IEEE Journal of Microwaves

MTT-S
MHz to THz Community

Received 4 September 2020; revised 12 October 2020; accepted 13 October 2020.
Date of current version 7 January 2021.

Digital Object Identifier 10.1109/JMW.2020.3032399

Sommerfeld Integrals and Their Relation to the Development of Planar Microwave Devices

JUAN R. MOSIG [1] (Life Fellow, IEEE), AND KRZYSZTOF A. MICHALSKI [2] (Life Fellow, IEEE)

(Invited Paper)

[1] École Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland
[2] Electrical and Computer Engineering Department, Texas A&M University, College Station, TX 77843-3128 USA

CORRESPONDING AUTHOR: Juan R. Mosig (e-mail: juan.mosig@epfl.ch).

ABSTRACT This paper deals with the mathematical expressions called Sommerfeld integrals. Introduced by A. Sommerfeld in 1909, they are mathematically equivalent to inverse Hankel transforms. The original historical goal of these integrals was to provide an accurate mathematical description of the electromagnetic phenomena involved in long-distance wireless radio and telegraphy. However, their scope was quickly enlarged thanks to the so-called spectral-domain stratified theory, and now they are ubiquitous in the mathematical models associated to many electromagnetic technologies, ranging from EMC lightning modelization and ground penetrating radar to optical and plasmonic integrated devices and going through the familiar microwave and millimeter-wave planar structures using printed circuit technology. In all these areas, Sommerfeld integrals can provide direct evaluations of the involved electromagnetic fields or they can be used as Green's functions in the frame of integral equation formulations. Other disciplines involving stratified media, like seismology and geological prospection, also benefit from these integrals. After discussing the most canonical Sommerfeld integral, appearing in the so-called Sommerfeld identity, this paper reviews three classical structures, namely, the original Sommerfeld problem involving two semi-infinite media, and the microstrip and stripline geometries. It is shown that Sommerfeld integrals provide a unifying treatment of these three problems and that their mathematical features have a direct translation in terms of the physical properties exhibited by the electromagnetic fields that can exist in them.

INDEX TERMS Stratified media, spectral domain, Sommerfeld integrals, Green's functions, stripline, microstrip.

I. INTRODUCTION

In 1909, Arnold Johannes Wilhem Sommerfeld (1868–1951) published in the prestigious German journal "Annalen der Physik" a 72-page paper [1], dealing with the propagation of electromagnetic (EM) waves above a lossy flat Earth. A mathematician by education, Sommerfeld became very interested in Applied Physics and Engineering during his tenure (1900-1906) as Associate Professor in Applied Mechanics at the Königliche Technische Hochschule Aachen (later RWTH Aachen University). His interest in electrodynamics can even be traced to earlier years with his research on EM waves along wires, that provided the base upon which his 1909 paper was built, as described in Eckert's detailed biographical book [2].

This interest for electromagnetic wave phenomena remained for all of his professional life, as witnessed by subsequent papers [3] and by the relevance given to them in his lectures during his 32 years of tenure as a Full Professor of Theoretical Physics at the University of Munich (1906-1938), where he supervised PhD theses on this subject. His lectures were translated into English and published in 1949 as a 6-volume book series, the popular "Lectures in Theoretical Physics" [4], which have inspired several generations of researchers.

Sommerfeld's 1909 paper contained a cornerstone equation [1, eq. (9), pp. 682], whose physical meaning was the expansion of a spherical wave in terms of a continuous spectrum of cylindrical waves. This equation, that can be also found, with a slightly modified notation, in the sixth volume of his book series ([4], §31.14), was soon to be known in our community as “the Sommerfeld identity”. This amazing formula became the starting point of many theoretical developments in Electromagnetics and is nowadays a textbook subject [5], [6], having even its own Wikipedia page [7].

The way of attack and the mathematical formulation that Sommerfeld used for his lossy Earth problem can nowadays be understood in terms of double Fourier transforms and their equivalent for rotationally-invariant problems, the Hankel transforms. Because of Sommerfeld’s pioneer work, radioengineers started to call inverse Hankel transform by the familiar name of Sommerfeld integrals (SI), whose canonical example is provided by the Sommerfeld identity.

Indeed, SIs are simply the mathematical expressions defining the EM fields and potentials created by a point source inside a planar stratified medium, thus providing the Green’s functions of the problem [8]. SIs express electromagnetic quantities in terms of a continuous spectrum of elementary cylindrical waves; the Sommerfeld identity is just the cylindrical wave expansion of the simplest scalar spherical wave generated by a point source radiating into an open infinite isotropic homogeneous space.

SIs and the associated stratified media theory have a strong educational appeal. Sommerfeld himself already saw them as the ideal playground for applying many mathematical tools, like differential equations, integral transforms, vectorial decompositions, complex integration, asymptotic expansions, and approximation theory, to a basic problem in Electrodynamics. Yet, he was also aware of the need to always conclude the theoretical developments with some results of immediate practical application. Indeed, his 1909 paper, introducing a rigorous theoretical model for the EM propagation above Earth in wireless telegraphy problems, was immediately followed by a complementary publication focused on the practical needs of the incipient technology of his time [2, §6.4, pp. 168–169].

In this paper, we aim at providing a tutorial explanation of the generation and use of SIs in the context of stratified media theory, complemented with some useful information about the algorithms commonly used for their numerical evaluation and the discussion of some typical results that can be obtained with these algorithms.

II. SURVEY OF SOME PRACTICAL APPLICATIONS

The first practical example of a stratified medium, was provided in the beginning of the 20th century by the success of wireless radio and telegraphy, empirically developed at that time by Guglielmo Marconi and others [9]. This is the so-called Sommerfeld problem, involving two half-spaces filled with air and a lossy earth (or sea water). The frequencies being first used in the wireless telegraphy were in the MHz range and many associated antennas were electrically small. Therefore, the EM fields created by these antennas could be well approximated by SIs and this gave a strong impetus to their study and calculation.

Whereas, in general, SIs can be only accurately evaluated through numerical computations, analytical asymptotic expansions can always be derived using only pen&paper. Since in the Earth propagation problem, the involved distances were much larger than the wavelength, far field approximations, obtained by asymptotic methods, were good enough. This also explains the early interest for the SIs well before the existence of digital computers.

The original 1909 Sommerfeld paper was soon followed by an overwhelming series of publications mostly concerning the asymptotical evaluations he provided for his SIs and discussing the relevance of the so-called Zenneck wave [10]–[13]. This happened to be a very controversial subject, including mathematical and semantic aspects. Some echoes of these controversies still reverberate in our days, more than a century after Sommerfeld’s paper [14], [15], although, in Sommerfeld’s own words [4] “the recurrent discussion in the literature on the reality of the Zenneck waves seems immaterial to us”.

The mathematical tools created by Sommerfeld were soon extended to more general multilayered geometries, resulting in the so-called stratified media theory, that was codified since the years 1960s in a series of cornerstone textbooks [5], [6], [16]–[20] authored by a distinguished list of researchers. This theory has found very interesting applications in areas like geological prospecting and seismology [21], [22], ground penetrating radar [23] and lightning [24], [25]. But the paramount application of stratified media theory became already evident in the 1950’s with the introduction of the first microwave planar transmission lines and waveguides using printed circuit technology [26]. Striplines and microstrip lines were soon followed by more sophisticated structures, like suspended striplines, coplanar waveguides, slotlines, substrate integrated waveguides, finlines, image lines... all forming the bountiful realm of planar integrated lines [27]. Of particular interest to us is the microstrip structure introduced in 1952 [28], [29] and analyzed with the Sommerfeld formalism only a couple of years after [30], [31].

Today the trend continues, as integrated circuit technology and 3D printing allow the possibility of more complicated and miniaturized structures thus extending the range of frequencies to millimeter waves, Terahertz and up to Optics. At these higher frequencies, negative permittivities become ubiquitous. This fact, combined with the increasing technological mastery for creating metamaterials and metasurfaces have generated a revival of the old Sommerfeld problem, with the Zenneck wave transformed into a surface plasmon polariton [32]–[35].

Obviously, most planar structures are not necessarily small compared to wavelength. Therefore, SIs cannot provide a direct solution. But they can be used as Green’s functions in an integral equation formulation of the problem. This fact gave new impetus to research on SIs and prompted the development of powerful numerical techniques for them, since Green’s functions near-field values are necessary. Probably, the Integral Equation technique better adapted to stratified media

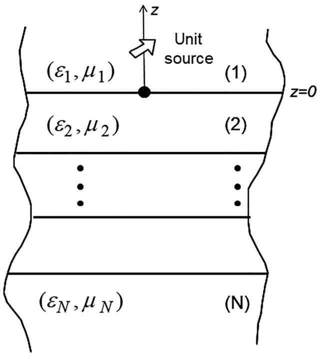


FIGURE 1. **A generic stratified media with an embedded unit source.**

problems is the Mixed Potential Integral Equation (MPIE). This technique was first explicitly mentioned in 1985 [36], although a preliminary version for the study of wire antennas residing in free space and called "Potential integral formulation" was introduced in 1968 by Harrington [37, Sec. 4.6] and later extended in 1982 to microstrip problems [38]. The MPIE was generalized in 1990 to arbitrary stratified media [39] using as Green's functions the SIs for the scalar and vector potential [40]. Since then, this combined formulation MPIE-SI has been integrated in the most popular computational electromagnetics software packages [41].

It can be safely said that the success of all the above-mentioned technologies is largely due to the ability to compute efficiently and accurately the EM fields existing inside all its layers. This can be fully or partially achieved by using the mathematical tools provided by SIs and stratified media theory, as witnessed by the recent successes in modeling with SIs optical metallo-dielectric structures [42] and graphene monoatomic layers [43].

III. STRATIFIED MEDIA THEORY: THE SPECTRAL DOMAIN

The general problem to be considered in this paper is depicted in Fig. 1, with the stratified medium composed by N stacked parallel planar layers. The first and last layers (at top and bottom) are either semi-infinite or finite-thickness layers, terminated by walls where the electromagnetic field fulfills some well-defined boundary conditions (impedance walls [44]). Although the inner layers will usually have a well-defined finite thickness, some layers could eventually be considered as having a zero thickness, thus becoming what is known as impedance sheets [45]. This is a good approximation for real-life situations like embedded sheets of solar cells, graphene layers and metasurfaces.

Embedded in this stratified medium there is a time-harmonic point source (an infinitesimal Hertz dipole). A time dependence $\exp(+j\omega t)$ is assumed and suppressed throughout this paper. Every layer is described by its material properties, namely a complex permittivity ε and a complex permeability μ. Staying within the classical theory, and for the sake of simplicity, all the layers will be considered homogeneous and isotropic (hence described by constant scalar permittivities and permeabilities). Recently, the introduction of metamaterials, now ubiquitous in most areas of electromagnetics, gave a new impetus to stratified media theory, prompting the reconsideration of its formulation and its extension to negative real parts of(ε, μ). Indeed, any universally valid theory for stratified media should consider arbitrary complex values of(ε, μ), eventually restricted to passive media, where these quantities must have negative or zero imaginary parts.

Stratified media properties change only along the normal z coordinate and exhibit a translational symmetry along the transverse x,y coordinates. Therefore the traditional way of solving stratified media problems is to apply a two-dimensional Fourier transform (2D-FT), moving from the space domain (x, y, z) to the spectral domain (k_x, k_y, z). We will use the symmetrical definition of the 2D-FT [46]. With this definition, it can be easily shown [46] that for functions depending on the space coordinates (x, y) only through the radial polar coordinate $\rho = \sqrt{x^2 + y^2}$, the 2D-FT reduces mathematically to a Hankel transform of zero-th order, formulated in terms of a radial spectral coordinate $k_\rho = \sqrt{k_x^2 + k_y^2}$. The Hankel transform pair is given by

$$\begin{aligned} \tilde{f}(k_\rho, z) &= \int_0^\infty J_n(k_\rho \rho)\, f(\rho, z)\, \rho\, d\rho \\ f(\rho, z) &= \int_0^\infty J_n(k_\rho \rho)\, \tilde{f}(k_\rho, z)\, k_\rho\, dk_\rho = \mathrm{S}_n[\tilde{f}]\,. \end{aligned} \tag{1}$$

The inverse Hankel transform in (1) is what is universally recognized in our community [4] as a Sommerfeld integral $\mathrm{S}_n[\tilde{f}]$.

The space coordinate ρ is obviously a real variable. Therefore, to enforce the symmetry of the Hankel transforms in (1), we can consider, for a start, that its spectral counterpart, the variable k_ρ is also real and that the integration path for the Sommerfeld integrals in (1) is the real positive axis, with k_ρ taking all the real values between zero and infinity. But the real variable k_ρ can be easily extended to the complex plane and the original integration path of SIs can be quickly deformed within this complex plane, thanks to the unlimited mathematical possibilities provided by complex integration theory.

For $n = 0$, the pair $f \leftrightarrow \tilde{f}$ is strictly equivalent to a 2D-FT pair. In this case, with $f = \mathrm{S}_0[\tilde{f}]$, we will need some mathematical properties related to derivatives with respect to transverse coordinates (x, y), that are easily demonstrated by performing the derivations under the integral sign in both the 2D-FT and the Hankel transform, with the result:

$$\begin{aligned} \partial f/\partial x &= \cos\varphi\, \partial f/\partial \rho = -\cos\varphi\, \mathrm{S}_1[k_\rho \tilde{f}] \;\leftrightarrow\; -jk_x \tilde{f} \\ \partial f/\partial y &= \sin\varphi\, \partial f/\partial \rho = -\sin\varphi\, \mathrm{S}_1[k_\rho \tilde{f}] \;\leftrightarrow\; -jk_y \tilde{f}\,. \end{aligned} \tag{2}$$

Now, applying the Hankel transform of zeroth order to a generic Helmholtz wave equation:

$$(\nabla^2 + k^2) f = g \tag{3}$$

with $k = \omega\sqrt{\mu\varepsilon}$, we obtain an ordinary differential equation in the spectral domain:

$$(d^2/dz^2 + k^2 - k_\rho^2)\tilde{f} = \tilde{g}\,. \tag{4}$$

Here, a word must be said on a notation choice, more relevant and of far-reaching consequences than it could be expected. A logical step, would be now to introduce a new spectral variable k_z such as $k_x^2 + k_y^2 + k_z^2 = k^2$ and hence $k_z^2 = k^2 - k_\rho^2$. However, following Sommerfeld, we will rather introduce the variable

$$u^2 = -k_z^2 = k_\rho^2 - k^2 \quad \Rightarrow \quad u = \sqrt{k_\rho^2 - k^2} \tag{5}$$

where we will define the square root sign as being the principal square root value, which makes u to behave asymptotically as k_ρ when k vanishes (static limit, see also Section IV).

With this choice, the solution of the spectral wave equation in a layer #n, devoid of sources (homogeneous case, $\tilde{g} = 0$), is simply

$$\tilde{f}_n = a_n \exp(-u_n z) + b_n \exp(u_n z) \tag{6}$$

with $u_n = \sqrt{k_\rho^2 - \omega^2 \mu_n \varepsilon_n}$, and the parameters a_n, b_n to be determined in every layer by forcing $\tilde{f}_n$ to satisfy the adequate boundary conditions at the interfaces.

It is worth mentioning that the "u-notation" leads to solutions that can be formally thought of as a combination of exponentially increasing and decreasing waves while a "k_z-notation" will lead to a combination of progressive and regressive waves $\exp(\pm jk_z z)$. It can be said that the choice of "u" or "k_z" amounts to think in terms, respectively, of either amplitude or phase. However, this is just a matter of appearance (and in the last resort of personal taste), since both quantities k_z and u can be in general complex quantities, depending on the values of k_ρ and k.

In the spectral domain, it is also convenient to separate any vector into its normal and transverse components. When this decomposition is applied to the spectral-domain Maxwell equations, we can show that in the spectral domain, the transverse components of the electromagnetic field can be univocally expressed in terms of the normal components and their normal derivatives as

$$\begin{aligned} k_\rho^2 \tilde{E}_x &= -jk_x\, \partial \tilde{E}_z/\partial z - \omega\mu k_y\, \tilde{H}_z \\ k_\rho^2 \tilde{E}_y &= -jk_y\, \partial \tilde{E}_z/\partial z + \omega\mu k_x\, \tilde{H}_z \\ k_\rho^2 \tilde{H}_x &= -jk_x\, \partial \tilde{H}_z/\partial z + \omega\varepsilon k_y\, \tilde{E}_z \\ k_\rho^2 \tilde{H}_y &= -jk_y\, \partial \tilde{H}_z/\partial z - \omega\varepsilon k_x\, \tilde{E}_z\,. \end{aligned} \tag{7}$$

Traditionally, most EM problems are solved by computing first some potentials and then deriving the fields from them. This approach is rooted in the fact that the source term g in the Helmholtz equations (3-4) is usually simpler for the potentials than for the fields. However, it turns out that the vector potential in stratified media is no longer colinear with the electric currents creating it. Moreover, the boundary conditions at the interfaces couple the vector potential components through rather involved equations.

On the other hand, at the field level, the boundary conditions at the interface between layers, demand simply the continuity of the tangential components of the electromagnetic field. Interesting enough, according to (7), this amounts, to require the continuity at the interfaces of $\varepsilon \tilde{E}_z$ and $\partial \tilde{E}_z/\partial z$ on the electric side and of $\mu \tilde{H}_z$ and $\partial \tilde{H}_z/\partial z$ on the magnetic side. Therefore, the normal fields obey to simple uncoupled boundary conditions at the interfaces. This strongly points out [5] to the possibility of considering the normal fields as basic unknowns of the problem, playing the role of potentials, and to the strategy of solving the problem by computing first $\tilde{E}_z$ and $\tilde{H}_z$ independently, and by deriving subsequently the transverse field components from the normal components.

Indeed, when classic vector and scalar potentials are needed in the context of a stratified media problem (for instance to set up a Mixed Potential Integral Equation), the simplest approach could be to derive these potentials from the normal fields in the spectral domain!

IV. THE SOMMERFELD IDENTITY

Obtaining the fields and potentials created by a point source embedded in a stratified medium is the basic problem of stratified media theory and amounts mathematically to finding the solution of the spectral Helmholz equation (4) using the adequate source term $\tilde{g}$. Then the corresponding spatial quantities, expressed in terms of SIs, can be either directly used as Green's functions in integral equation models, or considered as useful approximations in problems involving bodies whose size is much smaller than the working wavelength.

We will start by considering first an infinite homogeneous medium of properties (ε, μ) and finding the vector potential created by a unit point electric source located at the origin of coordinates (magnetic sources can be treated by using EM duality principles). By comparing the results obtained using both classical EM theory and stratified media theory, we will also build a simple proof of the Sommerfeld identity.

Let us assume the unit point source directed along, say, the x-coordinate (the direction is quite irrelevant in an infinite homogeneous isotropic medium). Classical EM theory tells us that the vector potential can be simply written as

$$\mathbf{A} = \mathbf{e}_x A_x = \mathbf{e}_x \frac{\mu}{4\pi} \frac{\exp(-jkr)}{r}\,. \tag{8}$$

Let us now find again this vector potential in the framework of stratified media theory. To this end we consider that the infinite space is divided in two semi-infinite spaces, #1 and #2, of identical properties and separated by the plane $z = 0$ containing the source. According to stratified media theory the solution for the vector potential in every half-space must be of the form given by (6). More precisely, we can postulate the expressions:

$$\begin{aligned} \tilde{A}_{x1} &= c \exp(-uz) \quad ; \quad z > 0 \\ \tilde{A}_{x2} &= c \exp(uz) \quad\;\; ; \quad z < 0\,. \end{aligned} \tag{9}$$

This choice is justified to avoid unbounded waves (exponentially increasing for large values of the normal coordinate) and to guarantee a converging Sommerfeld integral, which will include a continuous spectrum of real values of k_ρ, going from zero to infinity. Since the variable u behaves asymptotically as

k_ρ, the mandatory choice in (6) is $a_2 = b_1 = 0$. And the symmetry of the problem with respect to the normal coordinate requires also that $a_1 = b_2 = c$.

In order to find the unknown parameter c, we will enforce the classical boundary condition $\mathbf{e}_z \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$, which can be written here as

$$\frac{1}{\mu}\left(\partial A_{x1}/\partial z - \partial A_{x2}/\partial z\right) = \delta(x)\delta(y) \tag{10}$$

or, in the spectral domain,

$$\frac{1}{\mu}\left(d\tilde{A}_{x1}/dz - d\tilde{A}_{x2}/dz\right) = \frac{1}{2\pi}\,. \tag{11}$$

Introducing in the above equation the expressions given by (9), we easily obtain the value of c and the vector potential in the spectral domain, which can be written in a compact form as

$$\tilde{A}_x = \frac{\mu}{4\pi}\frac{\exp(-u|z|)}{u}\,. \tag{12}$$

Going back to the space domain:

$$A_x = \frac{\mu}{4\pi}\int_0^\infty J_0(k_\rho\rho)\,\frac{\exp(-u|z|)}{u}\,k_\rho\,dk_\rho\,. \tag{13}$$

Now, comparing this solution, cast in cylindrical coordinates (ρ, z), with the classical solution (8) depending on the spherical coordinate r, we get the identity:

$$\frac{\exp(-jkr)}{r} = \int_0^\infty J_0(k_\rho\rho)\,\frac{\exp(-u|z|)}{u}\,k_\rho\,dk_\rho \tag{14}$$

with $r = \sqrt{\rho^2 + z^2}$ and $u = \sqrt{k_\rho^2 - k^2}$.

This equation (14) is the famous Sommerfeld identity, first stated by Sommerfeld in his 1909 paper [1]. Obviously, the path used here to arrive at it is not exactly the one employed by Sommerfeld (as shown in his book [4]). But the above developments can be viewed as the "stratified media theory demonstration" of this far-reaching identity.

Care must be exerted with the choice of the square root giving u, if we want the Sommerfeld identity to be correct. If we start considering a lossless medium with k real, u must be taken as imaginary positive for values $k_\rho < k$ and as real positive for $k_\rho > k$. This choice can be generalized to arbitrary complex values of k_ρ and to lossy media, in which cases the choice of the square root for u must fulfill the condition

$$-\pi/2 < \arg(u) \leq +\pi/2\,. \tag{15}$$

This condition perfectly matches the mathematical definition of the so-called principal value of the square root and is the choice adopted in the most usual computer mathematical tools, like Mathematica and MatLab. From a complex plane perspective, every choice of the square root sign corresponds to a possible Riemann sheet [19]. In this paper we will only consider the "proper Riemann sheet" defined by (15).

The fields created by our point source inside an infinite homogeneous medium can be now obtained from the vector potential, according to classical EM theory. Anticipating the usual needs encountered in practical stratified problems, we should consider both "horizontal sources" (x-directed, parallel to the layers' interfaces), and "vertical sources" (z-directed, perpendicular to the interfaces). The following results are readily obtained in the spectral domain:

- z-directed source (VED):

$$\tilde{E}_z = \frac{1}{4\pi\,j\omega\varepsilon}k_\rho^2\frac{\exp(-u|z|)}{u};\quad \tilde{H}_z = 0 \tag{16}$$

- x-directed source (HED):

$$\tilde{E}_z = \frac{jk_x}{4\pi\,j\omega\varepsilon}\mathrm{sgn}(z)\exp(-u|z|);\quad \tilde{H}_z = \frac{jk_y}{4\pi}\frac{\exp(-u|z|)}{u}\,. \tag{17}$$

The above spectral expressions are valid for an infinite homogeneous medium and can be used to provide the "extra source term" to be added to the solution (6) of the homogeneous Helmholtz equation in the layers where sources are present.

Some of the mathematical features in (16–17) will remain unchanged when solving actual stratified media and are worth mentioning. For vertical z-directed sources, the z- spectral fields depends only on k_ρ and hence their spatial counterparts will show a perfect symmetry of revolution, being independent of the azimuthal angle φ. There is no vertical magnetic field, which translates in standard microwave engineering vocabulary as saying that the fields are purely "transverse magnetic (TM) with respect to the z-direction". For horizontal sources, both electric and magnetic fields have non-zero z-components and hence the total fields are a combination of transverse magnetic (TM) and transverse electric (TE) parts. In addition, the vertical spectral fields created by horizontal sources (17) include explicit dependences with k_x and k_y. In the space domain, according to (2), this will result in vertical components of the fields depending on the sine and cosine of the azimuthal angle.

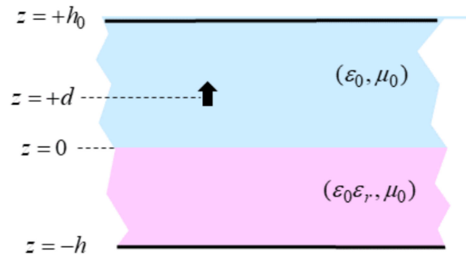


FIGURE 2. The basic stripline structure under study.

V. A FRUITFUL EXAMPLE: THE ASYMMETRIC STRIPLINE PROBLEM

As an interesting and "fruitful" example, we consider the asymmetric stripline structure depicted in Fig. 2. This is a stratified media problem formed by a sandwich of two non-magnetic layers, with thickness h_0, h and properties (ε_0, μ_0), $(\varepsilon_0\varepsilon_r, \mu_0)$, terminated at the top and the bottom sides by two perfect electric conducting sheets (ground planes). The interface between the layers defines the origin of coordinates and we consider a point unit source existing in the first layer at a height d above the interface.

We qualify this problem as "fruitful" because it includes the microstrip problem when $h_0 \to \infty$ and also the original Sommerfeld half-space problem when simultaneously $h_0 \to \infty$ and $h \to \infty$. This fact also justifies our rather non-standard notation, because in these two derived geometries, the upper layer will be usually free space and then our results will read as the ones encountered in the classical treatments of these problems. But it must be pointed out that the following developments are valid for any value of the layers' permittivities.

Let us consider a vertical source in the upper layer at a height d above the interface (Fig. 2). Then the normal magnetic field is zero, the total field is of TM-type and we can concentrate on the calculation of the normal electric field.

According to (6) and (16), we postulate the following expressions in the spectral domain:

$$\tilde{E}_{z0} = A_0 \exp(-u_0 z) + B_0 \exp(u_0 z) + k_\rho^2 \frac{\exp(-u_0|z-d|)}{4\pi j\omega\varepsilon_0 u_0}$$
$$\tilde{E}_z = A \exp(-uz) + B \exp(uz) \quad (18)$$

where we have introduced in (18) the "source term", adapted from (16).

The calculation of the four parameters in (18) is made by enforcing first the boundary condition $\partial\tilde{E}_z/\partial z = 0$ at the two ground planes and then the continuity of $\varepsilon\tilde{E}_z$ and $\partial\tilde{E}_z/\partial z$ in the interface ($z = 0$).

The involved algebraic manipulations are cumbersome but straightforward. When the final general results are particularized to the important case where both the source and the calculation point are at the interface ($d = 0$ and $z = 0$), we obtain the final result:

$$4\pi j\omega\varepsilon_0 \tilde{E}_{z0} = 2\varepsilon_r k_\rho^2 / D_{\mathrm{TM}} \quad (19)$$

where

$$D_{\mathrm{TM}} = \varepsilon_r u_0 \tanh u_0 h_0 + u \tanh uh \, . \quad (20)$$

This is a pertinent notation for the denominator appearing in the spectral expressions of the TM-fields. As is well known [19], the equation $D_{\mathrm{TM}} = 0$ is the characteristic equation for the TM-modes existing in a stripline waveguide.

So finally, the spatial value of the vertical electric field at the interface is given by the Sommerfeld integral:

$$4\pi j\omega\varepsilon E_{z0} = \int_0^\infty J_0(k_\rho \rho) \frac{2\varepsilon_r}{D_{\mathrm{TM}}} k_\rho^3 \, dk_\rho \, . \quad (21)$$

As anticipated when presenting this stripline geometry, the microstrip and the Sommerfeld half-spaces cases are now easily obtained, observing that the behavior of our variables allows us to define $\lim_{h_0\to\infty} \tanh u_0 h_0 = \lim_{h\to\infty} \tanh uh = 1$.

Then, the denominator (20) becomes for the microstrip geometry:

$$D_{\mathrm{TM}} = \varepsilon_r u_0 + u \tanh uh \quad (22)$$

and for the Sommerfeld problem:

$$D_{\mathrm{TM}} = \varepsilon_r u_0 + u \, . \quad (23)$$

Finally, if we go back to the infinite homogeneous medium problem by letting $\varepsilon_r = 1$ (and hence $u_0 = u$), we get the result $D_{\mathrm{TM}} = 2u$ and therefore we recover the spectral electric field given by (16) in the case $z = 0$.

A final interesting particular case is the homogeneous stripline ($\varepsilon_r = 1$), where we get

$$D_{\mathrm{TM}} = u_0(\tanh u_0 h_0 + \tanh u_0 h)$$
$$= u_0(1 + \tanh u_0 h_0 \tanh u_0 h) \tanh u_0(h_0 + h) \quad (24)$$

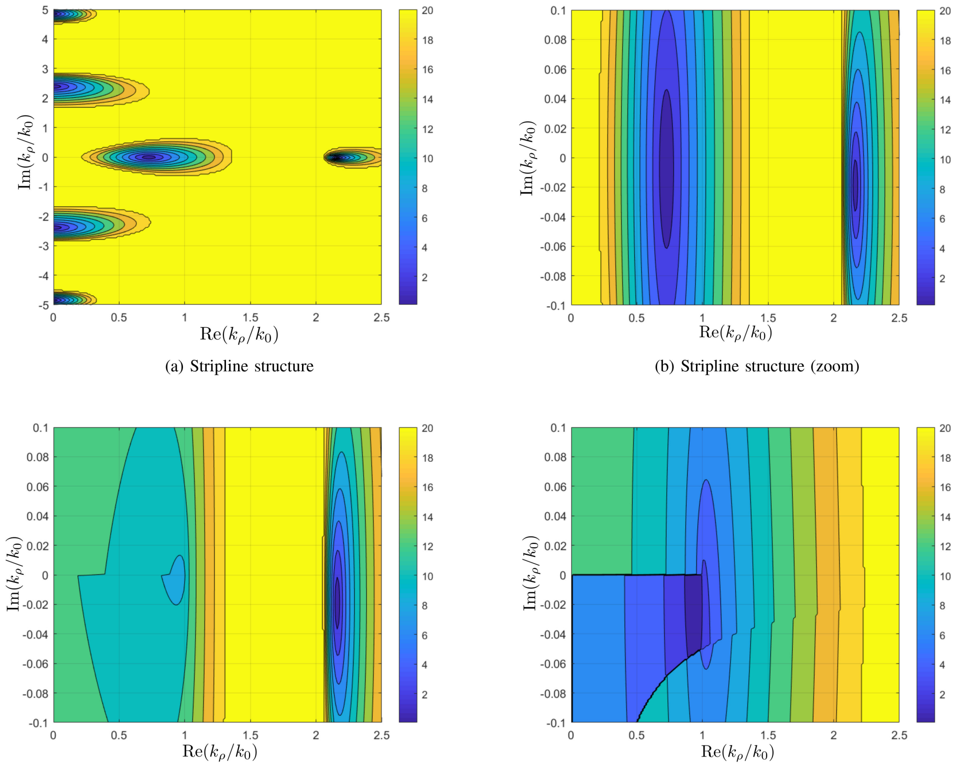
clearly showing that in this case the zeros will depend on the total thickness of the stripline.

A similar strategy can be followed to study the case of a horizontal source. No additional theoretical complexity arises, but both components $\tilde{E}_z$ and $\tilde{H}_z$ must be obtained using (17). They will produce, respectively, the TM- and TE-parts of the fields. Due to space limitations, these calculations will not be included here. But, evidently, a new denominator, D_{TE}, associated to the TE fields, will appear adding eventually new singularities.

Indeed, it is obvious that the "manual" approach described in this Section becomes quickly too cumbersome, especially in problems with more than two layers, and should be superseded by a systematic procedure easily implemented in a computer. This goal has been achieved with two strategies. In the first approach [47], a chain matrix connecting the coefficients (a_n, b_n) and (a_{n+1}, b_{n+1}) in two contiguous layers is defined by imposing the adequate boundary conditions. The full problem is then solved by matrix multiplication, giving an specific treatment to the source layer through the use of the corresponding source terms (16-17). The second approach, more popular, relies of the equivalent transmission line formulation of Maxwell equations introduced by Schelkunoff [48] and sees the stratified medium as a series of connected transmission line sections to which all the power of transmission line theory can be applied [49].

VI. BEHAVIOR OF THE SPECTRAL QUANTITIES IN THE COMPLEX PLANE

In order to develop well-adapted numerical integration techniques for evaluating SIs like that in (21), the behavior of the integrands (essentially the spectral fields times a Bessel function and a power of the spectral variable) must first be ascertained. In principle, a study of the integrand over the real axis $\mathrm{Re}(k_\rho)$ should suffice. But then, we could fail to notice the singularities frequently exhibited by the spectral fields in the complex plane (branch points, poles associated to the zeros of D_{TM}). These singularities could be very close to the real axis (in fact, they will be *on* the real axis in lossless problems) and could affect the strategies used for the integration along the real axis, not to mention along any other convenient path obtained by a valid deformation of the real axis.



(a) Stripline structure (b) Stripline structure (zoom)

(c) Microstrip structure (zoom) (d) Sommerfeld problem (zoom)

FIGURE 3. Level lines for the values of the modulus of the normalized TM-denominator D_{TM}/k_0 in the normalized complex plane k_ρ/k_0, for stripline, microstrip and Sommerfeld problem structures. $\varepsilon_r = 10 - j0.1$, $h = 0.1\lambda_0$, $h_0 = 2h$.

Moreover, the nature of these singularities provides interesting hints about the physics of the problems, quite different in the stripline (closed), microstrip (semi-open) and Sommerfeld (open) problems. Here, we concentrate on the singularities existing on the proper Riemann sheet, when the square roots are computed according to (15). However, it must be pointed out that the spectral fields may also exhibit singularities in other Riemann sheets. These improper singularities, like the so-called leaky-wave poles, can play an important role when the Sommerfeld integration path is deformed, searching for field representations that might be useful and accurate in some ranges of permittivities and distances [50].

Let us study the behavior of the modulus of the normalized denominator D_{TM}/k_0 in the complex normalized plane k_ρ/k_0, under the idea that any zero of this function will become a pole of the spectral field and hence of the SI integrand. To make our study more general, we consider a slightly lossy lower substrate with a relative permittivity $\varepsilon_r = 10 - j0.1$. Its normalized thickness will be selected as $h = 0.1\lambda_0$ ($k_0h = 0.2\pi$). We start with the stripline case, taking $h_0 = 2h$. Here, it must be pointed out that the denominator (20) is an even function on u and u_0. Therefore it will not depend on the choice of the square roots' sign in (5). No branch cuts must be expected in the complex plane. A straightforward depiction of the level curves of $|D_{TM}/k_0|$ (Fig. 3(a)) confirms the absence of branch cuts (the curves would be discontinuous when crossing a branch cut). It also reveals zeros very close to the imaginary axis (stripline modes under cut-off), a zero just below the real axis with $\mathrm{Re}[k_\rho/k_0] < 1$ (stripline mode propagating as a fast wave) and another zero just below the real axis but with $\mathrm{Re}[k_\rho/k_0] > 1$ (a slow stripline surface wave mode). If we concentrate now on the real axis region, a zoom (Fig. 3(b)) reveals that indeed, in the lossy situation considered, the zeros are just below the real axis with approximate values $0.725 - j0.008$ and $2.165 - j0.019$.

The correctness of these observations can be confirmed by studying analytically the simpler case of a homogeneous stripline (24) with no losses. We observe that for a thin structure, there is an infinity of zeros in the imaginary axis (stripline modes under cutoff). When the thickness increases, the zeros start moving into the real axis segment $0 < \mathrm{Re}[k_\rho/k_0] < 1$ and the corresponding modes start to propagate. In the homogeneous stripline case, there are no real zeros with $\mathrm{Re}[k_\rho/k_0] > 1$. As it is well known, surface waves exist only on interfaces where the permittivity changes.

The situation is completely different in the microstrip case ($h_0 \to \infty$), with a denominator even on u but odd on u_0. Hence a branch cut should appear starting in the branch point $k_\rho/k_0 = 1$. Indeed, what happens is that all the zeros corresponding to guided modes in the stripline case coalesce into the microstrip branch cut and only the surface wave zero remains (in a position practically identical to the stripline case, since it is essentially associated to the air-dielectric interface). Fig. 3(c) confirms these predictions, with no guided

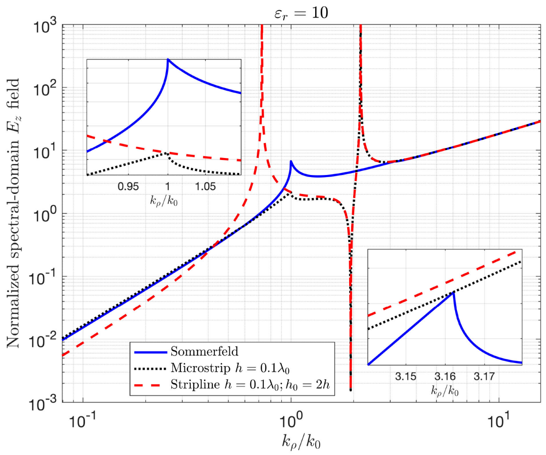


FIGURE 4. The normalized spectral E_z field computed on the real axis of the spectral plane.

modes and the branch cut appearing as slight discontinuities in the level curves when crossing the segment of the real axis $\mathrm{Re}[k_\rho/k_0] < 1$.

Finally, if we move to the Sommerfeld problem, the denominator (23) is now odd on both u and u_0 and two branch cuts should be expected. This is clearly visible in Fig. 3(d). The surface wave zero has disappeared but a new zero appears between the two branch cuts. This zero, which corresponds to the famous Zenneck wave, was analytically obtained by Sommerfeld (solving the equation $D_{\mathrm{TM}} = 0$) as

$$k_{\mathrm{ZW}} = \sqrt{\varepsilon_r/(\varepsilon_r + 1)}\,. \tag{25}$$

Despite the closeness of this zero to the real axis (in this particular case, we have $k_{\mathrm{ZW}} = 0.953 - j0.0004$), its effects are limited to the area between the two branch cuts. The "upper side" of the real axis remains practically unaffected by the zero, as clearly seen in Fig. 3(d). In fact, in a lossless situation, the two branch cuts coalesce on the real axis and the zero is "squeezed" into a null-area surface. So, the question of the existence of a Zenneck wave pole in a lossless case is completely irrelevant from the point of view of a real axis numerical integration. The situation changes dramatically in the case of a low-loss metamaterial with $\mathrm{Re}[\varepsilon_r] < 0$. In this case, according to (25), the zero appears close to real axis but in the region $\mathrm{Re}[k_\rho/k_0] > 1$ and it is no longer shielded by the branch cut. Hence its effect is akin to the surface wave seen in the microstrip case and it can be observed as a surface plasmon-polariton [14], [51].

If we abandon now the complex plane and concentrate on the original Sommerfeld integration path (the real positive axis defined as $(\mathrm{Re}[k_\rho] \geq 0\,;\ \mathrm{Im}[k_\rho] = 0^+)$, we can summarize all our empirical observations in the Fig. 4 where we have plotted the modulus of the normalized spectral E_z-field. We can observe (see also the figure insets) the three different behaviors. In the Sommerfeld problem, the two branch points $k_\rho/k_0 = 1$ and $k_\rho/k_0 = \sqrt{\varepsilon_r}$ produces derivative discontinuities but no pole effect is seen. In the microstrip problem, the surface wave pole with $\mathrm{Re}[k_\rho/k_0] > 1$ is well present and the discontinuity of the derivative at the unique branch point$k_\rho/k_0 = 1$ can be observed. And in the stripline problem, a second pole with $\mathrm{Re}[k_\rho/k_0] < 1$ (a guided mode) appears in addition to the surface wave pole, whereas no derivative discontinuities (no branch cuts) are observed.

VII. NUMERICAL INTEGRATION STRATEGIES FOR SOMMERFELD INTEGRALS

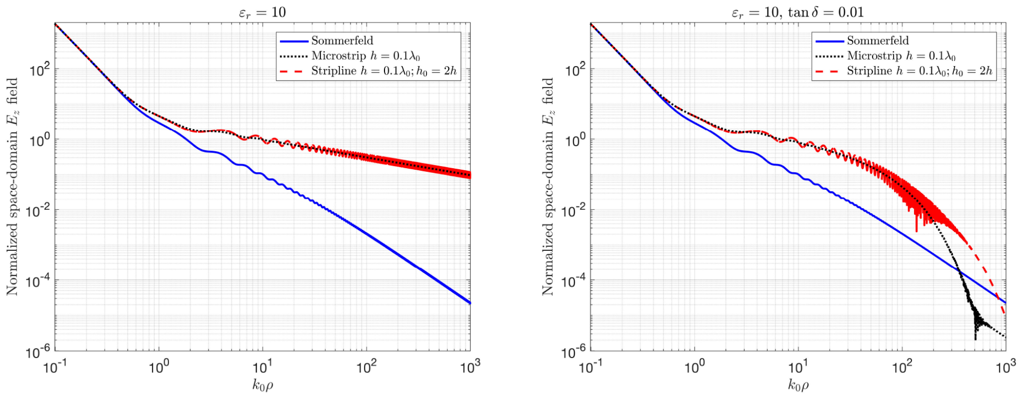
To finally obtain the true fields in the space domain, we must now compute the SIs given in (21), whose integrands are the spectral functions depicted in Fig. 4 multiplied by the factor $k_\rho J_0(k_\rho \rho)$. The literature describing the possible strategies for the computation of SIs is very abundant and well known. See, for instance, the comprehensive report in [52]. Here we will cite only a few representative works, centering on those able to efficiently produce accurate results for both small and large ranges of radial distances, without entering into the bountiful area of asymptotic methods, usually intended to provide far-field approximations.

One approach is to manipulate the spectral field expressions and to cast them in terms whose SIs correspond to classical point images, albeit of complex value and at complex distances from the source. This original idea, introduced in the eighties, is now a robust and well-established approach known as the "discrete complex images method" (DCIM) [53]–[55]. Other useful manipulations of the spectral expressions (combined sometimes) include the use of pencil-of-function methods [56], [57] and the rational function fitting method (RFFM) [58]–[62]. SIs can be also considered in their original form of inverse 2D-Fourier transforms and we can apply to them FFT algorithms [63] or original ideas like the expansion wave concept [64].

However, since the introduction in the 1970's of the Lawrence Livermore Lab pioneer algorithms [65], pure numerical integration in the space domain remains the more robust approach, being applied to all kinds of SIs and producing the more precise results over a wider range of distances. The following decade saw the publication of quite a number of practical algorithms, mostly based in the real axis integration [66]–[69] and frequently following the strategy described here below.

If we concentrate on the most difficult situation, where source and observation point are at the same level, the most common approach is to split the real axis into a finite part $[0, a]$ (the "head") and an infinite part $[a, \infty]$ (the "tail"). The threshold value a is chosen to ensure that all the integrand singularities affect only the head and that the tail is a smooth function, simply showing the divergent oscillations imposed by the factor $k_\rho J_0(k_\rho \rho)$.

For integrating the head segment, a possibility is to remain in the real axis of the k_ρ plane, and to apply well-known numerical devices to the existing (or nearby) singularities: adequate changes of variables for the branch points and extraction procedures for the poles [66], [69]–[71]. But a more robust strategy, that will work for the three types of structures considered and where the nature and exact position of the



(a) Lossless substrate (b) Lossy substrate

FIGURE 5. Normalized space-domain values of the Ez field for the three structures considered.

singularities can be ignored, is to deform the path of integration in the complex plane. The finite segment $[0, a]$ of the real axis is replaced by a contour joining its two ends and going inside the first quadrant of the complex plane. The form of the new path is quite irrelevant (semi-elliptical, triangular, half-sine, pulse-shaped...) [72]–[74], but in all cases a radial distance "ρ-adaptative" strategy must be adopted, controlling the depth of the deformation inside the complex plane, in particular to avoid extreme behaviors and abrupt changes of the complex Bessel function $J_0(k_\rho \rho)$. The integral is then evaluated using well-established numerical quadrature rules like Gauss-Jacobi, Gauss-Kronrod, Clenshaw-Curtis, Patterson or tanh-sinh double-exponential. An extreme path deformation, analogous to the so-called Schelkunoff formulation, includes the integration along the positive imaginary axis and provides a very interesting alternative [75].

As for the tail segment, the integration of oscillating functions over an infinite interval is a classical problem in numerical mathematics [76], and the research on the Sommerfeld integrals tails [77] has provided some solid benchmarks to recent mathematical books [78]. Currently, weighted averages in its many forms is considered the best strategy for obtaining fast and accurate numerical results in a wide range of radial distances [79]. The results of Fig. 5 have been obtained by using a half-sine path strategy for the head and a standard weighted averages algorithm for the tail.

This way, a unique, robust computer code has been used to obtain numerical values for the three structures, Sommerfeld, microstrip and stripline. In Fig. 5 a the lower medium is lossless ($\varepsilon_r = 10$), whereas small losses ($\varepsilon_r = 10 - j0.1$) have been introduced in Fig. 5(b). As expected, the numerical results confirm that, in all the cases, the E_z-field decreases as $1/\rho^3$ in the near field region. Then, in the lossless case, there is a transition region around $k_0\rho = 1$. After it, the field decreases as $1/\rho^2$ in the Sommerfeld problem and only as $1/\sqrt{\rho}$ in the microstrip and stripline cases, thus showing a typical surface wave behavior. The ripples observed in the three curves in the far field region are due to interferences between the waves that can be associated to the field singularities. However, these waves are just part of the total fields and usually cannot be observed independently. Theoretically, it is always possible to isolate them as well-defined parts of the total field by adequately deforming the integration path and separating the contribution around the poles and the branch cuts. This is an exciting and advanced topic that cannot be developed here (see, for instance, [51], [79]–[81]). In the stripline case, the strong ripple is obviously due to the beat between the surface wave and a propagating mode. In the microstrip case, the responsible singularities for the much weaker ripple are the surface wave pole and the branch point. In the Sommerfeld problem, the ripple is due to the interaction between the waves associated to the two branch points and it is always present although frequently too weak to be noticed.

The presence of losses (Fig. 5(b)) has very different effects on the far-field behavior in each structure. In the Sommerfeld problem, the behavior $1/\rho^2$ is practically unaffected by the losses. In the microstrip problem, losses affect the surface wave behavior, that after $k_0\rho = 10$ changes from a $1/\sqrt{\rho}$ behavior to an exponential decay. Then, after a second transition zone at around $k_0\rho = 600$, the total field shows a $1/\rho^2$ behavior. Finally, in the stripline case, no term $1/\rho^2$ is observed and the total field ends in the far-field with the exponential decay associated to guided modes and surface waves.

VIII. CONCLUSION

More than one century has elapsed since Arnold Sommerfeld published the amazing equation that the posterity would know as the "Sommerfeld identity". His research, made in the first decade of the 20th century, was motivated by the first successes at that time of the wireless radio and telegraphy, a subject of uppermost practical relevance. But, as Sommerfeld himself said in the introduction to his 1909 paper *"it is unfortunate that mere opinions are expressed in this discussion, and no one tackles the subject mathematically."* So he started his paper asking himself the question: *"To which type the waves in wireless telegraphy can be ascribed? Are they comparable to the Hertz waves in the air, or to the electrodynamic waves along wires?"* In answering mathematically these questions, Arnold Sommerfeld opened one of the more fruitful areas in modern Electromagnetics, the stratified media theory.

In this paper, we have tried to revisit the Sommerfeld identity and to make a brief survey of the stratified media theory. Unfortunately, the adopted level for the paper has forced us to ignore some of the most important and recent topics. The presented theoretical developments should be complemented with the study of horizontal sources, of alternative formulations using potentials, of proper and improper Riemann sheets with its associated singularities, and of the direct connections between Sommerfeld denominators and Fresnel reflection coefficients. In addition, the new technological developments (graphene, plasmonic structures, metamaterials and metasurfaces, magnetized films) call for considering arbitrary values of both permittivities and permeabilities, eventually considered as tensorial quantities. These generalizations drastically modify the topology of the complex spectral plane and the nature of its singularities. It is hoped that this paper will encourage the readers to plunge deeper into the existing literature.

The paper proposes an example, demonstrating how the transition between stripline, microstrip and Sommerfeld structures is achieved in the mathematical model by the simple presence or absence of some hyperbolic tangent functions. Moreover, it is seen how this mathematical feature fully determines (and explains) the physical behavior of the electromagnetic field. This is completely in line with Sommerfeld's scientific strategy. He always recommended to his students to start the solution of any physical problem with a "mathematical attack" : the graphics and numbers needed for practical engineering applications should be obtained only after the theoretical mathematical model had been fully elaborated and understood.

ACKNOWLEDGMENT

The authors would like to thank Dr. I. Stevanovic (OFCOM, Switzerland) for his comments, corrections, and the generation of the numerical results in Figs. 4 and 5.

REFERENCES

[1] A. Sommerfeld, "Über die Ausbreitung der Wellen in der drahtlosen Telegraphie," [On the propagation of waves in wireless telegraphy], *Ann. Phys.*, vol. 28, pp. 665–736, 1909.
[2] M. Eckert, trans. T. Artin, *Arnold Sommerfeld: Science, Life and Turbulent Times, 1868-1951*. New York, NY, USA: Springer, 2013.
[3] A. Sommerfeld, "Über die Ausbreitung der Wellen in der drahtlosen Telegraphie," [On the propagation of waves in wireless telegraphy, *Ann. Phys.*, vol. 81, pp. 1135–1153, 1926.
[4] A. Sommerfeld, *Partial Differential Equations in Physics. Vol. VI*, (Lectures on Theoretical Physics). New York, NY, USA: Academic Press, 1949.
[5] J. A. Kong, *Electromagnetic Wave Theory*, 2nd ed. New York, NY, USA: Wiley, 1990.
[6] W. C. Chew, *Waves and Fields in Inhomogeneous Media*. New York, NY, USA: IEEE Press, 1995.
[7] Wikipedia, "Sommerfeld identity." Accessed: Nov. 4, 2020. [Online]. Available: https://en.wikipedia.org/wiki/Sommerfeld_identity
[8] A. Poggio and E. Miller, "Integral equation solution of three-dimensional scattering problems," in *Computer Techniques for Electromagnetics*, R. Mittra, Ed. New York, NY, USA: Pergamon Press, 1973, ch. 4, pp. 159–264.
[9] T. K. Sarkar, R. Mailloux, A. A. Oliner, M. Salazar-Palma, D. L. Sengupta, *History of Wireless*. Hoboken, NJ, USA: Wiley, 2006.
[10] K. A. Norton, "The physical reality of space and surface waves in the radiation field of radio antennas," *Proc. Inst. Radio Eng.*, vol. 25, no. 9, Sep. 1937, pp. 1192–1202.
[11] W. H. Wise, "The physical reality of Zenneck's surface wave," *Bell Syst. Tech. J.*, vol. 16, no. 1, pp. 35–44, 1937.
[12] C. J. Bouwkamp, "On Sommerfeld's surface wave," *Phys. Rev.*, vol. 80, p. 294, 1950.
[13] S. Schelkunoff, "Anatomy of a surface wave," *IRE Trans. Antennas Propag.*, vol. 7, no. 5, pp. 133–139, Dec. 1959
[14] K. A. Michalski and J. R. Mosig, "The Sommerfeld halfspace problem redux: Alternative field representations, role of Zenneck and surface plasmon waves," *IEEE Trans. Antennas Propag.*, vol. 63, no. 12, pp. 5777–5790, Dec. 2015.
[15] T. K. Sarkar, M. N. Abdallah, M. Salazar-Palma, and W. M. Dyab, "Surface plasmons-polaritons, surface waves, and Zenneck waves: Clarification of the terms and a description of the concepts and their evolution," *IEEE Antennas Propag. Mag.*, vol. 59, no. 3, pp. 77–93, Jun. 2017.
[16] L. Brekhovskikh, *Waves Layered Media*, 2nd ed. New York, NY, USA: Academic, 1976.
[17] J. R. Wait, *Electromagnetic Waves in Stratified Media*. New York, NY, USA: Pergamon Press, 1962.
[18] A. Baños, *Dipole Radiation in the Presence of a Conducting Half-Space*. New York, NY, USA: Pergamon Press, 1966.
[19] L. B. Felsen and N. Marcuvitz, *Radiation and Scattering of Waves*. Englewood Cliffs, NJ, USA: Prentice-Hall, 1973.
[20] A. Ishimaru, *Electromagnetic Wave Propagation, Radiation, and Scattering*, 2nd ed. New York, NY, USA: IEEE Press, 2017.
[21] L. Tsang, R. Brown, and J. Kong, "Numerical evaluation of electromagnetic fields due to dipole antennas in the presence of stratified media," *J. Geophys. Res.*, vol. 79, no. 14, pp. 2077–2080, 1974.
[22] A. Caboussat and G.K. Miers, "Numerical approximation of electromagnetic signals arising in the evaluation of geological formations," *Comput. Math. Appl.*, vol. 59, no. 1, pp. 338–351, 2010.
[23] S. Lambot, E. C. Slob, H. Vereecken, "Fast evaluation of zero-offset Green's function for layered media with application to ground-penetrating radar," *Geophys. Res. Lett.*, vol. 34, no. 21, pp. L21405-1–L21405-6, 2007.
[24] J. Zou, T. N. Jiang, J. B. Lee, S. H. Chang, "Fast calculation of the electromagnetic field by a vertical electric dipole over a lossy ground and its application in evaluating the lightning radiation field in the frequency domain," *IEEE Trans. Electromagn. Compat.*, vol. 52, no. 1, pp. 147–154, Feb. 2010.
[25] F. Delfino, R. Procopio, M. Rossi, A. Shoory, and F. Rachidi, "Lightning electromagnetic radiation over a stratified conducting ground: Formulation and numerical evaluation of the electromagnetic fields," *J. Geophys. Res., Atmos.*, vol. 116, no. D4, 2011, Art. no. D04101.
[26] A. A. Oliner, "The evolution of electromagnetic waveguides," in *History of Wireless*, T. K. Sarkar, R. J. Mailloux, A. A. Oliner, M. Salazar-Palma, D. L. Sengupta, Eds. New York, NY, USA: Wiley, 2006, ch. 16, pp. 543–566.
[27] Wikipedia, "Planar transmission line." Accessed: Nov. 4, 2020. [Online]. Available: https://en.wikipedia.org/wiki/Planar_transmission_line
[28] D. D. Grieg and H. F. Engelmann, "Microstrip-A new transmission technique for the kilomegacycle range," *Proc. IRE*, vol. 40, no. 12, pp. 1644–1650, Dec. 1952.
[29] F. Assadourian and E. Rimai, "Simplified theory of microstrip transmission systems," *Proc. IRE*, vol. 40, no. 12, Dec. 1952.
[30] Y. T. Lo, "Electromagnetic field of a dipole source above a grounded dielectric slab," *J. Appl. Phys.*, vol. 25, 733, 1954.
[31] D. B. Brick, "The radiation of a Hertzian dipole over a coated conductor," *Proc. Inst. Elect. Eng. C*, vol. 102, pp. 104–121, 1955.
[32] T. I. Jeon and D. Grischkowsky, "THz Zenneck surface wave (THz surface plasmon) propagation on a metal sheet," *Appl. Phys. Lett.*, vol. 88, pp. 061113-1–061113-3, 2006.
[33] R. Gordon, "Surface plasmon nanophotonics: A tutorial," *IEEE Nanotechnol. Mag.*, vol. 2, no. 3, pp. 13–18, Sep. 2008.
[34] A. Sihvola, J. Qi, I. V. Lindell, "Bridging the gap between plasmonics and Zenneck wave," *IEEE Antennas Propag. Mag.*, vol. 52, no. 1, pp. 124–136, Feb. 2010.
[35] R. D. Nevels and K. A. Michalski, "On the behavior of surface plasmons at a metallo-dielectric interface," *J. Lightw. Technol.*, vol. 32, pp. 3299–3305, 2014.
[36] K. A. Michalski, "The mixed-potential electric field integral equation for objects in layered media," *Arch. Elek. Ubertragung.*, vol. 39, pp. 317–322, 1985

[37] R. F. Harrington, *Field Computation by Moment Methods*, (1968, reissued in IEEE Press Series on Electromagnetic Wave Theory). New York, NY, USA: IEEE Press, 1993.
[38] J. R. Mosig and F. E. Gardiol, "A dynamical radiation model for microstrip structures," *Adv. Electron. Electron Phys.*, vol. 59, pp. 139–237, 1982.
[39] K. A. Michalski and D. Zheng, "Electromagnetic scattering and radiation by surfaces of arbitrary shape in layered media, part I: Theory," *IEEE Trans. Antennas Propag.*, vol. 38, no. 3, pp. 335–344, Mar. 1990.
[40] K. A. Michalski and J. R. Mosig, "Multilayered media Green's functions in integral equation formulations," *IEEE Trans. Antennas Propag.*, vol. 45, no. 3, pp. 508–519, Mar. 1997.
[41] U. Jakobus, M. Bingle, M. Schoeman, J. J. van Tonder, F. Illenseer, "Tailoring FEKO for microwave problems," *IEEE Microw. Mag.*, vol. 9, no. 6, pp. 76–85, Dec. 2008.
[42] B. Ung and Y. Sheng, "Optical surface waves over metallo-dielectric nanostructures: Sommerfeld integrals revisited," *Opt. Exp.*, vol. 16, no. 12, pp. 9073–9086, 2008.
[43] G. Hanson, "Dyadic Green's functions for an anisotropic non-local model of biased graphene," *IEEE Trans. Antennas Propag.*, vol. 56, no. 3, pp. 747–757, Mar. 2009.
[44] D. J. Hoppe and Y. Rahmat-Samii, *Impedance Boundary Conditions in Electromagnetics*. Washington, DC, USA: Taylor & Francis, 1995.
[45] R. F. Harrington and J. R. Mautz, "An impedance sheet approximation for thin dielectric shells," *IEEE Trans. Antennas Propag.*, vol. 23, no. 4, pp. 531–534, Jul. 1975.
[46] A. Papoulis, *Systems and Transforms With Applications to Optics*. FL, USA: Krieger, 1981, pp. 140–175.
[47] J. R. Mosig, "Integral equation technique," in *Numerical Techniques for Microwave and Millimeter-Wave Passive Structures*, T. Itoh, Ed. New York, NY, USA: Wiley, 1989, pp. 133–213.
[48] S. A. Schelkunoff, "Conversion of Maxwell's equations into generalized telegraphist's equations," *Bell. Syst. Tech. J.*, vol. 34, no. 5, pp. 995–1043, Sep. 1955.
[49] K. A. Michalski, "Electromagnetic field computation in planar multilayers," in *Encyclopedia of RF and Microwave Engineering*, K. Chang, Ed. Hoboken, NJ, USA: Wiley-Interscience, 2005, vol. 2, pp. 1163–1190.
[50] F. Mesa and D. R. Jackson, "Investigation of integration paths in the spectral-domain analysis of leaky modes on printed circuit lines," *IEEE Trans. Microw. Theory Techn.*, vol. 50, no. 10, pp. 2267–2275, Oct. 2002.
[51] K. A. Michalski and J. R. Mosig, "The Sommerfeld half-space problem revisited: From radio frequencies and Zenneck waves to visible light and Fano modes," *J. Electromagn. Waves Appl.*, vol. 30, no. 1, pp. 1–42, 2016.
[52] D. Chatterjee and S. D. Walker, "Study of Sommerfeld and phase-integral approaches for Green's functions for PEC-terminated inhomogeneous media," Rpt. NRL/MR5310–10-92400, Naval Res. Lab, Washington DC, USA, Jan. 2010.
[53] D. G. Fang, J. J. Yang and G. Y. Delisle, "Discrete image theory for horizontal electric dipoles in a multilayered medium," *Proc. Inst. Elect. Eng. H, Microw. Antennas Propag.*, vol. 135, no. 5, pp. 297–303, Oct. 1988.
[54] M. I. Aksun and G. Dural, "Clarification of issues on the closed-form Green's functions in stratified media," *IEEE Trans. Antennas Propag.*, vol. 53, no. 11, pp. 3644–3653, Nov. 2005
[55] A. Alparslan, M. I. Aksun, and K. A. Michalski, "Closed-form Green's functions in planar layered media for all ranges and materials," *IEEE Trans. Microw. Theory Techn.*, vol. 58, no. 3, pp. 602–613, Mar. 2010.
[56] M.Yuan and T. K. Sarkar, "Computation of the Sommerfeld integral tails using the matrix pencil method," *IEEE Trans. Antennas Propag.*, vol. 54, no. 4, pp. 1358–1362, Apr. 2006.
[57] D. G. Kurup, "Spatial domain Green's functions of layered media using a new method for Sommerfeld integrals," *IEEE Microw. Wireless Compon. Lett.*, vol. 22, no. 4, pp. 161–163, Apr. 2012.
[58] V. I. Okhmatovski and A. C. Cangellaris, "A new technique for the derivation of closed-form electromagnetic Green's functions for unbounded planar layered media," *IEEE Trans. Antennas Propag.*, vol. 50, no. 7, pp. 1005–1016, Jul. 2002.
[59] V. N. Kourkoulos and A. C. Cangellaris, "Accurate approximation of Green's functions in planar stratified media in terms of a finite sum of spherical and cylindrical waves," *IEEE Trans. Antennas Propag.*, vol. 54, no. 5, pp. 1568–1576, May. 2006.
[60] F. Mesa, R. R. Boix, and F. Medina, "Closed-form expressions of multilayered planar Green's functions that account for the continuous spectrum in the far field," *IEEE Trans. Microw. Theory Techn.*, vol. 56, no. 7, pp. 1601–1614, Jul. 2008.
[61] T. N. Kaifas, "Direct rational fitting method for accurate evaluation of Sommerfeld integrals in stratified media," *IEEE Trans. Antennas Propag.*, vol 60, no. 1, pp. 282–291, Jan. 2012.
[62] D. G. Kurup, "Analytical expressions for spatial-domain Green's functions in layered media," *IEEE Trans. Antennas Propag.*, vol. 63, no. 11, pp. 4944–4951, Nov. 2015.
[63] S. L. Dvorak, "Application of the fast Fourier transform to the computation of the Sommerfeld integral for a vertical electric dipole above a half-space," *IEEE Trans. Antennas Propag.*, vol. 40, no. 7, pp. 798–805, Jul. 1992.
[64] F. J. Demuynck, G. A. E. Vandenbosch and A. R. Van de Capelle, "The expansion wave concept. I. Efficient calculation of spatial Green's functions in a stratified dielectric medium," *IEEE Trans. Antennas Propag.*, vol. 46, no. 3, pp. 397–406, Mar. 1998.
[65] R. J. Lytle and D. L. Lager, "Numerical evaluation of Sommerfeld integrals," Rep. No. UCRL-51688, Lawrence Livermore Lab., Univ. California, Livermore, CA, USA, 1974
[66] P. Parhami, Y. Rahmat-Samii and R. Mittra, "An efficient approach for evaluating Sommerfeld integrals encountered in the problem of a current element radiating over lossy ground," *IEEE Trans. Antennas Propag.*, vol. 28, no. 1, pp. 100–104, Jan. 1980.
[67] P. B. Katehi and N. G. Alexopoulos, "Real axis integration of Sommerfeld integrals with applications to printed circuit antennas," *J. Math. Phys.*, vol. 24, p. 527, 1983.
[68] J. R. Mosig and T. K. Sarkar, "Comparison of quasi-static and exact electromagnetic fields from a horizontal electric dipole above a lossy dielectric backed by an imperfect ground plane," *IEEE Trans. Microw. Theory Techn.*, vol. 34, no. 4, pp. 379–387, Apr. 1986.
[69] J. R. Mosig, R. C. Hall, and F. E. Gardiol, "Numerical analysis of microstrip patch antennas," in *The Handbook Microstrip Antennas*, J. R. James and P. S. Hall, Eds. Stevenage, London, U.K.: IEE Peter Peregrinus, 1989.
[70] M.-J. Tsai, C. Chen, and N. G. Alexopoulos, "Sommerfeld integrals in modeling interconnects and microstrip elements in multi-layered media," *Electromagnetics*, vol. 18, no. 3, pp. 267–288, 1998.
[71] K. A. Michalski and M. Mustafa, "Numerically stable and reliable computation of electromagnetic modes in multilayered waveguides using the Cauchy integration method with automatic differentiation," *IEEE Trans. Microw. Theory Techn.*, vol. 66, no. 9, pp. 3981–3992, Sep. 2018.
[72] E. Simsek, Q. H. Liu, and B. J. Wei, "Singularity subtraction for evaluation of Green's functions for multilayer media," *IEEE Trans. Microw. Theory Techn.*, vol. 54, no. 1, pp. 216–225, Jan. 2006.
[73] P. Gay-Balmaz and J. R. Mosig, "3-D planar radiating structures in stratified media," *Int. J. Microw. Millimeter Wave Comput. Aided Eng.*, vol. 7, pp. 330–343, 1997.
[74] A. Egel, S. W. Kettlitz, and U. Lemmer, "Efficient evaluation of Sommerfeld integrals for the optical simulation of many scattering particles in planarly layered media," *J. Opt. Soc. Amer. A*, vol. 33, pp. 698–706, 2016.
[75] J. R. Mosig and A. Alvarez-Melcon, "Green's functions in lossy layered media: Integration along the imaginary axis and asymptotic behavior," *IEEE Trans. Antennas Propag.*, vol. 51, no. 12, pp. 3200–3208, Dec. 2003.
[76] A. Sidi, *Practical Extrapolation Methods: Theory and Applications*. Cambridge, U.K.: Cambridge Univ. Press, 2003.
[77] K. A. Michalski and J. R. Mosig, "Efficient computation of Sommerfeld integral tails - methods and algorithms," *J. Electromagn. Waves Appl.*, vol. 30, no. 3, pp. 281–317, 2016.
[78] A. Deaño, D. Huybrechs and A. Iserles, *Computing Highly Oscillatory Integrals*. Philadelphia, PA, USA: SIAM, 2018.
[79] J. R. Mosig, "The weighted averages algorithm revisited," *IEEE Trans. Antennas Propag.*, vol. 60, no. 4, pp. 2011–2018, Apr. 2012.
[80] K. A. Michalski and J. R. Mosig, "On the surface fields excited by a Hertzian dipole over a layered halfspace: From radio to optical wavelengths," *IEEE Trans. Antennas Propag.*, vol. 63, no. 12, pp. 5741–5752, Dec. 2015.
[81] I. Koh, "A complete uniform asymptotic expansion of the Sommerfeld integral of an impedance plane for imperfectly homogeneous half-space," *IEEE Trans. Antennas Propag.*, vol. 68, no. 7, pp. 5568–5574, Jul. 2020.