near a given o.i.p. Z, uniformly in $u \in \mathbb{R}_+$ and $w \in \Omega$. This construction is valid when $\mathbb{R}_+^2$ is replaced by $\mathbb{R}_+^n$, but we will have no need for this generalization.

3.2.1 Algorithm. Suppose $Z \in \underline{\underline{Z}}$ is given. We define $\tilde{Z} = (\tilde{Z}_u)_{u \in \mathbb{D}_n}$ in the following manner. Set $\tilde{Z}_o = 0$, and suppose by induction that $\tilde{Z}_{k2^{-n}}$ has been defined. We then define $\tilde{Z}_{(k+1)2^{-n}}$ by

$$(2) \qquad \tilde{Z}_{(k+1)2^{-n}} = \begin{cases} \tilde{Z}_{k2^{-n}} + 2^{-n}(1,0) & \text{if} \quad \tau_1^k \leq \tau_2^k \\ \tilde{Z}_{k2^{-n}} + 2^{-n}(0,1) & \text{if} \quad \tau_2^k < \tau_1^k , \end{cases}$$

where $\tau_i^k = \inf \{u \in \mathbb{R}_+ : Z_u^i > \tilde{Z}_{k2^{-n}}^i\}$ , $i = 1,2$. □

This algorithm is illustrated by the picture in Figure 2.

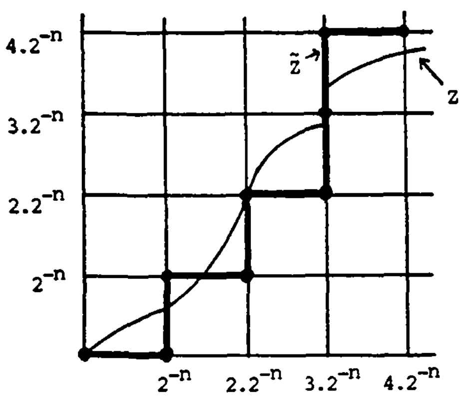



Figure 2.

It is not difficult to see that the event $\{\tau_1^k \leq \tau_2^k\}$ is the event "Z leaves the rectangle $[0, \tilde{Z}_{k2^{-n}}]$ through the segment $\{s : s_1 = \tilde{Z}^1_{k2^{-n}}, s_2 \leq \tilde{Z}^2_{k2^{-n}}\}$". Since $(\underline{F}_t)_{t\in\mathbb{R}^2_+}$ is right-continuous, this implies that $\{\tau_1^k \leq \tau_2^k\} \in \underline{F}_{\tilde{Z}_{k2^{-n}}}$, and so Algorithm 3.2.1 does define a $\mathbb{D}_n$-valued o.i.p., such that $\tilde{Z}_{(k+1)2^{-n}}$ is $\underline{F}_{\tilde{Z}_{k2^{-n}}}$-measurable, $\forall k \in \mathbb{D}_n$.

Thus, without requiring Hypothesis F4, CQI or the Intersection Property (1), we can extend $\tilde{Z}$ to an element of $\underline{Z}_n$, again denoted $\tilde{Z}$. It is clear (see Figure 2) that

$$(3) \qquad |Z_u - \tilde{Z}_u| \leq 2^{-n+1}, \quad \forall u \in \mathbb{R}_+,$$

and so $\tilde{Z}$ is a uniform approximation of Z. For further details and proofs of these statements, the reader is referred to [Ma], Theorem 7.

The path $\tilde{Z}$ has the following important property.

3.2.2 <u>Proposition.</u> Let $Z \in \underline{Z}$, and let $X : \Omega \longrightarrow \overline{I^2}$ be a random point that lies on Z. Fix $n \in \mathbb{N}$, let $\tilde{Z} \in \underline{Z}_n$ be the approximation of Z given by Algorithm 3.2.1, and $X^{[n]}$ be the smallest random point with values in $\overline{\mathbb{D}_n^2}$ which is greater that X. Then $X^{[n]}$ lies on Z (neither Hypothesis F4 nor CQI is required).

<u>Proof.</u> Suppose $t \in \mathbb{D}_n^2$ and $X^{[n]}(\omega) = t$, for some $\omega \in \Omega$. Then $t - 2^{-n}(1,1) \ll X(\omega) \leq t$. Since $X(\omega)$ lies on $Z(\omega)$,

$$Z_{|t|-2^{-n}}(\omega) \in ]\, t - 2^{-n}(1,1), t] \cup \{(t_1 - 2^{-n}, t_2), (t_1, t_2 - 2^{-n})\} .$$

Suppose to begin with that

$$Z_{|t|-2^{-n}}(\omega) \notin \{(t_1 - 2^{-n}, t_2), (t_1, t_2 - 2^{-n})\} .$$

Then (3) implies that

$$\tilde{Z}_{|t|-2^{-n}}(\omega) \in \{(t_1-2^{-n}, t_2), (t_1, t_2-2^{-n})\} .$$

Whichever the value of $\tilde{Z}_{|t|-2^{-n}}(\omega)$, $\tilde{Z}_{|t|}(\omega) = t$ by the definition of $\tilde{Z}$.

Now suppose for example that $Z_{|t|-2^{-n}}(\omega) = (t_1-2^{-n}, t_2)$. Then the definition of $\tilde{Z}$ implies that $\tilde{Z}_{|t|-2^{-n}}(\omega) = (t_1-2^{-n}, t_2)$. Since we must have $t_1-2^{-n} < X_1(\omega) \leq t_1$, $X_2(\omega) = t_2$, and since X lies on Z, $\tau_1^k(\omega) \leq \tau_2^k(\omega)$, and so the definition of $\tilde{Z}$ implies that $\tilde{Z}_{|t|}(\omega) = t$. □

## 3.3 Cluster points of a sequence of optional increasing paths.

Given a sequence $(Z^n)_{n\in\mathbb{N}}$ of optional increasing paths, we would like to determine an o.i.p. Z such that $Z_.(\omega)$ is a cluster point of $Z^n_.(\omega)$, for almost all $\omega \in \Omega$. If "cluster point" is interpreted with respect to the topology of uniform convergence (respectively pointwise convergence) on $\underline{\underline{z}}$, we call Z a uniform (respectively pointwise) cluster point of $(Z^n)_{n\in\mathbb{N}}$.

Walsh's attempted proof of the fact that all stopping points lie on some o.i.p. under Hypothesis F4 requires the existence of uniform cluster points of sequences of o.i.p.'s. The existence of uniform cluster points would also be useful in solving the multi-armed bandit problem (see [Ma]). Unfortunately, it is not at all clear whether uniform cluster points always exist. The object of this section is to show that pointwise cluster points do exist. This is sufficient to complete Walsh's proof, but does not seem to apply to the multi-armed bandit problem.

3.3.1 Theorem. Let $(Z^n)_{n\in\mathbb{N}}$ be a sequence of o.i.p.'s. Then there is an o.i.p. Z such that for all $u \in \mathbb{R}_+$ and $\omega \in \Omega$, $Z_u(\omega)$ is a cluster point of the sequence $(Z^n_u(\omega))_{n\in\mathbb{N}}$ (neither Hypothesis CQI nor F4 is assumed).

Proof. For fixed $u \in \mathbb{R}_+$, the sequence $(Z^n_u)$ takes values in the compact set $\{(s_1,s_2) : s_1+s_2 = u\}$. For $0 \leq r \leq u$, $n \geq 2$, set

$$A_{n,u,r} = \{Z^n_u \underline{\wedge} (r,u-r)\} \text{ and } B_{u,r} = \limsup_{n\in\mathbb{N}} A_{n,u,r} .$$

Observe that $A_{n,u,r} = \bigcap_{m\in\mathbb{N}} A^m_{n,u,r}$, where

$$A^m_{n,u,r} = \bigcup_{s\in\mathbb{Q}_+} \{Z^n_s \in [(0,u-r), (r,u-r+\tfrac{1}{m})]\} \in \underline{\underline{F}}_{r,u-r+\frac{1}{m}}$$

and so $A_{n,u,r} \in \underline{\underline{F}}_{r,u-r}$ since the filtration is right-continuous.

Set $Z_u(\omega) = (Z^1_u(\omega), u - Z^1_u(\omega))$, where

$$Z^1_u(\omega) = \inf \{r : \omega \in B_{u,r}\} .$$

From the definition of $B_{u,r}$, it is not difficult to see that $Z_u(\omega)$ is a cluster point of the sequence $(Z^n_u(\omega))_{n\in\mathbb{N}}$. Furthermore, for all $u \in \mathbb{R}_+$,

$$\{Z_u \leq (t_1,t_2)\} = \bigcap_{n\in\mathbb{N}} (B_{u,t_1+\frac{1}{n}} \cap B^c_{u,u-(t_2+\frac{1}{n})}) \in \underline{\underline{F}}_{t_1,t_2} ,$$

$\forall t \in \mathbb{R}^2_+$, so $Z_u$ is a stopping point. Since $u \leq u'$ implies

$$A_{n,u',r} \subset A_{n,u,r} \subset A_{n,u',u'-u+r}$$

and thus

$$B_{u',r} \subset B_{u,r} \subset B_{u',u'-u+r} ,$$

it follows that

$$\{Z_{u'} \leq (t_1,t_2)\} \subset \{Z_u \leq (t_1,t_2)\} , \quad u \leq u' .$$

This implies that $Z_\cdot(\omega)$ is an increasing path, and so $Z$ is an o.i.p. with the desired properties. □

## 3.4 Necessary and sufficient conditions for every stopping point to lie on an optional increasing path.

The following theorem generalizes the corresponding discrete case result of Krengel and Sucheston (see Theorem 3.1.7).

3.4.1 Theorem. Every stopping point lies on some o.i.p. if and only if the filtration $(\underline{F}_t)_{t\in\mathbb{R}_+^2}$ satisfies Hypothesis CQI.

Proof. We first prove that Hypothesis CQI is sufficient. Let T be a stopping point, and let $T^{[n]}$ be the smallest stopping point with values in $\mathbb{D}_n^2$ which is greater than T. Using Theorem 3.1.7, we see that there is an o.i.p. $Z^n$ with values in $\mathbb{D}^n$ which passes through $T^{[n]}$. Using Remark 3.1.5 and the fact that Hypothesis CQI implies the Intersection Property 1.4.4 (see Theorem 7.4.3), we extend $Z^n$ to a step o.i.p., again denoted $Z^n$. By Theorem 3.3.1., let Z be a pointwise cluster point of $(Z^n)_{n\in\mathbb{N}}$. We shall show that T lies on Z. Since by Proposition 3.1.2 (a),

$$\begin{aligned} |Z^n_{|T|} - T| &\leq |Z^n_{|T|} - Z^n_{|T^n|}| + |Z^n_{|T^n|} - T| \\ &\leq ||T|-|T^n|| + |T^n - T| \\ &\leq 2^{-n+2} , \end{aligned}$$

the sequence $(Z^n_{|T|})_{n\in\mathbb{N}}$ converges to T, and so T(w) is the unique cluster point of $(Z^n{}_{|T|}(\omega))_{n\in\mathbb{N}}$. Since Z is a pointwise cluster point of $(Z^n)_{n\in\mathbb{N}}$, $Z_{|T|} = T$, and so sufficiency is proven.

In order to see that Hypothesis CQI is necessary, suppose that every stopping point lies on some o.i.p. We begin by showing that for each $n\in\mathbb{N}$, the filtration $(\underline{F}_t)_{t\in\mathbb{D}_n^2}$ satisfies Hypothesis CQI. In view of Theorem 3.1.7, we need only show that every stopping point with values in $\overline{\mathbb{D}_n^2}$ lies on an o.i.p. with values in $\overline{\mathbb{D}_n^2}$.

Let T be a $\overline{\mathbb{D}_n^2}$-valued stopping point. By hypothesis, there is an o.i.p. Z passing through T. Let $\tilde{Z}$ be the discrete o.i.p. given by Algorithm 3.2.1. By Proposition 3.2.2, $\tilde{Z}$ passes through T. Thus $(\underline{F}_t)_{t\in \mathbb{D}_n^2}$ satisfies Hypothesis CQI. Using Lemma 3.4.2 below, the proof of the theorem is complete. □

3.4.2 Lemma. Suppose $(\underline{F}_t)_{t\in \mathbb{R}_+^2}$ is such that $(\underline{F}_t)_{t\in \mathbb{D}_n^2}$ satisfies Hypothesis CQI, $\forall n \in \mathbb{N}$. Then $(\underline{F}_t)_{t\in \mathbb{R}_+^2}$ satisfies Hypothesis CQI.

Proof. Fix $s,t \in \mathbb{R}_+^2$. We need to show that $CQI(\underline{F}_{s_1,s_2+t_2}, \underline{F}_{s_1+t_1,s_2}, \underline{F}_{s_1,s_2})$ holds. Let $s^n$ (resp. $t^n$) be a sequence of elements of $\mathbb{D}$ decreasing to s (resp. t). Let $X_1$ be $\underline{F}_{s_1,s_2+t_2}$ -measurable, and $X_2$ be $\underline{F}_{s_1+t_1,s_2}$-measurable. We use the conditional supremum operator of Chapter 7. Since

$$CQI(\underline{F}_{s_1^n,s_2^n+t_2^n}, \underline{F}_{s_1^n+t_1^n,s_2^n}, \underline{F}_{s_1^n,s_2^n})$$

holds,

$$S(X_1+X_2|\underline{F}_{s^n}) = S(X_1|\underline{F}_{s^n}) + S(X^2|\underline{F}_{s^n})$$

by Proposition 7.3.1 (c). Using Lemma 7.2.4 (g) and right-continuity of $(\underline{F}_t)_{t\in \mathbb{R}_+^2}$, we see that

$$S(X_1+X_2|\underline{F}_s) = S(X_1|\underline{F}_s) + S(X_2|\underline{F}_s) .$$

By Proposition 7.3.1 (c), this completes the proof. □

## 3.5 Examples of stopping points lying on a unique optional increasing path : discrete case.

As we have seen in the preceeding section, every stopping point lies on an o.i.p. when Hypothesis CQI holds. Somewhat unexpectedly, the o.i.p.

passing through a stopping point T may be unique on the stochastic interval $[0, |T|]$. We shall give two examples (with $|T| > 0$ a.s.) of this situation. In the first case the set of values of T will be totally ordered for $\leq$, and in the second totally ordered for $\underline{\wedge}$. These examples will be extended to the continuous case in Section 7.

The following lemma is a key ingredient for our examples.

3.5.1 Lemma. Let K be a compact metrisable space, and let $X = (X_t)_{t\in K}$, $Y = (Y_t)_{t\in K}$ be two processes with values in $\overline{\mathbb{R}}$, defined on some probability space $(\Omega, \underline{\underline{F}}, P)$. Suppose $\underline{\underline{G}}$ is a sub-$\sigma$-algebra of $\underline{\underline{F}}$ such that X is $\underline{\underline{G}}$-measurable and Y is independent of $\underline{\underline{G}}$. For a map $g : K \longrightarrow \overline{\mathbb{R}}$, set $F(g) = P(Y_t \geq g(t),\ \forall t \in K)$. Then :

(a) $P(Y_t \geq X_t,\quad \forall t \in K | \underline{\underline{G}}) = F(X_.)$.

(b) If $F(X_.) > 0$ a.s., then $P(G \cap \{Y_t \geq X_t,\ \ \forall t \in K\}) > 0$, for all $G \in \underline{\underline{G}}$ such that $P(G) > 0$.

Proof.

(a) Fix $G \in \underline{\underline{G}}$. Then

$$(4)\qquad \int_G dP(\omega)\ F(X_.(\omega)) = \int_G dP(\omega)\ P\{\omega' \in \Omega : Y_.(\omega') \geq X_o(\omega)\}.$$

Suppose for a moment that X is simple (but not necessarily continuous), that is

$$X_t(\omega) = \sum_{k\in\mathbb{N}} g_k(t)\ I_{A_k}(\omega) \quad , \text{ where } A_k \in \underline{\underline{G}}\ ,\ g_k : K \longrightarrow \overline{\mathbb{R}}.$$

Then the right-hand side of (4) becomes

$$\int_G dP(\omega) \sum_{k\in\mathbb{N}} P\{\omega' : Y_.(\omega') \geq g_k\}\ I_{A_k}(\omega) = \sum_{k\in\mathbb{N}} P(Y_. \geq g_k)\ P(G \cap A_k)$$

$$= \sum_{k\in\mathbb{N}} P(\{Y_. \geq g_k\} \cap G \cap A_k) = P(\{Y_. \geq X_.\} \cap G)\ .$$

In the second equality above, we have used the fact that Y and $\underline{\underline{G}}$ are independent.. This shows that (a) holds when X is simple. Thus (a) also holds for arbitrary continuous X since for any such X there is an increasing sequence $(X^n)_{n\in\mathbb{N}}$ of simple processes such that

$$X_t(\omega) = \lim_{n\to\infty} X_t^n(\omega) \ , \quad \forall t\in K, \quad \forall \omega \in \Omega \ .$$

Finally, statement (b) follows from (a), since

$$P(G\cap\{Y_\cdot \geq X_\cdot\}) = \int_G F(X_\cdot)dP > 0 \ . \quad \square$$

The following theorem presents the two examples we have announced.

3.5.2 Theorem. Let $(X_t)_{t\in\mathbb{N}^2}$ be a family of independent random variables. Set $\underline{\underline{F}}_t = \sigma(X_s, s\leq t)$, and observe that the filtration $(\underline{\underline{F}}_t)_{t\in\mathbb{N}^2}$ satisfies Hypothesis CQI.

(a) Suppose $P(X_t > k) > 0$, $\forall k\in\mathbb{R}$, $\forall t\in\mathbb{N}^2$, and let $z = (z_n)_{n\in\mathbb{N}}$ be an increasing path in $\mathbb{N}^2$. Set

$$T = \inf\{z_n : n\in\mathbb{N}, \sum_{t\leq z_n} X_t \geq 1\} \ .$$

Then T is a stopping point and the graph of any o.i.p. which passes through T coincides with z on $[0, |T|]$.

(b) Suppose $P(X_t = 0) + P(X_t = 1) = 1$ and $0 < P(X_t = 0) < 1$, $\forall t\in\mathbb{N}^2$. Let $Z = (Z_n)$ be the o.i.p. defined by

$$Z_{n+1} = \begin{cases} Z_n + (1,0) & \text{if } X_{Z_n} = 1 \ , \\ Z_n + (0,1) & \text{otherwise,} \end{cases}$$

and fix $k\in\mathbb{N}$. Then the graph of any o.i.p. which passes through $T' = Z_k$ coincides with the graph of Z on $[0, |T'|]$.

Proof.

(a) Clearly, it is sufficient to show that if S is a stopping point such that $S \leq T$, then $z_{|S|} = S$. Suppose for instance that

$$P(z_{|S|} \trianglerighteq (S_1 - 1, S_2 + 1)) > 0.$$

Then there exists $s = (s_1, s_2) \in \mathbb{N}^2$ such that $P(S = s) > 0$ and $z_{|s|} \trianglerighteq (s_1 - 1, s_2 + 1)$. By the definition of T, we have

$$(5) \qquad \{S = (s_1, s_2)\} \subset \{ \sum_{t \leq \tilde{z}} X_t < 1\} = \{Y < 1 - X\} ,$$

where $\tilde{z} = z_{|s|}$ and

$$X = \sum_{t_1 \leq \tilde{z}_1, t_2 \leq s_2} X_{t_1, t_2} \quad \text{and} \quad Y = \sum_{t_1 \leq \tilde{z}_1, s_2 < t_2 \leq \tilde{z}_2} X_{t_1, t_2}$$

Since $1 - X$ is $\underline{\underline{F}}_s$-measurable, Y is independent of $\underline{\underline{F}}_s$ and $P(Y>k) > 0$, $\forall k \in \mathbb{R}$, by hypothesis, Lemma 3.5.1 in the case where K is a singleton implies

$$\begin{aligned} P(S = s,\ Y < 1 - X) &= P(S = s) - P(S = s,\ Y \geq 1 - X) \\ &< P(S = s). \end{aligned}$$

This contradicts the inclusion in (5).

(b) Again, let S be a stopping point such that $S \leq T'$ . We want to show that $P(Z_{|S|} \neq S) = 0$. Observe that

$$\{|S| = k\} \subset \{S = T'\} \subset \{S = Z_{|S|}\},$$

and so $P(Z_{|S|} \neq S) > 0$ implies $P(|S| < k) > 0$. Thus we may suppose that for some $s = (s_1, s_2) \in \mathbb{N}^2$ such that $|S| < k$, $P(S = s,\ Z_{|S|} \wedge s) > 0$, and show that this leads to a contradiction. Now

$$\{S = s,\ Z_{|s|} \wedge s\} \subset \bigcup_{t_1 < s_1} \{S = s,\ Z_{t_1 + s_2} = (t_1, s_2),\ X_{t_1, s_2} = 0\},$$

and so there is some $t_1 < s_1$ such that

$$P(G \cap \{X_{t_1,s_2} = 0\}) \quad 0 \text{ , where } G = \{S = s,\ Z_{t_1+s_2} = (t_1,s_2)\} \text{ .}$$

On the other hand we have

$$(6) \qquad P(G \cap \{T' = (t_1,k-t_1)\}) = P(G \cap \{X_{t_1,s_2} = 0, \ldots, X_{t_1,k-t_1-1} = 0\})$$

$$= P(G \cap \{X_{t_1,s_2} = 0\})\ P(X_{t_1,s_2+1} = 0) \ldots P(X_{t_1,k-t_1-1} = 0)$$

$$> 0 \text{ .}$$

But since $S \leq T'$ , we have

$$G \cap \{T' = (t_1,k-t_1)\} \subset \{T' \wedge S\} = \emptyset \text{ ,}$$

which contradicts (6). □

## 3.6 Optional increasing paths as solutions of random differential equations.

The purpose of this section is to show that the solution to a random differential equation of the form

$$(7) \qquad \begin{cases} \dot{y}_u(\omega) = f(u,\ y_u(\omega),\omega) \quad , \quad u \in \mathbb{R}_+ ,\quad \omega \in \Omega \text{ ,} \\ \\ y_o \equiv 0 \end{cases}$$

is an o.i.p, provided f satisfies appropriate regularity and measurability conditions. This will enable us to extend the example in Theorem 3.5.2 (b) to the continuous case.

3.6.1' Theorem. Let $f : \mathbb{R}_+ \times \mathbb{R} \times \Omega \longrightarrow [0,1]$ be a map such that

(a) $(u,x) \longmapsto f(u,x,\omega)$ continuous, for all $\omega \in \Omega$ ;

(b) for each $n \in \mathbb{N}$ , there is a non-negative random variable $C_n$ such that

$$|f(u,x,\omega) - f(u,x',\omega)| \leq C_n(\omega)\ |x-x'|\ , \quad \forall u \leq n,\ x,x' \in \mathbb{R},\ \omega \in \Omega\ .$$

(c) for all $u \in \mathbb{R}_+$ and $t = (t_1,t_2)$ such that $|t| = u$,

$$\omega \longmapsto f(u,t_2,\omega)$$

is $\underline{F}_t$-measurable.

Then the random differential equation (7) has a unique solution, and this solution is such that $Z = (Z_u)_{u\in\mathbb{R}_+}$ is an o.i.p., where

$$Z_u(\omega) = (u-y_u(\omega),\ y_u(\omega))\ , \quad \forall u \in \mathbb{R}_+\ , \quad \forall \omega \in \Omega\ .$$

3.6.2 Lemma. Let f be as in Theorem 3.6.1. If $T = (T_1,T_2)$ is a stopping point with values in $\{s : |s| = u\}$, then $f(u,T_2(.),.)$ is $\underline{F}_T$-measurable.

Proof. Suppose to begin with that $u \in \mathbb{D}_n$ and T takes values in $\mathbb{D}_n^2 \cap \{s : |s| = u\}$. Let $B \subset \mathbb{R}$ be a Borel set. Then for $t \in \mathbb{R}_+^2$, the event

$$\{f(u,T_2(.),.) \in B\} \cap \{T \leq t\} = \bigcup_{\substack{r\in\mathbb{D}_n^2 \\ |r|=u,\, r\leq t}} (\{f(u,r_2,.) \in B\} \cap \{T = r\})$$

belongs to $\underline{F}_t$ , and so the statement of the lemma holds in this case, by the definition of $\underline{F}_T$ .

Now suppose $u \in \mathbb{R}_+$ is arbitrary and T is a stopping point with values in $\{s : |s| = u\}$. Let $u^{[n]}$ be the smallest element of $\mathbb{D}_n$ which is not greater than u, und let $T^{[n]}$ denote the smallest stopping point with values in $\mathbb{D}_n^2$ that is greater than T. Then $f(u^{[n]},\ T_2^{[n]}(.),.)$ is $\underline{F}_T[n]$-measurable by the first part of the proof. Since $f(.,.,\omega)$ is continuous,

$$f(u^{[n]},T_2^{[n]}(.),.) \xrightarrow[n\to\infty]{} f(u,T_2(.),.)\ ,$$

and so $f(u,T_2(.),.)$ is measurable with respect to $\bigcap_{n\in\mathbb{N}} \underline{F}_T[n]$. Since the filtration $(\underline{F}_t)_{t\in\mathbb{R}_+^2}$ is right-continuous, this σ-algebra is equal to $\underline{F}_T$. The lemma is proven. □

Proof of Theorem 3.6.1. The existence and unicity of solutions to an equation of type (7) has been established in far more general settings (see [DP]). Furthermore, the map $u \mapsto Z_u$ is increasing for $\leq$, since $0 \leq \dot{y}_u \leq 1$. In order to prove that Z is optional, we apply the Euler approximation scheme to (7). Under hypothesis (a) and (b), the approximate solutions $(Z^n_{k2^{-n}})_{k\in\mathbb{N}}$ converge uniformly for each $\omega\in\Omega$ to the solution of (7) on $[0,m]$, $\forall m\in\mathbb{N}$ (see [CM], Th. 4.2). In order to prove the theorem, it is thus sufficient to show that $(k2^{-n} - z^n_{k2^{-n}})$ is a stopping point, where $z^n_{k2^{-n}}$ is defined by

$$z^n_0 = (0,0) \text{ and } z_{k2^{-n}}(\omega) = z^n_{(k-1)2^{-n}} + 2^{-n} f((k-1)2^{-n}, z^n_{(k-1)2^{-n}}(\omega),\omega).$$

This is easily proved by induction on n using hypothesis (c) and Lemma 3.6.2. □

## 3.7 Examples of stopping points lying on a unique optional increasing path : continuous case.

In this section we present two continuous case examples analogous to those of Theorem 3.5.2 (a) and (b).

Let us suppose that $(W_t)_{t\in\mathbb{R}^2_+}$ is a continuous process with independent planar increments, vanishing on the coordinate axes. Suppose further that

$$P(\Delta_{]s,t]} W > k) > 0 , \quad \forall k\in\mathbb{R} , \quad s \ll t , \quad s,t\in\mathbb{R}^2_+ .$$

Set $\underline{F}_t = \sigma(W_s, s\leq t)$, and observe that the filtration $(\underline{F}_t)_{t\in\mathbb{R}^2_+}$ satisfies Hypothesis F4, hence also Hypothesis CQI.

3.7.1 Theorem. Let z be an increasing path, and set

$$T = \inf \{z_u : u\in\mathbb{R}_+ , W_{z_u} \geq 1\} .$$

Then T is a stopping point for $(\underline{F}_t)_{t\in\mathbb{R}^2_+}$ , and the graph of any o.i.p.

which passes through T coincides with z on $[0, |T|]$.

Proof. As in the proof of Theorem 3.5.2, we only need show that if S is a stopping point such that $S \leq T$, then $z_{|S|} = S$. Suppose ab absurdo that $P(z_{|S|} \neq S) > 0$. Then we may suppose that for some $\varepsilon > 0$,

$$P(z_{|S|} \triangleq (S_1 - \varepsilon, S_2 + \varepsilon)) > 0.$$

Fix $n \in \mathbb{N}$ such that $2^{-n+3} < \varepsilon$, and let $S^{[n]}$ (resp. $T^{[n]}$) be the smallest stopping point with values in $\mathbb{D}_n$ that is greater than S (resp. T). $S^{[n]}$ is not greater than $T^{[n]}$, and there is $s = (s_1, s_2) \in \mathbb{D}_n$ such that $P(S^{[n]} = s) > 0$ and

$$z_{|s|} \triangleq (s_1 - \frac{\varepsilon}{2}, s_2 + \frac{\varepsilon}{2})$$

Set

$$X = W_{z_m^1, s_2}, \quad Y = W_{z_m} - W_{z_m^1, s_2},$$

where $m = |s| - 2^{-n}$ and $z_m = (z_m^1, z_m^2)$, and observe by the definition of T that

$$\text{(8)} \qquad W_{z_m} = X + Y < 1 \quad \text{on} \quad \{S^n = s\}.$$

Since $1 - X$ is $\underline{F}_s$-measurable, Y is independent of $\underline{F}_s$ and by Hypothesis $P(Y > k) > 0$, $\forall k \in \mathbb{R}$, Lemma 3.5.1 in the case where K is a singleton implies

$$\begin{aligned} P(S = s, Y < 1 - X) &= P(S = s) - P(S = s, Y \geq 1 - X) \\ &< P(S = s), \end{aligned}$$

contradicting (8). □

3.7.2 Remark. The past in the sense of Fouque [F] of the stopping point T of Theorem 3.7.1 is contained in the deterministic path z. This result had been conjectured by Fouque in the special case where $(W_t)$ is a Brownian sheet and $z_u = (u/2, u/2)$, for all u. □

In order to give a continuous time example analogous to that in Theorem 3.5.2 (b), we suppose now that $W = (W_t)_{t\in\mathbb{R}_+^2}$ is a measurable process with stationary independent planar increments, vanishing on the coordinate axes and such that

$$\sup_{|s|\leq n} |W_s| < +\infty \text{ a.s.}, \quad \forall n \in \mathbb{N}.$$

We suppose further that the following statement holds.

3.7.3 Hypothesis. For all $r \in \mathbb{R}_+^2$, there is $M > 0$ such that for each $p \in C_0^\infty(\mathbb{R}_+^2, \mathbb{R})$ that vanishes on the coordinate axes,

$$P(|W_t - p(t)| \leq M, \quad \forall t \leq r) > 0.$$

($C_0^\infty$ denotes the set of indefinitely differentiable functions with compact support).

Let $(\underline{F}_t = \sigma(X_s, s \leq t))_{t\in\mathbb{R}_+^2}$ be the natural filtration of W, completed by P-null sets and made right-continuous. Observe that this filtration satisfies Hypothesis F4.

Define a process $(X_t)_{t\in\mathbb{R}_+^2}$ by

$$X_{t_1,t_2} = \int_0^{t_1} ds_1 \int_0^{t_2} ds_2\, W_{s_1,s_2}.$$

Set $g(x) = e^x$ and $h(x) = g(x)/(1+g(x))$. Since

$$|h(x) - h(x')| \leq |x-x'|,$$

the map

$$f(u,x,\omega) = \begin{cases} h(X_{u-x,x}(\omega)), & x \leq u, \quad \omega\in\Omega \\ 1/2, & x > u, \quad \omega \in \Omega \end{cases},$$

satisfies the hypothesis of Theorem 3.6.1 with

$$C_n = 2n \sup_{|s| \leq n} |W_s| .$$

So we let Z denote the o.i.p. whose existence is proved in Theorem 3.6.1.

3.7.4 Theorem. Let $(W_t)_{t \in \mathbb{R}_+^2}$ , $(\underline{\underline{F}}_t)_{t \in \mathbb{R}_+^2}$ and Z be as above. Fix $u^* \in \mathbb{R}_+$, and set $T = Z_{u^*}$. Then the graph of any o.i.p. which passes through T coincides with the graph of Z on $[0, |T|]$.

Proof. Let S be a stopping point such that $S \leq T$. We must show that $Z_{|S|} = S$, so we suppose that $P(Z_{|S|} \neq S) > 0$ and show that this leads to a contradiction. As in the proof of Theorem 3.7.1, we may suppose that there is $\varepsilon > 0$ such that

$$P(Z_{|S|} \mathrel{\underline{\wedge}} (S_1 - \varepsilon,\ S_2 + \varepsilon),\ |S| < u^* - \varepsilon\} > 0.$$

Fix $n \in \mathbb{N}$ such that $2^{-n+2} < \varepsilon$, and let $S^{[n]}$ be the smallest stopping point with values in $\mathbb{D}_n$ that is greater than S. Then there is $s^o = (s_1^o, s_2^o) \in \mathbb{D}_n^2$ such that $|s^o| < u^* - 2^{-n+1}$ and

$$P(S^{[n]} = s^o\ ,\ Z_{|S|} \mathrel{\underline{\wedge}} s^*) > 0\ ,\ \text{where } s^* = (s_1^o - 2^{-n+1},\ s_2^o).$$

This implies that

$$G = \{S \in ]s^o - 2^{-n}(1,1),\ s^o]\ ,\ \tau < +\infty\} \in \underline{\underline{F}}_{s^o}$$

has positive probability, where

$$\tau = \begin{cases} \inf\{u : Z_u \in \{s' : s' \mathrel{\underline{\wedge}} s^*\}\} & \text{if } \{\ \} \neq \emptyset\ , \\ \infty & \text{otherwise.} \end{cases}$$

Let $(z_u = (z_u^1\ ,\ z_u^2))_{u \in \mathbb{R}_+}$ be an increasing path that follows the parabola $\Gamma$ with summit $s^*$ and passing through $t^* = (s_1^o - 2^{-n},\ u^* - s_1^o + 2^{-n})$ (see Figure 3). Then $\dot{z}^2_{|s^*|} = 0$. Set

$$H = G \cap \{\omega \in \Omega : X_t(\omega) \geq \mathrm{Ln}(\dot{z}^2_{|t|} / (1 - \dot{z}^2_{|t|}))\ ,\ \forall t \in C\}\ ,$$

where $C = \{t : t \,\underline{\wedge}\, \Gamma,\ t \in [s^*, t^*]\}$. Since the map h is monotone increasing, using the definition of Z we see that $\omega \in H$ and $Z_u(\omega) \in C$ for some u implies $\dot{Z}^2_v(\omega) \geq \dot{z}^2_v$ for all $u \leq v \leq u^*$, and thus $Z_{u^*}(\omega) \,\underline{\wedge}\, t$, for all $\omega \in H$. This implies that $H \subset \{T \wedge S\}$, so in order to obtain the desired contradiction, it suffices to show that $P(H) > 0$. For this we first prove the following lemma.

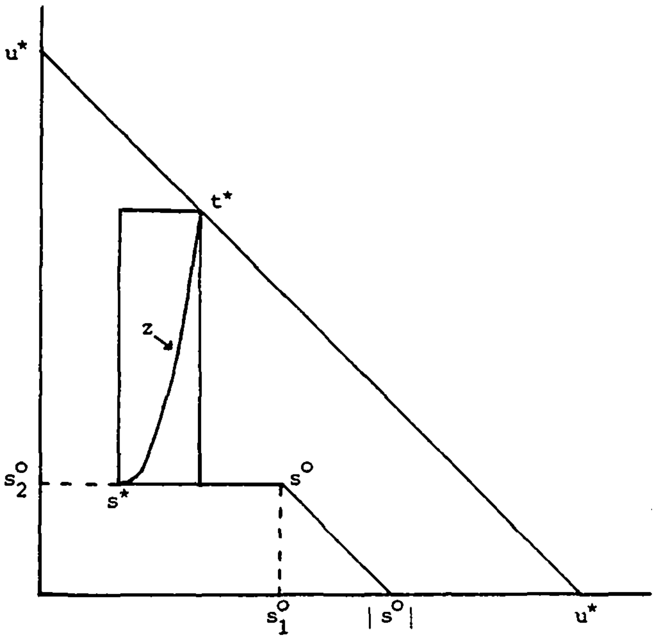



Figure 3.

3.7.5 Lemma. Let $s^*$, $t^*$, $(z_u = (z^1_u, z^2_u))_{u \in \mathbb{R}_+}$ and C be as in the proof above, and let q be a continuous function on $C \setminus \{s^*\}$ such that $\lim_{t \to s^*} q(t) = -\infty$. Under Hypothesis 3.7.3,

$$P\left(\int_0^{t_1} ds_1 \int_{s_2^*}^{t_2} ds_2\, (W_{s_1,s_2} - W_{s_1,s_2^*}) \geq q(t), \quad \forall t \in C\right) > 0.$$

Proof. Choose $\bar{u} > |s^*|$ such that

$$(9) \qquad q(t) < -M\, t_1^*(t_2^* - s_2^*) \quad \text{on} \quad [s^*, z_{\bar{u}}] ,$$

where M is the constant in Hypothesis 3.7.3. Choose $d > 0$ small enough and D large enough so that

$$-M(t_1^* + t_2^* - s_2^*)d + D(s_1^* - d)(z_{\bar{u}}^2 - d) \geq \sup_{t\in C} q(t).$$

Let $p \in C_0^\infty$ be positive and vanish on the horizontal and vertical segments passing through $(0,s_2^*)$, and such that $p - M > D$ on $R = [d,t_1^*] \times [s_2^* + d, t_2^*]$. Then for $t \in \mathbb{R}$,

$$(10) \qquad \int_0^{t_1} ds_1 \int_{s_2^*}^{t_2} ds_2\, (p(s_1,s_2) - M) \geq -Mt_1^* d - M(t_2^* - s_2^*)d + D(t_1 - d)(t_2 - s_2^* - d)$$

By (9) and (10),

$$(11) \qquad \int_0^{t_1} ds_1 \int_{s_2^*}^{t_2} ds_2\, (p(s_1,s_2) - M) \geq q(t) , \qquad \forall t \in C.$$

Since $(W_t)_{t\in\mathbb{R}_+^2}$ has stationary increments, Hypothesis 3.7.3 implies that $P(F) > 0$, where

$$F = \{(W_{s_1,s_2} - W_{s_1,s_2^*}) \geq p(s_1,s_2) - M , \qquad \forall s \in [(0,s_2^*), t^*]\}.$$

Together with (11), this proves the lemma. □

End of the proof of Theorem 3.7.4. Observe that

$$(12) \qquad X_t = \tilde{X}_t + \int_0^{t_1} ds_1 \int_{s_2^*}^{t_2} ds_2\, (W_{s_1,s_2} - W_{s_1,s_2^*}), \qquad t \in C,$$

where $(\tilde{X}_t)$ is a continuous $\underline{F}_{s^*}$-measurable process. In particular, $(\tilde{X}_t)$ is $\underline{F}_{s^0}$-measurable. Since

$$X_t(\omega) \geq \text{Ln}(\dot{z}_{|t|}^2/(1 - \dot{z}_{|t|}^2)) \iff Y_t \geq \text{Ln}(\dot{z}_{|t|}^2 / (1 - \dot{z}_{|t|}^2)) - \tilde{X}_t ,$$

where $Y_t$ denotes the double integral in (12), and $G \in \underline{F}_{s^o}$, $P(G) > 0$, we can use Lemmas 3.7.5 and 3.5.1 to conclude that $P(H) > 0$, since

$$\lim_{\substack{t \to s^* \\ t \in C}} \operatorname{Ln}(\dot{z}^2_{|t|} / (1 - \dot{z}^2_{|t|})) - \tilde{X}_t(\cdot) = -\infty \quad \text{a.s.} \qquad \square$$

3.7.6 <u>Theorem.</u> The Brownian sheet satisfies the hypothesis made on $(W_t)_{t \in \mathbb{R}^2_+}$ in Theorem 3.7.4.

Before proving this theorem, we recall some classical terminology. The second order moment function $m(\cdot,\cdot)$ of the Brownian sheet is

$$m(f,g) = E(\int_{\mathbb{R}^2_+} W_t \, f(t) \, dt \int_{\mathbb{R}^2_+} W_t \, g(t) \, dt) ,$$

for all $f,g : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$ for which this expression is defined (see [GS], V. §5). If $f,g \in C_0^\infty(\mathbb{R}^2_+)$, then

$$\begin{aligned} m(f,g) &= E(\int_{\mathbb{R}^2_+} F(f) \, dW \int_{\mathbb{R}^2_+} F(f) \, dW) \\ &= \int_{\mathbb{R}^2_+} F(f) \, F(g) \, dt , \end{aligned}$$

where

$$F(f)(t) = \int_{t_1}^\infty ds_1 \int_{t_2}^\infty ds_2 \, f(s_1,s_2).$$

(see [W2], p. 284). Thus the <u>correlation operator</u> B of the Brownian sheet, defined to be the unique bounded non-negative symmetrical linear operator B such that

$$\langle B(f), g \rangle = m(f,g) \quad , \quad \forall f, g ,$$

(see [GS], V. §5) satisfies

$$B^{\frac{1}{2}}(f) = F(f) \quad , \quad \forall f \in C_0^\infty(\mathbb{R}^2_+).$$

The inverse of this operator is

$$B^{-\frac{1}{2}}(f) = \frac{\partial^2 f}{\partial t_1 \partial t_2} .$$

Proof of Theorem 3.7.6. The only hypothesis that requires checking is Hypothesis 3.7.3. If $(W_t)_{t\in\mathbb{R}_+^2}$ is a Brownian sheet, then for fixed $t^* \in \mathbb{R}_+^2$, there is $M > 0$ such that

$$P(|W_t| \leq M \quad , \quad \forall t \leq t^*) > 0.$$

But then the above considerations, together with Theorem 1 of [GS], Chapter VII. §4, imply that

$$P(|W_t - p(t)| \leq M \ , \quad \forall t \leq t^*) =$$

$$= \int_{\{|W_t| \leq M, \forall t \leq t^*\}} dP \quad \exp\left(-\int_{\mathbb{R}_+^2} \frac{\partial^2 p}{\partial s_1 \partial s_2}\, dW - \frac{1}{2}\int_{\mathbb{R}_+^2} \left(\frac{\partial^2 p}{\partial s_1 \partial s_2}\right)^2 ds_1\, ds_2\right),$$

which is clearly strictly positive, $\forall p \in C_0^\infty(\mathbb{R}_+^2, \mathbb{R})$. □

CHAPTER 4

# COMBINATORIAL ASPECTS OF TWO-PARAMETER OPTIMAL STOPPING

The object of this chapter is to describe the optimal stopping problem of Chapter 1.6 when the σ-algebra $\underline{\underline{F}}$ is finite. In this case, we will show that it is natural to associate a matrix and a graph to each two-parameter filtration. When the filtration satisfies Hypothesis CQI, this graph is of quite particular nature (it is perfect, according to Definition 4.2.1 (e)). Since the set of randomized stopping points is isomorphic to a finite-dimensional polytope, we will be able to apply combinatorial results to the study of Question 1.8.2. In particular, we will give necessary and sufficient conditions for an affirmative answer to this question, and we will show that Hypothesis CQI implies that all extremal randomized stopping points are stopping points.

## 4.1 The fundamental matrix of a two-parameter filtration.

We shall suppose throughout this chapter that $\underline{\underline{F}}$ is finite. This leads us to make the following observations.

Without loss of generality, we may suppose that $\underline{\underline{F}}$ is generated by finitely many atoms $H^1, \ldots, H^m$, which form a partition of $\Omega$ (an atom of an algebra or σ-algebra is an element of the algebra which is minimal with respect to set inclusion among elements of the algebra with positive probability). Of course, each $\underline{\underline{F}}_t$ is also generated by finitely many atoms, denoted $H_t^1, \ldots, H_t^{n_t}$. Again without loss of generality, we may restrict ourselves to a finite parameter set $I^2 = \{t \in \mathbb{N}^2 : t \leq \tilde{t}\}$, for some fixed $\tilde{t} \in \mathbb{N}^2$, with $\underline{\underline{F}} = \underline{\underline{F}}_{\tilde{t}}$. These notations will be used throughout this chapter.

4.1.1 Definition.

(a) The fundamental matrix of the algebra $\underline{\underline{F}}_t$ is the $m \times n_t$ matrix $Q_t$ with coefficients $q_t^{i,j} = 1$ if $H^i \subset H_t^j$ and $q_t^{i,j} = 0$ otherwise (that is, when $H^i \cap H_t^j = \emptyset$).

(b) The fundamental matrix Q of the filtration $(\underline{\underline{F}}_t)_{t\in I^2}$ is

$$Q = (Q_{0,0} \,\vdots\, Q_{0,1} \,\vdots\, \cdots \,\vdots\, Q_{0,\tilde{t}_2} \,\vdots\, Q_{1,0} \,\vdots\, \cdots \,\vdots\, Q_{\tilde{t}}).$$

Observe that the matrix Q gives a complete description of the filtration and that $Q_{\tilde{t}}$ is the $m \times m$ identity matrix.

4.1.2 Example. For $i = 1,2$, define $\Omega^i = [0,1]$, let $\underline{\underline{F}}^i$ be the algebra generated by $\{[0,1/3[, [1/3,2/3[, [2/3,1]\}$, let $\underline{\underline{F}}_0^i = \{\emptyset, \Omega^i\}$ and $\underline{\underline{F}}_1^i$ be the algebra generated by $\{[0,2/3[, [2/3,1]\}$, and set $\underline{\underline{F}}_2^i = \underline{\underline{F}}^i$. Consider the measure space $(\Omega, \underline{\underline{F}})$, where $\Omega = \Omega^1 \times \Omega^2$ and $\underline{\underline{F}} = \underline{\underline{F}}^1 \times \underline{\underline{F}}^2$, together with the two-parameter filtration

$$(\underline{\underline{F}}_{t_1,t_2} = \underline{\underline{F}}_{t_1}^1 \times \underline{\underline{F}}_{t_2}^2)_{0\leq(t_1,t_2)\leq(2,2)} .$$

This filtration is represented in Figure 4 : the square in position $(t_1,t_2)$ represents $\underline{\underline{F}}_{t_1,t_2}$ with its atoms indexed to correspond to the columns of the related matrix $\tilde{Q}$ that appears in Figure 5. Matrix $\tilde{Q}$ is obtained from the fundamental matrix Q of $(\underline{\underline{F}}_t)_{0\leq t\leq(2,2)}$ by removing repeated copies of each column and the identity matrix $Q_{2,2}$.

4.1.3 Theorem.

(a) The set of randomized stopping points is isomorphic to the polytope $\underline{\underline{P}} = \{\underline{x} \in \mathbb{R}^M : Q\,\underline{x} = \underline{1}\ ,\ \underline{x} \geq \underline{0}\}$, where M is the number of columns of Q, and $\underline{1}$ denotes a column vector of m ones.

(b) The set of stopping points is isomorphic to $\underline{\underline{P}} \cap \{0,1\}^M$ .

(c) Suppose $(X_t)_{t\in I^2}$ is a two-parameter process. Then the optimal stopping problem reduces to the integer programming problem

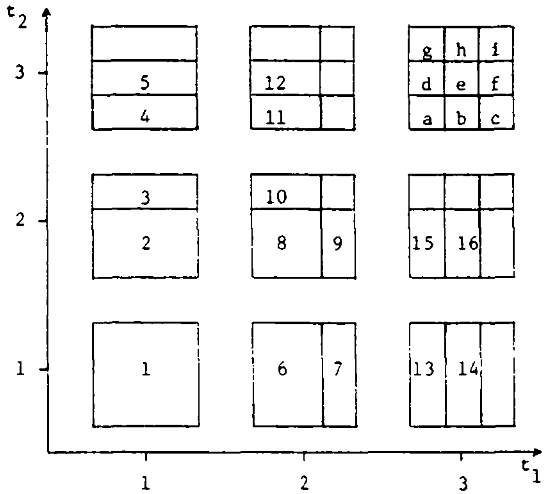



Figure 4.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| b | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| c | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| d | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 |
| e | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 |
| f | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| g | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| h | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| i | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Figure 5.

$$\text{maximize } \underline{c}^T \cdot \underline{x}$$
$$\text{subject to } Q\underline{x} = 1 , \quad \underline{x} \in \{0,1\}^M ,$$

where $\underline{c}^T = (c_{0,0}^T \mid \dots \mid c_{0,\tilde{t}_2}^T \mid c_{1,0}^T \mid \dots \mid c_{\tilde{t}}^T)$, and the $j^{th}$ coefficient of the column vector $\underline{c}_t$ is

$$c_t^j = \int_{H_t^j} X_t \, dP .$$

Proof. Each randomized stopping point can be identified with a process $q = (q_t)_{t \in I^2}$ such that $q_t$ is $\underline{\underline{F}}_t$-measurable, $q_t \geq 0$ a.s. and

$$\sum_{t \in I} q_t = 1 \quad \text{a.s.}$$

A stopping point T identifies with the $\{0,1\}$-valued process $(q_t = I_{\{T=t\}})_{t \in I^2}$, and clearly any $\{0,1\}$-valued element of $\underline{\underline{U}}$ identifies with a stopping point.

Since $q_t$ is $\underline{\underline{F}}_t$-measurable and $q_t \geq 0$,

$$q_t = \sum_{j=1}^{n_t} x_t^j \, I_{H_t^j} , \text{ where } x_t^j \geq 0 .$$

So the condition $\Sigma_{t \in I^2} q_t = 1$ becomes

$$\sum_{t \in I^2} \sum_{j=1}^{n_t} x_t^j \, I_{H_t^j} = 1 ,$$

which, expressed on each atom $H^i$ of $\underline{\underline{F}}$, translates into

$$\sum_{t \in I^2} \sum_{j=1}^{n_t} q_t^{i,j} \, x_t^j = 1 , \quad i = 1, \dots, m .$$

Since a stopping point leads to a $\{0,1\}$-vector $(x_t^j)$, and vice-versa, (a) and (b) are proved.

As for (c), observe that if a stopping point T corresponds to $(q_t)$ and $(x_t^j)$, then

$$E(X_T) = \sum_{t \in I^2} E(X_t \, q_t) = \sum_{t \in I} \sum_{j=1}^{n_t} c_t^j \, x_t^j ,$$

where the $x_t^j$ belong to $\{0,1\}$. The conclusion then follows from (b). □

Determining whether $\underline{\underline{T}} = \text{ext } \underline{\underline{U}}$ is thus reduced to determining whether or not all extremal elements of $\underline{\underline{P}}$ are integral, i.e.

$$\text{ext } \underline{\underline{P}} = \underline{\underline{P}} \cap \{0,1\}^M .$$

This question is clearly of practical importance, since in this case the solution to the integer program in Theorem 4.1.3 (c) can be obtained by solving the linear program

$$\text{maximize } \underline{c}^T \cdot \underline{x}$$
$$\text{subject to } Q\underline{x} = 1 , \quad \underline{x} \geq \underline{0} .$$

## 4.2 Necessary and sufficient conditions for all extremal randomized stopping points to be stopping points (finite case).

Before showing how Question 1.8.2 can be translated in terms of graphs, we recall some definitions.

4.2.1 Definition. (see [Go])

(a) A graph is a couple $G = (V,E)$ where $V$ is a set whose elements are termed vertices, and $E$ is a set of pairs of distinct elements of $V$, termed edges. In this chapter, we will suppose that $V$ is finite. (These graphs are in fact undirected graphs without multiple edges or loops).

(b) Two vertices $v_1, v_2 \in V$ are neighbors if $\{v_1,v_2\} \in E$. A clique in $G$ is a subset $K$ of $V$ such that $\{v_1,v_2\} \in E$, $\forall v_1,v_2 \in K$, $v_1 \neq v_2$. An n-clique is a clique with n distinct vertices.

(c) Given $W \subset V$, the subgraph induced by $W$ is the graph $G_W = (W,E_W)$, where $E_W = \{\{v_1,v_2\} \in E : v_1,v_2 \in W\}$.

(d) If $V$ is finite, we let $\omega(G)$ denote the cardinality of the largest clique in $G$, and $\chi(G)$ denote the fewest number of colors needed to color the vertices of G in such a way that no two neighbors receive the same color.

(e) A finite graph $G = (V,E)$ is <u>perfect</u> if $\omega(G_W) = \chi(G_W)$, for each $W \subset V$.

There is a natural way to relate graphs and matrices with $\{0,1\}$-valued entries. This is the object of the following definition.

4.2.2 <u>Definition.</u> (see [Go])

(a) Given an $m \times n$ matrix $A = (a_{i,j})$, where $a_{i,j} \in \{0,1\}$, $\forall i, j$, the <u>derived graph</u> of A has vertices $v_1,\ldots,v_n$ corresponding to the columns of A, with an edge $\{v_i,v_j\}$ when there is a row of A, say row k, such that $a_{k,i} = a_{k,j} = 1$.

(b) Given a graph G, let $\underline{\underline{K}}$ be the set of all maximal cliques in G (maximal with respect to set inclusion). The <u>clique matrix</u> of G is the matrix $A = (a_{K,v})_{K\in\underline{\underline{K}},v\in V}$, where $a_{K,v} = 1$ if $v \in K$ and $a_{K,v} = 0$ otherwise.

(c) An <u>augmented clique matrix</u> of a graph G is obtained by adding rows to its clique matrix, each new row being the indicator vector of a (not necessarily maximal) clique in G.

These definitions are of interest for Question 1.8.2 because of the following result, due to Fulkerson [Fu] and Chvátal [Cl].

4.2.3 <u>Theorem.</u> Let A be a matrix with $\{0,1\}$-valued entries. Consider the polytope $\underline{\underline{P}}(A) = \{\underline{x} : A\underline{x} \leq 1, \underline{x} \geq \underline{0}\}$. Then $\underline{\underline{P}}(A)$ has only integer-valued extreme points if and only if the following two conditions hold :

(a) A is an augmented clique matrix of its derived graph;

(b) the derived graph of A is perfect.

A proof of this result can be found in [Go], Theorem 3.19. In order to use this theorem to determine when $\underline{T} = \text{ext } \underline{U}$, we need the following definition.

4.2.4 <u>Definition.</u> Give a two-parameter filtration $(\underline{F}_t)_{t\in I^2}$, let $\underline{H}_t$ denote the set of all atoms of $\underline{F}_t$. The <u>intersection graph</u> of $(\underline{F}_t)_{t\in I^2}$ is the graph $G = (V,E)$, where

- $V = \bigcup_{t\in I^2 \setminus \{\tilde{t}\}} (\{t\} \times \underline{H}_t)$ ;
- if $v_1 = (t^1, H_1), v_2 = (t^2, H_2)$, where $H_i \in \underline{H}_{t^i}$ , then

  $\{v_1, v_2\} \in E \iff P(H_1 \cap H_2) > 0.$

(The cartesian product with $\{t\}$ appears above in order to distinguish identical atoms appearing in two or more algebras).

The following theorem gives necessary and sufficient conditions for an affirmative answer to Question 1.8.2.

4.2.5 <u>Theorem.</u> Let $\underline{H}$ denote the set of all atoms of all the $\underline{F}_t$, $t \neq \tilde{t}$. Then $\underline{T} = \text{ext } \underline{U}$ if and only if the following two conditions hold :

(a) $\underline{H}$ has Helly's intersection property;

(b) the intersection graph of $(\underline{F}_t)_{t\in I^2}$ is perfect.

(Recall that Helly's intersection property is that every family of pairwise intersecting elements has a non-empty common intersection).

<u>Proof.</u> Let Q be the fundamental matrix of $(\underline{F}_t)_{t\in I^2}$ . Since Q contains the identity matrix corresponding to the atoms of $\underline{F}_{\tilde{t}}$, $Q = (\tilde{Q} \mid I_m)$, and thus

$$\underline{P} = \{\underline{x} : Q\, \underline{x} = 1 \ , \ x \geq 0\}$$

is isomorphic to

$$\tilde{\underline{P}} = \{\underline{x} : \tilde{Q}\,\underline{x} \leq 1,\ \underline{x} \geq 0\}.$$

The property $\underline{T} = \text{ext}\ \underline{U}$ is equivalent to $\tilde{P} \cap \{0,1\}^{M-m} = \text{ext}\ \tilde{\underline{P}}$ , and so we only need to prove the following :

(1) $\tilde{Q}$ is an augmented clique matrix of its derived graph if and only if $\underline{H}$ has Helly's intersection property;

(2) the derived graph of $\tilde{Q}$ is isomorphic to the intersection graph of $(\underline{F}_t)_{t \in I^2}$ .

To see (2) it is sufficient to compare the definitions. As for (1), suppose $\tilde{Q}$ is an augmented clique matrix of its derived graph, and let $H_1,\ldots,H_n \in \underline{H}$ be such that $H_i \cap H_j \neq \emptyset$, $\forall i, j$. Suppose $H_i$ is an atom of $\underline{F}_{t^i}$ . Then $\{(t^1,H_1),\ldots,(t^n,H_n)\}$ forms a clique in the derived graph of $\tilde{Q}$, and since $\tilde{Q}$ is an augmented clique matrix, there is a row of $\tilde{Q}$, say row k, which is the indicator function of a maximal clique containing $\{(t^1,H_1),\ldots,(t^n,H_n)\}$. Thus $H_1 \cap \ldots \cap H_n$ contains the atom of $\underline{F}$ corresponding to the $k^{th}$ row, hence is not empty.

To see the converse, suppose $\underline{H}$ has Helly's intersection property, and let $K = \{v_1,\ldots,v_n\}$ be a maximal clique in the derived graph of $\tilde{Q}$. Suppose $v_i = (t^i,H_i)$, $H_i \in \underline{H}_{t^i}$. By (2), $H_i \cap H_j \neq \emptyset$, $\forall i, j$. Thus $H_1 \cap \ldots \cap H_n \neq \emptyset$, and so this set contains an atom of $\underline{F}$. If this atom of $\underline{F}$ corresponds to row k of $\tilde{Q}$, this row is the indicator vector of K. □

4.2.6 Remark. There seems to be no available probabilistic interpretation of conditions (a) and (b) in Theorem 4.2.5.

## 4.3 A new class of perfect graphs and sufficiency of Hypothesis CQI.

We shall show that if $(\underline{F}_t)_{t \in I^2}$ satisfies Hypothesis CQI, then conditions (a) and (b) in Theorem 4.2.5 hold. In order to do this we begin by translating Hypothesis CQI in terms of the intersection graph of the filtration.

4.3.1 Theorem. Given a two-parameter filtration $(\underline{\underline{F}}_t)_{t\in I^2}$, with its intersection graph $G = (V,E)$, define a function $\psi : V \longrightarrow \mathbb{N}^2$ by $\psi(t,H) = t$, $t \in I^2$, $H \in \underline{\underline{H}}_t$. Then $(\underline{\underline{F}}_t)_{t\in I^2}$ satisfies Hypothesis CQI if and only if its intersection graph satisfies

$$\text{(3)}\qquad \left.\begin{array}{l} v_1,v_2,v_3 \in V,\ \psi(v_1) \underline{\wedge} \psi(v_2) \underline{\wedge} \psi(v_3) \\ \text{and } \{v_1,v_2\} \in E,\ \{v_2,v_3\} \in E \end{array}\right\} \Rightarrow \{v_1,v_3\} \in E.$$

Proof. Suppose $(\underline{\underline{F}}_t)_{t\in I^2}$ satisfies Hypothesis CQI, and $v_1,v_2,v_3 \in V$ are as in the hypothesis of (3). If $v_i$ corresponds to $(t^i,H^i)$, $H_i \in \underline{\underline{H}}_{t^i}$, then by definition of $\psi$, $t^1 \underline{\wedge} t^2 \underline{\wedge} t^3$. Furthermore, $H_1 \cap H_2 \neq \emptyset \neq H_2 \cap H_3$. This implies that

$$H_2 \subset \{P(H_1|\underline{\underline{F}}_{t^2}) > 0\} \cap \{P(H_3|\underline{\underline{F}}_{t^2}) > 0\} = \{P(H_1 \cap H_3|\underline{\underline{F}}_{t^2}) > 0\}$$

by Hypothesis CQI. Thus $P(H_1 \cap H_3) > 0$, and so $H_1 \cap H_3 \neq \emptyset$, meaning $\{v_1,v_3\} \in E$.

Suppose now that (3) holds, and let $s,t \in I^2$ with $s \underline{\wedge} t$. Fix $H_1 \in \underline{\underline{H}}_s$, $H_2 \in \underline{\underline{H}}_t$, and set $u = (s_1,t_2)$. Then $G_i = \{P(H_i|\underline{\underline{F}}_u) > 0\}$ is the (unique) atom of $\underline{\underline{F}}_u$ that contains $H_i$. Suppose further, to begin with, that $G_1 \cap G_2 = \emptyset$. Then $H_1 \cap H_2 \subset G_1 \cap G_2 = \emptyset$, and so

$$\emptyset = \{P(H_1 \cap H_2|\underline{\underline{F}}_u) > 0\} = \{P(H_1|\underline{\underline{F}}_u) > 0\} \cap \{P(H_2|\underline{\underline{F}}_u) > 0\}.$$

Now suppose $G_1 \cap G_2 \neq \emptyset$. Then $G_1 = G_2 = G$, say, since both $G_1$ and $G_2$ are atoms of $\underline{\underline{F}}_u$. Since $H_1 \cap G \neq \emptyset \neq G \cap H_2$, we obtain by (3) and the definition of the intersection graph that $H_1 \cap H_2 \neq \emptyset$. Thus

$$G = \{P(H_1 \cap H_2|\underline{\underline{F}}_u) > 0\},$$

and so

$$\text{(4)}\qquad \{P(H_1 \cap H_2|\underline{\underline{F}}_u) > 0\} = \{P(H_1|\underline{\underline{F}}_u) > 0\} \cap \{P(H_2|\underline{\underline{F}}_u) > 0\}\ .$$

We have now shown that equality (4) holds when $H_1$ (resp. $H_2$) is an atom of $\underline{\underline{F}}_s$ (resp. $\underline{\underline{F}}_t$). By the general relationship

$$\{P(\bigcup_{i=1}^{k} B_i \mid \underline{\underline{G}}) > 0\} = \bigcup_{i=1}^{k} \{P(B_i \mid \underline{\underline{G}}) > 0\}$$

(see Lemma 7.1.3), we see that (4) is valid for arbitrary $H_1 \in \underline{\underline{F}}_s$, $H_2 \in \underline{\underline{F}}_t$. This shows that Hypothesis CQI holds. □

4.3.2 Theorem. If $(\underline{\underline{F}}_t)_{t\in I^2}$ satisfies Hypothesis CQI, then $\underline{\underline{H}}$ has Helly's intersection property.

Proof. If H is an atom of $\underline{\underline{F}}_s$, H' an atom of $\underline{\underline{F}}_t$, and $s \leq t$, then $\underline{\underline{F}}_s \subset \underline{\underline{F}}_t$ and so either $H \cap H' = \emptyset$ or $H' \subset H$. Thus, in order to verify Helly's intersection property, we need only show the following :

$$(5) \qquad \left.\begin{array}{l} H_i \in \underline{\underline{H}}_{t^i}\ ,\ i = 1,\dots,n\ ,\ \text{where} \\ t^1 \underline{\wedge} \dots \underline{\wedge}\, t^n \ \text{ and }\ H_i \cap H_j \neq \emptyset\ ,\ \forall i,j \end{array}\right\} \Rightarrow H_1 \cap \dots \cap H_n \neq \emptyset\,.$$

Set $\tilde{H}_i = H_1 \cap \dots \cap H_i$. $\tilde{H}_2$ is non-empty by hypothesis. We show by induction that $\tilde{H}_i$ is not empty, $2 \leq i \leq n$. Suppose $\tilde{H}_i \neq \emptyset$. Since $t^1 \underline{\wedge} \dots \underline{\wedge}\, t^i$, $\tilde{H}_i \in \underline{\underline{F}}_{t^i_1, t^1_2}$. Now $\tilde{H}_i \cap H_i \neq \emptyset$, and $H_i \cap H_{i+1} \neq \emptyset$. Using Theorem 4.3.1 and the definition of the intersection graph, we obtain $\tilde{H}_{i+1} = \tilde{H}_i \cap H_{i+1} \neq \emptyset$. □

4.3.3 Definition. A graph $G = (V,E)$ is a two-parameter graph if there is an index function $\psi : V \longrightarrow \mathbb{N}^2$ satisfying the following two conditions :

(T1) $\psi(u) \leq \psi(v)$ with $\{u,v\} \in E$ and $\{v,w\} \in E \Rightarrow \{u,w\} \in E$;

(T2) $\psi(u) \underline{\wedge} \psi(v) \underline{\wedge} \psi(w)$ with $\{u,v\} \in E$ and $\{v,w\} \in E \Rightarrow \{u,w\} \in E$,

for all $u,v,w \in V$.

4.3.4 Proposition. If $(\underline{\underline{F}}_t)_{t\in I^2}$ satisfies Hypothesis CQI, then its intersection graph is a two-parameter graph.

Proof. (T2) is satisfied by Theorem 4.3.1, and (T1) holds because if $\psi(u) \leq \psi(v)$ and $u = (s,H), v = (t,H')$, then $s \leq t$. Thus $\{u,v\} \in E$ means that $H' \subset H$, and so any atom that meets $H'$ meets $H$. □

We shall now prove that two-parameter graphs are perfect. In fact, we shall establish a stronger property, which requires the following definition :

4.3.5 Definition. (see [C]). A graph $G = (V,E)$ is perfectly orderable provided the set $V$ admits a total order $\rho$ such that

(P1) no four vertices $v_1, v_2, v_3, v_4 \in V$ with exactly the three edges $\{v_1,v_2\}$, $\{v_2,v_3\}$, $\{v_3,v_4\}$ have $v_1 \rho v_2$ and $v_4 \rho v_3$.

If, in addition, the following condition holds, then $G$ is termed strongly perfectly orderable ([C2]) :

(P2) no four vertices $v_1, v_2, v_3, v_4 \in V$ with exactly the four edges $\{v_1,v_2\}$, $\{v_2,v_3\}$, $\{v_3,v_4\}$, $\{v_4,v_1\}$ have $v_1 \rho v_2$ and $v_4 \rho v_3$.

Four vertices such that (P1) or (P2) holds will be called an obstruction. The order $\rho$ is perfect if (P1) holds and strongly perfect if both (P1) and (P2) hold.

4.3.6 Theorem. Every perfectly orderable graph is perfect. (for a proof, see Chvátal [C]).

4.3.7 Theorem. Every two-parameter graph is strongly perfectly orderable.

Proof. Suppose $G = (V,E)$ is a two-parameter graph with index function $\psi = (\psi_1,\psi_2) : V \longrightarrow \mathbb{N}^2$. Let $\rho$ be any total order on $V$ such that $v_1 \rho v_2 \Rightarrow \psi_1(v_1) \leq \psi_1(v_2)$ (for instance a lexicographic order). Suppose $\{v_1,v_2,v_3,v_4\}$ constitutes an obstruction for $\rho$. By property (T1), $\psi(v_2) \leq \psi(v_3)$ cannot hold, for in this case $\{v_2,v_4\}$ would belong to E.

Similarly $\psi(v_3) \leq \psi(v_2)$ cannot hold. But then we can suppose by symmetry that $\psi(v_2) \triangleq \psi(v_3)$. Now since $v_1 \rho v_2$, $\psi_1(v_1) \leq \psi_1(v_2)$, and thus either $\psi(v_1) \leq \psi(v_2)$ or $\psi(v_1) \triangleq \psi(v_2)$.

However, in the first case, (T1) would imply that $\{v_1,v_3\} \in E$, and in the second case, $\{v_1,v_3\} \in E$ would be implied by (T2). This is a contradiction. □

Two-parameter graphs form a class that is distinct from other known classes of perfectly orderable graphs. Many of their properties and their relationship with other classes of graphs are described in [DTW].

An important consequence of the preceeding results is the following :

4.3.8 <u>Corollary.</u> Set $I^2 = \{t \in \mathbb{N}^2 : t \leq \tilde{t}\}$. If $\underline{F}$ is finite and $(\underline{F}_t)_{t \in I^2}$ satisfies Hypothesis CQI, then all extremal elements of the set of randomized stopping points are stopping points.

<u>Proof.</u> By Theorem 4.3.2, the set $\underline{H}$ of all atoms of all the $\underline{F}_t$ has Helly's property. By Proposition 4.3.4 and Theorems 4.3.7 and 4.3.6, the intersection graph of $(\underline{F}_t)_{t \in I^2}$ is perfect. So the result follows from Theorem 4.2.5. □

Even though there are many perfect graphs which are not two-parameter graphs (see [DTW]), it is not immediately obvious that Hypothesis CQI is not necessary for the statement in Corollary 4.3.8. We will however give an example that shows that this is the case. Furthermore, there does not seem to be any available probabilistic hypothesis weaker than CQI which implies $\underline{T} = \text{ext}\ \underline{U}$.

4.3.9 <u>Example.</u> Set $\Omega = \{1,2,3,4,5,6,7\}$, $\underline{F}_t = \{\emptyset,\Omega\}$, for $t \in \{(0,0),(0,1),(1,0)\}$,

$$\underline{\underline{F}}_{0,2} = \sigma(\{1,3\},\ \{2,5\},\ \{4,7\},\ \{6\}),$$

$$\underline{\underline{F}}_{1,1} = \sigma(\{1,4\},\ \{3,6\},\ \{2\},\ \{5\},\ \{7\}),$$

$$\underline{\underline{F}}_{2,0} = \sigma(\{1,2\},\ \{3\},\ \{4\},\ \{5\},\ \{6\},\ \{7\}),$$

and $\underline{\underline{F}}_t = \underline{\underline{B}}(\Omega)$, $t \in \{(1,2),(2,1),(2,2)\}$. Then it is not difficult to check that $\underline{\underline{H}}$ has Helly's property, and that the intersection graph of $(\underline{\underline{F}}_t)_{t \leq (2,2)}$, shown in Figure 6, is perfect. Thus $\underline{\underline{T}} = \text{ext}\ \underline{\underline{U}}$.

However, if P is any probability on $\Omega$ such that $P\{\omega\} > 0$, $\forall \omega \in \Omega$, then Hypothesis CQI does not hold for this filtration, since

$$H_1 = \{1,3\} \in \underline{\underline{F}}_{0,2}\ , \quad H_2 = \{4\} \in \underline{\underline{F}}_{2,0}\ ,$$

but

$$\{P(H_1|\underline{\underline{F}}_{0,0}) > 0\} \cap \{P(H_2|\underline{\underline{F}}_{0,0}) > 0\} = \Omega$$
$$\neq \emptyset$$
$$= \{P(H_1 \cap H_2|\underline{\underline{F}}_{0,0}) > 0\}. \quad \square$$

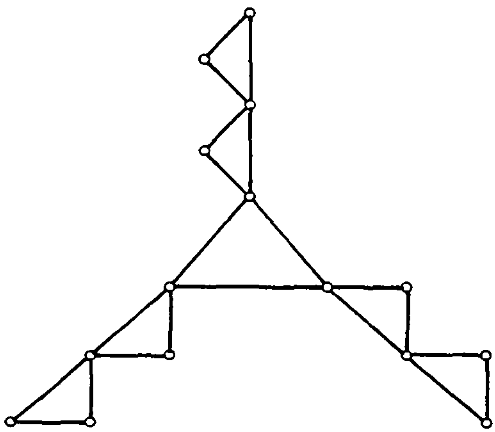


Figure 6.

## 4.4 Counterexamples for filtrations with the Intersection Property.

Given two single-parameter filtrations

$$(\underline{F}^1_{t_1})_{0\leq t_1\leq \tilde{t}_1} \quad \text{and} \quad (\underline{F}^2_{t_2})_{0\leq t_2\leq \tilde{t}_2} ,$$

one can construct a two-parameter filtration by setting

$$\underline{F}_t = \underline{F}^1_{t_1} \cap \underline{F}^2_{t_2} , \quad t = (t_1,t_2).$$

Such a filtration clearly satisfies the Intersection Property 1.4.4 when $\underline{F}^1_{\tilde{t}_1} = \underline{F}^2_{\tilde{t}_2} = F$.

We shall give two examples of such filtrations, in order to show that either condition (a) or (b) of Theorem 4.2.5 may fail.

4.4.1 Example. Set $\Omega = \{1,2,3,4,5,6\}$, and

$$\underline{F}^1_1 = \sigma(\{1,3\}, \{2,4\}, \{5,6\}) \quad , \quad \underline{F}^2_1 = \sigma(\{1,2\}, \{3,6\}, \{4,5\}),$$

$$\underline{F}^1_2 = \sigma(\{1\}, \{3\}, \{2,4\}, \{5\}, \{6\}) \; , \quad \underline{F}^2_2 = \sigma(\{1\},\{2\},\{3\},\{4,5\},\{6\}),$$

$$\underline{F}^1_3 = 2^\Omega \quad , \quad \underline{F}^2_3 = 2^\Omega .$$

Then

$$\underline{F}_{2,2} = \underline{F}^1_2 \cap \underline{F}^2_2 = \sigma(\{1\}, \{3\}, \{6\}, \{2,4,5\}),$$

$$\underline{F}_{1,3} = \underline{F}^1_1 \cap \underline{F}^2_3 = \underline{F}^1_1 ,$$

$$\underline{F}_{3,1} = \underline{F}^1_3 \cap \underline{F}^2_1 = \underline{F}^2_1 .$$

Set $v_1 = ((2,2),\{2,4,5\})$, $v_2 = ((1,3),\{5,6\})$ , $v_3 = ((3,1),\{3,6\})$,

$v_4 = ((1,3),\{1,3\})$ , $v_5 = ((3,1),\{1,2\})$.

Then the subgraph of the intersection graph of $(\underline{F}_t)_{t\in\{1,2,3\}^2}$ induced by $W = \{v_1,v_2,v_3,v_4,v_5\}$, shown in Figure 7, is a 5-cycle, which is not perfect.

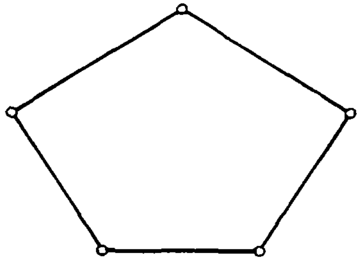


Figure 7.

4.4.2 Example. Set $\Omega = \{1,2,3,4\}$, and

$$\underline{F}_1^1 = \sigma(\{1,2\}, \{3,4\}) \quad , \quad \underline{F}_1^2 = \sigma(\{1,3\}, \{2,4\}),$$

$$\underline{F}_2^1 = \sigma(\{1\}, \{2\}, \{3,4\}), \quad \underline{F}_2^2 = \sigma(\{1,3\}, \{2\}, \{4\}),$$

$$\underline{F}_3^1 = 2^\Omega \quad , \quad \underline{F}_3^2 = 2^\Omega .$$

Then $\{1,2\} \in \underline{F}_{1,3}$, $\{1,3,4\} \in \underline{F}_{2,2}$ and $\{2,4\} \in \underline{F}_{3,1}$ are elements of $\underline{H}$ and are pairwise intersecting, but have an empty common intersection.

4.4.3 Remark. A more graphical description of the above examples is given in [DTW].

## 4.5 A counterexample for three-parameter filtrations.

For $i = 1,2,3$, set $\Omega^i = \{a,b\}$, $\underline{F}_0^i = \{\emptyset,\Omega^i\}$, $\underline{F}_1^i = 2^{\Omega^i}$. On the index set $I = \{0,1\}^3$, set

$$\underline{F}_{t_1,t_2,t_3} = \underline{F}_{t_1}^1 \times \underline{F}_{t_2}^2 \times \underline{F}_{t_3}^3 .$$

This defines a three-parameter filtration on $\Omega = \Omega^1 \times \Omega^2 \times \Omega^3$.

Set $K_1 = \{a,b\}$, $K_2 = \{a\}$, $K_3 = \{b\}$, and $a_1 = 0$, $a_2 = a_3 = 1$. Then

$$K_{\pi(1)} \times K_{\pi(2)} \times K_{\pi(3)} \in \underline{\underline{F}}_{a_{\pi(1)},a_{\pi(2)},a_{\pi(3)}}$$

for every function $\pi : \{1,2,3\} \longrightarrow \{1,2,3\}$, and the intersection graph of $(\underline{\underline{F}}_t)_{t\in\{0,1\}^3}$ contains the subgraph induced by

$$\{K_1 \times K_3 \times K_3,\ K_2 \times K_3 \times K_1,\ K_1 \times K_1 \times K_2,\ K_3 \times K_2 \times K_1,\ K_3 \times K_1 \times K_3\ ,$$

which is isomorphic to the graph in Figure 7, hence is not perfect.

## 4.6 Product filtrations, normal products of graphs and canonical two-parameter filtrations.

Given two single-parameter filtrations

$$(\underline{\underline{F}}^1_{t_1})_{0\leq t_1\leq \tilde{t}_1} \quad \text{and} \quad (\underline{\underline{F}}^2_{t_2})_{0\leq t_2\leq \tilde{t}_2}\ ,$$

defined respectively on $\Omega^1$ and $\Omega^2$, we can define a two-parameter filtration on $\Omega = \Omega^1 \times \Omega^2$ by setting

$$\underline{\underline{F}}_t = \underline{\underline{F}}^1_{t_1} \times \underline{\underline{F}}^2_{t_2}\ , \quad \textit{where } t = (t_1,t_2).$$

This <u>product filtration</u> satisfies Hypothesis CQI when the probability on $\Omega$ is such that each atom has positive probability.

The intersection graph of $(\underline{\underline{F}}_t)_{t\leq\tilde{t}}$ is closely related to that of $(\underline{\underline{F}}^1_{t_1})_{0\leq t_1\leq \tilde{t}_1}$ and $(\underline{\underline{F}}^2_{t_2})_{0\leq t_2\leq \tilde{t}_2}$. To see this, we need the following definition.

4.6.1 <u>Definition.</u> (see [Go]). If $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are two graphs, their <u>normal product</u> $G_1 \cdot G_2$ (see [Go]) is the graph $G = (V,E)$, where $V = V_1 \times V_2$ and $\{w^1,w^2\} \in E$ *for* $w^1 = (v^1_1,v^1_2)$, $w_2 = (v^2_1,v^2_2)$ if either $(v^1_1 = v^2_1$ and $\{v^1_2, v^2_2\} \in E_2)$ or $(\{v^1_1, v^2_1\} \in E_1$ and $v^1_2 = v^2_2)$ or

($\{v_1^1, v_1^2\} \in E_1$ and $\{v_2^1, v_2^2\} \in E_2$).

4.6.2 Proposition. The intersection graph of

$$(\underline{\underline{F}}_{t_1,t_2} = \underline{\underline{F}}^1_{t_1} \times \underline{\underline{F}}^2_{t_2})_{(t_1,t_2) \leq \tilde{t}} .$$

is the normal product of the intersection graphs of

$$(\underline{\underline{F}}^1_{t_1})_{0 \leq t_1 \leq \tilde{t}_1} \quad \text{and} \quad (\underline{\underline{F}}^2_{t_2})_{0 \leq t_2 \leq \tilde{t}_2} .$$

Proof. Each atom of $\underline{\underline{F}}_{t_1,t_2}$ is the cartesian product of an atom of $\underline{\underline{F}}^1_{t_1}$ and of $\underline{\underline{F}}^2_{t_2}$. Now if $A_1^i \subset \Omega_1$, $A_2^i \subset \Omega_2$, $i = 1,2$ then $(A_1^1 \times A_2^1) \cap (A_1^2 \times A_2^2) \neq \emptyset$ if and only if $A_1^1 \cap A_1^2 \neq \emptyset$ and $A_2^1 \cap A_2^2 \neq \emptyset$. Thus the conclusion follows from the definition of a normal product. □

4.6.3 Definition. Canonical two-parameter filtrations.

Let $\underline{R} \subset \mathbb{R}$ be a finite set of real numbers, set $\underline{R}^t = \underline{R}$ for $t \leq \tilde{t}$, and $\Omega^c = \Pi_{t \leq \tilde{t}} \underline{R}^t$. For $t \leq \tilde{t}$, let $\underline{H}_t^c$ be the partition of $\Omega^c$ into sets of the form $\Pi_{s \leq \tilde{t}} S_s$ , where $S_s$ is a singleton for $s \leq t$ and $S_s = \underline{R}$ otherwise.

Two atoms $H = \Pi_{s \leq \tilde{t}} S_s \in \underline{F}_t$ and $H' = \Pi_{s \leq \tilde{t}} S'_s \in \underline{F}_{t'}$, have a non-empty intersection if and only if $S_s = S'_s$ for $s \leq \min(t,t')$. It is easy to see that the filtration $(\underline{F}_t^c = \sigma(\underline{H}_t^c)_{t \leq \tilde{t}}$ satisfies Hypothesis CQI provided the probability on $(\Omega, \underline{F} = \underline{F}_{\tilde{t}}^c)$ is such that all elements of $\underline{F}$ have positive probability. Such a filtration is termed a canonical two-parameter filtration.

Using canonical two-parameter filtrations, it is easy to provide examples of two-parameter graphs. This will be useful in the next chapter. In particular, though $\underline{G}_1 \vee \underline{G}_2$ , where $\underline{G}_1$ and $\underline{G}_2$ are independent σ-algebras, is always isomorphic to a product σ-algebra (see Stroock [S]), a two-parameter filtration with CQI is not always isomorphic to a product filtration (see Proposition 5.6.3).

4.7 A constructive algorithm for a class of perfect polytopes.

In order to give conditions under which $\underline{T} = \text{ext}\ \underline{U}$, we applied Theorem 4.2.3 to a particular polytope of the form $\underline{P}(A) = \{x : A\underline{x} \leq 1,\ \underline{x} \geq 0\}$, where A is a matrix whose coefficients are all either 0 or 1. Though this theorem gives conditions for $\underline{P}(A)$ to have only integer-valued extreme points, it does not at all indicate how, given a non-integral element $\underline{x} \in \underline{P}(A)$, to determine two distinct elements $\underline{x}^1$, $\underline{x}^2 \in \underline{P}(A)$ such that $\underline{x} = 1/2\ \underline{x}^1 + 1/2\ \underline{x}^2$ .

We shall indicate here how this can be accomplished when A is the clique-matrix of a strongly perfectly orderable graph. In particular, this construction is valied for the fundamental matrix of a two-parameter filtration that satisfies Hypothesis CQI and will have a generalization in certain infinite-dimensional settings (see Chapter 7.5).

4.7.1 Algorithm. Given a graph $G = (V,E)$ and a total order $\prec$ on V, let $v_1,\dots,v_n$ be the enumeration of elements of V in increasing order for $\prec$, let $\underline{K}$ denote the set of all maximal cliques in $\underline{G}$ and $\underline{K}_i$ the set of cliques containing $v_i$. Let A be the clique matrix of G, the $i^{th}$ column of A corresponding to the vertex $v_i$, $1 \leq i \leq n$. For each $\underline{x} \in \mathbb{R}^n$, we set

$$f_1(\underline{x}) = \underline{x}^1 \ , \ f_2(\underline{x}) = \underline{x}^2 \ ,$$

where the components of $\underline{x}^1$, $\underline{x}^2$ and an auxiliary vector $(C_1,\dots,C_n)$ are defined inductively in the following manner.

Set $C_1 = 0$, and define $x_1^1$ and $x_1^2$ by formula (6) below with $i = 1$. Then suppose $x_j^1$, $x_j^2$ and $C_j$ have been defined for $j < i$, and use formula (7) below to define $C_i$, and then formula (6) to define $x_i^1$ and $x_i^2$.

$$(6) \quad \begin{array}{llll} x_i^1 = 2x_i \ , & x_i^2 = 0 & \text{if} & C_i + 2x_i < 1 \ , \\ x_i^1 = 1 - C_i \ , & x_i^2 = 2x_i - 1 + C_i & \text{if} & C_i < 1 \ , \ C_i + 2x_i \geq 1 \ , \\ x_i^1 = 0 \ , & x_i^2 = 2x_i & \text{if} & C_i \geq 1 \ . \end{array}$$

$$(7) \quad C_i = \max_{K \in \underline{K}_i} \sum_{\{j<i \,:\, v_j \in K\}} x_j^1 \ .$$

4.7.2 Lemma.

(a) If $\underline{x}^1 = f_1(\underline{x})$, $\underline{x}^2 = f_2(\underline{x})$, then $x_i^1 \geq 0$, $x_i^2 \geq 0$, $\forall i$, and

$$x = \frac{1}{2}\underline{x}^1 + \frac{1}{2}\underline{x}^2 \ .$$

(b) Suppose $\underline{x} \in \underline{P}(A)$ has $0 < x_i < 1$ for some i. Then $f_1(\underline{x}) \neq \underline{x} \neq f_2(\underline{x})$.

(c) Suppose $\underline{x} \in \underline{P}(A) \Rightarrow (f_1(\underline{x}) \in \underline{P}(A)$ and $f_2(\underline{x}) \in \underline{P}(A))$. Then $\prec$ is a perfect order on V.

<u>Proof.</u> (a) is clear by the definition of $\underline{x}^1$ and $\underline{x}^2$.

(b) Let i be the smallest integer such that $0 < x_i < 1$. Then for each $j < i$, $x_j = 0$. But then (by induction)

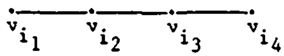
$$\sup_{K \in \underline{\underline{K}}_i} \sum_{\{j<i \,:\, v_j \in K\}} x_j^1 = 0 .$$

This implies that $C_i = 0$, and since $0 < x_i < 1$,

$$x_i^1 = \min(2x_i, 1) \neq x_i \neq \max(2x_i - 1, 0) = x_i^2 ,$$

proving (b).

(c) Suppose $W = \{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}\}$ induces an obstruction, that is, $v_{i_1} \prec v_{i_2}$ , $v_{i_4} \prec v_{i_3}$ , and the subgraph induced by W is



.

Then $\underline{x} = (x_1, \ldots, x_n) \in \underline{\underline{P}}(A)$, where

$$x_i = \frac{1}{2} \text{ if } i \in \{i_1, i_2, i_3, i_4\} \text{ and } x_i = 0 \text{ otherwise.}$$

However, $\underline{x}^j = f_j(\underline{x})$, $j = 1,2$, and $(C_1, \ldots, C_n)$ given by Algorithm 4.7.1 are such that $C_{i_1} = C_{i_4} = 0$ , so by (6),

$$x_{i_1}^1 = 1 - C_{i_1} = 1 , \quad x_{i_4}^1 = 1 - C_{i_4} = 1 ,$$

by (7), $C_{i_2} = C_{i_3} = 1$ , and thus

$$x_{i_2}^2 = 2x_{i_2} = 1 , \quad x_{i_3}^2 = 2x_{i_3} = 1 .$$

This implies that $\underline{x}^2 \notin \underline{\underline{P}}(A)$, since if K is a maximal clique containing $\{v_{i_2}, v_{i_3}\}$, then

$$\sum_{\{j:v_j \in K\}} x_j^2 \geq x_{i_2}^2 + x_{i_3}^2 = 2 . \quad \square$$

4.7.3 Lemma. Let $G = (V,E)$ be a graph, and let $<$ be a strong perfect order on $V$. Let $K$ be a clique in $G$, and suppose :

(8) for each $v \in K$, there is a clique $L_v$ of neighbors of $v$, such that $u \in L_v \Rightarrow u < v$.

Then for some $w \in K$, $L_w \cup K$ is a clique.

Proof. It is a generalization of that of Lemma 1 of [C]. Suppose to begin with that $K$ has only two elements $v^1$ and $v^2$, and that the assertion of the lemma is false. That is, for $i = 1,2$, there is $u^i \in L_{v^i}$ such that $\{u^i, v^{3-i}\} \notin E$. But then $\{u^1, v^1, v^2, u^2\}$ is an obstruction by (8), a contradiction.

We now proceed by induction on card $(K)$. Suppose that for each $v \in K$, there is $\tilde{v} \in K$ such that $L_{\tilde{v}} \cup (K \setminus \{v\})$ is a clique. Now we may assume that some vertex $u_{v,\tilde{v}} \in L_{\tilde{v}}$ is such that $\{u_{v,\tilde{v}}, v\} \notin E$, for otherwise we are done.

Now the map $v \longmapsto \tilde{v}$ is necessarily injective in this case, so it is onto.

Let $\bar{w}$ be the $<$-minimal element of $K$, and let $w_2, w_3 \in K$ be such that $\tilde{w}_2 = \bar{w}$ and $\tilde{w}_3 = w_2$. Set $w_1 = u_{w_3,w_2}$ and $w_4 = u_{w_2,\bar{w}}$. Since $\{w_1,w_2\} \in E$ but $\{w_1,w_3\} \notin E$, and by construction $\{w_4,w_3\} \in E$ but $\{w_4,w_2\} \notin E$, the four vertices $w_1,w_2,w_3,w_4$ induce and obstruction, a contradiction. $\square$

4.7.4 Theorem. Let $G = (V,E)$ be a graph with a strong perfect order $<$, let $A$ be the clique matrix of $G$ with columns ordered according to $<$, and set $\underline{P}(A) = \{\underline{x} : A\,\underline{x} \leq 1,\ \underline{x} \geq 0\}$. Then :

$$\underline{x} \in \underline{P}(A) \Rightarrow (f_1(\underline{x}) \in \underline{P}(A) \text{ and } f_2(\underline{x}) \in \underline{P}(A)).$$

Proof. Fix $\underline{x} \in \underline{P}(A)$. We use the notations of 4.7.1 and begin by showing that

(9) $\quad C_i \leq 1 \, , \; \forall i \, .$

Observe that $C_1 = 0 \leq 1$. Now suppose $C_i \leq 1$ for $i < i_o$, and show that $C_{i_o} \leq 1$. Consider $\tilde{K} \in \underline{K}_{i_o}$ such that

$$C_{i_o} = \sum_{\{j<i_o : v_j \in \tilde{K}\}} x_j^1$$

and let $j_o = \sup\{j \in \tilde{K} : j < i_o, \, x_j^1 > 0\}$ (if this set is empty, $C_{i_o} = 0$ and we are done). Then

$$C_{i_o} = x_{j_o}^1 + \sum_{\{j<j_o : v_j \in \tilde{K}\}} x_j^1 \leq x_{j_o}^1 + C_{j_o} \leq 1$$

by the definition of $x_{j_o}^1$ (see (6)) and the induction hypothesis. This proves (9).

Now for each $K \in \underline{K}$,

$$\sum_{\{j : v_j \in K\}} x_j^1 = \sup_{\{j : v_j \in K\}} \sum_{\{i \leq j : v_i \in K\}} x_i^1$$

$$\leq \sup_{\{j : v_j \in K\}} (C_j + x_j^1)$$

$$\leq 1$$

by the definition of $x_j^1$ and (9). This shows that $f_1(\underline{x}) \in \underline{P}(A)$.

In order to prove that $f_2(\underline{x}) \in \underline{P}(A)$, it remains to be proven that

(10) $$\sum_{\{j:v_j \in K\}} x_j^2 \leq 1 , \quad \forall K \in \underline{\underline{K}} .$$

Fix $K_o \in \underline{\underline{K}}$ .

Case 1. $C_j + 2x_j < 1$ when $v_j \in K_o$. Then $x_j^2 = 0$ whenever $v_j \in K_o$ (see (6)), and so

$$\sum_{\{j:v_j \in K_o\}} x_j^2 = 0 \leq 1 .$$

Case 2. $W = \{v_j \in K_o : C_j + 2\,x_j \geq 1\} \neq \emptyset$.

Then for each j such that $v_j \in W$, there is a clique $L_j$ of neighbors of $v_j$ such that

(11) $$v \in L_j \Rightarrow v \prec v_j$$

(12) $$\sum_{\{k:v_k \in L_j \cup \{v_j\}\}} x_k^1 = 1 .$$

By Lemma 4.7.3, there is $v_{j_o} \in W$ such that $L_{j_o} \cup W$ is a clique, and so, since $\underline{x} \in \underline{\underline{P}}(A)$,

(13) $$\sum_{\{j:v_j \in L_{j_o} \cup W\}} 2x_j \leq 2 .$$

Subtracting (12) (with $j = j_o$) from (13) and using the fact that $2x_j - x_j^1 = x_j^2$ by Lemma 4.7.2 (a), we get

$$\sum_{\{j:v_j \in L_{j_o} \cup \{v_{j_o}\}\}} x_j^2 + \sum_{\{j:v_j \in W \setminus (L_{j_o} \cup \{v_{j_o}\})\}} 2x_j \leq 1 .$$

But since $2x_j \geq x_j^2$ by Lemma 4.7.2 (a), this implies

$$\sum_{\{j:v_j\in K_o\}} x_j^2 = \sum_{\{j:v_j\in W\}} x_j^2 \leq 1 ,$$

and thus $f_2(\underline{x}) \in \underline{\underline{P}}(A)$. □

4.7.5 Remark. It is not known whether Theorem 4.7.4 remains valid if $\prec$ is a perfect order which is not a strong perfect order.

CHAPTER 5

# COMBINATORIAL STRUCTURE OF SINGLE AND TWO-PARAMETER FILTRATIONS

## 5.1 Objectives.

The intersection graph of a filtration describes the relationships between the atoms of different $\sigma$-algebras of the filtration, and thus represents the combinatorial structure of the filtration. The object of this chapter is to address the following questions : what graphs are intersection graphs of

- single-parameter filtrations ?
- two-parameter filtrations ?
- two-parameter filtrations that satisfy Hypothesis CQI ?

In fact, we will show that intersections graphs of single-parameter filtrations have a very simple structure, whereas any graph is a subgraph of a two-parameter filtration. Furthermore, the structure of two-parameter filtrations that satisfy Hypothesis CQI is quite complex, and no complete characterization is available.

5.1.1 Definition. Given a class $\underline{G}$ of graphs, a minimal forbidden induced subgraph for $\underline{G}$ is a graph G such that

(a) no graph in $\underline{G}$ contains G as an induced subgraph;
(b) every proper induced subgraph of G is contained in some graph in $\underline{G}$.

Knowledge of the minimal forbidden induced subgraphs of a class of filtrations gives an idea of their structure, and can also be used as a tool for recognizing whether or not a given graph is in the class. Our structural study will be centered on such forbidden subgraphs.

## 5.2 Intersection graphs of single-parameter filtrations.

Let us introduce some classical definitions.

**5.2.1 Definition.** ([Go])

(a) A simple path on n vertices is a graph with distinct vertices $v_1, \ldots, v_n$ and edges $\{v_1,v_2\}, \{v_2,v_3\}, \ldots, \{v_{n-1},v_n\}$. We say that $(v_1, \ldots, v_n)$ is a $\underline{P}_n$.

(b) A simple cycle on n vertices is a graph with distinct vertices $v_1, \ldots, v_n$ and edges $\{v_1,v_2\}, \{v_2,v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n,v_1\}$. We say that $(v_1, \ldots, v_n, v_1)$ is a $\underline{C}_n$.

If $(\underline{\underline{F}}_t)_{1\leq t\leq n}$, $n \in \mathbb{N}$, is a single-parameter filtration, then $\underline{\underline{F}}_s \subset \underline{\underline{F}}_t$ when $s \leq t$, and so the set $\underline{\underline{H}}_t$ of atoms of $\underline{\underline{F}}_t$ is refinement of $\underline{\underline{H}}_s$. Thus the set $\underline{\underline{H}}$ of all atoms of all the $\underline{\underline{F}}_t$ has the structure of a "tree".

**5.2.2 Definition.** ([Wo])

(a) A tree is an ordered set $(V,\rho)$ such that whenever $v_1$ and $v_2$ are incomparable elements of v, there is no $v \in V$ with $v_1 \rho v$ and $v_2 \rho v$.

(b) The comparability graph of an ordered set $(V,\rho)$ is the graph $G = (V,E)$, where $\{v_1,v_2\} \in E \iff v_1\rho v_2$ or $v_2\rho v_1$.

**5.2.3 Proposition.** A graph $G = (V,E)$ is isomorphic to an induced subgraph in the intersection graph of a single parameter filtration if and only if it is the comparability graph of a tree.

Proof. Let $(\underline{\underline{F}}_t)_{1\leq t\leq n}$ be a single-parameter filtration, $G = (V,E)$ its intersection graph. The set V can be ordered in the following way :

$$(s,H)\,\rho\,(t,H') \iff s \leq t \text{ and } H' \subset H.$$

Since $(\underline{\underline{F}}_t)_{1\leq t\leq n}$ is increasing, $(V,\rho)$ is a tree. Furthermore, the comparability graph of $(V,\rho)$ is $G$, and so every subgraph of $G$ is also the comparability graph of a tree (a sub-tree of $(V,\rho)$).

Now suppose $G = (V,E)$ is the comparability graph of a tree $(V,\rho)$. It is then easy to embed $(V,\rho)$ into a larger tree $(\tilde{V},\tilde{\rho})$ where each totally ordered subset of $\tilde{V}$ has the same cardinality $p$. Let $\tilde{V}_1$ be the set of $\tilde{\rho}$-minimal elements of $\tilde{V}$, $\tilde{V}_i$ the set of $\tilde{\rho}$-minimal elements of $\tilde{V} \setminus (\tilde{V}_1 \cup \ldots \cup \tilde{V}_{i-1})$, $i = 2, \ldots, p$.

Set $\Omega = \tilde{V}_p$ and $\underline{F}_t = \sigma(\{v \in \tilde{V}_p : u \tilde{\rho} v\}\ ,\ u \in \tilde{V}_t)$. Then $(\underline{F}_t)_{1 \leq t \leq p}$ is a single-parameter filtration, and $G$ is clearly a subgraph of the intersection graph of this filtration. $\square$

$\underline{P}_4$ $\underline{C}_4$

Figure 8.

5.2.4 Theorem. *For an arbitrary graph $G = (V,E)$, the following are equivalent :*

(a) $G$ is an induced subgraph of the intersection graph of some single-parameter filtration.

(b) $G$ contains no induced $\underline{P}_4$ or $\underline{C}_4$ (see Figure 8).

Proof. Wolk [Wo] has shown that a graph is the comparability graph of a tree if and only if it contains no induced $\underline{P}_4$ or $\underline{C}_4$. Thus the result follows from Proposition 5.2.3. □

Theorem 5.2.4 shows that intersection graphs of single-parameter filtrations give rise to a very restricted class of graphs. This is in sharp contrast with the results in the following sections concerning intersection graphs of two-parameter filtrations.

## 5.3 Intersection graphs of general two-parameter filtrations.

5.3.1 Theorem. Given an arbitrary graph $G = (V,E)$, there is a two-parameter filtration (which generally does not satisfy Hypothesis CQI) whose intersection graph contains $G$ as an induced subgraph.

In order to prove this theorem, we recall the following definition.

5.3.2 Definition. ([Go]). Given a graph $G = (V,E)$, a stable set $S$ is a subset of $V$ such that

$$\{v_1,v_2\} \notin E , \quad \forall v_1,v_2 \in S.$$

5.3.3 Lemma. Given a graph $G = (V,E)$, there is a set $\Omega$, an integer $n$ and a family $M_1, \ldots, M_n$ of partitions of $\Omega$ such that $G$ is a subgraph of the intersection graph of sets in $M_1 \cup \ldots \cup M_n$.

Proof. Let $S_1, \ldots, S_n$ be a cover of $G$ by stable sets, and let $K^1, \ldots, K^m$ be the family of all maximal cliques in $G$. If $G$ were the intersection graph of a family of partitions, each partition corresponding to one of the $S_i$ , then $S_i \cap K^j$ would be nonempty, for all $i,j$. Since this need not be the case, we add vertices to $V$ and edges to $E$, in the following manner.

Set $S_i^1 = S_i$ , $K_1^j = K^j$. For $i = 1, \ldots, n$ and $j = 1, \ldots, m$ , if $S_i^j \cap K_i^j \neq \emptyset$, set $S_i^{j+1} = S_i^j$, $K_{i+1}^j = K_i^j$ ,

otherwise : introduce into G a new vertex $v_{i,j}$ such that

$N(v_{i,j}) = K_i^j$, and set : $S_i^{j+1} = S_i^j \cup \{v_{i,j}\}$, $K_{i+1}^j = K_i^j \cup \{v_{i,j}\}$.

This algorithm yields a graph $G' = (V',E')$, that contains G as a subgraph, for which $\{S_i^{m+1}, i = 1, \ldots, n\}$ is a stable set cover, and whose maximal cliques are $K_{n+1}^1, \ldots, K_{n+1}^m$. Furthermore, $S_i^{m+1} \cap K_{n+1}^j \neq \emptyset$, for all i,j.

Set $\Omega = \{K_{n+1}^1, \ldots, K_{n+1}^m\}$. For $v \in V'$, set $H_v = \{K_{n+1}^j : v \in K_{n+1}^j\}$, and define a partition $M_i$ of $\Omega$ by $M_i = \{H_v, v \in S_i^{m+1}\}$. Observe that if $v_1, v_2 \in V$,

$$H_{v_1} \cap H_{v_2} \neq \emptyset \iff \exists j : v_1 \in K_{n+1}^j \text{ and } v_2 \in K_{n+1}^j \iff v_1,v_2 \in E ,$$

and thus G is a subgraph of the intersection graph of sets in $M_1 \cup \ldots \cup M_n$.

□

Proof of Theorem 5.3.1. Let $\Omega, M_1, \ldots, M_n$ be given by Lemma 5.3.3. For $s = (s_1,s_2)$, set $\underline{\underline{F}}_s = \{\emptyset,\Omega\}$ if $s_1 + s_2 < n-1$, $\underline{\underline{F}}_s = \sigma(M_{s_1+1})$ if $s_1 + s_2 = n-1$, $\underline{\underline{F}}_s = 2^\Omega$ if $s_1 + s_2 > n-1$. Then $(\underline{\underline{F}}_s)_{s \leq (n,n)}$ is a two-parameter filtration, and G is a subgraph of its intersection graph. □

## 5.4 Minimal forbidden induced subgraphs of two-parameter graphs.

5.4.1 Definition. A graph which is isomorphic to an induced subgraph of the intersection graph of a two-parameter filtration that satisfies Hypothesis CQI will be termed a CQI-graph.

Since CQI-graphs are perfect (see Theorem 4.3.7 and Proposition 4.3.4), it is of course not possible to obtain a result comparable to Theorem 5.3.1. However, we will show that the situation is much more complex than for single-parameter filtrations. Not only will there be infinitely many minimal forbidden induced subgraphs, but these subgraphs may have arbitrarily large clique size.

Rather than work with two-parameter filtrations that satisfy Hypothesis CQI, we shall begin by examining minimal forbidden induced subgraphs in two-parameter graphs.

Since bipartite graphs are transitively orientable (see [Go] for the corresponding definitions), they are two-parameter graphs ([DTW], Proposition 3.8). Thus every forbidden subgraph must contain an odd cycle. Since odd cycles of length greater than five are clearly minimal forbidden induced subgraphs, we begin by defining a class of forbidden induced subgraphs that contain 3-cliques.

5.4.2 Definition.

(a) Consider a 3-clique $K = \{v,u_1,u_2\}$ in a graph $G = (V,E)$. We will say that property $M(v,K,G)$ holds if

(a1) there is a vertex $w_i \in N(v) \setminus N(u_i)$, $i = 1,2$. ($N(v)$ denotes the set of neighbors of $v$).

(a2) there is a vertex $x_i$ such that $(v,w_i,x_i)$ is a $P_3$.

(a3) either $w_1 = w_2$ or $\{w_1,w_2\} \notin E$.

(b) We will say that $G$ contains a triple-$P_3$ configuration if there is a 3-clique $K=\{v_1,v_2,v_3\}$ in $G$ such that $M(v_i,K,G)$ holds for $i = 1,2,3$.

5.4.3 Theorem. Two-parameter graphs contain no triple-$P_3$ configurations.

Before proving this theorem, we need the following lemma.

5.4.4 Lemma. If $G = (V,E)$ is a two-parameter graph with index function $\psi : V \longrightarrow \mathbb{N}^2$, and $K = \{v_1,v_2,v_3\}$ is a 3-clique such that $M(v_3,K,G)$ holds, then neither $\psi(v_1) \trianglelefteq \psi(v_3) \trianglelefteq \psi(v_2)$ nor $\psi(v_2) \trianglelefteq \psi(v_3) \trianglelefteq \psi(v_1)$ can hold.

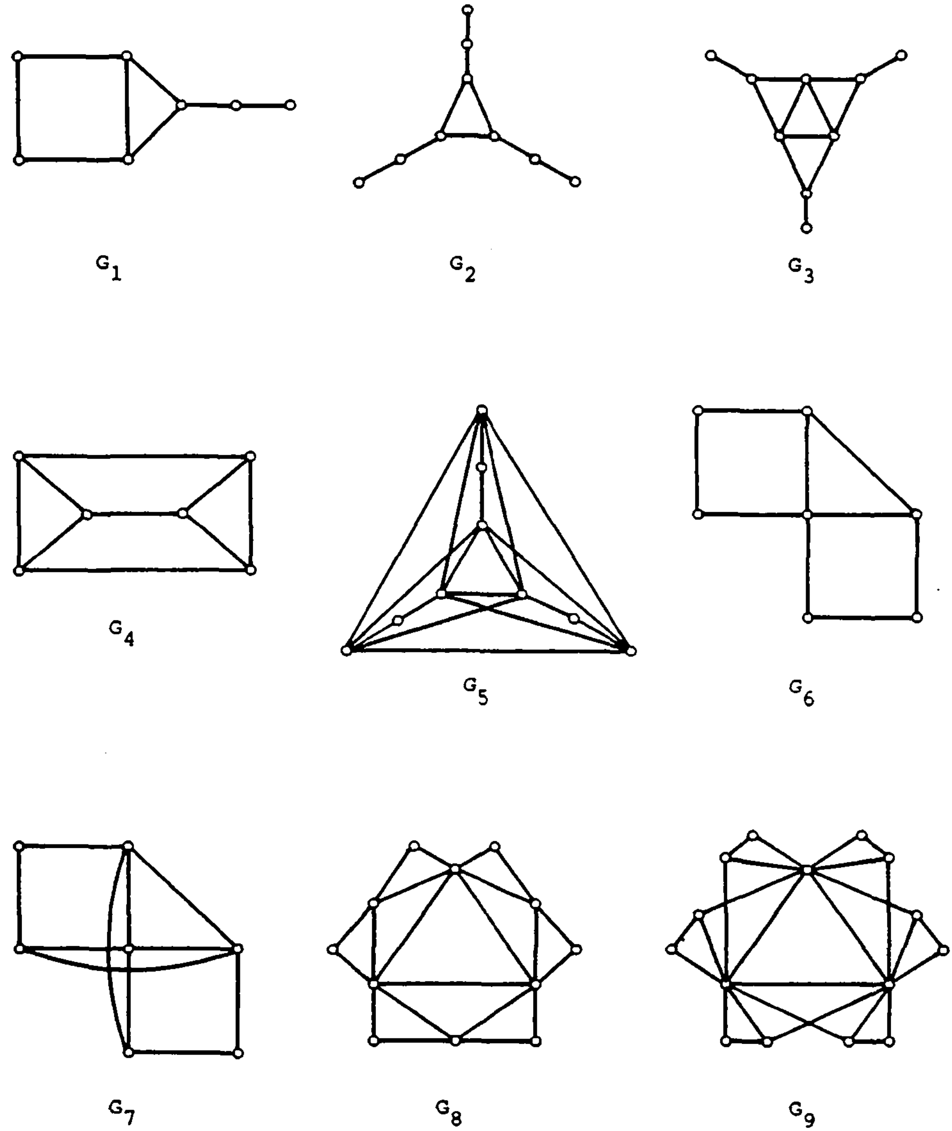



Figure 9. Some triple-$\underset{\sim}{P}_3$ configurations.

Proof. Let us suppose for example that

(1) $\psi(v_1) \underline{\wedge} \psi(v_3) \underline{\wedge} \psi(v_2)$.

Consider $w_i$ and $x_i$ associated with $v_3$ as in Definition 5.4.2. Since $x_i \in N(w_i) \setminus N(v_3)$ and $v_i \in N(v_3) \setminus (w_i)$, $i = 1,2$, $\psi(w_i)$ and $\psi(v_3)$ are not $\leqslant$-comparable. Since $\{v_i, w_i\}$ is not an edge, (1) and 4.3.3 (T2) imply $\psi(w_1) \wedge \psi(v_3) \wedge \psi(w_2)$. This is a contradiction if $w_1 = w_2$, as well as if $w_1 \neq w_2$, because in this last case, Definition 5.4.2 (a3) states that $w_1$ and $w_2$ are not neighbors, contradicting 4.3.3 (T2). □

Proof of Theorem 5.4.3. Suppose the contrary : that is, in a two-parameter graph $G = (V,E)$ with index function $\psi : V \longrightarrow \mathbb{N}^2$, there is a 3-clique $K = \{v_1, v_2, v_3\}$ and $w_j^i$ given by property $M(v_i, K, G)$, $j \neq i$. Note that $w_j^i \in N(v_i) \setminus N(v_j)$, so $\psi(v_i)$ and $\psi(v_j)$ are not $\leqslant$-comparable for $i \neq j$. Thus $\psi(v_1)$, $\psi(v_2)$ and $\psi(v_3)$ are totally ordered for $\underline{\wedge}$. Suppose by symmetry that $\psi(v_1) \underline{\wedge} \psi(v_3) \underline{\wedge} \psi(v_2)$. Since $M(v_3, K, G)$ holds, Lemma 5.4.4 leads to a contradiction. □

It is easy to construct graphs that contain triple-$\underline{P}_3$ configurations. Several examples, each a minimal forbidden induced subgraph for two-parameter graphs, are shown in Figure 9.

5.4.5 Remark. It is straightforward, but lengthy, to check that the graphs in Figure 9 are minimal forbidden induced subgraphs, not only for two-parameter graphs, but also for CQI-graphs. In order to illustrate how this can done, consider the subgraph $\tilde{G}_3$ of $G_3$ obtained by removing one of the vertices with a single neighbor. We shall show that there is a canonical two-parameter filtration whose intersection graph contains $\tilde{G}_3$. For this we use the notations of 4.6.3. Set $\underline{R} = \{1,2\}$, and $\tilde{t} = (4,4)$. Then $H = \prod_{s \leqslant \tilde{t}} S_s$ is an atom of $\underline{F}_t$ if and only if $S_s$ is a singleton when $s \leqslant t$, and $S_s = \{1,2\}$ otherwise. We can represent A schematically by labelling each $s \leqslant t$ with the (unique) element of $S_s$, and other points s with a dot.

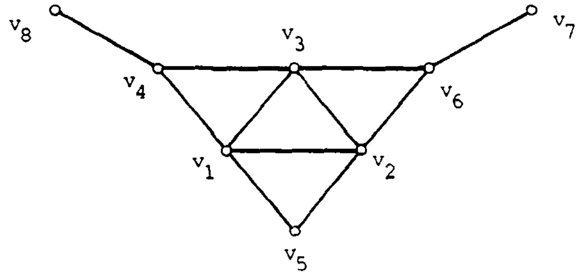



Figure 10.

| | | | | |
|---|---|---|---|---|
| 1 | · | · | · | · |
| 1 | · | · | · | · |
| 1 | · | · | · | · |
| 1 | · | · | · | · |
| 1 | · | · | · | · |

$v_1$

| | | | | |
|---|---|---|---|---|
| · | · | · | · | · |
| · | · | · | · | · |
| 1 | 1 | 1 | · | · |
| 1 | 1 | 1 | · | · |
| 1 | 1 | 1 | · | · |

$v_2$

| | | | | |
|---|---|---|---|---|
| · | · | · | · | · |
| · | · | · | · | · |
| · | · | · | · | · |
| · | · | · | · | · |
| 1 | 1 | 1 | 1 | 1 |

$v_3$

| | | | | |
|---|---|---|---|---|
| · | · | · | · | · |
| · | · | · | · | · |
| 1 | 2 | 2 | · | · |
| 1 | 1 | 2 | · | · |
| 1 | 1 | 1 | · | · |

$v_4$

| | | | | |
|---|---|---|---|---|
| · | · | · | · | · |
| 1 | 1 | 1 | 1 | · |
| 1 | 1 | 1 | 1 | · |
| 1 | 1 | 1 | 1 | · |
| 1 | 1 | 1 | 2 | · |

$v_5$

| | | | | |
|---|---|---|---|---|
| · | · | · | · | · |
| 2 | 1 | · | · | · |
| 1 | 1 | · | · | · |
| 1 | 1 | · | · | · |
| 1 | 1 | · | · | · |

$v_6$

| | | | | |
|---|---|---|---|---|
| · | · | · | · | · |
| 2 | 1 | 1 | · | · |
| 1 | 1 | 1 | · | · |
| 1 | 1 | 1 | · | · |
| 1 | 1 | 2 | · | · |

$v_7$

| | | | | |
|---|---|---|---|---|
| · | · | · | · | · |
| 2 | 2 | 2 | 1 | · |
| 1 | 2 | 2 | 1 | · |
| 1 | 1 | 2 | 1 | · |
| 1 | 1 | 1 | 2 | · |

$v_8$

Figure 11.

Figure 10 represents $\tilde{G}_3$ and Figure 11 represents eight atoms whose intersection graph is $\tilde{G}_3$.

Unfortunately, consideration of 3-cliques is far from exhausting the class of minimal forbidden subgraphs in two-parameter graphs, as the following theorem shows.

5.4.6 <u>Theorem.</u> For $n \geq 2$, there is a minimal forbidden induced subgraph for two-parameter graphs that contains an $(n+1)$-clique.

<u>Proof.</u> Define a graph $L_n$ consisting of exactly the following distinct vertices and edges :

(2) - an $(n+1)$-clique $\{v_1, \ldots, v_{n+1}\}$ ;

(3) - $(n-1)$ vertices $w_1, \ldots, w_{n-1}$, such that $v_i \neq w_j$, $\forall$ i,j, and $\{v_{n+1}, w_1, \ldots, w_{n-1}\}$ is an n-clique;

(4) - $\{v_i, w_j\}$ is an edge $\Leftrightarrow 1 \leq i < j$ or $j+1 < i \leq n+1$;

(5) - $x_1, \ldots, x_{n-1}$ are vertices such that $(v_{n+1}, w_i, x_i)$ is a $\underline{P}_3$, $i \leq n$;

(6) - $v_i'$ is a neighbor of $v_i$ but of no other vertex, $i = 2, \ldots, n-1$;

(7) - for $i \in \{1,n\}$, there are vertices $v_i'$, $v_i''$ such that $(v_i, v_i', v_i'')$ is a $\underline{P}_3$, and $v_i'$ has no neighbors other than $v_i$ and $v_i''$.

(For example, $G_2$ in Figure 9 is an $L_2$; an $L_3$ and an $L_4$ are presented in Figure 12). Suppose $\psi$ is a function from the vertices of $L_n$ into $\mathbb{N}^2$ which satisfies conditions 4.3.3 (T1) and (T2). We will show that this leads to a contradiction.

Note that for $n = 2$, conditions (4), (5) and (7) imply that $L_2$ is a triple-$\underline{P}_3$. So we suppose $n \geq 3$.

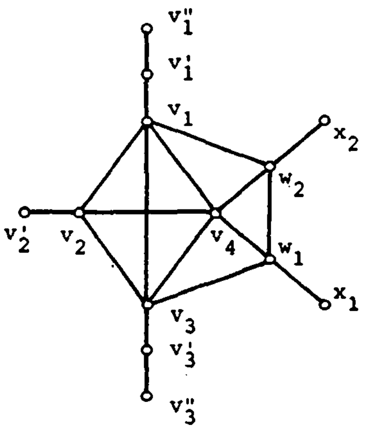
By (6) and (7), $v_i' \in N(v_i) \setminus N(v_j)$, $i \neq j$, $i \leq n$, and by (3) and (4), $w_i \in N(v_{n+1}) \setminus (N(v_i) \cup N(v_{i+1}))$, $i < n$, so by (T1), $\psi(v_j) \leq \psi(v_i)$ cannot hold for any pair $\{i,j\}$. Thus the set $\{\psi(v_i) : 1 \leq i \leq n+1\}$ is totally ordered with respect to $\underline{\wedge}$ .

By (7), $M(v_i, \{v_i,v_j,v_k\}, L_n)$ holds for $i \in \{1,n\}$ and $j,k \in \{2,\ldots,n-1\} \cup \{n+1\}$, $j \neq k$. Invoking Lemma 5.4.4, we can suppose by symmetry that

(8) $\psi(v_1) \triangleq \psi(v_i)$, $i \geq 2$, and $\psi(v_i) \triangleq \psi(v_n)$, $i \in \{1,\ldots n-1\} \cup \{n+1\}$.

By (4) and (5), $M(v_{n+1}, \{v_1,v_2,v_{n+1}\}, L_n)$ holds, so (8) and Lemma 5.4.4 imply $\psi(v_2) \triangleq \psi(v_{n+1})$. Thus if we set $i_o = \sup\{i \leq n : \psi(v_i) \triangleq \psi(v_{n+1})\}$, $i_o$ is such that $2 \leq i_o \leq n-1$. Now $\psi(v_{i_o}) \triangleq \psi(v_{n+1}) \triangleq \psi(v_{i_o+1})$. But again by (4) and (5), $M(v_{n+1},\{v_{i_o},v_{i_o+1},v_{n+1}\}, L_n)$ holds, contradicting Lemma 5.4.4 and proving that an $L_n$ is forbidden in two-parameter graphs.

Using the above construction it is easy to build $\psi$ if one removes $v_i'$, $x_i$, or $v_i''$ from $L_n$, and thus any subgraph of $L_n$ is permissible. Details are let to the reader. □



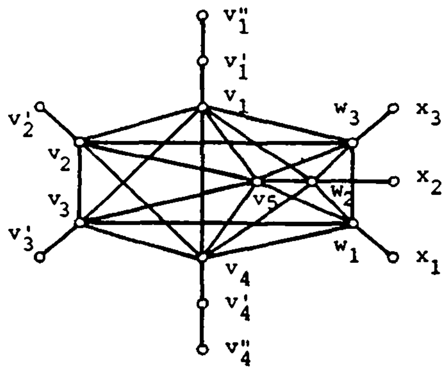




Figure 12.

5.4.7 Remark. As in Remark 5.4.5, it is straightforward to check that any proper subgraph of an $L_n$ can be obtained in the intersection graph of a canonical two-parameter filtration.

## 5.5 Minimal forbidden induced subgraphs for CQI-graphs.

In the previous section, we have shown that triple-$P_3$ configurations and the graphs $L_n$ of Theorem 5.4.6 are minimal forbidden configurations, both for two-parameter graphs and CQI-graphs. However, the class of two-parameter graphs properly contains the class of CQI-graphs. In this section, we show that even cycles of length greater than six (which are two-parameter graphs since they are bipartite) do not appear in CQI-graphs.

5.5.1 Theorem. For $n \geq 3$, the intersection graph of a two-parameter filtration satisfying Hypothesis CQI contains no induced $\underline{C}_{2n}$.

Proof. Suppose the contrary, that is, suppose some two-parameter filtration with CQI contains a $C_{2n}$ with consecutive vertices $H^{(1)},\dots,H^{(2n)}$, $H^{(2n+1)} = H^{(1)}$. Each $H^{(i)}$ represents some atom of an $\underline{\underline{F}}_t$, say of $\underline{\underline{F}}_{(k_i,\ell_i)}$.

Without restriction of generality we can suppose that $k_1 = \min_i k_i$. Note that $(k_i,\ell_i)$ and $(k_{i+1},\ell_{i+1})$ cannot be $\leq$-comparable, as this would create a chord (an edge between non-consecutive vertices of the cycle).

So suppose $(k_i,\ell_i) \triangleq (k_{i+1},\ell_{i+1})$ for some i. By Hypothesis CQI, if $(k_{i+1},\ell_{i+1}) \triangleq (k_{i+2},\ell_{i+2})$, then $\{H^{(i)}, H^{(i+2)}\}$ would be an edge. Thus

$$(k_i,\ell_i) \triangleq (k_{i+1},\ell_{i+1}) \iff (k_{i+2},\ell_{i+2}) \triangleq (k_{i+1},\ell_{i+1}).$$

Note that the choice of $k_1$ implies that $(k_1,\ell_1) \triangleq (k_2,\ell_2)$, and so

(9) $(k_{2i\pm 1},\ell_{2i\pm 1}) \triangleq (k_{2i},\ell_{2i})$, $i = 1,\dots,n$.

Suppose we have proved the following implications :

(10) $k_{2i-1} \leq k_{2i+1} \implies \ell_{2i} < \ell_{2i+2}$,

(11) $\ell_{2i} \leq \ell_{2i+2} \implies k_{2i+1} < k_{2i+3}$.

Then, since by definition of $k_1$, $k_1 \leq k_3$, we would get successively

$\ell_2 < \ell_4$, $k_3 < k_5$, $\ell_4 < \ell_6, \ldots$, so $k_1 \leq k_3 < k_5 \ldots < k_{2n+1}$, contradicting the fact that $k_1 = k_{2n+1}$, and the conclusion of the theorem would follow.

Let us prove (10). The proof of (11) is similar and will be omitted. Suppose that $k_{2i-1} \leq k_{2i+1}$ and $\ell_2 \geq \ell_{2i+2}$. Since $H^{(2i)}$ and $H^{(2i+1)}$ have a nonempty intersection, by (9) there is an atom U of $\underline{F}_{(k_{2i+1}, \ell_{2i})}$ containing both $H^{(2i)}$ and $H^{(2i+1)}$. Similarly, there is an atom V of $\underline{F}_{(k_{2i-1}, \ell_{2i})}$ that contains both $H^{(2i-1)}$ and $H^{(2i)}$. Since by hypothesis $k_{2i-1} \leq k_{2i+1}$, necessarily $U \subset V$.

Note that $H^{(2i+2)} \cap H^{(2i+1)}$ is not empty, so $H^{(2i+2)} \cap U \neq \emptyset$, implying $H^{(2i+2)} \cap V \neq \emptyset$. By hypothesis

$$\ell_{2i+2} \leq \ell_{2i} \leq \ell_{2i-1} \text{ and } k_{2i-1} \leq k_{2i+1} \leq k_{2i+2} ,$$

so

$$(k_{2i-1}, \ell_{2i-1}) \triangleq (k_{2i-1}, \ell_{2i}) \triangleq (k_{2i+2}, \ell_{2i+2}).$$

Thus Hypothesis CQI for $H^{(2i-1)}$, V and $H^{(2i+2)}$ gives $H^{(2i-1)} \cap H^{(2i+2)} \neq \emptyset$, meaning $H^{(2i-1)}$, $H^{(2i+2)}$ is an edge, a contradiction since $n \geq 3$. □

The following theorem summarizes the known forbidden configurations for CQI-graphs.

5.5.2 Theorem. Intersection graphs of two-parameter filtrations that satisfy Hypothesis CQI contain

(a) no simple cycles of length 5 or more;

(b) no triple-$P_3$ configurations;

(c) no $L_n$, $n \geq 2$ (defined in the proof of Theorem 5.4.6).

Proof. (a) holds by Theorem 5.5.1 and the fact that odd cycles of length 5 or more are not perfect. (b) is the statement of Theorem 5.4.3 and (c) follows from the proof of Theorem 5.4.6. □

## 5.6 A characterization of CQI-graphs.

In the previous sections, when we needed to check that a graph was a CQI-graph, we showed each time that this graph was an induced subgraph of the intersection graph of some canonical two-parameter filtration. In this section, we show that, in such a case, it is never necessary to consider other two-parameter filtrations that satisfy Hypothesis CQI. More precisely, every CQI-graph is isomorphic to an induced subgraph of the intersection graph of some canonical two-parameter filtration.

5.6.1 Theorem. *Let $(\Omega, \underline{\underline{F}}, P)$ be a finite probability space, and $(\underline{\underline{F}}_t)_{t\in I^2}$ a two-parameter filtration that satisfies Hypothesis CQI. Then the intersection graph of $(\underline{\underline{F}}_t)_{t\in I^2}$ is isomorphic to an induced subgraph of the intersection graph of a canonical two-parameter filtration.*

Proof. Suppose $\underline{\underline{F}}$ has m atoms. Using the notations of 4.6.3, set $\underline{\underline{R}}^t = \underline{\underline{R}} = \{1,\dots,m\}$, $\forall t\in I^2$, and $\Omega^{\underline{c}} = \underline{\underline{R}}^{I^2}$. If $\underline{\underline{H}}_t = \{H_t^1,\dots,H_t^{n_t}\}$ is the set of atoms of $\underline{\underline{F}}_t$, define $g_t : \underline{\underline{H}}_t \longrightarrow 2^{\underline{\underline{R}}}$ by $g_t(H_t^i) = \{i\}$, $1\leq i\leq n_t$ (this is possible since $n_t \leq m$). Let $(\underline{\underline{F}}_t^{\underline{c}})_{t\in I^2}$ be the canonical two-parameter filtration on $\Omega^{\underline{c}}$, and $\underline{\underline{H}}_t^{\underline{c}}$ the set of atoms of $\underline{\underline{F}}_t^{\underline{c}}$. Let $G = (V,E)$ (respectively $G^{\underline{c}} = (V^{\underline{c}}, E^{\underline{c}})$) be the intersection graph of $(\underline{\underline{F}}_t)_{t\in I^2}$ (respectively $(\underline{\underline{F}}_t^{\underline{c}})_{t\in I^2}$).

We define a function $g : V \longrightarrow V^{\underline{c}}$ by setting

$$g(t, H_t^i) = (t, \tilde{g}_t(H_t^i)) , \quad \forall t\in I^2 , \quad 1\leq i\leq n_t ,$$

where

$$\tilde{g}_t(H_t^i) = (\prod_{s<t} g_s(\{P(H_t^i|\underline{\underline{F}}_s)>0\})) \times \prod_{s\not\leq t} \underline{\underline{R}}^s .$$

Observe that $\tilde{g}_t(H_t^i) \in \underline{\underline{H}}_t^{\underline{c}}$, and that g and $\tilde{g}_t$ are injective, since $g_t$ is, $\forall t\in I^2$. So the theorem will be proved if we show that

$$(12) \quad \tilde{g}(H_t^i) \cap \tilde{g}(H_s^j) \neq \emptyset \iff P(H_t^i\cap H_s^j)>0 .$$

Suppose to begin with that $s\leq t$. Since $H_t^i$ and $H_s^j$ are atoms, $P(H_t^i)>0$

and $P(H_s^j) > 0$, and so

$$P(H_t^i \cap H_s^j) > 0 \iff H_t^i \subset H_s^j$$

$$\iff \{P(H_t^i | \underline{F}_u) > 0\} = \{P(H_s^j | \underline{F}_u) > 0\}, \quad \forall u \leq s$$

$$\iff \tilde{g}(H_t^i) \cap \tilde{g}(H_s^j) \neq \emptyset.$$

Thus (12) holds in this case.

Now suppose that $s \underline{\wedge} t$, and set $u = \min(s,t)$. Then by Hypothesis CQI,

$$P(H_t^i \cap H_s^j) > 0 \iff \{P(H_t^i | \underline{F}_u) > 0\} = \{P(H_s^j | \underline{F}_u) > 0\}$$

$$\iff \{P(H_t^i | \underline{F}_v) > 0\} = \{P(H_s^j | \underline{F}_v) > 0\}, \quad \forall v \leq u$$

$$\iff \tilde{g}(H_t^i) \cap g(H_s^j) \neq \emptyset.$$

This proves the theorem. □

An obvious corollary to Theorem 5.6.1 is :

5.6.2 Corollary. A graph $G = (V,E)$ is a CQI-graph if and only if it is isomorphic to an induced subgraph of the intersection graph of some canonical two-parameter filtration.

In contrast to Corollary 5.6.2, the following proposition gives an example of a CQI-graph which is not an induced subgraph of a product filtration.

5.6.3 Proposition. Let $G_{10}$ be the graph shown in Figure 13. Then $G_{10}$ is a CQI-graph, but $G_{10}$ is not an induced subgraph of the intersection graph of a product filtration.

Proof. $G_{10} = (V,E)$ is a proper subgraph of $G_3$ in Figure 9, and so

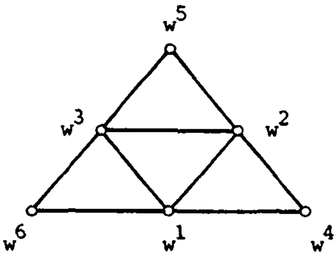



Figure 13.

$G_{10}$ is a CQI-graph. Now suppose ab absurdo that

$$(\underline{\underline{F}}^1_{t_1})_{t_1 \in I} \quad \text{and} \quad (\underline{\underline{F}}^2_{t_2})_{t_2 \in I}$$

are two single-parameter filtrations, such that $G_{10}$ is an induced subgraph of the intersection graph of the product filtration

$$(\underline{\underline{F}}_{t_1,t_2} = \underline{\underline{F}}^1_{t_1} \times \underline{\underline{F}}^2_{t_2})_{(t_1,t_2) \in I^2} \,.$$

Let $w^1, \ldots, w^6$ be the vertices of $G_{10}$, as shown in Figure 13. Each $w^j$ corresponds to an atom of some $\underline{\underline{F}}_t$, say to $H^j_1 \times H^j_2 \in \underline{\underline{H}}_{(k_j,\ell_j)}$, $(k_j,\ell_j) \in I^2$.

Now $(k_1,\ell_1)$ and $(k_2,\ell_2)$ cannot be $\leq$-comparable, since $w^6 \in N(w^1) \setminus N(w^2)$ and $w^5 \in N(w^2) \setminus N(w^1)$. So we can suppose by symmetry that

(13) $\quad (k_1,\ell_1) \;\underline{\wedge}\; (k_2,\ell_2) \;\underline{\wedge}\; (k_3,\ell_3)$.

Thus

(14) $\quad H^1_1 \supset H^2_1 \supset H^3_1 \quad ,$

(15) $H_2^1 \subset H_2^2 \subset H_2^3$ .

Now since $\{w^6,w^3\} \in E$, $H_1^6 \cap H_1^3 \neq \emptyset$, so (14) implies that

(16) $H_1^6 \cap H_1^2 \neq \emptyset$.

Similarly, since $\{w^6,w^1\} \in E$, $H_2^6 \cap H_2^1 \neq \emptyset$, and so (15) implies that

(17) $H_2^6 \cap H_2^2 \neq \emptyset$.

But then (16) and (17) would imply that $\{w^6,w^2\}$ would be an edge, a contradiction. This proves the proposition. □

CHAPTER 6

# INFINITE SYSTEMS OF LINEAR INEQUALITIES AND PERFECT GRAPHS

In Chapter 4, we showed that when the σ-algebra $\underline{\underline{F}}$ on the set $\Omega$ is finite, the set of randomized stopping points associated with a two-parameter filtration $(\underline{\underline{F}}_t)_{t\in I^2}$ on $(\Omega, \underline{\underline{F}})$ is isomorphic to a finite-dimensional polytope

(1) $\underline{\underline{P}}(A) = \{\underline{x}\in \mathbb{R}^n : A\underline{x} \leqslant 1,\ \underline{x} \geqslant 0\}$ ,

where $A = (a_{i,j})$ is a matrix, all of whose coefficients are 0 or 1 (see proof of Theorem 4.2.5). Of course, if $\underline{\underline{F}}$ were generated by a countable partition of $\Omega$, the set of randomized stopping points would again be isomorphic to a set as in (1), but the matrix A would have infinitely many rows and columns. One would like to know whether the graph-theoretical characterizations of Chapter 4 remain valid in this setting.

Extremal elements of sets defined by an infinite system of linear inequalities have been studied by several authors, primarily in connection with doubly stochastic (or substochastic) matrices (see [K], [I], [Mau]). Recall that a doubly substochastic matrix is a matrix $(m_{i,j})$ such that

(2) $m_{i,j} \geqslant 0$ , $\sum_j m_{i,j} \leqslant 1$ , $\sum_i m_{i,j} \leqslant 1$ , $\forall i, j$ .

The set of doubly substochastic measures is clearly a "polyhedron" as in (1).

Another situation where one would like to know the extremal elements of an infinite system of linear inequalities is the following, closely related to a well-known theorem of Dilworth (see [Go], Chap. 5.7).

Let $(J, \leqslant)$ be a finite ordered set, and let $\underline{\underline{K}}$ be the family of all maximal totally ordered subsets of J. Define a matrix $A = (a_{K,j})_{K\in\underline{\underline{K}}, j\in J}$ by

$$a_{K,j} = 1 \text{ if } j \in K, \quad a_{K,j} = 0 \text{ otherwise},$$

and consider the polytope $\underline{\underline{P}}(A)$ defined in (1). As was observed in [TW],

$$\text{ext } \underline{\underline{P}}(A) = \underline{\underline{P}}(A) \cap \{0,1\}^J .$$

The same characterization would be valid if we had defined $\underline{\underline{K}}$ to be the family of all maximal sets of pairwise noncomparable elements of J (see again [TW]).

A question of interest is whether the two above statements are valid when J is infinite. We will show that this is indeed the case.

The questions stated above will be answered using a generalization of the theorem of Fulkerson and Chvátal (see Theorem 4.2.3), through a topological argument akin to that presented in [K] in an addendum by J.C. Kiefer. We shall also apply this result to the characterization of the extremal elements of a set of "non-measurable randomized stopping points".

## 6.1 Infinite-dimensional polytopes and perfect graphs.

In this section, we consider a matrix $A = (a_{i,j})_{i\in I, j\in J}$, where I and J are two arbitrary sets, and $a_{i,j} \in \{0,1\}$, $\forall i \in I$, $j \in J$. The (possibly infinite-dimensional) "polyhedron" $\underline{\underline{P}}(A)$ is defined by

$$\underline{\underline{P}}(A) = \left\{ x = (x_j)_{j\in J} \in \mathbb{R}^J \ : \ \sum_{j\in J} a_{i,j}\, x_j \leqslant 1, \quad \forall i \in I, \right.$$
$$\left. \text{and } x_j \geqslant 0 , \quad \forall j \in J \right\} .$$

(For a given $x \in \underline{\underline{P}}(A)$ and $i \in I$, there will of course be at most countably many $j \in J$ such that $a_{i,j}\, x_j > 0$).

The main result of this section states that the necessary and sufficient condition for this polyhedron to have only integer-valued extreme points is that every "subpolyhedron" supported by finitely many columns of A have the same property. This motivates the following natural definition of infinite perfect graphs.

6.1.1 <u>Definition.</u> A (not necessarily finite) <u>graph</u> is a couple $G = (V,E)$, where V is an arbitrary set, whose elements are termed <u>vertices</u>, and E is a set of pairs of (distinct) elements of V, called <u>edges</u>.

(b) An infinite graph is <u>perfect</u> if the subgraph induced by each finite set of vertices is perfect (see Definition 4.2.1).

The notions of derived graph and augmented clique matrix (see Definition 4.2.2) can be defined in this setting as follows.

We will say that $(x_j)_{j\in J} \in \mathbb{R}^J$ <u>dominates</u> $(y_j)_{j\in J} \in \mathbb{R}^J$ if $y_j \leq x_j$, $\forall j \in J$. Given a matrix $A = (a_{i,j})_{i\in I, j\in J}$ with $\{0,1\}$-valued entries, the <u>derived graph</u> of A has vertices $v_j$, $j \in J$, corresponding to the columns of A, and edges $\{v_j, v_{j'}\}$ when there is some $i \in I$ such that $a_{i,j} = a_{i,j'} = 1$. Given a graph $G = (V,E)$, an <u>augmented clique matrix</u> of G is a matrix A with $\{0,1\}$-valued entries such that G is the derived graph of A, and such that the indicator function of each finite (not necessarily maximal) clique in G is dominated by some row of A. When I and J are finite, these definitions are consistent with those of Definition 4.2.2.

The following theorem generalizes the result of Fulkerson and Chvátal (see Theorem 4.2.3).

6.1.2 <u>Theorem.</u> All extreme points of $\underline{P}(A)$ are integer-valued if and only if (a) and (b) below hold :

(a) A is an augmented clique matrix of its derived graph;

(b) the derived graph of A is perfect.

The starting point to the proof of this theorem is Theorem 4.2.3

which states that Theorem 6.1.2 is valid when I and J are finite. The extension to arbitrary I and J is similar to that contained in [K] for countably infinite doubly stochastic matrices, and uses the following variation on the Krein-Milman Theorem (see [B], II.7.1).

6.1.3 Theorem. In a locally convex Hausdorff vector space, consider a compact convex set $\underline{P}$ and a closed subset $\underline{T}$ of $\underline{P}$. Then (a) and (b) are equivalent :

(a) $\underline{P}$ is the closed convex hull of $\underline{T}$;
(b) $\underline{T}$ contains all extreme points of $\underline{P}$.

For a finite sequence $(j_1,\dots,j_n)$ of elements of J, let $A_{(j_1,\dots,j_n)}$ be the submatrix of A consisting of columns $j_1,\dots,j_n$ of A. Note that $A_{(j_1,\dots,j_n)}$ only has finitely many distinct rows. Set

$$\underline{P}_{(j_1,\dots,j_n)}(A) = \{\underline{x} \in \mathbb{R}^n : A_{(j_1,\dots,j_n)}\underline{x} \leq 1,\ \underline{x} \geq 0\}.$$

In view of our definitions of infinite perfect graph and augmented clique matrix, it is straightforward to see that the statement of Theorem 6.1.2 is equivalent to the following :

6.1.4 Theorem. The following two conditions are equivalent :

(a) $\text{ext}\ \underline{P}(A) = \underline{P}(A) \cap \{0,1\}^J$ ;

(b) $\forall n \in \mathbb{N},\ (j_1,\dots,j_n) \in J^n$ ,

$$\text{ext}\ \underline{P}_{(j_1,\dots,j_n)}(A) = \underline{P}_{(j_1,\dots,j_n)}(A) \cap \{0,1\}^n .$$

In order to be able to use Theorem 6.1.3 to prove Theorem 6.1.4, we equip $\mathbb{R}^J$ with the product topology $\tau$. This topology makes $\mathbb{R}^J$ a locally convex Hausdorff vector space.

6.1.5 Lemma. Suppose A contains no zero-column. Then :

(a) $\underline{P}(A)$ is $\tau$-compact;

(b) if

$$\text{ext } \underline{P}_{(j_1,\ldots,j_n)}(A) = \underline{P}_{(j_1,\ldots,j_n)}(A) \cap \{0,1\}^n ,$$

for all $n \in \mathbb{N}$ and $(j_1,\ldots,j_n) \in J^n$, then $\underline{P}(A)$ is the closed convex hull of $\underline{P}(A) \cap \{0,1\}^J$.

<u>Proof.</u> If A contains no zero-column, then $\underline{P}(A) \subset [0,1]^J$, so to prove (a) we only need to show that $\underline{P}(A)$ is $\tau$-closed. Observe that

$$\underline{P}(A) = \left( \bigcap_{i\in I} \bigcap_{n\in\mathbb{N}} \bigcap_{(j_1,\ldots,j_n)\in J^n} \left\{ (x_j)_{j\in J} \in \mathbb{R}^J : \sum_{k=1}^{n} a_{i,j_k} x_{j_k} \leq 1 \right\} \right)$$

$$\cap \left( \bigcap_{j_o\in J} \left\{ (x_j)_{j\in J} \in \mathbb{R}^J : x_{j_o} \geq 0 \right\} \right),$$

so $\underline{P}(A)$, as the intersection of closed sets, is closed.

To prove (b), consider $x \in \underline{P}(A)$, and let $\mathcal{O}$ be a $\tau$-open set containing x. Without loss of generality, we may suppose that

$$\mathcal{O} = \prod_{j\in J} \mathcal{O}^j ,$$

where $\mathcal{O}^j$ is an open interval and $\mathcal{O}^j \neq \mathbb{R}$ only for j in some finite subset $\{j_1,\ldots,j_n\}$ of J. Note that $(x_{j_1},\ldots,x_{j_n}) \in \underline{P}_{(j_1,\ldots,j_n)}(A)$, and so by hypothesis and Theorem 6.1.3,

$$(x_{j_1},\ldots,x_{j_n}) \in \text{conv } (\underline{P}_{(j_1,\ldots,j_n)}(A) \cap \{0,1\}^n).$$

It follows that $y \in \mathbb{R}^J$ defined by

$$y_j = y_{j_k} \quad \text{for } j = j_k , \quad y_j = 0 \quad \text{if } j \notin \{j_1,\ldots,j_n\}$$

is an element of $\mathcal{O} \cap \text{conv}(\underline{P}(A) \cap \{0,1\}^J)$, and the proof is complete. □

Proof of Theorem 6.1.4.

(a) $\Rightarrow$ (b) : We only need to show that

$$\text{ext}\ \underline{P}_{(j_1,\dots,j_n)}(A) \subset \underline{P}_{(j_1,\dots,j_n)}(A) \cap \{0,1\}^n ,$$

since the converse inclusion is clear. Fix

$$(x_1,\dots,x_n) \in \text{ext}\ \underline{P}_{(j_1,\dots,j_n)}(A).$$

Then $(y_j)_{j\in J}$, defined by $y = x_k$ for $j = j_k$ , $y_j = 0$ if $j \notin \{j_1,\dots,j_n\}$, is an element of ext $\underline{P}(A)$. By (a),

$$(y_j)_{j\in J} \in \underline{P}(A) \cap \{0,1\}^J ,$$

and thus

$$(x_1,\dots,x_n) \in \underline{P}_{(j_1,\dots,j_n)}(A) \cap \{0,1\}^n .$$

This proves (b).

(b) $\Rightarrow$ (a) : Suppose to begin with that A contains no zero-column. Then by Lemma 6.1.5, $\underline{P}(A)$ is the closed convex hull of $\underline{P}(A) \cap \{0,1\}^J$. By Theorem 6.1.3, this implies that ext $\underline{P}(A) \subset \underline{P}(A) \cap \{0,1\}^J$. Since the converse inclusion is trivial, (a) is proven.

Suppose now that A contains some zero-columns. Then coordinates corresponding to these columns will be zero for each extreme point of $\underline{P}(A)$, and so we can reduce to the previous situation. □

## 6.2 Application. Randomized stopping points on a discrete probability space.

Let $(\Omega, \underline{F}, P)$ be a discrete probability space, that is, $\underline{F}$ is generated by a countable partion $(H^i)_{i\in\mathbb{N}}$ of $\Omega$. Without loss of generality, we suppose $P(H^i) > 0$, $\forall i \in \mathbb{N}$ .

Let $(\underline{F}_t)_{t\in\mathbb{N}^2}$ be a two-parameter filtration. Then each $\underline{F}_t$ is also generated by a countable partition of $\Omega$ (since each atom of $\underline{F}_t$ will contain $H^i$ for some $i \in \mathbb{N}$, there cannot be more than countably many).

All the notions of Chapter 4 still make sense; in particular we can speak of the <u>fundamental matrix</u> and the <u>intersection graph</u> of $(\underline{F}_t)_{t\in\mathbb{N}^2}$. It is also not difficult to see that Theorem 4.1.3 remains valid. More importantly, we have the following generalisation of Theorem 4.2.5.

6.2.1 <u>Corollary.</u> ($\underline{F}$ discrete). Let $\underline{H}$ denote the set of all atoms of all the $\underline{F}_t$, $t \in \mathbb{N}^2$. Then $\underline{T} = \text{ext } \underline{U}$ if and only if the following two conditions hold :

(a) $\underline{H}$ has Helly's intersection property (meaning every <u>finite</u> family of pairwise intersecting elements of $\underline{H}$ has a non-empty common intersection);

(b) the intersection graph of $(\underline{F}_t)_{t\in I^2}$ is perfect (in the sense of Definition 6.1.1).

<u>Proof.</u> As in the first part of the proof of Theorem 4.2.5, we can identify the set of randomized stopping points with "polytope"

$$P(\tilde{Q}) = \{\underline{x} : \tilde{Q}\, \underline{x} \leq 1,\ \underline{x} \geq 0\},$$

where $\tilde{Q}$ is a submatrix of the fundamental matrix of $(\underline{F}_t)_{t\in\mathbb{N}^2}$.

By comparing their respective definitions, we see that the derived graph of $\tilde{Q}$ is isomorphic to the intersection graph of $(\underline{F}_t)_{t\in\mathbb{N}^2}$. So in order to conclude the proof, we need only show that (a) above is equivalent to (a) in Theorem 6.1.2. This can be done in the same way as in the proof of Theorem 4.2.5. □

6.2.2 <u>Remark.</u> When $\underline{F}$ is discrete, Corollary 6.2.1 can be used to show that $\underline{T} = \text{ext } \underline{U}$ when $(\underline{F}_t)_{t\in\mathbb{N}^2}$ satisfies Hypothesis CQI. We do not go into details here because a more general result will be proved in Chapter 7, using purely probabilistic methods.

## 6.3 Application. Infinite doubly stochastic matrices and Dilworth's theorem.

Let I be an arbitrary index set, and let $\underline{M}$ denote the set of all doubly stochastic $I \times I$-matrices, that is :

$$\underline{M} = \left\{ (m_{i,j})_{i,j \in I} : m_{i,j} \geq 0 , \sum_{i \in I} m_{i,j} = 1, \sum_{j \in I} m_{i,j} = 1, \ \forall i,j \in I \right\}.$$

6.3.1 Theorem. ext $\underline{M} = \underline{M} \cap \{0,1\}^I$.

When I is finite, this is the well-known Birkhof - von Neuman Theorem (see [Bi], Problem 111), and when $I = \mathbb{N}$, this result was proved by Kendall [K], and a purely algebraic proof was given by J. Isbell [I]. The finite result, together with Theorem 6.1.4, shows that Theorem 6.3.1 is valid for arbitrary I.

6.3.2 Remark. Extensions in a different direction of the Birkhof - von Neuman Theorem have also been considered by several authors, primarily in connection with doubly stochastic measures (see [DS1], [SS], [L]). It seems that the extreme point problem is as yet unsolved in this case. A review of probabilistic aspects in infinite-dimensional convexity theory is contained in [WW].

6.3.3 Theorem. Let $(J, \leq)$ be an ordered set (of arbitrary cardinality). Let $\underline{K}^1$ be the family of all maximal totally ordered subsets of J, and $\underline{K}^2$ the family of all maximal pairwise non-comparable elements of J. Define $A^i = (a^i_{K,j})_{K \in \underline{K}^i, j \in J}$ by $a^i_{K,j} = 1$ if $j \in K$ and $a^i_{K,j} = 0$ otherwise, $i = 1,2$. Then each extremal element of $\underline{P}(A^i)$ is the indicator function of a set of pairwise non-comparable elements of J when $i = 1$, and the indicator function of a totally ordered subset of J when $i = 2$.

In the above, the derived graph of $A^1$ is transitively orientable, hence perfect (see [Go], Theorem 5.3.4), and condition (a) of Theorem 6.1.2 is clearly satisfied since $\underline{K}^1$ contains all maximal totally ordered subsets of J. The derived graph of $A^2$ is a cocomparability graph, hence perfect (see [Go], Theorem 5.3.5, and it is again not difficult to see that condition (a) of Theorem 6.1.2 is satisfied.

## 6.4 Application : non-measurable randomized stopping points.

Let $\Omega = \mathbb{R}^{\mathbb{N}^2}$ be the canonical space for discrete time two-parameter processes, $Y_t : \Omega \longrightarrow \mathbb{R}$ be the $t^{th}$ coordinate mapping, $(\underline{F}_t = \sigma(Y_s, s \leq t))_{t\in\mathbb{N}^2}$ be the canonical filtration, and

$$\underline{F} = \bigvee_{t\in\mathbb{N}^2} \underline{F}_t$$

be the standard $\sigma$-algebra on $\Omega$. Let P be some probability measure on $(\Omega, \underline{F})$, and let $\omega = (\omega_s)_{s\in\mathbb{N}^2}$ denote the generic element of $\Omega$.

In this setting, the set of randomized stopping points should be the set

$$\underline{U} = \Big\{(a_t)_{t\in\mathbb{N}^2} : \quad \begin{array}{l} - a_t : \Omega \longrightarrow \mathbb{R} \text{ is a measurable function that only depends on coordinates } \omega_s,\ s \leq t\ ; \\ - a_t \geq 0 \text{ a.s. ;} \\ - \sum_{t\in\mathbb{N}^2} a_t \leq 1 \text{ a.s.} \end{array} \Big\}.$$

*Consider the set*

$$\underline{V} = \Big\{(f_t)_{t\in\mathbb{N}^2} : \quad \begin{array}{l} - f_t : \Omega \longrightarrow \mathbb{R} \text{ only depends on coordinates } \omega_s, \\ \quad s \leq t\ ; \\ - f_t \geq 0\ ; \\ - \sum_{t\in\mathbb{N}^2} f_t \leq 1 \end{array} \Big\}.$$

(The difference between $\underline{U}$ and $\underline{V}$ is that in $\underline{V}$, the measurability requirement has been dropped, as have been the "a.s".) Requiring that $f_t$ only depend on coordinates $\omega_s$, $s \leq t$, is equivalent to requiring that $f_t$ be constant on each atom of $\underline{F}_t$ (here, the appropriate definition of an atom is that of [DM], I.9).

It is natural by analogy to call elements of $\underline{V}$ <u>"non-measurable randomized stopping points.</u>

6.4.1 <u>Theorem.</u> $(f_t)_{t\in\mathbb{N}^2}$ is an extremal element of $\underline{V}$ if and only if $f_t(\omega)\in\{0,1\}$, $\forall\omega\in\Omega$, $\forall t\in\mathbb{N}^2$.

<u>Proof.</u> Define a matrix $A = (a_{\omega,H})_{\omega\in\Omega,H\in\underline{H}}$ where $\underline{H}$ is the set of all atoms of all the $\underline{F}_t$'s, $t\in\mathbb{N}^2$, and $a_{\omega,H} = 1$ if $\omega\in H$, $a_{\omega,H} = 0$ otherwise. Then $\underline{V}$ can clearly be identified with the polytope $\underline{P}(A)$.

If one considers the matrix $\underline{P}_{(H_1,\dots,H_n)}$ associated with a finite set of atoms $H_1,\dots,H_n$ of $\underline{H}$, it is easy to see that its derived graph is a <u>two-parameter graph</u> (see Definition 4.2.3), thus is perfect, and that all extreme points of $\underline{P}_{(H_1,\dots,H_n)}$ are integer-valued. So the conclusion follows from Theorem 6.1.4. □

One might expect from this result that extremal elements of $\underline{U}$ are $\{0,1\}$-valued for any probability measure P. However, this depends on the null sets of P, which may be so large that $\underline{U}$ and $\underline{V}$ will be very different.

6.4.2 <u>Example.</u> Set

$$\omega^1 = (\omega^1_t = I_{\{(0,2)\}}(t))_{t\in\mathbb{N}^2} \,,$$

$$\omega^2 = (\omega^2_t = I_{\{(1,1)\}}(t))_{t\in\mathbb{N}^2} \,,$$

$$\omega^3 = (\omega^3_t = I_{\{(2,0)\}}(t))_{t\in\mathbb{N}^2} \,,$$

and let P be the probability measure on $\Omega$ defined by $P\{\omega^1\} = P\{\omega^2\} = P\{\omega^3\} = 1/3$. It is then easy to see that for any randomized stopping point $a = (a_t)_{t\in\mathbb{N}^2}$, there are constants $\beta,\beta',\beta'',\gamma,\gamma',\gamma''$, such that

$$a_{(0,2)} = \beta\, I_{\{\omega^1\}} + \gamma\, I_{\{\omega^2,\omega^3\}} \quad \text{P-a.s.}$$

$$a_{(1,1)} = \beta' I_{\{\omega^2\}} + \gamma' I_{\{\omega^1,\omega^3\}} \quad \text{P-a.s.}$$

$$a_{(2,0)} = \beta'' I_{\{\omega^3\}} + \gamma'' I_{\{\omega^1,\omega^2\}} \quad \text{P-a.s.}$$

Since the sets $\{\omega^2,\omega^3\}$, $\{\omega^3,\omega^1\}$ and $\{\omega^1,\omega^2\}$ are pairwise intersecting but have an empty common intersection, they do not satisfy Helly's property. Thus there is a non-integral extremal element in $\underline{\underline{U}}$ (see Theorem 4.2.5). In fact, setting $\beta = \beta' = \beta'' = 0$ and $\gamma = \gamma' = \gamma'' = 1/2$, and $a_t \equiv 0$ if $t \notin \{(0,2), (1,1), (2,0)\}$ gives an extremal element of $\underline{\underline{U}}$. □

Example 6.4.2 illustrates the explicite dependence of the extremal elements of $\underline{\underline{U}}$ on the null sets of P.

6.4.3 Remark. The characterization of the extremal elements of the set of non-measurable randomized stopping points stated in Theorem 6.4.1 extends without difficulty to non-measurable randomized stopping points on other countable index sets, such as $\mathbb{Q}_+^2$ for example. This will not be the case of the probabilistic method of Chapter 7.

CHAPTER 7

# THE CONDITIONAL SUPREMUM OPERATOR AND RANDOMIZED STOPPING POINTS ON $\mathbb{N}^2$

The objective of this chapter is to show that when the index set is $\mathbb{N}^2$ and the two-parameter filtration satisfies Hypothesis CQI, all extremal elements of the set of randomized stopping points are stopping points. We shall adopt a constructive approach which is in fact an extension of that for perfect polytopes associated with a strongly perfectly orderable graph (see 4.7.1). An application of these considerations is an existence result for optimal stopping of discrete-time two-parameter processes taking values in a Banach space.

The construction mentioned above requires the use of a "conditional supremum" operator, which we shall introduce in Section 7.2. This non-linear operator has many properties of conditional expectation, and furthermore, it is invariant under a change of equivalent measure. Hypothesis CQI becomes a commutation property of this operator, in the same way that Hypothesis F4 is the commutation property of conditional expectation operators.

We begin by recalling some elementary properties of conditional probabilities.

## 7.1 Some elementary lemmas.

Throughout this chapter, $\underline{\underline{G}}$ will denote a complete sub $\sigma$-algebra of $\underline{\underline{F}}$. The notation $B \subset C$ a.s (we will often omit the "a.s") will mean $P(B \cap C^c) = 0$.

7.1.1 Lemma.

(a) $B \in \underline{\underline{F}}$ , $C \in \underline{\underline{G}}$ , $B \subset C$ $\Rightarrow$ $\{P(B|\underline{\underline{G}}) > 0\} \subset C$ ;

(b) $B \in \underline{\underline{F}}$ , $X \in \underline{\underline{G}}$ , $I_B \leq X$ $\Rightarrow$ $I_{\{P(B|\underline{\underline{G}})>0\}} \leq X$ .

Proof.

(a) $B \subset C$ $\Rightarrow$ $\{P(B|\underline{\underline{G}}) > 0\} \subset \{P(C|\underline{\underline{G}}) > 0\} = C$.

(b) $I_B \leq X$ implies $X \geq 0$, and $B \subset \{X \geq 1\}$. By (a), $\{P(B \mid \underline{\underline{G}})>0\} \subset \{X \geq 1\}$. □

7.1.2 Lemma. If $X \geq 0$, then

(a) $\{X>0\} \subset \{E(X|\underline{\underline{G}}) > 0\}$ ;

(b) $\{E(X|\underline{\underline{G}}) > 0\} = \{P(X > 0|\underline{\underline{G}}) > 0\}$ .

Proof.

(a) $E(XI_{\{E(X|\underline{\underline{G}})=0\}}) = E(E(X|\underline{\underline{G}})I_{\{E(X|\underline{\underline{G}})=0\}}) = 0$, so $P(X>0, E(X|\underline{\underline{G}})=0)=0$.

(b) $0 \leq E(E(X|\underline{\underline{G}})I_{\{P(X>0|\underline{\underline{G}})=0\}}) = E(XI_{\{P(X>0|\underline{\underline{G}})=0\}}) \leq E(XI_{\{X=0\}}) = 0$,

where the last inequality is a consequence of (a). Thus $\{E(X|\underline{\underline{G}})>0\} \subset \{P(X>0|\underline{\underline{G}})>0\}$. On the other hand, by (a) and Lemma 7.1.1 (a), $\{P(X>0|\underline{\underline{G}})>0\} \subset \{E(X|\underline{\underline{G}})>0\}$, and (b) is proven. □

7.1.3 Lemma. For $B$, $C$, $B_n \in \underline{\underline{F}}$ ,

(a) $\{P(B|\underline{\underline{G}})>0\} \supset B \supset \{P(B^c|\underline{\underline{G}}) = 0\}$ ;

(b) $\{P(B \cup C|\underline{\underline{G}})>0\} = \{P(B|\underline{\underline{G}})>0\} \cup \{P(C|\underline{\underline{G}})>0\}$ ;

(c) $\bigcup_{n \in \mathbb{N}} B_n = B \Rightarrow \bigcup_{n \in \mathbb{N}} \{P(B_n|\underline{\underline{G}})>0\} = \{P(B|\underline{\underline{G}})>0\}$ .

Proof. (a) is a special case of Lemma 7.1.2 (a).

(b) Suppose $B \cap C = \emptyset$. Then $P(B \cup C|\underline{\underline{G}}) = P(B|\underline{\underline{G}}) + P(C|\underline{\underline{G}})$ a.s., so (b) holds in this case. If $B \cap C \neq \emptyset$, write $B \cup C = (B \cap C^c) \cup (B \cap C) \cup (B^c \cap C)$ to get the result.

(c) By (b),

$$\{P(\bigcup_{n \leq m} B_n|\underline{\underline{G}})>0\} = \bigcup_{n \leq m} \{P(B_n|\underline{\underline{G}})>0\} \uparrow \bigcup_{n \in \mathbb{N}} \{P(B_n|\underline{\underline{G}})>0\}$$

as $m \longrightarrow \infty$. But then (c) follows from the fact that

$$P(\bigcup_{n \leq m} B_n|\underline{\underline{G}}) \uparrow P(B|\underline{\underline{G}})$$

as $m \longrightarrow \infty$. □

Note that one cannot replace $\cup$ by $\cap$ in Lemma 7.1.3 (b) or (c).

We shall need the notion of "essential infimum" of a family of random variables (see Neveu [N], § VI. 1). Given a family $(X_i)_{i \in I}$ of $\underline{\underline{F}}$-measurable random variables, there is a unique $\underline{\underline{F}}$-measurable random variable Y, denoted $\operatorname{essinf}_{i \in I} X_i$, that takes values in $\overline{\mathbb{R}}$ and is such that

(a) $Y \leq X_i$, $\forall i \in I$,

(b) $(Z \in \underline{\underline{F}}, Z \leq X_i, \forall i \in I) \Rightarrow Z \leq Y$.

The following is an elementary property of essential infima.

7.1.4 <u>Lemma.</u> Let $(X_i)_{i \in I}$ be a family of random variables, and consider $Z \geq 0$. Then

$$\operatorname{essinf}_{i \in I} (Z X_i) = Z \operatorname{essinf}_{i \in I} X_i .$$

<u>Proof.</u> Since $Z \operatorname{essinf}_{i \in I} X_i \leq Z X_i$, $\forall i \in I$, we have

$$Z \operatorname*{essinf}_{i\in I} X_i \le \operatorname*{essinf}_{i\in I} (Z X_i).$$

On the other hand, following [N], VI. 6.1, let $(i_k)_{k\in\mathbb{N}}$ be a sequence of elements of I such that $\operatorname{essinf}_{i\in I} X_i = \inf_{k\in\mathbb{N}} X_{i_k}$ . Then

$$\operatorname*{essinf}_{i\in I} (ZX_i) \le Z X_{i_k} \ , \quad \forall k \in \mathbb{N} \ ,$$

and so

$$\operatorname*{essinf}_{i\in I} (ZX_i) \le \inf_{k\in\mathbb{N}} (ZX_{i_k}) = Z \inf_{k\in\mathbb{N}} X_{i_k}$$

$$= Z \operatorname*{essinf}_{i\in I} X_i \ .$$

This conclusdes the proof. □

## 7.2 The conditional supremum operator.

7.2.1 Definition. Given a random variable Y, and a $\sigma$-algebra $\underline{\underline{G}}$, the conditional supremum of Y given $\underline{\underline{G}}$ is the random variable with values in $\overline{\mathbb{R}}$, denoted $S(Y|\underline{\underline{G}})$, defined by

$$S(Y|\underline{\underline{G}}) = \operatorname*{essinf}_{Z\in\underline{\underline{G}},\ Z\ge Y} Z \ .$$

Similar to conditional expectation, this (non-linear) operator is defined on the space of equivalence classes of random variables, yielding an equivalence class (we will have no need for the analogous notion of "conditional infimum"). Furthermore, its definition is clearly invariant under a change of equivalent measure. Before examining its properties, let us consider some examples.

7.2.2 Examples.

(a) $Y = I_B$ , $B \in \underline{\underline{F}}$ . Then $S(Y|\underline{\underline{G}}) = I_{\{P(B|\underline{\underline{G}})>0\}}$ (by 7.1.1).

(b) Suppose $\underline{\underline{G}}$ is the σ-algebra of null sets and their complements. Then $S(Y|\underline{\underline{G}}) = \|Y\|_\infty$, $\forall Y \geq 0$.

(c) Let $\Omega' = \{\omega_1',\ldots,\omega_n'\}$, $\Omega'' = \{\omega_1'',\ldots,\omega_m''\}$, $\Omega = \Omega' \times \Omega''$, $\underline{\underline{F}} = \underline{\underline{B}}(\Omega)$, and let P be a probability on Ω such that $P\{(\omega_i',\omega_j'')\} > 0$, $\forall i,j$. Set $G = \underline{\underline{B}}(\Omega') \times \{\emptyset,\Omega''\}$. Then for $Y \in \underline{\underline{F}}$,

$$S(Y|\underline{\underline{G}})(\omega_i,\omega_j') = \sup_{k=1,\ldots,m} Y(\omega_i',\omega_k'') \quad , \quad \forall i,j \ .$$

The following proposition establishes a fundamental relationship between conditional supremum and conditional probability.

7.2.3 Proposition. Given $Y \in \underline{\underline{F}}$ and $k \in \mathbb{R}$,

$$\{S(Y|\underline{\underline{G}})>k\} = \{P(Y>k|\underline{\underline{G}})>0\} \ .$$

Proof. To check "⊂", let $M \in \mathbb{R} \cup \{+\infty\}$ be such that $Y \leq M$. Using Lemma 7.1.3 (a), we see that

$$Y \leq kI_{\{P(Y>k|\underline{\underline{G}})=0\}} + MI_{\{P(Y>k|\underline{\underline{G}})>0\}} \ .$$

Thus $S(Y|\underline{\underline{G}}) \leq k$ a.s. on $\{P(Y>k|\underline{\underline{G}})=0\}$, proving

$$\{S(Y|\underline{\underline{G}})>k\} \subset \{P(Y>k|\underline{\underline{G}})>0\} \ .$$

To see the converse inclusion, consider $Z \in \underline{\underline{G}}$, $Z \geq Y$. For $r > 0$, $\{Y \geq k+r\} \subset \{Z \geq k+r\}$, so by Lemma 7.1.1 (a), $B = \{P(Y \geq k+r|\underline{\underline{G}})>0\} \subset \{Z \geq k+r\}$. It follows by Lemma 7.1.4 that

$$I_B S(Y|\underline{\underline{G}}) = I_B \operatorname*{essinf}_{Z \in G, Z \geq Y} Z = \operatorname*{essinf}_{Z \in G,\ Z \geq Y} (Z\ I_B)$$

$$\geq \operatorname*{essinf}_{Z \in G,\ Z \geq Y} ((k+r)I_B) = (k+r)I_B \ .$$

Thus $S(Y|\underline{\underline{G}}) \geq k+r$ on B, meaning $\{P(Y \geq k+r|\underline{\underline{G}})>0\} \subset \{S(Y|\underline{\underline{G}}) \geq k+r\}$. This implies that

$$\{S(Y|\underline{G})>k\} = \bigcup_{r\in\mathbb{Q}_+} \{S(Y|\underline{G})\geq k+r\} \supset \bigcup_{r\in\mathbb{Q}_+} \{P(Y\geq k+r|\underline{G})>0\}$$

$$= \{P(Y>k|\underline{G})>0\} ,$$

where the last equality follows from Lemma 7.1.3 (c). The proposition is proven. □

In the following lemma, we establish basic properties of the operator $S(\cdot|\underline{G})$.

7.2.4 Lemma.

(a) Measurability — $S(Y|\underline{G}) \in \underline{G}$ ;
— $Y \in \underline{G} \Rightarrow S(Y|\underline{G}) = Y$.

(b) Minimality — $Y \leq S(Y|\underline{G})$ ;
— $Z \in \underline{G},\ Z \geq Y \Rightarrow Z \geq S(Y|\underline{G})$ .

(c) Monotonicity — $Y_1 \leq Y_2 \Rightarrow S(Y_1|\underline{G}) \leq S(Y_2|\underline{G})$ ;
— $\underline{G}_1 \subset \underline{G}_2 \Rightarrow S(Y|\underline{G}_1) \geq S(Y|\underline{G}_2)$ .

(d) Iteration — $\underline{G}_1 \subset \underline{G}_2 \Rightarrow S(S(Y|\underline{G}_2)|\underline{G}_1) = S(Y|\underline{G}_1)$ .

(e) $\underline{G}$-additivity — $X \in \underline{G} \Rightarrow S(X+Y|\underline{G}) = X + S(Y|\underline{G})$ .

(f) $\underline{G}$-multiplicativity — $X \in \underline{G},\ X,Y \geq 0 \Rightarrow S(XY|\underline{G}) = XS(Y|\underline{G})$ .

(g) Monotone convergence — $Y_n \uparrow Y \Rightarrow S(Y_n|G) \uparrow S(Y|\underline{G})$ ;
— $\underline{G}_n \downarrow G \Rightarrow S(Y|\underline{G}_n) \uparrow S(Y|\underline{G})$ .

(h) Stopped filtration — suppose $(\underline{F}_n)_{n\in\mathbb{N}}$ is a standard filtration, and $T : \Omega \longrightarrow \mathbb{N}$ is a stopping time. Then

$$S(Y|\underline{F}_T) = \sum_{n\in\mathbb{N}} S(Y|\underline{F}_n)I_{\{T=n\}} .$$

Proof. (a), (b) and (c) follow immediately from the definition.

(d) Observe that $S(Y|\underline{G}_2) \geq Y$, so by (c), $S(S(Y|\underline{G}_2)|\underline{G}_1) \geq S(Y|\underline{G}_1)$.

On the other hand, $S(Y|\underline{G}_1) \geq Y$ and $S(Y|\underline{G}_1) \in \underline{G}_2$, so by (b), $S(Y|\underline{G}_1) \geq S(Y|\underline{G}_2)$. But then again by (b),

$$S(Y|\underline{G}_1) \geq S(S(Y|\underline{G}_2)|\underline{G}_1) ,$$

and equality follows.

(e) Consider $Z \in \underline{G}$ such that $Z \geq X + Y$. Then $Z - X \geq Y$, $Z - X \in \underline{G}$, so $Z - X \geq S(Y|\underline{G})$, or equivalently $Z \geq X + S(Y|\underline{G})$. Thus $S(X + Y|\underline{G}) \geq X + S(Y|\underline{G})$. Since $X + S(Y|\underline{G})$ is $\underline{G}$-measurable and greater than $X + Y$, (b) implies the desired equality.

(f) Using Proposition 7.2.3, we have for $k \geq 0$ :

$$\begin{aligned}
\{XS(Y|\underline{G}) > k\} &= \bigcup_{r\in\mathbb{Q}_+} (\{X > r\} \cap \{P(Y > \tfrac{k}{r} \mid G) > 0\}) \\
&= \bigcup_{r\in\mathbb{Q}_+} \{I_{\{X>r\}} P(Y > \tfrac{k}{r} \mid G) > 0\} \\
&= \bigcup_{r\in\mathbb{Q}_+} \{P(X > r, Y > \tfrac{k}{r} \mid G) > 0\} \\
&= \{P(\bigcup_{r\in\mathbb{Q}_+} \{X > r, Y > \tfrac{k}{r}\} \mid \underline{G}) > 0\} \\
&= \{S(XY|\underline{G}) > k\} .
\end{aligned}$$

The fourth equality follows from Lemma 7.1.3 (c); (f) is proven.

(g) Consider the first implication. $S(Y_n|\underline{G})$ is increasing in n by (c). Furthermore, $S(Y_n|\underline{G}) \leq S(Y|G)$. Now for all $k \in \mathbb{R}$,

$$\begin{aligned}
\{S(Y|\underline{G}) > k\} = \{P(Y > k \mid \underline{G}) > 0\} &= \bigcup_{n\in\mathbb{N}} \{P(Y_n > k|\underline{G}) > 0\} \\
&= \bigcup_{n\in\mathbb{N}} \{S(Y_n|\underline{G} > k\} ,
\end{aligned}$$

thus proving $S(Y_n|G) \uparrow S(Y|\underline{G})$.

As for the second implication, observe that since

$$P(Y>k|\underline{G}_n) \longrightarrow P(Y>k|\underline{G})$$

as $n \longrightarrow \infty$, we can use Proposition 7.2.3 to get

$$\{S(Y|\underline{\underline{G}})>k\} = \{P(Y>k|\underline{\underline{G}})>0\} \subset \bigcup_{n\in\mathbb{N}} \{P(Y>k|\underline{\underline{G}}_n)>0\}$$

$$= \bigcup_{n\in\mathbb{N}} \{S(Y|\underline{\underline{G}}_n)>k\} .$$

Since by (c), $S(Y|\underline{\underline{G}}) \geq S(Y|\underline{\underline{G}}_n)$, the above inclusion is in fact an equality, and thus $S(Y|\underline{\underline{G}}_n) \uparrow S(Y|\underline{\underline{G}})$.

(h) Observe, using Proposition 7.2.3, that

$$\left\{\sum_{n\in\mathbb{N}} S(Y|\underline{\underline{F}}_n)I_{\{T=n\}}>k\right\} = \bigcup_{n\in\mathbb{N}} \{T=n,\ S(Y|\underline{\underline{F}}_n)>k\}$$

$$= \bigcup_{n\in\mathbb{N}} \{T=n,\ P(Y>k|\underline{\underline{F}}_n)>0\}$$

$$= \bigcup_{n\in\mathbb{N}} \{T=n,\ P(Y>k|\underline{\underline{F}}_T)>0\}$$

$$= \{P(Y>k|\underline{\underline{F}}_T)>0\}$$

$$= \{S(Y|\underline{\underline{F}}_T)>k\} .$$

This proves (h). $\square$

7.2.5 Remark. When $Y_n \downarrow Y$, it is easy to see that $S(Y_n|\underline{\underline{G}})$ does not necessarily decrease to $S(Y|\underline{\underline{G}})$. Similarly, if $\bigcup_n \underline{\underline{G}}_n = \underline{\underline{G}}$, $S(Y|\underline{\underline{G}}_n)$ does not necessarily decrease to $S(Y|\underline{\underline{G}})$. In particular, $S(\cdot|\underline{\underline{G}})$ does not have sufficient regularity to apply to it Monotone Class Theorems ([DM], I. 21) or Dynkin's $\pi$-$\lambda$ Theorem. It is often possible to get around this by using Proposition 7.2.3.

## 7.3 Some equivalent formulations of conditional qualitative independence.

In this section, we show that conditional qualitative independence of two $\sigma$-algebras given a third can be expressed using the conditional supremum operator in the same way that conditional independence is expressed using the conditional expectation operator. Let $\underline{\underline{F}}^1$ and $\underline{\underline{F}}^2$ denote two sub-$\sigma$-algebras of $\underline{\underline{F}}$.

7.3.1 Proposition. The following statements are equivalent :

(a) $CQI(\underline{\underline{F}}^1, \underline{\underline{F}}^2, \underline{\underline{G}})$ holds ;

(b) $X^1 \in \underline{\underline{F}}^1,\ X^2 \in \underline{\underline{F}}^2,\ X^1,\ X^2 \geq 0 \Rightarrow S(X^1X^2|\underline{\underline{G}}) = S(X^1|\underline{\underline{G}})S(X^2|\underline{\underline{G}})$ ;

(c) $X^1 \in \underline{\underline{F}}^1,\ X^2 \in \underline{\underline{F}}^2 \Rightarrow S(X^1+X^2|\underline{\underline{G}}) = S(X^1|\underline{\underline{G}}) + S(X^2|\underline{\underline{G}})$ .

Proof.

(b) ⇒ (a). Set $X^i = I_{B^i}$, where $B^i \in \underline{\underline{F}}^i$. By Example 7.2.2 (a),

$$S(X^i|G) = I_{\{P(B^i|\underline{\underline{G}})>0\}},\ i = 1,2,\ S(X^1X^2|\underline{\underline{G}}) = I_{\{P(B^1\cap B^2|\underline{\underline{G}})>0\}} ,$$

so (b) means in this case that

$$\{P(B^1\cap B^2|\underline{\underline{G}})>0\} = \{P(B^1|\underline{\underline{G}})>0\} \cap \{P(B^2|\underline{\underline{G}})>0\} ,$$

which is the definition of $CQI(F^1, F^2, \underline{\underline{G}})$.

(a) ⇒ (b). Using Proposition 7.2.3, we see that

$$\{S(X^1|\underline{\underline{G}})S(X^2|\underline{\underline{G}}>k\} = \bigcup_{r\in\mathbb{Q}_+} (\{S(X^1|\underline{\underline{G}})>r\} \cap \{S(X^2|\underline{\underline{G}}) > \tfrac{k}{r}\})$$

$$= \bigcup_{r\in\mathbb{Q}_+} (\{P(X^1>r|\underline{\underline{G}})>0\} \cap \{P(X^2> \tfrac{k}{r} \mid G)>0\}).$$

By (a), this is equal to

$$\bigcup_{r\in\mathbb{Q}_+} \{P(X^1>r, X^2 > \tfrac{k}{r} \mid G)>0\} ,$$

which by Lemma 7.1.3 (c) is

$$\{P(X^1X^2>k|\underline{\underline{G}})>0\} = \{S(X^1X^2|\underline{\underline{G}})>k\} .$$

This proves the desired implication.

(c) => (a). Consider $B^i \in F^i$ = 1,2. Then, where $\Delta$ denotes symmetric difference,

$$I_{B^1} + I_{B^2} = 2\, I_{B^1 \cap B^2} + I_{B^1 \Delta B^2}$$

$$\leq 2\, I_{\{P(B^1 \cap B^2|\underline{G})>0\}} + I_{\{P(B^1 \Delta B^2|\underline{G})>0\}}\ .$$

This last expression is $\underline{G}$-measurable, so by (c) and Example 7.2.2 (a),

$$I_{\{P(B^1|\underline{G})>0\}} + I_{\{P(B^2|\underline{G})>0\}} \leq 2\, I_{\{P(B^1 \cap B^2|\underline{G})>0\}} + I_{\{P(B^1 \Delta B^2|\underline{G})>0\}}$$

This inequality implies that

$$\{P(B^1|\underline{G})>0\} \cap \{P(B^2|\underline{G})>0\} \subset \{P(B^1 \cap B^2|\underline{G})>0\}\ ,$$

since elements of the left-hand side of this inclusion give value 2 in the left-hand side of the last inequality above.

Since the converse inclusion always holds, (a) follows.

(a) => (c). Since

$$X^1 + X^2 \leq S(X^1|\underline{G}) + S(X^2|\underline{G}),$$

the inequality

$$S(X^1 + X^2|\underline{G}) \leq S(X^1|\underline{G}) + S(X^2|\underline{G})$$

follows from Lemma 7.2.4 (b). To see the converse inequality, we consider $Z \in \underline{G}$ such that $Z \geq X^1 + X^2$, and show that $Z \geq S(X^1|\underline{G}) + S(X^2|\underline{G})$.

By Proposition 7.2.3,

$$\{S(X^1|\underline{G}) + S(X^2|\underline{G})>Z\} = \bigcup_{k\in\mathbb{Q}} (\{S(X^1|\underline{G}) + S(X^2|\underline{G}) > k\} \cap \{Z \leq k\})$$

$$= \bigcup_{k\in\mathbb{Q}} ( \bigcup_{r\in\mathbb{Q}} (\{S(X^1|\underline{G})>r\} \cap \{S(X^2|\underline{G})>k-r\}) \cap \{Z\leq k\})$$

$$= \bigcup_{k\in\mathbb{Q}} ( \bigcup_{r\in\mathbb{Q}} (\{P(X^1>r|\underline{G})>0\} \cap \{P(X^2>k-r|G)>0\}) \cap \{Z\leq k\}).$$

By (a), this set is equal to

$$\bigcup_{k\in\mathbb{Q}} \left( \left( \bigcup_{r\in\mathbb{Q}} \{P(X^1 > r,\ X^2 > k-r \mid \underline{\underline{G}}) > 0\} \right) \cap \{Z \leq k\} \right),$$

which by Lemma 7.1.3 (c) is equal to

$$\bigcup_{k\in\mathbb{Q}} (\{P(X^1 + X^2 > k \mid \underline{\underline{G}}) > 0\} \cap \{Z \leq k\}).$$

Since $X^1 + X^2 \leq Z$. this last set is included in

$$\bigcup_{k\in\mathbb{Q}} (\{P(Z > k \mid \underline{\underline{G}}) > 0\} \cap \{Z \leq k\}),$$

which, since $Z \in G$, is equal to

$$\bigcup_{k\in\mathbb{Q}} (\{Z > k\} \cap \{Z \leq k\}) = \emptyset ,$$

proving the desired implication. □

7.4 <u>Hypothesis CQI is a commutation property of conditional suprema.</u>

As has been stated in Chapter I, the conditional independence hypothesis F4 of Cairoli and Walsh (see 1.4.1) can be expressed as a commutation property of certain conditional expectations. Given the formal properties of conditional suprema established in the previous sections, it is not surprising that Hypothesis CQI is a commutation property of conditional suprema. The proof of this statement is the object of this section.

We first prove two lemmas.

7.4.1 <u>Lemma.</u>

$$CQI(\underline{\underline{F}}^1, \underline{\underline{F}}^2, \underline{\underline{G}}) \Rightarrow \underline{\underline{F}}^1 \cap \underline{\underline{F}}^2 \subset \underline{\underline{G}} .$$

<u>Proof.</u> Let $B \in \underline{\underline{F}}^1 \cap \underline{\underline{F}}^2$. Then by CQI,

$$\{P(B|\underline{\underline{G}})>0\} \cap \{P(B^c|\underline{\underline{G}})>0\} = \{P(B\cap B^c|\underline{\underline{G}})>0\} = \emptyset ,$$

so by Lemma 7.1.3 (a),

$$B \supset \{P(B^c|\underline{\underline{G}})=0\} \supset \{P(B|\underline{\underline{G}})>0\} \supset B .$$

This implies that $B = \{P(B|\underline{\underline{G}})>0\}$, so $B \in \underline{\underline{G}}$. □

7.4.2 Lemma. Let $(\underline{\underline{F}}_t)_{t\in\mathbb{N}^2}$ be a two-parameter filtration. Then

(a) $s \leq t \Rightarrow S(\cdot|\underline{\underline{F}}_s) \geq S(\cdot|\underline{\underline{F}}_t)$ .

(b) Suppose $(\underline{\underline{F}}_t)_{t\in\mathbb{N}^2}$ satisfies Hypothesis CQI. Then

(b1) $s \underline{\wedge} u \underline{\wedge} t$, $Y \in \underline{\underline{F}}_s \Rightarrow S(Y|\underline{\underline{F}}_u) \leq S(Y|\underline{\underline{F}}_t)$ ;

(b2) suppose $(s_n)_{n\in\mathbb{N}} \subset \mathbb{N}^2$ is such that $s_n \leq s_{n+1}$ and $s_n \underline{\wedge} u \underline{\wedge} t$, $\forall n \in \mathbb{N}$, and $Y = \lim_{n\to\infty} \uparrow Y_n$, where $Y_n \in \underline{\underline{F}}_{s_n}$. Then $S(Y|\underline{\underline{F}}_u) \leq S(Y|\underline{\underline{F}}_t)$.

Proof. (a) is an immediate consequence of Lemma 7.2.4 (c) and the fact that $\underline{\underline{F}}_s \subset \underline{\underline{F}}_t$. To see (b1), observe that

$$\{S(Y|\underline{\underline{F}}_u) > S(Y|\underline{\underline{F}}_t)\} = \bigcup_{k\in\mathbb{Q}} (\{S(Y|\underline{\underline{F}}_u) > k\} \cap \{S(Y|\underline{\underline{F}}_t) \leq k\}).$$

By Proposition 7.2.3 and Lemma 7.1.3 (a), this is included in

$$\bigcup_{k\in\mathbb{Q}} (\{P(Y > k|\underline{\underline{F}}_u) > 0\} \cap \{P(S(Y|\underline{\underline{F}}_t) \leq k|\underline{\underline{F}}_u) > 0\}).$$

Since $Y \in \underline{\underline{F}}_s$, we can use Hypothesis CQI to see that this is equal to

$$\bigcup_{k\in\mathbb{Q}} (\{P(Y > k,\ S(Y|\underline{\underline{F}}_t) \leq k|\underline{\underline{F}}_u) > 0\} = \emptyset ,$$

and (b1) is proven.

As for (b2), it is straightforward consequence of (b1) and Lemma 7.2.4 (g). □

7.4.3 Theorem. A two-parameter filtration $(\underline{F}_t)_{t\in\mathbb{N}^2}$ satisfies Hypothesis CQI if and only if

$$\forall s,\ t\in\mathbb{N}^2,\ s \underline{\wedge} t \Rightarrow \underline{F}_{\min(s,t)} = \underline{F}_s \cap \underline{F}_t \text{ and } S(\cdot|\underline{F}_s|\underline{F}_t) = S(\cdot|\underline{F}_s|\underline{F}_t).$$

Proof. Set $u = \min(s,t)$, and suppose $(\underline{F}_t)_{t\in\mathbb{N}^2}$ satisfies Hypothesis CQI. Then $\underline{F}_u = \underline{F}_s \cap \underline{F}_t$ by Lemma 7.4.1.

To show that $S(\cdot|\underline{F}_s|\underline{F}_t) = S(\cdot|\underline{F}_t|\underline{F}_s)$, it is sufficient by symmetry to show that $S(\cdot|\underline{F}_s|\underline{F}_t) = S(\cdot|\underline{F}_u)$. By Lemma 7.2.4 (d), $S(Y|\underline{F}_u) = S(S(Y|\underline{F}_s)|\underline{F}_u)$. But by Lemma 7.4.2 (a) and (b1), this is equal to $S(S(Y|\underline{F}_s)|\underline{F}_t)$, and the implication is proven.

To see the converse, observe that $S(\cdot|\underline{F}_s|\underline{F}_t) = S(\cdot|\underline{F}_t|\underline{F}_s)$ implies $S(Y|\underline{F}_s|\underline{F}_t) \in \underline{F}_s \cap \underline{F}_t = \underline{F}_u$ by hypothesis. Since by Lemma 7.2.4 (b) and (c), $Y \leq S(Y|\underline{F}_t) \leq S(S(Y|\underline{F}_s)|\underline{F}_t)$, we get $S(Y|\underline{F}_u) \leq S(Y|\underline{F}_s|\underline{F}_t)$. Furthermore, $S(Y|\underline{F}_u) \geq Y$, so by Lemma 7.2.4 (b), $S(Y|\underline{F}_u) \geq S(Y|\underline{F}_s|\underline{F}_t)$, proving $S(Y|\underline{F}_u) = S(Y|\underline{F}_s|\underline{F}_t)$.

We now show that $CQI(\underline{F}_s, \underline{F}_t, \underline{F}_u)$ holds by checking the equivalent form 7.3.1 (b). Fix $X^1 \in \underline{F}_s$, $X^2 \in \underline{F}_t$, $X^1$, $X^2 \geq 0$. Then by Lemma 7.2.4 (f),

$$S(X^1X^2|\underline{F}_u) = S(X^1X^2|\underline{F}_s|\underline{F}_t) = S(X^1S(X^2|\underline{F}_s)|\underline{F}_t).$$

Since by Lemma 7.4.2 (b1), $S(X^2|\underline{F}_s) = S(X^2|\underline{F}_u)$, we have

$$S(X^1X^2|\underline{F}_u) = S(X^1|\underline{F}_t)\ S(X^2|\underline{F}_u) = S(X^1|\underline{F}_u)\ S(X^2|\underline{F}_u)\ ,$$

by Lemma 7.2.4 (f) and 7.4.2 (b1). □

## 7.5 A constructive algorithm for randomized stopping points on $\mathbb{N}^2$.

We shall suppose throughout this section that $(\underline{F}_t)_{t\in\mathbb{N}^2}$ is a two-parameter filtration on some (complete) probability space $(\Omega, \underline{F}, P)$, that satisfies Hypothesis CQI.

Let $a = (a_t)_{t\in\mathbb{N}^2}$ be a randomized stopping point. The object of this section is to construct two randomized stopping points $a^i = (a^i_t)_{t\in\mathbb{N}^2}$, $i = 1,2$, such that

(1) $a = \frac{1}{2} a^1 + \frac{1}{2} a^2$

and

(2) $P\{\omega\in\Omega : \exists t\in\mathbb{N}^2 \text{ with } 0 < a_t(\omega) < 1\} > 0 \Rightarrow a^1 \neq a \neq a^2$.

The randomized stopping points $a^1$ and $a^2$ will be defined by induction, in a manner similar to that used for perfect polytopes in Chapter 4.7. Though both constructions rely on the same intuition, the one we give here is purely probabilistic and can be read independently of the former.

Recall that $\preceq$ denotes the lexicographic order on $\mathbb{N}^2$, and the notation $s \prec t$ means $s \preceq t$ and $s \neq t$.

7.5.1 <u>The algorithm.</u> We define successively, in increasing order for $\preceq$, two processes $(a^1_t)_{t\in\mathbb{N}^2}$ and $(a^2_t)_{t\in\mathbb{N}^2}$ (we will show further on that these processes are randomized stopping points), and a process $(C_t)_{t\in\mathbb{N}^2}$, in the following manner. Set

(3) $C_{0,0} \equiv 0$.

Then use formulas (4) below with $t = (0,0)$ to define $a^1_{0,0}$ and $a^2_{0,0}$.

Now suppose by induction that $a^1_s$, $a^2_s$ and $C_s$ have been defined for $s \prec t$. Then use formula (5) to define $C_t$, and then formulas (4) to define $a^1_t$ and $a^2_t$.

This defines the processes $(a^i_t)_{t\in\mathbb{N}^2}$, $i = 1,2$, and $(C_t)_{t\in\mathbb{N}^2}$.

(4)
(a) $a^1_t = 2a_t \quad , a^2_t = 0 \qquad \text{if } C_t + 2a_t < 1$ ;

(b) $a^1_t = 1 - C_t \quad , a^2_t = 2a_t - 1 + C_t \quad \text{if } C_t < 1,\ C_t + 2a_t \geq 1$ ;

(c) $a^1_t = 0 \quad , a^2_t = 2a_t \qquad \text{if } C_t \geq 1$ ;

(5) $$C_t = S\left(\sum_{s<t} a_s^1 \mid \underline{\underline{F}}_t\right). \quad \square$$

Observe that $a_t^i$ is $\underline{\underline{F}}_t$-measurable, $a_t^i \geq 0$, $i = 1,2$, $C_t \geq 0$ and

$$a_t = \frac{1}{2} a_t^1 + \frac{1}{2} a_t^2 , \quad \forall t \in \mathbb{N}^2 .$$

The following lemma shows that when the third case in (4) holds, $C_t = 1$.

7.5.2 Lemma. $C_t \leq 1, \quad \forall t \in \mathbb{N}^2$.

Proof. By (3), $C_{0,0} \leq 1$. So we suppose by induction that $C_t \leq 1$, $t = (t_1,t_2)$, and show that $C_{t_1,t_2+1} \leq 1$. Using Lemma 7.2.4 (e) gives

$$C_{t_1,t_2+1} = S\left(\sum_{s\leq t} a_s^1 \mid \underline{\underline{F}}_{t_1,t_2+1}\right) = S\left(\sum_{s<t} a_s^1 + a_t^1 \mid \underline{\underline{F}}_{t_1,t_2+1}\right)$$

$$= a_t^1 + S\left(\sum_{s<t} a_s^1 \mid \underline{\underline{F}}_{t_1,t_2+1}\right)$$

Using Lemma 7.4.2 (a), we see that this expression is less than

$$a_t^1 + S\left(\sum_{s<t} a_s^1 \mid \underline{\underline{F}}_t\right) = a_t^1 + C_t .$$

Checking the three cases in (4) shows that this last expression is less than 1, since $C_t \leq 1$.

We now suppose that $C_{t_1,t_2} \leq 1$, for all $t_2 \in \mathbb{N}$, and show that $C_{t_1+1,0} \leq 1$. Now by Lemma 7.2.4 (b), $C_{t_1,t_2} \leq 1$ implies that

$$\sum_{s<(t_1,t_2)} a_s^1 \leq 1.$$

Since this holds for all $t_2$,

$$\sum_{s<(t_1+1,0)} a_s^1 \leq 1,$$

implying $C_{t_1+1,0} \leq 1$ by definition of C. This concludes the proof. □

In order to prove that $a^i = (a^i_t)_{t\in\mathbb{N}^2}$, $i = 1,2$, defined by Algorithm 7.5.1, are randomized stopping points, it only remains to be shown that $\Sigma_{s\in\mathbb{N}^2}\ a^i_s \leq 1$ a.s. For $i = 1$, this follows easily from Lemma 7.5.2, and most of the remainder of this section will be devoted to proving this inequality for $i = 2$.

Before doing this, we examine the relationship between the supports of $a^1$, $a^2$ and a. In fact, we show that there is an unbounded optional increasing path $Z = (Z_p)_{p\in\mathbb{N}}$ (see Definition 3.1.1) such that the support of $a^1$ is to the left of Z, whereas the support of $a^2$ will be to its right. The fact that this path can be built in an optional way will be a direct consequence of Hypothesis CQI. This construction will be used in Chapter 8.

Let us define the following sets :

$$U_t = \{C_t + 2a_t \geq 1\},\quad V_t = \bigcup_{k\geq 1} U_{t_1,t_2+k},\quad t\in\mathbb{N}^2 ,$$

and, by induction, an optional increasing path $Z = (Z_p)_{p\in\mathbb{N}}$ such that $Z_0 \equiv (0,0)$ and

$$(6)\qquad Z_{p+1} = \begin{cases} (t_1,t_2+1) & \text{on } \{P(V_t|\underline{F}_t) > 0\}\cap\{Z_p = t\} \\[2ex] (t_1+1,t_2) & \text{on } \{P(V_t|\underline{F}_t) = 0\}\cap\{Z_p = t\} \end{cases} ,\quad p\geq 0 .$$

For $s\in\mathbb{N}^2$, set $R_s = \{t\in\mathbb{N}^2 : t\leq s\}$, $R^-_s = \{t\in\mathbb{N}^2 : t_1 < s_1,\ t_2 > s_2\}$, $R^+_s = \{t\in\mathbb{N}^2 : t_1 > s_1,\ t_2 \leq s_2\}$ (note the lack of symmetry in the definitions of $R^-_s$ and $R^+_s$).

7.5.3 <u>Lemma.</u> If $t\in R^+_s$, then

(a) $U_s \cap \{C_t < 1\} = \emptyset$ ;

(b) $\{P(V_s|\underline{F}_s) > 0\} \cap \{C_t < 1\} = \emptyset$ .

Proof. Concerning (a), observe that by Lemma 7.5.2,

$$1 \geq C_t = S\left(\sum_{u<t} a_u^1 \mid \underline{F}_t\right) \geq S\left(\sum_{u\leq s} a_u^1 \mid \underline{F}_t\right).$$

By Lemma 7.4.2 (b2), the last expression is not less than $S(\Sigma_{u\leq s}\, a_u^1 | \underline{F}_s)$. Since on $U_s$, we are in either of the last two cases in (4), Lemma 7.2.4 (e) implies that this quantity is equal to

$$a_s^1 + S\left(\sum_{u<s} a_u^1 \mid \underline{F}_s\right) = a_s^1 + C_s = 1 \quad \text{on } U_s .$$

This proves that $C_t = 1$ on $U_s$, and (a) follows.

As for (b), observe that (a) implies that $U_{s_1,s_2+k} \cap \{C_t < 1\} = \emptyset$, for all $k \in \mathbb{N}$, so by CQI,

$$\{P(U_{s_1,s_2+k} \mid \underline{F}_s) > 0\} \cap \{C_t < 1\} = \emptyset , \quad \forall k \in \mathbb{N} .$$

Taking the union over $k \in \mathbb{N}$ of these sets and using Lemma 7.1.3 (c) and the definition of $V_s$, we get the desired result. □

7.5.4 Proposition. For $t \in \mathbb{N}^2$, set $L_t = \{\omega : t \wedge Z_{|t|}(\omega)\}$, $M_t = \{\omega : Z_{|t|}(\omega) = t\}$ and $N_t = \{\omega : Z_{|t|}(\omega) \wedge t\}$. Then $\{L_t, M_t, N_t\}$ is a partition of $\Omega$, and

(a) $a_t^1 = 2a_t$ , $\quad a_t^2 = 0 \quad$ on $L_t$ ;

(b) $a_t^1 = 0$ , $\quad a_t^2 = 2a_t \quad$ on $N_t$ .

Proof. For almost all $\omega \in L_t$, there is $s \in \mathbb{N}^2$ such that $s_1 = t_1$, $s_2 < t_2$, $Z_{|s|}(\omega) = s$, and $Z_{|s|+1}(\omega) = (s_1 + 1, s_2)$. So by the definition of Z (see (6)), $\omega \in \{P(V_s|\underline{F}_s) = 0\}$. Thus $C_t + 2a_t < 1$ on $L_t$ by the definition of $V_s$, implying (a) by the definition of $a^1$ and $a^2$ (see (4)).

Now for almost all $\omega \in N_t$, there is $s \in \mathbb{N}^2$ such that $s_1 < t_1$, $s_2 = t_2$, $Z_{|s|}(\omega) = s$ and $Z_{|s|+1}(\omega) = (s_1, s_2 + 1)$. So by the definition of Z,

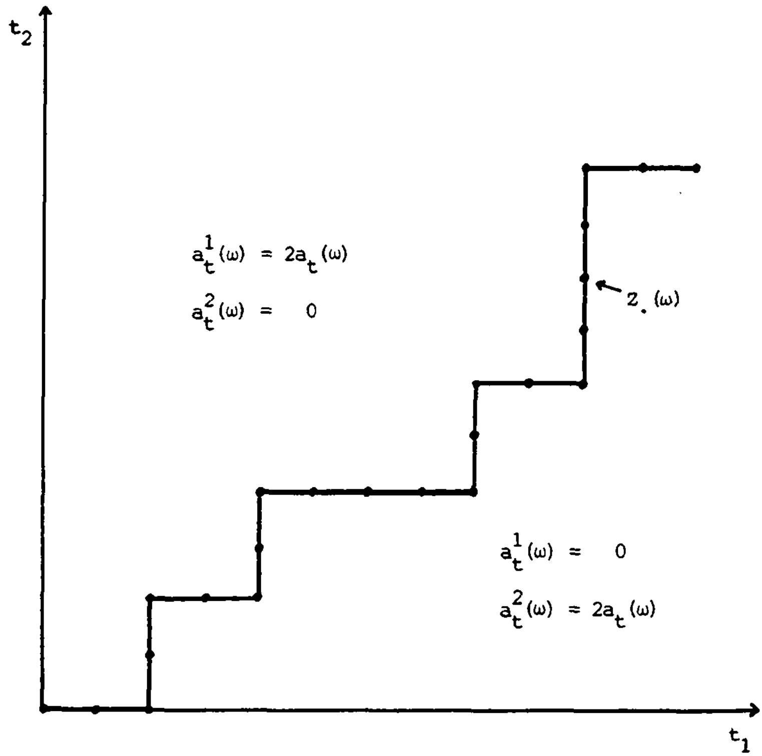

$t_2$
$a_t^1(\omega) = 2a_t(\omega)$
$a_t^2(\omega) = 0$
$Z_\cdot(\omega)$
$a_t^1(\omega) = 0$
$a_t^2(\omega) = 2a_t(\omega)$
$t_1$

Figure 14.

$\omega \in \{P(V_s | \underline{F}_s) > 0\}$. Thus, by Lemma 7.5.3 (b), $C_t(\omega) = 1$, and this implies (b), again by the definition of $a^1$ and $a^2$. □

The above proposition shows that the support of $a^1_\cdot$ is on the left of the path $p \longrightarrow Z_p$, whereas the support of $a^2_\cdot$ is to the right (see Figure 14). On the path itself, all three cases in (4) may occur.

7.5.5 Theorem. $a^1$ and $a^2$ defined by Algorithm 7.5.1 are randomized stopping points.

Proof. We have already observed that $a^i_t \geq 0$, $a^i_t \in \underline{F}_t$, $\forall t \in \mathbb{N}^2$, $i = 1,2$, and that $\Sigma_{s<t}\ a^1_s \leq C_t \leq 1$ by Lemma 7.5.2, for all $t \in \mathbb{N}^2$, so $\Sigma_{s\in\mathbb{N}^2}\ a^1_s \leq 1$, and thus $a^1$ is a randomized stopping point. So it only remains to be shown that

$$(7) \qquad \sum_{s \in \mathbb{N}^2} a^2_s \leq 1 .$$

Suppose we have proved the following inequality for all $t \in \mathbb{N}^2$ :

$$(8) \qquad \sum_{s \in R_t \cup R^+_t} a^2_s \leq 1 \quad \text{on } U_t .$$

Then there would be a null set N such that

$$\omega \in U_t \setminus N \implies \sum_{s \in R_t \cup R^+_t} a^2_s(\omega) \leq 1 , \quad \forall t \in \mathbb{N}^2 .$$

Set $U(\omega) = \{t : \omega \in U_t\}$. By the definition of $a^2$, $U(\omega) \supset \{t : a^2_t(\omega) > 0\}$, so

$$\sum_{s \in \mathbb{N}^2} a^2_s(\omega) = \sum_{s \in U(\omega)} a^2_s(\omega) = \sup_{t \in U(\omega)} \sum_{s \in R_t \cup R^+_t} a^2_s(\omega) \leq 1, \ \omega \in \Omega \setminus N.$$

So we only need to prove (8).

To this end, observe that since a is a randomized stopping point,

$$\sum_{s \in R_t} 2a_s + \sum_{s \in R_t^-} 2a_s + \sum_{s \in R_t^+} 2a_s \leq 2 .$$

Using Lemma 7.2.4 (g) to approach the last two sums on the left hand side by finite sums, we apply CQI in the equivalent form of Proposition 7.3.1 (c) to get

$$\sum_{s \in R_t} 2a_s + S\left(\sum_{s \in R_t^-} 2a_s \middle| \underline{\underline{F}}_t\right) + S\left(\sum_{s \in R_t^+} 2a_s \middle| \underline{\underline{F}}_t\right) \leq 2 .$$

Since $a_t^i \leq 2a_t$ , $\forall t \in \mathbb{N}^2$ , $i = 1,2$, this implies

$$\text{(9)} \qquad \sum_{s \in R_t} 2a_s + S\left(\sum_{s \in R_t^-} a_s^1 \middle| \underline{\underline{F}}_t\right) + S\left(\sum_{s \in R_t^+} a_s^2 \middle| \underline{\underline{F}}_t\right) \leq 2 .$$

On the other hand, $C_t + a_t^1 = 1$ on $U_t$ (since we are in the second or third case in (4)), so

$$\text{(10)} \qquad \sum_{s \in R_t} a_s^1 + S\left(\sum_{s \in R_t^-} a_s^1 \middle| \underline{\underline{F}}_t\right) = 1 \quad \text{on } U_t .$$

Subtracting (10) from (9) gives

$$\sum_{s \in R_t} (2a_s - a_s^1) + S\left(\sum_{s \in R_t^+} a_s^2 \middle| \underline{\underline{F}}_t\right) \leq 1 \quad \text{on } U_t .$$

Since $\frac{1}{2}(a_s^1 + a_s^2) = a_s$ , $\forall s \in \mathbb{N}^2$, this implies that

$$\sum_{s \in R_t} a_s^2 + S\left(\sum_{s \in R_t^+} a_s^2 \middle| \underline{\underline{F}}_t\right) \leq 1 \quad \text{on } U_t ,$$

and (8) follows. □

7.5.6 <u>Theorem.</u> The randomized stopping points $a^1$ and $a^2$ defined by Algorithm 7.5.1 satisfy (2), that is

$$P\{\omega \in \Omega : \exists t \in \mathbb{N}^2 \text{ with } 0 < a_t(\omega) > 1\} > 0 \implies a^1 \neq a \neq a^2 .$$

Proof. Suppose $P\{\omega \in \Omega : \exists t \in \mathbb{N}^2 \text{ with } 0 < a_t(\omega) < 1\} > 0$, and let $t^o \in \mathbb{N}^2$ be the smallest $t \in \mathbb{N}^2$ (with respect to $\preceq$) such that $P(0 < a_t < 1) > 0$.

For all $s \in \mathbb{N}^2$ such that $s \prec t^o$, there is $B_s \in \underline{F}_s$ such that $a_s = I_{B_s}$. Let $B = \{0 < a_{t^o} < 1\}$. Since $\Sigma_{t \in \mathbb{N}^2}\ a_t \leq 1$, $B_s \subset B^c$, implying by Lemma 7.1.1 (a) that $\{P(B_s | \underline{F}_{t^o}) > 0\} \cap B = \emptyset$ . Using Lemma 7.1.3 (c), we get

$$(11) \quad \{P(\bigcup_{s \prec t^o} B_s | \underline{F}_{t^o}) > 0\} \cap B = \emptyset .$$

Observe that

$$\bigcup_{s \prec t^o} B_s = \left\{\sum_{s \prec t^o} a_s = 1\right\} = \left\{\sum_{s \prec t^o} a_s > 0\right\} ,$$

so by Proposition 7.2.3, (11) is equivalent to

$$\left\{S\left(\sum_{s \prec t^o} a_s | \underline{F}_{t^o}\right) > 0\right\} \cap B = \emptyset .$$

Thus $C_{t^o} = 0$ on B, and so

$$a^1_{t^o} = \min(2a_{t^o} , 1) \neq a_{t^o} \neq \max(2a_{t^o} - 1, 0) = a^2_{t^o}$$

on B by (4). Since $P(B) > 0$, we are done. □

## 7.6 Solution to the extremal question on $\mathbb{N}^2$.

The following theorem is a direct consequence of the results of the preceeding section.

7.6.1 Theorem. Let $(\Omega, \underline{F}, P)$ be a complete probability space, and $(\underline{F}_t)_{t \in \mathbb{N}^2}$ be a two-parameter filtration that satisfies Hypothesis CQI. Then all extremal elements of the set of randomized stopping points on $\mathbb{N}^2$

are stopping points.

Proof. If $(a_t)_{t\in\mathbb{N}^2}$ is a randomized stopping point which is not a stopping point, then

$$P\{\omega\in\Omega : \exists t\in\mathbb{N}^2 \text{ with } 0<a_t(\omega)<1\}>0 .$$

By Theorem 7.5.5, we can construct, using Algorithm 7.5.1, two randomized stopping points $a^1$ and $a^2$ such that

$$a=\frac{1}{2}a^1+\frac{1}{2}a^2 .$$

By Theorem 7.5.6, $a^1\neq a\neq a^2$ , and so a is not an extremal element of the set of randomized stopping points. □

## 7.7 Application : an optimal stopping problem for processes with values in a Banach space.

Let B be a Banach space with norm $\|\cdot\|$, and suppose $(X_t)_{t\in\overline{\mathbb{N}^2}}$ is a B-valued process defined on $(\Omega, \underline{F}, P)$. Given a two-parameter filtration $(\underline{F}_t)_{t\in\mathbb{N}^2}$ , is there a stopping point $T_o$ which would maximize a (convex) function $\varphi$ of the expected value $E(X_T)$, i.e.

$$\varphi(E(X_{T_o}))=\sup_T \varphi(E(X_T)) \text{ ?}$$

Using results of the preceeding sections, we will show that the answer is affirmative when the filtration satisfies Hypothesis CQI and the process is continuous at infinity. We do not require $(X_t)$ to be adapted to $(\underline{F}_t)_{t\in\mathbb{N}^2}$.

The corresponding problem for real-valued processes was solved by Mazziotto and Szpirglas [MS], using Snell's enveloppe. As pointed out by Edgar, Millet and Sucheston [EMS], who studied the single-parameter B-valued case both in discrete and continuous time, there is no analogue of Snell methods in this case.

Our proof follows that of [EMS]. However, we remove an unnecessary separability assumption on the σ-algebra $\underline{F}$, and replace a cumbersome "de-randomization" argument with a more direct approach. Furthermore, we only require φ to be upper-semicontinuous instead of continuous.

7.7.1 **Lemma.** Suppose Y is a B-valued Bochner integrable random variable, and $f : \overline{\mathbb{N}^2} \to \mathbb{R}$ is continuous. Then the function

$$(a_t)_{t\in\overline{\mathbb{N}^2}} \longmapsto E\left(Y \sum_{t\in\overline{\mathbb{N}^2}} f(t)a_t\right)$$

from $\underline{U} \to B$ is continuous (for the weak topology on $\underline{U}$ and the norm topology on B).

Proof. Suppose to begin with that Y is simple, that is

$$Y = \sum_{i=1}^{k} x_i \, I_{A_i} \;, \; x_i \in B.$$

Then

$$E\left(Y \sum_{t\in\overline{\mathbb{N}^2}} f(t)a_t\right) = \sum_{i=1}^{k} x_i \, E\left(I_{A_i} \sum_{t\in\overline{\mathbb{N}^2}} f(t)a_t\right) .$$

Since $(t,\omega) \mapsto I_{A_i}(\omega)\, f(t)$ is a continuous real-valued process, the function

$$(a_t)_{t\in\mathbb{N}^2} \longmapsto E\left(Y \sum_{t\in\overline{\mathbb{N}^2}} f(t)a_t\right)$$

is continuous.

Suppose now that Y is arbitrary Bochner integrable. Fix $\varepsilon > 0$ and let $(a_t)_{t\in\overline{\mathbb{N}^2}}$ be a randomized stopping point. Since simple random variables are dense in $L^1$-norm, let Z be a simple random variable such that

$$E(\|Y-Z\|) < \frac{\varepsilon}{3\|f\|_\infty}$$

(if $\|f\|_\infty = 0$, the statement of the lemma is trivial). Set

$$\psi(Y, (a_t)) = E\left(Y \sum_{t\in\overline{\mathbb{N}^2}} f(t)a_t\right),$$

and let V be the open set of randomized stopping points $(a'_t)$ such that

$$\|\psi(Z, (a'_t)) - \psi(Z, (a_t))\| < \frac{\varepsilon}{3}.$$

Then if $(a'_t) \in V$,

$$\|\psi(Y, (a'_t)) - \psi(Y, (a_t))\| \leq \|\psi(Y, (a'_t)) - \psi(Z, (a'_t))\|$$

$$+ \|\psi(Z, (a'_t)) - \psi(Z, (a_t))\| + \|\psi(Z, (a_t)) - \psi(Y, (a_t))\|$$

$$\leq E\left(\|(Y-Z) \sum_{t\in\overline{\mathbb{N}^2}} f(t)\ a'_t\|\right) + \frac{\varepsilon}{3} + E\left(\|(Z-Y) \sum_{t\in\overline{\mathbb{N}^2}} f(t)\ a_t\|\right)$$

The first and third terms in the last expression are bounded by $E(\|Y-Z\|)\|f\|_\infty < \frac{\varepsilon}{3}$. This shows that

$$(a_t) \longmapsto \psi(Y, (a_t))$$

is continuous, and so we are done. □

7.7.2 Lemma. Suppose $(X_t)_{t\in\overline{\mathbb{N}^2}}$ is a B-valued process such that

(11) $\quad E(\sup_{t\in\overline{\mathbb{N}^2}} \|X_t\|) < +\infty$;

(12) $\quad \lim_{t\to\infty} E(\sup_{s\in\overline{\mathbb{N}^2}\setminus R_t} \|X_t - X_\infty\|) = 0.$

Then

$$(a_t)_{t\in\overline{\mathbb{N}^2}} \longmapsto E\left(\sum_{t\in\overline{\mathbb{N}^2}} X_t\ a_t\right)$$

from $\underline{\underline{U}}$ into B is continuous (for the weak topology on $\underline{\underline{U}}$ and the norm topology on B).

Proof. Observe that

$$(13)\quad \Big\|E\Big(\sum_{s\in\overline{\mathbb{N}^2}} X_s\, a_s\Big) - E\Big(\sum_{s\in\overline{\mathbb{N}^2}} X_s\, a'_s\Big)\Big\| \le \Big\|E\Big(\sum_{s\in R_t} X_s\, a_s\Big) - E\Big(\sum_{s\in R_t} X_s\, a'_s\Big)\Big\|$$

$$+ \Big\|E\Big(\sum_{s\in R_t^c} X_\infty\, a_s\Big) - E\Big(\sum_{s\in R_t^c} X_\infty\, a'_s\Big)\Big\| + \Big\|E\Big(\sum_{s\in R_t^c} (X_s - X_\infty) a_t\Big)\Big\|$$

$$+ \Big\|E\Big(\sum_{s\in R_t^c} (X_s - X_\infty) a'_s\Big)\Big\| .$$

The sum of the last two terms is bounded by

$$2\, E\big( \sup_{s\in\overline{\mathbb{N}^2}\setminus R_t} \|X_s - X_\infty\| \big) .$$

By (12), we can choose t so that this quantity is less than $\varepsilon/2$.

Since

$$\sum_{s\in R_t} X_s\, a_s = \sum_{s\in R_t} X_s \sum_{u\in\mathbb{N}^2} f_s(u)\, a_u ,$$

where $f_s(u) = I_{\{s\}}(u)$ is a continuous function from $\overline{\mathbb{N}^2}$ into $\mathbb{R}$, there is, by Lemma 7.7.1, a weakly open set V such that the sum of the first two terms in the right-hand side of (13) is less than $\frac{\varepsilon}{2}$, whenever $(a'_t) \in V$. This proves the lemma. □

7.7.3 Theorem. Suppose $(\Omega, \underline{F}, P)$ is a complete probability space, and $(\underline{F}_t)_{t\in\mathbb{N}^2}$ is a two-parameter filtration that satisfies Hypothesis CQI. Let $(X_t)_{t\in\overline{\mathbb{N}^2}}$ be a process with values in a Banach space B, such that (11) and (12) hold, and let $\varphi : \{y\in B : \|y\| \le E(\sup_{t\in\mathbb{N}^2}\|X_t\|)\} \longrightarrow \mathbb{R}$ be upper-semi-continuous and convex. Then there is a stopping point $T_o$ such that

$$\varphi(E(X_{T_o})) = \sup_{T\in\underline{T}} \varphi(E(X_T)).$$

<u>Proof.</u> The function $\varphi$ is only defined on a ball in B, but since

$$\left\| E\left( \sum_{t\in\overline{\mathbb{N}^2}} X_t\, a_t \right) \right\| \leq E\left( \sup_{t\in\overline{\mathbb{N}^2}} \|X_t\| \right),$$

we can define a map $\Phi$ from $\underline{\underline{U}}$ into $\mathbb{R}$ by

$$(a_t)_{t\in\overline{\mathbb{N}^2}} \longmapsto \varphi\left( E\left( \sum_{t\in\overline{\mathbb{N}^2}} X_t\, a_t \right) \right).$$

This map is weakly upper-semicontinuous by Lemma 7.7.2, is convex, and is an extension of

$$T \longmapsto \varphi(E(X_T)).$$

The map $\Phi$ attains its maximum at some extremal randomized stopping point ([B], II. §7, Prop. 1). By Theorem 7.6.1, this is a stopping point, which is clearly optimal. □

CHAPTER 8

# INFINITESIMAL METHODS IN THE THEORY OF TWO-PARAMETER PROCESSES

The objective of this section is to extend the extremal result of the preceeding section (Theorem 7.6.1) to randomized stopping points on $\overline{\mathbb{R}_+^2}$, or equivalently on $[0,1]^2$. A natural idea is to build a path with properties similar to those of the path defined in (6) of Chapter 7.5. There, since the parameter set was discrete, the path was constructed by a step by step procedure.

In the continuous case, one would imagine that the path $(Z_u)_{u\in\mathbb{R}_+}$ could be defined as the solution of a (random) differential equation of the form

$$(1)\qquad \frac{dZ_u}{du}(\omega) = f(u,(Z_v)_{v\leq u}, \omega).$$

However, no regularity is to be expected from the function $f(.,.,w)$, and in particular, $f(u,.,w)$ will not be Lipschitz. Now certain stochastic differential equations with non-Lipschitz continuous coefficients are known not to have any (strong) solution (see Barlow [Ba]), and so it is improbable that (1) would have a solution in any useful sense. On the other hand, Keisler ([Kl]), Theorems 5.2 and 5.5) has shown under minimal regularity assumptions that stochastic differential equations have a (strong) solution when the probability space is hyperfinite.

In view of these considerations, it seems natural to study Question 1.8.2 when the probability space is hyperfinite. In order to do this, we define a hyperfinite filtered two-parameter probability space, which is the natural extension of that used in the single-parameter theory (see for instance the book by Stroyan and Bayod [SB]). The two-parameter filtration satisfies Hypothesis F4 for the Loeb measure on $\Omega$. It is then possible, using the Transfer Principle (see [K2], [HL] and [SL] for general results in non-standard analysis), to extend the discrete construction of Chapter 7 to randomized stopping points on $[0,1]^2$.

## 8.1 Preliminaries on non-standard analysis.

This introduction follows that of [K1]. Let $\beta(A)$ denote the power set of all subsets of a set A. Given a set S, the superstructure $V(S)$ over S is

$$V(S) = \bigcup_{n\in\mathbb{N}} V_n(S) ,$$

where

$$V_0(S) = S , \quad V_{n+1}(S) = V_n(S) \cup \beta(V_n(S)) , \quad n \geq 0.$$

Most of classical Analysis can be done within $V(\mathbb{R})$. For example, $\mathbb{R}^2 \in V_3(\mathbb{R})$, the function $\sin(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ belongs to $V_3(\mathbb{R})$, the set $C([0,1])$ of continuous real-valued functions belongs to $V_4(\mathbb{R})$, the topology of uniform convergence on $C([0,1])$ belongs to $V_5(\mathbb{R})$ (see [SB], Section 0.1).

The starting point to non-standard analysis is Robinson's construction of an ordered field extension ${}^*\mathbb{R}$ of $\mathbb{R}$ called the set of hyperreal numbers, and a mapping

$$* : V(\mathbb{R}) \longrightarrow V({}^*\mathbb{R})$$

with three basic properties. To state these properties, we need two notions from mathematical logic.

An elementary statement is a statement $\mathcal{s}$ built up from the relations $=$ and $\in$, the connectives "and", "or", "not", and bounded quantifiers $(\forall u \in v)$, $(\exists u \in v)$.

An internal object is an element of the set

$$\bigcup_{S \in V(\mathbb{R})} {}^*S .$$

Thus the set of internal objects is a subset of $V({}^*\mathbb{R})$. An object in $V({}^*\mathbb{R})$ which is not internal is called <u>external</u>.

We can now state the three basic properties of $*$ and ${}^*\mathbb{R}$.

8.1.1 <u>Extension Principle.</u> ${}^*\mathbb{R}$ is a proper ordered field extension of $\mathbb{R}$. Each element of ${}^*\mathbb{R}$ of the form ${}^*r$, with $r \in \mathbb{R}$, is again denoted $r$.

8.1.2 <u>Transfer Principle.</u> Let $S_1, \ldots, S_n \in V(\mathbb{R})$. Any elementary statement $\mathscr{s}$ which is true of $S_1, \ldots, S_n$ is true of ${}^*S_1, \ldots, {}^*S_n$.

8.1.3 <u>Saturation Principle.</u> If $A_1 \supset A_2 \supset \ldots$ is a countable decreasing chain of nonempty internal sets, then

$$\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset .$$

8.1.4 <u>Theorem.</u> (Robinson [R]). There exists a set ${}^*\mathbb{R}$ and a mapping $* : V(\mathbb{R}) \longrightarrow V({}^*\mathbb{R})$ satisfying 8.1.1, 8.1.2 and 8.1.3.

(see [K2], [HL] or [SL] for a proof and further details).

The construction of ${}^*\mathbb{R}$ makes explicit use of the Axiom of Choice. ${}^*\mathbb{R}$ can be defined as a set of equivalence classes of sequences of real numbers, in a way which is similar to the definition of $\mathbb{R}$ using equivalence classes of sequences of rational numbers. The specific details of this construction are not essential for our purposes. We only recall the following facts.

Elements $x \in {}^*\mathbb{R}$ are of two kinds :

(a) <u>infinite</u>, that is , $r < |x|$, $\forall r \in \mathbb{R}$;

(b) <u>finite</u>, that is, there is $r \in \mathbb{R}$ such that $|x| < r$.

In this case, there is a unique $s \in \mathbb{R}$ such that $|x-s| < \varepsilon$, $\forall \varepsilon > 0$, $\varepsilon \in \mathbb{R}$. The real number s is called the <u>standard part</u> of x and is denoted st(x).

When $x,y \in {}^*\mathbb{R}$ are such that $|x-y| < \varepsilon$, $\forall \varepsilon > 0$, $\varepsilon \in \mathbb{R}$, we write $x \approx y$. Clearly, if $x,y \in {}^*\mathbb{R}$ are finite and $x \approx y$, then $st(x) = st(y)$. We will use the following elementary properties of st :

$$st(x+y) = st(x) + st(y) \ ; \quad st(xy) = st(x)\, st(y) \ ;$$

$$st(\max(x,y)) = \max(st(x),st(y)) \ ; \quad st(\min(x,y)) = \min(st(x),st(y)).$$

If $s = (s_1,s_2)$, $t = (t_1,t_2)$ belong to $({}^*\mathbb{R})^2$, we write $s \approx t$ when $s_1 \approx t_1$ and $s_2 \approx t_2$, and set $st(s) = (st(s_1), st(s_2))$.

An important subset of ${}^*\mathbb{R}$ is ${}^*\mathbb{N}$, called the set of <u>hyperintegers</u>. This set "looks like" $\mathbb{N}$ followed by infinitely many infinite elements of ${}^*\mathbb{R}$.

The Saturation Principle is known to be equivalent to the following (see [K1], Theorem 7.1).

8.1.5 <u>Countable Comprehension Principle.</u> Let A be an internal set, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of elements of A. Then there exists an internal sequence $(b_n)_{n\in{}^*\mathbb{N}}$ of elements of A, such that $b_n = a_n$, $\forall n \in \mathbb{N}$.

Of paramount importance is the ability to recognize whether or not an object in $V({}^*\mathbb{R})$ is internal. This is usually done with the help of the following result, due to Keisler (see [K2]).

8.1.6 <u>Internal Definition Principle.</u> Let $A_1,\dots,A_n$ be internal objects, and let $\delta(u_1,\dots,u_n,v)$ be an elementary statement. Then

$$\{v \in A_1 : \delta(A_1,\dots,A_n,v)\}$$

is internal.
(for a proof, see [HL], Theorem II.6.4).

An important class of internal sets is the class of hyperfinite sets : an internal set $\Omega \in V({}^*\mathbb{R})$ is termed hyperfinite if there is an internal one-to-one function f mapping $\Omega$ onto $\{k \in {}^*\mathbb{N} : k \leq L\}$, for some $L \in {}^*\mathbb{N}$. L is called the internal cardinality of $\Omega$ and is denoted $L = |\Omega|$.

Let $\Omega$ be a hyperfinite set. We assign to each internal subset $A \subset \Omega$ the internal "uniform counting measure"

$$\bar{P}(A) = \frac{|A|}{|\Omega|} .$$

$\bar{P}(A)$ is a hyperreal number between 0 and 1, and $\bar{P}$ is finitely additive. The following theorem due to Loeb, is the starting point to hyperfinite probability theory.

8.1.7 Theorem. Let $\Omega$ be a hyperfinite set, $\underline{\underline{B}}(\Omega)$ the (external) $\sigma$-algebra generated by the internal subsets of $\Omega$ ($\underline{\underline{B}}(\Omega)$ is called the Borel $\sigma$-algebra of $\Omega$). There is a unique $\sigma$-additive probability measure P on $\underline{\underline{B}}(\Omega)$, such that for each internal $A \subset \Omega$, $P(A) = st(\bar{P}(A))$.

The completion of P is called the uniform Loeb measure on $\Omega$.

## 8.2 A hyperfinite probability space for two-parameter processes.

The object of this section is to build a hyperfinite probability space $\Omega$ which is convenient for the study of two-parameter processes. This space is a direct extension of that for single-parameter processes. Moreover, the two-parameter filtration that we will construct will satisfy Hypothesis F4 with respect to the Loeb measure on $\Omega$.

Throughout this chapter, we fix $n_o \in {}^*\mathbb{N} \setminus \mathbb{N}$, and set $L = n_o!$ and $\Delta u = 1/L$. Let

$$T = \{0, \Delta u, 2\Delta u, \dots, 1\},$$

and observe that since L is an infinite factorial, every element of

$\mathbb{Q} \cap [0,1]$ belongs to T. Let

$$T^2 = \{\Delta u(k,\ell) : 0 \leq k,\ell \leq L, \quad k,\ell \in {}^*\mathbb{N} \cup \{0\}\},$$

and fix a hyperfinite set $\Omega_o$ with at least two elements.

Our probability space $\Omega$ will be the hyperfinite set

$$\Omega = \Omega_o^{T^2}$$

of all internal functions from $T^2$ into $\Omega_o$, together with its (completed) Borel $\sigma$-algebra and uniform Loeb measure P.

In order to define a "canonical" two-parameter filtration on $\Omega$, we need the following definition.

8.2.1 <u>Definition.</u>

(a) For $t \in T^2$, we denote by $\approx_t$ the internal equivalence relation on $\Omega$ defined by

$$\omega \approx_t \omega' \iff \omega(s) = \omega'(s) , \quad \forall s \leq t, \quad s \in T^2 .$$

The equivalence class of $\omega$ for $\approx_t$ is denoted $\rho_t(w)$.

(b) For $t \in [0,1]^2$, $\sim_t$ denotes the (external) equivalence relation on $\Omega$ defined by

$$\omega \sim_t \omega' \iff \omega \approx_s \omega' , \quad \forall s \in T^2 , \quad s \approx t .$$

The following lemma establishes a relationship between these different equivalence relations.

8.2.2 <u>Lemma.</u> Suppose $\omega \sim_t \omega'$ , for some $t \in [0,1]^2$. Then there is $s \in T^2$ such that $st(s) \gg t$ and $\omega \approx_s \omega'$.

<u>Proof.</u> Consider $u \in T^2$ such that $u \gg t$ and $u \approx t$, and set

$B = \{x \in T : \omega \approx_{u+(x,x)} \omega'\}$. By the Internal Definition Principle, B is internal. Since B is non-empty and hyperfinite, B contains a maximal element $x_o$ ([HL], II.6.8). This element must satisfy $st(x_o) > 0$, and so $s = u + x_o$ has the desired properties. □

8.2.3 Remark. If $t = (1,t_2)$, $0 \leq t_2 < 1$, and $\omega \sim_t \omega'$, then, similar to above, there is $s = (1,s_2) \in T^2$ such that $st(s_2) > t_2$ and $\omega \approx_s \omega'$.

8.2.4 Theorem. For $t \in [0,1]^2$, let $\underline{F}_t^o$ be the set of elements of $\underline{B}(\Omega)$ which are closed under the relation $\sim_t$.

(a) $\underline{F}_t^o$ is a $\sigma$-algebra, $\forall t \in [0,1]^2$ .

(b) $(\underline{F}_t^o)_{t\in[0,1]^2}$ is a right-continuous two-parameter filtration.

Proof. (a) follows from Exercise (1.5.8) of [SB]. As for (b), observe that $s \leq t \Rightarrow \underline{F}_s^o \subset \underline{F}_t^o$ follows from the definitions; to see right-continuity, fix $t \in [0,1[^2$, and $U \in \bigcap_{u \gg t} \underline{F}_u^o$ . Let $\omega \in U$. To show that $U \in \underline{F}_t^o$, it suffices to show that $\omega' \sim_t \omega \Rightarrow \omega' \in U$. Fix $\omega' \sim_t \omega$. Using Lemma 8.2.2 we see that there is $s \in T^2$ such that $st(s) \gg t$ and $\omega \approx_s \omega'$. Set $u = (st(s) + t)/2$. Then $u \gg t$ and $\omega \sim_u \omega'$ . Since $U \in \underline{F}_u^o$ , $\omega' \in U$. Since for $t \in [0,1]^2 \setminus [0,1[^2$, the proof is similar, the theorem is proved. □

Let $\underline{F}_t$ denote the complete $\sigma$-algebra generated by $\underline{F}_t^o$ and all sets of Loeb measure zero : $(\underline{F}_t)_{t\in[0,1]^2}$ is a two-parameter filtration.

8.2.5 Definition. The filtered probability space $(\Omega, \underline{B}(\Omega), P, (\underline{F}_t)_{t\in[0,1]^2})$ is called the canonical hyperfinite probability space for two-parameter processes.

Unless otherwise mentioned, all random variables and processes considered throughout this chapter will be defined on this canonical probability space.

## 8.3 Basic lifting theorems and the commutation property.

Let $M \in V(\mathbb{R})$ be a complete separable metric space with metric $d(.,.)$. M is equipped with its Borel σ-algebra. The *-transform $^*d$ of d is again denoted d. An element $x \in {}^*M$ is near-standard provided there is some $y \in M$ such that $d(x,y) \approx 0$. In this case, y is unique and we write $y = st(x)$, the standard part of x.

8.3.1 Definition. A lifting of a random variable $X : \Omega \longrightarrow M$ is an internal function $\tilde{X} : \Omega \longrightarrow {}^*M$ such that $st(\tilde{X}(\omega)) = X(\omega)$, for almost all $\omega \in \Omega$.

The object of this section is to prove the existence of certain liftings and to deduce property F4 from them. The basic lifting theorem, due to Loeb, is stated below. Though we will mainly be concerned with real-valued random variables, we would like to point out that working with random variables with values in a complete separable metric space presents no extra difficulty.

8.3.2 Theorem. $X : \Omega \longrightarrow M$ is a random variable if and only if it has a lifting (here, Ω is an arbitrary hyperfinite set with its Borel σ-algebra).

(For a proof, see [SB], (5.1.6)).

8.3.3 Theorem. Fix $t \in [0,1]^2$. An M-valued random variable X is $\underline{\underline{F}}_t$-measurable if and only if X has a lifting $\tilde{X}$ which satisfies the following condition :

$$\exists s \in T^2 \ , \ s \approx t \ , \text{ such that } \omega \approx_s \omega' \ \Rightarrow \ \tilde{X}(\omega) = \tilde{X}(\omega').$$

Proof. We follow that of [K1], Theorem 2.10. We first prove that the condition is sufficient. Let $\mathcal{O}$ be an open subset of M. and set $F = \{\omega \in \Omega : st(\tilde{X}(\omega)) \in \mathcal{O}\}$. By Theorem 8.3.2, F is a Borel set in Ω. Thus, in order to prove that $F \in \underline{\underline{F}}_t$, we need only show that F is closed for $\sim_t$.

Suppose $\omega \in F$, and $\omega' \sim_t \omega$. Then $\omega' \approx_s \omega$ and so $\omega' \in F$. Thus $F \in \underline{F}_t^o$, and so $st(\tilde{X})$ is $\underline{F}_t^o$-measurable. Since $X = st(\tilde{X})$ a.s., $X$ is $\underline{F}_t$-measurable.

To see that the condition of the theorem is necessary, suppose $X$ is $\underline{F}_t$-measurable. Let $Y$ be $\underline{F}_t^o$-measurable such that $X = Y$ a.s., and let $\tilde{Y}$ be a lifting of $Y$ (and $X$). For each rational $q \gg t$, let $\Omega_q = \{\rho_q(\omega) : \omega \in \Omega\}$ be the set of equivalence classes of the relation $\rho_q$. This set is internal by the Internal Definition Principle, thus hyperfinite since $\Omega$ is. Since $Y$ is $\underline{F}_t^o$-measurable and $q \gg t$ ,

$$\omega \approx_q \omega' \implies Y(\omega) = Y(\omega') ,$$

so we can define a random variable $Y_q$ on $\Omega_q$ by

$$Y_q(\rho_q(\omega)) = Y(\omega) .$$

By Theorem 8.3.2, $Y_q$ has a lifting $\tilde{Y}_q$ on $\Omega_q$. Let $\tilde{X}_q : \Omega \longrightarrow {}^*M$ be the internal function such that

$$\tilde{X}_q(\omega) = \tilde{Y}_q(\rho_q(\omega)).$$

Since $\tilde{Y}_q$ is a lifting of $Y_q$, for each $n \in \mathbb{N}$ there is an internal set $A_n \subset \Omega_q$ such that

$$\bar{P}_q(A_n) < \frac{1}{n} \text{ and } \rho_q(\omega) \notin A_n \implies Y_q(\rho_q(\omega)) = st(\tilde{Y}_q(\rho_q(\omega)))$$

($\bar{P}_q$ denotes the uniform counting measure on $\Omega_q$). Set $B_n = \{\omega \in \Omega : \rho_q(\omega) \in A_n\}$. Since

$$|\rho_q(\omega)| = |\rho_q(\omega')| , \quad \forall \omega, \omega' \in \Omega ,$$

we have

$$\bar{P}(B_n) < \frac{1}{n} \text{ and } st(\tilde{X}_q(\omega)) = Y(\omega) , \quad \forall \omega \in \Omega \setminus B_n .$$

Thus

$$\bar{P}(d(\tilde{X}_q, \tilde{Y}) < \frac{1}{n}) \geq 1 - \frac{1}{n} , \quad \forall n \in \mathbb{N} .$$

Note that $\tilde{X}_q$ satisfies

$$\omega \approx_q \omega' \implies \tilde{X}_q(\omega) = \tilde{X}_q(\omega').$$

Let $(q_k)_{k\in\mathbb{N}}$ be a sequence of rational elements of $T^2$ such that $q_k \gg q_{k+1} \gg t$, and $\lim_{k\to\infty} q_k = t$. By the Countable Comprehension Principle, we can extend $(q_k,\tilde{X}_{q_k})_{k\in\mathbb{N}}$ to an internal sequence $(q_k,\tilde{X}_{q_k})_{k\in{}^*\mathbb{N}}$. The internal set

$$\{k\in{}^*\mathbb{N} \ : \ \bar{P}(d(\tilde{X}_{q_k},\tilde{Y}) < \tfrac{1}{k}) \geq 1 - \tfrac{1}{k}\ ,\ q_\ell \gg q_k \gg t \text{ and}$$

$$\omega \approx_{q_\ell} \omega' \implies \tilde{X}_{q_k}(\omega) = \tilde{X}_{q_k}(\omega')\ , \quad \forall \ell \leq k\}$$

contains $\mathbb{N}$. Since $\mathbb{N}$ is external ([HL], Corollary II.6.10), this set contains some infinite $\bar{k}$. Set $\tilde{X} = \tilde{X}_{q_{\bar{k}}}$. $\tilde{X}$ satisfies

$$(2) \qquad \bar{P}(d(\tilde{X},\tilde{Y}) < \tfrac{1}{\bar{k}}) \geq 1 - \tfrac{1}{\bar{k}}\ ,$$

$$\omega \approx_{q_\ell} \omega' \implies \tilde{X}(\omega) = \tilde{X}(\omega')\ , \quad \forall \ell \leq \bar{k}\ .$$

By (2), $\tilde{X}$ is a lifting of X. Since $q_{\bar{k}} \approx t$, the theorem is proven. □

8.3.4 <u>Remark.</u> In this proof we have made explicit use of the fact that P is the uniform Loeb measure on $\Omega$.

8.3.5 <u>Example.</u> Given two random variables X and Y such that

$$X + Y \leq 1, \quad X \geq 0,\ Y \geq 0 \quad \text{a.s.}\ ,$$

find two internal functions $\tilde{X},\tilde{Y} : \Omega \longrightarrow {}^*[0,1]$ such that $\tilde{X}(\omega) + \tilde{Y}(\omega) \leq 1$, $\forall\omega\in\Omega$. A simple way of doing this (though certainly not the only one) is to consider the random vector (X,Y) with values in the separable metric space

$$M = \{s = (s_1,s_2)\in\mathbb{R}^2 \ : \ s_1 + s_2 \leq 1,\ s_1,s_2 \geq 0\}\ ,$$

and to use Theorem 8.3.3 to get a lifting $(\tilde{X},\tilde{Y})$ with the desired properties. □

For $t \in T^2$, let $\underline{\underline{G}}_t$ denote the (external) σ-algebra of Borel sets closed under $\approx_t$. As follows from Exercise (1.5.8) of [SB], $\underline{\underline{G}}_t$ is in fact the σ-algebra generated by the internal sets of Ω which are closed under $\approx_t$.

The following proposition relates conditional expectations with respect to $\underline{\underline{F}}_t$ and $\underline{\underline{G}}_s$, $s \approx t$.

8.3.6 <u>Proposition.</u> Let X be a random variable, and $t \in [0,1]^2$. Then there is $u \approx t$, $u \in T^2$ (depending on X), such that for $s \geq u$, $s \approx t$,

$$E(X|\underline{\underline{G}}_s) = E(X|\underline{\underline{F}}_t) \text{ a.s.}$$

<u>Proof.</u> Observe that if $s \in T^2$, $s \geq t$, $s \approx t$, then $\underline{\underline{G}}_s \subset \underline{\underline{F}}_t$. Thus $E(X|\underline{\underline{G}}_s) = E(E(X|\underline{\underline{F}}_t)|\underline{\underline{G}}_s)$. Set $Y = E(X|\underline{\underline{F}}_t)$. Since Y is $\underline{\underline{F}}_t$-measurable, Theorem 8.3.3 affirms the existence of an internal function $\tilde{Y} : \Omega \longrightarrow {}^*\mathbb{R}$ such that $Y = st(\tilde{Y})$ a.s. and

$$\exists u \in T^2,\ u \approx t : \omega' \approx_u \omega \Rightarrow \tilde{Y}(\omega) = \tilde{Y}(\omega) \ ,$$

Thus, for any $s \in T^2$, $s \geq u$, Y is measurable with respect to the completion of $\underline{\underline{G}}_s$, and so $E(Y|\underline{\underline{G}}_s) = Y$. But then, when we also have $s \approx t$,

$$E(X|\underline{\underline{G}}_s) = E(Y|\underline{\underline{G}}_s) = Y = E(X|\underline{\underline{F}}_t) \text{ a.s.} \quad \Box$$

8.3.7 <u>Definition.</u> Let ρ be an internal equivalence relation on Ω. For $\omega \in \Omega$, $\rho(\omega)$ denotes the equivalence class of ω for ρ. If $\tilde{X} : \Omega \longrightarrow {}^*\mathbb{R}$ is internal, then the <u>internal conditional expectation</u> of $\tilde{X}$ given ρ is the (internal) function $\omega \longrightarrow \bar{E}(\tilde{X}|\rho(\omega))$, where

$$\bar{E}(\tilde{X}|\rho(\omega)) = \frac{1}{|\rho(\omega)|} \sum_{\omega'\in\rho(\omega)} \tilde{X}(\omega').$$

The following proposition relates internal conditional expectations and ordinary conditional expectations. The boundedness hypothesis is not necessary, but avoids introducing more definitions.

8.3.8 Proposition. Let X be a bounded real-valued random variable and $\tilde{X}$ a lifting of X. Then, where $\rho_t$ is the equivalence relation defined in 8.2.1, $\bar{E}(\tilde{X}|\rho_t(.))$ is a lifting of $E(X|\underline{\underline{G}}_t)$, $\forall t\in T^2$.
For a proof, see [SB], Proposition (1.5.10).

8.3.9 Lemma. Fix $s,t\in T^2$ such that $s \underline{\wedge} t$, and set $u = \min(s,t)$. If $B\in\underline{\underline{G}}_s$ and $C\in\underline{\underline{G}}_t$ are internal, then

$$\bar{P}(B\cap C|\rho_u(\omega)) = \bar{P}(B|\rho_u(\omega))\ \bar{P}(C|\rho_u(\omega)) \ , \quad \forall\omega\in\Omega \ .$$

Proof. By the Transfer Principle, every internal element of $\underline{\underline{G}}_s$ (resp. $\underline{\underline{G}}_t$) is a hyperfinite union of disjoint equivalence classes $\rho_s(\omega)$ (resp. $\rho_t(\omega)$). Thus it is sufficient to show that

(3) $\quad \bar{P}(\rho_s(\omega')\cap\rho_t(\omega'')|\rho_u(\omega)) = \bar{P}(\rho_s(\omega')|\rho_u(\omega))\ \bar{P}(\rho_s(\omega'')|\rho_u(\omega)),$

$\forall\omega,\ \omega',\omega''\in\Omega$ .

Observe that both sides of (3) are zero unless $\omega'\approx_u\omega\approx_u\omega''$. In this case, (3) is equivalent to

(4) $\quad |\rho_s(\omega')\cap\rho_t(\omega'')| = \dfrac{|\rho_s(\omega')|\ |\rho_t(\omega'')|}{|\rho_u(\omega)|}$

By the definition of $\Omega$, the left-hand side of (4) is equal to

$$|\Omega_o|^{L^2((1-s_2) + (1-t_1)s_2 + (s_2-t_2)(t_1-s_1))}$$

and the right-hand side to

$$\frac{|\Omega_o|^{L^2(1-s_1s_2)}\ |\Omega_o|^{L^2(1-t_1t_2)}}{|\Omega_o|^{L^2(1-u_1u_2)}} .$$

Since $u_1u_2 = s_1t_2$, these two quantities are equal, concluding the proof. □

8.3.10 Theorem. The canonical hyperfinite probability space has property F4.

Proof. Fix $s,t \in T^2$ such that $s \,\underline{\wedge}\, t$, and set $u = \min(s,t)$. Let $B \in \underline{\underline{F}}_s$ and $C \in \underline{\underline{F}}_t$ be internal. For sufficiently large $v \approx u$, $v \in T^2$, we have by Propositions 8.3.6 and 8.3.8, and Lemma 8.3.9 :

$$\begin{aligned} P(B\cap C|\underline{\underline{F}}_u) &= P(B\cap C|\underline{\underline{G}}_v) \\ &= st(\bar{P}(B\cap C|\rho_v(.))) \\ &= st(\bar{P}(B|\rho_v(.))\ \bar{P}(C|\rho_v(.))) \\ &= P(B|\underline{\underline{G}}_v)\ P(C|\underline{\underline{G}}_v) \\ &= P(B|\underline{\underline{F}}_u)\ P(C|\underline{\underline{F}}_u) . \end{aligned}$$

This proves that $\underline{\underline{F}}_s \perp_P \underline{\underline{F}}_t \mid \underline{\underline{F}}_u$ . □

## 8.4 The Simultaneous Lifting Theorem.

The simplest example of a simultaneous lifting is that of Example 8.3.5. The object of this section is the following theorem, which allows the simultaneous lifting of several random variables, each of which is measurable with respect to some given σ-algebra.

8.4.1 Theorem. Fix $n \in \mathbb{N}$, and set $I = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\right\}$. Let $(a_t)_{t\in I^2}$ be a family of real random variables such that

(5) $a_t$ is $\underline{\underline{F}}_t$-measurable , $\forall t \in I^2$ ;

(6) $a_t \geq 0$ a.s. , $\forall t \in I^2$ ;

(7) $\sum_{t \in I^2} a_t = 1$ a.s.

Then there is a family $(\tilde{a}_t)_{t \in I^2}$ of internal functions from $\Omega$ into ${}^*\mathbb{R}$ such that

(8) $st(\tilde{a}_t) = a_t$ a.s. ;

(9) for each $t \in I^2$, there is $s \in T^2$, $s \approx t$ such that

$$\omega \approx_s \omega' \Rightarrow \tilde{a}_t(\omega) = \tilde{a}_t(\omega') ;$$

(10) $\tilde{a}_t(\omega) \geq 0$ , $\forall \omega \in \Omega$ , $\forall t \in I^2$ ;

(11) $\sum_{t \in I^2} \tilde{a}_t(\omega) = 1$ , $\forall \omega \in \Omega$ .

8.4.2 <u>Remark.</u> Though the proof of this theorem seems non-trivial already for $n = 2$, and uses the conditional supremum operator of Chapter 7, its proof would be quite simple in the single-parameter case, when $I^2$ is replaced by I. We briefly indicate how the theorem can be proved in this case.

Let $\tilde{b}_t$ be a lifting of $\Sigma_{s \leq t} a_s$ , such that

$$0 \leq \tilde{b}_t(\omega) \leq 1 , \quad \forall \omega \in \Omega ;$$

$$\text{for some } s^t \approx t , \quad \omega \approx_{s^t} \omega' \Rightarrow \tilde{b}_t(\omega) = \tilde{b}_t(\omega').$$

Set $\tilde{c}_t = \sup_{s \leq t} \tilde{b}_s$ , and

$$\tilde{a}_o = \tilde{c}_o ,$$

$$\tilde{a}_t = \tilde{c}_t - \tilde{c}_{t-1/n} , \ t \in I \setminus \{0,1\},$$

$$\tilde{a}_1 = 1 - \tilde{c}_{(n-1)/n} .$$

Then $(\tilde{a}_k)_{k\in I}$ has the desired properties. □

<u>Proof of Theorem 8.4.1.</u> For $t\in I^2$, set $R_t^- = \{(s_1,s_2)\in I^2 : s_1\leq t_1, s_2\geq t_2\}$ (a slightly different definition was given in Chapter 7.5), and

$$A_t^- = \sum_{s\in R_t^-} a_s .$$

Observe that $0\leq S(A_t^-|\underset{=}{F}_t)\leq 1$ a.s., since $0\leq A_t^-\leq 1$ a.s. (see Lemma 7.2.4 (b)). Since $a_t$ and $S(A_t^-|\underset{=}{F}_t)$ are $\underset{=}{F}_t$-measurable, there exists by Theorem 8.3.3 two internal functions $\tilde{b}_t, \tilde{S}_t : \Omega \longrightarrow {}^*[0,1]$ such that

(12) $\quad st(\tilde{b}_t) = a_t$ a.s., $\quad st(\tilde{S}_t) = S(A_t^-|\underset{=}{F}_t)$ a.s.,

(13) $\quad$ for some $s^t \approx t$ , $\quad \omega \approx_{s^t} \omega' \Rightarrow (\tilde{b}_t(\omega) = \tilde{b}_t(\omega'), \tilde{S}_t(\omega) = \tilde{S}_t(\omega'))$.

We can now define algorithmically $\tilde{a}_t$ , $t\in I^2$, in increasing order for $\preceq$ (the lexicographic order on $I^2$). Throughout this proof, k and $\ell$ will denote elements of I. Set

$$\tilde{a}_{0,0} = \max(0, \min(\tilde{b}_{0,0} , \tilde{S}_{0,0})) ,$$

and suppose by induction that $\tilde{a}_s$ has been defined, $s\prec t$. Then set

(14) $$\tilde{a}_t = \max(0, \min(\tilde{b}_t, \min_{\substack{0\leq k<t_1 \\ 0\leq \ell\leq t_2}} (\tilde{S}_{t_1,\ell} - \tilde{S}_{k,t_2} - \sum_{\substack{u<t \\ u\in R^-_{t_1,\ell}\setminus R^-_{k,t_2}}} \tilde{a}_u)))$$

if $t \neq (1,1)$, and

$$\tilde{a}_{(1,1)} = 1 - \sum_{t\in I^2\setminus\{(1,1)\}} \tilde{a}_t .$$

Then property (11) is trivially satisfied. Before showing that properties (8), (9) and (10) hold, we prove the following lemmas.

8.4.3 Lemma. Fix $t \in I^2$, and $0 \leq k < t_1$, $0 \leq \ell \leq t_2$ . Then

$$a_t \leq S(A^-_{t_1,\ell} | \underline{F}_{t_1,\ell}) - S(A^-_{k,t_2} | \underline{F}_{k,t_2}) - \sum_{\substack{u<t \\ u \in R^-_{t_1,\ell} \setminus R^-_{k,t_2}}} a_u \quad \text{a.s.}$$

Proof. Observe that since $(\underline{F}_t)_{t \in I^2}$ satisfies Hypothesis F4 by Theorem 8.3.10, it also satisfies Hypothesis CQI, and so Lemma 7.4.2 (a) and (b1) imply that

$$S(A^-_{k,t_2} | \underline{F}_{k,t_2}) + \sum_{\substack{u \leq t \\ u \in R^-_{t_1,\ell} \setminus R^-_{k,t_2}}} a_u = S(A^-_{k,t_2} | \underline{F}_t) + \sum_{\substack{u \leq t \\ u \in R^-_{t_1,\ell} \setminus R^-_{k,t_2}}} a_u ,$$

which, by Lemma 7.2.4 (e) is equal to

$$S(A^-_{k,t_2} + \sum_{\substack{u \leq t \\ u \in R^-_{t_1,\ell} \setminus R^-_{k,t_2}}} a_u \mid \underline{F}_t).$$

By Lemma 7.2.4 (c), this is not greater than

$$S(A^-_{t_1,\ell} | \underline{F}_t) \leq S(A^-_{t_1,\ell} | \underline{F}_{t_1,\ell}) .$$

This clearly implies the statement of the lemma. □

8.4.4 Lemma. Fix $t_1, \ell \in I$, and $\omega \in \Omega$ , and suppose $\tilde{a}_{t_1,t_2}(\omega) > 0$, for some $t_2 \geq \ell$ with $(t_1,t_2) \neq (1,1)$. Then

$$\sum_{\substack{s \in R^-_{t_1,\ell} \\ s \neq (1,1)}} \tilde{a}_s(\omega) \leq \tilde{S}_{t_1,\ell}(\omega) .$$

Proof. We first show that the statement of the lemma holds when

$t_1 = 0$. Suppose $\tilde{a}_{0,t_2}(\omega) > 0$ for some $t_2 \geq \ell$. Let $t_2 \in I$ be maximal with this property. Then

$$(15) \qquad \sum_{s \in R^-_{0,\ell}} \tilde{a}_s(\omega) = \sum_{\ell \leq s_2 \leq t_2} \tilde{a}_{0,s_2}(\omega) .$$

Now by (14), $\tilde{a}_{0,t_2}(\omega) > 0$ implies

$$\tilde{a}_{0,t_2}(\omega) \leq \tilde{S}_{0,\ell}(\omega) - \sum_{\ell \leq s_2 < t_2} \tilde{a}_{0,s_2}(\omega) ,$$

and thus

$$\sum_{\ell \leq s_2 \leq t_2} \tilde{a}_{0,s_2}(\omega) \leq \tilde{S}_{0,\ell}(\omega) .$$

By (15), the lemma holds for $t_1 = 0$.

Suppose now by induction that the statement of the lemma holds for $0 \leq t_1' < t_1$ , and show that it holds for $t_1$. Fix $\ell \in I$, and suppose $\tilde{a}_{t_1,t_2}(\omega) > 0$, for some $t_2 \geq \ell$, with $(t_1,t_2) \neq (1,1)$. Let $t_2$ be maximal with this property.

<u>Case 1 : $\tilde{a}_{t_1',t_2'}(\omega) = 0, \quad \forall t_1' < t_1, \quad t_2' \geq t_2$</u> . Then

$$\sum_{\substack{s \in R^-_{t_1,\ell} \\ s \neq (1,1)}} \tilde{a}_s(\omega) = \sum_{\substack{s \leq (t_1,t_2) \\ s \in R^-_{t_1,\ell} \setminus R^-_{t_1-1,t_2}}} \tilde{a}_s(\omega)$$

$$= \tilde{a}_{t_1,t_2}(\omega) + \sum_{\substack{s < (t_1,t_2) \\ s \in R^-_{t_1,\ell} \setminus R^-_{t_1-1,t_2}}} \tilde{a}_s(\omega) .$$

By (14), $\tilde{a}_{t_1,t_2}(\omega) > 0$ implies that the last expression above is not greater than

$$\tilde{S}_{t_1,\ell}(\omega) - \tilde{S}_{t_1-1,t_2}(\omega) \leq \tilde{S}_{t_1,\ell}(\omega) ,$$

which implies the desired property.

Case 2 : $\exists\, k < t_1,\ \exists\, t_2' \geq t_2 : \tilde{a}_{k,t_2'}(\omega) > 0$. Let k be maximal with this property. Then

$$(16) \qquad \sum_{s \in R^-_{t_1,\ell}} \tilde{a}_s(\omega) = \sum_{s \in R^-_{k,t_2}} \tilde{a}_s(\omega) + \sum_{\substack{s < (t_1,t_2) \\ s \in R^-_{t_1,\ell} \setminus R^-_{k,t_2}}} \tilde{a}_s(\omega) + \tilde{a}_{t_1,t_2}(\omega) .$$

Applying the induction hypothesis, to the first term on the right-hand side of (16) and using the fact that $\tilde{a}_{t_1,t_2}(\omega) > 0$, we see by (14) that this last expression is not greater than

$$\tilde{S}_{k,t_2}(\omega) + \tilde{S}_{t_1,\ell}(\omega) - \tilde{S}_{k,t_2}(\omega) = \tilde{S}_{t_1,\ell}(\omega) .$$

This completes the proof of the lemma. $\square$

End of the proof of Theorem 8.4.1. To see (8), we proceed by induction in increasing order for $\lesssim$ . Use Lemma 8.4.3 and (12) and (14) to see that

$$st(\tilde{a}_t) = \max(0, \min(st(\tilde{b}_t), \min_{\substack{0 \leq k < t_1 \\ 0 \leq \ell \leq t_2}} st(\tilde{S}_{t_1,\ell} - \tilde{S}_{k,t_2} - \sum_{\substack{u < t \\ u \in R^-_{t_1,\ell} \setminus R^-_{k,t_2}}} \tilde{a}_u)))$$

$$= st(\tilde{b}_t) = a_t \text{ a.s.}$$

Again proceeding by induction in increasing order for $\lesssim$ , we see that (9) is implied by (13) and (14). Now (10) clearly holds for all $t \in I^2 \setminus \{(1,1)\}$

by (14). To see that (10) holds for $t = (1,1)$, we must show that

$$\sum_{s<(1,1)} \tilde{a}_s(\omega) \le 1 \ , \quad \forall \omega \in \Omega \ .$$

Let $t \in I^2 \setminus \{(1,1)\}$ be $\lesssim$-maximal such that $\tilde{a}_t(\omega) > 0$. Using Lemma 8.4.4, we see that

$$\sum_{s<(1,1)} \tilde{a}_s(\omega) = \sum_{s \in R^-_{t_1,0} \setminus \{(1,1)\}} \tilde{a}_s(\omega) \le \tilde{S}_{t_1,0}(\omega) \le 1.$$

This concludes the proof of the theorem. □

## 8.5 A lifting theorem for randomized stopping points.

8.5.1 Definition. An internal weight process is an internal function $\delta\alpha : \Omega \times T^2 \longrightarrow {}^*[0,1]$. Such a weight process defines a random internal additive measure $\bar{\alpha}$ on the internal algebra of internal subsets of $T^2$ by the formula

$$\bar{\alpha}(\omega, B) = \sum_{t \in B} \delta\alpha(\omega,t) \ ,$$

where $\omega \in \Omega$ and $B$ is an internal subset of $T^2$. If $\bar{\alpha}$ is finite a.s., the $\sigma$-additive extension of $st(\bar{\alpha}(\omega,.))$ to the Borel $\sigma$-algebra on $T^2$ is denoted $\alpha(\omega,.)$.

The object of this section is to lift each randomized stopping point to an internal weight process, so as to be able to apply, via the Transfer Principle, the methods used in the discrete case in Chapter 7. Our lifting method relies on the Simultaneous Lifting Theorem 8.4.1, and is quite different from the single-parameter lifting theorem of [SB], Chapter 7.1, which uses Skorohod's topology on right-continuous processes with left limits.

8.5.2 <u>Theorem.</u> Let $(A_t)_{t\in[0,1]^2}$ be a randomized stopping point. Then there is $h \in {}^*\mathbb{N}\setminus\mathbb{N}$, an internal weight process $\delta\alpha$, and an (external) P-null set $N\subset\Omega$ such that

(a) $t \in T^2$ , $\omega \approx_{t+(\frac{1}{h},\frac{1}{h})} \omega' \Rightarrow \delta\alpha(\omega,t) = \delta\alpha(\omega',t)$ ;

(b) $\Delta_{]s,t]}A(\omega) = \alpha(\omega, {}^*]s,t]\cap T^2)$ , $\forall s,t\in\mathbb{D}^2$, $s<t$, $\forall\omega\in\Omega\setminus N$;

(c) $\bar{\alpha}(\omega,T^2) = 1$ , $\forall\omega\in\Omega$ .

($\mathbb{D}$ denotes the dyadics in $[0,1]$. Throughout this section we use the following convention :

$$t + (\frac{1}{h},\frac{1}{h}) = (\min(t_1+\frac{1}{h},1), \min(t_2+\frac{1}{h},1)) ).$$

<u>Proof.</u> Set $k = (k_1,k_2)$, $k^- = (k_1-1, k_2-1)$, $k^+ = (k_1+1, k_2+1)$. Using Theorem 8.4.1, we see that for each $n\in\mathbb{N}$ and $0\leq k_1,k_2\leq 2^n$ , there is a null set $N^n_{k_1,k_2}$ and an internal function $\delta\alpha^n_{k_1,k_2} : \Omega \longrightarrow {}^*[0,1]$ such that

(17) $$\omega\in\Omega\setminus N^n_{k_1,k_2} \Rightarrow st(\delta\alpha^n_{k_1,k_2}(\omega)) = \Delta_{2^{-n}]k^-,k]} A(\omega) ,$$

(18) $$\omega \approx_{2^{-n}k^+} \omega' \Rightarrow \delta\alpha^n_{k_1,k_2}(\omega) = \delta\alpha^n_{k_1,k_2}(\omega') ,$$

(19) $$\sum_{0\leq k_1,k_2\leq 2^n} \delta\alpha^n_{k_1,k_2}(\omega) = 1 , \quad \forall\omega\in\Omega .$$

Let B denote the set of internal functions from $\Omega\times T^2$ into ${}^*[0,1]$. B is internal (see [HL], Ex. II.6.12). For $n\in\mathbb{N}$, we define an element $\delta\alpha^n$ of B by setting

(20) $$\delta\alpha^n(\omega,t) = \begin{cases} \delta\alpha^n_{k_1,k_2}(\omega) & \text{if } t = k\,2^{-n} \text{ , for some } 0\leq k_1,k_2\leq 2^n \text{ ;} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that by (18),

(21) $t\in {}^*]k^-2^{-n},\ k2^{-n}]\cap T^2,\ \omega \approx_{2^{-n}k^+} \omega' \ \Rightarrow\ \delta\alpha^n(\omega,t) = \delta\alpha^n(\omega',t).$

Set $\tilde{N} = \bigcup_{n\in\mathbb{N}} \bigcup_{0\le k_1,k_2\le 2^n} N^n_{t_1,k_2}$. Then there is a sequence $(N^n)_{n\in\mathbb{N}}$ of internal subsets of $\Omega$ such that $N^n \supset \tilde{N}$ and

(22) $\bar{P}(N^n) < \frac{1}{n}\ ;\ N^m \supset N^n\ ,\quad \forall m \le n\ ,$

$0 \le k_1,k_2 \le 2^n\ ,\ m \le n\ ,\ \omega\in\Omega\smallsetminus N^n$

(23) $\Rightarrow$

$$\Big|\sum_{t\in {}^*]k^-2^{-m},k2^{-m}]\cap T^2} \delta\alpha^m(\omega,t) - \sum_{t\in {}^*]k^-2^{-m},k2^{-m}]\cap T^2} \delta\alpha^n(\omega,t)\Big| < \frac{1}{n}\ ,$$

(24) $\sum_{t\in T^2} \delta\alpha^n(\omega,t) = 1\ ,\quad \forall\omega\in\Omega\ .$

*Using the Countable Comprehension Principle, we can extend the sequence* $(\delta\alpha^n,N^n)_{n\in\mathbb{N}}$ to an internal sequence $(\delta\alpha^n,N^n)_{n\in{}^*\mathbb{N}}$. Set

$C = \{n\in{}^*\mathbb{N}\ :\ (21),\ (22),\ (23)\ \text{and}\ (24)\ \text{hold}\}.$

By the Internal Definition Principle, C is an internal set, which contains the (external) set $\mathbb{N}$. Hence there is $\underline{m}\in C\smallsetminus\mathbb{N}$. We may choose $\underline{m}\le n_o$ where $n_o$ is the hyperinteger used in the construction of $\Omega$ (see Section 8.2).

We set $\delta\alpha = \delta\alpha^{\underline{m}}$. Observe that (c) is satisfied by (24), and (a) holds by (21), with $h = 2^{\underline{m}-1}$. Set $N = \tilde{N}\cup N^{\underline{m}}$. Then $P(N) = 0$ by (22), and for all $m\in\mathbb{N}$, $0\le k_1,k_2\le 2^m$, (23) implies that

$$\omega\in\Omega\smallsetminus N \ \Rightarrow\ \Big|\sum_{t\in{}^*]k^-2^{-m},k2^{-m}]\cap T^2} \delta\alpha^{\underline{m}}(\omega,t) - \bar{\alpha}(\omega,{}^*]k^-2^{-m},k2^{-m}]\cap T^2)\Big| < \frac{1}{\underline{m}}$$

By (17) and (20), this implies that for all $n\in\mathbb{N}$ and $0\le k_1,k_2\le 2^n$ ,

$$\alpha(\omega,{}^*]k^-2^{-m},k2^{-m}]\cap T^2) = st(\bar{\alpha}(\omega,{}^*]k^-2^{-m},k2^{-m}]\cap T^2))$$

$$= \Delta_{2^{-m}]k^-,k]}A(\omega).$$

This proves (b), and concludes the proof. $\square$

8.5.3 Corollary. Let $(A_t)_{t\in[0,1]^2}$ be a randomized stopping point, and let $\delta\alpha$ be the internal weight process and N the null set given by Theorem 8.5.2 For any Borel set $B\subset[0,1]^2$ ,

(a) $st^{-1}(B)\cap T^2$ is a Borel subset of $T^2$ ;

(b) $\int_B d_tA_t(\omega) = \alpha(\omega,st^{-1}(B)\cap T^2)$ , $\forall\omega\in\Omega\setminus N$ .

Proof. (a) is a consequence of Theorem (2.2.6) of [SB]. Furthermore, by a classical Monotone Class argument, it is sufficient to prove (b) when B is a rectangle with dyadic edges, $B = ]s,t[$, $s < t$, $s,t\in \mathbb{D}^2$ . We fix $\omega\in\Omega\setminus N$, and only consider the case $s \ll t$.

Let $\mu_\omega$ be the random measure on $[0,1]^2$ whose distribution function is $t\longmapsto A_t(\omega)$. By Theorem 8.5.2 (b),

$$(25)\qquad \mu_\omega(]s,t]) = \alpha(\omega,{}^*]s,t]\cap T^2).$$

The remainder of the proof follows that of Lemma (2.3.2) of [SB]. Since

$$st^{-1}(]s,t[)\subset {}^*]s,t]\subset st^{-1}([s,t]) ,$$

we get by (25) that

$$\alpha(\omega,st^{-1}(]s,t[)\cap T^2) \leq \mu_\omega(]s,t]) \leq \alpha(\omega,st^{-1}([s,t])\cap T^2).$$

Now

$$\alpha(\omega,st^{-1}(]s,t[\cap T^2)) = \lim_{n\to\infty}\alpha(\omega,st^{-1}([s+(\tfrac{1}{n},\tfrac{1}{n}),t-(\tfrac{1}{n},\tfrac{1}{n})])\cap T^2)$$

$$= \mu_\omega(]s,t[)$$

since

$$\mu_\omega(]s,t[) = \lim_{n\to\infty} \mu_\omega(]s + (\frac{1}{n},\frac{1}{n}),\ t - (\frac{1}{n},\frac{1}{n})])$$

$$\leq \lim_{n\to\infty} \alpha(\omega, st^{-1}([s + (\frac{1}{n},\frac{1}{n}),\ t - (\frac{1}{n},\frac{1}{n})]) \cap T^2)$$

$$\leq \lim_{n\to\infty} \alpha(\omega, st^{-1}(]s + (\frac{1}{2n},\frac{1}{2n}),\ t - (\frac{1}{2n},\frac{1}{2n})[) \cap T^2)$$

$$\leq \lim_{n\to\infty} \mu_\omega(]s + (\frac{1}{2n},\frac{1}{2n}),\ t - (\frac{1}{2n},\frac{1}{2n})])$$

$$= \mu_\omega(]s,t[) .$$

This completes the proof. □

## 8.6 Extension of the discrete construction of Chapter 7.5 to the continuous case.

Throughout this section we work with a fixed randomized stopping point $A = (A_t)_{t\in[0,1]^2}$ . Using the lifting theorem 8.5.2, together with the construction of Chapter 7.5 and the Transfer Principle, we shall build two randomized stopping points $A^i = (A^i_t)_{t\in[0,1]^2}$ , $i = 1,2$, and an optional increasing path Z such that

(26) $A = \frac{1}{2} A^1 + \frac{1}{2} A^2$

and Z splits $[0,1]^2$ into two parts, one of which contains the support of the random measure associated with $A^1$, and the other the support of the random measure associated with $A^2$ (of course, if A is a stopping point, the supports of A, $A^1$ and $A^2$ will be contained in Z).

As in Chapter 7, this will lead immediately to an affirmative answer to Question 1.8.2 for the canonical filtered hyperfinite probability space for two-parameter processes.

Let $\delta\alpha$ be the internal weight process given by Theorem 8.5.2, together with $h \in {}^*\mathbb{N} \setminus \mathbb{N}$ and the P-null set N. We let $\underline{\underline{G}}_t^h$ be the internal algebra of internal events closed under the relation $\approx_{t+(1/h,1/h)}$. Then $\delta\alpha$ is <u>adapted to the internal filtration</u> $\underline{\underline{G}}^h = (\underline{\underline{G}}_t^h)_{t\in T^2}$ in the following sense :

$$\omega \approx_{t+(\frac{1}{h},\frac{1}{h})} \omega' \implies \delta\alpha(\omega,t) = \delta\alpha(\omega',t).$$

Let $T + T = \{0,\Delta u, 2\Delta u,\dots,2\}$. The Transfer Principle, applied to the results of Proposition 7.5.4 and Theorem 7.5.5 in the case of a finite canonical two-parameter filtration, affirms the existence of an internal function $Z : \Omega \times (T+T) \longrightarrow T^2$ and of two internal weight processes $\delta\alpha^1, \delta\alpha^2$ : $\Omega \times T^2 \longrightarrow {}^*[0,1]$ with the following properties for all $t \in T^2$, $\omega \in \Omega$, $p \in T+T$ :

(27) $\delta\alpha(\omega,t) = \frac{1}{2}\,\delta\alpha^1(\omega,t) + \frac{1}{2}\,\delta\alpha^2(\omega,t)$ ;

(28) $\omega' \approx_{t+(\frac{1}{h},\frac{1}{h})} \omega \implies \delta\alpha^i(\omega',t) = \delta\alpha^i(\omega,t),\ i = 1,2$ ;

(29) $\sum_{s\in T^2} \delta\alpha^i(\omega,s) = 1 , \quad i = 1,2$ ;

(30) $Z(\omega,p+\Delta u) \in \{Z(\omega,p) + (\Delta u,0),\ Z(\omega,p) + (0,\Delta u)\}$ ;

(31) $\omega' \approx_{t+(\frac{1}{h},\frac{1}{h})} \omega,\ Z(\omega,p) \leq t \implies Z(\omega',p) \leq t$ ;

(32) $t \wedge Z(\omega,|t|) \implies (\delta\alpha^1(\omega,t) = 2\delta\alpha(\omega,t),\ \delta\alpha^2(\omega,t) = 0)$ ;

(33) $Z(\omega,|t|) \wedge t \implies (\delta\alpha^1(\omega,t) = 0,\ \delta\alpha^2(\omega,t) = 2\delta\alpha(\omega,t))$ .

The randomized stopping points $A^1$ and $A^2$ shall be obtained from $\delta\alpha^1$ and $\delta\alpha^2$ via the standard part map. Before doing this, we recall the following definition.

8.6.1 <u>Definition.</u> (see [SB], App. 1.4). A mapping $f : T \longrightarrow T^2$ is <u>S-continuous</u> if $x \approx y \implies f(x) \approx f(y)$.

8.6.2 Lemma.

(a) For $\omega \in \Omega$, $p \longmapsto Z(\omega,p)$ is S-continuous.

(b) Set $\tilde{Z}_{st(p)}(\omega) = st(Z(\omega,p))$ , $\omega \in \Omega$ , $p \in T+T$ .

Then $(\tilde{Z}_u)_{u\in[0,2]}$ is an optional increasing path.

Proof. Property (a) is a consequence of the equality

$$|Z(\omega,p) - Z(\omega,q)| = |p-q| \ , \quad \forall\omega \in \Omega \quad , \ p,q \in T+T \ .$$

As for (b), observe that $\tilde{Z}_u$ is well-defined by (a), since if $U = st(p) = st(\tilde{p})$, then $p \approx \tilde{p}$, thus

$$Z(\omega,p) \approx Z(\omega,\tilde{p}) \quad \text{and so} \quad st(Z(\omega,p)) = st(Z(\omega,\tilde{p})).$$

Furthermore, $u \longmapsto \tilde{Z}_u(.)$ is increasing by (30), and if $p \in T+T$ is such that $p \approx u$, then

$$|\tilde{Z}_u(\omega)| = st(|Z(\omega,p)|) = st(p) = u \ ,$$

also by (30). Now fix $u \in [0,2]$ and $t \in [0,1]^2$. We must show that

$$\{\omega \in \Omega : \tilde{Z}_u(\omega) \leq t\} \in \underline{\underline{F}}_t \ .$$

Since $(\underline{\underline{F}}_t)_{t\in[0,1]^2}$ is right-continuous, it is sufficient to show that for $t \in [0,1]^2 \cap \mathbb{D}^2$ ,

$$F = \{\omega \in \Omega : st(Z(\omega,p)) \ll t\} \in \underline{\underline{F}}_t \ ,$$

where $p \in T+T$ is such that $p \approx u$. By Theorem 8.3.2, we see that F is a Borel set. Since $t \in T^2$, observe that by (31),

$$\omega \approx_{t+(\frac{1}{h},\frac{1}{h})} \omega' \ , \quad \omega \in F \implies \omega' \in F \ ,$$

so F is closed under $\approx_{t+(1/h,1/h)}$. By Theorem 8.3.3, this proves that $F \in \underline{\underline{F}}_t$, and concludes the proof. □

8.6.3 Proposition. For $t \in [0,1]^2$, set

$$A_t^i(\omega) = \inf_{q \in \mathbb{D}^2 \cap ]t,(1,1)]} \alpha^i(\omega, {}^*[0,q] \cap T^2), \quad t \neq (1,1),$$

$$A_{(1,1)}^i(\omega) = 1 ,$$

where $\alpha^i$ is defined from $\delta\alpha^i$ as in Definition 8.5.1. Then for $i = 1, 2$, $A^i = (A_t^i)_{t \in [0,1]^2}$ is a randomized stopping point, and

$$A = \frac{1}{2} A^1 + \frac{1}{2} A^2 .$$

Proof. For $i = 1,2$, the definition of $A^i$ clearly implies that $A_\cdot^i$ is right-continuous and has positive planar increments. Since $A_{(1,1)}^i \equiv 1$, $A^i$ will be a randomized stopping point provided $A_t$ is $\underline{\underline{F}}_t$-measurable, $\forall t \in [0,1]^2$. This is the case since $(\underline{\underline{F}}_t)$ is right-continuous and the random variable $\alpha^i(.,[0,q])$ is $\underline{\underline{F}}_q$-measurable by Theorem 8.3.3 and (28).

The fact that $A = \frac{1}{2} A^1 + \frac{1}{2} A^2$ is an immediate consequence of the equality

$$\alpha(\omega, {}^*[0,q] \cap T^2) = \frac{1}{2} \alpha^1(\omega, {}^*[0,q] \cap T^2) + \frac{1}{2} \alpha^2(\omega, {}^*[0,q] \cap T^2),$$

which holds by (27), and the fact that $\alpha(., {}^*[0,q]) \cap T^2) = A_q(.)$ a.s., $\forall q \in \mathbb{D}^2$, by Theorem 8.5.2. □

The following proposition is similar to Proposition 7.5.4.

8.6.4 Proposition. Fix $\omega \in \Omega$, and $s,t \in \mathbb{D}^2$ such that $s \leq t$.

(a) Suppose $(t_1,s_2) \wedge \tilde{Z}_{t_1+s_2}(\omega)$. Then $\Delta_{]s,t]} A^2(\omega) = 0$ .

(b) Suppose $\tilde{Z}_{s_1+t_2}(\omega) \wedge (s_1,t_2)$. Then $\Delta_{]s,t]} A^1(\omega) = 0$ .

Proof. We only prove (a). By the hypothesis and (32), there is

$\varepsilon > 0$, $\varepsilon \in \mathbb{R}$ such that

$$\delta\alpha^2(\omega,u) = 0 \ , \quad \forall u \in {}^*]s,t+(\varepsilon,\varepsilon)] \cap T^2 \ .$$

So if $p,q \in {}^*]s,t+(\varepsilon,\varepsilon)] \cap \mathbb{D}^2$, $p \leq q$,

$$\sum_{p<u\leq q} \delta\alpha^2(\omega,u) = 0 \ .$$

Thus, where $\alpha^2(\omega,[a,b])$ is an abbreviation of $\alpha^2(\omega,{}^*[a,b] \cap T^2)$,

$$\alpha^2(\omega,[0,q]) - \alpha^2(\omega,[0,(p_1,q_2)]) - \alpha^2(\omega,[0,(q_1,p_2)]) + \alpha^2(\omega,[0,p]) = 0.$$

Taking the limit as $q \downarrow t$, $p \downarrow s$ gives the desired result. □

## 8.7 Extremal randomized stopping points on the canonical hyperfinite probability space for two-parameter processes.

If $(Z_u)_{u\in[0,2]}$ is an optional increasing path, we set,

$$\text{Im } Z_.(\omega) = \{Z_u(\omega) \ : \ 0 \leq u \leq 2\} \ .$$

8.7.1 Lemma. Suppose $\mu_\omega(.)$ is the random measure whose distribution function is the randomized stopping point $(A_t)_{t\in[0,1]^2}$ considered in the preceeding section, and suppose

$$P\{\omega \in \Omega \ : \ \text{supp } \mu_\omega(.) \subset \text{Im } Z_.(\omega)\} < 1$$

for all optional increasing paths $(Z_u)_{u\in[0,2]}$. Then $A^1 \neq A \neq A^2$, where $A^i$ is the randomized stopping point defined in Proposition 8.6.3, $i = 1,2$.

Proof. Let $(\tilde{Z}_u)_{u\in[0,2]}$ be the optional increasing path defined in Lemma 8.6.2, and set

$$\tilde{F} = \{\omega \in \Omega \ : \ \text{supp } \mu_\omega(.) \not\subset \text{Im } \tilde{Z}_.(\omega)\} \ .$$

Since $P(\tilde{F}) > 0$, we may suppose for example that $P(F) > 0$, where

$$F = \{\omega \in \Omega : \exists s,\ t \in \mathbb{D}^2,\ s \leq t, \text{ such that } (t_1,s_2) \wedge \tilde{Z}_{t_1+s_2}(\omega)$$
$$\text{and } \Delta_{]s,t]}A(\omega) > 0\} .$$

Now for each $\omega \in F$, we have by Propositions 8.6.3 and 8.6.4 :

$$\Delta_{]s,t]}A^1(\omega) = 2\ \Delta_{]s,t]}A(\omega)$$
$$\neq \Delta_{]s,t]}A(\omega)$$

for some couple $(s,t)$ with $s \leq t$. This implies that the sample paths

$$t \longmapsto A_t(\omega) \ , \quad t \longmapsto A_t^i(\omega) \ , \quad i = 1,2,$$

are distinct for $\omega \in F$. Since $P(F) > 0$, $A^1 \neq A \neq A^2$. □

The following lemma is a straightforward extension of the single-parameter result of Lemma 1.8.3.

8.7.2 <u>Lemma.</u> Let $(\Omega, \underline{F}, P)$ be an arbitrary (complete) probability space, and $(\underline{F}_t)_{t\in[0,1]^2}$ an arbitrary two-parameter filtration (with or without CQI or F4). Suppose $A = (A_t)_{t\in[0,1]^2}$ is a randomized stopping point and $(Z_u)_{u\in[0,2]}$ is an optional increasing path, such that

$$P\{\omega \in \Omega : \text{supp } \mu_\omega(.) \subset \text{Im } Z_.(\omega)\} = 1 .$$

Then A is an extremal element of the set of randomized stopping points if and only if A is a stopping point.

<u>Proof.</u> Set $A_t^1 = \min(2A_t,1)$, $A_t^2 = \max(2A_t-1,0)$. Clearly $A_t = \frac{1}{2} A_t^1 + \frac{1}{2} A_t^2$, and the sample paths

$$t \longmapsto A_t(\omega) \quad \text{and} \quad t \longmapsto A_t^i(\omega) \quad , \quad i = 1,2 ,$$

are distinct if and only if $0 < A_t(\omega) < 1$ for some t. If $s,t \in [0,1]^2$ are such that $s \leqslant t$, it is easy to see that $\Delta_{]s,t]}A^i \geqslant 0$ a.s. by examining the relative positions of s,t and the path $u \longmapsto Z_u$ (see Figure 15).

Since $A^i_{(1,1)} \equiv 1$, this implies that $A^1$ and $A^2$ are randomized stopping points. Thus if A is extremal, we must have

$$A_t \in \{0,1\} \text{ a.s.}$$

But then A is a stopping point. □

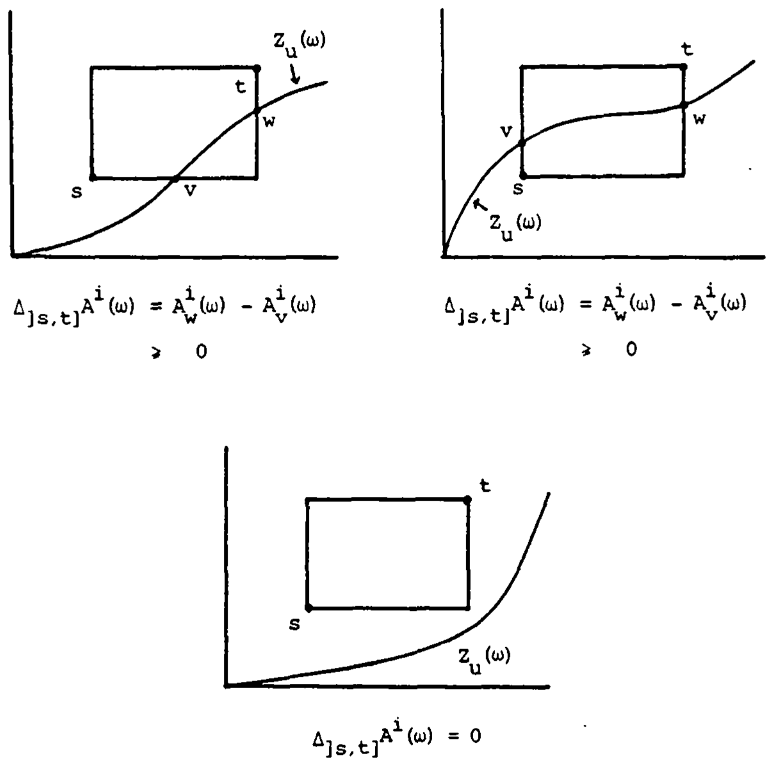



Figure 15.

The following theorem gives a continuous time extension of Theorem 7.6.1.

8.7.3 Theorem. On the canonical hyperfinite probability space for two-parameter processes, all extremal elements of the set of randomized stopping points on $[0,1]^2$ are stopping points.

Proof. Let $A = (A_t)_{t\in[0,1]^2}$ be a randomized stopping point. Suppose

$$P\{\omega\in\Omega : \operatorname{supp}\mu_\omega(.) \subset \operatorname{Im} Z_.(\omega)\} < 1$$

for all optional increasing paths Z, where $\mu_\omega(.)$ is the measure on $[0,1]^2$ whose distribution function is $t \longmapsto A_t(\omega)$. Then by Proposition 8.6.3 and Lemma 8.7.1, A is the midpoint of two distinct randomized stopping points, and thus is not an extremal element of $\underline{\underline{U}}$. This implies that any extremal randomized stopping point must satisfy

$$P\{\omega\in\Omega : \operatorname{supp}\mu_\omega(.) \subset \operatorname{Im} Z_.(\omega)\} = 1$$

for some optional increasing path Z. But then the statement of the theorem is a consequence of Lemma 8.7.2. $\square$

CHAPTER 9

# FUNCTIONALS OF RANDOMIZED STOPPING POINTS

As we stated in Chapter 1.6, there are two difficulties when using randomized stopping points to prove the existence of optimal stopping points for two or multi-parameter processes. The first is that all extremal randomized stopping points should be stopping points. This question has been discussed in Chapters 4, 7 and 8. The second is that the extension of $T \mapsto E(X_T)$ to the function $\Phi : \underline{\underline{U}} \longrightarrow \mathbb{R}$ defined by

$$(1) \qquad \Phi((A_t)_{t\in\overline{\mathbb{R}^2}}) = E(\int_{\overline{\mathbb{R}^2}} X_t(.)\ d_tA_t(.))$$

should be sufficiently regular for a maximum over $\underline{\underline{U}}$ to exist. A weak condition would be upper-semicontinuity of $\Phi$. The primary objective of this section is to show that upper-semicontinuity of the sample paths of the reward process X implies upper-semicontinuity of $\Phi$.

This result is based on an approximation of upper-semicontinuous processes by continuous processes, which is simpler than the one presented in [D]. This approximation has turned out to be useful in several contexts (see [MM],[MM1]). In particular, it will enable us to show the existence of optimal stopping points on the canonical hyperfinite probability space for two-parameter processes.

The regularity results are independent of the filtration, and so we state and prove them directly for multi-parameter processes, that is processes indexed by $\overline{I^n} = \overline{\mathbb{R}^n_+}$ .

## 9.1 Continuous and upper-semicontinuous functions.

Except where otherwise mentioned, we shall be working on an arbitrary complete probability space $(\Omega, \underline{\underline{F}}, P)$. Recall that the set $\underline{\underline{U}}$ is equipped with the weak topology (see Chapter 1.7).

9.1.1 Lemma. Suppose $(X_t)_{t\in\overline{I^n}}$ is a continuous process such that $E(\sup_{t\in\overline{I^n}} |X_t|) < +\infty$. Then the mapping $\Phi$ defined in (1) is continuous.

Proof. $\Phi$ is continuous by definition of the weak topology on $\underline{\underline{U}}$ (see Chapter 1.7). □

There are no useful necessary and sufficient conditions for a real-valued function on a set to attain its maximum. A weak "almost necessary" condition (see [EK]) is upper-semicontinuity.

9.1.2 Definition. (see [Bl], IV. §6.2)

If E is a topological space, then a function $f : E \longrightarrow \mathbb{R}$ is upper-semicontinuous on E provided for each $x \in E$ and $\varepsilon > 0$, there is a neighborhood V of x such that

$$y \in V \Rightarrow f(y) < f(x) + \varepsilon .$$

The word "upper-semicontinuous" will be abreviated to "u.s.c."

We have already encountered an u.s.c. function in Chapter 7.7. Here, we begin by giving sufficient conditions on X for the function $\Phi$ defined in (1) to be u.s.c.

Recall (see [DM], IV. 8) that an evanescent set in $\Omega \times \overline{I^n}$ is a subset N of $\Omega \times \overline{I^n}$ such that

$$P\{\omega \in \Omega : \exists t \in \overline{I^n}, (\omega, t) \in N\} = 0 .$$

9.1.3 Proposition. Let $X = (X_t)_{t\in\overline{I^n}}$ be a measurable process. Suppose that there exists a non-increasing sequence $(Y^k)_{k\in\mathbb{N}}$ of measurable processes such that

(a) the mapping $\Phi^k : \underline{U} \longrightarrow \mathbb{R}$ defined by

$$\Phi^k((A_t)_{t\in\overline{I^n}}) = E(\int_{\overline{I^n}} Y_t^k(.)\, d_tA_t(.))$$

is upper-semicontinuous;

(b) $X_t(\omega) = \lim_{k\to\infty} \downarrow Y_t^k(\omega)$ outside an evanescent set.

Then the mapping $\Phi$ defined by (1) is upper-semicontinuous.

Proof. Observe that

$$\Phi((A_t)_{t\in\overline{I^n}}) = E(\int_{\overline{I^n}} X_t(.)\, d_tA_t(.))$$

$$= \lim_{k\to\infty} \downarrow E(\int_{\overline{I^n}} Y_t^k(.)\, d_tA_t(.))$$

$$= \lim_{k\to\infty} \downarrow \Phi^k((A_t)_{t\in\overline{I^n}}) .$$

Since each $\Phi^k$ is upper-semicontinuous, this implies that $\Phi$ is also (see Theorem 4 of [B1], IV. 6.2). □

Thus, given a class of processes satisfying condition (a) of Proposition 9.1.3, this class can be extended by taking monotone decreasing limits. This proves that $\Phi$ defined by (1) is upper-semicontinuous whenever X is a monotone decreasing limit of continuous processes (see Lemma 9.1.1). Since every u.s.c. function defined on a metric space is a decreasing limit of continuous functions (see [B1], IX. §2.7), it is not difficult to provide examples of processes for which $\Phi$ is u.s.c.

9.1.4 Examples.

(a) $Y = f(X^1,\ldots,X^k)$, where f is an u.s.c. bounded function from $\mathbb{R}$ into $\mathbb{R}$ and the $X^i$ are continuous processes.

(b) $Y_t(\omega) = \sum_{i=1}^{m} Z_i(\omega)\; g_i(t)$ ,

where the $Z_i$ are random variables and the $g_i : \overline{I^n} \longrightarrow \mathbb{R}$ are u.s.c and bounded.

(c) $Y_t(\omega) = \sum_{i=1}^{\infty} \alpha_i\; g_i(t)\; I_{\{S^i \leq t\}}(\omega)$ ,

where $(\alpha_i)_{i\in\mathbb{N}}$ is a sequence of real numbers such that $\Sigma_{i=1}^{\infty}|\alpha_i| < +\infty$, the $g_i : \overline{I^n} \longrightarrow \mathbb{R}$ are u.s.c and bounded, and the $S^i$ are random variables with values in $\overline{I^n}$.

9.2 Approximation of upper-semicontinuous processes.

Since u.s.c. functions can always be approximated by continuous ones, the approximation of an u.s.c. process $(X_t)_{t\in\overline{I^n}}$ by continuous processes could be attempted in the following manner : for each $\omega \in \Omega$, there is a sequence $Y^{n,\omega}_{\cdot}$ of continuous functions decreasing to $X_{\cdot}(\omega)$. However, a priori, there is no reason for $\omega \longmapsto Y^{n,\omega}_t$ to be measurable for each fixed t. This difficulty is resolved in the following theorem.

9.2.1 Theorem. *Let $X = (X_t)_{t\in\overline{I^n}}$ be a bounded measurable process with upper-semicontinuous sample paths, defined on a complete probability space $(\Omega, \underline{F}, P)$. Then there is a non-increasing sequence $(Y^k)_{k\in\mathbb{N}}$ of continuous bounded processes such that*

$$\lim_{k\to\infty} \downarrow Y^k_t(\omega) = X_t(\omega) , \quad \forall t \in \overline{I^n} , \quad \forall \omega \in \Omega .$$

The proof of this theorem uses the following lemma.

9.2.2 Lemma. Consider $F \subset \underline{F} \times \underline{B}(\overline{I^n})$ such that for each $\omega \in \Omega$, the section $F_\omega = \{t \in \overline{I^n} : (\omega,t) \in F\}$ is closed. Then the mapping

$$\omega \longmapsto \mathrm{dist}(t,F_\omega)$$

is $\underline{F}$-measurable ($\mathrm{dist}(t,F_\omega)$ denotes the distance between t and the set $F_\omega$ for the usual metric on $\overline{I^n}$).

Proof. For $r > 0$,

$$A = \{\omega \in \Omega : \mathrm{dist}(t,F_\omega) < r\} = \{\omega \in \Omega : \exists s \in F_\omega \ , \ d(s,t) < r\} \ ,$$

so A is the projection on $\Omega$ of the $\underline{F} \times \underline{B}(\overline{I^n})$-measurable set $F \cap (\Omega \times B(t,r))$, where $B(t,r)$ denotes the open ball centered at t with radius r. Thus A is $\underline{F}$-analytic by Theorem III.13 of [DM], and since $\underline{F}$ is complete, $F \in \underline{F}$ by III.33 of [DM]. This proves the lemma. □

Proof of Theorem 9.2.1. This proof follows that of Proposition 11 in [B1], IX. §2.7. Since X is bounded, we may suppose without loss of generality that X takes its values in the interval [-1,0]. Set

$$X_t^n(\omega) = \frac{-1}{2^n} \sum_{k=1}^{2^n} I_{U^{k,n}}(\omega,t) \ ,$$

where

$$U^{k,n} = \{(\omega,t) \in \Omega \times \overline{I^n} : X_t(\omega) < -\frac{k}{2^n}\} \ ,$$

and observe that $(X^n)_{n \in \mathbb{N}}$ is a non-increasing sequence which converges to X. Now since X is u.s.c., the section $U_\omega^{k,n}$ of $U^{k,n}$ is open for each $\omega \in \Omega$ .

For each fixed k and n, set

$$Z_t^{k,n,\ell}(\omega) = \min(1,\ell\,\mathrm{dist}(t,\overline{I^n} \setminus U_\omega^{k,n})) \ .$$

Then $\omega \longmapsto Z^{k,n,\ell}(\omega)$ is a measurable map by Lemma 9.2.2, $t \longmapsto Z_t^{k,n,\ell}(\omega)$ is continuous and

$$(t\in \overline{I^n}\setminus U^{k,n}_\omega \quad\text{or}\quad \mathrm{dist}(t,\overline{I^n}\setminus U^{k,n})>\tfrac{1}{\ell}) \implies Z^{k,n,\ell}_t(\omega)=I_{U^{k,n}}(\omega,t) ,$$

so

$$\lim_{\ell\to\infty}\uparrow Z^{k,n,\ell}_t(\omega)=I_{U^{k,n}}(\omega,t) ,\quad \forall t\in\overline{I^n} ,\quad \forall\omega\in\Omega .$$

Thus if we define a continuous process $X^{n,\ell}$ by setting

$$X^{n,\ell}_t(\omega) = -\frac{1}{2^n}\sum_{k=1}^{2^n} Z^{n,k,\ell}_t(\omega) ,$$

we have

$$\lim_{\ell\to\infty}\downarrow X^{n,\ell}_t(\omega)=X^n_t(\omega) ,\quad \forall t\in\overline{I^n} ,\quad \forall\omega\in\Omega .$$

But then the sequence $(Y^k)_{k\in\mathbb{N}}$ of continuous processes defined by

$$Y^k_t(\omega)=\min_{n,\ell\leq k} X^{n,\ell}_t(\omega)$$

satisfies the conditions of the theorem. □

## 9.3 Upper-semicontinuous functionals of randomized stopping points.

9.3.1 Theorem. Let $(\Omega, \underline{\underline{F}}, P)$ be a complete probability space, and $(X_t)_{t\in\overline{I^n}}$ a measurable process with u.s.c. sample paths such that $E(\sup_{t\in\overline{I^n}}|X_t|)<+\infty$. Then the map $\Phi : \underline{\underline{U}} \longrightarrow \mathbb{R}$ defined by

$$\Phi((A_t)_{t\in\overline{I^n}}) = E(\int_{\overline{I^n}} X_t(.)\, d_tA_t(.))$$

is u.s.c. (for the weak topology on $\underline{\underline{U}}$).

Proof. Observe that $\omega\longmapsto \sup_{t\in\overline{I^n}}|X_t(\omega)|$ is measurable (see the proof of [DM], IV.33.a). Now suppose to begin with that $X=(X_t)_{t\in\overline{I^n}}$ is bounded.

Then we can use Theorem 9.2.1, Proposition 9.1.3 and Lemma 9.1.1 to see that $\Phi$ is u.s.c.

Now suppose that X is as in the hypothesis of the theorem and let $X^+$ (respectively $X^-$) denote the positive (respectively negative) part of X. Observe that

$$- X_t^-(\omega) = \lim_{n\to\infty} \downarrow \max(-X_t^-(\omega), -n) ,$$

and $\max(-X_t^-, -n)$ is u.s.c. and bounded. Thus

$$(A_t)_{t\in\overline{I^n}} \longmapsto E(\int_{\overline{I^n}} (-X_t^-(.))\, d_tA_t(.))$$

is u.s.c. by Proposition 9.1.3. So it only remains to be shown that $\tilde{\Phi}$ is u.s.c., where

$$\tilde{\Phi}((A_t)_{t\in\overline{I^n}}) = E(\int_{\overline{I^n}} X_t^+(.)\, d_tA_t(.)) .$$

To this end, observe that $\Phi^m : \underline{U} \longrightarrow \mathbb{R}$ defined by

$$\Phi^m((A_t)_{t\in\overline{I^n}}) = E(\int_{\overline{I^n}} \min(X_t^+(.),m)\, d_tA_t(.))$$

is u.s.c. by the first part of the proof. So $\tilde{\Phi}$ will be u.s.c. if we show that

$$\lim_{m\to\infty} \Phi^m = \tilde{\Phi} \text{ uniformly on } \underline{U} .$$

Now observe that

$$\begin{aligned} |\tilde{\Phi}((A_t)_{t\in\overline{I^n}}) - \Phi^m((A_t)_{t\in\overline{I^n}})| &= |E(\int_{\overline{I^n}} I_{\{X>m\}}\, (X-m)\, d_tA_t)| \\ &= E(\int_{\overline{I^n}} I_{\{X>m\}}\, (X-m)\, d_tA_t) \\ &\leq E(\int_{\overline{I^n}} I_{\{X>m\}}\, X\, d_tA_t) \end{aligned}$$

$$\leq E(\int_{I^n} I_{\{|X|>m\}} \sup_{t\in I^n} |X_t| \, d_t A_t)$$

$$\leq E(I_{\{\sup_{t\in I^n} |X_t|>m\}} \sup_{t\in I^n} |X_t|)$$

$$\xrightarrow[m\to\infty]{} 0 ,$$

thus $\Phi^m \longrightarrow \tilde{\Phi}$ uniformly on $\underline{U}$, so $\tilde{\Phi}$ is u.s.c., and the proof is complete. □

## 9.4 Application : existence of optimal stopping points.

In this section we prove the existence of optimal stopping points for upper-semicontinuous two-parameter processes defined on the canonic hyperfinite probability space of Chapter 8.2. Note that the existence of optimal stopping points for such processes has been proved by different methods in [MM1].

9.4.1 __Theorem.__ Let $(\Omega, \underline{B}(\Omega), P, (\underline{F}_t)_{t\in[0,1]^2})$ be as in Definition 8.2.5, and let $X = (X_t)_{t\in[0,1]^2}$ be a measurable process with u.s.c. sample paths *defined on this space, such that*

$$E(\sup_{t\in[0,1]^2} |X_t|) < +\infty .$$

Then there is a stopping point $T_o$ such that

$$E(X_{T_o}) = \sup_{T\in\underline{T}} E(X_T) .$$

__Proof.__ Consider the functional $\Phi : \underline{U} \longrightarrow \mathbb{R}$ defined by

$$\Phi((A_t)_{t\in[0,1]^2}) = E(\int_{[0,1]^2} X_t(.) \, d_t A_t(.)) .$$

By Theorem 9.3.1, this functional is u.s.c. on $\underline{U}$. Since $\Phi$ is affine, it attains its maximum on $\underline{\underline{U}}$ at an extremal element $A^o \in \text{ext } \underline{\underline{U}}$ ([B], II. §7 , Prop. 1). By Theorem 8.7.3, $A^o$ is in fact a stopping point, which we denote $T_o$. This stopping point is clearly optimal. □

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# INDEX

adapted, 8, 166
atom, 63, 114
augmented clique matrix, 68, 107
bipartite graph, 92
Borel, 3, 147
canonical
  two-parameter filtration, 80
  hyperfinite probability space, 149
clique, 67
clique matrix, 68
cluster point, 45
cocomparability graph, 113
Commutation Property, 7
comparability graph, 88
complete, 3
conditional
  independence, 3
  qualitative independence, 4
  supremum, 120
configuration
  triple-$\underline{P}_3$-, 92
correlation operator, 60
Countable Comprehension Principle, 146
countably conditionally qualitatively independent, 30
CQI-graph, 91
  Hypothesis —, 7
cycle, 88
derived graph, 68, 107
dominate, 107
edge, 67, 107
elementary statement, 144
equivalent, 3
essential infimum, 119
evanescent, 174
Extension Principle, 145
external, 145
F4,
  Hypothesis —, 6
filtered probability space, 37
filtration, 5
finite, 145
fundamental
  matrix, 64
  solution, 18
graph, 67, 107
  bipartite —, 92
  cocomparability —, 113
  comparability —, 88
  CQI-, 91
  derived —, 68, 107
  intersection —, 69
  perfect —, 68, 107
  perfectly orderable —, 73
    strongly —, 73
  transitively orientable —, 92
  two-parameter —, 72
hyperfinite, 147
hyperinteger, 146
hyperreal, 144
Hypothesis CQI, 7

Hypothesis F4, 6

increasing path, 40
  optional —, 40
indistinguishable, 8
infinite, 145
Internal Definition Principle, 146
internal, 144
  cardinality, 147
  conditional expectation, 153
  weight process, 161
intersection graph, 69
Intersection Property, 8

lie on, 42
lifting, 150
Lusin, 37

measurable
  map, 3
  process, 9
minimal forbidden induced
  subgraph, 87

near-standard, 150
neighbor, 67
non-measurable randomized
  stopping point, 114
normal product, 78

object, 144
obstruction, 73
o.i.p., 40
optional increasing path, 40
  step —, 42

parameter set, 1
pass through, 42
path,
  increasing —, 40
    optional —, 40
perfect graph, 68, 107
perfectly orderable, 73
process
  adapted —, 8, 166
  measurable —, 9
  single-parameter —, 8
  two-parameter —, 8
product
  filtration, 78
  normal —, 78

random
  distribution function, 15
  measure, 15
  variable, 3
randomized
  point, 10
  stopping point, 12
regular, 37

Saturation Principle, 145
S-continuous, 166
semi-continuous, 174
simple
  cycle, 88
  path, 88
single-parameter
  filtration, 5
  process, 8
stable set, 90
standard part, 146, 150
step o.i.p., 42

stopping
  point, 5
  time, 5
strongly perfectly orderable, 73
subgraph, 67
superstructure, 144
supremum
  conditional —, 120

time, 1
Transfer Principle, 145
transitively orientable graph, 92
tree, 88
triple-$\underline{P}_3$ configuration, 92
two-parameter
  filtration, 5
  graph, 72
  process, 8

uniform
  counting measure, 147
  Loeb measure, 147
upper semi-continuous, 174
u.s.c., 174

vertex, 67, 107

weak topology, 12

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"A Prediction Problem for the Brownian Sheet" (with F. Russo), accepted for publication in the J. of Mult. Analysis.

"On Randomized Stopping Points and Perfect Graphs" (with L.E. Trotter Jr. and D. de Werra), to appear in the J. Comb. Theory (B).

"On Infinite Perfect Graphs and Randomized Stopping Points on the Plane", submitted for publication.

"Existence of optional increasing paths passing through a stopping point in the plane", submitted for publication.