The time-domain Cartesian multipole expansion of electromagnetic fields

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ABSTRACT

Time-domain solutions of Maxwell's equations in homogeneous and isotropic media are paramount to studying transient or broadband phenomena. However, analytical solutions are generally unavailable for practical applications, while numerical solutions are computationally intensive and require significant memory. Semi-analytical solutions (e.g., series expansion), such as those provided by the current theoretical framework of the multipole expansion, can be discouraging for practical case studies. This paper shows how sophisticated mathematical tools standard in modern physics can be leveraged to find semi-analytical solutions for arbitrary localized time-varying current distributions thanks to the novel time-domain Cartesian multipole expansion. We present the theory, apply it to a concrete application involving the imaging of an intricate current distribution, and verify our results with an existing analytical approach. Thanks to the concept of current "pixels" introduced in this paper, we derive time-domain semi-analytical solutions of Maxwell's equations for arbitrary planar geometries.

1 Introduction

Solutions of Maxwell's equations for arbitrary charge and current densities are often impossible to derive analytically, and numerical solutions require long computation times and significant memory. However, semi-analytic closed-form solutions (e.g., solutions involving a series expansion) are sometimes needed to derive fundamental results, allow model fitting, or provide insight into the underlying physics. For example, essential properties of electromagnetic time reversal can be evaluated using time-domain solutions of Maxwell's equations. This paper presents a systematic approach to derive time-domain semi-analytical expressions for the electromagnetic fields radiated by time-varying localized sources. This time-domain approach is valid for any frequency independently of the source size. The sources are modeled by Schwartz distributions,[1,2] which offer a generalization of functions to objects such as the Dirac δ distribution and its derivatives. Locally, distributions can be represented as derivatives of continuous functions.

The multipole expansion is a well-known approach to finding series representations of solutions of Maxwell's equations. It has been thoroughly studied in the time and frequency domain[3–10]. Some issues involving the definitions of point sources and Green's functions have been studied within the scope of distribution theory[9,11,12]. In particular, there have been mentions[13–16] of the link between the spherical and Cartesian multipole expansions and derivatives of the Dirac δ distribution, including a detailed analysis[17] of this link in frequency domain. In frequency domain and spherical coordinates, the multipole expansion relies on the separation of variables[18], decomposing scalar solutions of the Helmholtz equation into a product of angle-dependent functions (the spherical harmonics, i.e., the restriction to the sphere of harmonic and homogeneous polynomials in $\mathbb{R}^3$), and radius-dependent functions (spherical Bessel or Hankel functions or both). As already studied within the framework of pseudopotentials[19] or distributions in spherical coordinates[20–22], the singular behavior of the spherical multipole expansion is not trivial. This singular behavior is slightly easier in Cartesian coordinates, as the Cartesian derivatives of the Dirac δ distribution are well-defined.

This paper builds on the existing approaches involving Schwartz distributions and introduces the time-domain Cartesian multipole expansion. Such an expansion cannot trivially be derived from its frequency-domain counterpart. Indeed, performing a Taylor series expansion of the Green's function in the time domain is formally impossible, as this function is represented by a singular Schwartz distribution. Also, in contrast with the existing literature, we show how to apply the technique to realistic configurations involving an intricate current distribution, thanks to the proposed concept of current pixel. The results are validated by comparing to an existing approach[8]. In contrast with the latter reference, our work is based on a different class of time-varying moment, which involves a three- (instead of four-) dimensional integration and offers a straightforward implementation for rectangular domains, such as printed circuit boards. Moreover, it is self-contained in that it can be entirely derived using Schwartz distributions. We also provide an open-source Python implementation of the theory, allowing the reader to experiment with custom current distributions. Finally, the contribution of the charge is explicit, offering a straightforward interpretation at low frequencies.

This paper is organized as follows. First, Section 2 presents the method, that is, the time-domain Cartesian multipole expansion, and applies it to electromagnetic fields. This method is then illustrated and validated in Section 3 by computing the electric field radiated by broadband intricate current distributions. Finally, Section 4 closes with some remarks and conclusions.

2 The time-domain Cartesian multipole expansion of electromagnetic fields

2.1 A generalized Cartesian multipole expansion

We are interested in electromagnetic radiation from localized sources in free space in a homogeneous and isotropic medium described by constant electric permittivity $\varepsilon = \varepsilon_r\varepsilon_0$ and magnetic permeability $\mu = \mu_r\mu_0$. We use the following convention for the partial derivative ∂_i, $i = 0,1,2,3$: $\partial_0 = \frac{\partial}{\partial t}$, $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$, $\partial_3 = \frac{\partial}{\partial x_3}$, and use the multi-index notation $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{N}^3$, with $|\alpha| = \alpha_1+\alpha_2+\alpha_3$, $\alpha! = \alpha_1!\,\alpha_2!\,\alpha_3!$, $D^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$, and $\mathbf{x}^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}$.

In classical electrodynamics, the radiation of a point charge distribution can be evaluated by computing the Taylor series of the Green's function. Here, we show that this procedure is equivalent to taking linear combinations of derivatives of the Dirac δ distribution.

Suppose that a scalar distribution f satisfies the scalar differential equation

$$\begin{cases} \mathscr{L} f(t,\mathbf{x}) = \xi(t,\mathbf{x}) \\ \text{boundary and initial conditions} \end{cases} \tag{1}$$

where we suppose that the boundary and initial conditions are sufficient for the solution to exist and be unique. Here, $\mathscr{L}$ is a linear differential operator with constant coefficients, and $\xi(t,\mathbf{x})$ is the source term. For example, for $f = E_1$ (the first electric field component), we have that $\mathscr{L} = \square = \mu\varepsilon\partial_0^2 - \nabla^2$ and $\xi = -1/\varepsilon\partial_1\rho - \mu\partial_0 J_1$. The solution can be represented by a convolution with the appropriate Green's distribution G[23],

$$f(t,\mathbf{x}) = [G(s,\mathbf{y}) * \xi(s,\mathbf{y})]\,(t,\mathbf{x}) \tag{2}$$

where the convolution $*$ is meant over $\mathbb{R}\times\mathbb{R}^3$. Unfortunately, in general, there are no analytical formulas for such a four-dimensional convolution. Instead, we might replace G by some approximation $\hat{G}$ with a simpler formulation. If the application $G \mapsto f = G * \xi$ is continuous under some appropriate metric, the fact that $\hat{G}$ is close to G implies that the approximate solution

$$\hat{f}(t,\mathbf{x}) = \left[\hat{G}(s,\mathbf{y}) * \xi(s,\mathbf{y})\right](t,\mathbf{x}) \tag{3}$$

is close to the true solution f.

One possible option is to replace G with its truncated spatial Taylor series $\hat{G}$ of order n. In this case, we obtain the classical results for the multipole expansion in Cartesian coordinates. Here, we show that the obtained approximate solution $\hat{f}$ can also be derived by approximating the source ξ by a sum of derivatives of Dirac δ distributions.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^3$ be an open set where G is spatially analytic and suppose that ξ is a measure with compact support whose spatial moments up to order n are finite. Next,*

- *let $\tilde{f}$ be the solution of Equation (1) when the source term is*

$$\tilde{\xi}(t,\mathbf{x}) = \sum_{|\alpha|\le n} \frac{(-1)^{|\alpha|}}{\alpha!} C_\alpha(t) D^\alpha \delta^3(\mathbf{x}) = C_{\mathbf{0}}(t)\delta^3(\mathbf{x}) - C_{(1,0,0)}(t)\partial_1\delta^3(\mathbf{x}) + \dots \quad + C_{(1,0,1)}(t)\partial_{13}^2\delta^3(\mathbf{x}) + \dots \tag{4}$$

  where $C_\alpha(t) = \iiint \mathbf{y}^\alpha \xi(t,\mathrm{d}^3\mathbf{y})$,
- *and let $\hat{f}$ be the multipole expansion in Cartesian coordinates: $\hat{f}(t,\mathbf{x}) = \left[\hat{G}(s,\mathbf{y}) * \xi(s,\mathbf{y})\right](t,\mathbf{x})$.*

  Then $\hat{f} = \tilde{f}$ almost everywhere in $\mathbb{R}\times\Omega$.

The proof of Theorem 2.1 is in Appendix A. The compact support of ξ means that the source is localized and that we turn the sources on during a finite duration. This hypothesis holds for many systems of interest in physics and engineering. Also, Theorem 2.1 offers a generalization of the multipole expansion approach. Indeed, even when G is not analytical, such as with the wave equation in the time domain, it is still possible to convolve it with a point source as in Equation (4). Moreover, this result is not constrained to electromagnetic fields but applies to any field solution of a differential equation similar to Equation (1).

2.2 Application to electromagnetic fields

Directly from Maxwell's equations, we derive the electric and magnetic field wave equations:

$$\Box \mathbf{E}(t,\mathbf{x}) = -\frac{1}{\varepsilon}\nabla\rho(t,\mathbf{x}) - \mu\partial_0\mathbf{J}(t,\mathbf{x}) \tag{5}$$

$$\Box \mathbf{B}(t,\mathbf{x}) = \mu\nabla\times\mathbf{J}(t,\mathbf{x}) \tag{6}$$

where $\Box = \mu\varepsilon\partial_0^2 - \nabla^2$ is the d'Alembert operator. Since the solutions of the wave equations are well-known, it is interesting to start from equations (5) and (6), in which the electric and the magnetic fields are seemingly decoupled. Of course, this coupling exists, and it is embedded in the continuity equation:

$$0 = \partial_0\rho(t,\mathbf{x}) + \nabla\cdot\mathbf{J}(t,\mathbf{x}) \tag{7}$$

Now, notice that the ith component ($i = 1,2,3$) of the wave equation for the electric field (Equation (5)) can be written as

$$\Box E_i(t,\mathbf{x}) = -\frac{1}{\varepsilon}\partial_i\rho(t,\mathbf{x}) - \mu\partial_0 J_i(t,\mathbf{x}) \tag{8}$$

Let us focus on the right-hand side. By performing a multipole expansion in Cartesian coordinates and by including the continuity equation, both the current and the charge densities consist of sums of derivatives of Dirac δ^3 distributions, where each term of the sum may be symbolically written as $C_\alpha(t)D^\alpha\delta^3(\mathbf{x})$ for some appropriate time-dependent function C_α and some appropriate derivative D^α. This type of distribution models a time-dependent point source. Explicitly, the distribution $C_\alpha(t)D^\alpha\delta^3(\mathbf{x})$ acts on any test function $\psi\in\mathscr{D}$ as

$$\left\langle C_\alpha(t)D^\alpha\delta^3(\mathbf{x}),\psi(t,\mathbf{x})\right\rangle = (-1)^{|\alpha|}\int_{\mathbb{R}} C_\alpha(t)(D^\alpha\psi)(t,\mathbf{0})\,\mathrm{d}t \tag{9}$$

by definition of the derivative of a distribution, and where $\mathscr{D}$ denotes the set of compactly-supported smooth functions mapping space-time coordinates to real numbers. In Equation (9), we can treat the time variable t independently from the space variable $\mathbf{x}$. Since C_α is a regular distribution, the effect of C_α on ψ is a time integral. On the other hand, the Dirac δ distribution evaluates (the derivative of) ψ at $\mathbf{x}=\mathbf{0}$, which is why the integrand in Equation (9) is $(-1)^{|\alpha|}C_\alpha(t)(D^\alpha\psi)(t,\mathbf{0})$ In light of the above, the right-hand side of Equation (8) can be written as a finite sum of time-dependent point sources, say $\Box E_i(t,\mathbf{x}) = \sum_{j=1}^n C_j(t)D^{\alpha_j}\delta^3(\mathbf{x})$ for some integer n.

Now, suppose that we know a solution to the simpler differential equation $\Box E_i^j(t,\mathbf{x}) = C_j(t)D^{\alpha_j}\delta^3(\mathbf{x})$. Then, by the linearity of the wave equation and since the initial conditions are zero for $t<0$, the field $E_i(t,\mathbf{x}) = \sum_{j=1}^n E_i^j(t,\mathbf{x})$ is a solution of the original differential equation for the ith component of the electric field, namely, $\Box E_i(t,\mathbf{x}) = \sum_{j=1}^n C_j(t)D^{\alpha_j}\delta^3(\mathbf{x})$, with $\alpha\in\mathbb{N}^3$. Of course, the same applies to the magnetic field.

By this reasoning, the solution of the electromagnetic wave equations with time-dependent point source densities can be determined by the generic solution $f_\alpha(t,\mathbf{x};C)$ of the scalar differential equation

$$\Box f_\alpha(t,\mathbf{x};C) = C(t)D^\alpha\delta^3(\mathbf{x}) \tag{10}$$

where the function C describes the time dependence. Next, we will show that the solution of Equation (10) can be computed recursively. To this end, let us define the auxiliary function $g_\alpha(t,\mathbf{x};C)$ as

- If $\alpha = \mathbf{0}$: $g_{\mathbf{0}}(t,\mathbf{x};C) = \pm\frac{C(t)}{4\pi|\mathbf{x}|}$
- Else, for $|\alpha| > 0$, let j be the first nonzero dimension of α, $c^2 = (\mu\varepsilon)^{-1}$, and we have the recursion relation

$$g_\alpha(t,\mathbf{x};C) = (\partial_j g_{\alpha-\mathbf{e}_j})(t,\mathbf{x};C) \mp (\partial_0 g_{\alpha-\mathbf{e}_j})(t,\mathbf{x};C)\frac{x_j}{c|\mathbf{x}|} \tag{recursion relation}$$

The signs correspond to causal (+) and anti-causal (−) solutions. As an example, the causal recursion relation for $\alpha = (1,0,0)$ reads

$$g_{(1,0,0)}(t,\mathbf{x};C) = (\partial_1 g_{\mathbf{0}})(t,\mathbf{x};C) - (\partial_0 g_{\mathbf{0}})(t,\mathbf{x};C)\frac{x_1}{c|\mathbf{x}|} = -\frac{x_1}{4\pi}\left[\frac{C(t)}{|\mathbf{x}|^3} + \frac{\mathrm{d}C(t)}{\mathrm{d}t}\frac{1}{c|\mathbf{x}|^2}\right] \tag{11}$$

We can explicitly compute the expression of $g_\alpha(t,\mathbf{x};C)$ using the recursion relation with any symbolic programming language. Note that g_α is a real-valued function defined on $\mathbb{R}\times(\mathbb{R}^3\backslash\{0\})$. It depends only on the space coordinate $\mathbf{x}$ and the derivatives of the time-dependence C. Also, it always contains a term in $1/|\mathbf{x}|$, linked to a time derivative of order $|\alpha|$.

We are now ready to solve Equation (10):

Theorem 2.2. *Let f_α be a solution of the scalar differential equation (10). Then*

$$\langle f_\alpha(t,\mathbf{x};C), \psi(t,\mathbf{x})\rangle = \langle g_\alpha(t \mp |\mathbf{x}|/c, \mathbf{x}; C), \psi(t,\mathbf{x})\rangle$$

for every smooth ψ with compact support such that its derivatives of any order vanish at $\mathbf{x} = \mathbf{0}$. This implies that $g_\alpha(t \mp |\mathbf{x}|/c, \mathbf{x}; C)$ can be considered in a suitable weak sense as a solution of Equation (10)*.*

The weaker statement chosen in the theorem allows to cirumvent subtleties around the singular behavior of the solution at the origin.

3 Application to intricate current distributions

In this Section, we apply the aforementioned method to the electric field radiated by intricate current distributions. To simplify the derivation, we assume that the time dependence of the current density is separable, i.e., $\mathbf{J}(t,\mathbf{x}) = h(t)\mathbf{J}(\mathbf{x})$. If this assumption does not hold, the only different result in what follows is the formula for the moments in equations (12) and (13). In both cases, we first determine the charge density ρ, given by Equation (7). By the separability assumption, the charge can be written as $\rho(t,\mathbf{x}) = \partial_0^{-1}h(t)\rho(\mathbf{x})$ where $\partial_0^{-1}h(t)$ is an antiderivative of h. Second, we compute the time-domain moments corresponding to the right-hand side of Equation (8). These moments can be separated into current moments

$$C_\alpha^{J_i}(t) = -\mu h'(t) \iiint_{\mathbb{R}^3} J_i(\mathbf{y})\mathbf{y}^\alpha \, \mathrm{d}^3\mathbf{y} \tag{12}$$

and charge moments

$$C_\alpha^{\rho,i}(t) = -\frac{1}{\varepsilon}\partial_0^{-1}h(t) \iiint_{\mathbb{R}^3} \partial_i\rho(\mathbf{y})\mathbf{y}^\alpha \, \mathrm{d}^3\mathbf{y} \tag{13}$$

Third, we recursively compute the auxiliary function g_α up to the desired order n, thanks to the recursion relation. Finally, by Theorem 2.2, the components of the electric field are given by

$$E_i(t,\mathbf{x}) = \sum_{|\alpha|\le n} g_\alpha\left(t - |\mathbf{x}|/c, \mathbf{x}; C_\alpha^{J_i}(t) + C_\alpha^{\rho,i}(t)\right) \tag{14}$$

Similar to how on-screen pixels represent smooth curves by concatenations of rectangles, it is reasonable to assume that any planar current density can be decomposed into an appropriately large set of rectangular current "pixels", with a negligible effect on the radiated field. To find such a decomposition, we first determine the radiation of such a pixel, aligned to the x_1- and x_2-axes, centered at the coordinates $(x_1^0, x_2^0, 0)$, of width σ_1 and height σ_2. We suppose that a time-varying current density h travels across it in the direction of $\mathbf{e}_1$. Such a current distribution corresponds to

$$\mathbf{J}(t,\mathbf{x}) = h(t)\delta(x_3)\Pi\left(\frac{x_1 - x_1^0}{\sigma_1}\right)\Pi\left(\frac{x_2 - x_2^0}{\sigma_2}\right)\mathbf{e}_1 \tag{15}$$

where Π is the Heaviside Π function, equal to 1 for an argument between $-\frac{1}{2}$ and $\frac{1}{2}$, and zero otherwise. To simplify the derivation, we focus on the first component of the electric field, which is the most difficult since the current is polarized along the first axis. Of course, what follows can be generalized to the other components of the electric or magnetic fields. Equation (8) shows that we also need to compute the derivative of the charge density, which we express as a function of the current density thanks to the continuity equation, Equation (7):

$$\partial_1\rho(t,\mathbf{x}) = -\partial_1\partial_0^{-1}\partial_1 J_1(t,\mathbf{x}) = -\partial_0^{-1}h(t)\delta(x_3)\partial_1^2\left[\Pi\left(\frac{x_1 - x_1^0}{\sigma_1}\right)\right]\Pi\left(\frac{x_2 - x_2^0}{\sigma_2}\right) \tag{16}$$

A detailed computation of the moments is presented in Appendix A.2. Next, any intricate current distributions on the x_1x_2-plane, polarized along the first axis, can be decomposed into a sum of rectangular pixels: $\mathbf{J}(t,\mathbf{x}) = \sum_{i=1}^m \mathbf{J}^{(i)}(t,\mathbf{x})$ where each disjoint rectangular current pixel $\mathbf{J}^{(i)}$ can be expressed according to Equation (15). By linearity of the wave equation, the total field is given by the sum of the contribution of every current pixel.

To illustrate and validate the presented approach, we present two case studies:

1. First, we validate the method by computing the electric field radiated by a broadband Gaussian pulse[8] of length $T = 3.06\,\mathrm{ns}$ and smallest wavelength $\lambda = cT$, flowing through the geometry illustrated in Figure 1 (b) (a disc of radius 9λ). We then compare the obtained approximate result to the alternative method[8], which uses a multipole expansion in spherical coordinates and a different class of time-domain moments (Figure 1 (a)).

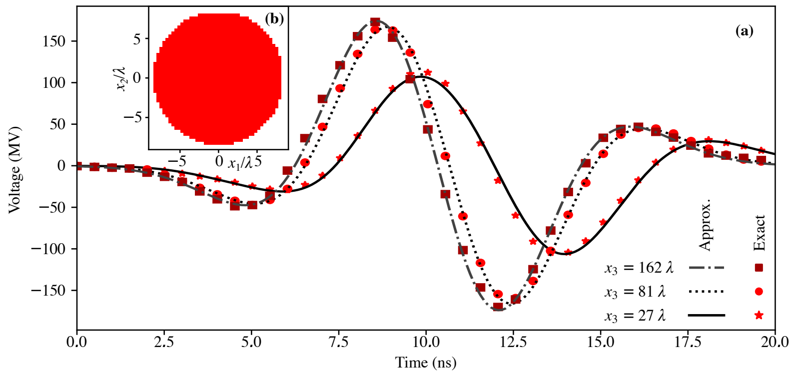

Figure 1. Comparison of the semi-analytical result with the literature[8]. **(a)** The voltage x_3E_1 corresponding to the component of the electric field radiated by a disc at the points $\mathbf{x} = (0,0,27\lambda)$, $\mathbf{x} = (0,0,81\lambda)$ and $\mathbf{x} = (0,0,162\lambda)$. The solid, solid-dashed, and dashed lines give the field for a 24th-order multipole expansion of the approximate disc (b), while the stars, circles, and squares indicate the exact results[8]. **(b)** Distribution of the current density approximating a disc across the x_1x_2-plane. The disc is formed by approximately 1500 square current pixels.

2. Second, we illustrate the method in an imaging experiment. We consider an intricate current density forming the letter "E" carrying the pulse $h(t) = e^{-(t/T)^2}$, where T is such that the 3 dB cutoff frequency corresponds to the wavelength λ_0. The imaging experiment is obtained by superposing the causal and anti-causal solutions (see Equation (14))

$$E_i(t,\mathbf{x}) = \sum_{|\alpha|\leq n} g_\alpha\left(t - |\mathbf{x}|/c, \mathbf{x}; C_\alpha^{J_i}(t) + C_\alpha^{\rho,i}(t)\right) - g_\alpha\left(t + |\mathbf{x}|/c, \mathbf{x}; C_\alpha^{J_i}(t) + C_\alpha^{\rho,i}(t)\right) \tag{17}$$

   and analyzing the electric field at $t = 0$ s. This corresponds to an ideal time-reversal cavity[24]. The current density and corresponding fields are illustrated in Figure 2.

The results are discussed in the next section.

4 Discussion and conclusion

We presented the time-domain Cartesian multipole expansion, a method to derive the electromagnetic fields radiated by any localized time-varying current density in homogeneous and isotropic media. The technique is self-contained thanks to Schwartz distributions and easily implemented in any programming language thanks to its formulation in Cartesian coordinates. Moreover, to the authors' best knowledge, it is the first explicit time-domain approach in Cartesian coordinates. Moreover, we showed how to apply the method to intricate current densities and validated the result by comparing it to an existing and comparable method[8] (see Figure 1).

The illustration case study in Figure 2 shows how the proposed method can be used, together with time-reversal[24], to predict imaging experiment results. For electrically small sources ($w = 0.08\lambda_0$, Figure 2 (a-c)), low truncation orders n are enough to represent the field accurately (e.g., Figure 2 (b)). However, the current distribution geometry is lost in the radiated field. By increasing the electrical size (up to $w = 0.75\lambda_0$, Figure 2 (d-i)), details such as sharp corners become visible (Figure 2 (i)), and the letter becomes legible (Figure 2 (e-f, i)). As a tradeoff, the truncation order needs to be increased to $n = 60$; below, the truncated field is invalid (Figure 2 (a, d, g-h)).

Compared to the latter reference, the proposed time-domain moments rely on a three-dimensional integral, whereas the existing method needs a four-dimensional integral. Having one less integral to perform simplifies numerical implementations. Otherwise, the only difference lies in using the Cartesian multipole expansion versus the spherical multipole expansion. Theoretically, both methods are comparable, although the singular behavior of the Cartesian multipole expansion is free from the subtleties appearing when using spherical coordinates[19,25]. If the geometry can be described by rectangles (such as current pixels), the Cartesian multipole moments are easier to compute. Moreover, the Cartesian formulation enables the computation of the radiation of intricate current distributions described by square current pixels of arbitrary polarization.

Compared to numerical methods, the proposed approach has several advantages. First, the memory requirement and computation time depend only on the number of observation points. In contrast, finite-difference time-domain approaches are constrained by the size of the domain and the smallest feature in the domain. Finite-element approaches depend on the required mesh and domain sizes, even for a single observation point. Second, the result can be displayed in a human-readable

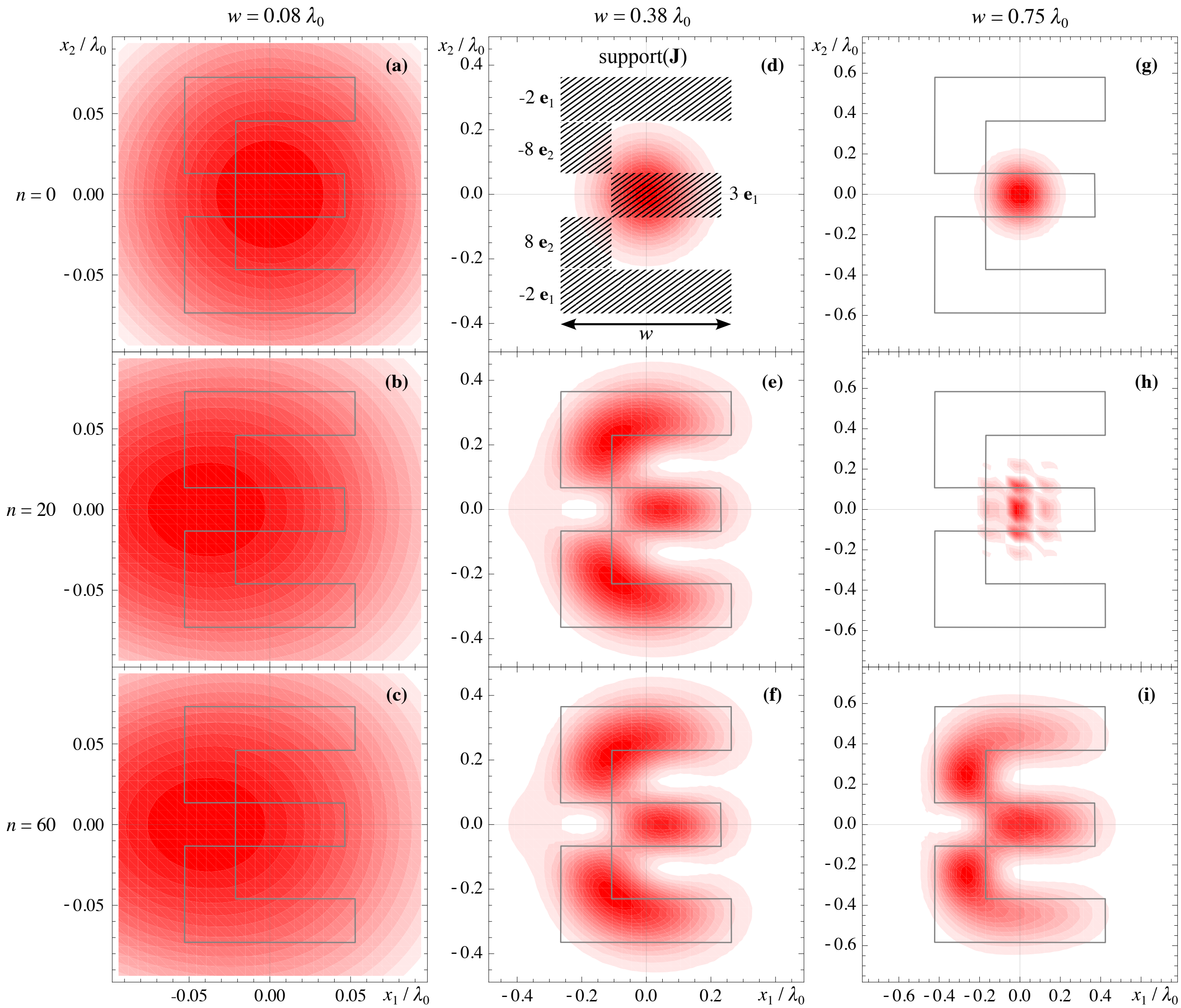

Figure 2. Normalized norm of the electric field obtained from the imaging experiment at $t = 0$ and $x_3 = 0$. Three electrical sizes are illustrated along the three columns. The rows indicate the truncation order n. The current density, whose outline is illustrated, is obtained by splitting the letter "E" into five rectangles whose amplitudes and polarities are also indicated in **(d)**.

series expansion (a semi-analytical result), allowing for finer interpretation and predictions. Finally, the primary constraint for these multipole expansion approaches is the electrical size of the source[8]. Indeed, even in the far field, the expansion order is proportional to the electrical size of the source (ratio of the source diameter and the smallest significant wavelength).

Contrary to numerical methods, once an expression for the multipole expansion is obtained, it can be evaluated efficiently for varying parameters (e.g., the amplitude of the moments, the shape of the time-domain excitation h, the origin of the multipole expansion, etc.), opening the door to physics-informed machine learning approaches for model identification. Other applications of interest include problems involving wideband radiation, such as impulse radiating antennas or electrostatic discharges.

Supplement

An open-source implementation of the proposed theory is available at `https://github.com/eliasleb/pynoza`. After installation, the validation result can be obtained by running `python tests/test_EPFL_logo.py --order 8 --case logo --method python --cname x1x2x3` from the root directory.

A Proof of Theorem 2.1

Proof. By the integrability assumption on ξ and since the Taylor series of G exists, we have that

$$\hat{f}(t,\mathbf{x}) = \int_0^T \iiint_{B(\mathbf{0},R)} \hat{G}(t-s,\mathbf{x}-\mathbf{y})\xi(s,\mathrm{d}^3\mathbf{y})\,\mathrm{d}s \tag{18}$$

where the support of ξ is contained in the time interval $[0,T]$ and the ball $B(\mathbf{0},R)$. Next, the Taylor series of $G(t-s,\mathbf{x}-\mathbf{y})$ around $\mathbf{y}=\mathbf{0}$ reads

$$\hat{G}(t-s,\mathbf{x}-\mathbf{y}) = \sum_{|\alpha|\le n} \frac{(-1)^{|\alpha|}}{\alpha!}(D^\alpha G)(t-s,\mathbf{x})\mathbf{y}^\alpha \tag{19}$$

The term $(-1)^{|\alpha|}$ appears because we differentiate with respect to $\mathbf{y}$ but express the Taylor series as a function of the derivatives of G, $(D^\alpha G)$. Therefore, by the linearity of the integral in Equation (18),

$$\hat{f}(t,\mathbf{x}) = \int_0^T \iiint_{B(\mathbf{0},R)} \hat{G}(t-s,\mathbf{x}-\mathbf{y})\xi(s,\mathrm{d}^3\mathbf{y})\,\mathrm{d}s = \int_0^T \iiint_{B(\mathbf{0},R)} \sum_{|\alpha|\le n} \frac{(-1)^{|\alpha|}}{\alpha!}(D^\alpha G)(t-s,\mathbf{x})\mathbf{y}^\alpha\xi(s,\mathrm{d}^3\mathbf{y})\,\mathrm{d}s \tag{20}$$

$$= \int_0^T \sum_{|\alpha|\le n} \frac{(-1)^{|\alpha|}}{\alpha!}(D^\alpha G)(t-s,\mathbf{x}) \iiint_{B(\mathbf{0},R)} \mathbf{y}^\alpha\xi(s,\mathrm{d}^3\mathbf{y})\,\mathrm{d}s = \int_0^T \sum_{|\alpha|\le n} \frac{(-1)^{|\alpha|}}{\alpha!}(D^\alpha G)(t-s,\mathbf{x})C_\alpha(s)\,\mathrm{d}s \tag{21}$$

where we interchanged integration and sum and recognized the definition of C_α.

On the other hand, let us compute the field $\tilde{f}$ radiated by the source $\tilde{\xi}$ of Equation (4). To this end, let ψ be a test function whose support is in $\mathbb{R}\times\Omega$. Hence

$$\left\langle \tilde{f}(t,\mathbf{x}),\psi(t,\mathbf{x})\right\rangle = \left\langle \left[G(s,\mathbf{y}) * \tilde{\xi}(s,\mathbf{y})\right](t,\mathbf{x}),\psi(t,\mathbf{x})\right\rangle = \left\langle G(t,\mathbf{x}),\left\langle \tilde{\xi}(s,\mathbf{y}),\psi(t+s,\mathbf{x}+\mathbf{y})\right\rangle\right\rangle \tag{22}$$

(by definition of the convolution, see [2, § 6.2])

$$= \sum_{|\alpha|\le n} \frac{(-1)^{|\alpha|}}{\alpha!}\left\langle G(t,\mathbf{x}),\left\langle C_\alpha(s)(D^\alpha\delta^3)(\mathbf{y}),\quad \psi(t+s,\mathbf{x}+\mathbf{y})\right\rangle\right\rangle = \sum_{|\alpha|\le n} \frac{1}{\alpha!}\left\langle G(t,\mathbf{x}),\left\langle C_\alpha(s)\delta^3(\mathbf{y}),(D^\alpha\psi)(t+s,\mathbf{x}+\mathbf{y})\right\rangle\right\rangle \tag{23}$$

(by definition of $\tilde{\xi}$ and linearity of the convolution product, then by definition of the derivative of a distribution)

$$= \sum_{|\alpha|\le n} \frac{1}{\alpha!}\left\langle G(t,\mathbf{x}),\int_0^T C_\alpha(s)(D^\alpha\psi)(t+s,\mathbf{x})\,\mathrm{d}s\right\rangle \tag{24}$$

by the regularity of $C_\alpha(s)$ and by definition of the Dirac δ distribution. Next, since G is a regular distribution, we get

$$= \sum_{|\alpha|\le n} \frac{1}{\alpha!}\int_{\mathbb{R}}\int_0^T \iiint_{\mathbb{R}^3} G(t,\mathbf{x})C_\alpha(s)(D^\alpha\psi)(t+s,\mathbf{x})\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}s\,\mathrm{d}t \tag{25}$$

$$= \sum_{|\alpha|\le n} \frac{(-1)^{|\alpha|}}{\alpha!}\int_{\mathbb{R}}\int_0^T \iiint_{\mathbb{R}^3} (D^\alpha G)(t,\mathbf{x})C_\alpha(s)\psi(t+s,\mathbf{x})\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}s\,\mathrm{d}t \tag{26}$$

by integration by parts and the compact support of ψ. Now, we introduce the change of coordinates $\begin{bmatrix} t & s & \mathbf{x}\end{bmatrix} \mapsto \begin{bmatrix} u & s & \mathbf{x}\end{bmatrix} = \begin{bmatrix} t+s & s & \mathbf{x}\end{bmatrix}$. The determinant of the Jacobian matrix of this change of coordinates is 1. We thus obtain

$$= \sum_{|\alpha|\le n} \frac{(-1)^{|\alpha|}}{\alpha!}\int_{\mathbb{R}}\int_0^T \iiint_{\mathbb{R}^3} (D^\alpha G)(u-s,\mathbf{x})C_\alpha(s)\psi(u,\mathbf{x})\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}s\,\mathrm{d}u \tag{27}$$

Here, we recognize the expression of the regular distribution

$$\tilde{f}(t,\mathbf{x}) = \int_0^T \sum_{|\alpha|\le n} \frac{(-1)^{|\alpha|}}{\alpha!}(D^\alpha G)(t-s,\mathbf{x})C_\alpha(s)\,\mathrm{d}s \tag{28}$$

Comparing with Equation (21), it is clear that $\langle \hat{f},\psi\rangle = \langle \tilde{f},\psi\rangle$ for all test functions ψ such that support$(\psi)\subset\mathbb{R}\times\Omega$. By [2, Theorem 6.4], this means $\hat{f}=\tilde{f}$ almost everywhere in $\mathbb{R}\times\Omega$. $\square$

A.1 Proof of Theorem 2.2

Proof. We will show the result by recursion on the derivative order $|\alpha|$. First, we show the result for $|\alpha| = 0$, i.e., $\alpha = \mathbf{0}$. Second, even though it is not formally needed, we also show the result for $|\alpha| = 1$. We show this case because it introduces almost all the mechanisms we need for the recursion step while keeping a relatively short notation. We finally show the recursion step.

The goal of the proof is to show that for any test function $\psi \in \mathscr{D}$ whose derivatives vanish at $\mathbf{x} = \mathbf{0}$,

$$\langle f_\alpha(t, \mathbf{x}; C), \psi(t, \mathbf{x})\rangle = \langle g_\alpha(t \mp |\mathbf{x}|/c, \mathbf{x}; C), \psi(t, \mathbf{x})\rangle \tag{29}$$

The requirement $D^\alpha \psi(t, \mathbf{0}) = 0$ for any α comes from the singularity of g_α at $\mathbf{x} = \mathbf{0}$. It ensures that all integrals below are well-defined. With this in mind, let us proceed to the proof of the recursion relation, as described above.

First, let us show the result for $\alpha = \mathbf{0}$. We need to find $f_\mathbf{0}$ satisfying $\Box f_\mathbf{0}(t, \mathbf{x}; C) = C(t)\delta^3(\mathbf{x})$. To this end, we use the fact[23,26] that the corresponding Green's distribution is uniquely determined by $G(t, \mathbf{x}) = \pm\frac{1}{4\pi}\frac{\delta(t \mp |\mathbf{x}|/c)}{|\mathbf{x}|}$. Hence, f is given by the convolution product $G(s, \mathbf{y}) * [C(s)\delta^3(\mathbf{y})]$. Since neither the Green's distribution nor the source term $C(s)\delta^3(\mathbf{y})$ is regular, we must use the most general definition of the convolution product [2, Section 6.2], which reads, for any test function $\psi \in \mathscr{D}$ whose derivatives vanish at $\mathbf{x} = \mathbf{0}$,

$$\langle f_\mathbf{0}(t, \mathbf{x}; C), \psi(t, \mathbf{x})\rangle = \left\langle [G(s, \mathbf{y}) * (C(s)\delta^3(\mathbf{y}))](t, \mathbf{x}), \psi(t, \mathbf{x})\right\rangle = \left\langle G(t, \mathbf{x}), \left\langle C(s)\delta^3(\mathbf{y}), \psi(t+s, \mathbf{x}+\mathbf{y}\right\rangle\right\rangle \tag{30}$$

(by [2, 6.2])

$$= \left\langle G(t, \mathbf{x}), \int_\mathbb{R} C(s)\psi(t+s, \mathbf{x})\,\mathrm{d}s\right\rangle = \left\langle \frac{\pm 1}{4\pi}\frac{\delta(t \mp |\mathbf{x}|/c)}{|\mathbf{x}|}, \int_\mathbb{R} C(s)\psi(t+s, \mathbf{x})\,\mathrm{d}s\right\rangle \tag{31}$$

(by Equation (9) and the definition of G)

$$= \iiint_{\mathbb{R}^3}\int_\mathbb{R} \frac{\pm 1}{4\pi}\frac{C(s)}{|\mathbf{x}|}\psi(s \pm |\mathbf{x}|/c, \mathbf{x})\,\mathrm{d}s\,\mathrm{d}^3\mathbf{x} \tag{32}$$

Now, we make the change of coordinates

$$\begin{bmatrix} s & \mathbf{x}\end{bmatrix} \mapsto \begin{bmatrix} t & \mathbf{x}\end{bmatrix} = \begin{bmatrix} s \pm |\mathbf{x}|/c & \mathbf{x}\end{bmatrix} \tag{33}$$

Since the determinant of the Jacobian matrix of this change of coordinates is 1, we get

$$\langle f_\mathbf{0}(t, \mathbf{x}; C), \psi(t, \mathbf{x})\rangle = \iiint_{\mathbb{R}^3}\int_\mathbb{R} \frac{\pm 1}{4\pi}\frac{C(t \mp |\mathbf{x}|/c)}{|\mathbf{x}|}\psi(t, \mathbf{x})\,\mathrm{d}t\,\mathrm{d}^3\mathbf{x} \tag{34}$$

At this point, we recognize the expression of the regular distribution

$$f_\mathbf{0}(t, \mathbf{x}; C) = \frac{\pm 1}{4\pi}\frac{C(t \mp |\mathbf{x}|/c)}{|\mathbf{x}|} = g_\mathbf{0}(t \mp |\mathbf{x}|/c, \mathbf{x}; C) \tag{35}$$

Second, let us show the result for $|\alpha| = 1$. Let $\alpha = (1, 0, 0)$ and ψ be as before. We proceed as for $\alpha = \mathbf{0}$:

$$\begin{aligned}\langle f_\alpha(t, \mathbf{x}; C), \psi(t, \mathbf{x})\rangle &= \left\langle G(t, \mathbf{x}), \left\langle C(s)D^{(1,0,0)}\delta^3(y), \psi(t+s, \mathbf{x}+\mathbf{y})\right\rangle\right\rangle \\ &= \left\langle G(t, \mathbf{x}), \left\langle C(s)\partial_1\delta^3(\mathbf{y}), \psi(t+s, \mathbf{x}+\mathbf{y})\right\rangle\right\rangle = -\left\langle G(t, \mathbf{x}), \left\langle C(s)\delta^3(\mathbf{y}), (\partial_1\psi)(t+s, \mathbf{x}+\mathbf{y})\right\rangle\right\rangle \end{aligned} \tag{36}$$

by definition of the derivative of a distribution. Using equations (30) to (32),

$$= -\iiint_{\mathbb{R}^3}\int_\mathbb{R} g_\mathbf{0}(s, \mathbf{x}; C)(\partial_1\psi)(s \pm |\mathbf{x}|/c, \mathbf{x})\,\mathrm{d}s\,\mathrm{d}^3\mathbf{x} \tag{37}$$

Now, we use the following result from differentiation:

$$\partial_1[\psi(s \pm |\mathbf{x}|/c, \mathbf{x})] = (\partial_1\psi)(s \pm |\mathbf{x}|/c, \mathbf{x}) \pm (\partial_0\psi)(s \pm |\mathbf{x}|/c, \mathbf{x})\frac{x_1}{c|\mathbf{x}|} \tag{38}$$

Notice the similarity with the recursion relation. Using Equation (38), we obtain

$$\langle f_\alpha(t, \mathbf{x}; C), \psi(t, \mathbf{x})\rangle = -\iiint_{\mathbb{R}^3}\int_\mathbb{R} g_\mathbf{0}(s, \mathbf{x}; C)\partial_1[\psi(s \pm |\mathbf{x}|/c, \mathbf{x})]\,\mathrm{d}s\,\mathrm{d}^3\mathbf{x} \pm \iiint_{\mathbb{R}^3}\int_\mathbb{R} g_\mathbf{0}(s, \mathbf{x}; C)\frac{x_1}{c|\mathbf{x}|}(\partial_0\psi)(s + |\mathbf{x}|/c, \mathbf{x})\,\mathrm{d}s\,\mathrm{d}^3\mathbf{x} \tag{39}$$

Since the integrands are continuous and have compact support, we can use Fubini's theorem and integrate successively over each space-time coordinate. We can, for example, start by integrating with respect to x_1 over $(-\infty,\infty)$. Using integration by parts, we get

$$= -\iint_{\mathbb{R}^2}\int_{\mathbb{R}} g_{\mathbf{0}}(s,\mathbf{x};C)\,\psi(s\pm|\mathbf{x}|/c,\mathbf{x})|_{x_1=-\infty}^{\infty}\mathrm{d}s\,\mathrm{d}x_2\,\mathrm{d}x_3 + \iiint_{\mathbb{R}^3}\int_{\mathbb{R}}(\partial_1 g_{\mathbf{0}})(s,\mathbf{x};C)\,\psi(s\pm|\mathbf{x}|/c,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}^3\mathbf{x}$$
$$\pm\iiint_{\mathbb{R}^3} g_{\mathbf{0}}(s,\mathbf{x};C)\frac{x_1}{c|\mathbf{x}|}\psi(s\pm|\mathbf{x}|/c,\mathbf{x})|_{s=-\infty}^{\infty}\mathrm{d}^3\mathbf{x} \mp \iiint_{\mathbb{R}^3}\int_{\mathbb{R}}(\partial_0 g_{\mathbf{0}}(s,\mathbf{x};C))\frac{x_1}{c|\mathbf{x}|}\psi(s\pm|\mathbf{x}|/c,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}^3\mathbf{x} \quad (40)$$

Since ψ has compact support, $\psi(s\pm|\mathbf{x}|/c,\mathbf{x})|_{x_1=-\infty}^{\infty}=0$ for any fixed s, and $\psi(s\pm|\mathbf{x}|/c,\mathbf{x})|_{s=-\infty}^{\infty}=0$ for any fixed $\mathbf{x}$. Hence

$$\langle f_\alpha(t,\mathbf{x};C),\psi(t,\mathbf{x})\rangle = \iiint_{\mathbb{R}^3}\int_{\mathbb{R}}\left[(\partial_1 g_{\mathbf{0}})(s,\mathbf{x};C)\mp(\partial_0 g_{\mathbf{0}})(s,\mathbf{x};C)\frac{x_1}{c|\mathbf{x}|}\right]\psi(s\pm|\mathbf{x}|/c,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}^3\mathbf{x}$$
$$= \iiint_{\mathbb{R}^3}\int_{\mathbb{R}} g_\alpha(s,\mathbf{x};C)\,\psi(s\pm|\mathbf{x}|/c,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}^3\mathbf{x} = \langle g_\alpha(t\mp|\mathbf{x}|/c,\mathbf{x};C),\psi(t,\mathbf{x})\rangle \quad (41)$$

by definition of g_1 and the change of coordinates in Equation (33). We have also rewritten the integral using the bracket notation $\langle\cdot,\cdot\rangle$. Notice that the proof is valid for any $\alpha\in\{(1,0,0),(0,1,0),(0,0,1)\}$, which shows the result for $|\alpha|=1$.

We finish the proof by showing the recursion step: we assume that for $|\alpha|\geq 1$, $f_\alpha(t,\mathbf{x};C)=g_\alpha(t\mp|\mathbf{x}|/c,\mathbf{x};C)$ and want to show the result for the order $|\alpha|+1$. Given any multi-index α, all multi-indices of order $|\alpha|+1$ equal $\alpha+\mathbf{e}_i$ for some $i\in\{1,2,3\}$. Hence, we need to show that $f_{\alpha+\mathbf{e}_i}(t,\mathbf{x};C)=g_{\alpha+\mathbf{e}_i}(t\mp|\mathbf{x}|/c,\mathbf{x};C)$. Let us take a test function $\psi\in\mathscr{D}$ whose derivatives vanish at $\mathbf{x}=\mathbf{0}$ and notice that

$$\langle f_\alpha(t,\mathbf{x};C),(\partial_i\psi)(t,\mathbf{x})\rangle = \left\langle G(t,\mathbf{x}),\left\langle C(s)(D^\alpha\delta^3(\mathbf{y})),(\partial_i\psi)(t+s,\mathbf{x}+\mathbf{y})\right\rangle\right\rangle \quad (42)$$

(by definition of the convolution and f_α)

$$= (-1)^{|\alpha|}\left\langle G(t,\mathbf{x}),\left\langle C(s)\delta^3(\mathbf{y}),(D^{\alpha+\mathbf{e}_i}\psi)(t+s,\mathbf{x}+\mathbf{y})\right\rangle\right\rangle = (-1)^{|\alpha|}\left\langle G(t,\mathbf{x}),\int_{\mathbb{R}} C(s)(D^{\alpha+\mathbf{e}_i}\psi)(t+s,\mathbf{x})\,\mathrm{d}s\right\rangle \quad (43)$$

(by definition of the derivative of a distribution, then by Equation (9))

$$= (-1)^{|\alpha|}\left\langle \frac{\pm 1}{4\pi}\frac{\delta(t\mp|\mathbf{x}|/c)}{|\mathbf{x}|},\int_{\mathbb{R}} C(s)(D^{\alpha+\mathbf{e}_i}\psi)(t+s,\mathbf{x})\,\mathrm{d}s\right\rangle \quad (44)$$

(by definition of the Green's distribution)

$$= (-1)^{|\alpha|}\iiint_{\mathbb{R}^3}\int_{\mathbb{R}} g_{\mathbf{0}}(s,\mathbf{x};C)(D^{\alpha+\mathbf{e}_i}\psi)(s\pm|\mathbf{x}|/c,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}^3\mathbf{x} \quad (45)$$

by definition of $g_{\mathbf{0}}$. On the other hand, since $\partial_i\psi$ vanishes at $\mathbf{x}=\mathbf{0}$, we can compute

$$\langle f_\alpha(t,\mathbf{x};C),(\partial_i\psi)(s,\mathbf{x})\rangle = \iiint_{\mathbb{R}^3}\int_{\mathbb{R}} f_\alpha(t,\mathbf{x};C)(\partial_i\psi)(t,\mathbf{x})\,\mathrm{d}t\,\mathrm{d}^3\mathbf{x} = \iiint_{\mathbb{R}^3}\int_{\mathbb{R}} g_\alpha(t\mp|\mathbf{x}|/c,\mathbf{x};C)(\partial_i\psi)(t,\mathbf{x})\,\mathrm{d}t\,\mathrm{d}^3\mathbf{x} \quad (46)$$

(by the recursion assumption)

$$= \iiint_{\mathbb{R}^3}\int_{\mathbb{R}} g_\alpha(s,\mathbf{x};C)(\partial_i\psi)(s\pm|\mathbf{x}|/c,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}^3\mathbf{x} \quad (47)$$

by making the inverse change of coordinates described in Equation (33). We will need this later.

Now, let us again use the convolution to find $f_{\alpha+\mathbf{e}_i}$:

$$\langle f_{\alpha+\mathbf{e}_i}(t,\mathbf{x};C),\psi(t,\mathbf{x})\rangle = (-1)^{|\alpha+\mathbf{e}_i|}\iiint_{\mathbb{R}^3}\int_{\mathbb{R}} g_{\mathbf{0}}(s,\mathbf{x};C)(D^{\alpha+\mathbf{e}_i}\psi)(s\pm|\mathbf{x}|/c,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}^3\mathbf{x} \quad (48)$$

(by the exact same procedure that led to Equation (45))

$$= -(-1)^{|\alpha|}\iiint_{\mathbb{R}^3}\int_{\mathbb{R}} g_{\mathbf{0}}(s,\mathbf{x};C)(D^\alpha\partial_i\psi)(s\pm|\mathbf{x}|/c,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}^3\mathbf{x} = -\langle f_\alpha(t,\mathbf{x};C),\partial_i\psi(t,\mathbf{x})\rangle \quad (49)$$

(again by the procedure that led to Equation (45))

$$= -\iiint_{\mathbb{R}^3}\int_{\mathbb{R}} g_\alpha(s,\mathbf{x};C)(\partial_i\psi)(s\pm|\mathbf{x}|/c,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}^3\mathbf{x} \tag{50}$$

(by Equation (47))

$$= \iiint_{\mathbb{R}^3}\int_{\mathbb{R}} \left[\partial_i g_\alpha(s,\mathbf{x};C) \mp (\partial_0 g_\alpha)(s,\mathbf{x};C)\frac{x_i}{c|\mathbf{x}|}\right]\psi(s\pm|\mathbf{x}|/c,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}^3\mathbf{x} \tag{51}$$

(by integration by parts and the compact support of ψ – recall Equation (38))

$$= \iiint_{\mathbb{R}^3}\int_{\mathbb{R}} \left[D^{\mathbf{e}_i} g_\alpha(s,\mathbf{x};C) \mp (\partial_0 g_\alpha)(s,\mathbf{x};C)\frac{\mathbf{x}^{\mathbf{e}_i}}{c|\mathbf{x}|}\right]\psi(s\pm|\mathbf{x}|/c,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}^3\mathbf{x} \tag{52}$$

(by definition of the multi-index $\mathbf{e}_i$)

$$= \iiint_{\mathbb{R}^3}\int_{\mathbb{R}} g_{\alpha+\mathbf{e}_i}(s,\mathbf{x};C)\psi(s\pm|\mathbf{x}|/c,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}^3\mathbf{x} = \iiint_{\mathbb{R}^3}\int_{\mathbb{R}} g_{\alpha+\mathbf{e}_i}(t\mp|\mathbf{x}|/c,\mathbf{x};C)\psi(t,\mathbf{x})\,\mathrm{d}t\,\mathrm{d}^3\mathbf{x} \tag{53}$$

(by definition of $g_{\alpha+\mathbf{e}_i}$, see the recursion relation, then by the change of coordinates described in Equation (33))

$$= \langle g_{\alpha+\mathbf{e}_i}(t\mp|\mathbf{x}|/c,\mathbf{x};C),\psi(t,\mathbf{x})\rangle \tag{54}$$

which shows the statement and concludes the proof. □

A.2 Computation of the time-domain multipole moments

Computing the time-domain moments linked to the current density amounts to computing the integral

$$\int_{\mathbb{R}} x_1^{\alpha_1}\Pi\left(\frac{x_1-x_1^0}{\sigma_1}\right)\mathrm{d}x_1 = \left.\frac{x_1^{\alpha_1+1}}{\alpha_1+1}\right|_{x_1=x_1^0-\sigma_1/2}^{x_1=x_1^0+\sigma_1/2} \tag{55}$$

The charge density involves the second-order derivative with respect to x_1:

$$\partial_1^2\Pi\left(\frac{x_1-x_1^0}{\sigma_1}\right) = \partial_1\left[\delta(x_1-x_1^0+\sigma_1/2)-\delta(x_1-x_1^0-\sigma_1/2)\right] = \delta'(x_1-x_1^0+\sigma_1/2)-\delta'(x_1-x_1^0-\sigma_1/2) \tag{56}$$

where we have used the scaling property of the Dirac δ distribution. The moments give

$$\int_{\mathbb{R}}\left(\delta'(x_1-x_1^0+\sigma_1/2)-\delta'(x_1-x_1^0-\sigma_1/2)\right)x_1^{\alpha_1}\,\mathrm{d}x_1 = \left.\alpha_1 x_1^{\alpha_1-1}\right|_{x_1=x_1^0-\sigma_1/2}^{x_1=x_1^0+\sigma_1/2} \tag{57}$$

if $\alpha_1 > 0$, and zero otherwise.

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Author contributions statement

E.L. derived the results and proofs, ran the simulations, and implemented the method; E.R. verified the proofs and updated the statements of the theorems; E.L., C.K., N.M., F.R., M.R., F.V. analyzed the results; C.K., N.M., F.R., M.R., F.V. provided additional context. All authors reviewed the manuscript.

Additional information

The authors declare no competing interests.

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