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Bernhard Riemann 1861 revisited: existence of flat coordinates for an arbitrary bilinear form

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Abstract

We generalize the celebrated results of Bernhard Riemann and Gaston Darboux: we give necessary and sufficient conditions for a bilinear form to be flat. More precisely, we give explicit necessary and sufficient conditions for a tensor field of type $(0, 2)$ which is not necessary symmetric or skew-symmetric, and is possibly degenerate, to have constant entries in a local coordinate system.

Keywords Flat coordinates · Degenerate metrics · Symplectic structure · Poisson structure · Hamiltonian vector fields · Curvature · Pfaffian systems · Darboux theorem · Pullback equation · Hartman Theorem

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1 Introduction

In the paper [22] of 1861 Bernhard Riemann considered what is now called a Riemannian metric, that is, a symmetric positive definite 2-form $g = g_{ij}(x)$. He asked and answered the question under what conditions there exists a coordinate system such that g is given by a constant matrix. He proved that such coordinates exist locally if and only if what is now called the Riemann curvature tensor is identically zero. This result was announced in Riemann's famous inaugural lecture in 1854, see [24, Abschnitt 4]. Both the inaugural lecture and the paper [22] are viewed nowadays as the starting points of Riemannian Geometry. Note that [22] is written in Latin and its first part is not relevant to this question. An English translation of the relevant second part, with a detailed discussion, can be found in [25, pp. 179–182]. In particular it is explained there (and was known before) that the assumption of positive definiteness is not essential for the proof of Riemann: it is sufficient that the symmetric form is nondegenerate. See also [23].

The case when the bilinear form is skew-symmetric was considered and solved by Gaston Darboux [12]: he has shown that a nondegenerate differential 2-form $\omega = \omega_{ij}(x)$ is given by a constant matrix in a certain local coordinate system, if and only if it is closed. This result lays at the foundation of Symplectic Geometry.

In the present paper we ask and give a complete answer to the same question for an arbitrary bilinear form, that is a tensor field of type (0, 2), which may have nontrivial symmetric and skew-symmetric parts that can be degenerate. Note that the case where the symmetric part is nondegenerate can easily be reduced to the methods of Riemann (see e.g. [6] for a proof and a discussion of boundary, smoothness and global issues). Indeed, the existence of coordinates such that the components of the bilinear form $g_{ij} + \omega_{ij}$ are constant implies the existence of a symmetric (torsion free) connection $\nabla = (\Gamma^i_{jk})$ whose curvature is zero and such that the bilinear form is parallel. If the symmetric part g is nondegenerate, the only candidate for the connection is the Levi-Civita connection; the necessary condition is then that its curvature tensor vanishes. The other necessary condition is that the skew-symmeric part ω is parallel with respect to the Levi-Civita connection of g. These conditions are also sufficient. Therefore, the results in the present paper are new only in the case where g is degenerate and ω is arbitrary.

Our results are formulated in a way that the hypothesis on g and ω can effectively be checked using only differentiation and algebraic manipulations, as was the case in the results of Riemann and Darboux (in particular, if the entries of the bilinear forms are explicitly given by elementary functions, or as solutions of explicit systems of algebraic equations with rational coefficients, then the necessary and sufficient conditions for the the existence of flat coordinates can be checked using a computer algebra system).

Our paper is organized as follows: in Sect. 2 we treat the case when $\omega = 0$ and g is (possibly) degenerate, see Theorem 2.2 and Theorem 2.9. In Sect. 3, we consider in Theorems 3.1 and 3.3 the case where the skew-symmetric part is nondegenerate; and the symmetric part may be degenerate. In Sect. 4 we first treat the known case when the symmetric part is zero (and the skew-symmetric part may be degenerate), see Theorem 4.1, and then the general case, when both g and ω are allowed to be degenerate, see Theorem 4.4.

Sections 2.1 and 3.2 are about regularity issues; the reader who is only interested in smooth tensors can ignore them without any loss.

Our proofs use a variety of ideas and methods coming from different areas of differential geometry and the final Sect. 5 is an outlook of those methods.

Note that our investigation is mostly local (with the exception of the global statements in Corollaries 2.7 and 2.8 and the related global questions discussed in the outlook Sect. 5). Whenever possible, we give two proofs. The first proof assumes that all objects are sufficiently smooth, which allows for simpler and more geometric arguments and allows us to use the simplest possible mathematical language. Such proofs would be understood by Bernhard Riemann and mathematicians coming shortly after him, such as Sophus Lie, Gregorio Ricci-Curbastro, Gaston Darboux, Tullio Levi-Civita and Ferdinand Georg Frobenius. We recommend [25, Chaps. 4 and 5] or [13, Chaps. 3 and 4] for some background on the notations we use and relation to other notations commonly used in differential geometry. We also tried to give, whenever possible, a proof in a lower regularity.

2 The degenerate symmetric case

We consider a bilinear symmetric form $g = g_{ij}(x)$ and call it a (possibly, degenerate) metric on a domain in $\mathbb{R}^n$ with coordinates $x^1, ..., x^n$. We view g as a covariant tensor field, meaning that if $y^1, ..., y^n$ are a different coordinate system, then in these coordinates g has coefficients

$$\tilde{g}_{ij}(y) = \sum_{r,s} g_{rs}(x) \frac{\partial x^r}{\partial y^i} \frac{\partial x^s}{\partial y^j}. \tag{2.1}$$

Here, and throughout the paper, unless otherwise specified, all indexes run from 1 to n. A coordinate system is called *flat*, if in this coordinate system g is given by a constant matrix; our goal in this section is to give necessary and sufficient conditions for the existence of local flat coordinate systems for a given degenerate metric g. Our first result will play a key role in building such coordinates.

Theorem 2.1 *For every i, j, s consider*

$$\Gamma_{ij,s} := \frac{1}{2}\left(\frac{\partial g_{js}}{\partial x^i} + \frac{\partial g_{is}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^s}\right) \tag{2.2}$$

(we call them Christoffel symbols of the first kind). Then, at a point x there exist numbers Γ^i_{jk} with $\Gamma^i_{jk} = \Gamma^i_{kj}$ (we call them Christoffel symbols of the second kind) satisfying

$$\sum_s \left(\Gamma^s_{jk} g_{is} + \Gamma^s_{ik} g_{js}\right) = \frac{\partial g_{ij}}{\partial x^k} \tag{2.3}$$

if and only if the following condition holds:

$$\sum_s \Gamma_{ij,s} v^s = 0 \quad \text{for every } v^s \in \mathcal{R}, \tag{2.4}$$

where

$$\mathcal{R} := \mathcal{R}_g(x) := \mathrm{Kernel}(g) := \{v \in T_x M \mid g(v, \cdot) = 0\}. \tag{2.5}$$

If such numbers Γ^i_{jk} exist, the "freedom" in choosing them is the addition of possibly several terms of the form

$$v^i T_{jk} \text{ with } v \in \mathcal{R} \text{ and } T_{jk} = T_{kj}. \tag{2.6}$$

Moreover, if the rank of g is constant and (2.4) holds for every point x, then there exist smooth functions $\Gamma^i_{jk}(x)$ with $\Gamma^i_{jk} = \Gamma^i_{kj}$ satisfying (2.3).

Proof We fix a point x and view (2.3) as a system of linear equations on unknowns Γ^i_{jk}; the coefficients of this system come from g and partial derivatives of g. Remember now that a linear system of equation

$$Ay = b \tag{2.7}$$

(where A is a $N \times N$-matrix, $y = (y_1, .., y_N)$ is an unknown vector and $b = (b_1, ..., b_N) \in \mathbb{R}^N$ is a known vector) has a solution if and only if for every vector $a = (a_1, ..., a_N) \in \mathbb{R}^N$ such that $a^t A = \mathbf{0}$ we have $a^t b = 0$. We observe that the Eq. (2.3) is of the form (2.7) with $N = \frac{n^2(n+1)}{2}$. By standard algebraic manipulations (known at least to Levi-Civita) one reduces (2.3) to the system of equations

$$\sum_s g_{sk}\Gamma^s_{ij} = \Gamma_{ij,k}. \tag{2.8}$$

Indeed, replacing $\sum_s g_{si}\Gamma^s_{jk}$ by $\Gamma_{jk,i}$ and $\sum_s g_{sj}\Gamma^s_{ik}$ by $\Gamma_{ik,j}$ in (2.3) we see that any solution Γ^i_{jk} of (2.3) solves (2.8) and vice versa, thus there are two equivalent linear systems. It remains to observe that the condition $a^t b = 0$ applied to (2.8) is just the condition (2.4), and then for a linear system of equations (2.7) such that the coefficient matrix A and the free terms b smoothly depend on x one can find a smooth solution provided a solution exists at every point and the rank of A is constant. □

Remark The Christoffel symbols $\Gamma^i_{jk}(x)$ from the previous Theorem will always be considered to be the coefficients of an affine symmetric (torsion free) connection. This means that if $y^1, \ldots, y^n$ is a different coordinate system, then the corresponding Christoffel symbols $\tilde{\Gamma}^i_{jk}(y)$ should by definition be given by

$$\tilde{\Gamma}^k_{ij}(y) = \sum_{a,b,c} \frac{\partial y^k}{\partial x^c}\left(\Gamma^c_{ab}(x)\frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j} + \frac{\partial^2 x^c}{\partial y^i \partial y^j}\right).$$

This rule for the change of coordinate guarantees that the covariant derivative is a well defined operation on any tensor field, independently of the chosen coordinates, that is if $P = P^{i_1 \ldots i_k}_{j_1 \ldots j_m}$ is a tensor field of type (k, m), then

$$\begin{aligned} \nabla_i P^{i_1\ldots i_k}_{j_1\ldots j_m} &= \frac{\partial}{\partial x^i} P^{i_1\ldots i_k}_{j_1\ldots j_m} + \sum_s \left(P^{s i_2 \ldots i_k}_{j_1\ldots j_m}\Gamma^{i_1}_{si} + P^{i_1 s i_3\ldots i_k}_{j_1\ldots j_m}\Gamma^{i_2}_{si} + \cdots + P^{i_1 i_3\ldots i_{k-1} s}_{j_1\ldots j_m}\Gamma^{i_k}_{si}\right) \\ &\quad - \sum_s \left(P^{i_1\ldots i_k}_{s j_2\ldots j_m}\Gamma^s_{ij_1} + P^{i_1\ldots i_k}_{j_1 s j_3\ldots j_m}\Gamma^s_{ij_2} + \cdots + P^{i_1\ldots i_k}_{j_1\ldots j_{m-1} s}\Gamma^s_{ij_m}\right) \end{aligned}$$

is a well defined tensor field of type $(k, m+1)$. This tensor field is called the *covariant derivative* of P and denoted by ∇P, and we say that P is *parallel* if $\nabla P = 0$. For instance (2.3) just says that g is parallel with respect to ∇. The covariant derivative depends on the freedom (2.6), but by construction the condition $\nabla g = 0$ does not.

Our first main result is the following

Theorem 2.2 *Suppose rank of g is constant and assume (2.4) is fulfilled at any point. Then, for any smooth functions Γ^i_{jk} with $\Gamma^i_{jk} = \Gamma^i_{kj}$ satisfying (2.3) the functions*

$$R_{ijk\ell} := \sum_s g_{is}\left(\frac{\partial}{\partial x^k}\Gamma^s_{j\ell} - \frac{\partial}{\partial x^\ell}\Gamma^s_{jk} + \sum_a \left(\Gamma^s_{ka}\Gamma^a_{\ell j} - \Gamma^s_{\ell a}\Gamma^a_{jk}\right)\right) \tag{2.9}$$

do not depend on the freedom (2.6). *Moreover, there exist flat coordinates for g if and only if there exist smooth functions $\Gamma^i_{jk}(x)$ with $\Gamma^i_{jk}=\Gamma^i_{kj}$ satisfying* (2.3) *such that*[1]

$$R_{ijk\ell}=0 \text{ for every } i,j,k,\ell. \tag{2.10}$$

Proof In order to show that $R_{ijk\ell}$ does not depend on the freedom in choosing Γ, let us plug $\tilde{\Gamma}^i_{jk}=\Gamma^i_{jk}+v^iT_{jk}$ with $v\in\mathcal{R}$ instead of Γ in the formula (2.9) for $R_{ijk\ell}$. The terms of the form $v^s\frac{\partial}{\partial x^m}T_{jk}$, $v^s\frac{\partial}{\partial x^k}T_{jm}$, $v^sT_{ka}\tilde{\Gamma}^a_{\ell j}$, $v^sT_{\ell a}\tilde{\Gamma}^a_{kj}$ vanish after contracting with g_{is} so the result differs from the initial formula for $R_{ijk\ell}$ by

$$\sum_s g_{is}T_{j\ell}\frac{\partial v^s}{\partial x^k}+\sum_a v^a\Gamma_{ka,i}T_{j\ell}-\sum_s g_{is}T_{k\ell}\frac{\partial v^s}{\partial x^j}-\sum_a v^a\Gamma_{\ell a,i}T_{jk}. \tag{2.11}$$

Next, in view of condition $\sum_s g_{is}v^s=0$ we have that $\sum_s g_{is}T_{j\ell}\frac{\partial v^s}{\partial x^k}=-\sum_s T_{j\ell}v^s\frac{\partial g_{is}}{\partial x^k}$, which together with (2.2) imply that the sum of the first two terms of (2.11) is equal to $-\sum_s T_{j\ell}\Gamma_{ki,s}v^s\overset{(2.4)}{=}0$. Similarly, the sum of the last two terms is zero. The argument proves that the freedom in choosing Γ does not affect $R_{ijk\ell}$ and therefore the condition (2.10).

By the standard argument (due already to classics, see e.g. [25, Prop. 5 in Chapter 4]) we know that $R_{ijk\ell}$ is a tensor field. Then, its vanishing in one coordinate system implies its vanishing in any other coordinate system. Then, the existence of flat coordinates implies that $R_{ijk\ell}=0$, so the conditions listed in Theorem 2.2 are necessary.

Let us prove that they are sufficient. We first observe that for any smooth vector field $v\in\mathcal{R}$ the metric g is preserved by its flow. Indeed, the Lie derivative of the metric is given by

$$\begin{aligned}(\mathcal{L}_v g)_{ij}&=\sum_s\left(v^s\frac{\partial g_{ij}}{\partial x^s}+g_{is}\frac{\partial v^s}{\partial x^j}+g_{js}\frac{\partial v^s}{\partial x^i}\right)\\&=\sum_s\left(v^s\frac{\partial g_{ij}}{\partial x^s}-v^s\frac{\partial g_{is}}{\partial x^j}-v^s\frac{\partial g_{js}}{\partial x^i}\right)\\&=-2\sum_s v^s\Gamma_{ij,s}=0.\end{aligned}$$

Next, let us show that the distribution $\mathcal{R}$ is integrable, that is, for any two vector fields v,u from this distribution its commutator $[u,v]$ lies in the distribution. We obtain it by direct calculations:

$$\begin{aligned}\sum_i g_{ij}[u,v]^i&=\sum_{s,i}\left(g_{ij}u^s\frac{\partial v^i}{\partial x^s}-g_{ij}v^s\frac{\partial u^i}{\partial x^s}\right)=-\sum_{s,i}\left(v^iu^s-u^iv^s\right)\frac{\partial g_{ij}}{\partial x^s}\\&=\sum_{s,i}\left(v^iu^s-u^iv^s\right)(\Gamma_{is,j}+\Gamma_{js,i})=0.\end{aligned}$$

Then, there exist coordinates $(x^1,...,x^k,y^1,...,y^{n-k})$ such the distribution is spanned by $\frac{\partial}{\partial y^1},...,\frac{\partial}{\partial y^{n-k}}$. In these coordinates the metric has the form

$$g=\sum_{i,j}^k g_{ij}dx^idx^j.$$

[1] We stress that, unless g is non degenerate, Condition (2.10) is of course *not* equivalent to the vanishing of $R^i_{jk\ell}=\frac{\partial}{\partial x^k}\Gamma^i_{j\ell}-\frac{\partial}{\partial x^\ell}\Gamma^i_{jk}+\sum_a\left(\Gamma^i_{ka}\Gamma^a_{\ell j}-\Gamma^i_{\ell a}\Gamma^a_{jk}\right)$.

Since the vector fields $\frac{\partial}{\partial y^i} \in \mathcal{R}$ and therefore their flows preserve g, the components g_{ij} are independent of y-coordinates. We then may view g as a metric on a k-dimensional manifold with local coordinate system $x^1, ..., x^k$. Equation (2.3) implies that $\left(\Gamma^i_{jm}\right)_{i,j,m=1,...,k}$ are coefficients of the Levi-Civita connection of this metric (of dimension k). Without loss of generality, because of the freedom (2.6), we may assume that all Γ^i_{jm} with $i > k$ are equal to zero. Then, the formula for the components $R_{ij\ell m}$ of the curvature tensor (with lower indexes) of this k-dimensional metric coincides, for $i, j, \ell, m \leq k$, with (2.9). Then, the problem is reduced to the case when g is nondegenerate, which was already solved by Riemann (see e.g. [23, Sect. 4.4.7]). □

As the following example shows, the condition (2.10) almost everywhere does not imply that the rank of g is constant.

Example 2.3 We consider the function

$$\phi : \mathbb{R}^2 \to \mathbb{R}, \quad \phi(x, y) = x^2 + y^2$$

and as g we take $(d\phi)^2$. Locally, in a neighbourhood of any point different from $(0, 0)$ the degenerate metric g has constant coefficients in any coordinate system such that ϕ is the first coordinate. Its rank falls to zero at the point $(0, 0)$ and is one otherwise. By direct calculation one sees that any continuous solution $\Gamma^i_{jk}(x)$ of (2.3) (assuming $\Gamma^i_{jk}(x) = \Gamma^i_{kj}(x)$) is not bounded when approaches $(0, 0)$.

The example can easily be generalised for any dimension and any rank. On the other hand, the existence of continuous functions Γ^i_{jk} satisfying (2.3) implies that the rank of g is constant.

Remark 2.4 The book [15] of D. Kupeli studies degenerate metrics (Kupeli calls them "singular metrics"), the corresponding affine connections and their curvature tensors. The condition (2.4) is equivalent to the *stationarity condition* [15, Def. 3.1.3]. This author did not study the existence of flat coordinates but the invariance of $R_{ijk\ell}$ with respect to the freedom and the Condition (2.6) are implicitly contained in his book.

Corollary 2.5 *Assume g admits flat coordinates. Consider the following system of PDE:*

$$0 = \nabla_j u_i := \frac{\partial u_i}{\partial x^j} - \sum_s \Gamma^s_{ij} u_s \tag{2.12}$$

on the unknown functions $u_1(x), ..., u_n(x)$, where Γ^s_{ij} is a (smooth) solution of (2.3)*. Then, for every point $\hat{x}$ and for any initial data $(\hat{u}_1, ..., \hat{u}_n) \in \mathbb{R}^n$ such that for every $v \in \mathcal{R}(\hat{x})$ we have $\sum_s v^s \hat{u}_s = 0$ there exists a unique solution $u_1, ..., u_n$ of* (2.12) *with the initial conditions $u_i(\hat{x}) = \hat{u}_i$. This solution has the property $\sum_s v^s u_s = 0$ at every x and for every $v \in \mathcal{R}(x)$. Furthermore, for any such a solution $u_1, ..., u_n$ the 1-form $u_1 dx^1 + ... + u_n dx^n$ is closed so there exists locally a function f such that $\frac{\partial f}{\partial x^i} = u_i$. Moreover, if a solution vanishes at one point, it vanishes at every point.*

Proof The equation (2.12) means that the 1-form $u_1 dx^1 + ... + u_n dx^n$ is parallel with respect to the connection $\nabla = (\Gamma^i_{jk})$. In particular, the equation is invariant with respect to the coordinate changes. Because $\mathcal{R}$ is invariant under parallel transport, if $\mathcal{R} \subseteq \mathrm{Kernel}(u_1 dx^1 + ... + u_n dx^n)$ at the point $\hat{x}$, then $\mathcal{R} \subseteq \mathrm{Kernel}(u_1 dx^1 + ... + u_n dx^n)$ at every point. In the flat coordinates $x^1, ..., x^n$ such that $g = \sum_{s=1}^r \varepsilon_i (dx^i)^2$ (with $\varepsilon_i \in \{-1, 1\}$) the Eq. (2.12) reads $\frac{\partial u_i}{\partial x^j} = 0$. Then, if the initial data satisfy $\sum_s v^s \hat{u}_s = 0$, then for any solution we have

$u_{r+1} = \cdots = u_n = 0$ and first r functions u_1, ..., u_r satisfy $\frac{\partial u_i}{\partial x^j} = 0$ which implies that they are arbitrary constants. □

Remark 2.6 The case of rank one metric is special, the following statement is true: *If g has rank 1, then $g = \pm\theta \otimes \theta$ for a locally defined (non zero) 1-form θ. Furthermore* (2.4) *is equivalent to $d\theta = 0$, and this holds if and only if g admits flat coordinates.*

To see this, recall that g_{ij} is symmetric of rank one if and only if there exists $(a_1, \ldots, a_n)$, non vanishing, such that $g_{ij} = \pm a_i a_j$. Suppose (2.3) holds for $g = \pm\theta \otimes \theta$ with $\theta = a_1 dx^1 + \ldots + a_n dx^n$. Clearly, in the flat coordinate system for g the components a_i are constant and θ is closed. In the other direction, if $\nabla g = 0$ then $\nabla(\theta) = 0$ implying $d\theta = 0$.

So far we have worked on (an open subset of) $\mathbb{R}^n$, but because the conditions (2.3) and (2.10) are coordinate invariant, they have a meaning globally on a smooth manifold M and we can state the following

Corollary 2.7 *If M is smooth closed manifold such that $H^1_{dR}(M) = 0$, then it does not admit a degenerate metric g_{ij} of constant* $\mathrm{Rank}(g) \geq 1$ *such that $R_{ijk\ell} = 0$.*

Proof If $H^1_{\mathrm{dR}}(M) = 0$, any closed 1-form is exact so the form $\sum_i u_i dx^i$ given by Corollary 2.5 is the differential of a function. Then, it vanishes at the points where the function takes its extremal values which gives a contradiction. □

Corollary 2.8 *If the smooth closed manifold M admits a degenerate metric g of rank* 1 *such that* (2.4) *holds, then M or its double cover is a fiber bundle over a circle.*

Proof By Remark 2.6, we know that locally $g = \pm\theta \otimes \theta$ for a nowhere vanishing closed 1-form θ. Then θ is either well defined globally on M, or it is well defined on a double cover. The claim follows then from [26, Theorem 1]. □

2.1 Optimal C^r-regularity for Theorem 2.2

It is known that for a non degenerate metric, the following optimal regularity holds: if g is of class C^r with $r \in \mathbb{N}$ and satisfies (2.4) and(2.10), then there exist flat coordinate systems of class C^{r+1} (if $r = 1$, then the curvature has to be interpreted in the sense of distributions[2]). We refer to [18] or [6, Theorem 8 and Remark 9] for a proof of this optimality result. In the degenerate case, our proof of Theorem 2.2 loses one degree of regularity when we "factor out" the kernel of g. Thus our proof of Theorem 2.2 assumes g to be of class C^r with $r \geq 2$ and produces a flat coordinate system of class C^r. Our next result states the existence of flat coordinates in optimal regularity:

Theorem 2.9 *Suppose g has constant rank and assume* (2.4) *holds at any point. If $g \in C^r$ for some $r \in \mathbb{N}$, then one can find Γ^i_{jk} of class C^{r-1} such that $\Gamma^i_{jk} = \Gamma^i_{kj}$ and* (2.3) *holds. Moreover, there exist flat coordinates of class C^{r+1} if and only if* (2.10) *is fulfilled.*

Remark 2.10 In our convention the set $\mathbb{N}$ starts with 1. When $r = 1$, the condition (2.10) has to be understood in the weak sense, see [14, Sect. VI.I.6]. In the present situation, this

[2] Note that a similar result cannot hold for $r = 0$. In [9, Sect. 6], E. Calabi and P. Hartman have given an example of a continuous metric which is locally isometric to the Euclidean metric but admits no flat coordinates of class C^1.

conditions means that for any $k, \ell \in \{1, \ldots, n\}$ and any smooth 1-form $u = (u_i) = \sum_i u_i dx^i$ with compact support such that $\mathcal{R}_g \subseteq \mathrm{Kernel}(u)$, we have

$$\int \sum_s \left(-\Gamma^s_{j\ell} \frac{\partial u_s}{\partial x^k} + \Gamma^s_{jk} \frac{\partial u_s}{\partial x^\ell} + u_s \sum_a \left(\Gamma^s_{ka} \Gamma^a_{\ell j} - \Gamma^s_{\ell a} \Gamma^a_{jk} \right) \right) dx = 0.$$

This condition is independent of the freedom (2.6).

Proof The proof that (2.10) holds if there exist flat coordinates is similar to the proof of the analogous statement in Theorem 2.2. Also the proof that Γ^i_{jk} can be chosen of regularity C^{r-1} is the same as in Theorem 2.2. In order to prove the existence and smoothness of flat coordinates assuming (2.10), let us consider a $n \times (n-m)$-matrix-valued function $B(x)$ such that its columns are basis vectors of $\mathcal{R}_g$. Since $\mathcal{R}_g$ is given by a system of linear equations of constant rank whose coefficients are of class C^r, we may assume that B is of regularity C^r. Next, without loss of generality we may assume that the last $n-m$ rows of B form a nondegenerate matrix (of dimension $(n-m) \times (n-m)$). Then, there exists a unique $m \times (n-m)$-matrix-valued function F such that for every x the vector $(u_1, ..., u_n)$ whose first components $u_1, ..., u_m$ are arbitrary and the other components $u_{m+1}, ..., u_n$ are constructed by $u_1, ..., u_m$ via matrix-multiplication

$$(u_{m+1}, ..., u_n) = (u_1, ..., u_m) F \tag{2.13}$$

the following condition[3] is fulfilled:

$$(u_1, ..., u_n) B = 0. \tag{2.14}$$

The matrix F can be explicitly constructed as follows: if we denote by B' the submatrix of B containing the first m rows of B and by B'' the submatrix of B containing the last $n-m$ rows by B'', then B'' is an invertible square matrix by hypothesis and F is explicitly given by $F = -B'(B'')^{-1}$. In particular F is of class C^r.

In what follows we denote the i^{th} component of the left hand side of (2.13) by $F(u)_{m+i}$. and we consider the following system of $m \times n$ PDEs on m unknown functions $u_1, ..., u_m$ of the variables $(x^1, \ldots, x^n)$:

$$\frac{\partial u_i}{\partial x^j} = \sum_{s=1}^{m} \Gamma^s_{ij} u_s + \sum_{s=m+1}^{n} \Gamma^s_{ij} F(u)_s. \tag{2.15}$$

where $1 \le i \le m$ and $1 \le j \le n$. It follows from (2.6) that the system (2.15) is independent of the choice of connection Γ^k_{ij} satisfying (2.3). We observe the following facts concerning the system (2.15):

(i) The system (2.15) is of Pfaff–Frobenius–Cauchy type, in the sense that all derivatives of unknown functions are linear expressions of unknown functions whose coefficients are functions of the position.

(ii) If g is of class C^r, with $r \ge 1$, then the coefficient of (2.15) are of class C^{r-1}. This is due to the fact that (2.3) is a linear system of constant rank with coefficient of class C^r (see the proof of Theorem 2.1 for an explanation). One can therefore find Γ^i_{jk} of class C^{r-1} satisfying (2.3).

(iii) The compatibility conditions for (2.15) are equivalent to (2.10) (see e.g., [14, Sect. VI.I.6]).

[3] Geometrically, $(u_1, ..., u_n)$ should be viewed as a covector, i.e., as the 1-form $u_1 dx^1 + ... + u_n dx^n$. The condition (2.14) is just the condition $\mathrm{Kernel}(u_1 dx^1 + ... + u_n dx^n) \supseteq \mathcal{R}_g$.

(iv) If the compatibility conditions are satisfied, there exists, for any point p and any initial condition $u_1(p), ..., u_m(p)$, a unique (local) solution of (2.15) with this initial condition. Furthermore, if the coefficients of (2.15) are of class C^{r-1} for some $r \in \mathbb{N}$, then this solution is of class C^r (if $r = 1$ the compatibility condition has to be interpreted n the weak sense). This statement is proved in [14, Chap. VI, Corollary 6.1].

Let us now show that if $u_1(x), ..., u_m(x)$ is a solution of (2.15), then the differential form whose first components are $u_1(x), ..., u_m(x)$ and the remaining $(n-m)$ components are given by (2.13) is parallel with respect to any symmetric connection $\nabla = (\Gamma^i_{jk})$ whose coefficients satisfy (2.3). Indeed. for $i \in \{1, ..., m\}$ the condition $\nabla_j u_i = 0$ is clearly equivalent to (2.15). To deal with the case $i \in \{m+1, ..., n\}$ we need the following additional statement: for any vector field $v = v^i \in \mathcal{R}_g$ of class C^r and any vector field z^j we have

$$\sum_j \left(z^j \nabla_j v^i \right) \in \mathcal{R}_g. \tag{2.16}$$

Indeed,

$$0 = \sum_j z^j \tfrac{\partial}{\partial x^j} \left(\sum_{s,r} g_{sr} v^s w^r \right) = \sum_{s,r,j} g_{sr} w^r z^j \nabla_j v^s + \underbrace{\sum_{s,r,j} g_{sr} v^s z^j \nabla_j w^r}_{=0 \text{ for } v \in \mathcal{R}_g}$$

so $\sum_j z^j \nabla_j v^s$ is a linear combination of the vectors from $\mathcal{R}_g$.

Using (2.16), we obtain that for any $v \in \mathcal{R}_g$ and any z (both of class C^r) we have

$$\sum_{i,j} v^i z^j \nabla_j u_i = \sum_j z^j \nabla_j \left(\sum_i u_i v^i \right) - \sum_{i,j} u_i z^j \nabla_j v^i = 0 - 0 = 0.$$

Then, the covector whose components are given by

$$\left(\sum_j z^j \nabla_j u_1, ..., \sum_j z^j \nabla_j u_n \right)$$

satisfies (2.14), so its last $n - m$ components are determined by its first m components via (2.13). Since the first m components are zero, as we proved above, also the last $n - m$ components are zero.

Thus, we have shown that for any point p and for any initial values $u_1(p), ..., u_n(p)$ such that $\mathrm{Kernel}(u_1 dx^1 + ... + u_n dx^n) \supseteq \mathcal{R}_g(p)$ there exists a unique 1-form $u_i(x) = u_1(x)dx^1 + ... + u_n(x)dx^n$ of class C^r such that it is $\nabla-$parallel, moreover, this form has the condition $\mathrm{Kernel}(u_1 dx^1 + ... + u_n dx^n) \supseteq \mathcal{R}_g(p)$ at every point. This form is automatically closed. We take m linearly independent 1-forms of such type and denote by $f^1, ..., f^m$ their primitive functions. At the point p, there exists a $m \times m$ symmetric nondegenerate matrix c_{ij} such that at p we have $g = \sum_{i,j=1}^m c_{ij} df^i df^j$. Since by construction g and each of the forms df^i are parallel, this condition holds at any point so every coordinate system such that the first m coordinates are the functions $f^1, ..., f^m$ is flat for this metric. □

Corollary 2.11 *Suppose g has constant rank and satisfies (2.4) everywhere. Suppose also $g \in C^{r,\alpha}$ with $r \in \mathbb{N}$ and $0 \le \alpha \le 1$, then there exists a flat coordinate system of class $C^{r+1,\alpha}$ if and only if (2.10) holds.*

Proof Arguing as in proof of Theorem 2.9, we consider the system (2.15), whose solutions correspond to the differentials of the first m flat coordinates. We know that the solutions are of class C^r. We also see that the derivatives of the solutions are linear expression in the solutions with coefficients at least of class $C^{r-1,\alpha}$. Therefore, the derivatives of the solutions are of class $C^{r-1,\alpha}$ and the solutions of (2.15) are therefore of class $C^{r,\alpha}$. This implies that the flat coordinates are of class $C^{r+1,\alpha}$. □

Remark 2.12 The proof of Theorem 2.9 shows that the metric g has a flat coordinate system if and only there exist functions $f^1, ..., f^m$ (with $m = \mathrm{Rank(g)}$) such that $g = \sum_{ij=1}^m c_{ij} df^i df^j$ with constant c_{ij}, furthermore the 1-forms df^i are parallel and the flat coordinate system $x^1, \ldots, x^n$ can be chosen such that $x^i = f^i$ for $1 \leq i \leq m$. Furthermore, if g of of class $C^{r,\alpha}$, then on can chose f^i of class $C^{r+1,\alpha}$

3 On flat coordinates for the pair (degenerate metric, symplectic structure)

3.1 Existence of flat coordinates

In this section we obtain necessary and sufficient conditions for the existence of flat coordinates for the bilinear form $g + \omega$ with nondegenerate skew-symmetric part ω. Obvious necessary conditions are that g has flat coordinates and ω is a closed form. We will prove the following result:

Theorem 3.1 *Let g be a a symmetric (possibly degenerate) bilinear form such that there exist flat coordinates for it and $\omega = \omega_{ij}$ be a symplectic form. Then, there exists a coordinate system such that the components of both g and ω are constant if and only if the equation*

$$\sum_{a,b,c,d} g_{ia} P^{ab} P^{cd} g_{dj} \nabla_k \omega_{bc} = 0 \tag{3.1}$$

holds for every i, j, k, where P^{ij} is the inverse matrix of ω_{ij}

$$\sum_s P^{is} \omega_{sj} = \delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and ∇ is any connection compatible with g, i.e. satisfying (2.3). Condition (3.1) does not depend on the chosen connection.

Remark 3.2 (i) The matrix $P = P^{ij}$, inverse of ω_{ij} represents a *contravariant* tensor field. This means that under a change of coordinates, the transformation rule is given by the rule dual to (2.1):

$$\tilde{P}^{ij}(y) = \sum_{r,s} P^{rs}(x) \frac{\partial y^i}{\partial x^r} \frac{\partial y^j}{\partial x^s}.$$

(ii) Another possible formulation of Condition (3.1) can be written using the $(1, 1)$ tensor J such that $g(X, Y) = \omega(JX, Y)$, that is $J^j_i = -\sum_k g_{ik} P^{kj}$. Using this tensor, we define a differential (skew symmetric) 2-form α by $\alpha(u, v) := g(u, Jv)$. Condition (3.1) is then equivalent to $\nabla\alpha = 0$.

Before proving Theorem 3.1, we first give necessary and sufficient conditions for the existence of a local coordinate system in which both g and P have constant components:

Theorem 3.3 *Let $g = g_{ij}$ be a symmetric (possibly degenerate) bilinear form such that there exist flat coordinates for it near a point $p \in \mathbb{R}^n$ and $P = P^{ij}$ be a skew-symmetric tensor field of rank n at p. Then, there exists a local coordinate system near p such that both g and P have constant components if and only if the following conditions hold:*

(1) *P^{ij} generates a Poisson structure, that is*

$$\sum_s P^{sk} \frac{\partial}{\partial x^s} P^{ij} + P^{si} \frac{\partial}{\partial x^s} P^{jk} + P^{sj} \frac{\partial}{\partial x^s} P^{ki} = 0. \tag{3.2}$$

(2) *The following holds for every i, j, k:*

$$\sum_{a,b} g_{ai} g_{bj} \frac{\partial P^{ab}}{\partial x^k} + \sum_s \left(P_i^{\ s} \Gamma_{ks,j} + P^s_{\ j} \Gamma_{ks,i} \right) = 0, \tag{3.3}$$

where $P_i^{\ s} = \sum_c g_{ic} P^{cs}$ and $P^s_{\ j} = \sum_c g_{jc} P^{sc}$, and $\Gamma_{ij,s}$ are as in (2.2).

Proof Let us first observe that conditions (3.2) and (3.3) are geometric. Indeed (3.2) is just the condition that the bilinear operation $\{\cdot, \cdot\}$ defined on functions by

$$\{f, h\} := \sum_{i,j} \frac{\partial f}{\partial x^i} \frac{\partial h}{\partial x^j} P^{ij}, \tag{3.4}$$

satisfies the Jacobi identity and is therefore a Poisson bracket. The condition (3.3) says that the tensor obtained by lowering both upper indexes in

$$\nabla_k P^{ij} = \frac{\partial}{\partial x^k} P^{ij} + \sum_s \left(\Gamma^i_{sk} P^{sj} + \Gamma^j_{sk} P^{is} \right), \tag{3.5}$$

by g_{ij} vanishes. In particular (3.3) does not depend on the choice of the connection Γ^i_{jk} satisfying (2.3). Furthermore, both (3.2) and (3.3) are tensorial conditions, that are obviously satisfied in a flat coordinate system. So if there exists flat coordinates for both g and P, then then (3.3) and (3.2) hold in any coordinate system.

In order to prove Theorem 3.3 in the other direction, let us consider smooth functions $f^1, ..., f^m$ such that $g = \sum_{i=1}^m \varepsilon_i (df^i)^2$ with $\varepsilon_1, ..., \varepsilon_m \in \{-1, \ 1\}$. We assume that the differentials of these functions are linearly independent in every points which implies $m = \mathrm{rank}(g)$. Furthermore $\nabla(df^i) = \left(\nabla_k \frac{\partial f^i}{\partial x^j} \right) = 0$. The existence of such functions follows from the existence of flat coordinates.

We claim that (3.3) is equivalent to the condition that for any $i, j \in \{1, \dots, m\}$ the Poisson bracket $\{f^i, f^j\}$ is a constant. Indeed, using (3.5) and

$$\nabla_k \left(\frac{\partial f^i}{\partial x^a} \right) = \frac{\partial^2 f^i}{\partial x^k \partial x^a} - \Gamma^b_{ka} \frac{\partial f^j}{\partial x^b} = 0,$$

we obtain

$$\begin{aligned}
\nabla_k\{f^i, f^j\} &= \sum_{a,b} \nabla_k \left(P^{ab} \frac{\partial f^i}{\partial x^a} \frac{\partial f^j}{\partial x^b} \right) \\
&= \sum_{a,b} \nabla_k (P^{ab}) \frac{\partial f^i}{\partial x^a} \frac{\partial f^j}{\partial x^b} + \sum_{a,b} P^{ab} \nabla_k \left(\frac{\partial f^i}{\partial x^a} \right) \frac{\partial f^j}{\partial x^b} + \sum_{a,b} P^{ab} \frac{\partial f^i}{\partial x^a} \nabla_k \left(\frac{\partial f^j}{\partial x^b} \right) \\
&= 0.
\end{aligned}$$

Next, consider the vector fields $X_{f^1}, ..., X_{f^m}$ whose components are given by:

$$X^{\,i}_{f^j} = \sum_s P^{si} \tfrac{\partial f^j}{\partial x^s},$$

(they are called the *Hamiltonian vector fields of* f^j). The condition (3.2) implies that they commute. Indeed, the commutator of the vector fields X_{f^μ} and X_{f^ν} is given by

$$\begin{aligned}
[X_{f^\mu}, X_{f^\nu}]^i &= \sum_{a,b,s} \left(P^{as} \frac{\partial f^\mu}{\partial x^a} \frac{\partial}{\partial x^s} \left(P^{bi} \frac{\partial f^\nu}{\partial x^b} \right) - P^{as} \frac{\partial f^\nu}{\partial x^a} \frac{\partial}{\partial x^s} \left(P^{bi} \frac{\partial f^\mu}{\partial x^b} \right) \right) \\
&= \sum_{a,b,s} \left(P^{as} \frac{\partial f^\mu}{\partial x^a} P^{bi} \frac{\partial^\nu f^\nu}{\partial x^b \partial x^s} + P^{as} \frac{\partial f^\mu}{\partial x^a} \frac{\partial P^{bi}}{\partial x^s} \frac{\partial f^\nu}{\partial x^b} \right. \\
&\qquad \left. - P^{as} \frac{\partial f^\nu}{\partial x^a} P^{bi} \frac{\partial^\nu f^\mu}{\partial x^b \partial x^s} - P^{as} \frac{\partial f^\nu}{\partial x^a} \frac{\partial P^{bi}}{\partial x^s} \frac{\partial f^\mu}{\partial x^b} \right) \\
&\overset{(3.2)}{=} \sum_s P^{si} \frac{\partial}{\partial x^s} \{f^\mu, f^\nu\} = 0.
\end{aligned}$$

Let us show that there exists a function f^{m+1} such that the differential df^{m+1} is linearly independent (at the point in whose small neighbourhood we are working in) from the differentials of the functions $df^1, ..., df^m$ and such that for every $i = 1, ..., m$ the function $df^{m+1}(X_{f^i})$ is a constant. In order to do it, we consider the coordinates $(t^1, ..., t^m, z^{m+1}, ..., z^n)$ such that in these coordinates for every $i = 1, ..., m$ the vector field X_{f^i} is equal to $\frac{\partial}{\partial t^i}$. The coordinates exist by the (simultaneous) Rectification Theorem.

Chose now an arbitrary 1-form θ with constant entries in this coordinate system which is linearly independent from $df^1, ..., df^m$. Clearly $d\theta$ is closed and we can choose f^{m+1} such that $df^{m+1} = \theta$. It is clear from the construction that $\{f^{m+1}, f^i\} = \theta(X_{f^i})$ is constant for all i.

We consider then the symmetric bilinear form

$$g_{ext} := g + (df^{m+1})^2.$$

It has constant rank equal to $m + 1$ and its entries are constant in the coordinate system $(x^1 = f^1, ..., x^{m+1} = f^{m+1}, x^{m+2}, ..., x^n)$. Moreover, the (natural analog of the) condition (3.3) is satisfied for this metric. Indeed, this condition is equivalent to the condition that

$$\{f^i, f^j\} = \sum_{a,b} \frac{\partial f^i}{\partial x^a} \frac{\partial f^j}{\partial x^b} P^{ab}$$

is constant for every $i, j = 1, ..., m + 1$, which is clearly the case by the construction.

Then, we can enlarge the rank of g further and in $n - m$ such steps come to the coordinate system $f^1, ..., f^n$ in which both the metric and the tensor P have constant components. □

We can now prove the main Theorem of the section.

Proof of Theorem 3.1 It is well known that the dual P of a symplectic form ω is a Poisson structure, thus condition (3.2) is satisfied. We claim that (3.1) and (3.3) are equivalent conditions. To prove this claim, recall that $\delta^i_j = \sum_s P^{is}\omega_{sj}$, is a parallel tensor for any connection, therefore

$$0 = \nabla_k(\delta^a_t) = \nabla_k\left(\sum_s P^{as}\omega_{st}\right) = \sum_s(\nabla_k P^{as})\omega_{st} + \sum_s P^{as}\nabla_k\omega_{st},$$

we thus have

$$\nabla_k P^{ab} = \sum_{t,s} P^{bt}(\nabla_k P^{as})\omega_{st} = -\sum_{t,s} P^{as}P^{bt}\nabla_k\omega_{st}.$$

Lowering both upper indexes in this identity by g gives the equivalence (3.1) $\Leftrightarrow$ (3.3). Theorem 3.3 gives us now the existence of coordinates in which both g and P have constant entries. Clearly ω is also constant in these coordinates. □

The following example provides a simple instance where Theorem 3.3 implies the existence of flat coordinates for $g + \omega$. However, directly establishing the existence of such coordinates may not be straightforward.

Example 3.4 Let us consider the following tensors in $\mathbb{R}^4$:

$$g = (dx^1)^2 + (dx^2)^2 \quad \text{and} \quad \omega = dx^1 \wedge dx^2 + a \cdot dx^2 \wedge dx^3 + dx^3 \wedge dx^4,$$

where $a = a(x^2, x^3)$ is a smooth, non constant function of x^2 and x^3. Since g is constant we will choose ∇ to be the standard connection on $\mathbb{R}^4$. A tensor is then parallel for ∇ if and only its entries are constant. In matrix notations, the tensors g, ω and P are

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -a & 0 \\ 0 & a & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad P = \Omega^{-1} = \begin{pmatrix} 0 & 1 & 0 & a \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -a & 0 & -1 & 0 \end{pmatrix}.$$

The tensor

$$GPG = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is constant. By Theorem 3.3, we know that there exists a local coordinate system in some neighborhood of any point of $\mathbb{R}^4$ such that g, P and ω have constant components.

One should note however that

$$J = -PG = (GP)^\top = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix}.$$

is *not* constant, hence not parallel for the connection ∇.

Remark 3.5 In the proof of Theorem 3.3, we have used several times the (simultaneous) *Rectification Theorem*, which states that if $X_1, \dots, X_k$ are k linearly independent vector fields in a domain of $\mathbb{R}^n$ such that $[X_i, X_j] = 0$, then there exist local coordinates $x^1, \dots, x^n$ in a neighborhood of any points such that $X_i = \frac{\partial}{\partial x^i}$ for $i = 1, \dots, k$. Furthermore, if the fields are of class $C^{r,\alpha}$ with $r \geq 1$ and $0 \leq \alpha \leq 1$, then the coordinates are also of class $C^{r,\alpha}$. Indeed, by standard results from the theory of ordinary differential equations, we know that a vector field of class $C^{r,\alpha}$ generates a flow of class $C^{r,\alpha}$ (see e.g. [11, Theorem 12.2]). Therefore, the proof of Theorem 3.1, shows that if one assumes that g and ω are of class $C^{r,\alpha}$ with $r \geq \max\{(n-m), 1\}$, then there exists a coordinate system of class $C^{r+1+m-n,\alpha}$ that is flat for both g and ω. The reason is that in the proof of Theorem 3.3, we loose one class of regularity at each step of the construction (note that by Remark 2.12, the functions $f^1, \dots, f^m$ are of class $C^{r+1,\alpha}$ and the proof requires $n-m$ steps, so the resulting coordinates are indeed of class $C^{r+1+m-n,\alpha}$). A better regularity result will be given in next section.

Finally, note that the arguments in our previous proof also show that the following statement is true:

Theorem 3.6 *Let ω be a symplectic form of class defined on a domain $U \subset \mathbb{R}^n$. Suppose there exists $f^1, \dots, f^m \in C^r(U)$ such that $df^1, \dots, df^m$ are everywhere linearly independent and the Poisson brackets $\{f^i, f^j\}$ are constant on U for any $i, j \in \{1, \dots, m\}$. If $r \geq p = n - m$, then there exists a coordinate system $y^1, \dots, y^n$ of class in some neighborhood of any point in U such that $y^i = f^i$ for $i = 1, \dots, m$ and ω has constant coefficients ω_{ij} in these coordinates.*

Note in particular that the case $m = 0$ gives an alternative proof of Darboux' Theorem. We are not aware of such a proof in the literature.

3.2 On the regularity of flat coordinates for the pair (degenerate metric, symplectic structure)

By Theorem 2.9 and Corollary 2.11, if the (degenerate) metric g is of class $C^{r,\alpha}$, then the flat coordinate system, if it exists, is of class $C^{r+1,\alpha}$. A similar phenomenon holds in the purely skew-symmetric case, when $g = 0$ and ω is nondegenerate. Indeed, it has been proved in [4, Theorem 18] that given a symplectic form ω of class $C^{r,\alpha}$ with $0 < \alpha < 1$ and $r \in \mathbb{N} \cup \{0\}$, there exists local coordinate systems of class $C^{r+1,\alpha}$ in which ω has constant entries. In view of these results, one might hope that if g and ω are of class $C^{r,\alpha}$, then a flat coordinate system of class $C^{r+1,\alpha}$ should exists for $g + \omega$. The following example ruins such hope.

Example 3.7 We consider $\mathbb{R}^2$ with the coordinates (x, y) and the bilinear form $g + \omega$ with $g = dx^2$ and $\omega = h(x) dx \wedge dy$ with $h \neq 0$. Then, the condition (3.1) holds, and up to a $C^{r,\alpha}$-coordinate change, the flat coordinates are given by $(x, u(x, y))$ with the function u satisfying the equation $\frac{\partial u}{\partial y} = h(x)$. The general solution of this equation is $u(x, y) = \hat{u}(x) + yh(x)$ with an arbitrary function $\hat{u}(x)$. If h is not of class $C^{r,\alpha}$, then $u(x, y)$ is also not of class $C^{r,\alpha}$, which implies that flat coordinates cannot be of class $C^{r+1,\alpha}$.

The next result improves Theorem 3.1: if the bilinear form is of class $C^{r,\alpha}$ with $3 \leq r \in \mathbb{N}$ and $0 < \alpha < 1$, then one can find flat coordinates of class $C^{r-2,\alpha}$.

Theorem 3.8 *Under the hypothesis of Theorem 3.1, if the condition (3.1) is fulfilled and the bilinear form $g + \omega$ is of class $C^{r,\alpha}$ with $3 \leq r \in \mathbb{N}$ and $0 < \alpha < 1$, then there exists a flat coordinate system of class $C^{r-2,\alpha}$.*

The rest of the section is devoted to proving this Theorem; the proof is quite involved and can be omitted with no damage for the understanding of the rest of the article. For the proof, we will need the following two statements, which are known in folklore, but for which we did not find explicit references. We sketch the ideas leading to the proof.

Lemma 3.9 (Poincaré Lemma with parameters) *Let ω_s be a family of closed m-forms on a ball U^n with coordinates $x^1, ..., x^n$, where $s = (s^1, ..., s^k)$ are some parameters. Assume that the dependence of the components of ω_s on x and on s is of class $C^{r,\alpha}$ with $r \in \mathbb{N}$ and $0 \leq \alpha \leq 1$, then, there exists a family θ_s of $(m-1)$-forms, such that their dependence on (x, s) is of class $C^{r,\alpha}$ and such that for every s we have $d\theta_s = \omega_s$.*

Indeed, the standard proof of the Poincaré Lemma (such as written in [1]) is based on a purely algebraic construction followed by an integration along a selected coordinate. The first operation obviously does not affect the regularity of the form with respect to any set of parameters and the integration also preserves the $C^{r,\alpha}$ regularity, thanks to the Lebesgue dominated convergence Theorem.

Lemma 3.10 (Darboux Theorem with parameters) *Let ω_s be a family of symplectic 2-forms on a ball U^{2n} with coordinates $x^1, ..., x^{2n}$, where $s = (s^1, ..., s^k)$ are some parameters. Assume that the dependence of the components of ω_s on x and on s is of class $C^{r,\alpha}$ with $r \in \mathbb{N}$ and $0 \leq \alpha \leq 1$. Then, there exists a family ϕ_s of local diffeomorphisms $\phi_s : U^{2n} \to \mathbb{R}^{2n}$ such that their dependence on (x, s) is of class $C^{r,\alpha}$ and such that for every s the form ω_s is the pullback under ϕ_s of the standard symplectic form on $\mathbb{R}^{2n}$.*

Idea of the proof: The proof via the "Moser trick" requires the Poincaré Lemma and standard facts about the existence and regularity of systems of ordinary differential equations. This allows one to keep track of how the change of coordinate system depends on the parameter s. Indeed, the Moser trick is based on a construction of a (time depending) vector field such that its flow at time $t = 1$ gives us the required diffeomorphism. The construction of the vector field uses the Poincaré Lemma, and applying the previous Lemma one can check that the vector field and its flow are of class $C^{r,\alpha}$ with respect to both the space variables x and the parameter s. See [21, Sect. 3.2] for more details on Moser's proof.

Proof of Theorem 3.8 By Corollary 2.11 there exist functions $f^1, ..., f^m$ of class $C^{r+1,\alpha}$ with $m = \text{Rank}(g)$ such that $g = \sum_{i,j=1}^{m} c_{ij} df^i df^j$ for some constant nondegenerate symmetric $m \times m$-matrix (c_{ij}). By (3.3), the Poisson bracket of any two these functions is constant. We may assume without loss of generality that there exist k', k'' with $2k' + k'' = m$ such that

$$-\{f^i, f^{i+k'}\} = \{f^{i+k'}, f^i\} = 1 \text{ for } i \leq k'$$

and such that for any other pair of functions f^i its Poisson bracket is zero. Next, as in Sect. 3, we consider the commuting vector fields X_{f^i}; they are of class $C^{r,\alpha}$. By the Rectification Theorem, there exists a coordinate system $(x^1, ..., x^n)$ of class $C^{r,\alpha}$ such that the following holds:

(A) The first $2k' + k''$ coordinates are $x^1 = f^1, ..., x^{2k'+k''} = f^{2k'+k''}$.
(B) The first k' vector fields X_{f^i}, $i = 1, ..., k'$, are given by: $X_{f^i} = -\frac{\partial}{\partial x^{k'+i}}$.
(C) The next k' vector fields X_{f^i}, $i = k'+1, ..., 2k'$, are given by: $X_{f^i} = \frac{\partial}{\partial x^{i-k'}}$.
(D) The next k'' vector fields X_{f^i}, $i = 2k'+1, ..., 2k'+k''$, are given by: $X_{f^i} = -\frac{\partial}{\partial x^{i+k''}}$.

Let us explain the existence of this coordinate system. Consider the local action of $\mathbb{R}^{2k'+k''}$ generated by the flows of commutative linearly independent vector fields $X_{f^1}, ..., X_{f^{2k'+k''}}$. Take a transversal $n-2k'-k''$-dimensional submanifold to the orbits of this action such that on this transversal the values of $f^1, \dots, f^{2k''}$ are equal to zero. We may do it without loss of generality since adding a constant to f^i changes nothing.

The functions $f^{2k'+1}, ..., f^{2k'+k''}$ restricted to any transversal have linearly independent differentials since they are constant on the orbits of the action of $\mathbb{R}^{2k'+k''}$. We take a coordinate system on the transversal such that its first k'' coordinates are $f^{2k'+1}, ..., f^{2k'+k''}$.

Next, consider the coordinates $(t^1, ..., t^{2k'+k''}, y^{2k'+k''+1}, ..., y^n)$ coming from the Rectification Theorem, constructed by these vector fields, by this transversal, and by this choice of the coordinates on the transversal. Recall that these coordinates have the following properties: The vector fields X_{f^i} are the vectors $\frac{\partial}{\partial t^i}$.

The coordinates $(t^1, ..., t^{2k'+k''}, y^{2k'+k''+1}, ..., y^n)$, after the following reorganisation and proper changing the signs are as we require in (A–D): we consider the coordinates

$$\begin{aligned}(x^1 &= t^{k'+1}, ..., x^{k'} = t^{2k'}, x^{k'+1} = -t^1, ..., x^{2k'} = -t^{k'}, x^{2k'+1} = y^{2k'+k''+1}, ..., x^{2k'+k''}\\ &= y^{2k'+2k''},\\ x^{2k'+k''+1} &= -t^{2k'+1}, ..., x^{2k'+k''} = -t^{2k'+k''}, x^{2k'+2k''+1} = y^{2k'+2k''+1}, ..., x^n = y^n).\end{aligned}$$

Let us explain that by the construction of the coordinates the first $m = 2k' + k''$ coordinates are the functions $f^1, ..., f^m$ as we require in (A). Indeed, at our transversal the values of the coordinates $x_1, ..., x_{2k'}$ are zero and therefore coincide with that of $f^1, ..., f^{2k'}$. Next, by the assumptions

$$X_{f^i}(f^j) = P^{ij} = \{f^i, f^j\} = \begin{cases} -1 & \text{if } 1 \le i \le k' \text{ and } j = k'+i,\\ \ \ 1 & \text{if } 1 \le j \le k' \text{ and } i = k'+j,\\ \ \ 0 & \text{otherwise.}\end{cases}$$

implying (A). Finally, observe that the ith column of P is the vector $-X_{x^i}$, which gives us (B, C, D).

In this coordinate system the matrix of the Poisson structure P is given as follows (since P is skew-symmetric it is sufficient to describe the entries P^{ij} with $i > j$ only): Its first k' columns are the vectors $\frac{\partial}{\partial x^{k'+1}}, ..., \frac{\partial}{\partial x^{2k'}}$. The next k' columns are the vectors $-\frac{\partial}{\partial x^1}, ..., -\frac{\partial}{\partial x^{k'}}$. The next k'' columns are $\frac{\partial}{\partial x^{2k'+k''+1}}, ..., \frac{\partial}{\partial x^{2k'+2k''}}$. Moreover, all entries of the matrix P^{ij} areof class $C^{r-1,\alpha}$ and independent of the coordinates $x^1, ..., x^{2k'}$ and of the coordinates $x^{m+1}, ..., x^{m+k''}$. Indeed, it is known and follows from the Jacobi identity that any Poisson structure is preserved by the flow of any Hamiltonian vector field. Then, our Poisson structure P is preserved by the flows of the vector fields $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^{2k'}}$ and $\frac{\partial}{\partial x^{2k'+k''+1}}, ..., \frac{\partial}{\partial x^{2k'+2k''}}$ implying that its entries are independent of $x^1, ..., x^{2k'}$ and of $x^{2k'+k''+1}, ..., x^{2k'+2k''}$.

For example, the general form of such a matrix P^{ij} with $k' = 1$, $k'' = 2$, $n = 8$ is as follows:

$$P^{ij} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & P^{56} & P^{57} & P^{58}\\ 0 & 0 & 0 & 1 & -P^{56} & 0 & P^{67} & P^{68}\\ 0 & 0 & 0 & 0 & -P^{57} & -P^{67} & 0 & P^{78}\\ 0 & 0 & 0 & 0 & -P^{58} & -P^{68} & -P^{78} & 0 \end{pmatrix} \tag{3.6}$$

where the components P^{ij} with $4 < i < j \le 8$ are functions of the variables x^3, x^4, x^7, x^8 only.

Calculating the inverse matrix $(\omega_{ij}) = (P^{ij})^{-1}$ we see that in these coordinates it is given by

$$\begin{aligned}\omega = &\textstyle\sum_{i=1}^{k'} dx^i \wedge dx^{k'+i} + \sum_{i=1+2k'}^{2k'+k''} dx^i \wedge dx^{k''+i} + \sum_{i,j=1+2k'}^{2k'+k''} u_{ij} dx^i \wedge dx^j \\ &+ \textstyle\sum_{i=2k'+1}^{2k'+k''} \sum_{\mu=2k'+2k''+1}^{n} v_{i\mu} dx^i \wedge dx^\mu + \sum_{\mu,\nu=2k'+2k''+1}^{n} w_{\mu\nu} dx^\mu \wedge dx^\nu.\end{aligned} \tag{3.7}$$

The functions u_{ij}, $v_{i\alpha}$ and $w_{\alpha\beta}$ are explicit algebraic expressions in the entries of P^{ij}. Therefore, they are of class $C^{r-1,\alpha}$ and are independent of the coordinates $x^1, \ldots, x^{2k'}$ and of the coordinates $x^{2k'+k''+1}, \ldots, x^{2k'+2k''}$. Note also that the $(n-2k'-2k'') \times (n-2k'-2k'')$-matrix $w_{\alpha\beta}$ is skew-symmetric and nondegenerate.

For example, the inverse of the matrix (3.6) is as follows:

$$\omega_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u & 1 & 0 & v_{37} & v_{38} \\ 0 & 0 & -u & 0 & 0 & 1 & v_{47} & v_{48} \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -v_{37} & -v_{47} & 0 & 0 & 0 & w \\ 0 & 0 & -v_{38} & -v_{48} & 0 & 0 & -w & 0 \end{pmatrix}, \tag{3.8}$$

where the functions v_{ij}, u, w may depend on x^3, x^4, x^7, x^8 only and are of class $C^{r-1,\alpha}$.

Let us view now the last sum in (3.7), namely $\tilde{\omega} = \sum_{\alpha,\beta=2k'+2k''+1}^{n} w_{\mu\nu} dx^\mu \wedge dx^\nu$, as a 2-form on a $(n-2k'-2k'')$-dimensional neighborhood with local coordinates $(x^{2k'+2k''+1}, \ldots, x^n)$. The coordinates $(x^1, \ldots, x^{2k'+2k''})$ are now viewed as parameters and we actually know that the form ω does not depend on the coordinates $x^1, \ldots, x^{2k'}$ and on the coordinates $x^{2k'+k''+1}, \ldots, x^{2k'+2k''}$, so effectively the parameters are $x^{2k'+1}, \ldots, x^{2k'+k''}$. The form $\tilde{\omega}$ is closed and non-degenerate; i.e., it is a symplectic form.

By Lemma 3.10, there exists a coordinate change (depending on the parameters)

$$\begin{aligned} x_{old}^{2k'+2k''+1} &= x_{new}^{2k'+2k''+1}(x^{2k'+1}, \ldots, x^{2k'+k''}, x_{old}^{2k'+2k''+1}, \ldots, x_{old}^{n}), \cdots \\ x_{old}^{n} &= x_{new}^{n}(x^{2k'+1}, \ldots, x^{2k'+k''}, x_{old}^{2k'+2k''+1}, \ldots, x_{old}^{n}) \end{aligned} \tag{3.9}$$

such that after this coordinate change $\tilde{\omega}$ has constant entries. Then, after the coordinate change of class $C^{r-1,\alpha}$ which leaves the first coordinates $(x^1, \ldots, x^{2k'+2k''})$ unchanged and transforms the remaining coordinates $(x^{2k'+2k''+1}, \ldots, x^n)$ by the rule (3.9), we achieve that the components $w_{\mu,\nu}$ in (3.7) are now constants. Note that this coordinate change does otherwise not affect the structure of ω given by (3.7).

Next, we use that the form (3.7) is closed. The coefficient of $dx^i \wedge dx^\mu \wedge dx^\nu$ in $d\omega$ is given by $\left(\frac{\partial v_{i\mu}}{\partial x^\nu} - \frac{\partial v_{i\nu}}{\partial x^\mu}\right)$ for any $i \in \{2k'+1, \ldots, 2k'+k''\}$ and $\mu, \nu \in \{2k'+2k''+1, \ldots, n\}$. We thus see that for every $i = 2k'+1, \ldots, 2k'+k''$ the 1-form $\theta_i := \sum_{\mu=2k'+2k''+1}^{n} v_{i\mu} dx^\mu$, viewed as a 1-form on a neighborhood of dimension $n-2k'-2k''$ with coordinates $(x^{2k'+2k''+1}, \ldots, x^n)$, with coefficients depending on parameters $x^{2k'+1}, \ldots, x^{2k'+k''}$, is closed. Then, by Lemma 3.9, there exist functions V_i of class $C^{r-2,\alpha}$ such that $\frac{\partial V_i}{\partial x^\mu} = v_{i\mu}$.

We consider now the following coordinate change of class $C^{r-2,\alpha}$. The coordinates $x^1, \ldots, x^{2k'+k''}$ and $x^{2k'+2k''+1}, \ldots, x^n$ remain unchanged and the coordinates $x^{2k'+k''+1}, \ldots,$

$x^{2k'+2k''}$ are changed by the rule:

$$x_{old}^{2k'+k''+i} = x_{new}^{2k'+k''+i} + V_i. \tag{3.10}$$

This coordinate change does not affect the previous improvements; that is: The structure of ω is still given by (3.7) and the terms $w_{\mu\nu}$ are still constant. But now the terms $v_{i\mu}$ are zero.

Finally, we consider the term $\sum_{i,j=2k'+1}^{2k'+k''} u_{ij} dx^i \wedge dx^j$ of (3.7). Calculating the differential of the 2-form ω we see that u_{ij} may depend on the coordinates $x^{2k'+1}, ..., x^{2k'+k''}$ only, and that the 2-form $\sum_{i,j=2k'+1}^{2k'+k''} u_{ij} dx^i \wedge dx^j$ viewed as a 2-form on a k''-dimensional neighborhood with coordinates $x^{2k'+1}, ..., x^{2k'+k''}$ is closed. The components of this form are of class $C^{r-3,\alpha}$, and by [11, Theorem 8.3], there exist functions $U_{2k'+1}, ..., U_{2k'+k''}$, of class $C^{r-2,\alpha}$, depending on the coordinates $x^{2k'+1}, ..., x^{2k'+k''}$ such that

$$\sum_{i,j=2k'+1}^{2k'+k''} u_{ij} dx^i \wedge dx^j = \sum_{i=2k'+1}^{2k'+k''} d\left(U_i dx^i\right).$$

We use the functions U_i in the last coordinate change: the coordinates $x^1, ..., x^{2k'+k''}$ and $x^{2k'+2k''+1}, ..., x^n$ remain unchanged and the coordinates $x^{2k'+k''+1}, ..., x^{2k'+2k''}$ are changed by the rule:

$$x_{old}^{2k'+k''+i} = x_{new}^{2k'+k''+i} + U_i. \tag{3.11}$$

This coordinate change does not affect the previous improvements; i.e., the structure of ω is still given by (3.7), the terms $w_{\mu\nu}$ are still constant, the terms $v_{i\mu}$ are still zero but now also the terms u_{ij} are zero. Thus, the coordinates are flat for g and for ω. This completes the proof of the Theorem. □

4 The general case

In this section, we consider a bilinear form $g+\omega$ where both the symmetric and skew-symetric part may be degenerate.

4.1 The case when the symmetric part is zero.

We first assume ω is degenerate and $g = 0$, and discuss the existence of a flat coordinate system.

Theorem 4.1 *There exists a smooth flat coordinate system for a given smooth skew-symmetric 2-form $\omega = \omega_{ij} = \sum_{i<j} \omega_{ij} dx^i \wedge dx^j$ if and only if ω has constant rank and $d\omega = 0$.*

Although this theorem is known, see e.g. [1, Theorem 5.1.3], we give a short proof for selfcontainedness and because we use certain ideas of the proof later.

Proof If ω has maximal rank, then this result is the classical Darboux Theorem. Let us reduce to it the case of smaller rank. We denote by

$$\mathcal{R}_\omega = \{v \in T_x M \mid \omega(v, \cdot) = 0\} \tag{4.1}$$

the kernel of ω. Because ω has constant rank, $\mathcal{R}_\omega$ is a smooth distribution. Furthermore, the condition $d\omega = 0$ implies that it is integrable; indeed, for any vector fields $u, v \in \mathcal{R}_\omega$ and arbitrary vector field w we have

$$0 = \mathcal{L}_v\,(\omega(u, w)) = (\mathcal{L}_v\omega)\,(u, w) + \omega([v, u], w) + \omega(u, [v, w]).$$

The first term on the right hand side vanishes because of the Cartan magic formula, the third term because $u, w \in \mathcal{R}_\omega$. Then, $[v, u] \in \mathcal{R}_\omega$ and the distribution is integrable.

Assume now the distribution has dimension $n - p$, where $p = \operatorname{rank}(\omega)$, and consider a coordinate system $x^1, ..., x^p, y^1, ..., y^{n-p}$ such that the distribution $\mathcal{R}_\omega$ is spanned by the vector fields $\frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^{n-p}}$. In this coordinate system, we have

$$\omega = \sum_{i<j\le p} \omega_{ij}(x)\, dx^i \wedge dx^j.$$

Since $d\omega = 0$, the components ω_{ij} do not depend on the variables $y^1, ..., y^{n-p}$ (indeed, suppose for instance that ω_{12} depends on y^1 then $d\omega$ would contain the nonzero term $\frac{\partial\omega_{12}}{\partial y^1}\, dy^1 \wedge dx^1 \wedge dx^2$ which does not cancel with any other term). The problem is then reduced to the classical Darboux Theorem in dimension p, which completes the proof of the Theorem. $\square$

Remark 4.2 It has been proved in [4, Theorem 18], see also [11, Theorem 14.1], that if ω is a symplectic form (that is non degenerate and close) of class $C^{r,\alpha}$ with $r \in \mathbb{N}$ and $0 < \alpha < 1$, then the previous result still holds and the obtained flat coordinates are of class $C^{r+1,\alpha}$.

For a closed 2-form of constant rank $< n$, one can still find flat coordinates of class $C^{r,\alpha}$. See [5, Theorem 3.2] and the extended discussion in [11, §14.3]. The degenerate case is proved by reducing it to the symplectic case, taking into account that factoring out the kernel of ω reduces one degree of regularity,

4.2 A necessary and sufficient condition in the general case

We consider the tensor field $g_{ij} + \omega_{ij}$ with g_{ij} symmetric and ω_{ij} skew-symmetric and study the existence of a flat coordinate system. This is equivalent to the existence of a symmetric affine connection $\nabla = (\Gamma^i_{jk})$ such that its curvature is zero and such that both g and ω are parallel, meaning that

$$\frac{\partial g_{ij}}{\partial x^k} = \sum_s g_{sj}\Gamma^s_{ik} + g_{is}\Gamma^s_{jk} \tag{4.2}$$

$$\frac{\partial \omega_{ij}}{\partial x^k} = \sum_s \omega_{sj}\Gamma^s_{ik} + \omega_{is}\Gamma^s_{jk}. \tag{4.3}$$

We view (4.2, 4.3) as a linear inhomogeneous system of equations where the unknown quantities are the Γ^i_{jk}. Algebraic compatibility conditions of each of the Eqs. (4.2) and (4.3) have a clear geometric interpretation. Indeed, as we understood in Sect. 2, the algebraic consistency condition of (4.2) is (2.4) and the freedom in choosing Γ satisfying (4.2) once (2.4) is satisfied is the addition of (possibly several expressions of the form)

$$v^i T_{jk} \quad \text{with } v \in \mathcal{R}_g \text{ and } T_{jk} = T_{kj}. \tag{4.4}$$

Concerning the second set of equations, we have the following

Lemma 4.3 *Suppose ω is of class C^1, then there exists Γ^k_{ij} such that $\Gamma^k_{ij} = \Gamma^k_{ji}$ and (4.3) holds if and only if ω is a closed 2-form.*

Proof If ω is of class $C^{1,\alpha}$ for some $0 < \alpha < 1$, then the Lemma immediately follows from Theorem 4.1. Since we only assume the C^1-regularity of ω, a purely algebraic argument is needed. Observe first that a necessary condition is

$$\frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{jk}}{\partial x^i} + \frac{\partial \omega_{ki}}{\partial x^j} = 0. \tag{4.5}$$

Indeed, if one relabels the index in (4.3) by the schemes ($i \to j \to k \to i$) and ($i \to k \to j \to i$), and add the obtained equations to the initial equation, one obtains (4.5). The geometric interpretation of (4.5) is clear: it holds at every point if and only if ω is a closed form.

Observe now that, assuming (4.5) holds, the system (4.3) is algebraically equivalent to the following system of linear equations:[4]

$$\sum_s \omega_{is}\Gamma^s_{jk} = \frac{1}{3}\left(\frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{ik}}{\partial x^j}\right) + T_{ijk}, \tag{4.6}$$

where T_{ijk} is totally symmetric. This linear system is always compatible if (4.5) holds. Indeed, the compatibility condition for the equations (4.6) is as follows: for any $v \in \mathcal{R}_\omega$ the expression

$$\sum_s v^s \left(\frac{\partial \omega_{sj}}{\partial x^k} + \frac{\partial \omega_{sk}}{\partial x^j}\right)$$

should be symmetric in $j \longleftrightarrow k$. We see that this condition is always fulfilled. We conclude that (4.5) are sufficient conditions for compatibility of (4.3). □

Unfortunately, we do not have an easy geometric interpretation for compatibility conditions of the whole system (4.3, 4.2).

We now state our main result:

Theorem 4.4 *Let $g + \omega$ be a smooth (here we assume C^∞ for simplicity) bilinear form on a domain $U \subset \mathbb{R}^n$ (where g is symmetric and ω is skew-symmetric). Suppose there is a flat coordinate system for g and ω, then there exist smooth functions Γ^i_{jk} such that both (4.2) and (4.3) are fulfilled; in particular ω is closed and has constant rank. Moreover, (2.10) holds. Conversely, if there exist smooth functions Γ^i_{jk} such that (4.2) and (4.3) are fulfilled and (2.10) holds, then there exists a flat coordinate system.*

Proof The direction "⇒" is clear. Indeed, the conditions (4.2) and (4.3) are geometric and are trivially satisfied in a flat coordinate system for $\Gamma^i_{jk} = 0$, therefore they hold in any coordinate system. Let us prove the non trivial direction.

We assume the existence of smooth functions Γ^i_{jk} defined on U, such that (4.2) and (4.3) hold. We view these functions as coefficients of a connection ∇. The parallel transport with respect to this connection preserves g and ω. In particular g and ω have constant rank. We set $m = \mathrm{rank}(g)$ and $p = \mathrm{rank}(\omega)$. We also assume that condition (2.10) holds.

Our first step is to show that one may assume without loss of generality, $\mathcal{R}_g \cap \mathcal{R}_\omega = \{0\}$ at one and therefore at every point. Indeed, it is integrable and we can consider a coordinate system $x^1, ..., x^k, y^1, ..., y^{n-k}$ such that $\mathcal{R}_g \cap \mathcal{R}_\omega$ is spanned by $\frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^{n-k}}$. We know

[4] In the symplectic case (when ω is nondegenerate) (4.6) is known [7].

that both g and ω are preserved along the flow of any vector field $v \in \mathcal{R}_g \cap \mathcal{R}_\omega$. Indeed, for g we proved this in Sect. 2 and for ω in Sect. 4.1. Then, in the coordinate system g and ω are given by

$$g = \sum_{i,j=1}^{k} g_{ij} dx^i dx^j, \quad \omega = \sum_{i<j\le k} \omega_{ij} dx^i \wedge dx^j$$

such that g_{ij} and ω_{ij} do not depend on the y-coordinates. We see that the situation is reduced to an analogous situation on a k-dimensional manifold such that $\mathcal{R}_g \cap \mathcal{R}_\omega$ is trivial. Note that the existence of smooth functions Γ^i_{jk} satisfying (4.3) and (4.2) is not affected by this reduction since the freedom (4.4) with $v \in \mathcal{R}_g \cap \mathcal{R}_\omega$, affects neither (4.2) nor (4.3). For the rest of the proof we may and will assume that $\mathcal{R}_g \cap \mathcal{R}_\omega$ is trivial.

Because of (4.3), the distribution $\mathcal{R}_\omega$ is integrable and invariant under parallel transport. We assume that ω has rank p, so $\mathcal{R}_\omega$ has dimension $n - p$. Taking in account (2.10) and Theorem 2.2 we obtain the local existence of functions f^1, ..., f^m, where $m = \text{rank}(g)$ and such that the differentials df^i are linearly independent and parallel, and

$$g = \sum_{i,j=1}^{m} c_{ij} df^i df^j, \tag{4.7}$$

where $c = (c_{ij})$ is a constant nondegenerate symmetric $m \times m$ matrix. Without loss of generality, we may also assume that

(a) The functions f^1, ..., f^r have the property $\text{Kernel}(df^i) \supseteq \mathcal{R}_\omega$.
(b) No nontrivial linear combination of the remaining functions f^{r+1}, ..., f^m has this property.

Indeed, if a function f has property $\nabla_i \nabla_j f = 0$ at all points, then the property $\text{Kernel}(df) \supseteq \mathcal{R}_\omega$ at one point x implies this property at all points. To see it, we chose a smooth path $c(t)$ joining a base point x to an arbitrary point y and denote by $v(t) \in \mathcal{R}_\omega(c(t))$ the parallel transport of the vector $v \in \mathcal{R}_\omega(x)$ along this curve. We then have

$$\frac{d}{dt}\left(df_{c(t)}(v(t))\right) = \sum_k \frac{\partial}{\partial x^k}(df(v))\,\frac{dc^k}{dt} = \sum_{s,k}\left(v^s \nabla_k \nabla_s f + \frac{\partial f}{\partial x^s}\nabla_k v^s\right)\frac{dc^k}{dt} = 0 + 0 = 0.$$

Observe now that the hypothesis $\mathcal{R}_g \cap \mathcal{R}_\omega = \{0\}$ implies that $r \le p$ and $n = p + m - r$. Furthermore the functions f^{r+1}, ..., f^m restricted to any integral submanifold of $\mathcal{R}_\omega$ define local coordinates on this submanifold. Indeed, no nontrivial linear combination of their differentials annihilates $\mathcal{R}_\omega$.

We denote by $\hat{U} = U/\mathcal{R}_\omega$ the quotient manifold of U by the flow of all vector fields in $\mathcal{R}_\omega$ (we identify points of U lying on the same integral submanifold of the distribution $\mathcal{R}_\omega$). The manifold $\hat{U}$ is of dimension $p = \text{rank}(\omega)$, let us fix some coordinates $(z^1, \ldots, z^p)$ on $\hat{U}$ (concretely they are provided by any coordinate system on a manifold transverse to $\mathcal{R}_\omega$).

Observe that the functions $f^1, \ldots, f^r$ are constant on any integral manifold of $\mathcal{R}_\omega$ and therefore induce well defined functions on $\hat{U}$; we denote them by $\hat{f}^1, \ldots, \hat{f}^r$. Likewise, the form ω induces a well defined 2-form $\hat{\omega}$ on $\hat{U}$, which is clearly a symplectic form on $\hat{U}$. We denote by $\hat{P}$ the dual Poisson structure of $\hat{\omega}$. We claim that for any $1 < \mu, \nu \le r$, the Poisson bracket

$$\{\hat{f}^\mu, \hat{f}^\nu\} = \sum_{i,j=1}^{p} \hat{P}^{ij}\,\frac{\partial \hat{f}^\mu}{\partial z^j}\frac{\partial \hat{f}^\mu}{\partial z^j}$$

is constant. Indeed, this quantity is scalar and constructed by linear algebraic operations from the triple (ω, df^μ, f^ν) (viewed now as objects on U) and all the objects in this triple are parallel with respect to ∇.

We then know from Theorem 3.1, that there exists a coordinate system $y^1, \ldots, y^p$ on $\hat{U}$ such that $y^j = \hat{f}^j$ for $j = 1, \ldots, r$ and $\hat{\omega}$ has constant components in this coordinate. We thus have proved that the coordinate system on U defined by

$$(x^1, \ldots, x^n) = (f^1, \ldots, f^r, y^{r+1}, \ldots, y^p, f^{r+1}, \ldots, f^m)$$

is flat for both g and ω. □

We conclude this section with a few remarks:

Remark 4.5 (i) Let us stress that verifying the hypothesis of Theorem 4.4 requires only differentiation and linear algebraic operations. The main computational difficulty is to decide if the combined linear system containing (4.2) and (4.3) is solvable.

(ii) In the proof of Theorem 4.4 we assumed that all objects are as smooth as we need for the proof. We need them to be $C^{r,\alpha}$ with $r \geq 4$ and $0 < \alpha < 1$. The flat coordinate system is then of class $C^{r-3,\alpha}$. We do not have an example demonstrating that the regularity is optimal, and in fact rather tend to believe that it is not optimal.

(iii) The proof of Theorem 4.4 shows that if g has constant rank 1, then there locally exists flat coordinates for $g + \omega$ if and only if the following conditions are satisfied:

(a) $g = \pm\theta \otimes \theta$ for a closed 1-form θ.
(b) ω is closed and has constant rank.
(c) $\mathcal{R}_\omega \cap \mathcal{R}_g$ has constant dimension.

5 Ideas used in our proofs, conclusion and outlook

We solved, for an arbitrary bilinear form, the problem stated by Riemann: we found necessary and sufficient conditions for a bilinear form to have constant entries in a local coordinate system. Our results generalize the special cases solved by Riemann himself (when the bilinear form is symmetric and nondegenerate) and by Darboux (when it is skew-symmetric and nondegenerate).

Our proofs in the smooth case use methods and, whenever possible, notations which were available to, and used by, Riemann, Darboux and other fathers of differential geometry. These methods include basic real analysis, basic linear algebra and the standard results on the existence and uniqueness of solutions of systems of ordinary differential equations.

We also employ a fundamental idea used in particular by Riemann in [22], and which is one of the main reasons for many successful applications of differential geometry in mathematical physics: *if one works with geometric (covariant, in the language used in physics) objects, then one can work with them in a coordinate system which is best adapted to the geometric situation*.

The ideas behind the proofs are based on concepts that appeared later. Let us comment on them and relate our proofs to these concepts.

The first one is the concept of *parallel transport*, it was introduced by Levi-Civita and was effectively used by Elie Cartan. Recall that for any connection $\nabla = (\Gamma^i_{jk})$ the parallel transport along the curve $c : [0, 1] \to M$ is a linear mapping $\tau_c : T_{c(0)}M \to T_{c(1)}M$. It it defined via the differential equation $\sum_s \frac{dc^s(t)}{dt} \nabla_s V^i(c(t)) = 0$ and can also be extended to arbitrary tensors replacing the differential equation by $\sum_s \frac{dc^s(t)}{dt} \nabla_s P^{i_1..i_k}_{j_1...j_m}(c(t)) = 0$. The parallel transport is compatible with all geometric operations on tensors.

The condition that a (possibly, degenerate) metric g is parallel with respect a given connection $\nabla = (\Gamma^i_{jk})$ is equivalent to (2.3), and it means that the parallel transport preserves the metric. This implies that the distribution $\mathcal{R}_g = \ker(g)$ is invariant by parallel transport. It is then integrable and the flow generated by any vector fields belonging to this distribution preserves g (in other words, the vector fields in $\mathcal{R}_g$ are Killing vector fields). This was a key argument to reduce the proofs of Theorems 2.2 to the nondegenerate case, which was solved already by Riemann.

A similar reasoning shows that in the situation discussed in Theorem 4.4 one can "quotient out" first the joint kernel of ω and g and then the kernel of ω, so the situation is reduced to the one discussed in Theorem 3.3. Indeed, the parallel transport preserves $\mathcal{R}_g$ ($\mathcal{R}_\omega$, respectively) so the distributions of $\mathcal{R}_g$ ($\mathcal{R}_\omega$, respectively) are integrable; moreover, g (ω,respectively) is preserved along the flow of any vector fields lying in $\mathcal{R}_g$ ($\mathcal{R}_\omega$, respectively). This allowed us to reduce the proofs of 4.1 and Theorems 3.3 to the Darboux Theorem and to Theorem 3.3.

The second concept is the idea of the holonomy (group). This concept was successfully used already by Cartan and is still an active object of study. For an affine connection $\nabla = (\Gamma^i_{jk})$ and a fixed point p, the holonomy group generated by parallel transports along curves $c : [0, 1] \to M$ starting and ending at p (the so-called loops). The situation studied in Theorem 2.2 suggests that we consider the holonomy group restricted to the anihilator

$$\mathcal{R}^o(p) := \{\xi \in T^*_p M \ \mid \ \text{Kernel } (\xi) \supseteq \mathcal{R}_g(p)\}.$$

This space is invariant with respect to parallel transport along the loops since it is defined via $\mathcal{R}_g$ which is parallel and therefore is invariant. The Ambrose-Singer Theorem [2], states that the holonomy group is generated by the curvature and is trivial if the curvature is zero. Now, (2.10) implies that the curvature (of the connection ∇ viewed as the connection on the subbundle $\mathcal{R}^o$ of T^*M) vanishes. This implies the existence of sufficiently many parallel 1-forms belonging to this bundle. They are automatically closed and give rise to flat coordinates.

The third concept came from the theory of integrable Hamiltonian systems and was crystallized in the 1970's; the standard references are [1, 3]. The key observation is that for any two functions f, h we have $[X_f, X_h] = X_{\{f,h\}}$, where $\{\ ,\ \}$ is a Poisson structure and X_h , X_f are the Hamiltonian vector fields corresponding to f and h. The condition that $\{f, h\}$ is constant implies then that vector fields X_f and X_h commute, which was the key point in the proof of Theorem 3.3.

We have mostly used the "old-fashioned" language and notations for two reasons. We wish our proofs to be available to any mathematician, even without special training in differential geometry and integrable systems. Our declared goal is to present the proofs in the form the fathers of Riemannian Geometry and Symplectic Geometry could understand them, and we believe that we achieved this goal, at least partially. In addition, we expect that our results may have applications outside of differential geometry.

The second reason is that we aim at understanding the lowest regularity assumptions on g and ω under which our results holds. The "modern" differential geometrical ideas touched in this section require, as a rule, higher regularity than it is necessary. The point is that the so-called "invariant notations" that are highly successful in dealing with global differential geometry on manifolds are, by nature, non-transparent about regularity.

For example, the proof of Riemann works under the assumption that the metric is of class C^2 (of course for Riemann himself all functions were real analytic by definition). Later, alternative proofs appeared which allowed to find the optimal regularity assumption for the result of Riemann, see e.g. [6, 16, 18, 19]. Other examples include the Darboux theorem (under optimal regularity assumptions it was proved in [4] and [11]) and also the optimal regularity results for isometries of Riemannian (see e.g. the appendix to [20] for an overview) and Finsler metrics [17, 20].

As an illustration, our proof of Theorem 3.3 requires the bilinear forms to be of rather high regularity, see Remark 3.5. By contrast, the proof of Theorem 3.8 produces flat coordinates of class $C^{r-2,\alpha}$.

Though our results are local, they may open a door to a global investigation of flat bilinear forms. We already have several relatively easy global results, Corollaries 2.7 and 2.8. We also allow ourself to formulate the following conjecture:

Conjecture 5.1 *Suppose a closed manifold M has a flat (possibly degenerate) non-negative definite metric g of rank m. Then, it is finitely covered by a manifold which is diffeomorphic to a fiber bundle over a m-dimensional torus.*

Note that in the nondegenerate case $m = n = \dim M$, the Conjecture is equivalent to Bieberbach's Theorem, see e.g. [8]. In this situation one can find m parallel forms $\theta_1, ..., \theta_m$ on a finite cover $\tilde{M}$ of M such that the lifted metric $\tilde{g}$ writes as $g = \sum_{i,j}^{m} c_{ij}\theta_i\theta_j$, with a constant symmetric positively definite matrix c_{ij}. Note also that by [10], if a manifold admits m closed forms such that in every point they are linearly independent, the manifold is diffeomorphic to a fibre bundle over a m-torus.

Note also that some of our results can be easily generalized for the nonflat case. Say, one can define degenerate metrics of constant curvature $\kappa \in \mathbb{R}$ by the equation $R_{ijk\ell} = \kappa(g_{i\ell}g_{jk} - g_{ik}g_{j\ell})$, and degenerate symmetric space by the formula $\nabla_m R_{ijk\ell} = 0$. Neither formula depends on the freedom (2.6).

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