Koopman-based Data-driven Robust Control of Nonlinear Systems Using Integral Quadratic Constraints

Mert Eyüboğlu and Alireza Karimi

Abstract— **This paper presents a novel method for data-driven robust control of nonlinear systems using the Koopman operator and Integral Quadratic Constraints (IQCs). Koopman operator theory enables a linear representation of nonlinear dynamics in a higher-dimensional space. Data-driven Koopman-based models inherently offer only approximate representations due to various factors. We focus on characterizing modeling errors effectively to ensure closed-loop guarantees. Non-parametric IQC multipliers are identified to characterize modeling errors in a data-driven manner through frequency domain (FD) linear matrix inequalities (LMIs), treating them as additive uncertainty for robust control design. These multipliers provide a convex set representation of stabilizing robust controllers. The optimal controller within this set is obtained by solving a different set of FD LMIs. Finally, we propose an iterative approach alternating between IQC multiplier identification and robust controller synthesis, ensuring monotonic convergence of a robust performance index.**

I. INTRODUCTION

Koopman operator [1] has gained popularity for offering a global linear representation of nonlinear systems [2], [3]. It focuses on the evolution of observable functions, expressing nonlinear system dynamics linearly in a higher-dimensional space. Achieving global linearization often requires lifting the system to an infinite-dimensional space. Therefore, in practice, a finite-dimensional truncation of the operator is considered, providing a linear but approximate representation of the dynamics. The Extended Dynamic Mode Decomposition (EDMD) algorithm [4] facilitates the computation of such approximations from data. For non-autonomous systems, linearity in observables does not extend to linearity in inputs. Some works, like [5], impose linearity in inputs by additional constraints on observable functions, while others consider bilinear models, balancing accuracy and control design ease [6]. However, practical lifted models are never exact due to finite-dimensional truncation and data-driven approximation. Thus, characterizing modeling error for data-driven lifted models is crucial for closed-loop guarantees.

Probabilistic error bounds for EDMD-based bilinear approximate models of input-affine systems are derived in [7]. In a complementary effort, [8] reformulates these error bounds and designs state feedback controller in the lifted space with closed-loop guarantees. These works focus on continuous-time systems, which necessitates state derivative measurements. To alter this, the discrete-time counterpart of the error bounds, along with robust controller synthesis, is worked out in [9]. Considering general nonlinear systems, a data-driven characterization of the model error in terms of the worst-case ℓ_2-gain is proposed in [10], where probabilistic guarantees are given by the scenario approach [11]. Utilizing these error bounds, [10] further presents robust controller synthesis tailored for LTI and open-loop stable LPV models. However, these works consider disc-shaped static error bounds, which are likely to yield conservative control design.

Introduced by [12], IQC approach provides an effective tool for analysis and control of uncertain dynamical systems by its flexibility of representing general nonlinearities. While most of the existing literature on IQCs focuses on the analysis of uncertain systems, an iterative algorithm alternating between nominal $\mathcal{H}_\infty$ controller synthesis and robustness analysis was presented in [13]. More recently, IQC synthesis methods based on non-smooth optimization for $\mathcal{H}_\infty$ and $\mathcal{H}_2$ performance are developed, respectively, in [14] and [15], yielding local certificates of optimality. Despite that, all these works can only handle IQC multipliers with a specific parametrization. Such a parametrization constraints the set of feasible IQC multipliers which results in conservative control design. Recently, a controller synthesis method robust against uncertainties characterized by non-parametric IQCs is proposed in [16]. This method performs controller synthesis by solving FD LMIs and enables less conservative designs through compatibility with non-parametric IQC multipliers.

In this work, we propose a novel robust controller synthesis approach for nonlinear systems using Koopman operator and IQCs. We focus on LTI lifted models of nonlinear systems obtained via EDMD solely from system data. To address inherent modeling errors, we propose characterizing model error using IQCs. Through FD LMIs, we identify non-parametric IQC multipliers for modeling error. Employing the control design method of [16], we design controllers with robust performance guarantees. Since the set of robustly stabilizing controllers depend on the identified IQC multipliers, we propose an iterative algorithm alternating between IQC multiplier identification and controller synthesis, ensuring monotonic convergence of a chosen performance index.

The paper is organized as follows: preliminaries are provided in Section II. In Section III, the proposed method is presented, composed of IQC-based characterization of model error and synthesis of robust controllers. Frequency sampling approach for implementation of the optimization problems is discussed, followed by the iterative algorithm. Application of the proposed algorithm on a simulation example is presented in Section IV. A brief conclusion is offered in Section V.

The authors are with Laboratoire d'Automatique, École Polytechnique Fédérale de Lausanne (EPFL), Switzerland (e-mail: mert.eyuboglu@epfl.ch; alireza.karimi@epfl.ch).

This work has been supported by the Swiss National Science Foundation under NCCR Automation (grant agreement 51NF40 180545).

II. PRELIMINARIES

Notations: $\mathbb{R}$ and $\mathbb{C}$ are used to denote the sets of real and complex numbers respectively. ℓ_2^p denotes the space of p dimensional square integrable signals while $\mathcal{RH}_\infty$ represents the set of real rational stable transfer functions with bounded ∞-norm. Identity matrix of an appropriate size is represented by I. $S \succ (\succeq) 0$ and $S \prec (\preceq) 0$ indicate that the matrix S is positive (-semi) definite and negative (-semi) definite respectively. The conjugate transpose of a complex matrix S is denoted by S^* and the pseudo-inverse of S is denoted by $S^\dagger$. If $S \in \mathbb{C}$ is full row rank, the right inverse is denoted as $S^{\mathsf{R}} = S^*(SS^*)^{-1}$. If $S \in \mathbb{C}$ is full column rank, the left inverse is denoted as $S^{\mathsf{L}} = (S^*S)^{-1}S^*$. The frequency response of a discrete-time system G is denoted by $G(e^{j\omega})$.

A. Koopman operator

Consider the discrete-time nonlinear system,

$$H : \left\{ x_{k+1} = f(x_k, u_k), \right. \tag{1}$$

where $x \in \mathbb{X} \subseteq \mathbb{R}^{n_x}$ is the state variable, $u \in \mathbb{U} \subseteq \mathbb{R}^{n_u}$ is the input and $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ is the nonlinear state transition map. The Koopman operator $\mathcal{K} : \mathcal{F} \to \mathcal{F}$ is a linear operator that advances an observable function $\xi(x_k, u_k)$ one-step ahead in time,

$$\xi(x_{k+1}, u_{k+1}) = \mathcal{K}\xi(x_k, u_k) = \xi(f(x_k, u_k), u_{k+1}), \tag{2}$$

where $\mathcal{F}$ is a Banach space of observable functions that is invariant under the action of the Koopman operator. Therefore, Koopman operator $\mathcal{K}$ globally maps the nonlinear dynamics in the state space to linear dynamics in the lifted space of observables. In general, the Koopman operator is defined on an infinite-dimensional space. In practice, however, a finite-dimensional approximation of the Koopman operator denoted by $\mathbb{K}$, is used where a finite set of observable functions $\mathcal{D} = \{\xi_j\}_{j=1}^d$ called a dictionary is considered.

Due to the availability of well established tools for LTI systems, identifying such a model is often desirable. To obtain a lifted LTI representation we consider a dictionary structured as $\mathcal{D} = \begin{bmatrix} \boldsymbol{\xi}(x_k) & u_k \end{bmatrix}^T$ with $\boldsymbol{\xi}(x_k) = \begin{bmatrix} \xi_1(x_k) & \xi_2(x_k) & \dots & \xi_{d-1}(x_k) \end{bmatrix}^T$ yielding,

$$\begin{bmatrix} \boldsymbol{\xi}(x_{k+1}) \\ u_{k+1} \end{bmatrix} \approx \begin{bmatrix} \mathbb{K}_{11} & \mathbb{K}_{12} \\ \mathbb{K}_{21} & \mathbb{K}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}(x_k) \\ u_k \end{bmatrix}. \tag{3}$$

Since predicting the future values of the input is not of interest we discard the last n_u rows of $\mathbb{K}$ resulting in,

$$\boldsymbol{\xi}(x_{k+1}) = A\boldsymbol{\xi}(x_k) + Bu_k + \varepsilon_k, \tag{4}$$

where $A = \mathbb{K}_{11}$, $B = \mathbb{K}_{12}$ and ε_k denotes the one step ahead prediction error. The prediction error ε_k is introduced by the restriction of the Koopman operator to a finite dimensional space as well as the structure imposed on the dictionary.

EDMD [4] enables the computation of the matrices A and B in (4) by solving a least-squares problem as follows. Based on a set of data trajectories with N samples $\{x_k, u_k\}_{k=0}^{N-1}$ and a selected dictionary of observable functions $\boldsymbol{\xi}$, the matrices $Z := \begin{bmatrix} \boldsymbol{\xi}(x_0) & \dots & \boldsymbol{\xi}(x_{N-2}) \end{bmatrix}$, $Z^+ := \begin{bmatrix} \boldsymbol{\xi}(x_1) & \dots & \boldsymbol{\xi}(x_{N-1}) \end{bmatrix}$ and $U := \begin{bmatrix} u_0 & \dots & u_{N-2} \end{bmatrix}$, are constructed. Then, A and B in (4) are obtained by solving,

$$\min_{A,B} \quad \left\| Z^+ - \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} Z \\ U \end{bmatrix} \right\|. \tag{5}$$

B. Problem Formulation

Consider data $\{\{x_k^m, u_k^m\}_{k=0}^{N-1}\}_{m=1}^M$ collected from a general discrete-time nonlinear system (1) with sampling time T_s, as M trajectories of N samples. The data is collected from bounded sets such that $x \in \mathbb{X}$, $u \in \mathbb{U}$. Using the data and a predetermined set of observable functions $\boldsymbol{\xi}(x_k)$, the discrete-time nonlinear dynamics can be approximated in the lifted space as H_0, defined by:

$$\hat{\boldsymbol{\xi}}_{k+1} = A\hat{\boldsymbol{\xi}}_k + Bu_k, \tag{6}$$

where $\hat{\boldsymbol{\xi}}_k \approx \boldsymbol{\xi}(x_k)$ and the matrices A, B are calculated by EDMD. Due to the prediction error term in (4), the LTI system H_0 is only an approximation of the true system H such that $H = H_0 + \Delta$, where Δ represents the error system to be treated as additive uncertainty for controller design. Thus, the interconnection of the nonlinear system H with a controller K can be represented as in Fig. 1. To proceed we need the following assumption:

Assumption 1. *The nonlinear system H can be represented as the sum of H_0 and a bounded causal operator Δ in the lifted space.*

This assumption can be ensured by appropriately selecting observable functions $\boldsymbol{\xi}(x_k)$ and their dimensions, such that the space spanned by $\boldsymbol{\xi}(x_k)$ yields a bounded Δ.

Based on these, we formulate the problem of designing a data-driven controller providing closed-loop guarantees for the nonlinear system H, as the following two subproblems,

1) Characterization of the error system Δ using non-parametric dynamic IQC multipliers.
2) Synthesis of a fixed-structure controller K for H_0 with guarantees of robust stability against Δ and robust performance with respect to Π_p on $w \to z$.

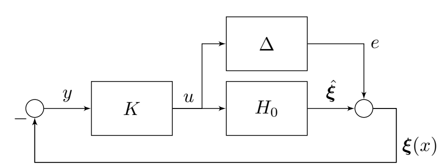


Fig. 1: Block diagram of the closed-loop system.

C. Integral Quadratic Constraints

Two discrete-time signals $p(k) \in \ell_2^{n_p}[0, \infty]$ and $q(k) \in \ell_2^{n_q}[0, \infty]$ with sampling time T_s are said to satisfy the IQC defined by Π if,

$$\int_{\omega \in \Omega} \begin{bmatrix} P(e^{j\omega}) \\ Q(e^{j\omega}) \end{bmatrix}^* \Pi(e^{j\omega}) \begin{bmatrix} P(e^{j\omega}) \\ Q(e^{j\omega}) \end{bmatrix} d\omega \geq 0, \tag{7}$$

where $P(e^{j\omega})$ and $Q(e^{j\omega})$ represent the discrete-time Fourier transforms of $p(k)$ and $q(k)$ respectively and $\Omega = (-\pi/T_s, \pi/T_s]$.

Let a performance metric on the channel $w \to z$ with respect to the multiplier $\Pi_p(\gamma)$ be defined such that, performance with index γ is achieved if w and z satisfy the IQC defined by $\Pi_p(\gamma)$. By the IQC theorem [17, Corollary 3]:

Theorem 1. *The feedback interconnection of a discrete-time stable LTI system T and a bounded causal operator Δ as depicted in Fig. 2a, is robustly stable and has robust performance on the channel $w \to z$ with respect to Π_p if,*

1) *interconnection of T and $\tau\Delta$ is well-posed, i.e. $(I - T\Delta)$ has a causal inverse, $\forall \tau \in [0,1]$;*
2) *the IQC defined by Π is satisfied by $\tau\Delta$, $\forall \tau \in [0,1]$;*
3) *for all $\omega \in \Omega$,*

$$\begin{bmatrix} T(e^{j\omega}) \\ I \end{bmatrix}^* \Pi_{rp}(e^{j\omega}) \begin{bmatrix} T(e^{j\omega}) \\ I \end{bmatrix} \prec 0; \tag{8}$$

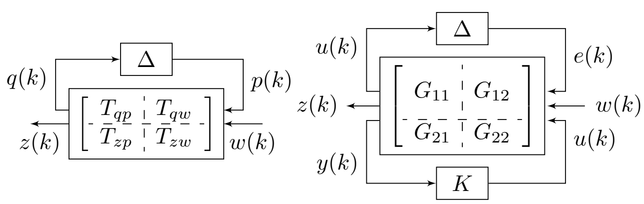
where,

$$\Pi_{rp} = \left[\begin{array}{cc|cc} \Pi_{11} & 0 & \Pi_{12} & 0 \\ 0 & \Pi_{p,11} & 0 & \Pi_{p,12} \\ \hline \Pi_{12}^* & 0 & \Pi_{22} & 0 \\ 0 & \Pi_{p,12}^* & 0 & \Pi_{p,22} \end{array}\right]. \tag{9}$$

By [17, Remark 3] if Π is partitioned as

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix},$$

with $\Pi_{11} \succeq 0$ and $\Pi_{22} \preceq 0$, then $\tau\Delta$ satisfies the IQC defined by Π for all $\tau \in [0,1]$ if and only if Δ satisfies the IQC.



(a) (b)

Fig. 2: (a) General feedback interconnection. (b) Generalized plant structure of the feedback interconnection.

III. Data-driven Robust Control Design

For any arbitrary channel $w \to z$ on which the performance objective is defined, it is fairly standard to transform the block diagram in Fig. 1 to a generalized plant structure as in Fig. 2b where $G_{22} = -H_0$. Then, by applying a lower linear fractional transformation to the generalized plant G and controller K, the closed-loop system can be represented as in Fig. 2a with $T = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$. Using A and B obtained by EDMD, the frequency response function (FRF) of the LTI model $H_0(e^{j\omega}) = (e^{j\omega}I - A)^{-1}B$ can be computed for any $\omega \in \Omega$. Based on $H_0(e^{j\omega})$ and following the corresponding generalized plant formulation G, the FRF $T(e^{j\omega})$ can be obtained similarly. For the generalized plant model G we assume that $G_{21}(e^{j\omega})$ has full rank and $G(e^{j\omega})$ is bounded, $\forall \omega \in \Omega$.

Next, we address robust controller synthesis against uncertainty Δ characterized by an IQC multiplier Π. To facilitate problem solving, we discuss the error characterization problem afterward, enabling formulation in terms of decomposed elements of Π. Thus, solution of the IQC-based error characterization problem can be directly used for controller synthesis. Throughout the paper, we interchangeably use Π and $\Pi(e^{j\omega})$ for simplicity.

A. Robust Controller Synthesis

The objective of the controller synthesis is to obtain a controller parametrized as $K = XY^{-1}$ where $X, Y \in \mathcal{RH}_\infty$ are linear in optimization variables. This controller guarantees robust stability against Δ and robust performance on the channel $w \to z$ with respect to $\Pi_p(\gamma)$ where γ denotes the achieved robust performance index. For some Π, with $\Pi_{11} \succeq 0$ and $\Pi_{22} \preceq 0$, such that the error system Δ satisfies the IQC defined by Π, this objective can be formulated as an optimization problem,

$$\begin{aligned} \min_K \quad & \gamma \\ \text{s.t.} \quad & \begin{bmatrix} T \\ I \end{bmatrix}^* \Pi_{rp}(\gamma) \begin{bmatrix} T \\ I \end{bmatrix} (e^{j\omega}) \prec 0, \ \forall \omega \in \Omega, \\ & T = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \text{ is stable.} \end{aligned} \tag{10}$$

The closed-loop transfer function T can be written as,

$$\begin{aligned} T &= G_{11} + G_{12}X\Phi^{\mathsf{L}} = G_{11}(\Phi\Phi^{\mathsf{L}} + \Psi) + G_{12}X\Phi^{\mathsf{L}} \\ &= (G_{11}\Phi + G_{12}X)\Phi^{\mathsf{L}} + G_{11}\Psi. \end{aligned} \tag{11}$$

where $\Phi = G_{21}^{\mathsf{R}}(Y - G_{22}X)$ and $\Psi = I - \Phi\Phi^{\mathsf{L}} = G_{21}^{\mathsf{R}}G_{21}$. Since Ψ is a hermitian idempotent matrix such that, $\Psi\Phi = \Phi - \Phi\Phi^{\mathsf{L}}\Phi = 0$ and $\Phi^{\mathsf{L}}\Psi = \Phi^{\mathsf{L}} - \Phi^{\mathsf{L}}\Phi\Phi^{\mathsf{L}} = 0$, we get,

$$\begin{bmatrix} T \\ I \end{bmatrix} = \begin{bmatrix} G_{11}\Phi + G_{12}X & G_{11}\Psi \\ \Phi & \Psi \end{bmatrix} \begin{bmatrix} \Phi^{\mathsf{L}} \\ \Psi \end{bmatrix} = L \begin{bmatrix} \Phi^{\mathsf{L}} \\ \Psi \end{bmatrix}, \tag{12}$$

with an evident definition for L. Then, by [18, Proposition 8.1.2] the first constraint in (10) can be replaced by $L^*\Pi_{rp}L \prec 0$. Using the fact that any square matrix accepts a factorisation $\Pi_{rp} = \Pi_{rp}^+ + \Pi_{rp}^-$ with $\Pi_{rp}^+ \succ 0$ and $\Pi_{rp}^- \preceq 0$, $L^*\Pi_{rp}L \prec 0$ can be written as $L^*\Pi_{rp}^+L - (-L^*\Pi_{rp}^-L) \prec 0$. By the Schur complement lemma, this yields the constraint,

$$\begin{bmatrix} (\Pi_{rp}^+)^{-1} & L \\ L^* & -L^*\Pi_{rp}^-L \end{bmatrix} \succ 0. \tag{13}$$

The quadratic component $-L^*\Pi_{rp}^-L$ in (13) can be convexified around a known controller $K_c = X_cY_c^{-1}$ such that,

$$L^*\Pi_{rp}^-L \preceq L^*\Pi_{rp}^-L_c + L_c^*\Pi_{rp}^-L - L_c^*\Pi_{rp}^-L_c \prec 0, \tag{14}$$

where

$$L_c = \begin{bmatrix} G_{11}\Phi_c + G_{12}X_c & G_{11}\Psi \\ \Phi_c & \Psi \end{bmatrix}$$

and $\Phi_c = G_{21}^{\mathsf{R}}(Y_c - G_{22}X_c)$. Then the optimisation problem (10) can be converted to the following convex optimisation problem [16, Theorem 2], if $K_c = X_c Y_c^{-1}$ is a stabilising controller:

$$\min_{\gamma, X, Y} \quad \gamma \tag{15}$$

$$\text{s.t.} \quad \begin{bmatrix} (\Pi_{rp}^{+}(\gamma))^{-1} & L \\ L^* & -\mathcal{L} \end{bmatrix} (e^{j\omega}) \succ 0, \ \forall \omega \in \Omega, \tag{16}$$

$$(\Phi^* \Phi_c + \Phi_c^* \Phi - \Phi_c^* \Phi_c)(e^{jw}) \succeq 0, \ \forall \omega \in \Omega, \tag{17}$$

where $\mathcal{L} = L^* \Pi_{rp}^{-} L_c + L_c^* \Pi_{rp}^{-} L - L_c^* \Pi_{rp}^{-} L_c$. The first constraint in (15) implies the first constraint in (10) and the second constraint ensures the stability of T. Thus, solving (15) for any $\Pi_{rp}^{+} \succ 0$ and $\Pi_{rp}^{-} \preceq 0$, we obtain the controller $K = XY^{-1}$ guaranteeing robust performance with index γ.

A Π_p defining a performance objective can be easily decomposed in to $\Pi_p^{+} \succ 0$ and $\Pi_p^{-} \preceq 0$ such that $\Pi_p = \Pi_p^{+} + \Pi_p^{-}$ for many conventional IQC multipliers. For examples we refer to [16]. By applying the structure in (9) to Π^{+}, Π_p^{+} and Π^{-}, Π_p^{-} respectively decomposition of Π_{rp} in to $\Pi_{rp}^{+} \succ 0$ and $\Pi_{rp}^{-} \preceq 0$ can also be obtained. Obtaining Π^{+} and Π^{-} from data is addressed in Section III-B.

Remark 1. *Both constraints in (10) are convexified around the initial controller K_c arriving at (15), resulting in convex inner approximations of convex-concave constraints. The conservatism due to this inner approximation can be reduced by an iterative approach [19], replacing the initial controller at each iteration by the optimal controller obtained in the previous one. This iterative approach guarantees monotonic convergence of the objective to a local minimum or saddle point of the non-convex problem in (10).*

B. Error characterization via non-parametric IQCs

The main goal in error characterization is to identify a dynamic IQC multiplier $\Pi(e^{j\omega})$ in a data-driven fashion, such that Δ satisfies the IQC defined by the resulting $\Pi(e^{j\omega})$. Since the control design method explained in Section III-A can handle non-parametric IQC multipliers, we aim for identifying non-parametric dynamic multipliers that can be treated as complex Hermitian matrices defined $\forall \omega \in \Omega$.

We first compute the frequency spectrum of the signals u and e using the available data. In order to do so, we simulate H_0 with the inputs used for data collection $\{\{u_k^m\}_{k=0}^{N-1}\}_{m=1}^{M}$ starting from the initial conditions $\hat{\boldsymbol{\xi}}_0^m = \boldsymbol{\xi}(x_0^m)$ for all $m \in [1, M]$. We obtain the corresponding trajectories of e as, $\{\{e_k^m\}_{k=0}^{N-1}\}_{m=1}^{M} = \{\{\boldsymbol{\xi}(x_k^m) - \hat{\boldsymbol{\xi}}_k^m\}_{k=0}^{N-1}\}_{m=1}^{M}$. Then, the frequency content of e for each trajectory can be computed,

$$E_m(e^{j\omega}) = \sum_{k=0}^{N-1} e_k^m e^{-j\omega T_s k}, \ \forall \omega \in \Omega, \ \forall m \in [1, M]. \tag{18}$$

Similarly, the frequency spectrum of the plant input u can also be computed $\forall \omega \in \Omega$ and $\forall m \in [1, M]$ as in (18).

Assumption 2. *The data $\{\{e_k^m, u_k^m\}_{k=0}^{N-1}\}_{m=1}^{M}$ is informative for IQC multiplier identification such that if the data satisfies the IQC defined by Π, it would also be satisfied for any other input-output trajectory of Δ with $u \in \ell_2$.*

This assumption can be met in practice by collecting more rich data. Based on this assumption, Δ can be characterized using the available data by finding a $\Pi(e^{j\omega})$ satisfying,

$$\int_{\omega \in \Omega} \begin{bmatrix} U_m \\ E_m \end{bmatrix}^* \Pi \begin{bmatrix} U_m \\ E_m \end{bmatrix} (e^{j\omega}) d\omega \geq 0, \ \forall m \in [1, M]. \tag{19}$$

This constraint is convex with respect to Π and defines the set of all Πs that can characterize Δ. Since our objectives also include providing robust stability and robust performance guarantees, we seek for the optimal Π that is within the set defined by (19), providing the best robust performance guarantee. Consider $\Pi_{rp}(e^{j\omega}, \gamma)$ composed as in (9), where $\Pi(e^{j\omega})$ satisfies (19) and $\Pi_p(e^{j\omega}, \gamma)$ defines a robust performance objective. For a known robustly stabilising controller K_c, the robust performance condition (8) should be satisfied by $\Pi_{rp}(e^{j\omega}, \gamma)$ for some γ. Thus, our goal is to find the IQC multiplier Π that satisfies (19) such that the resulting Π_{rp} fulfills (8) with the smallest possible γ, thereby providing the tightest robust performance guarantee. Since T is known for $K = K_c$, (8) becomes an FD LMI condition to be satisfied by $\Pi(e^{j\omega})$. By combining (19) and (8), for a known robustly stabilising initial controller K_c, an IQC multiplier characterizing the error system as well as the achieved robust performance index can be obtained by solving the following FD convex optimization problem:

$$\begin{aligned} \min_{\gamma, \Pi^{+}, \Pi^{-}} \quad & \gamma \\ \text{s.t.} \quad & \int_{\omega \in \Omega} \begin{bmatrix} U_m \\ E_m \end{bmatrix}^* \Pi \begin{bmatrix} U_m \\ E_m \end{bmatrix} (e^{j\omega}) d\omega \geq 0, \ \forall m \in [1, M], \\ & \begin{bmatrix} T \\ I \end{bmatrix}^* \Pi_{rp}(\gamma) \begin{bmatrix} T \\ I \end{bmatrix} (e^{j\omega}) \prec 0, \ \forall \omega \in \Omega, \\ & \Pi(e^{j\omega}) = \Pi^{+}(e^{j\omega}) + \Pi^{-}(e^{j\omega}), \ \forall \omega \in \Omega, \\ & \Pi_{11}(e^{j\omega}) \succeq 0, \ \Pi_{22}(e^{j\omega}) \preceq 0, \ \forall \omega \in \Omega, \\ & \Pi^{+}(e^{j\omega}) \succ 0, \ \Pi^{-}(e^{j\omega}) \preceq 0, \ \forall \omega \in \Omega, \end{aligned} \tag{20}$$

where the additional constraints on $\Pi(e^{j\omega})$ are imposed such that the controller synthesis method explained in Section III-A can be employed directly without computing an appropriate factorization of the multiplier.

Synthesis of a Robustly Stabilising Initial Controller: Note that robust stability of the closed-loop is imposed in (20), while the robust performance index is being optimized. Thus, (20) is not feasible if the initial controller K_c is not robustly stabilising. In that case, one should first obtain a robustly stabilising controller before optimizing for the robust performance index. To do so, the second constraint in (20) should be relaxed to obtain:

$$\begin{bmatrix} T \\ I \end{bmatrix}^* \Pi_{rp}(\gamma) \begin{bmatrix} T \\ I \end{bmatrix} (e^{j\omega}) \prec \gamma_{s1} I, \ \forall \omega \in \Omega, \tag{21}$$

and γ_{s1} should be minimized instead of the robust performance index γ. For a K_c that is not robustly stabilising this problem becomes feasible for some $\gamma_{s1} > 0$. Next, to obtain a robustly stabilising K, first constraint of (15) should be relaxed similarly resulting in:

$$\begin{bmatrix} (\Pi_{rp}^{+}(\gamma))^{-1} & L \\ L^{*} & -\mathcal{L} \end{bmatrix} (e^{j\omega}) + \gamma_{s2} I \succ 0, \ \forall \omega \in \Omega, \quad (22)$$

where this time γ_{s2} should be minimized instead. If this yields some $\gamma_{s2} \leq 0$ the resulting K is robustly stabilising. Otherwise, the relaxed versions of (15) and (20) should be solved iteratively until $\gamma_{s1} \leq 0$ or $\gamma_{s2} \leq 0$ is achieved. A similar iterative procedure to be followed to optimize the robust performance index γ once a robustly stabilising controller is found is presented in Algorithm 1.

C. Frequency Sampling

Both problems in this paper are framed as FD convex optimization problems with infinitely many constraints known as convex semi-infinite programs (SIPs). A common strategy for solving SIPs is to sample the infinite constraints in the FD at a sufficiently large set of finite frequencies $\Omega_g = \{\omega_1, \ldots, \omega_g\} \subset \Omega$. Since all constraints in (15) and (20) are imposed on Hermitian matrices, it suffices to consider frequencies only in the range $\Omega_g \in [0, \pi/T_s)$. As a result, we obtain the non-parametric IQC multipliers $\Pi(e^{j\omega})$ also at finite number of frequency points Ω_g, such that $\Pi(e^{j\omega})$ is defined as a set of Hermitian complex matrices $\forall \omega \in \Omega_g$.

While this sampling approach does not guarantee constraint satisfaction at all frequencies, probabilistic guarantees dependent on the number of finite frequency points g can be obtained by the scenario approach [11]. Consider that the SIPs are solved for a finite set of independent identically distributed (i.i.d.) samples Ω_g yielding a robust performance guarantee with index $\hat{\gamma}$. By the scenario approach [11], if,

$$g \geq \frac{2}{\epsilon} \left(\ln \frac{1}{\beta} + d \right), \quad (23)$$

then, with probability no smaller than $1 - \beta$, $\hat{\gamma}$ satisfies the constraints for all Ω but at most an ϵ-fraction where d denotes the number of optimization variables. For example, consider that (15) is solved for $g = 4000$ frequency points where K is defined as a static output feedback controller with $n_u = 1$ output and $n_y = 3$ inputs such that $d = 12$. Then, having a violation probability greater than $\epsilon = 0.01$ has a probability less than 3.36×10^{-4} while this upper bound exponentially goes to 0 with g. Since all optimization problems are convex and their complexity scales linearly with the number of frequencies in Ω_g, they can be solved for a large set of frequencies by modern semi-definite programming solvers. Thus, by choosing a sufficiently large set of frequency points a robust controller can be synthesized in practice.

D. Iterative Approach

To achieve the best possible performance we propose an iterative scheme between the two subproblems addressed in subsections III-A and III-B. Clearly, for the signals u and e there is not a unique IQC multiplier Π characterizing the error system. And since a particular multiplier Π determines a convex set of controllers that we can choose from during the controller synthesis, without iteratively updating the Π and K, it is very likely that the achieved performance indexes will be highly conservative. Thus, we propose Algorithm 1.

Algorithm 1: Iterative algorithm over error system characterization and robust controller synthesis

---

Data: measured trajectories: $\{\{x_k^m, u_k^m\}_{k=0}^{N-1}\}_{m=1}^{M}$,
lifting functions: $\boldsymbol{\xi}(x)$,
initial robustly stabilising controller: K_c

Preparation:
obtain A and B in (6) by EDMD.
compute $T(e^{j\omega})$ using $K = K_c, \ \forall \omega \in \Omega_g$.
compute $\{(U_m, E_m)(e^{j\omega})\}_{m=1}^{M}, \ \forall \omega \in \Omega_g$.

Iteration: set $i = 0$.

while γ, converges and $i \leq i_{\max}$ **do**
- update IQC multiplier Π:
  solve (20) for $\omega \in \Omega_g$, obtain $\Pi^+, \Pi^- \ \forall \omega \in \Omega_g$.
- update controller K:
  solve (15) for $\omega \in \Omega_g$, (iteratively as in [19]),
  obtain $K = XY^{-1}$ and update $T(e^{j\omega})$
- set $i = i + 1$.

end

Result: K, γ.

Algorithm 1 solves the joint problem of identification of Π and design of controller K yielding best performance index γ using a coordinate descent method. Since the objective function γ is bounded and the solution from the previous iteration is always a feasible solution, the algorithm yields monotonic convergence of the performance objective γ. However, optimality guarantees of the solution cannot be claimed [20]. Algorithm 1 yields, under Assumptions 1 and 2, a controller K that guarantees robust stability against Δ and robust performance on the channel $w \to z$ with respect to $\Pi_p(\gamma)$, by only using data collected from the system and a lifting dictionary. It should be noted that while the resulting controller is linear in the lifted space, thanks to Koopman lifting, this controller is nonlinear in the actual state space.

IV. Numerical Example

To demonstrate the proposed method on a simulation example we consider the nonlinear pendulum with dynamics:

$$\dot{x}_1(t) = x_2(t), \quad (24)$$

$$\dot{x}_2(t) = -\frac{g}{l} \sin x_1(t) - \frac{b}{ml^2} x_2(t) + \frac{1}{ml^2} u(t), \quad (25)$$

with $m = 1 \ kg$, $l = 1 \ m$, $b = 0.01$, and $g = 9.81 \ m/s^2$. We discretize the dynamics using the 4^{th}-order Runge-Kutta method with sampling time $T_s = 0.01 \ s$ and consider the discrete-time model as our true nonlinear system. To collect data, we simulate the discrete-time system for a single trajectory of $N = 5000$ samples with initial condition $x_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ and u_k randomly chosen from $\mathbb{U} = [-10, 10]$ with a uniform distribution for all $k \in [0, N-1]$. By also inferring some knowledge of the dynamics we choose the lifting functions $\boldsymbol{\xi}(x) = \begin{bmatrix} x_1 & x_2 & \sin(x_1) \end{bmatrix}^T$.

After applying the EDMD algorithm the lifted state matrices as in (6) are obtained, yielding a 3-dimensional stable LTI representation of the system. We consider the tracking problem where the pendulum angle x_1 is desired to track the reference w. The performance channel output is defined as $z = \begin{bmatrix}(W_1(w-x_1))^T & (W_2 u)^T\end{bmatrix}^T$, where we use a low-pass filter W_1 defined by the Matlab command `W1 = 1/makeweight(0.001,1,2,Ts)` and we set $W_2 = 0.1$. We select $\Pi_p = \mathrm{diag}(\gamma^2 I, -I)$ such that minimizing $\mathcal{H}_\infty$ norm of T_{zw} is our objective. Next, applying Algorithm 1 with initial controller $K_c = 0$, yields the state feedback controller $K = \begin{bmatrix}43.45 & 6.679 & -9.608\end{bmatrix}$ with robust performance index $\gamma^* = 4.7868$. Considering the resulting controller, the true value of the control design objective is computed as $\|T_{zw}\|_\infty = 4.7848$ verifying that the robust performance guarantee claimed by the algorithm based on finite frequency samples is attained $\forall\omega \in \Omega$.

a) Improvements by Koopman lifting: To observe the benefits of Koopman lifting, we consider the case where we did not employ lifting such that $\boldsymbol{\xi}(x) = \begin{bmatrix}x_1 & x_2\end{bmatrix}$, while the robust performance objective is kept the same. After identifying the system matrices by solving the EDMD problem, we use Algorithm 1 for robust controller synthesis. This approach yields a robust performance guarantee with index $\gamma_1^* = 31.89$ achieved by the linear state feedback controller $K_1 = \begin{bmatrix}281.6 & 22.88\end{bmatrix}$.

b) Improvements by the non-parametric IQC: To assess the advantage of non-parametric IQCs, we compare with the approach in [13] using parametric IQC multipliers and the same performance objective. We employ the lifting functions $\boldsymbol{\xi}(x) = \begin{bmatrix}x_1 & x_2 & \sin(x_1)\end{bmatrix}^T$ for consistency with the earlier lifted representation. Considering the single measured trajectory, we find a lower bound on the error systems worst case ℓ_2-gain by finding the minimum value of $\gamma_e > 0$ such that

$$\sum_{k=0}^{N-1} \|e_k\|^2 \leq \gamma_e^2 \sum_{k=0}^{N-1} \|u_k\|^2,$$

is satisfied. This yields the lower bound of $\gamma_e^* = 0.1186$ achieved on the worst case ℓ_2-gain of the error system. This allows us to define the error system as a norm bounded uncertainty which can be handled by the framework proposed in [13]. Next, by applying the iterative approach for controller synthesis from [13], a robust performance index of $\gamma_2^* = 380.4554$ is obtained.

While all three approaches yield robust controllers capable of tracking a reference within the entire operating range $x_1 \in [-\pi, \pi]$, the proposed method achieves significantly superior performance. Although we only demonstrate state feedback synthesis for simplicity, the proposed method also facilitates structured dynamic output feedback controller synthesis.

V. Conclusion

The method offers a promising approach to robustly control nonlinear systems by utilizing Koopman operator theory and IQCs. The use of non-parametric IQC multipliers for modeling error yields a tight uncertainty around the lifted LTI model, reducing conservatism for control design significantly. Overall, the algorithm enables data-driven control of nonlinear systems using linear control methods and solving convex problems. However, closed-loop guarantees rely on Assumptions 1 and 2. While Assumption 2 can be met by collecting sufficient data, quantification of data quality is a topic for future research to enhance a priori guarantees. The simulation example demonstrates the benefit of the proposed non-parametric IQC-based error characterization, highlighting the main contribution of this work.

References

[1] B. O. Koopman, "Hamiltonian systems and transformation in hilbert space," *Proc. Natl. Acad. Sci. U. S. A.*, vol. 17, no. 5, pp. 315–318, 1931.

[2] A. Mauroy, I. Mezić, and Y. Susuki, Eds., *The Koopman Operator in Systems and Control: Concepts, Methodologies, and Applications*, 1st ed., ser. Lecture Notes in Control and Information Sciences. Springer, Jan. 2020, vol. 484.

[3] S. L. Brunton, M. Budišić, E. Kaiser, and J. N. Kutz, "Modern koopman theory for dynamical systems," *SIAM Review*, vol. 64, no. 2, pp. 229–340, 2022.

[4] M. O. Williams, I. G. Kevrekidis, and C. W. Rowley, "A data–driven approximation of the koopman operator: Extending dynamic mode decomposition," *Journal of Nonlinear Science*, vol. 25, no. 6, pp. 1307–1346, 2015.

[5] M. Korda and I. Mezić, "Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control," *Automatica*, vol. 93, pp. 149–160, 2018.

[6] D. Bruder, X. Fu, and R. Vasudevan, "Advantages of bilinear koopman realizations for the modeling and control of systems with unknown dynamics," *IEEE Robotics and Automation Letters*, vol. 6, pp. 4369–4376, 2020.

[7] M. Schaller, K. Worthmann, F. Philipp, S. Peitz, and F. Nüske, "Towards reliable data-based optimal and predictive control using extended DMD," *IFAC-PapersOnLine*, vol. 56, no. 1, pp. 169–174, 2023.

[8] R. Strässer, M. Schaller, K. Worthmann, J. Berberich, and F. Allgöwer, "Koopman-based feedback design with stability guarantees," 2023.

[9] ——, "Safedmd: A certified learning architecture tailored to data-driven control of nonlinear dynamical systems," 2024.

[10] M. Eyuboglu, N. R. Powell, and A. Karimi, "Data-driven control synthesis using koopman operator: A robust approach," 2024. [Online]. Available: http://infoscience.epfl.ch/record/306594

[11] M. C. Campi, S. Garatti, and M. Prandini, "The scenario approach for systems and control design," *Annual Reviews in Control*, vol. 33, no. 2, pp. 149–157, 2009.

[12] A. Megretski and A. Rantzer, "System analysis via integral quadratic constraints," *IEEE Transactions on Automatic Control*, vol. 42, no. 6, pp. 819–830, 1997.

[13] J. Veenman and C. W. Scherer, "Iqc-synthesis with general dynamic multipliers," *International Journal of Robust and Nonlinear Control*, vol. 24, no. 17, pp. 3027–3056, 2014.

[14] V. M. G. B. Cavalcanti and A. M. Simões, "IQC-synthesis under structural constraints," *International Journal of Robust and Nonlinear Control*, vol. 30, pp. 4880 – 4905, 2020.

[15] M. Schütte, A. Eichler, and H. Werner, "Structured IQC synthesis of robust $\mathcal{H}_2$ controllers in the frequency domain," *IFAC-PapersOnLine*, vol. 56, no. 2, pp. 10 408–10 413, 2023, 22nd IFAC World Congress.

[16] V. Gupta, E. Klauser, and A. Karimi, "Non-parametric IQC multipliers in data-driven robust controller synthesis," 2024. [Online]. Available: http://infoscience.epfl.ch/record/307954

[17] J. Veenman, C. W. Scherer, and H. Köroğlu, "Robust stability and performance analysis based on integral quadratic constraints," *European Journal of Control*, vol. 31, pp. 1–32, 2016.

[18] D. S. Bernstein, *Matrix Mathematics: Theory, Facts, and Formulas*, 2nd ed. Princeton University Press, 2011.

[19] P. Schuchert, V. Gupta, and A. Karimi, "Data-driven fixed-structure frequency-based $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controller design," *Automatica*, vol. 160, p. 111398, 2024.

[20] M. J. D. Powell, "On search directions for minimization algorithms," *Mathematical Programming*, vol. 4, no. 1, pp. 193–201, 1973.