# Error estimates for SUPG-stabilised Dynamical Low Rank Approximations 

Fabio Nobile ${ }^{1}$ and Thomas Trigo Trindade ${ }^{1}$<br>${ }^{1}$ CSQI, École Polytechnique Fédérale de Lausanne, Switzerland

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#### Abstract

We perform an error analysis of a fully discretised Streamline Upwind Petrov Galerkin Dynamical Low Rank (SUPG-DLR) method for random time-dependent advection-dominated problems. The time integration scheme has a splitting-like nature, allowing for potentially efficient computations of the factors characterising the discretised random field. The method allows to efficiently compute a low-rank approximation of the true solution, while naturally "inbuilding" the SUPG stabilisation. Standard error rates in the $\|\cdot\|_{L^{2}}$ and $\|\cdot\|_{\text {SUPG }}$-norms are recovered. Numerical experiments validate the predicted rates.


## 1 Introduction

The simulation of random time-dependent advection-dominated problems

$$
\begin{equation*}
\partial_{t} u-\varepsilon \Delta u+\mathbf{b} \cdot \nabla u+c u=f, \quad \text { in } D \subset \mathbb{R}^{d}, \tag{1}
\end{equation*}
$$

with coefficients $\varepsilon, \mathbf{b}, c$ and data $f$ depending on some random parameter $\omega \in \Omega$, with probability measure $\mu$ on $\Omega$, remains a challenge for multiple reasons. These processes often have poorly decaying Kolmogorov $n$-widths in the time-space domain, even if at each point in time the solution profile is well-approximated by a small subspace. Furthermore, it is well-known that applying the standard Finite Element Method to such problems causes the numerical solution to display unphysical spurious oscillations, in particular when the solution has sharp gradients and/or boundary layers. For practical purposes, it becomes necessary to remove or alleviate these oscillations by using some stabilisation strategy.

The purpose of [11] was to introduce the generalised Petrov-Galerkin Dynamical Low Rank (PG-DLR) framework and its particularisation to the Streamline Upwind/Petrov-Galerkin (SUPG-DLR), which allows to simultaneously tackle both issues. The Dynamical Low Rank (DLR) 8 framework, in this work written in its Dynamically Orthogonal (DO) 13 formalism, consists in seeking an approximation of the form $u_{\mathrm{DLR}}=\sum_{i=1}^{R} U_{i}(t, x) Y_{i}(t, \omega)$ of the solution $u_{\text {true }}(t, x, \omega)$ of (1). The peculiar feature of this framework is that the physical $\left\{U_{i}(t, x)\right\}_{i=1}^{R}$ and the stochastic modes $\left\{Y_{i}(t, \omega)\right\}_{i=1}^{R}$ evolve in time to follow a (quasi-)optimal low-rank approximation of $u_{\text {true }}$, making it suited for the type of transport-dominated problems described above. As an extension of that framework, the PG-DLR framework allows to seamlessly import many stabilisation techniques that can be framed as generalised Petrov-Galerkin problems.

The focus of this paper is an error analysis of the SUPG-DLR framework. This work inscribes itself within a growing body of literature addressing the stabilisation of Reduced Order Models, including e.g. 15, 3 for SUPG-stabilised POD methods for advection-dominated problems. An error analysis for the SUPG-stabilised POD method was carried out in 4 for time-dependent advection-diffusion-reaction problems. In the DO setting, a noteworthy alternative to our method is the stabilisation based on Shapiro filters in [2], applied after each time step to smooth out the oscillations.

## 2 Problem setting \& SUPG-DLR approximations

Solutions to random PDEs are function-valued random variables. In this work, we consider the advection-diffusion-reaction problem 1 with homogeneous Dirichlet boundary conditions $u=0$ on $\partial D$ and initial condition $u_{\mid t=0}=u_{0} \in L_{\hat{\mu}}^{2}\left(H_{0}^{1}(D)\right)$. The coefficients verify the following the Coefficient Assumptions (CoefA): $\varepsilon>0, c \in L_{\hat{\mu}}^{\infty}\left(L^{\infty}(D)\right)$ and $c(x, \omega) \geq c_{0}>0$ for a.e. $\quad x \in D, \forall \omega \in \hat{\Omega}, f \in L_{\hat{\mu}}^{2}\left(L^{2}(D)\right), \mathbf{b} \in\left(L^{\infty}(D)\right)^{d}$, $\operatorname{div} \mathbf{b}(x)=0$. Therefore the solution $u_{\text {true }}(t, \cdot, \omega)$ belongs to $H_{0}^{1}(D)$ for (almost) every $t>0$ and $\omega \in \Omega$. The probability space is discretised via a collocation method (e.g., the Monte Carlo method), yielding the collocation points $\hat{\Omega}:=\left\{\omega_{i}\right\}_{i=1}^{N_{C}} \subset \Omega$ and a discrete measure $\hat{\mu}$. $L_{\hat{\mu}}^{2}(\hat{\Omega})$ denotes the space of random variables, with scalar product $\mathbb{E}_{\hat{\mu}}[Y Z]=\sum_{i=1}^{N_{C}} m_{i} Y_{i} Z_{i}$, where $\left\{m_{i}\right\}_{i=1}^{N_{C}}$ are positive weights summing up to 1 , and $Y_{i}=Y\left(\omega_{i}\right), Z_{i}=Z\left(\omega_{i}\right)$. The random solution $u_{\text {true }}(t, \cdot, \cdot)$ satisfies for almost every $t, u \in L_{\hat{\mu}}^{2}(\hat{\Omega}, X):=L_{\hat{\mu}}^{2}(X)$, where $X=H_{0}^{1}(D)$ (with standard $H_{0}^{1}$-scalar product) or $L^{2}(D)$. These Bochner spaces admit the scalar product $(u, v)_{L_{\hat{\mu}}^{2}(X)}=$ $\sum_{i=1}^{N_{C}} m_{i}\left\langle u\left(\omega_{i}\right), v\left(\omega_{i}\right)\right\rangle_{X}$. Hereafter, we use the shorthand notation $(\cdot, \cdot)$ and $\|\cdot\|$ to denote the $\left.L_{\hat{\mu}}^{2}=L^{2}(D)\right)$ inner product and norm.

We use the Finite Elements Method on a quasi-uniform mesh $\mathcal{T}_{h}$ with characteristic mesh size $h$, and consider the space of continuous piece-wise polynomials of degree $k, V_{h}:=\mathbb{P}_{k}^{C}\left(\mathcal{T}_{h}\right) \subset$ $H_{0}^{1}(D)$ where $k$ denotes the polynomial degree and $N_{h}:=\left|V_{h}\right|$. In this work, we will consider the advection-dominated regime with the condition $\|\mathbf{b}\|_{L^{\infty}} h>2 \varepsilon$ assumed true hereafter.

The numerical approximation $\tilde{u}_{h, \hat{\mu}}$ is sought in $V_{h} \otimes L_{\hat{\mu}}^{2}$. The inverse inequality from standard Finite Element theory can be extended to elements in $V_{h} \otimes L_{\hat{\mu}}^{2}$, yielding $\left\|\nabla \tilde{u}_{h, \hat{\mu}}\right\| \leq$ $C_{I} h^{-1}\left\|\tilde{u}_{h, \hat{\mu}}\right\|$ for some $C_{I}>0$ and every $\tilde{u}_{h, \hat{\mu}} \in V_{h} \otimes L_{\hat{\mu}}^{2}$, as the inequality holds pointwise in $\omega$. For the same reasons, the standard Poincaré inequality can be extended to $V_{h} \otimes L_{\hat{\mu}}^{2}$, yielding $\left\|\tilde{u}_{h, \hat{\mu}}\right\| \leq C_{P}\left\|\nabla \tilde{u}_{h, \hat{\mu}}\right\|$, where $C_{P}$ is the Poincaré constant. Hereafter, to lighten the notation, $\tilde{u} \equiv \tilde{u}_{h, \hat{\mu}} \in V_{h} \otimes L_{\hat{\mu}}^{2}$.

The DLR approximation belongs to the differential manifold of $R$-rank functions, defined as

$$
\begin{align*}
\mathcal{M}_{R}=\left\{\tilde{u} \in V_{h} \otimes L_{\hat{\mu}}^{2}(\hat{\Omega}): \tilde{u}=\right. & \sum_{i=1}^{R} U_{i} Y_{i}, \text { s.t. } \mathbb{E}_{\hat{\mu}}\left[Y_{i} Y_{j}\right]=\delta_{i j} \\
& \left.\left\{U_{i}\right\}_{i=1}^{R} \text { lin. ind. and }\left\{U_{i}\right\}_{i=1}^{R} \in V_{h},\left\{Y_{i}\right\}_{i=1}^{R} \in L_{\hat{\mu}}^{2}(\hat{\Omega})\right\} \tag{2}
\end{align*}
$$

Each point $u \in \mathcal{M}_{R}$ can be equipped with a tangent space, spanned by tangent vectors $\delta u=$ $\sum_{i=1}^{R} \delta u_{i} Y_{i}+U_{i} \delta y_{i}$, uniquely identified by imposing the Dual Dynamically Orthogonal (Dual $\mathrm{DO})$ condition [10], $\mathbb{E}\left[Y_{i} \delta y_{j}\right]=0$ for $i, j=1, \ldots, R$. This leads to the following characterisation

$$
\begin{align*}
& \mathcal{T}_{u} \mathcal{M}_{R}=\left\{\delta u=\sum_{i=1}^{R} \delta u_{i} Y_{i}+U_{i} \delta y_{i}, \text { such that }\left\{\delta u_{i}\right\}_{i=1}^{R} \in V_{h}\right. \\
& \left.\qquad\left\{\delta y_{i}\right\}_{i=1}^{R} \in L_{\hat{\mu}}^{2}(\hat{\Omega}), \mathbb{E}_{\hat{\mu}}\left[\delta y_{i} Y_{j}\right]=0, \forall 1 \leq i, j \leq R\right\} . \tag{3}
\end{align*}
$$

Given $U=\left(U_{1}, \ldots, U_{R}\right)$ and $Y=\left(Y_{1}, \ldots, Y_{R}\right)$ s.t. $u=U Y^{\top}$, the tangent space at $u$ is denoted by $\mathcal{T}_{U Y^{\top}} \mathcal{M}_{R}$. Furthermore, for an $L_{\hat{\mu}}^{2}$-orthonormal set $Y$, let $\mathcal{Y}:=\operatorname{span}\left(Y_{1}, \ldots, Y_{R}\right)$, and $\mathcal{P}_{\mathcal{Y}}[v]=\sum_{i=1}^{R} \mathbb{E}\left[v Y_{i}\right] Y_{i}$ and $\mathcal{P} \frac{\perp}{\mathcal{Y}}[v]=v-\mathcal{P}_{\mathcal{Y}}[v]$.

To recover dynamic equations for the physical and stochastic modes, the idea is to project Equation (1) onto the tangent space $\mathcal{T}_{U Y^{\top}} \mathcal{M}_{R}$ at each time instant. The SUPG-DLR framework proposes to solve the problem

$$
\begin{align*}
\left(\dot{u}_{\mathrm{DLR}}, \tilde{v}+\delta \mathbf{b} \cdot \nabla \tilde{v}\right)+a_{\mathrm{SUPG}}\left(u_{\mathrm{DLR}}, \tilde{v}\right)=(f, \tilde{v}+\delta \mathbf{b} \cdot \nabla \tilde{v}) & \forall \tilde{v} \in \mathcal{T}_{u_{\mathrm{DLR}}} \mathcal{M}_{R}, \text { a.e. } t \in(0, T]
\end{align*}
$$

with

$$
\begin{aligned}
& a_{\mathrm{SUPG}}(\tilde{u}, \tilde{v})=(\varepsilon \nabla \tilde{u}, \nabla \tilde{v})+(\mathbf{b} \cdot \nabla \tilde{u}, \tilde{v})+(c \tilde{u}, \tilde{v}) \\
&+\sum_{K \in \mathcal{T}_{h}} \delta_{K}(-\varepsilon \Delta \tilde{u}+\mathbf{b} \cdot \nabla \tilde{u}+c \tilde{u}, \mathbf{b} \cdot \nabla \tilde{v})_{K, L_{\tilde{\mu}}^{2}},
\end{aligned}
$$

where $(\cdot, \cdot)_{K, L_{\mu}^{2}}:=(\cdot, \cdot)_{L_{\mu}^{2}\left(L^{2}(K)\right)}$. Hereafter, we use a uniform stabilisation parameter $\delta \equiv \delta_{K}$ for each $K \in \mathcal{T}_{h}$.

Particularising the conditions in (11 to our setting, if CoefA and

$$
\begin{equation*}
\delta \leq \min _{K \in \mathcal{T}_{h}}\left\{\frac{1}{2\|c\|_{L_{\mu}^{\infty}\left(L^{\infty}\right)}}, \frac{h_{K}^{2}}{2 \varepsilon C_{I}^{2}}, \frac{h_{K}}{\|\mathbf{b}\|_{L^{\infty} C_{I}}}\right\} \tag{5}
\end{equation*}
$$

hold true, then

$$
\begin{equation*}
a_{\mathrm{SUPG}}(\tilde{u}, \tilde{u}) \geq \frac{1}{2}\|\tilde{u}\|_{\mathrm{SUPG}}^{2} \tag{6}
\end{equation*}
$$

where $\|\tilde{u}\|_{\text {SUPG }}^{2}=\varepsilon\|\nabla \tilde{u}\|^{2}+\delta \sum_{K \in \mathcal{T}_{h}}\|\mathbf{b} \cdot \nabla \tilde{u}\|_{K, L_{\tilde{\mu}}^{2}}^{2}+\left\|c^{1 / 2} \tilde{u}\right\|^{2}$. This norm is suitable for advection-dominated problems, as it offers a better control of the stream-line diffusion. As an immediate consequence of $[5],\|\tilde{v}+\delta \mathbf{b} \cdot \nabla \tilde{v}\| \leq 2\|\tilde{v}\|$. Two additional properties of the SUPG setting are summarised below :
Lemma 2.1. Assuming CoefA, it holds

$$
\begin{equation*}
a_{\mathrm{SUPG}}(\tilde{u}, \tilde{v}) \leq C_{1}\|\nabla \tilde{u}\|\|\tilde{v}\|, \quad\|\tilde{u}\| \leq c_{0}^{-1}\|\tilde{u}\|_{\mathrm{SUPG}}, \tag{7}
\end{equation*}
$$

where $C_{1}=\left(C_{I}+2\right)\|\mathbf{b}\|_{L^{\infty}}+2 C_{P}\|c\|_{L_{\mu}^{\infty}\left(L^{\infty}\right)}$.
Proof. We detail the proof for some terms, the bounds for the others being direct. Firstly, $\varepsilon|(\nabla \tilde{u}, \nabla \tilde{v})| \leq\|\nabla \tilde{u}\|\|\varepsilon \nabla \tilde{v}\| \leq \frac{C_{I}\|\mathbf{b}\|_{L^{\infty}}}{2}\|\nabla \tilde{u}\|\|\tilde{v}\|$, having used $\varepsilon<\frac{1}{2}\|\mathbf{b}\|_{L^{\infty}} h$ and the inverse inequality. Additionally, letting $C_{2}=\frac{C_{I}}{2}\|\mathbf{b}\|_{L^{\infty}}$,

$$
\left|\delta \sum_{K \in \mathcal{T}_{h}}(\varepsilon \Delta \tilde{u}, \mathbf{b} \cdot \nabla \tilde{v})_{K, L_{\tilde{\mu}}^{2}}\right| \leq C_{2} \sum_{K \in \mathcal{T}_{h}}\|\nabla \tilde{u}\|_{K, L_{\tilde{\mu}}^{2}}\|\tilde{v}\|_{K, L_{\tilde{\mu}}^{2}} \leq C_{2}\|\nabla \tilde{u}\|\|\tilde{v}\| .
$$

In [11, we use Algorithm 1 reproduced below to sequentially update the physical and stochastic modes in a (potentially) cheap fashion, resulting in a non-linear update on $\mathcal{M}_{R}$. The algorithm was originally proposed and analysed in [6] for random uniform coercive problems, and is very similar to the Projector-Splitting algorithm [9]. In this work, we focus on the implicit version of the scheme; however, semi-implicit and fully explicit versions are also possible.
Algorithm 1. Given the solution $u_{h, \hat{\mu}}^{n}=\sum_{i=1}^{R} U_{i}^{n} Y_{i}^{n}$ :

1. Find $\tilde{U}_{j}^{n+1}, j=1, \ldots, R$, such that

$$
\begin{align*}
\Delta t^{-1}\left(\tilde{U}_{j}^{n+1}-U_{j}^{n}, v_{h}+\delta \mathbf{b} \cdot \nabla v_{h}\right)_{L^{2}(D)} & +a_{\mathrm{SUPG}}\left(u_{h, \hat{\mu}}^{n+1}, v_{h} Y_{j}^{n}\right) \\
& =\left(f^{n+1}, v_{h} Y_{j}^{n}+\mathbf{b} \cdot \nabla v_{h} Y_{j}^{n}\right), \quad \forall v_{h} \in V_{h} . \tag{8}
\end{align*}
$$

2. Find $\tilde{Y}_{j}^{n+1}, j=1, \ldots, R$ such that $\left(\tilde{Y}_{j}^{n+1}-Y_{j}^{n}\right) \in \mathcal{Y}^{\perp}=\mathcal{P} \frac{\perp}{\mathcal{Y}}\left(L_{\hat{\mu}}^{2}\right)$ and

$$
\begin{align*}
\triangle t^{-1} \sum_{i=1}^{R} \mathbb{E}\left[\left(\tilde{Y}_{i}^{n+1}-Y_{i}^{n}\right) z\right] \tilde{W}_{i j}^{n+1} & +a_{\mathrm{SUPG}}\left(u_{h, \hat{\mu}}^{n+1}, \tilde{U}_{j}^{n+1} \mathcal{P} \mathcal{Y} z\right) \\
= & \left(f^{n+1}, \tilde{U}_{j}^{n+1} \mathcal{P} \frac{\perp}{\mathcal{Y}} z+\delta \mathbf{b} \nabla \tilde{U}_{j}^{n+1} \mathcal{P} \frac{\mathcal{Y}}{\perp} z\right), \quad \forall z \in L_{\hat{\mu}}^{2} . \tag{9}
\end{align*}
$$

where $\tilde{W}_{i j}^{n+1}=\left(\tilde{U}_{i}^{n+1}, \tilde{U}_{j}^{n+1}+\delta \mathbf{b} \nabla \tilde{U}_{j}^{n+1}\right)_{L^{2}(D)}$.
3. Reorthonormalise $\tilde{Y}^{n+1}$ such that $\mathbb{E}\left[Y_{i}^{n+1} Y_{j}^{n+1}\right]=\delta_{i j}$ and modify $\left\{\tilde{U}_{i}^{n+1}\right\}_{i=1}^{R}$ such that $\sum_{i=1}^{R} \tilde{U}_{i}^{n+1} \tilde{Y}_{i}^{n+1}=\sum_{i=1}^{R} U_{i}^{n+1} Y_{i}^{n+1}$.
4. The new solution is given by $u_{h, \hat{\mu}}^{n+1}=\sum_{i=1}^{R} U_{i}^{n+1} Y_{i}^{n+1}$.

When applying Algorithm 1 the update verifies a variational formulation (Proposition 2.1) which allows to analyse the scheme using variational methods and, among others, prove normstability of the scheme (Proposition 2.2.
Proposition 2.1. (from [11]) The numerical solution by Algorithm 1 satisfies

$$
\begin{align*}
\frac{1}{\triangle t}\left(u_{h, \hat{\mu}}^{n+1}-u_{h, \hat{\mu}}^{n}, v_{h, \hat{\mu}}+\delta \mathbf{b} \cdot \nabla v_{h, \hat{\mu}}\right)+a_{\mathrm{SUPG}}\left(u_{h, \hat{\mu}}^{n+1}, v_{h, \hat{\mu}}\right)=( & \left.f^{n+1}, v_{h, \hat{\mu}}+\delta \mathbf{b} \cdot \nabla v_{h, \hat{\mu}}\right) \\
& \forall v_{h, \hat{\mu}} \in \mathcal{T}_{\tilde{U}^{n+1}\left(Y^{n}\right)^{\top}} \mathcal{M}_{R} . \tag{10}
\end{align*}
$$

Proposition 2.2. (from [11]) Assuming $\delta$ verifies (5) and $\delta \leq \Delta t / 4$, then it holds for the numerical solution computed by Algorithm 1

$$
\begin{equation*}
\left\|u_{h, \hat{\mu}}^{N}\right\|^{2}+\sum_{n=1}^{N} \triangle t\left\|u_{h, \hat{\mu}}^{n}\right\|_{\mathrm{SUPG}}^{2} \leq\left\|u_{h, \hat{\mu}}^{0}\right\|^{2}+\triangle t\left(\frac{4}{c_{0}}+4 \delta\right) \sum_{j=1}^{N}\left\|f^{j}\right\|^{2} . \tag{11}
\end{equation*}
$$

## 3 Error estimate

The idea of the SUPG method is to skew the test space by $\mathcal{H}=(I+\delta \mathbf{b} \cdot \nabla)$. Its ajoint is given by $\mathcal{H}^{*}=I-\delta \mathbf{b} \cdot \nabla$ thanks to the zero-divergence of $\mathbf{b}$. Denote $\mathcal{P}_{\mathcal{H}^{*}}: V_{h} \otimes L_{\hat{\mu}}^{2} \rightarrow \mathcal{T}_{u} \mathcal{M}_{R}$ the oblique projection on the tangent space:

$$
\begin{equation*}
\left(\mathcal{P}_{\mathcal{H}^{*}} \tilde{u}, \mathcal{H}^{*} w\right)=\left(\tilde{u}, \mathcal{H}^{*} w\right) \quad \forall w \in \mathcal{T}_{u} \mathcal{M}_{R} . \tag{12}
\end{equation*}
$$

Its well-posedness is ensured by the coercivity of $\left(u, \mathcal{H}^{*} u\right)=\|u\|^{2}$ on $V_{h} \otimes L_{\hat{\mu}}^{2}$. Hereafter, we use the shorthand notation $\tilde{v}^{\perp}:=\mathcal{P}_{\mathcal{H}}{ }^{*} \tilde{v}=\tilde{v}-\mathcal{P}_{\mathcal{H}^{*}} \tilde{v}$ for any $\tilde{v} \in V_{h} \otimes L_{\hat{\mu}}^{2}(\hat{\Omega})$. By definition of the projection,

$$
\begin{equation*}
\left(\mathcal{P}_{\mathcal{H}^{*}}^{\perp} \tilde{v}, \mathcal{H}^{*} w\right)=0 \quad \forall w \in \mathcal{T}_{u} \mathcal{M}_{R} \tag{13}
\end{equation*}
$$

A useful property of the oblique projection is the following :

## Lemma 3.1.

$$
\begin{equation*}
\left\|I-\mathcal{P}_{\mathcal{H}^{*}}\right\|=\left\|\mathcal{P}_{\mathcal{H}^{*}}\right\| \leq 3 \tag{14}
\end{equation*}
$$

Proof. The first equality is a standard result of projectors [14. Consider the orthogonal projector $\Pi: V_{h} \otimes L_{\hat{\mu}}^{2} \rightarrow \mathcal{T}_{u} \mathcal{M}_{R}$ verifying $(\Pi \tilde{u}, w)=(\tilde{u}, w)$ for $w \in \mathcal{T}_{u} \mathcal{M}_{R}$, we have

$$
\begin{align*}
\left\|\left(\mathcal{P}_{\mathcal{H}^{*}}-\Pi\right) \tilde{u}\right\|^{2}=\left(\left(\mathcal{P}_{\mathcal{H}^{*}}\right.\right. & \left.-\Pi) \tilde{u},(I-\delta \mathbf{b} \nabla)\left(\mathcal{P}_{\mathcal{H}^{*}}-\Pi\right) \tilde{u}\right) \\
& =\left(\tilde{u}-\Pi \tilde{u},(I-\delta \mathbf{b} \nabla)\left(\mathcal{P}_{\mathcal{H}^{*}}-\Pi\right) \tilde{u}\right) \leq 2\left\|\Pi^{\perp} \tilde{u}\right\|\left\|\left(\mathcal{P}_{\mathcal{H}^{*}}-\Pi\right) \tilde{u}\right\| \tag{15}
\end{align*}
$$

From there, we conclude $\left\|\mathcal{P}_{\mathcal{H}^{*}} \tilde{u}\right\| \leq\left\|\left(\mathcal{P}_{\mathcal{H}^{*}}-\Pi\right) \tilde{u}\right\|+\|\Pi \tilde{u}\| \leq 3\|\tilde{u}\|$.
We will make use of the following assumptions to analyse the convergence of the SUPGDLR method. The first is the standard Model Error Assumption, particularised to the SUPGcontext. It asks that the dynamics neglected by the DLR approximation is negligible. This is a standard assumption made to analyse the convergence of DLR approximations [1, 7, 8,
Assumption 3.1. (Model Error Assumption) For $n=0, \ldots, N-1$, let $\hat{u}^{n}=\tilde{U}^{n+1} Y^{n}$ be the "intermediate" point obtained by Algorithm 1. For $\nu \ll 1$, it holds

$$
\begin{equation*}
\left|a_{\mathrm{SUPG}}\left(\hat{u}^{n}, v_{h, \hat{\mu}}^{\perp}\right)-\left(f, \mathcal{H} v_{h, \hat{\mu}}^{\perp}\right)\right| \leq \nu\|\tilde{v}\|, \quad \forall \tilde{v} \in V_{h} \otimes L_{\hat{\mu}}^{2}, \quad \text { for } \nu \ll 1 . \tag{16}
\end{equation*}
$$

The second is an assumption on the $H^{1}$-stability of the physical basis.
Assumption 3.2. (Local basis inverse inequality) Given the DLR iterates $\left\{u_{h, \hat{\mu}}^{n}=\tilde{U}^{n} \tilde{Y}^{n}\right\}_{n=1}^{N}$ obtained via Algorithm 1 , and denoting $\left(\mathbb{S}_{n}\right)_{i j}=\left(\nabla \tilde{U}_{i}^{n+1}, \nabla \tilde{U}_{j}^{n+1}\right)_{L^{2}(D)}$ and $\left(\mathbb{M}_{n}\right)_{i j}=\left(\tilde{U}_{i}^{n+1}, \tilde{U}_{j}^{n+1}\right)_{L^{2}(D)}$ the stiffness and mass matrices associated to the physical basis $\left\{\tilde{U}_{i}^{n}\right\}_{i=1}^{R}$, there exists a constant $C_{\text {lbi }}<\infty$ such that

$$
\begin{equation*}
\max _{n=0, \ldots, N}\left(\sup _{x \in \mathbb{R}^{R}} \frac{x^{\top} \mathbb{S}_{n} x}{x^{\top} \mathbb{M}_{n} x}\right) \leq C_{\mathrm{lbi}} \tag{17}
\end{equation*}
$$

The functions $\left(U_{1}^{n}, \ldots, U_{R}^{n}\right)$ are typically globally supported and display regularity, justifying a moderate value for $C_{\text {lbi }}$. Assumption 3.2 implies that, for any $n \geq 0$,

$$
\begin{equation*}
\left\|\nabla \tilde{U}^{n} Z^{\top}\right\| \leq C_{\mathrm{lbi}}\left\|\tilde{U}^{n} Z^{\top}\right\| \quad \text { for } Z \in\left[L_{\hat{\mu}}^{2}\right]^{R} \tag{18}
\end{equation*}
$$

The elliptic projection operator $\pi: L_{\hat{\mu}}^{2}\left(\hat{\Omega}, H_{0}^{1}(D)\right) \rightarrow V_{h} \otimes L_{\hat{\mu}}^{2}$ is defined by

$$
\begin{equation*}
\left(\nabla(u-\pi u), \nabla v_{h, \hat{\mu}}\right)=0, \quad \forall v_{h, \mu} \in V_{h} \otimes L_{\hat{\mu}}^{2} \tag{19}
\end{equation*}
$$

For brevity, denote $\pi^{n} u=\pi u\left(t_{n}\right)$. We split $u_{h, \hat{\mu}}^{n}-u\left(t_{n}\right)=\left(u_{h, \hat{\mu}}^{n}-\pi^{n} u\right)+\left(\pi^{n} u-u\left(t_{n}\right)\right)=$ $\tilde{e}^{n}+\eta^{n}$. The interpolation error $\eta^{n}$ is bounded using standard estimates which, assuming $u\left(t_{n}\right) \in L_{\hat{\mu}}^{2}\left(H^{k+1}\right)$ for any $n$, yields (see e.g. [12])

$$
\begin{equation*}
\mathcal{E}^{N}(\eta):=\left\|\eta^{N}\right\|^{2}+\frac{\Delta t}{4} \sum_{j=1}^{N}\left\|\eta^{j}\right\|_{\text {SUPG }}^{2} \lesssim h^{2 k+1} \tag{20}
\end{equation*}
$$

For the other error term, Proposition 2.1 allows to derive

$$
\begin{align*}
& \triangle t^{-1}\left(\tilde{e}^{n+1}-\tilde{e}^{n}, \tilde{v}\right)+a_{\mathrm{SUPG}}\left(\tilde{e}^{n+1}, \tilde{v}\right)=a_{\mathrm{SUPG}}\left(u\left(t_{n+1}\right)-\pi^{n+1} u, \tilde{v}\right) \\
& \quad+\left(\dot{u}\left(t_{n+1}\right)-\triangle t^{-1}\left(\pi^{n+1} u-\pi^{n} u\right), \mathcal{H} \tilde{v}\right)-\delta \triangle t^{-1}\left(\tilde{e}^{n+1}-\tilde{e}^{n}, \mathbf{b} \cdot \nabla \tilde{v}\right)+a_{\mathrm{SUPG}}\left(\hat{u}^{n}, \tilde{v}^{\perp}\right) \\
& -\left(f^{n+1}, \mathcal{H} \tilde{v}^{\perp}\right)-a_{\mathrm{SUPG}}\left(\hat{u}^{n}-u_{h, \hat{\mu}}^{n+1}, \tilde{v}^{\perp}\right)+\triangle t^{-1}\left(u_{h, \tilde{\mu}}^{n+1}-u_{h, \hat{\mu}}^{n}, \mathcal{H} \tilde{v}^{\perp}\right), \quad \forall \tilde{v} \in V_{h} \otimes L_{\hat{\mu}}^{2} \tag{21}
\end{align*}
$$

Note that the last term in vanishes by (13). One last technical lemma is needed before presenting the main result:

Lemma 3.2. Let $\tilde{\delta} Y^{n}:=\tilde{Y}^{n+1}-Y^{n}$. It holds

$$
\triangle t^{-1}\left\|\tilde{U}^{n+1} \tilde{\delta} Y^{n}\right\|^{2}=a_{\mathrm{SUPG}}\left(u_{h, \tilde{\mu}}^{n+1}, \tilde{U}^{n+1} \tilde{\delta} Y^{n}\right)+\left(f^{n+1}, \mathcal{H} \tilde{U}^{n+1} \tilde{\delta} Y^{n}\right)
$$

Proof. Start from (9). Using the definition of $\tilde{W}_{i j}^{n+1}$, we rewrite it as

$$
\begin{aligned}
& \Delta t^{-1}\left(\sum_{i=1}^{R} \tilde{U}_{i}^{n+1} \tilde{\delta} Y_{j}^{n}, \tilde{U}_{j}^{n+1} z_{j}+\delta \mathbf{b} \cdot \nabla \tilde{U}_{j}^{n+1} z_{j}\right)=a_{\mathrm{SUPG}}\left(u_{h, \hat{\mu}}^{n+1}, \tilde{U}_{j}^{n+1} \mathcal{P} \frac{\perp}{\mathcal{Y}} z_{j}\right) \\
&+\left(f^{n+1}, \tilde{U}_{j}^{n+1} \mathcal{P} \frac{\perp}{\mathcal{Y}} z_{j}+\delta \mathbf{b} \cdot \nabla \tilde{U}_{j}^{n+1} \mathcal{P} \frac{\mathcal{Y}}{} z_{j}\right) \quad \text { for } j \in 1, \ldots, R, \forall z_{j} \in L_{\hat{\mu}}^{2} .
\end{aligned}
$$

Set $z_{j}=\tilde{Y}_{j}^{n+1}-Y_{j}^{n}$, the result is obtained by summing over $j$ since $\tilde{Y}^{n+1}-Y^{n} \in\left(\mathcal{Y}^{n}\right)^{\perp}$ (the l.h.s. becomes $\left\|\tilde{U}^{n+1} \tilde{\delta} Y^{n}\right\|^{2}$ by zero-divergence of $\mathbf{b}$ ).

Theorem 3.1. Let $\mathbf{b} \in\left(L^{\infty}(D)\right)^{d}$ such that $\operatorname{div} \mathbf{b}=0, c \in L_{\hat{\mu}}^{\infty}\left(L^{\infty}(D)\right)$ and assume the true solution verifies $u, \partial_{t} u \in L^{\infty}\left(0, T ; L_{\hat{\mu}}^{\infty}\left(H^{k+1}(D)\right)\right)$, $\partial_{t}^{2} u \in L^{2}\left(0, T ; L_{\hat{\mu}}^{\infty}\left(H^{1}\right)\right)$. Under CoEFA), 16, 55 as well as $\delta \leq \Delta t / 4$, the DLR iterates $\left\{u_{h, \hat{\mu}}^{n}\right\}_{n=0}^{N}$ of Algorithm 1 satisfy

$$
\begin{align*}
&\left\|u\left(t_{N}\right)-u_{h, \hat{\mu}}^{N}\right\|+\left(\sum_{i=1}^{N} \triangle t\left\|u\left(t_{i}\right)-u_{h, \hat{\mu}}^{i}\right\|_{\text {SUPG }}^{2}\right)^{1 / 2} \\
& \quad \lesssim h^{k+1}+\triangle t+\delta^{1 / 2} h^{k}+\delta^{-1 / 2} h^{k+1}+\left\|\pi^{0} u-u_{h, \hat{\mu}}^{0}\right\|+\nu \tag{22}
\end{align*}
$$

Proof. The proof largely follows the structure of the proof in [5] pp. 10-12]. Testing against $\tilde{e}^{n+1}$, the first two terms in the r.h.s of 21 verify

$$
\begin{aligned}
a_{\mathrm{SUPG}}\left(u\left(t_{n+1}\right)-\pi^{n+1} u\right. & \left., \tilde{e}^{n+1}\right)+\left(\dot{u}\left(t_{n+1}\right)-\Delta t^{-1}\left(\pi^{n+1} u-\pi^{n} u\right), \mathcal{H} \tilde{e}^{n+1}\right) \\
& =\delta \sum_{K \in \mathcal{T}_{h}}\left(\tilde{T}_{\text {stab }, K}^{n+1}, \mathbf{b} \cdot \nabla \tilde{e}^{n+1}\right)_{K, L_{\tilde{\mu}}^{2}}+\left(T_{\text {zero }}^{n+1}, \tilde{e}^{n+1}\right)+\left(T_{\text {conv }}^{n+1}, \tilde{e}^{n+1}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
T_{\text {zero }}^{n+1} & =\left(\dot{u}\left(t_{n+1}\right)-\pi^{n+1} \dot{u}\right)+c\left(u\left(t_{n+1}\right)-\pi^{n+1} u\right)+\left(\pi^{n+1} \dot{u}-\frac{\pi^{n+1} u-\pi^{n} u}{\Delta t}\right), \\
T_{\text {conv }}^{n+1} & =\mathbf{b} \cdot \nabla\left(u\left(t_{n+1}\right)-\pi^{n+1} u\right) \\
\tilde{T}_{\text {stab }, K}^{n+1} & =\left(T_{\text {zero }}^{n+1}+T_{\text {conv }}^{n+1}+\varepsilon \Delta\left(\pi^{n+1} u-u\left(t_{n+1}\right)\right)\right)_{\mid K} .
\end{aligned}
$$

Counter-integrating ( $T_{\text {conv }}^{n+1}, \tilde{e}^{n+1}$ ) and using the zero-divergence of $\mathbf{b}$ yields

$$
\left(T_{\text {conv }}^{n+1}, \tilde{e}^{n+1}\right)=-\delta \sum_{K \in \mathcal{T}_{h}}\left(\delta^{-1}\left(\pi^{n+1} u-u\left(t_{n+1}\right)\right), \mathbf{b} \cdot \nabla \tilde{e}^{n+1}\right)_{K, L_{\tilde{\mu}}^{2}},
$$

which can then be included in $T_{\text {stab }, K}^{n+1}$, defining

$$
T_{\mathrm{stab}, K}^{n+1}=\tilde{T}_{\mathrm{stab}, K}^{n+1}-\delta^{-1}\left(\pi^{n+1} u-u\left(t_{n+1}\right)\right) .
$$

We then bound the terms via Young's inequality, suitably balancing the coefficients such that the $\tilde{e}^{n+1}$-quantities on the r.h.s can be absorbed by $\frac{1}{2}\left\|e^{n+1}\right\|_{\text {SUPG }}^{2}$ on the l.h.s. To this end, let
$0<\gamma \leq 1 / 16$. As $(2 \triangle t)^{-1}\left(\left\|\tilde{e}^{n+1}\right\|^{2}-\left\|\tilde{e}^{n}\right\|^{2}+\left\|\tilde{e}^{n+1}-e^{n}\right\|^{2}\right)+1 / 2\left\|\tilde{e}^{n+1}\right\|_{\text {SUPG }}^{2}$ lower-bounds the l.h.s of 21, it holds

$$
\begin{aligned}
& (2 \Delta t)^{-1}\left(\left\|\tilde{e}^{n+1}\right\|^{2}-\left\|\tilde{e}^{n}\right\|^{2}+\left\|\tilde{e}^{n+1}-\tilde{e}^{n}\right\|^{2}\right)+{ }^{1 / 2}\left\|\tilde{e}^{n+1}\right\|_{\text {SUPG }}^{2} \\
& \quad \leq \delta \sum_{K \in \mathcal{T}_{h}}\left(T_{\text {stab,K}}^{n+1}-\Delta t^{-1}\left(\tilde{e}^{n+1}-\tilde{e}^{n}\right), \mathbf{b} \cdot \nabla \tilde{e}^{n+1}\right)_{K, L_{\tilde{\mu}}^{2}}+\left(T_{\text {zero }}^{n+1}, \tilde{e}^{n+1}\right) \\
& \quad+a_{\mathrm{SUPG}}\left(u_{h, \hat{\mu}}^{n+1}-\hat{u}^{n},\left(\tilde{e}^{n+1}\right)^{\perp}\right)+a_{\mathrm{SUPG}}\left(\hat{u}^{n},\left(\tilde{e}^{n+1}\right)^{\perp}\right)-\left(f^{n+1}, \mathcal{H}\left(\tilde{e}^{n+1}\right)^{\perp}\right) \\
& \leq C \delta \sum_{K \in \mathcal{T}_{h}}\left\|T_{\text {stab }, K}^{n+1}\right\|_{K, L_{\hat{\mu}}^{2}}^{2}+\delta \gamma\left\|\mathbf{b} \cdot \nabla \tilde{e}^{n+1}\right\|_{K, L_{\hat{\mu}}^{2}}^{2}+C \triangle t^{-1}\left\|\tilde{e}^{n+1}-\tilde{e}^{n}\right\|^{2}+C\left\|T_{\text {zero }}^{n+1}\right\|^{2} \\
& \quad+\gamma\left\|\tilde{e}^{n+1}\right\|^{2}+a_{\operatorname{SUPG}}\left(u_{h, \hat{\mu}}^{n+1}-\hat{u}^{n},\left(\tilde{e}^{n+1}\right)^{\perp}\right)+a_{\operatorname{SUPG}}\left(\hat{u}^{n},\left(\tilde{e}^{n+1}\right)^{\perp}\right)-\left(f^{n+1}, \mathcal{H}\left(\tilde{e}^{n+1}\right)^{\perp}\right),
\end{aligned}
$$

having used $\delta \lesssim \triangle t$ in the last inequality, and where $C$ depends on $\gamma^{-1}$. Lemma 3.2 with 18 and (16) respectively yield

$$
\begin{aligned}
& a_{\operatorname{SUPG}}\left(\tilde{U}^{n+1} \tilde{\delta} Y,\left(\tilde{e}^{n+1}\right)^{\perp}\right) \lesssim\left\|\tilde{U}^{n+1} \tilde{\delta} Y\right\|\left\|\tilde{e}^{n+1}\right\| \leq C \triangle t^{2}\left(\left\|u_{h, \hat{\mu}}^{n+1}\right\|^{2}+\|f\|^{2}\right)+\gamma\left\|\tilde{e}^{n+1}\right\|^{2}, \\
& a_{\operatorname{SUPG}}\left(\hat{u}^{n},\left(\tilde{e}^{n+1}\right)^{\perp}\right)-\left(f^{n+1}, \mathcal{H}\left(\tilde{e}^{n+1}\right)^{\perp}\right) \leq \nu\left\|\tilde{e}^{n+1}\right\| \leq C \nu^{2}+\gamma\left\|\tilde{e}^{n+1}\right\|^{2}
\end{aligned}
$$

Note that $\sum_{j=0}^{N-1} \triangle t^{2}\left(\left\|u_{h, \mu}^{j}\right\|^{2}+\|f\|^{2}\right) \lesssim \Delta t$ by Proposition 2.2 Cancelling, rearranging a few terms and summing over $j=0, \ldots, N-1$, we obtain

$$
\left\|\tilde{e}^{n+1}\right\|^{2}+\sum_{n=1}^{N}\left\|\tilde{e}^{n}\right\|_{\mathrm{SUPG}}^{2} \lesssim\left\|\tilde{e}^{0}\right\|^{2}+\triangle t \sum_{n=1}^{N}\left\|T_{\mathrm{zero}}^{n}\right\|^{2}+\triangle t \sum_{n=1}^{N} \sum_{K \in \mathcal{T}_{h}} \delta_{K}\left\|T_{\mathrm{stab}, K}^{n}\right\|_{K}^{2}+\nu^{2}+\triangle t^{2}
$$

As in [5], the regularity assumptions on $u$ and its derivatives allow to bound

$$
\Delta t \sum_{n=1}^{N}\left\|T_{\mathrm{zero}}^{n}\right\|^{2}+\Delta t \sum_{n=1}^{N} \sum_{K \in \mathcal{T}_{h}} \delta\left\|T_{\mathrm{stab}, K}^{n}\right\|^{2} \lesssim h^{2 k+2}+\triangle t^{2}+\delta h^{2 k}+h^{2 k+2} \delta^{-1}
$$

Denoting $\gamma^{n}:=u_{h, \hat{\mu}}^{n}-u\left(t_{n}\right)$, the claim follows as $\mathcal{E}^{N}(\gamma) \lesssim \mathcal{E}^{N}(\tilde{e})+\mathcal{E}^{N}(\eta)$.

## 4 Numerical experiments

We solve problem (1) on $D=[0,1]$ with

$$
\varepsilon=10^{-8}, \quad \quad \mathbf{b}=1, \quad c(x, \omega)=1+\omega, \quad \omega \sim \mathcal{U}[0,1]
$$

and choose the right-hand-side such that the true solution is given by

$$
\begin{equation*}
u_{\text {true }}(t, x, \omega)=e^{x \sin (2 \pi \omega(t+1))} \sin (2 \pi x) . \tag{23}
\end{equation*}
$$

The stochasticity therefore resides in the initial conditions, reaction term and forcing term. The sample space $\Omega=[0,1]$ is then approximated with the discrete set $\hat{\Omega}=\left\{{ }^{i} / N_{C}\right\}_{i=1}^{N_{C}}$ with $N_{C}=15$ and equal probabilities in all the sample points. The physical space is discretised using a regular mesh with increasingly fine mesh size $h_{i} \sim 2^{-i}$. For the initial conditions, we compute a (generalised) SVD of 23) at time $t=0$.

As in [5], the terms $\triangle t, \delta^{1 / 2} h^{k}$ and $h^{k+1} \delta^{-1 / 2}$ in the error estimate need to be balanced to yield the best possible decay rate for a fixed $h$. Since Proposition 2.2 requires $\delta \sim \Delta t$, this imposes the condition $\Delta t \sim \mathcal{O}\left(h^{\frac{2(k+1)}{3}}\right)$.


Figure 1: SUPG error for $k=1,2$ and small approximation rank $R$.

For the simulations we do not use the implicit scheme, but a semi-implicit version close to it that is both more technical and practical (reyling on a slightly different parametrisation of the approximation manifold with isolated mean, see [11]. With some technical details, the results carry over for that time-stepping scheme too.

In the first numerical experiment, we choose a rank $R=6$ to ensure the error associated to the rank truncation is negligible. The rates observed in Figure 1 a are those predicted by Theorem 3.1. both for the $L_{\hat{\mu}}^{2}\left(L^{2}(D)\right)$ and the SUPG error. Figures 1 b and 1 c display the errors of DLR approximations computed with $R=1,2,3$. The error is quasi-optimal with respect to the error obtained when using the optimal rank- $R$ truncation.

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