Advances in Mathematics 442 (2024) 109594

Contents lists available at ScienceDirect

Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim

A family of exotic group C*-algebras

Maria Gerasimova [a], Nicolas Monod [b,*]

[a] *University of Münster, Mathematical Institute, Einsteinstraße 62, 48149 Münster, Germany*
[b] *École Polytechnique Fédérale de Lausanne (EPFL), CH–1015 Lausanne, Switzerland*

ARTICLE INFO

Article history:
Received 12 May 2023
Received in revised form 18 December 2023
Accepted 19 February 2024
Available online 1 March 2024
Communicated by Dan Voiculescu

MSC:
primary 22D25, 46L05

Keywords:
Group C*-algebra
Exotic algebra

ABSTRACT

We show that a large family of groups without non-abelian free subgroups satisfy the following strengthening of non-amenability: they each have a rich supply of irreducible representations defining exotic C*-algebras. The construction is explicit.

1. Introduction

1.1. *Group C*-algebras*

Let Γ be any group; in this note, we consider groups without topology. Two C*-algebras are canonically attached to Γ: the **maximal C*-algebra** $C^*_{\max}(\Gamma)$ and the **reduced C*-algebra** $C^*_{\mathrm{red}}(\Gamma)$. Moreover, there is a canonical epimorphism $C^*_{\max}(\Gamma) \to C^*_{\mathrm{red}}(\Gamma)$, which is an isomorphism if and only if Γ is amenable.

* Corresponding author.
E-mail addresses: mari9gerasimova@mail.ru (M. Gerasimova), nicolas.monod@epfl.ch (N. Monod).

https://doi.org/10.1016/j.aim.2024.109594
0001-8708/

Following [17], [4], [15], [16], [27] and [25], a C*-algebra A is called **exotic** if it lies strictly inbetween $C^*_{\mathrm{max}}(\Gamma)$ and $C^*_{\mathrm{red}}(\Gamma)$. That is, if $C^*_{\mathrm{max}}(\Gamma) \to C^*_{\mathrm{red}}(\Gamma)$ factors through two epimorphisms

$$C^*_{\mathrm{max}}(\Gamma) \longrightarrow A \longrightarrow C^*_{\mathrm{red}}(\Gamma)$$

and neither of them is an isomorphism. Thus, the existence of an exotic C*-algebra is a refinement of the non-amenability of Γ.

The purpose of this note is to describe an explicit situation where Γ admits an uncountable family of different exotic C*-algebras. Moreover, these algebras are defined by concrete *irreducible* representations of Γ. Specifically, we consider the quasi-regular representation algebras associated to uncountably many suitable subgroups of Γ.

In particular, we obtain an explicit and simple parametrization of a huge set in the **primitive dual** $\mathrm{Prim}(\Gamma)$ of the group Γ, reflecting layers upon layers of non-amenability in this dual. Moreover, in our situation Γ is known to be **C*-simple**, which by definition means that $C^*_{\mathrm{red}}(\Gamma)$ is simple, i.e. that the interval between $C^*_{\mathrm{max}}(\Gamma)$ and $C^*_{\mathrm{red}}(\Gamma)$ is a maximal interval (in the poset of quotients maps).

Of particular interest is the fact that our groups Γ do not contain non-abelian free subgroups. First, since exotic algebras constitute a refinement of non-amenability, the von Neumann–Day problem naturally challenges us to find such examples. Secondly, some early constructions of exotic algebras ([4], [24]) were precisely based on the analytical properties of non-abelian free groups and subgroups, namely on so-called L^p-representations (cf. also [9]).

Very different examples can already be found in [6] and [3]; as they are based on non-solvable Lie groups and respectively their lattices, they happen to contain non-abelian free subgroups too. A rather different approach with remarkable properties of quasi-regular C*-algebras can be found in [5]; see also [18, Thm. 7.6]. Finally, a completely general observation from [10, Rem. 2.2] is that whenever Γ is a non-amenable non-Kazhdan group, the (very much non-irreducible) representation $\lambda_\Gamma \oplus \mathbf{1}$ generates an exotic C*-algebra A. Indeed, A maps onto $C^*_{\mathrm{red}}(\Gamma)$ by construction. This projection is not an isomorphism since $\mathbf{1}$ is not weakly contained in λ_Γ by non-amenability (the Hulanicki–Reiter criterion). The fact that $A \cong C^*_{\mathrm{max}}(\Gamma)$ would imply Kazhdan's property is the so-called "Kazhdan projection" criterion, see [26, Lem. 3.1].

1.2. A family of groups

Let $S \subseteq \mathbf{N}$ be any set of prime numbers and denote by $\mathbf{Z}[1/S]$ the ring of S-integers. Following [21,22], we consider the group Γ_S of all piecewise-$\mathbf{SL}_2(\mathbf{Z}[1/S])$ homeomorphisms of the real line $\mathbf{R}$. More precisely, Γ_S consists of all homeomorphisms g for which $\mathbf{R}$ can be cut into finitely many intervals such that, on each interval, g coincides with a projective transformation $x \mapsto \dfrac{ax+b}{cx+d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbf{SL}_2(\mathbf{Z}[1/S])$.

We recall that Γ_S is a "free group free group"; in the notation of [21], $\Gamma_S = H(\mathbf{Z}[1/S])$, except for $S = \varnothing$ where the breakpoint conditions chosen in [21] define a smaller group, see [22, §2].

If we fix S, any subset $T \subseteq S$ provides a subgroup $\Gamma_T < \Gamma_S$. Starting with S infinite, this is an uncountable family of subgroups of Γ_S. This poset, the collection of all subsets $T \subseteq S$, will parametrise the region of interest in the primitive dual $\mathrm{Prim}(\Gamma_S)$ under the map $T \mapsto \mathrm{C}^*(\lambda_{\Gamma_S/\Gamma_T})$ which takes T to the C*-algebra generated by the quasi-regular representation $\lambda_{\Gamma_S/\Gamma_T}$ of Γ_S associated to Γ_T. In other words, $\lambda_{\Gamma_S/\Gamma_T}$ is the Γ_S-representation induced from the trivial Γ_T-representation.

Theorem. *For any non-empty sets $T \subsetneqq S$ of prime numbers, we have an exotic algebra*

$$\mathrm{C}^*_{\max}(\Gamma_S) \xrightarrow{\not\simeq} \mathrm{C}^*(\lambda_{\Gamma_S/\Gamma_T}) \xrightarrow{\not\simeq} \mathrm{C}^*_{\mathrm{red}}(\Gamma_S)$$

with $\lambda_{\Gamma_S/\Gamma_T}$ irreducible and $\mathrm{C}^*_{\mathrm{red}}(\Gamma_S)$ simple.

Moreover, given S, the corresponding quotients maps $\mathrm{C}^*_{\max}(\Gamma_S) \twoheadrightarrow \mathrm{C}^*(\lambda_{\Gamma_S/\Gamma_T})$ are pairwise non-isomorphic as T varies.

Since there is a correspondence between (non-degenerate) representations of $\mathrm{C}^*_{\max}(\Gamma_S)$ and unitary Γ_S-representations [8, §13], all the above statements on group C*-algebras can be reformulated in terms of weak containment and weak inequivalence of various Γ_S-representations. For instance, the last statement means that the various $\lambda_{\Gamma_S/\Gamma_T}$ are pairwise not weakly equivalent Γ_S-representations.

Remark. We do not know, however, whether the various $\mathrm{C}^*(\lambda_{\Gamma_S/\Gamma_T})$ are non-isomorphic *as C*-algebras*. Note that the very definition of exotic C*-algebra is defined by morphisms rather than by objects.

In the opposite direction, the only part of the Theorem that we stated in terms of Γ_S-representations is the irreducibility, which alows us to view $\lambda_{\Gamma_S/\Gamma_T}$ as points in the unitary dual $\widehat{\Gamma}_S$. In the C* language, this amounts to saying that the ideal of $\mathrm{C}^*_{\max}(\Gamma_S)$ defining $\mathrm{C}^*(\lambda_{\Gamma_S/\Gamma_T})$ is a *primitive* ideal.

In conclusion, we have faithfully embedded the entire collection subsets $T \subseteq S$ into the primitive dual $\mathrm{Prim}(\Gamma_S)$ of Γ_S and a fortiori in the unitary dual $\widehat{\Gamma}_S$ since $\mathrm{Prim}(\Gamma_S)$ can be viewed as the Kolmogorov T_0-quotient of $\widehat{\Gamma}_S$. Moreover, this region of $\mathrm{Prim}(\Gamma_S)$ consists entirely of exotic group C*-algebra of Γ_S.

2. Proof of the theorem

We begin by recording a general property of piecewise-projective groups; similar facts were already observed in [21], [7] and [2]. Given any ring $A < \mathbf{R}$, we denote by H_c the subgroup of *compactly supported* piecewise-$\mathbf{SL}_2(A)$ homeomorphisms of $\mathbf{R}$ and by H_c' the derived subgroup of H_c.

Lemma. *For any $h_0 \in \mathbf{SL}_2(A)$ and any compact interval $I \subseteq \mathbf{R}$ with $\infty \notin h_0 I$, there is $h \in H_c'$ such that h and h_0 coincide on I.*

Proof. We first claim that there is $h_1 \in H_c$ coinciding with h_0 on I. Write $I = [u, v]$; we shall construct h_1 on $[v, +\infty)$ with $h_1 v = h_0 v$, and then the same argument can be applied on $(-\infty, u]$ to complete the claim. If h_0 fixes v, we can continue with the identity. If $h_0 v > v$, we can pick $x \in (v, h_0 v)$ close enough to v that $h_0 x \in (h_0 v, +\infty)$. We can choose a hyperbolic element $q \in \mathbf{SL}_2(A)$ whose repelling/attracting fixed points ξ_-, ξ_+ are respectively in (v, x) and in $(h_0 x, +\infty)$. Indeed already for $\mathbf{SL}_2(\mathbf{Z})$ the pairs of fixed points are dense in $\mathbf{P}^1 \times \mathbf{P}^1$. Upon replacing q by a positive power of itself, $qx > h_0 x$. Since on the other hand $q\xi_- = \xi_- < h_0 v < h_0 \xi_-$, there is by continuity some $t \in (\xi_-, x)$ with $qt = h_0 t$. We can now define h_1 by h_0 on $[v, t]$, by q on $[t, \xi_+]$ and by the identity on $[\xi_+, +\infty)$.

The case $h_0 v < v$ is analogous: pick $x > v$ close enough that $h_0 x \in (h_0 v, v)$ and choose q such that $\xi_- \in (v, x)$ and $\xi_+ \in (-\infty, h_0 v)$. Replacing q by a suitable power, we have $qv < h_0 v$ but on the other hand $q\xi_- = \xi_- > v > h_0 x > h_0 \xi_-$. Thus there is $t \in (v, \xi_-)$ with $qt = h_0 t$ and we define h_1 by h_0 on $[v, t]$, by q on $[t, \xi_-]$ and by the identity on $[\xi_-, +\infty)$. The claim is established.

Let now J be a compact interval containing I and the support of h_1. We can choose an element $b_1 \in H_c$ with $b_1^{-1} J \cap J = \varnothing$; this exists e.g. by another application of the first claim, this time for J and a translation $b_0 \in \mathbf{SL}_2(A)$ that translates the left endpoint of J past its right endpoint. Then $b_1 h_1^{-1} b_1^{-1}$ is trivial on I and hence the commutator $h = h_1 b_1 h_1^{-1} b_1^{-1}$ in H_c' has the desired properties. □

2.1. Irreducibility

It is well-known that the representation $\lambda_{\Gamma_S/\Gamma_T}$ is irreducible if and only if Γ_T is self-commensurating in Γ_S. This is generally attributed to Mackey as it follows from Theorem 6' in [20]; we note that it was already proved by Godement in Appendice A p. 80 of [13].

Thus, given any element $g \in \Gamma_S$ not in Γ_T, we need to show that $\Gamma_T^g \cap \Gamma_T$ does not have finite index in both Γ_T and the conjugate Γ_T^g.

To this end, it suffices to find a subgroup $\Lambda < \Gamma_T$ without proper finite index subgroups, e.g. infinite and *simple*, such that Λ^g is not in Γ_T. We now proceed to show that the second derived subgroup $\Lambda = (\Gamma_T)''$ has the required properties.

The simplicity of $H(A)''$ and the identity $H(A)'' = H(A)_c'$ hold for any ring $A < \mathbf{R}$, see [7]. In fact, all this holds more generally for all "locally moving" groups of homeomorphisms of $\mathbf{R}$, see [2, §4]. In our case, $\Gamma_T = H(A)$ with $A = \mathbf{Z}[1/T]$.

Since $g \notin \Gamma_T$, there is an interval J on which g is represented by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbf{SL}_2(\mathbf{Z}[1/S])$ with at least one coefficient not in A. That is, some coefficient contains a negative power of some $p \in S \smallsetminus T$.

Suppose first that this coefficient is a or b. Choose any $q \in T$ and select $n \in \mathbf{N}$ large enough so that the matrix $h_0 = \begin{pmatrix} 1 & 0 \\ q^{-n} & 1 \end{pmatrix}$ satisfies $h_0 I \subseteq \mathrm{Int}(gJ)$ for some compact interval $I \subseteq gJ$. This is possible since h_0 converges to the identity as $n \to +\infty$. Applying the Lemma, we obtain $h \in \Lambda$ given by h_0 on I. The conjugate $g^{-1}hg$ is given on $g^{-1}I \subseteq J$ by a matrix of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ q^{-n} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & -b^2 q^{-n} \\ a^2 q^{-n} & * \end{pmatrix}.$$

Thus the negative power of p is still present in that case and hence $g^{-1}hg$ is not in Γ_T.

If the coefficient is c or d, then we argue similarly with $h_0 = \begin{pmatrix} 1 & q^{-n} \\ 0 & 1 \end{pmatrix}$ and this time the conjugate $g^{-1}h_g$ involves a matrix $\begin{pmatrix} * & d^2 q^{-n} \\ -c^2 q^{-n} & * \end{pmatrix}$, and hence again is not in Γ_T. This completes the proof of irreducibility.

2.2. Unconfinment

Let $\Lambda < \Gamma$ be a subgroup of a group Γ. Following [14], recall that Λ is called **unconfined** in Γ if the closure in the Chabauty space of subgroups of Γ of the conjugation Γ-orbit of Λ contains the trivial subgroup. Explicitly, this simply means that for any finite subset $E \subseteq \Gamma$ not containing the identity, there is $\gamma \in \Gamma$ such that the conjugate $\gamma \Lambda \gamma^{-1}$ does not meet E.

The relevance of this notion to our situation is that it implies that the quasi-regular representation $\lambda_{\Gamma/\Lambda}$ weakly contains the regular representation λ_Γ. Indeed, this follows from Fell's continuity of the induction map, see Theorem 4.2 in [12].

Given $T \subsetneqq S$, we shall now prove that Γ_T is unconfined in Γ_S. Equivalently, we produce a sequence (g_n) in Γ_S such that for every non-trivial $h \in \Gamma_S$, the conjugate $g_n^{-1} h g_n$ is outside Γ_T for all n large enough (depending on h).

To this end, let $p \in S \smallsetminus T$ and define g_n by the element $\begin{pmatrix} p^n & p^{-2n} \\ 0 & p^{-n} \end{pmatrix}$ of $\mathbf{SL}_2(\mathbf{Z}[1/S])$. Note that g_n fixes ∞ and hence defines an element of Γ_S. Consider now any non-trivial $h \in \Gamma_S$. There is some interval $I \subseteq \mathbf{R}$ on which h is represented by an element $h_I = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\mathbf{SL}_2(\mathbf{Z}[1/S])$ which is not $\pm\mathrm{Id}$. On $g_n^{-1}I$, the conjugate $g_n^{-1} h g_n$ is represented by the conjugate of h_I, whose top right corner is computed to be

$$bp^{-2n} + (a-d)p^{-3n} - cp^{-4n}.$$

If this number were in $\mathbf{Z}[1/T]$ for arbitrarily large n, then the three coefficients b, $a-d$ and c would vanish because the exponents of p^{-n} are different. Together with the determinant condition $ad-bc=1$, this implies $a^2=1$ and hence $h_I=\pm\mathrm{Id}$, contrary to our assumption.

2.3. Inequivalence

We now justify that whenever $T, T' \subseteq S$ are two different subsets, the quotient maps $\mathrm{C}^*_{\max}(\Gamma_S) \twoheadrightarrow \mathrm{C}^*(\lambda_{\Gamma_S/\Gamma_T})$ and $\mathrm{C}^*_{\max}(\Gamma_S) \twoheadrightarrow \mathrm{C}^*(\lambda_{\Gamma_S/\Gamma_{T'}})$ are non-isomorphic. Without loss of generality, we can assume $T \not\subseteq T'$ and we shall verify the following more precise statement: there is no vertical morphism for which the following diagram commutes.

$$\begin{array}{ccc} & \nearrow & \mathrm{C}^*(\lambda_{\Gamma_S/\Gamma_{T'}}) \\ \mathrm{C}^*_{\max}(\Gamma_S) & & \Big\downarrow \\ & \searrow & \mathrm{C}^*(\lambda_{\Gamma_S/\Gamma_T}) \end{array}$$

By the correspondence between representations of Γ_S and of $\mathrm{C}^*_{\max}(\Gamma_S)$, this is equivalent to the statement that $\lambda_{\Gamma_S/\Gamma_T}$ is not weakly contained in $\lambda_{\Gamma_S/\Gamma_{T'}}$.

Suppose for a contradiction that this weak containment holds. In particular, the restriction $(\lambda_{\Gamma_S/\Gamma_T})|_{\Gamma_T}$ to Γ_T is weakly contained in the restriction $(\lambda_{\Gamma_S/\Gamma_{T'}})|_{\Gamma_T}$. But $(\lambda_{\Gamma_S/\Gamma_T})|_{\Gamma_T}$ contains the trivial Γ_T-representation. Thus this trivial representation is weakly contained in $(\lambda_{\Gamma_S/\Gamma_{T'}})|_{\Gamma_T}$. This is equivalent to stating that $\Gamma_{T'}$ is *co-amenable to* Γ_T *relative to* Γ_S, see [22]. However, it is proved in [22] that this relative co-amenability does not hold, in fact not even relatively to the larger group $H(\mathbf{Q})$.

2.4. End of proof

The representation $\lambda_{\Gamma_S/\Gamma_T}$ defines a quotient $\mathrm{C}^*(\lambda_{\Gamma_S/\Gamma_T})$ of $\mathrm{C}^*_{\max}(\Gamma_S)$. The fact that the canonical map $\mathrm{C}^*_{\max}(\Gamma_S) \to \mathrm{C}^*_{\mathrm{red}}(\Gamma_S)$ factors through

$$\mathrm{C}^*_{\max}(\Gamma_S) \longrightarrow \mathrm{C}^*(\lambda_{\Gamma_S/\Gamma_T}) \longrightarrow \mathrm{C}^*_{\mathrm{red}}(\Gamma_S) = \mathrm{C}^*(\lambda_{\Gamma_S})$$

is equivalent to λ_{Γ_S} being weakly contained in $\lambda_{\Gamma_S/\Gamma_T}$, which is established in Section 2.2.

For $\mathrm{C}^*(\lambda_{\Gamma_S/\Gamma_T})$ to be exotic, we need to know that neither of the above two morphisms is an isomorphism.

If the first morphism is an isomorphism, then $\mathrm{C}^*(\lambda_{\Gamma_S/\Gamma_T})$ has an epimorphism to the scalar algebra $\mathbf{C}$ because $\mathrm{C}^*_{\max}(\Gamma_S)$ admits such a morphism. This means that the trivial representation is weakly contained in $\lambda_{\Gamma_S/\Gamma_T}$, which happens if and only if Γ_T

is co-amenable in Γ_S, see [11, No. 3, §2]. However, it is shown in [22] that Γ_T is not co-amenable in Γ_S.

The second morphism is an isomorphism if and only if $\lambda_{\Gamma_S/\Gamma_T}$ is weakly contained in λ_{Γ_S}. The latter holds if and only if Γ_T is an amenable group, see Proposition 4.2.1 in [1]. (That reference requires another condition which is trivially satisfied in the current setting of discrete groups.) Thus, using the non-amenability established in [21], we conclude that indeed $\mathrm{C}^*(\lambda_{\Gamma_S/\Gamma_T})$ is exotic.

The fact that $\lambda_{\Gamma_S/\Gamma_T}$ is irreducible was proved above in Section 2.1 and the simplicity of $\mathrm{C}^*_{\mathrm{red}}(\Gamma_S)$ was established in [19]. Finally, the inequivalence was proved in Section 2.3. □

3. Comments

If we only want a group Γ without non-abelian free subgroups but admitting *some* exotic group C*-algebra, then other easy examples from quasi-regular representations associated to subgroups $\Lambda < \Gamma$ can be constructed as follows. Of course, these simple examples will not enjoy the stronger properties listed in the Theorem, in particular the representations will be far from irreducible and therefore they do not describe anything in the dual or primitive dual of Γ.

For the reasons exposed in Section 2, we will have an exotic algebra

$$\mathrm{C}^*_{\mathrm{max}}(\Gamma) \xrightarrow{\not\cong} \mathrm{C}^*(\lambda_{\Gamma/\Lambda}) \xrightarrow{\not\cong} \mathrm{C}^*_{\mathrm{red}}(\Gamma)$$

provided the following three conditions are all satisfied:

(i) Λ is not amenable;
(ii) Λ is unconfined in Γ;
(iii) Λ is not co-amenable in Γ.

Start with any non-amenable group Λ without non-abelian free subgroups. Consider the "lamplighter" restricted wreath product

$$\Gamma = \Lambda \wr Z = \Big(\bigoplus_{z\in Z} \Lambda_z\Big) \rtimes Z$$

where Z is any infinite group without non-abelian free subgroups; e.g. $Z = \mathbf{Z}$ or $Z = \Lambda$. Here Λ_z denotes a copy of Λ for each $z \in Z$. Note that Γ still has no non-abelian free subgroups. View Λ as a subgroup of Γ, say $\Lambda = \Lambda_e$ at the coordinate $e \in Z$.

Condition (i) holds by construction. For (ii), let $(z_n)_{n\geq 1}$ be any sequence in Z leaving any finite subset, viewed as a sequence in Γ. Then Λ^{z_n} converges to the trivial subgroup in the Chabauty space because $\Lambda^{z_n} = \Lambda_{z_n^{-1}}$, while any given element of $\bigoplus_{z\in Z} \Lambda_z$ has finite support in Z.

Condition (iii) is immediate if Z is non-amenable, because if Λ were co-amenable in Γ, then so would be the normal subgroup $\bigoplus_{z\in Z}\Lambda_z$, which is equivalent to the quotient Z being amenable.

However, condition (iii) does indeed hold more generally as soon as Z is non-trivial, e.g. $Z = \mathbf{Z}$. Suppose indeed for a contradiction that μ is a Γ-invariant mean on Γ/Λ. Consider a general element $\gamma \in \Gamma$ with coordinates $\gamma = ((\lambda_z)_{z\in Z}, x)$. Given $y \in Z$, the stabiliser in Λ_y of the point $\gamma\Lambda \in \Gamma/\Lambda$ is $\Lambda_y \cap \gamma\Lambda_e\gamma^{-1}$. On the other hand, $\gamma\Lambda_e\gamma^{-1} = \Lambda_x$ and therefore this stabiliser is trivial whenever $x \neq y$. In conclusion, all orbits of the Λ_y-action on Γ/Λ are regular orbits except the orbits of the points $\gamma\Lambda$ where $\gamma = ((\lambda_z)_{z\in Z}, y)$. Since μ is Λ_y-invariant and $\Lambda_y \cong \Lambda$ is non-amenable, μ is supported on the complement of the union of regular Λ_y-orbits. That is, μ is supported on the union of the orbits of the form $(*, y)\Lambda$. Applying the same argument to any $y' \neq y$, which exists since Z is non-trivial, leads to a contradiction.

We observe that when $Z = \mathbf{Z}$, this non-co-amenability contrasts with the co-amenability of $\bigoplus_{n\geq 0}\Lambda_z$ in $\Lambda \wr \mathbf{Z}$, see [23].

Acknowledgments

We thank Mehrdad Kalantar and Tim de Laat for their comments on the context of exotic algebras. We are very grateful to the anonymous referee for their remarks and suggestions, particularly for pointing out that a comment made in an earlier version of this note was unfounded.

During this research M.G. was supported by the DFG, Project-ID 427320536, SFB 1442, and under Germany's Excellence Strategy EXC 2044 390685587, Mathematics Münster: Dynamics-Geometry-Structure.

References

[1] C. Anantharaman-Delaroche, On spectral characterizations of amenability, Isr. J. Math. 137 (2003) 1–33.
[2] J. Brum, N.M. Bon, C. Rivas, M. Triestino, Locally moving groups acting on the line and ℝ-focal actions, Preprint, arXiv:2104.14678v1, 2021.
[3] M. El Bachir Bekka, On the full C^*-algebras of arithmetic groups and the congruence subgroup problem, Forum Math. 11 (6) (1999) 705–715.
[4] N.P. Brown, E.P. Guentner, New C*-completions of discrete groups and related spaces, Bull. Lond. Math. Soc. 45 (6) (2013) 1181–1193.
[5] M. El Bachir Bekka, M. Kalantar, Quasi-regular representations of discrete groups and associated C^*-algebras, Trans. Am. Math. Soc. 373 (3) (2020) 2105–2133.
[6] M. El Bachir Bekka, E. Kaniuth, A. To-Ming Lau, G. Schlichting, On C^*-algebras associated with locally compact groups, Proc. Am. Math. Soc. 124 (10) (1996) 3151–3158.
[7] J. Burillo, Y. Lodha, L.D. Reeves, Commutators in groups of piecewise projective homeomorphisms, Adv. Math. 332 (2018) 34–56.
[8] J. Dixmier, Les C^*-algèbres et leurs représentations, Les Grands Classiques Gauthier-Villars, Éditions Jacques Gabay, Paris, 1996, Reprint of the second (1969) edition.
[9] T. de Laat, T. Siebenand, Exotic group C^*-algebras of simple Lie groups with real rank one, Ann. Inst. Fourier (Grenoble) 71 (5) (2021) 2117–2136.

[10] S. Echterhoff, J.C. Quigg, Induced coactions of discrete groups on C^*-algebras, Can. J. Math. 51 (4) (1999) 745–770, MR 1701340.
[11] P. Eymard, Moyennes invariantes et représentations unitaires, Lecture Notes in Mathematics, vol. 300, Springer-Verlag, Berlin, 1972.
[12] J.M.G. Fell, Weak containment and induced representations of groups. II, Trans. Am. Math. Soc. 110 (1964) 424–447.
[13] R. Godement, Les fonctions de type positif et la théorie des groupes, Trans. Am. Math. Soc. 63 (1948) 1–84.
[14] B. Hartley, A.E. Zalesskiĭ, On simple periodic linear groups—dense subgroups, permutation representations, and induced modules, Isr. J. Math. 82 (1–3) (1993) 299–327.
[15] S.P. Kaliszewski, M.B. Landstad, J.C. Quigg, Exotic group C^*-algebras in noncommutative duality, N.Y. J. Math. 19 (2013) 689–711, MR 3141810.
[16] S.P. Kaliszewski, M.B. Landstad, J.C. Quigg, Exotic coactions, Proc. Edinb. Math. Soc. (2) 59 (2) (2016) 411–434.
[17] D. Kyed, P.M. Sołtan, Property (T) and exotic quantum group norms, J. Noncommut. Geom. 6 (4) (2012) 773–800.
[18] M. Kalantar, E. Scarparo, Boundary maps, germs and quasi-regular representations, Adv. Math. 394 (2022) 108130.
[19] A. Le Boudec, N. Matte Bon, Subgroup dynamics and C^*-simplicity of groups of homeomorphisms, Ann. Sci. Éc. Norm. Supér. (4) 51 (3) (2018) 557–602.
[20] G.W. Mackey, On induced representations of groups, Am. J. Math. 73 (1951) 576–592.
[21] N. Monod, Groups of piecewise projective homeomorphisms, Proc. Natl. Acad. Sci. USA 110 (12) (2013) 4524–4527.
[22] N. Monod, Some comments on piecewise-projective groups of the line, Preprint, https://doi.org/10.48550/arXiv.2305.00796, 2023.
[23] N. Monod, S. Popa, On co-amenability for groups and von Neumann algebras, C. R. Math. Acad. Sci. Soc. R. Can. 25 (3) (2003) 82–87.
[24] R. Okayasu, Free group C^*-algebras associated with ℓ_p, Int. J. Math. 25 (7) (2014) 1450065.
[25] Z.-J. Ruan, M. Wiersma, On exotic group C*-algebras, J. Funct. Anal. 271 (2) (2016) 437–453.
[26] A. Valette, Minimal projections, integrable representations and property (T), Arch. Math. (Basel) 43 (5) (1984) 397–406.
[27] M. Wiersma, Constructions of exotic group C^*-algebras, Ill. J. Math. 60 (3–4) (2016) 655–667.