# Self-sustained dynamics and forced resonant oscillations in flows: cross-junction jets and sloshing liquids 

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par

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The world little knows
how many of the thoughts and theories
which have passed through the mind of a scientific investigator,
have been crushed in silence and secrecy by his own severe criticism and adverse examination.

- Michael Faraday

To my parents,
who made this journey possible in the first place,
and to all those who, after me,
will be inspired by it.

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## Abstract

In engineering, oscillatory instabilities and resonances are often considered undesirable flow features and measures are taken to avoid them. This may include avoiding certain parametric regions or implementing control and mitigation strategies. However, the examples considered in this thesis illustrate a different perspective: self-sustained or driven flow oscillations can be harnessed in the design of a wide spectrum of engineering devices ranging from microfluidic circuitry, and bioreactors for cell cultivation, to liquid-based templates for assembling microscale materials. The key to an effective design of these fluidic devices lies in having an adequate predictive understanding of the hydrodynamic processes at stake.
Microfluidic oscillators based on interacting jets, sloshing waves and parametric Faraday waves belong to these flows for which time-oscillations, manifesting spontaneously or as a consequence of external forcing, can be seen as beneficial. Although the emergence of an oscillatory response in these systems can be predicted by linear analysis, the observed flow dynamics and features are typically dependent on the oscillation amplitude through nonlinear mechanisms and may deviate from the anticipated patterns due to the interaction of multiple modes.
This thesis uses the tools of global linear stability analysis and asymptotic techniques to provide a comprehensive theoretical framework that can rationalize some of these oscillatory dynamics. To achieve this, the work draws upon direct numerical simulations as well as existing and new dedicated experiments.
In Part I, we describe a microfluidic oscillator based on two laminar impinging jets. After determining the space of control parameters for which self-sustained oscillations appear, linear stability and sensitivity analysis are used to identify a shear instability located in the jet's interaction region, as the main candidate for the emergence of the oscillatory regime. Further nonlinear features are also described by means of the multiple-scales weakly nonlinear theory.
In Part II, we study the harmonic and super-harmonic resonant sloshing dynamics in orbitally-shaken cylindrical reservoirs. We develop amplitude equations models capable of predicting the wave amplitude saturation and wave patterns associated with various wave regimes, such as planar, irregular, swirling and counter-swirling motions, experimentally under elliptic-like shaking conditions.
In Part III, we consider parametric Faraday waves in two different configurations, which are linked to each other for the importance of the sidewall boundary conditions. First, we describe the weakly nonlinear coupling of sub-harmonic parametric waves and harmonic capillary waves produced by an axisymmetric oscillating meniscus. Successively, we propose


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a modified gap-averaged Darcy model for Faraday waves in Hele-Shaw cells that translates into a new damping coefficient, whose complex value is a function of the ratio between the Stokes boundary layer thickness and the cell's gap.

To conclude, in Part IV, we develop a mathematical model based on successive linear eigenmode projections to solve the relaxation dynamics of viscous capillary-gravity waves subjected to an experimentally inspired nonlinear contact line model that accounts for nonlinear Coulomb solid-like friction. We show that each projection eventually induces a rapid loss of total energy in the liquid motion and contributes to its nonlinear damping.

Keywords: oscillations, resonance, forcing, stability, amplitude equations, nonlinear dynamics, microfluidics, sloshing


## Résumé

En ingénierie, les instabilités oscillatoires et les résonances dans les écoulements sont souvent perçus comme indésirables. Pour s'en défaire, on peut par exemple éviter certaines régions paramétriques ou mettre en oeuvre des stratégies de contrôle et d'atténuation. Cependant, les exemples examinés dans cette thèse illustrent une perspective différente : les oscillations auto-entretenues ou forcées des écoulements peuvent être exploitées dans de nombreuses applications, allant des circuits microfluidiques et bioréacteurs pour la culture cellulaire, à l'assemblage de matériaux à l'échelle microscopique. Pour une conception efficace de ces dispositifs fluidiques, une compréhension prédictive de l'hydrodynamique en jeu est nécessaire.
Les jets microfluidiques en interaction, les ondes de balottement et les ondes paramétriques de Faraday sont associés à des écoulements pour lesquels les oscillations, qui se manifestent spontanément ou à la suite d'un forçage externe, peuvent être exploitées avantageusement. Bien que l'apparition d'une réponse oscillatoire dans ces systèmes puisse être prédite par l'analyse linéaire, la dynamique observée dépend généralement de l'amplitude de l'oscillation du fait de mécanismes non-linéaires, et peut s'écarter des motifs prédits linéairement en raison de l'interaction de multiples modes.
Cette thèse utilise des techniques de stabilité linéaire et d'analyse asymptotique pour fournir un cadre théorique complet qui rationalise certaines de ces dynamiques oscillatoires. Pour ce faire, le travail s'appuie sur des simulations numériques directes ainsi que sur des expériences dédiées, existantes et nouvelles.
Dans la Partie I, nous décrivons un oscillateur microfluidique reposant sur l'interaction de deux jets laminaires. Après avoir déterminé les paramètres de contrôle pour lesquels des oscillations auto-entretenues apparaissent, nous identifions, par analyse de la stabilité linéaire et de la sensibilité, une instabilité de cisaillement, située dans la région d'interaction des jets, comme principale responsable de l'émergence du régime oscillatoire.

Dans la Partie II, nous étudions la dynamique du balottement harmonique et superharmonique dans les réservoirs cylindriques agités orbitalement. En particulier, nous dérivons des équations d'amplitude qui peuvent prédire la saturation et la forme d'onde associées à différents régimes, notamment planaire, irrégulier, rotatif et contre-rotatif.
Dans la Partie III, nous étudions les ondes de Faraday paramétriques. Tout d'abord, nous décrivons le couplage non linéaire d'ondes paramétriques sous-harmoniques et d'ondes capillaires harmoniques produites par les oscillations d'un ménisque axisymétrique. Ensuite, nous proposons une modification du modèle de Darcy pour les ondes de Faraday dans les cellules de Hele-Shaw. Ce modèle aboutit à coefficient d'amortissement dont la valeur complexe est
fonction du rapport entre l'épaisseur de la couche limite de Stokes et celle du récipient.
Pour conclure, dans la Partie IV, nous développons un modèle mathématique basé sur des projections successives des vecteurs propres du système pour résoudre la dynamique de relaxation des ondes gravito-capillaires, en utilisant un modèle de ligne de contact non linéaire qui tient compte du frottement Coulombien le long de la paroi. Nous montrons que chaque projection induit une perte rapide d'énergie totale dans le mouvement du liquide et contribue à son amortissement non linéaire.

Mots clés : oscillations, résonance, forçage, stabilité, équations d'amplitude, dynamique non-linéaire, microfluidique, ballottement

## Riassunto

In molti flussi industriali, instabilità oscillatorie e risonanze sono spesso caratteristiche indesiderate, mitigate da specifiche contromisure quali l'esclusione di intere regioni parametriche o l'implementazione di strategie di controllo. Gli esempi considerati in questa tesi illustrano una prospettiva diversa: le oscillazioni di flusso autosostenute o alimentate da forzanti esterne possono essere sfruttate nella progettazione di un ampio spettro di dispositivi ingegneristici che vanno da circuiti microfluidici, a bioreattori per la coltivazione di cellule, all'assemblaggio di materiali microstrutturati. La chiave per una progettazione efficace di questi dispositivi risiede in un'adeguata comprensione dei processi idrodinamici in gioco.
Tra i flussi per i quali le oscillazioni, spontanee o prodotto di una forzante esterna, possono essere sfruttate vantaggiosamente, annoveriamo quelli generati dall'interazione tra getti microfluidici, da onde di "sloshing" e da onde parametriche di Faraday. Sebbene la comparsa di una risposta oscillatoria in questi sistemi possa essere predetta da un'analisi lineare, le reali dinamiche e caratteristiche di flusso osservate dipendono tipicamente dall'ampiezza delle oscillazioni attraverso meccanismi non lineari e possono deviare dalle predizioni dei modelli lineari a causa dell'interazione di più modi globali.
Questa tesi sfrutta tecniche di stabilità lineare e di analisi asintotica per fornire un quadro teorico completo in grado di razionalizzare alcune di queste dinamiche oscillatorie. A tal fine, il lavoro trae ispirazione da quanto osservato in simulazioni numeriche ed in esperimenti dedicati, sia esistenti che nuovi.
Parte I descrive un oscillatore microfluidico basato sull'interazione di due getti laminari. Dopo aver determinato i parametri di controllo per i quali compaiono oscillazioni autosostenute, l'analisi di stabilità lineare e la sensitività sono utilizzate per identificare un'instabilità di "taglio" situata nella regione di interazione dei getti, come principale responsabile dell'emergere del regime oscillatorio.
Parte II studia le dinamiche armoniche e super-armoniche di sloshing in serbatoi cilindrici agitati orbitalmente. In particolare, deriviamo delle equazioni d'ampiezza in grado di prevedere la saturazione e la forma d'onda associata a vari regimi, tra cui planari, irregolari, rotativi e contro-rotativi.

Parte III considera le onde parametriche di Faraday. In primo luogo, descriviamo l'accoppiamento non lineare di onde parametriche sub-armoniche e onde capillari armoniche prodotte da un menisco asimmetrico oscillante. Successivamente, proponiamo una modifica del modello mediato di Darcy per onde di Faraday in contenitori di tipo Hele-Shaw. Tale modello si traduce in un coefficiente di smorzamento, il cui valore complesso è funzione del rapporto tra
lo spessore dello strato limite di Stokes e quello del contenitore.
Per concludere, Parte IV tratta lo sviluppo di un modello matematico basato su proiezioni successive degli autovettori di un determinato sistema per risolvere la dinamica di rilassamento viscoso di onde gravito-capillari, sottoposte a un modello di linea di contatto non lineare che tiene conto dell'attrito solido alla parete. È infine dimostrato che ogni proiezione induce una rapida perdita di energia totale nel moto del liquido e contribuisce al suo smorzamento non lineare.
Parole chiave: oscillazioni, risonanza, forzante, stabilità, equazioni dell'ampiezza, dinamica nonlineare, microfluidica, "sloshing"

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[^1]
## 1 Introduction

### 1.1 Classification of oscillatory fluid systems via linear stability

The transition from one flow state to another with the appearance of new patterns and unsteadiness is ubiquitous in fluid mechanics. A few representative examples of interesting oscillatory flows are shown in figure 1.1, namely (a,b) the famous von Kármán vortex street, which manifests in the wake of bluff bodies when the flow advection is sufficiently high, (c,d) a laminar-to-turbulent transition in jet flowing out of a circular nozzle and (e) surface waves in an agitated glass of water. These examples of oscillations in fluids are not merely academic but are rather fundamentally relevant to a broad spectrum of industrial applications, e.g. in the design of turbojet nozzles (d) or in the structural and dynamical analysis of skyscrapers and tanker ships ( f ), for which resonant vortex- or sloshing-induced vibrations could lead to catastrophic failures.
At the core of their fluid dynamic descriptions are the Navier-Stokes equations, which are a direct consequence of mass conservation and Newton's second law applied to an incompressible volume of fluid and govern the fluid velocity and pressure fields, $(\mathbf{u}, p)^{T}$,

$$
\begin{equation*}
\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u} \cdot \nabla) \mathbf{u}=-\nabla p+\frac{1}{R e} \Delta \mathbf{u}, \quad \nabla \cdot \mathbf{u}=0 \tag{1.1}
\end{equation*}
$$

and where $R e$ is the Reynolds number, a nondimensional number that quantifies the relative importance between flow advection and diffusion. The Navier-Stokes equations, supplemented with proper case-dependent initial and boundary conditions, have been demonstrated capable of successfully englobing and describing the interplay of multiple physical mechanisms, such as advection, dissipation, external body forces, capillary and geometrical effects, turbulence, etc., in a large variety of experiments and applications. This incredible complexity often complicates the understanding of the individual physical mechanisms behind the transitions between different flow states and patterns and the emergence of unsteadiness.
For such reasons, one area of fluid mechanics that witnessed explosive growth at the end of the $20^{\text {th }}$ century and that is still full of life is the study of hydrodynamic instabilities and the


Figure 1.1 - (a) von Kármán vortex street behind a circular solid cylinder (photograph by Jürgen Wagner: https://commons.wikimedia.org/wiki/File:Karmansche_Wirbelstr_kleine_Re.JPG) and (b) around industrial cylindrical cables (inset from Seyed-Aghazadeh et al. (2021)). (c) Smoke visualisation of a jet flow (Huck, 2017) with industrial application in turbojets (d) (https://defencyclopedia.com/2015/05/13/explained-how-jet-engines-work/). (e) Snapshot of sloshing waves in a partially filled container with important applications in liquid transport, i.e., tanker ships' safety (f) (Credit: alexyz3d/AdobeStock) (inset from Pastoor et al. (2005)).
associated nonlinear phenomena, through linear stability and asymptotic theories.
Stability theory is indisputably the most classical approach to describe instability and state transitions, e.g. steady-to-unsteady, through bifurcations (Charru, 2011; Chomaz, 2005; Drazin and Reid, 2004; Huerre and Monkewitz, 1990; Huerre and Rossi, 1998; Schmid et al., 2002; Theofilis, 2011). The regime transition affecting a flow when a control parameter, such as $R e$, is varied, can be investigated by computing the linear stability of a base flow, $\mathbf{q}_{0}=\left(\mathbf{u}_{0}, p_{0}\right)^{T}$, representing an equilibrium solution of (1.1) for fixed control parameters, to infinitesimal time-dependent perturbations, i.e. $\mathbf{q}_{1}=\left(\mathbf{u}_{1}, p_{1}\right)$, such that the total flow is decomposed as $\mathbf{q}=\mathbf{q}_{0}+\mathbf{q}_{1}$, with $\left\|\mathbf{q}_{1}\right\| \ll\left\|\mathbf{q}_{0}\right\|$. After introducing this flow decomposition in (1.1) and, successively, discretizing the linearized system of governing equations into an algebraic system, one can adopt the formalism of the dynamical system theory so as to write down the linearised Navier-Stokes equations in a compact form as

$$
\begin{equation*}
\mathscr{M} \frac{d \mathbf{q}_{1}}{d t}=\mathscr{L} \mathbf{q}_{1} \tag{1.2}
\end{equation*}
$$

where $\mathscr{M}$ is a mass matrix and the linearized Navier-Stokes operator $\mathscr{L}$ depends on the equilibrium state $\mathbf{q}_{0}$ computed for a fixed $R e$.

One can seek a linear eigensolution of (1.2) in the standard normal form $\mathbf{q}_{1}=\hat{\mathbf{q}}_{1 n}(\mathbf{x}) e^{\left(\sigma_{n}+\mathrm{i} \omega_{n}\right) t}$, where the natural mode $\hat{\mathbf{q}}_{1 n}(\mathbf{x})$ and its associated eigenvalue $\lambda_{n}=\sigma_{n}+\mathrm{i} \omega_{n}$, are eigensolutions of the generalized eigenvalue problem

$$
\begin{equation*}
\lambda_{n} \mathscr{M} \hat{\mathbf{q}}_{1 n}=\mathscr{L} \hat{\mathbf{q}}_{1 n} . \tag{1.3}
\end{equation*}
$$

Let us suppose that at some threshold value, e.g., of the Reynolds number, $R e=R e_{c r}$, the system becomes unstable to infinitesimal perturbations with growth rate, $\sigma_{n}=0$, and frequency, $\omega_{n}$. This is, for instance, the case of the cylinder flow of figure 1.2(a). Then, linear stability analysis will predict the exact value of $R e=R e_{c r}$ at which the instability first manifests. As for $R e>R e_{c r}$ the growth rate is positive, $\sigma_{n}>0$, we expect the unstable infinitesimal perturbation (eigenmode) $\hat{\mathbf{q}}_{1 n}$ to grow exponentially in time until the system progressively evolves towards a limit cycle with a large-time finite perturbation amplitude saturated by nonlinear effects, as depicted in figure $1.2(\mathrm{~b})$. Moreover, if the value of $\omega_{n}$ at $R e_{c r}$ differs from zero as in the cylinder flow, then the instability is oscillatory and a steady-to-unsteady regime transition occurs in the flow via a Hopf bifurcation (Jackson, 1987; Provansal et al., 1987; von Kármán, 1921; Williamson, 1988), found to be supercritical in this case (see figure 1.2(c)).

In the following, three archetypal flow problems are used to categorised unsteady oscillatory flows into three main families, namely oscillators, amplifiers and resonators, on the basis of the stability properties of their corresponding linearized Navier-Stokes operator, $\mathscr{L}$.
We will use $\omega_{n}$ to denote a natural frequency of the system, $\omega$ to indicate the actual frequency of the nonlinear system's response, whereas $\Omega=2 \pi / T$ will refer to an external driving frequency of oscillation period $T$.


Figure 1.2 - Sketch of the supercritical Hopf bifurcation and steady-to-unsteady state transition in the cylinder flow (modified figure from (Mantič-Lugo, 2015)). (a) Sketch of the transition from stable to unstable given by a positive growth rate $\sigma_{n}$ at the critical Reynolds number $R e_{c r}$. (b) Sketch of the evolution with Reynolds number $R e$ of the saturated finite amplitude $A$ of the periodic fluctuations, which is modelled by the Stuart-Landau amplitude equation (Stuart, 1960). (c) Flow visualization of the steady base flow (bottom) and a time-snapshot of the unsteady oscillatory regime for $R e>R e_{c r}$ (top).


Figure 1.3-Oscillators: cylinder flow at $R e=140$ (figure modified from Dyke (1982)); (a) linear stability analysis (LSA) of an unstable base-flow ( $R e>R e_{c r}$ ) in the cylinder flow showing an eigenvalue spectrum with a single unstable mode of natural frequency $\omega_{n}$; (d) Power spectral density function (PSD) extracted from a signal in the cylinder wake, showing a clear peaked frequency associated with the von Kàrmàn vortex street (Pier, 2002); (g) Time-dependent and local signal of the horizontal velocity extracted from a DNS of the cylinder flow at $R e=$ 100 (Mantič-Lugo, 2015), showing the initial exponential growth, $\sim \exp \left(\sigma_{n} t\right)$, as well as the finite-amplitude saturation to a limit cycle with oscillation frequency $\omega_{L C} \neq \omega_{n}$. Amplifiers: turbulent jet at $R e=10000$ (from Dyke (1982)); (b) LSA on the mean flow of a turbulent jet displaying a flat, stable spectrum (Nichols and Lele, 2010); (e) PSD of signals extracted at various streamwise locations $x$ in a turbulent jet and showing a broad frequency response to noise (Bogey et al., 2007). Resonators: sloshing waves in a rectangular cell (Bäuerlein and Avila, 2021); (c) linear spectrum displaying a series of discrete and slightly damped eigenmodes; (f) PSD experimentally measured for a longitudinal time-periodic container motion of frequency $\Omega$ and showing a main peak at $\omega / \Omega=1$; (h) The linear response peaks around $\Omega \approx \omega_{n}$ with an amplification $\propto 1 / \sigma_{n}$. The nonlinear response saturates at lower values and bends the resonance curve, a feature successfully modelled by the Duffing equation (Duffing, 1918).

### 1.1.1 Oscillators

Figure 1.3(a) shows the eigenvalue spectrum obtained by performing the linear stability analysis (LSA) of a base-state for the famous cylinder flow, already discussed at the end of the previous section. The spectrum at $R e>R e_{c r}$ displays a well-isolated eigenvalue with natural oscillation frequency $\omega_{n}$ and with a positive growth rate $\sigma_{n}$, meaning that the equilibrium (steady) solution considered, $\mathbf{q}_{0}$, is unstable. When looking at the power spectral density (PSD) of a time series extracted from a nonlinear flow field experimentally or numerically computed (see figure 1.3(d)), a dominant and clear peaked frequency associated with the von Kármán vortex street, together with a few higher-order harmonics of lower PSD, is well identifiable. This is a consequence of the presence of an "outstanding" eigenvalue that dictates the longterm behaviour of an initial small perturbation. More precisely, the unstable eigenmode and eigenvalue of $\mathscr{L}$ describe the initial structure and growth of the perturbation (Theofilis, 2011) before its amplitude becomes too big and nonlinear interactions come into play (Barkley, 2006; Sipp and Lebedev, 2007). If the natural frequency, $\omega_{n}$, differs from zero, as in figure 1.3(a), the flow becomes unsteady and oscillates spontaneously in a self-sustained manner. For this reason, unstable flows are typically referred to as oscillators (Huerre and Rossi, 1998).
Although oscillations naturally emerge without the need for external driving, the latter can be applied for control strategies. For instance, in certain cases, the vortex-shedding phenomenon can cause concerning structural vibrations and drag increases (Choi et al., 2008). When dealing with fluid-induced vibration issues, it's crucial to note that unstable flow frequencies can only really become hazardous when they align with the structural modes. Therefore, adjusting the flow's frequency slightly could be a viable solution to resolve the problem. This can be achieved in open-loop control by imposing to the unstable flow an external harmonic forcing of frequency $\Omega$, amplitude $f$ and with a proper spatial structure, $\mathbf{q}_{f}(\mathbf{x})$ (Sipp, 2012). Indeed, if the forcing frequency $\Omega$ is chosen close to the natural frequency $\omega_{n}$, then the flow oscillations will lock onto $\Omega$, so as to shift the frequency and move it away from the resonance (see Fauve (1998), Bender et al. (1999) and Chapter 8 of Charru (2011) for further details on the lock-in phenomenon).

### 1.1.2 Amplifiers

In figure 1.3(b) we report the eigenvalue spectrum of an amplifier flow, i.e. a laminar jet of air flows exiting a circular tube and whose edges, moving downstream, develop axisymmetric oscillations, rolls up into vortex rings, and then abruptly becomes turbulent. Despite the fact that the eigenvalue-spectrum is fully stable, the PSD function of a local time-series in the jet flow, reported figure 1.3(e), shows that small harmonic external excitations result in a large amplification of the system responses (Crow and Champagne, 1971). Moreover, the system response has a rather broad or mildly selective frequency selection mechanism, with a frequency of maximum amplification which does not necessarily match one of the least stable modes. These features can be better understood by drawing attention to the non-normality of the linear operator $\mathscr{L}$. A linear operator $\mathscr{L}$ is said to be non-normal if it does not commute
with its adjoint operator, $\mathscr{L}^{\dagger}$, i.e. $\mathscr{L} \mathscr{L}^{\dagger} \neq \mathscr{L}^{\dagger} \mathscr{L}$. It is important to note that the definition of the adjoint operator is not univocal, but rather depends on the introduction of an arbitrary inner product, although the latter is very often chosen so as to represent an energy norm for the system, e.g. the total or kinetic energy. If the operator is non-normal, the eigenmode basis is not orthogonal. It follows that, even for a stable operator, whose eigenmodes all decay at large times, small initial perturbations may experience very large transient growth. Strong non-normality also possibly implies high sensitivity to small operator perturbations and a large response to harmonic forcing away from eigenfrequencies (Ducimetière et al., 2022a,b; Trefethen et al., 1993), hence arguing for the importance of sensitivity analysis and flow control of these systems (Bottaro et al., 2003; Boujo and Gallaire, 2015; Camarri, 2015; Sipp et al., 2010). In the case of the Navier-Stokes operator, non-normality is generally produced by strong streamwise advection of the base flow (Chomaz, 2005; Farrell and Ioannou, 1996; Schmid, 2007; Schmid et al., 2002; Trefethen et al., 1993). Therefore, the linear stability analysis, capable of predicting the instability onset in oscillators with a dominant unstable eigenmode, appears almost irrelevant in these scenarios and it thus fails in describing the dynamics of strongly non-normal flows like noise amplifiers.

### 1.1.3 Resonators

An example of a resonator is represented by sloshing, a term used to denote any motion of the free liquid surface in a partially filled reservoir subjected to horizontal motions, i.e. perpendicular to the direction of gravity. Figure 1.3(c) shows the eigenvalue spectrum of a typical sloshing system. The eigenvalues correspond to the natural sloshing modes, i.e. free surface capillary-gravity waves, for a container, e.g. rectangular, partially filled with a liquid and undergoing a longitudinal harmonic motion at a driving frequency $\Omega=2 \pi / T$ (see also figure 1.4). In absence of external forcing, the equilibrium or base-state configuration for this flow is a liquid column stably at rest under the effect of gravity. Thus, similarly to the amplifier system of figure $1.3(\mathrm{~b})$, the linear spectrum is stable, although the eigenvalues are here well separated from each other, as a result of the lateral confinement, which, through necessary boundary conditions, only allows for some specific modal perturbations (Faltinsen and Timokha, 2009; Ibrahim, 2005). In our classification, what fundamentally discerns resonators from amplifiers is the normal nature of the linearized Navier-Stokes operator. Indeed, for resonators like that of figure 1.3, the linearized operator $\mathscr{L}$ is typically normal, meaning that it commutes with its adjoint (Viola et al., 2018; Viola and Gallaire, 2018), i.e. $\mathscr{L}_{\mathscr{L}^{\dagger}}=\mathscr{L}^{\dagger} \mathscr{L}$ (and $\mathscr{L}$ is said to be self-adjoint). As a consequence, the eigenmode basis is fully orthogonal. These features have strong implications for the system's response to perturbations and harmonic forcing in resonators. Given the stability and self-adjointness of $\mathscr{L}$, the linear evolution of initial perturbations, which is given by the superposition of eigenvectors, shows a decaying large-time behaviour without experiencing any transient growth. Furthermore, a sustained oscillatory response can only be achieved by externally driving the system, e.g. at a forcing frequency $\Omega$. If the system is subjected to white noise or, more simply, to a harmonic forcing of varying frequency $\Omega$, the maximum amplification is achieved in the neighbourhood of a
natural frequency, i.e. for $\Omega \approx \omega_{n}$, with a linear amplitude response $\sim 1 / \sigma_{n}$ (see figure $1.3(\mathrm{~h})$ ), hence showing a very precise frequency selection mechanism, in contradistinction with the broad frequency response of amplifiers.

Within the family of resonators, we can further distinguish among three sub-classes of oscillatory responses depending on their nature, namely driven oscillations, parametric oscillations and natural transient oscillations resulting from a non-zero initial condition.


Figure 1.4 - Top: A sketch of the experimental apparatus of Bäuerlein and Avila (2021) and snapshots of different sloshing states observed. Bottom: Sloshing liquid in a horizontally oscillated rectangular tank over one oscillation period (Bäuerlein and Avila, 2021). The tank has a width of $\mathrm{w}=500 \mathrm{~mm}$, is filled with water to the height $h=400 \mathrm{~mm}$ and is driven with the frequency $\Omega=2 \pi / T$, with $T=0.88 \mathrm{~s}(1 / T=1.13 \mathrm{~Hz})$. Nonlinear resonances amplify periodic surface waves (marked as a red line) and produce oscillations of the liquid's centre of mass (indicated by red circles). Stereoscopic particle image velocimetry measurements of the in-plane velocity (displayed as arrows) show that the maximum velocities are reached when the surface elevation is lowest. The excitation frequency is close to the first system's natural frequency resonance, $\Omega / \omega_{n}=0.917$ (harmonic resonance). The excitation amplitude is $f=a \Omega^{2}$, with $a=a_{x} / w=0.64$ ( $a_{x}$ is the peak amplitude of the horizontal tank displacement).

## Driven oscillations

When a resonator like the sloshing system of figures 1.3 and 1.4 is externally driven at a frequency $\Omega$, the large-time response is generally characterized by a finite amplitude, set by the saturation resulting from the system's dissipation and nonlinear mechanisms, and by an oscillation frequency coinciding with that of the external forcing. Indeed, the PSD function shows a main peak centred around $\omega / \Omega \approx 1$ and a series of super-harmonics triggered by
nonlinear effects. The PSD function of figure 1.3(f) is reminiscent of that typical of oscillators (see figure 1.3(d)), although here, oscillations are not self-sustained but are rather maintained by the external driving.


Figure 1.5 - (a) A vertically vibrating liquid layer that spontaneously excites sub-harmonic surface standing waves (Benjamin and Ursell, 1954; Faraday, 1831; Kumar and Tuckerman, 1994a) (modified figure from Sampara and Gilet (2016)). (b) Power spectral density (PSD) computed numerically by simulating the response of a liquid layer in a cylindrical container vertically excited at a frequency $\Omega$ (Bongarzone et al., 2021b). The PSD shows a dominant peak at $\omega / \Omega=1 / 2$, thus indicating a sub-harmonic parametric response.

## Parametric oscillations

Parametric resonators are systems where an oscillatory system's response can be induced by time-modulating one or more internal parameters of the system at some frequencies that possibly differ from its natural frequencies, $\Omega \neq \omega_{n}$. Such a modulation can be achieved in several ways; a simple archetypal example is given by the parametric pendulum, whose pivot position is vertically modulated by imposing an external forcing. This translates into a modulation of the gravity acceleration acting on the system, which can then be parametrically and resonantly pumped by frequency modulations with $\omega_{n} / \Omega \approx p / 2(p \in N)$ (Kovacic et al., 2018a). The strongest amplification is typically achieved for $\omega_{n} / \Omega \approx 1 / 2$, and it is referred to as sub-harmonic resonance. This parametric amplification also occurs in continuous media. For instance, the flat interface of a liquid contained in a vertically vibrating tank (see figure 1.5(a)) may be parametrically excited, leading to the generation of standing waves oscillating at a frequency (see figure $1.5(\mathrm{~b})$ ) that is half that of the external driving, leading to the so-called Faraday instability (Benjamin and Ursell, 1954; Faraday, 1831; Kuhlmann and Rath, 1998; Kumar and Tuckerman, 1994a).
Since the two examples of parametric oscillations mentioned here both involve the use of an external forcing, the distinction between driven and parametric resonators may still appear somewhat vague at this stage. Nevertheless, such a distinction becomes much clearer at the level of the governing equations, particularly by noticing that, while the external forcing in driven resonators appears as an additive extra term, in parametric resonators the forcing is multiplicative and it appears in front of one or more state variables. Explicative archetypal
examples of this differentiation are offered in the following by the Duffing equation (1.7) (driven) and by the Mathieu equation (1.15).

## Natural oscillations

If, for instance, the external driving is eventually turned off as in figure 1.6(b), the stable nature of these resonators no more allows for sustained oscillations and the system enters a new dynamical phase during which it relaxes towards the original equilibrium solution through natural, free, oscillations. The amplitude response decreases with time and the system, initially oscillating at the driving frequency $\Omega$, progressively adjusts its free oscillation frequency, which will tend to the least damped natural frequency $\omega_{n}$ at large times. The relaxation dynamics is ideally exponential with a decay rate possibly dictated by the damping rate of the least damped natural mode, i.e. $\sim \exp \left(\sigma_{n} t\right)\left(\sigma_{n}<0\right)$, although nonlinear phenomena, such as friction or free surface and contact line capillary effects in confined liquid oscillations (see figure 1.6(a) and Viola (2016)), becoming more and more important as the wave amplitude decreases, may alter the features of the decaying behaviour. As a side comment, we note that, in order to observe the natural evolution dynamics, the system does not necessarily have to start from sustained oscillations; it could start from any initial condition, e.g. an impulsive perturbation.


Figure 1.6 - (a) Nonlinear friction in sloshing dynamics is induced by one or more layers of foam placed at the free surface (Zhang et al., 2019). As a consequence, the sloshing wave does not relax exponentially (Sauret et al., 2015; Viola et al., 2016c). (b) Relaxation dynamics of a harmonically driven sloshing wave following the suppression of the external driving at time $t=0$ (modified figure from Bäuerlein and Avila (2021)). In the. absence of nonlinear effects, the relaxation dynamics is exponential, with a decay rate defined by the damping coefficient $\sigma_{n}$ of the previously excited natural mode. However, nonlinear friction acting at the contact line may affect the relaxation dynamics provoking the motion arrest at finite times (Cocciaro et al., 1993; Dollet et al., 2020; Viola et al., 2018) (see purple line in connection with panel (a)). In both scenarios, the system, initially oscillating non-parametrically at the driving frequency $\Omega$, progressively adjusts its oscillation frequency, which equals the natural frequency $\omega_{n}$ at large times.

### 1.2 Nonlinear effects and envelope equations

The physical problems investigated in this thesis are all attributable to the two categories of oscillators and resonators, for which linear stability analysis provides relevant and useful pieces of information about the initial evolution of a perturbation and the response to weak external forcing. Nevertheless, even when non-normal effects are not important or completely absent, as for most of the oscillators and resonators, the linear modal behaviour, used for the classification outlined in the previous section, does not always fully capture the entire dynamics of the perturbation.
In figure $1.3(\mathrm{~g})$ and (h), we have already anticipated the role that nonlinearities play in the cylinder flow and for resonantly driven sloshing waves in a rectangular container. For instance, in an unstable cylinder flow, i.e. $R e>R e_{c r}$, the perturbation, initially oscillating at the natural frequency $\omega_{n}$, grows exponentially until the amplitude becomes large enough and nonlinear mechanisms kick in. The oscillation frequency progressively increases, while the perturbation amplitude saturates and the system eventually settles into a limit cycle, with self-sustained oscillations at $\omega=\omega_{L C} \neq \omega_{n}$ and finite amplitude. The amplitude saturation and the frequency modulation are direct consequences of nonlinear mechanisms. Similar saturation and frequency detuning effects, as well as other nonlinear effects, happen in sloshing and Faraday waves, although the wave motion needs to be triggered and permanently sustained by external driving. We have also shown, in figure 1.6, how some kind of nonlinear effects (sub-linear (Viola, 2016)), such as capillary-induced friction in confined surface waves, can nonlinearly damp the oscillations and, becoming particularly effective at small amplitudes, eventually induce the arrest of the interface motion at finite times.
Generally speaking, a high-fidelity description and prediction of nonlinear phenomena observed in real-life experiments are only achievable by solving the fully nonlinear governing equations, which often do not admit closed-form analytical solutions. Accurate approximated solutions can be computed via direct numerical simulations (DNS), which are, however, computationally costly. Hence, the formalization of reduced models involving lower degrees-offreedom, such as envelope (also called amplitude) equations, derived by means of asymptotic theories and englobing the relevant nonlinear flow features, constitutes an attractive alternative to DNS whenever applicable, e.g. when nonlinear effects are only weak. With regards to oscillators and resonators, nonlinearities are generally small close to bifurcation points and for small external forcing amplitudes.
For example, Provansal et al. (1987) and Dušek et al. (1994) observed that, in the case of the first instability in the cylinder wake, the complex amplitude of the perturbation, $A=|A| e^{\mathrm{i} \Phi}$, close to the bifurcation ( $R e \approx R e_{c r}$ ) is governed by the Stuart-Landau equation (Stuart, 1960),

$$
\begin{equation*}
\frac{d A}{d t}=\lambda A+v|A|^{2} A \tag{1.4}
\end{equation*}
$$

which describes the saturation mechanism in this super-critical Hopf bifurcations (Kuznetsov et al., 1998) (see figure $1.2(\mathrm{~b})$ ) as the steady base flow passes from stable to unstable, providing an estimation of the time evolution of the instability amplitude.

The complex coefficients $\lambda=\lambda_{r}+\mathrm{i} \lambda_{i}$ and $v=v_{r}+\mathrm{i} v_{i}$, originally determined experimentally, have been computed in a rigorous manner by Sipp and Lebedev (2007) using weakly nonlinear analysis for the Navier-Stokes equations in the neighbourhood of the critical Reynolds number, $R e_{c r}$ (Stuart, 1958). They showed that the Stuart-Landau equation naturally appears as a compatibility condition in the asymptotic scheme. In the same spirit, a weakly nonlinear mode expansion for different flows (precessing vortex breakdown, wakes of disks and spheres) has been carried out by Meliga et al. (2009a, 2012a).
At the core of these perturbative analyses is the multiple-scales method (Cole, 1968), which has been widely used to obtain amplitude equations describing the slow dynamics of the large-scale modulation of a basic structure predetermined by $a$-priori calculations, e.g. from global or local linear stability (Newell and Whitehead, 1969; Segel, 1969). The method lies within the family of asymptotic techniques and it assumes, after non-dimensionalization of the governing equations, the existence of a small non-dimensional parameter $\epsilon \ll 1$ in the underlying problem and that can be taken, for instance, as a measure of the departure from criticality in terms of control parameters, e.g. $R e^{-1}-R e_{c r}^{-1} \sim \epsilon$, or as the amplitude of a small external forcing, $f \sim \epsilon$, if any. It is then meaningful to seek for a solution $\mathbf{q}$ as a formal power series in the small parameter, i.e. $\mathbf{q}=\mathbf{q}_{0}+\epsilon \mathbf{q}_{1}+\epsilon^{2} \mathbf{q}_{2}+\ldots+\epsilon^{k} \mathbf{q}_{k}+\mathrm{O}\left(\epsilon^{k+1}\right)$, where in most cases, retaining only the first few terms of the series is sufficient to describe the small $\epsilon$ behaviour of the actual solution. In fact, the multiple scales approach consists in postulating that the system's functions vary on two (or more) temporal and/or spatial scales, so that some functions, e.g. the perturbation amplitude, depend on time $t$ and space $\mathbf{x}$, only through the product $T_{i}=\epsilon^{i} t$ and $X_{j}=\epsilon^{j} \mathbf{x}$, e.g. $A\left(\epsilon t, \ldots, \epsilon^{i} t, \epsilon \mathbf{X}, \ldots, \epsilon^{j} \mathbf{x}\right)=A\left(T_{1}, \ldots, T_{i}, \mathbf{X}_{1}, \ldots, \mathbf{X}_{j}\right)$, with $i, j<n$. Requiring, through the imposition of a solvability condition, the suppression of unphysical secular terms in the standard expansion eventually fixes the ensuing arbitrariness by providing a governing equation for $A$.
As a more general example, the combined introduction of slow time and spatial scales is the starting point in the derivation of the famous nonlinear Schrödinger equation (NLS) (Ablowitz et al., 1991; Benjamin and Feir, 1967; Stoker, 1992; Whitham, 1974; Zakharov, 1972), as an envelope equation for gravity waves that describes the evolution of slowly modulated wavetrains:

$$
\begin{equation*}
\frac{\partial A}{\partial t}-\gamma \frac{\partial^{2} A}{\partial x^{2}}=v|A|^{2} A, \tag{1.5}
\end{equation*}
$$

(written in a non-dimensional form and in a coordinate system moving with the group velocity) with coefficients $\gamma=-\mathrm{i} \omega_{n} / 8 k_{n}^{2}, v=-\mathrm{i} \omega_{n} k_{n}^{2} / 2$ and where $k_{n}$ represents the wave number of the carrier wave, whereas $\omega_{n}=\sqrt{g k_{n}}$ (g: gravity acceleration) is the linear dispersion relation of gravity waves in the deep water regime (Lamb, 1993). For instance, an important issue in naval engineering is the phenomenon of rogue waves, extreme events occurring in systems characterized by the presence of many waves (Onorato et al., 2001); most of the models which have been developed so far have a weakly nonlinear nature and are based on the NLS. See Onorato et al. (2013) for a series of representative examples, where the main physical mechanisms at the origin of rogue waves are elucidated.
The NLS appears in different physical contexts, including plasma physics and nonlinear
optics, since it simply describes the interaction of dispersion and weak nonlinearity. Equation (1.5) is a special case of an amplitude equation for a conservative system. In the more common case where dissipation cannot be neglected, the usual amplitude equation is the so-called complex Ginzburg-Landau equation (Aranson and Kramer, 2002; Godrèche and Manneville, 1998),

$$
\begin{equation*}
\frac{\partial A}{\partial t}-\gamma \frac{\partial^{2} A}{\partial x^{2}}=\lambda A+v|A|^{2} A \tag{1.6}
\end{equation*}
$$

where coefficients $\gamma, \lambda$ and $v$ are not purely imaginary as in (1.5). To give a few examples, equation (1.6) has been used to describe the Benjamin-Feir phase instabilities, as well as other symmetry-breaking secondary instabilities of cellular flows, as the Eckhaus and the "zigzag" instabilities (Godrèche and Manneville, 1998).

### 1.2.1 Weakly nonlinear analysis via multiple time-scales method

A multiple scales expansion in space is commonly employed in WKBJ approaches (Bender et al., 1999; Gaster et al., 1985; Huerre and Rossi, 1998; Nayfeh, 2008a) for weakly nonparallel flows, in which the steady base or mean flow varies slowly on a long length scale when compared to the shorter instability waves (Charru, 2011; Chomaz, 2005; Schmid et al., 2002).
However, in the problems tackled in this thesis, the effect of the geometry and confinement on the flow is such that the instabilities and the base flow have no separated length scales: the dynamics of the perturbation result from the interactions of global modes extended over the whole physical domain and whose spatial structures valid at any spatial location can be computed by means of linear stability calculations. As a result, no slow spatial scales need to be introduced, and one only needs to account for slow time modulations of the perturbation amplitudes, which are governed by nonlinear ODEs, rather than PDEs as in the case of the nonlinear Schrödinger equation (1.5) or the Ginzburg-Landau equation (1.6).
Since the weakly nonlinear analysis via multiple time-scales method constitutes a fundamental theoretical building block of the present work, in the following we provide a quick overview of the method, using as examples a series of archetypal single-degrees-of-freedom systems.

## Asymptotic solution of the forced Duffing equation

Let us first consider the Duffing equation (Duffing, 1918), a popular single-degrees-of-freedom system often used to model the nonlinear response of externally driven resonators,

$$
\begin{equation*}
\ddot{x}+2 \sigma_{n} \dot{x}+\omega_{n}^{2} x+\beta x^{3}=f \cos \Omega t, \tag{1.7}
\end{equation*}
$$

where $\sigma_{n}$ is the damping coefficient, $\beta$ is the nonlinear coefficient, while $f$ and $\Omega$ are the driving amplitude and frequency. In the following, we only consider the limit of small forcing amplitudes, $f=\epsilon \hat{f}$, weak nonlinearities, $\beta=\epsilon \hat{\beta}$ and small damping, $\sigma_{n}=\epsilon \hat{\sigma}_{n}$, with the auxiliary parameters $\hat{f}, \hat{\beta}$ and $\hat{\sigma}_{n}$ assumed of order $\sim \mathrm{O}$ (1). Most generally speaking, $\epsilon$ represents
a small parameter, i.e. $0<\epsilon \ll 1$, which does not necessarily need to be explicitly defined, but it can rather be considered as an implicit separation of the different orders of magnitude at play. A straightforward perturbation-series approach to the problem proceeds by writing $x(t)=x_{0}(t)+\epsilon x_{1}(t)+\mathrm{O}\left(\epsilon^{2}\right)$ and substituting this into (1.7). Matching powers of $\epsilon$ gives the $\epsilon^{0}$-order equation

$$
\begin{equation*}
\ddot{x}_{0}+\omega_{n}^{2} x_{0}=0 \quad \longrightarrow \quad x_{0}=A e^{\mathrm{i} \omega_{n} t}+c . c . \tag{1.8}
\end{equation*}
$$

with c.c. denoting the complex conjugate, and $\epsilon$-order problem

$$
\begin{align*}
& \ddot{x}_{1}+\omega_{n}^{2} x_{1}=-2 \hat{\sigma}_{n} \dot{x}_{0}-\hat{\beta} x_{0}^{3}+\hat{f} \cos \Omega t=\frac{\hat{f}}{2} e^{\mathrm{i} \Omega t}-2 \hat{\sigma}_{n} \mathrm{i} A e^{\mathrm{i} \omega_{n} t}-\hat{\beta} A^{3} e^{\mathrm{i} 3 \omega_{n} t}-3 \hat{\beta}|A|^{2} A e^{\mathrm{i} \omega_{n} t}+c . c .,  \tag{1.9}\\
& x_{1}=A^{3} \frac{\hat{\beta}}{8 \omega_{n}^{2}} e^{\mathrm{i} 3 \omega_{n} t}-\left(|A|^{2} \frac{3 \hat{\beta}}{4 \omega_{n}^{2}}+\mathrm{i} \frac{\hat{\sigma}_{n}}{2 \omega_{n}^{2}}\right) A e^{\mathrm{i} \omega_{n} t}-\frac{\hat{f}}{\Omega^{2}-\omega_{n}^{2}} e^{\mathrm{i} \Omega t}+  \tag{1.10}\\
&+\underbrace{\left[\left.|\mathrm{i}| A\right|^{2} \frac{3 \hat{\beta}}{2 \omega_{n}}-\frac{\hat{\sigma}_{n}}{\omega_{n}}\right] A t e^{\mathrm{i} \omega_{n} t}}_{\alpha t}+\text { c.c., }
\end{align*}
$$

where the second-order homogeneous solution in (1.10) has been omitted for brevity. The most general solution of (1.10) is unbounded due to the linear terms in $t$ (see framed terms in (1.10)), which are classically referred to as secular terms. In particular, for $t=\mathrm{O}\left(\epsilon^{-1}\right)$, these terms become $\mathrm{O}(1)$ and have the same order of magnitude as the leading-order term, $x_{0}$. Because the asymptotic terms have become disordered, the series is no longer an asymptotic expansion of the solution, i.e. the straightforward perturbation expansion breaks down. Such a linear growth is obviously a spurious effect since it is clear that (1.7) conserves energy. This pathological behaviour is resolved by resorting to the multiple scales framework (Godrèche and Manneville, 1998; Nayfeh, 2008a). Let us introduce explicitly the slow time scale $T=\epsilon t$, which leads to

$$
\begin{equation*}
\frac{d}{d t}=\frac{\partial}{\partial t}+\epsilon \frac{\partial}{\partial T}+\mathrm{O}\left(\epsilon^{2}\right) \quad \frac{d^{2}}{d t^{2}}=\frac{\partial^{2}}{\partial t^{2}}+2 \epsilon \frac{\partial^{2}}{\partial t \partial T}+\epsilon^{2} \frac{\partial^{2}}{\partial T^{2}}+\mathrm{O}\left(\epsilon^{3}\right), \tag{1.11}
\end{equation*}
$$

as if $t$ and $T$ were independent variable. With these definitions, the $\epsilon^{0}$-order problem remains unchanged and has solution $x_{0}=A(T) e^{\mathrm{i} \omega_{n} t}+c . c$. The only, but fundamental, difference, consists in assuming that amplitude $A(T)$ is now a function of the slow time scale, it represents the slow wave-amplitude modulation of the fast wave oscillations, and it is still undetermined at this stage of the asymptotic expansion. The $\epsilon$-order problem is now forced by the following terms:

$$
\begin{array}{r}
\frac{\partial^{2} x_{1}}{\partial t^{2}}+\omega_{n}^{2} x_{1}=-2 \frac{\partial^{2} x_{0}}{\partial t \partial T}-2 \hat{\sigma}_{n} \frac{\partial x_{0}}{\partial t}-\hat{\beta} x_{0}^{3}+\hat{f} \cos \Omega t  \tag{1.12}\\
=\left(-2 \mathrm{i} \omega_{n} \frac{\partial A}{\partial T}-2 \mathrm{i} \hat{\sigma}_{n} \omega_{n} A-3 \hat{\beta}|A|^{2} A+\frac{\hat{f}}{2} e^{\mathrm{i} \hat{\Lambda} T}\right) e^{\mathrm{i} \omega_{n} t}+\text { c.c. }+\mathrm{NRT},
\end{array}
$$

where NRT stands for non-resonating terms, meaning terms that are not secular and that are not necessarily relevant for further analysis (unless one aims at pursuing the expansion to the next order). In (1.12), we have already considered the most dangerous scenario, in which the system is driven close to the natural frequency (resonant condition). This has been done by introducing a small detuning parameter $\Lambda$, i.e. $\Omega=\omega_{n}+\Lambda$ with $\Lambda=\epsilon \hat{\Lambda}$ (and $\hat{\Lambda} \sim \mathrm{O}$ (1), such that $\Omega t=\omega_{n} t+\hat{\Lambda} T$ in (1.12). The arbitrariness introduced by $A(T)$ is fixed by requiring that secular terms are not present in the solution (1.10), which implies cancelling out the harmonic forcing terms in $\omega_{n}$ appearing on the right-hand side of (1.12) or (1.9). Such a solvability condition prescribes the amplitude $A(T)$ to obey the following ordinary differential equations

$$
\begin{equation*}
\frac{\partial A}{\partial T}=\frac{1}{\epsilon} \frac{d A}{d t}=\hat{\lambda} A+\hat{v}|A|^{2} A+\mu \hat{f} \quad \underset{\text { time } t \text { by eliminating } \epsilon}{\text { reintroducing the physical }} \quad \frac{d A}{d t}=\lambda A+v|A|^{2} A+\mu f \tag{1.13}
\end{equation*}
$$

with coefficients $\lambda=\epsilon \hat{\lambda}=-\left(\sigma_{n}+\mathrm{i}\left(\Omega-\omega_{n}\right)\right), v=\epsilon \hat{v}=\mathrm{i} 3 \beta / 2 \omega_{n}$ and $\mu=-\mathrm{i} / 4 \omega_{n}$ and where the transformation $A \rightarrow A e^{\mathrm{i} \hat{\Lambda} T}$ has been used so as to make the amplitude equation autonomous. Note that (1.13) takes the form of a Stuart-Landau equation supplemented with an external driving term. Hence, the envelope equation (1.13) provides a governing equation for the perturbation's amplitude and the leading order solution, $x_{0}=A(t) e^{\mathrm{i}\left(\omega_{n}+\Lambda t\right)}+$ c.c. $=$ $2|A(t)| \cos (\Omega t+\Phi(t))$, represents a good approximation of (1.7) valid for small forcing in the vicinity of the resonance and for weak nonlinearities.
The close-to-resonant asymptotic approximation of the forced Duffing equation has been widely used in the modelling of resonant sloshing waves, e.g, in rectangular container (Bäuerlein and Avila, 2021; Ockendon and Ockendon, 2001, 1973), and it has been shown capable of describing the finite wave amplitude saturation through hardening- or softening-like behaviours. By properly fitting coefficients $\sigma_{n}$ and $\beta$ from experimental measurements by Bäuerlein and Avila (2021) (see also figure 1.4), the latter can be compared with the predictions from approximation (1.13). This is outlined in figure 1.7(a) in terms of non-dimensional steadystate wave amplitude (large-time dynamics) for different non-dimensional forcing amplitude. These steady-state solutions can be obtained, e.g., by time-integrating (1.13) for large time intervals. Alternatively, one can directly seek stationary solutions by setting $d A / d t=0$ and then study their stability to small perturbations in the form $A=A_{0}+\epsilon A_{1} e^{\left(s_{r}+\mathrm{is} s_{i}\right) t}$, so that the sign of $s_{r}$ establishes whether the steady solution $A_{0}$ is stable or unstable.

## Mathieu's equation with nonlinearities

With regard to this thesis, another relevant single-degree-of-freedom system, used to model the response of parametrically driven resonators, is the parametric pendulum, already introduced in the previous section. At the linear order, this simple system is described by the Mathieu equation (Mathieu, 1868)

$$
\begin{equation*}
\ddot{x}+2 \sigma_{n} \dot{x}+\omega_{n}^{2} x=x \omega^{2} \epsilon f \cos \Omega t . \tag{1.14}
\end{equation*}
$$

The parametrically unstable regions in the forcing parameter space $(\Omega, f)$ can be computed by means of the linear Floquet stability theory performed around one of the two possible


Figure 1.7 - (a) Asymptotic approximation (1.13) is compared with sloshing experiments in a rectangular container by (Bäuerlein and Avila, 2021). The comparison is outlined in terms of the non-dimensional steady-state wave amplitude (large-time dynamics) of the wave's center of mass for different non-dimensional forcing amplitude, $a$. Those amplitudes are re-scaled by the container's width, w (see figure 1.4). The forcing acceleration is $f=c_{1} a \Omega^{2}$, with $\Omega$ the driving frequency and $c_{1}=0.3183$ a characteristic system parameter. Coefficients $\sigma_{n}$ and $v$ are set to $8.4 \times 10^{-3}$ and -59.2 , respectively. (b) Asymptotic approximation (1.16) is compared with experiments by (Henderson and Miles, 1990) for single-mode Faraday waves in a small circular cylinder. The grey-shaded area represents the sub-harmonically unstable region in the $(\Omega, a)$ forcing parameter space (right-y axis), whereas lines correspond to the nonlinear amplitude saturation (left-y axis). Amplitudes are re-scaled by the container's radius $R$. The forcing acceleration is $f=c_{1} a \Omega^{2}$, with $c_{1}=1.0291$. Coefficients: $\sigma_{n}=0.0157$ and $\beta=-6$. (c) Asymptotic approximation (1.18) is compared with the measurements by (Dollet et al., 2020) of the relaxation dynamics of liquid oscillations in a $U$-shaped tube. The amplitude is rescaled by the initial non-dimensional elevation, $2 h_{0} / l$ with $l$ the overall tube length. Coefficients: $\sigma_{n}=0.06$ and $\Delta=0.0047$. In ( $\mathrm{a}, \mathrm{c}$ ), $\omega_{n}=1$, while $\omega_{n}=1.9641$ in (b). In (a,b,c) lines indicate the asymptotic approximations, whereas markers denote experiments. In (a,b), dashed lines designate unstable steady-state solutions of (1.13) and (1.16). In (a,b,c), parameters $\sigma_{n}, \beta$ and $\Delta$ are fitted in order to match the experiments.
equilibrium solutions (Kovacic et al., 2018a). Within such regions the equilibrium solution is unstable, and the perturbation grows exponentially. The most relevant parametric resonance is the sub-harmonic one, as it is the one that requires the lowest driving amplitude to be excited. Yet, equation (1.15) does not tell us anything about nonlinear mechanisms. Nonlinearities are partially reintroduced by accounting for a cubic term $\left(\sin x \approx x-x^{3} / 6+\ldots\right.$ ) (Kovacic et al., 2018a),

$$
\begin{equation*}
\ddot{x}+2 \sigma_{n} \dot{x}+\omega_{n}^{2} x+\beta x^{3}=x \omega_{n}^{2} f \cos \Omega t . \tag{1.15}
\end{equation*}
$$

Without going into the details (the derivation is similar to that for the Duffing equation), an asymptotic approximation of the fundamental sub-harmonic resonance can be obtained from (1.15) via multiple time-scales method in the limit of small forcing amplitude, $f=\epsilon \hat{f}$, weak damping, $\sigma_{n}=\epsilon \hat{\sigma}_{n}$, weak nonlinearities, $\beta=\epsilon \hat{\beta}$, and assuming a driving frequency to be $\Omega=2 \omega_{n}+\Lambda=2 \omega_{n}+\epsilon \hat{\Lambda}$, i.e. in the neighbourhood of the sub-harmonic resonance. Note that the introduced auxiliary parameter $\hat{f}, \hat{\sigma}_{n}, \hat{\beta}$ and $\hat{\Lambda}$ are all of order $\sim \mathrm{O}$ (1). The asymptotic procedure and the imposition of a solvability condition lead to the following amplitude equation

$$
\begin{equation*}
\frac{d A}{d t}=\lambda A+v|A|^{2} A+\mu \bar{A} f, \tag{1.16}
\end{equation*}
$$

with $\lambda=-\left(\sigma_{n}+\mathrm{i}\left(\Omega-2 \omega_{n}\right) / 2\right), v=\mathrm{i} 3 \beta / 2 \omega_{n}$ and $\mu=-\mathrm{i} / 4 \omega_{n}$.
The very same amplitude equation, originally derived by symmetry arguments, has been widely used for modelling the wave amplitude saturation of sub-harmonically unstable Faraday standing waves in lab-scale containers (see Douady (1990) and Henderson and Miles (1990) among many others). After fitting coefficients $\sigma_{n}$ and $\beta$ from experiments by Henderson and Miles (1990) of single-global-mode sub-harmonic Faraday waves in a small circular cylinder, approximation (1.16) is compared with those measurements in figure 1.7(b). The comparison is outlined in terms of the stability region associated with the sub-harmonic parametric resonance (grey-shaded area) and in terms of non-dimensional steady-state wave amplitude (large-time dynamics) at a fixed non-dimensional forcing amplitude.

## Solid friction in free pendulum dynamics

The Duffing equation and the nonlinear Mathieu equation are examples of oscillatory dynamics which experience nonlinear effects for increasing amplitudes. Those effects are responsible for the saturation mechanism of the perturbation amplitude at large times. On the contrary, here we propose an example of nonlinearity becoming important for decreasing amplitudes and that is induced by dry friction in a simple pendulum initially perturbed out of its stable equilibrium position (Butikov, 2015):

$$
\begin{equation*}
\ddot{x}+2 \sigma_{n} \dot{x}+\omega_{n}^{2} x+\Delta \operatorname{sgn} x=0, \quad x(0)=x^{i}, \quad \dot{x}(0)=\dot{x}^{i}(=0) . \tag{1.17}
\end{equation*}
$$

Assuming again small linear damping $\sigma_{n}=\epsilon \hat{\sigma}_{n}$ and small friction coefficient $\Delta=\epsilon \hat{\Delta}$, with $\hat{\sigma}_{n}$ and $\hat{\Delta}$ both of order $\sim \mathrm{O}(1)$, an asymptotic approximation can be obtained in the form of an
amplitude equation following Viola et al. (2018) and Dollet et al. (2020)

$$
\begin{equation*}
\frac{d A}{d t}=\lambda A+\chi A /|A| \tag{1.18}
\end{equation*}
$$

with $\lambda=-\sigma_{n}$ and $\chi=-\Delta / \pi \omega_{n}$.
Turning (1.18) into polar coordinates, $A=|A| e^{\mathrm{i} \Phi}$, allows one to readily obtain an analytical solution for the envelope module of $x_{0}=2|A| \cos \left(\omega_{n} t+\Phi\right)$,

$$
\begin{equation*}
|A(t)|=\left[-\frac{\Delta}{\pi \sigma_{n} \omega_{n}}+\left(\frac{x_{0}}{2}+\frac{\Delta}{\pi \sigma_{n} \omega_{n}}\right) e^{-\sigma_{n} t}\right] . \tag{1.19}
\end{equation*}
$$

Solution (1.19) suggests that the nonlinear term in (1.18) is such that there exist a finite time, $t=t^{*}$, for which the the motion arrests irreversibly as $\left|A\left(t=t^{*}\right)\right|$ becomes zero,

$$
\begin{equation*}
t^{*}=\frac{1}{\sigma_{n}} \log \left(\frac{\pi \sigma_{n} \omega_{n} x_{0}+2 \Delta}{2 \Delta}\right) \tag{1.20}
\end{equation*}
$$

a feature that has already been described in figure 1.6. This simple pendulum analogy can be used to model the nonlinear relaxation dynamics of small amplitude liquid oscillations induced by contact angle hysteresis (Dollet et al., 2020; Viola et al., 2018). In figure 1.7(c), after fitting coefficients $\sigma_{n}$ and $\chi$, the asymptotic prediction (1.19) is compared with the measurements by (Dollet et al., 2020) of the relaxation dynamics of liquid oscillations in a $U$-shaped tube and it is indeed shown to be in fairly good agreement.

### 1.2.2 Generalization to large systems: the emergence of secular terms and the imposition of a solvability condition via Fredholm alternative ${ }^{1}$

In the previous section, by considering a few archetypal one-degree-of-freedom systems, we have discussed the asymptotic breakdown provoked by the emergence of secular terms in the straightforward weakly nonlinear expansion. Specifically, we have shown how the employment of the multiple time-scales method, by assuming a slow time amplitude modulation of the perturbation, naturally leads to the imposition of a solvability condition that eliminates secular terms and prescribes the perturbation amplitude to obey a given normal form, i.e. an amplitude equation.
In the following, we briefly discuss how the concepts of the emergence of secular terms and the imposition of a solvability condition are generalizable to large systems.
To this end, let us seek an asymptotic solution of, e.g., the Navier-Stokes equation (1.1) also subjected to an external and time-dependent body or boundary force, $\mathbf{f}(\mathbf{x}, t)$,

$$
\begin{equation*}
\mathbf{q}(\mathbf{x}, t)=\mathbf{q}_{0}(\mathbf{x})+\epsilon \mathbf{q}_{1}(\mathbf{x}, t)+\epsilon^{2} \mathbf{q}_{2}(\mathbf{x}, t)+\ldots+\epsilon^{k} \mathbf{q}_{k}+\mathrm{O}\left(\epsilon^{k+1}\right), \quad\|\mathbf{f}(\mathbf{x}, t)\| \sim \epsilon^{k} \tag{1.21}
\end{equation*}
$$

[^2]where $\mathbf{q}_{0}$ represents an equilibrium solution (or steady base flow) of (1.1) and where the amplitude of the external forcing $\|\mathbf{f}(\mathbf{x}, t)\|$ is assumed small of order $\epsilon^{k}$. With the aim of giving a pedagogical example, in the following let us consider $k=3$. After linearization of the governing equations around $\mathbf{q}_{0}$, the $\epsilon$-order problem generally takes the form of a linear homogeneous problem, such as
\[

$$
\begin{equation*}
\frac{\partial \mathbf{q}_{1}}{\partial t}=\mathscr{L} \mathbf{q}_{1} . \tag{1.22}
\end{equation*}
$$

\]

As already discussed at the beginning of this introduction, one can then seek eigensolutions of (1.22) in the standard normal form

$$
\begin{equation*}
\mathbf{q}_{1}(\mathbf{x}, t)=A(T) \hat{\mathbf{q}}_{1 n}(\mathbf{x}) e^{\lambda_{n} t}+\text { c.c., } \quad T=\epsilon^{2} t, \tag{1.23}
\end{equation*}
$$

where $\hat{\mathbf{q}}_{1 n}$ is the $n$th eigenmode and $\lambda_{n}$ is the corresponding eigenvalue, solution of the generalized eigenvalue problem

$$
\begin{equation*}
\lambda_{n} \hat{\mathbf{q}}_{1 n}=\mathscr{L} \hat{\mathbf{q}}_{1 n} . \tag{1.24}
\end{equation*}
$$

The formalism of the multiple scales analysis requires the eigenvalue $\lambda_{n}=\sigma_{n}+\mathrm{i} \omega_{n}$ to be marginally stable (Godrèche and Manneville, 1998). How to relax this constraint to cases where the growth or decay rate $\sigma_{n}$ is much smaller than $\omega_{n}$ is discussed, e.g., in Meliga et al. (2009a). However, for the sake of simplicity, we assume hereinafter the marginal stability condition, i.e. $\sigma_{n}=0$ and $\lambda_{n}=\mathrm{i} \omega_{n}$. Note that, in the spirit of the multiple time-scales method, the perturbation amplitude $A(T)$ has been assumed to depend on the slow time scale $T$, as defined in (1.23); the use of the partial derivative symbol in (1.22) anticipated the decomposition of the physical time into two different time scales.
Before moving forward, let us also introduce an inner product, e.g. the Hermitian scalar product $\left\langle\hat{\mathbf{a}}, \hat{\mathbf{b}}>=\hat{\mathbf{a}}^{H} \hat{\mathbf{b}}\right.$, with $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ two generic vectors and the superscript $H$ denoting the Hermitian transpose. With respect to the considered scalar product, we can define an adjoint operator of $\mathscr{L}$, namely $\mathscr{L}^{\dagger}$,

$$
\begin{equation*}
\lambda_{m}^{\dagger} \hat{\mathbf{q}}_{1 m}^{\dagger}=\mathscr{L}^{\dagger} \hat{\mathbf{q}}_{1 m}^{\dagger} \tag{1.25}
\end{equation*}
$$

such that $\hat{\mathbf{q}}_{1 m}^{\dagger}$ and $\lambda_{m}^{\dagger}$ are, respectively, the $m$ th adjoint eigenmode and adjoint eigenvalue. Particularly, the direct, $\hat{\mathbf{q}}_{1 n}$ and adjoint, $\hat{\mathbf{q}}_{1 m}^{\dagger}$, eigenmodes form a bi-orthogonal basis, meaning that $\left\langle\hat{\mathbf{q}}_{1 m}^{\dagger}, \hat{\mathbf{q}}_{1 n}\right\rangle=\delta_{n m}$, with $\delta_{n m}$ the Kronecker delta. Hence, for $\left.n=m,<\hat{\mathbf{q}}_{1 n}^{\dagger}, \hat{\mathbf{q}}_{1 n}\right\rangle=1$ and $\lambda_{m}^{\dagger}=\bar{\lambda}_{n}\left(=-\mathrm{i} \omega_{n}\right.$ in the case here considered $)$.
The problem at order $\epsilon^{3}$ will typically take the form of an inhomogeneous linear problem, where the right-hand side contains forcing terms produced by the weakly nonlinear interactions of the previous order solutions and by the external body or boundary forces, e.g. a time-harmonic $\mathbf{f}(\mathbf{x}, t)=f \hat{\mathbf{f}}(\mathbf{x}) e^{\mathrm{i} \Omega t}+c . c$., whose amplitude $f$ has been assumed to be small of order $\epsilon^{3}$ and whose oscillation frequency is close to the natural frequency $\Omega \approx \omega_{n}$,

$$
\begin{equation*}
\frac{\partial \mathbf{q}_{3}}{\partial t}-\mathscr{L} \mathbf{q}_{k}=-\frac{\partial \mathbf{q}_{1}}{\partial T}+\mathscr{N}\left(\mathbf{q}_{0}, \mathbf{q}_{1}, \mathbf{q}_{2}, \ldots\right)+\mathbf{f}(\mathbf{x}, t)=-\frac{\partial \mathbf{q}_{1}}{\partial T}+\mathscr{F}(A, f, \ldots) . \tag{1.26}
\end{equation*}
$$

The forcing term $\mathscr{F}(A, f, \ldots)$ is generally a function of the perturbation amplitude, forcing amplitude, etc. We also notice that the right-hand side contains a forcing term associated with
the slow time derivative of the leading order perturbation $\mathbf{q}_{1}$. Let us suppose now a Fourier decomposition of the time-dependent forcing term $\mathscr{F}(A, f, \ldots)$ into a component gathering all the resonant terms oscillating at the natural frequency $\omega_{n}, \mathscr{F}_{\text {RT }}(A, f, \ldots)$ and a second component gathering all the non-resonant terms, $\mathscr{F}_{\mathrm{NRT}}(A, f, \ldots)$, which are not relevant for the further analysis and will be therefore simply ignored, so that (1.26) reduces to

$$
\begin{equation*}
\frac{\partial \mathbf{q}_{k}}{\partial t}=\mathscr{L} \mathbf{q}_{k}+\left(-\frac{\partial \mathbf{q}_{1}}{\partial T}+\mathscr{F}_{\mathrm{RT}}(A, f, \ldots)\right), \quad \mathscr{F}_{\mathrm{RT}}=\hat{\mathscr{F}}(A, f, \ldots) e^{\mathrm{i} \omega_{n} t}+c . c . \tag{1.27}
\end{equation*}
$$

subjected to a certain initial condition, e.g. $\mathbf{q}_{k}=\mathbf{0}$ at $t=0$.
In the most general form, the response of the system in time can be written by using the exponential matrix $e^{\mathscr{L} t}$, such that

$$
\begin{align*}
\mathbf{q}_{3}(\mathbf{x}, t) & =e^{\mathscr{L} t} \int_{0}^{t} e^{-\mathscr{L} s}\left[\left(-\frac{\partial A}{\partial T} \hat{\mathbf{q}}_{1 n}+\hat{\mathscr{F}}_{\mathrm{RT}}(A, f, \ldots)\right) e^{\mathrm{i} \omega_{n} s}\right] \mathrm{d} s=  \tag{1.28}\\
& =\left(-\frac{\partial A}{\partial T} \hat{\mathbf{q}}_{1 n}+\hat{\mathscr{F}}_{\mathrm{RT}}(A, f, \ldots)\right) e^{\mathscr{L} s} \int_{0}^{t} e^{-\mathscr{L} s} e^{\mathrm{i} \omega_{n} s} \mathrm{~d} s+c . c .
\end{align*}
$$

The exponential matrix can be decomposed as

$$
\begin{equation*}
e^{\mathscr{L} s}=\mathscr{Q} e^{\mathscr{D} s} \mathscr{Q}^{-1} \tag{1.29}
\end{equation*}
$$

where the matrix $\mathscr{Q}$ contains the eigenmodes of $\mathscr{L}$, whereas the diagonal matrix $\mathscr{D}$ contains the corresponding eigenvalues of $\mathscr{L}$, i.e. $\mathscr{D}=\operatorname{diag}\left(\mathrm{i} \omega_{n},-\mathrm{i} \omega_{n}, \lambda_{l}, \bar{\lambda}_{l}, \ldots\right)$ (with $\left.l \neq n\right)$. Hence,

$$
\begin{equation*}
e^{\mathscr{L} t} e^{-\mathscr{L} s} e^{\mathrm{i} \omega_{n} s}=\operatorname{diag}\left(e^{\mathrm{i} \omega_{n} t}, e^{-\mathrm{i} \omega_{n}(t-2 s)}, e^{\lambda_{l}(t-s)+\mathrm{i} \omega_{n} s}, e^{\bar{\lambda}_{l}(t-s)+\mathrm{i} \omega_{n} s}, \ldots\right) \tag{1.30}
\end{equation*}
$$

Using the decomposition (1.29), one can express

$$
\begin{equation*}
\int_{0}^{t} e^{\mathscr{L} t} e^{-\mathscr{L} s} e^{\mathrm{i} \omega_{n} s} \mathrm{~d} s= \tag{1.31}
\end{equation*}
$$

$$
=\mathscr{Q}\left[\begin{array}{ccccc}
t e^{\mathrm{i} \omega_{n} t} & 0 & 0 & 0 & \cdots \\
0 & \frac{1}{\mathrm{i} 2 \omega_{n}}\left(e^{\mathrm{i} \omega_{n} t}-e^{-\mathrm{i} \omega_{n} t}\right) & 0 & 0 & \cdots \\
0 & 0 & \frac{1}{\mathrm{i} \omega_{n}-\lambda_{l}}\left(e^{\mathrm{i} \omega_{n} t}-e^{\lambda_{l} t}\right) & 0 & \cdots \\
0 & 0 & 0 & \frac{1}{\mathrm{i} \omega_{n}-\bar{\lambda}_{l}}\left(e^{\mathrm{i} \omega_{n} t}-e^{\bar{\lambda}_{l} t}\right) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right] \mathscr{Q}^{-1},
$$

so that,

$$
\begin{align*}
\mathbf{q}_{3} & =\left(-\frac{\partial A}{\partial T} \hat{\mathbf{q}}_{1 n}+\hat{\mathscr{F}}_{\mathrm{RT}}(A, f, \ldots)\right) \int_{0}^{t} e^{\mathscr{L} t} e^{-\mathscr{L} s} e^{\mathrm{i} \omega_{n} s} \mathrm{~d} s+c . c .=  \tag{1.32}\\
& =\underbrace{\left(\hat{\mathbf{q}}_{1 n} \frac{<\hat{\mathbf{q}}_{1 n}^{\dagger},\left(-\frac{\partial A}{\partial T} \hat{\mathbf{q}}_{1 n}+\hat{\mathscr{F}}_{\mathrm{RT}}(A, f, \ldots)\right)>}{<\hat{\mathbf{q}}_{1 n}^{\dagger}, \hat{\mathbf{q}}_{1 n}>} e^{\mathrm{i} \omega_{n} t} t+c . c .\right)}_{\text {secular terms: linearly growing in time } \alpha t}+\mathrm{oscillating,} \tag{1.33}
\end{align*}
$$

in which we have used the fact that $\mathscr{Q}=\left(\hat{\mathbf{q}}_{1 n}, \hat{\mathbf{q}}_{1 m}, \ldots\right)^{T}$ and, therefore,

$$
\mathscr{Q}^{-1}\left(-\frac{\partial A}{\partial T} \hat{\mathbf{q}}_{1 n}+\hat{\mathscr{F}}_{\mathrm{RT}}(A, f, \ldots)\right)=\left[\begin{array}{c}
\frac{\left\langle\hat{\mathbf{q}}_{1 n}^{\dagger},\left(-\frac{\partial A}{\partial T} \hat{\mathbf{q}}_{1 n}+\hat{\mathscr{F}}_{\mathrm{RT}}(A, f, \ldots)\right)>\right.}{\left\langle\hat{\mathbf{q}}_{1 n}^{\dagger}, \hat{\mathbf{q}}_{1 n}\right\rangle}  \tag{1.34}\\
\frac{\left\langle\hat{\mathbf{q}}_{1 m}^{\dagger},\left(-\frac{\partial A}{\partial T} \hat{\mathbf{q}}_{n 1 n}+\hat{\mathscr{F}}_{\mathrm{RT}}(A, f, \ldots)\right)>\right.}{\left\langle\hat{\mathbf{q}}_{1 m}^{\dagger}, \hat{\mathbf{q}}_{1 m}>\right.} \\
\vdots
\end{array}\right],
$$

since

$$
\begin{array}{r}
\mathscr{Q} \mathscr{Q}^{-1}\left(-\frac{\partial A}{\partial T} \hat{\mathbf{q}}_{1 n}+\hat{\mathscr{F}}_{\mathrm{RT}}(A, f, \ldots)\right)=\sum_{n} \hat{\mathbf{q}}_{1 n} \frac{\left\langle\hat{\mathbf{q}}_{1 n}^{\dagger},\left(-\frac{\partial A}{\partial T} \hat{\mathbf{q}}_{1 n}+\hat{\mathscr{F}}_{\mathrm{RT}}(A, f, \ldots)\right)>\right.}{\left\langle\hat{\mathbf{q}}_{1 n}^{\dagger}, \hat{\mathbf{q}}_{1 n}>\right.}=  \tag{1.35}\\
=\left(-\frac{\partial A}{\partial T} \hat{\mathbf{q}}_{1 n}+\hat{\mathscr{F}}_{\mathrm{RT}}(A, f, \ldots)\right),
\end{array}
$$

according to the bi-orthogonality property of direct and adjoint modes.
Lastly, from (1.32), it appears clear that avoiding an algebraic growth implies requiring that

$$
\begin{equation*}
\frac{<\hat{\mathbf{q}}_{1 n}^{\dagger},\left(-\frac{\partial A}{\partial T} \hat{\mathbf{q}}_{1 n}+\hat{\mathscr{F}}_{\mathrm{RT}}(A, f, \ldots)\right)>}{<\hat{\mathbf{q}}_{1 n}^{\dagger}, \hat{\mathbf{q}}_{1 n}>}=0 . \tag{1.36}
\end{equation*}
$$

which is equivalent to asking that the forcing term must be orthogonal to the cokernel of $\mathscr{L}$, or, alternatively said, to the kernel of the adjoint operator $\mathscr{L}^{\dagger}$, as stated by the Fredholm alternative (Olver, 2014a).

The imposition of a solvability condition through the Fredholm alternative eventually fixes the arbitrariness introduced by the perturbation amplitude by prescribing a governing equation for $A(T)$, which constitutes our final amplitude equation:

$$
\begin{equation*}
\frac{\partial A}{\partial T}=\frac{1}{\epsilon^{2}} \frac{d A}{d t}=\frac{<\hat{\mathbf{q}}_{1 n}^{\dagger}, \hat{\mathscr{F}}_{\mathrm{RT}}(A, f, \ldots)>}{<\hat{\mathbf{q}}_{1 n}^{\dagger}, \hat{\mathbf{q}}_{1 n}>} \Longrightarrow \frac{d A}{d t}=F(A, f, \ldots) \tag{1.37}
\end{equation*}
$$

As a side comment, we note that in our starting point (1.22), we have implicitly assumed that the mass matrix $\mathscr{M}$ coincides with the identity matrix $\mathscr{I}$. In general, $\mathscr{M} \neq \mathscr{I}$ and $\mathscr{M}$ enters in the definition of the inner product, $<\hat{\mathbf{a}}, \mathscr{M} \hat{\mathbf{b}}>=\hat{\mathbf{a}}^{H}(\mathscr{M} \hat{\mathbf{b}})$.
Lastly, it appears now clearer as the single-degree-of-freedom systems previously examined constitutes the trivial limit of equation (1.36). Indeed, by taking $\hat{\mathbf{q}}_{1 n}=\hat{\mathbf{q}}_{1 n}^{\dagger}=1$, equation (1.36) simply means requiring that the resonant forcing terms are zero, i.e. $\frac{\partial A}{\partial T}-\hat{\mathscr{F}}_{\text {RT }}(A, f, \ldots)=0$.

### 1.2.3 In this thesis: derivation of normal form coefficients from first principles

In this section, we have introduced the multiple time-scale method, and we have illustrated, using a few single-degree-of-freedom archetypal examples, how to derive envelope equations for these systems. A generalization of the method to large systems has been then briefly discussed. Throughout this thesis, envelope equations (and their coefficients) for a series


Figure 1.8 - Structure of the present document. This thesis is divided into four main parts, each classified according to the type of the underlying oscillatory system's response: self-sustained, externally driven, externally driven, but parametrically and natural. Each chapter is devoted to the theoretical modelling and further understanding of these complex nonlinear fluid dynamics using the tools of linear stability and weakly nonlinear theories.
of complex fluidic oscillators and resonators will be formally derived (and computed) from first principles via weakly nonlinear multiple time-scales analyses of the full hydrodynamic systems. It will be shown that the weakly nonlinear dynamics of these oscillatory flows, ranging from self-sustained impinging-jets oscillators to driven sloshing-like resonator, is well described by slightly different and enriched versions of the envelope equations just introduced, i.e. (1.4), (1.13), (1.16) and (1.18), hence making possible the identification of the few degrees-of-freedom that are actually relevant to the overall dynamics.

### 1.3 Forewords

Despite the main focus on fundamental physics questions, the problems tackled in this thesis are directly relevant to several industrial applications. While in many engineering problems, as those of figure $1.1(\mathrm{~b}, \mathrm{~d}, \mathrm{f})$, oscillatory instabilities and resonances are seen as endangering features to be avoided at all costs, resulting in entire parametric regions to be avoided or in the need for efficient control and mitigation strategies, the examples discussed in this document, like many others, illustrate a different view: self-sustained or driven oscillations can be indeed harnessed for the design of a wide variety of engineering devices, ranging from microfluidic circuitry (hydrodynamic converters or switching devices), orbital-shaken bioreactors for cell cultivation and drug production to liquid-based template for the assembly of microscale materials. A proper predictive understanding and modelling of the hydrodynamic at stake is therefore essential in the design of all these processes.
With the support of existing and home-made experimental observations and measurements (see figure 1.9), the present research aims precisely at modelling and providing comprehensive
theoretical frameworks capable of rationalising some of these complex nonlinear oscillatory dynamics, most of which have not been fully elucidated yet.
As amplifier-like systems have not been studied in this thesis, let us recall the distinction made between oscillators and resonators on the basis of the nature of their oscillatory responses:

- Oscillators:
- Self-sustained oscillations
- Resonators:
- Driven oscillations
- Parametric oscillations
- Natural oscillations

Keeping in mind the distinction made above, the present document is organised as in figure 1.8 and figure 1.9. The thesis contains published or submitted material carried out in collaboration with other experienced researchers and my supervisor, to which I fundamentally contributed. If a Chapter contains published material where I do not appear as the first author, my personal contribution is explicitly specified at the beginning of the Chapter.

In the following, a general outline of this thesis with a short description of each part is provided, whereas more detailed and dedicated introductions are given at the beginning of each part.

## PART I Self-sustained oscillations

## Chapters 2-3: Feedback-free fluidic oscillators based on impinging jets

In Chapter 2, we describe a microfluidic oscillator based on facing impinging jets and operating in laminar flow conditions. Using appropriate cross-junction configurations with two intersecting inlets and outlets, pulsatile liquid flows are experimentally generated at the microscale from steady and equal inlet flow conditions and without moving parts or external stimuli. Experiments and DNS are used to determine the region in the control parameter space (device's geometry and Reynolds number, $R e$ ) where self-sustained oscillations manifest.

To better elucidate the physical mechanism behind these oscillations, in Chapter 3, we consider a simplified two-dimensional configuration. Advances in the understanding of such a mechanism are made by performing linear global stability and sensitivity analysis, which identify the Kelvin-Helmholtz instability, located in the jet's interaction region, as the main candidate for the origin of the oscillations observed in fluidic devices. Further interesting nonlinear flow features, involving symmetry-breaking and subcritical transitions, are also described by means of the weakly nonlinear theory.

## PART II Driven oscillations

Chapter 4-5-6: Harmonic and super-harmonic sloshing dynamics of orbital-shaken cylindrical reservoirs

The container motion along a planar circular trajectory at a constant angular velocity, i.e. circular shaking, is of interest in several industrial applications, e.g. for fermentation processes or in the cultivation of stem cells, where good mixing and efficient gas exchange are the main targets. Under this external forcing condition, the system always responds with a swirling wave co-directed with the container's direction of motion. Depending on the driving frequency and amplitude, the frequency response can be either harmonic or super-harmonic. In Chapter 4, existing experimental data are used to develop a weakly nonlinear model capable of describing the fundamental harmonic and super-harmonic resonances in terms of flow patterns and amplitude response.
From the perspective of hydrodynamic instability, the case of longitudinal container motions, i.e. longitudinal shaking, appears more interesting. In this configuration, the system exhibits a richer variety of wave regimes, such as planar, irregular and swirling motions. In Chapter 5, we extend the weakly nonlinear model previously developed in order to study harmonic and super-harmonic resonances under these forcing conditions. Our theoretical predictions are confirmed by dedicated experiments.
Lastly, in Chapter 6, with the main focus on harmonic resonances, we provide an experimental characterisation of the free liquid surface response for a generic, elliptic periodic container trajectory, i.e. elliptical shaking, so as to bridge the gap between the two diametrically opposed shaking conditions previously discussed. Experiments demonstrate for the first time the counter-intuitive existence of stable swirling waves travelling in the opposite direction of the container motion. These findings are then rationalized by using a slight variation of the theoretical tools developed in Chapters 4 and 5.

## PART III Parametric oscillations

## Chapter 7-8: Sub-harmonic Faraday waves in circular cylinders and thin annuli

In this Part, we consider the problem of Faraday waves, undoubtedly the most famous parametric resonator system in fluids. In particular, we tackle two very different system configurations, but which are linked to each other for the importance of the lateral wall and contact line boundary conditions.
In Chapter 7, we focus on the problem of the coupling and interaction of parametric waves and capillary meniscus waves, the latter being typically unwanted. Their suppression of the latter can be achieved by imposing a contact line pinned at the container brim. However, tunable meniscus waves are desired in some applications such as those of liquid-based biosensors, where they can be controlled by adjusting the shape of the static meniscus by slightly under/overfilling the vessel while keeping the contact line fixed at the brim. Considering this contact line condition in cylindrical containers, we formalize a weakly nonlinear analysis
which predicts the impact of static contact angle effects on the instability onset of viscous sub-harmonic Faraday waves. The theory is validated with previous experiments and DNS.
In Chapter 8 we instead consider the case of Faraday waves in Hele-Shaw cells, for which previous theoretical analyses typically rely on the Darcy approximation based on the parabolic flow profile assumption in the narrow direction. However, Darcy's model is known to be inaccurate whenever inertia is not negligible, e.g. in unsteady flows. In this work, we propose a revised gap-averaged linear model that accounts for inertial effects induced by the unsteady terms in the Navier-Stokes equations. The theory also includes a linear law for the dynamic contact angle that serves to reintroduce the contact line dissipation. The latter is indeed seen to be a critical contribution to the overall dissipation of the system. The stability of the system is studied by performing a Floquet analysis, whose predictions compare well with previous experiments in rectangular Hele-Shaw cells and with new dedicated experiments in thin annuli.

## PART IV Natural oscillations

Chapter 9-10: Nonlinear relaxation dynamics of free surface oscillations due to contact angle hysteresis

In Chapter 9, we present a physics-inspired mathematical model based on successive linear eigenmode projections to solve the relaxation (natural dynamics) of small-amplitude and twodimensional viscous capillary-gravity waves with a phenomenological and experimentallyinspired nonlinear contact line model accounting for Coulomb solid-like friction. We show that each projection eventually induces a rapid loss of total energy in the liquid motion and contributes to its nonlinear damping. This approach captures the transition from a contact line stick-slip (or nearly stick-slip) motion to a pinned (or nearly pinned) configuration, as well as the secondary fluid bulk motion following the arrest of the contact line, overlooked by previous asymptotic analyses.
In Chapter 10, the projection model formalized in Chapter 9 is applied to the more realistic case of liquid oscillation in a U-tube configuration. A comparison with existing experiments proves the predictive power of this method, although a fitting parameter is still required owing to the lack of information about the actual contact line dynamics.

See figure 1.9 for a visual illustration of the salient points pertaining to each Part.


Self-Sustained Oscillations
PART II : Harmonic and super-harmonic sloshing dynamics of orbital-shaken cylindrical reservoirs
Driven Oscillations

PART III : Sub-harmonic Faraday waves in circular cylinders and thin annuli


PART IV : Relaxation dynamics of liquid oscillations due to contact line nonlinearity


Figure 1.9 - Visual illustration of the salient points pertaining to the main Parts of this thesis. The classification is based on the nature of the fluid oscillations. Sketches, representing the various geometrical configurations considered, are given on the right, whereas a few examples of homemade experimental outputs are given on the left. Theoretical models have been built on the basis of these observations.

Feedback-free fluidic oscillators based Part I on impinging jets

## Introduction

Fluidic oscillators are devices that issue an oscillating jet of fluid when supplied with a continuous stream of pressurized gas or liquid; as such, they can be seen as fluidic DC/AC converters. They started to be studied in the 1960s, as well as other fluidic devices functioning with no moving parts, such as fluidic logic elements or fluidic amplifiers (Angrist, 1964; Glaettli, 1964; Tanney, 1970). There are two main types of fluidic oscillators: wall-attachment devices and jetinteraction devices (see figure I.1). The wall-attachment oscillators are based on the Coandă effect, where the fluid jet interacts with an adjacent wall, which results in its deflection. The jet-interaction devices also named "feedback-free" devices are based on the interactions of two jets inside an interaction chamber having a specific geometry (Raghu, 2001).

Only a few industrial applications of fluidic oscillators have emerged over the years, such as flow metering (Beale and Lawler, 1974) and windshield washer devices (Stouffer, 1985), however, with the development of microfluidics and its applications to lab-on-chip devices, a renewed interest for fluidic devices appeared, and in particular for fluidic devices with no moving parts, such as static micromixers (Bertsch et al., 2001) or fluidic diodes (Anduze et al., 2001; Fani et al., 2013; Haward et al., 2016).
Most research work performed so far on microfluidic oscillators operating with liquids aims either at the study of new types of static micromixers or at the implementation of fluidic logic circuits. A small number of such fluidic oscillators have been described in the scientific literature, and implement a variety of working principles.
Notwithstanding active microfluidic oscillators have been studied (Niu and Lee, 2003), here we mainly focus on passive oscillators, where a constant liquid flow is applied at the inlets and oscillations are generated by the design of the microfluidic network. One of the most studied types of microfluidic oscillators is based on the use of fluidic resistors, capacitors and valves, and uses the analogy between the electrical and fluidic domains, where voltage is replaced by pressure and electrical current is replaced by hydraulic volume flow. The microfluidic equivalents of electrical resistors are channels, microfluidic capacitors are chambers with membranes that store energy by membrane deformation, while equivalents of diodes, and transistors, are valves of diverse designs that can completely shut off the flow in given conditions. Based on this electronic-fluidic analogy, a fluidic astable multivibrator driven by a constant pressure flow was described by Lammerink et al. (1995). This concept was further developed later, taking advantage of the versatility of the microfabrication methods based on the use of polydimethylsiloxane (PDMS), an elastomeric material that renders the fabrication


Figure I. 1 - (a) A fluidic oscillator based on wall-attachment. The feedback mechanism is provided by two feedback channels. Four snapshots of the first half of the oscillation cycle are shown: $\varphi$ denote the corresponding flow phase-angle. Visualization as streamlines numerically computed (modified figure from Woszidlo et al. (2015)). (b) A fluidic oscillator based on two jets interacting within a mixing chamber. Four snapshots of the first half of the oscillation cycle are shown as streamlines numerically computed (modified figure from Tomac and Gregory (2014)).
of fluidic networks containing membranes very simple. Mosadegh et al. (2010) demonstrated a microfluidic oscillator and used it to perform flow-switching and clocking functions. Kim et al. $(2011,2013)$ fabricated a number of devices based on this type of microfluidic oscillator,
among which a micromixer and an autonomous pulsed flow generation system capable of generating on-demand and independently a range of flow rates and a range of flow oscillation frequencies (Li and Kim, 2017). (Kim et al., 2015) applied it in studying endothelial cell elongation response to fluidic flow patterns. Devaraju and Unger (2012) also demonstrated a fluidic oscillator, among many other fluidic logic functions and Nguyen et al. (2012) performed peristaltic pumping on chip using a control signal generated on chip through a fluidic oscillator circuit.
Xia et al. (2012) also developed a micromixer based on a vibrating elastomeric diaphragm trapped in a two-level cavity. Here, there is no need for a complex fluidic circuit as the deformation of the diaphragm directly creates the oscillating liquid flow, but the wear of the elastomeric material limited the use of this device. Simpler microfluidic oscillators containing no moving parts, no deformable membranes and no complex fluidic circuit have also been studied by several authors. These oscillators are based on jets interacting in a simple cavity and generating an oscillating flow (Gregory et al., 2007; Tomac and Gregory, 2012). Yang et al. (2007) demonstrated that feedback-driven microfluidic oscillators based on the Coandă effect can generate an oscillatory liquid flow at small Reynolds numbers. Their design used a micro-nozzle with a sudden expansion and asymmetric feedback channels and measured oscillatory frequencies of the flow below 1 Hz for Reynolds numbers between 1 and 100. Similar oscillator designs were later studied experimentally by Xu and Chu (2015) to develop feedback micromixers based on the Coandă effect. They demonstrated that there were three different oscillating mechanisms that resulted in mixing in such structures, depending on the magnitude of the Reynolds number: vortex mixing, internal recirculation mixing, and oscillation mixing. Xie and Xu (2017) simulated the fluidic behaviour of such devices using the Fluent $®$ CFD software.
Finally, Sun et al. (2017); Sun and Sun (2011) studied liquid mixing resulting from a microfluidic oscillator using an impinging jet on a concave semi-circular surface. This type of microfluidic oscillator is another example of the use of the Coandă effect. Oscillations were observed for Reynolds numbers as low as 70 , with the frequency of oscillations below 1 Hz .

In Chapter 2, we present a microfluidic oscillator that can be classified in the jet-interaction device category. It has a very simple configuration, i.e. X-shaped cross-junction where two incoming streams meet head-to-head and exhaust into two outlet channels (see figure I.2). Its oscillations depend on the jet interactions more than on the shape of the surrounding cavity. This device is based on facing impinging liquid jets and operates in laminar flow conditions. Observations of flow patterns obtained with micromixers having geometries similar to the ones presented in this paper but much larger dimensions were performed by Tesař (2009), however, the manufacturing method of these devices limited their aspect ratios and allowed to perform observations of only a limited part of the phenomenon. Impinging self-oscillating jets have been described in scientific literature by Denshchikov et al. (1978) using facing turbulent water jets, having dimensions in the centimeter range immersed in a 230 L water tank. In a follow-up paper (Denshchikov et al., 1983), the period of the auto-oscillating phenomenon
was empirically described by a set of equations. If the phenomena described in this work show some similarities with the jet configuration presented in (Denshchikov et al., 1978, 1983), the jets dimensions are orders of magnitude smaller and the flow conditions remain laminar (Lashgari et al., 2014).
We provide a detailed experimental and numerical description of the self-sustained oscillatory regime and we study the evolution of the self-oscillation frequency when the main geometric parameters of the cavity are changed. Most interestingly, the shedding frequency is shown to be proportional to the averaged flow velocity imposed at the symmetric inlets and inversely proportional to the distance between the jets, irrespective of many other parameters, such as channel width, depth, length, Reynolds number, etc.


Figure I.2- (a) Snapshot of the self-sustained oscillatory flow obtained experimentally and (c) numerically (via DNS) in our flaring X-junction (Center) for a Reynolds number greater than the critical one, $R e_{c r}$. See also https://doi.org/10.1103/APS.DFD.2019.GFM.V0036 for a Gallery of Fluid Motion award-winning video). The actual device is shown in (b), while an example of a computational domain for DNS is illustrated in (d).

By analogy with the cylinder flow discussed in Chapter 1, the oscillatory instability described in Chapter 2, is experimentally seen to be of supercritical nature with oscillations starting above a precise instability threshold, i.e. $R e>R e_{c r}$. Although several plausible candidates are proposed, at this stage no physical mechanism could be precisely identified from which the self-sustained oscillations would originate, thus calling for a significant interpretation effort.
Furthermore, cross-slot flows are also known to show hysteresis. For instance, Burshtein et al. (2019) experimentally showed that hysteretic behaviours due to symmetry-breaking transitions appear in X-junction flows with proper geometrical parameters, for which no oscillations are observed (see figure I.3). There are similarities in the microchannel geometries between the case described by Burshtein et al. (2019) and the one presented in this work, with microchannels crossing at right angle in both cases and liquid flows at relatively low values
of the Reynolds number. However, in the geometry considered by Burshtein et al. (2019) all channels have comparable dimensions, whereas here, there are two facing narrow channels which open into wider channels. Particularly, we observe oscillations only in the cases where the wider channels have dimensions at least 3 times larger than the narrow channels, which differs significantly from Burshtein et al. (2019). Such a consideration further underlines the importance of the distance separating the inlets in cross-slot geometries in the destabilization mechanism.


Figure I.3- Study of the vortex dynamics and interactions associated with a symmetry-breaking flow instability at a 4-way intersection (Burshtein et al., 2019). The merging and the splitting of vortices are connected with the symmetry-breaking transition and are affected by the degree of vortex confinement, i.e. by the geometry. (a) A schematic diagram of the experimental setup of Burshtein et al. (2019), which allows a direct observation of the $x=0$ plane on an inverted microscope. Inflow (along y) is indicated by the blue arrows, and outflow (along $x$ ) is indicated by the red arrows. (b) Schematic diagram of a vortex in the cross-slot device for flow at $R e>R e_{c r} ; d$ and $w$ are the channel depth and width, respectively. (c,e) $\mu$-PIV images of the vorticity field at $x=0$ : (c) a symmetric flow field with four cells of Dean vortices; (d) an asymmetric flow field where two intensified Dean vortices have commenced merging; and (e) a single steady, central streamwise vortex is formed by the merging of the two Dean vortices.

Hence, Chapter 3 aims at answering two main questions arising from different observations presented in Chapter 2 (Bertsch et al., 2020a,b) and by Burshtein et al. (2019): (i) to identify
the physical mechanism governing the self-sustained oscillatory regime studied in Bertsch et al. (2020a); (ii) to investigate the existence of a range of geometrical parameters in which steady symmetry-breaking conditions could directly interact with this dynamic instability.
With these objectives, we consider a two-dimensional (2D) X-junction with straight lateral channels and two symmetric inlets, where a fully developed flow is imposed, separated by a certain distance. Despite the simplistic geometry, a 2D flow not only allows one to perform a faster computational analysis but also often makes it possible to capture the main physical features of interest in the 3D problem. Particularly, since the main geometrical parameter, i.e. the distance between the two jets, is kept in this crude dimensional reduction from 3D to 2D, we may expect that our 2D analysis reveals the dominant physical mechanism behind the oscillatory instability observed in 3D. Steady symmetry-breaking instabilities are also expected in 2D (Liu et al., 2016; Pawlowski et al., 2006), even though their nature differs from the intrinsically 3D one presented in Burshtein et al. (2019). An exhaustive stability analysis is here conducted using the tools of the classic linear global stability and sensitivity analysis as well as the weakly nonlinear theory based on amplitude equations.
Precisely, global stability identifies a region of geometrical and flow parameters where two unstable global modes become simultaneously unstable. One of the modes is responsible for the self-oscillations, whereas the other induces a steady symmetry-breaking. The interaction of such modes is described by means of amplitude equations, whose resulting predictions are confirmed by DNS. The weakly nonlinear model also predicts an oscillation frequency that scales like $U / s$, with $U$ the mean inlet velocity and $s$ the distance between the inlets, hence confirming the two-dimensional nature of the oscillator-like dynamics. Lastly, the use of sensitivity analysis helps us in identifying the Kelvin-Helmholtz instability, located in the jet's interaction region, as the main candidate for the origin of the oscillations observed in these jet-interaction fluidic devices.

## 2 Feedback-free microfluidic oscillator with impinging jets

Remark: this chapter is largely inspired by the publication of the same name and by a short paper associated with a video winner of a 2019 American Physical Society's Division of Fluid Dynamics (DFD) Gallery of Fluid Motion Award for work presented at the DFD Gallery of Fluid Motion, available online at the Gallery of Fluid Motion:
https://doi.org/10.1103/APS.DFD.2019.GFM.V0036

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The present paper describes a microfluidic oscillator based on facing impinging jets and operating in laminar flow conditions. Using appropriate microchannel configurations, pulsatile liquid flows are generated at the microscale from steady and equal inlet flow conditions and without moving parts or external stimuli. An experimental campaign has been carried out, using oscillator structures manufactured in silicon using conventional microfabrication techniques. This allowed us to study in detail the impact of the main geometric parameters of these structures on the oscillation frequency. The observed range of regular oscillations was found to depend on the geometry of the output channels, with highly regular oscillations occurring over a very large range of Reynolds numbers ( $R e$ ) when an expansion of the output channel is added. The evolution of the self-oscillating frequency was shown to be dependent on the
distance separating the impinging jets and on the average speed of the jets. Direct numerical simulations (DNS) have been performed using a spectral element method. The computed dye concentration fields and non-dimensional self-oscillation frequencies compare well with the experiments. The simulations enable a detailed characterization of the self-oscillation phenomenon in terms of pressure and velocity fields.

### 2.1 Microfluidic devices, fabrication and experiment description

The oscillator structures presented in the present paper were fabricated using conventional microfabrication technologies in standard conditions. The fabrication process is very simple and did not require any particular development. It is based on the use of the Bosch process to create small components with a high aspect-ratio, but various other microfabrication processes could have been successfully used for fabricating such simple structures.
A 10 cm in diameter double-side polished silicon wafer was first bonded to a glass wafer by anodic bonding $\left(800 \mathrm{~V}, 420^{\circ} \mathrm{C}\right)$. The silicon part will be patterned to form the fluidic network, while the glass layer both supports these structures and will later allow observation using an inverted optical microscope. When necessary, the thickness of the silicon wafer was reduced by grinding. The silicon surface of the bonded wafers was then coated with a thick layer of positive photoresist (AZ9260, $10 \mu \mathrm{~m}$ ) and patterned by direct writing (MLA150, Heidelberg Instruments). The patterned silicon was etched using the Bosch process, until the glass layer is reached (Adixen AMS200, Alcatel Micro Machining Systems). During this step, the full thickness of the silicon wafer is etched, as well as part of the photoresist masking layer. The cavities created with the Bosch process will constitute the microchannels, inlets and outlets of the fluidic network. The remains of the photoresist mask are finally striped using $O_{2}$ plasma $(10 \mathrm{~min}, 500 \mathrm{~W}$ ) and the wafers are diced into chips. Each of the fabricated chips is closed by a 5 mm thick flat slab of polydimethylsiloxane (PDMS) in which inlet and outlet holes are made using a 0.75 mm in diameter puncher. The PDMS cover is placed on top of the silicon surface of the diced chips after submitting both components to an oxygen plasma, which results in an adequate bonding of the two components.

A schematic diagram of the design of the fabricated components is presented in figure 2.1. The liquid enters the device by two inlets and is pushed through long and narrow facing channels of width $w$ towards a wider transverse channel. The narrow entry channels, whose lengths are at least 2.3 mm each, act as two nozzles separated by a distance $s$ to create two facing liquid jets when they reach the larger lateral channel. Outlets are provided at both ends of the large channel, far away from the intersection. The outlet channel extends over the entire length of the manufactured chip and the liquid exits the chip at a distance of 8 mm from the facing nozzles. Within this geometry, the Reynolds number can be defined as:

$$
\begin{equation*}
R e=\frac{\rho U w}{\mu}, \tag{2.1}
\end{equation*}
$$



Figure 2.1 - Three-dimensional sketch of a general oscillator structure.
where $\rho$ is the fluid density, $\mu$ is its dynamic viscosity, $U$ is the average velocity of the liquid flow at the nozzles. In a certain range of Reynolds numbers, these colliding jets do self-oscillate transversally into the two output channels. Away from the nozzles, the width of the output channels quickly increases to a constant value $L$, in the most general case, but experiments have also been performed with two simple intersecting straight channels ( $L=s$ ). When an expansion of the outlet channel is provided $(L \neq s)$ the full width of the outlet channel is reached at a distance of 0.75 L away from the nozzle, and the wall profile in this area is a circular arc, tangent to the outlet channel wall and joining the nozzle. The height $h$ of the walls is constant for the whole device.
The microfluidic devices were placed with their glass side facing down on the stage of an inverted microscope. Fluidic connections were made through the PDMS top layer by inserting 0.79 mm in outer diameter PEEK tubes of equal length in the inlet holes. Deionized water colored with two different food dyes was pushed through the two inlet tubes using a syringe pump (PHD2000, Harvard apparatus). The syringe pump accommodated two identical syringes that were actuated simultaneously. The outlet holes on the PDMS cover were also fitted with PEEK tubes of identical length, and the liquid flow coming out of them was discarded. With the syringe pumps used, flow rates up to $20 \mathrm{~mL} / \mathrm{min}$ could be obtained in each of the entry channels, depending on the overall flow resistance of the studied microfluidic device. During experiments, the flow rates were changed abruptly, without ramps. Experiments were carried out in which the flow rates were first increased and later decreased, but no hysteresis in the evolution of the oscillation frequency with Reynolds was observed. Observations were made using a 10x microscope objective in bright field conditions, and recorded using a high-speed camera (Miro M 310, Phantom). The resolution, frame rate and gain of the camera were chosen for each experiment such that the frequency of the microfluidic oscillators could be clearly observed and measured using a large number of frames. As the microscope light source illuminates the complete microchannel height, the recorded light intensity provides a depth-averaged concentration field.

To evaluate the effect of the length of the entry channel on the oscillator behaviour and


Figure 2.2 - Images of the oscillating flow observed for increasing values of the Reynolds number for $h=525 \mu m, w=100 \mu m, s=800 \mu m, L=800 \mu m: a) R e=15$, no oscillations. b) $R e=23, F=27 \mathrm{~Hz}$, slow oscillations. c) $R e=31, F=43 \mathrm{~Hz}$, alternating jets. d) $R e=95$, Jets do not cross regularly. See movie S1 in the Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevFluids.5.054202.
make sure that the observed oscillations were not an artefact related to the inlet flow profile, multiple identical oscillator cavities differing only in the lengths of the inlet channel were manufactured, with an inlet channel varying between 0.45 mm and 8.35 mm in the length and an inlet width of $100 \mu m$ (a ratio between 4.5 and 83.5 respectively). The evolution of the frequency with $R e$ was measured in each configuration and showed no difference from chip to chip, indicating that the oscillator behaviour is not influenced by the inlet channel length, at least in the geometries investigated in the present paper. This justified conducting all other experiments with an inlet channel length of 2.3 mm .

### 2.2 Experimental results

### 2.2.1 Oscillations in simple straight channels and in channels with expansion

Figure. 2.2 shows images extracted from high-speed videos, visualizing the water flow colored with two food dyes in a structure made of straight channels crossing at right angle. Inlet channels are $w=100 \mu \mathrm{~m}$ in width, the two nozzles are $s=800 \mu \mathrm{~m}$ apart, outlet channels are $L=800 \mu \mathrm{~m}$ in width. The height of all channels is $h=525 \mu \mathrm{~m}$. There is no expansion of the output channels in this design $(s=L)$. For low values of the Reynolds number steady flow conditions are present (figure 2.2-a), the flow of both dyes is steady and the boundary between fluids is stable with time. When Re increases and reaches a value of about 20, the two flows start to oscillate in an antisymmetric way, with both jets first bifurcating in opposite directions, and later coming back towards one another until they collide and switch sides. Figure. 2.2-b shows oscillations at $R e=23$. They have a low frequency, are very regular temporally and spread widely in the lateral output channels. For larger values of $R e$, clear alternating arrowshaped jets oscillating very regularly can be observed (figure 2.2-c).
Their oscillating frequency increases with $R e$. When $R e$ reaches a threshold value of about $R e_{i r r}=90$, the regularity is lost and the flow evolves into a complex, irregular and aperiodic


Figure 2.3 - Images of the oscillating flow observed for different values of the Reynolds number for $h=525 \mu m, w=100 \mu m, s=800 \mu m, L=2000 \mu m$ : a) $R e=15$, no oscillations. b) $R e=23$, $F=32 \mathrm{~Hz}$ slow oscillations. c) $R e=31, F=53 \mathrm{~Hz}$ large oscillations resulting in a stretching and folding of the liquid flows. d) $R e=158, F=362 \mathrm{~Hz}$, fast oscillations with arrow-shaped jets, but resulting in an apparently less efficient mixing of the two liquids. See movie S2 in the Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevFluids.5.054202.
regime (figure 2.2-d). Little mixing occurs between the liquid coming from each of the two jets, and each output channel contains mostly the liquid originating from one of the jets only. From time to time oscillations of the jets do occur, but without following a regular temporal switching pattern.
Figure 2.3 illustrates the evolution of the flow with $R e$ in a configuration that is exactly the same as the one presented in figure 2.2, except for the exit channels that do present an expansion in their width: the two nozzles are still $800 \mu \mathrm{~m}$ apart, but the width of the exit channel quickly increases to $L=2 \mathrm{~mm}$. For low values of $R e$, Stokes flow conditions are observed (figure 2.3-a), with a steady boundary between the flows emerging from each inlet. When Re increases, symmetric oscillations still start to occur for a value of $R e$ of about 20. Figure 2.3-b shows oscillations observed for $R e=23$. When $R e$ is further increased, the oscillation frequency also increases. Large oscillations having dimensions similar to the distance between the jets are observed and result in a stretching and folding of the liquid flows (figure 2.3-c).

Regular oscillations of the two impinging jets were observed until $R e=630$, where the experiment was stopped as the used syringe pumps could not provide a larger flow rate. As shown in figure 2.3- $d$, high values of $R e$ induce fast oscillations of the two liquid flows, with arrow-shaped jets, but the oscillations lateral amplitude reduces. The stretching and folding of the fluid flow is of lesser magnitude than in the case of figure $2.3-c$, as most of the liquid issued from one nozzle is strongly pushed towards the opposite side of the exit channel. The


Figure 2.4 - Evolution of the oscillation frequency with $R e$ for the four channel designs presented in the insert, showing the same configuration except for the output channels that do present different expansions in their widths. For all designs, $h=380 \mu m, w=100 \mu m$, $s=800 \mu \mathrm{~m}$. The larger the output channel, the larger the range of $R e$ for which oscillations are stable. The dotted line is drawn only to guide the eye. The red arrows indicate the end of the stable oscillation regime for each value of the output channel width.
comparison of the oscillations resulting from identical designs with and without expansion in the exit channel shows that the threshold at which oscillations start in both cases, occurs for similar values of the Re number. The oscillation frequency observed is slightly higher when an expansion channel is present. More importantly, the impinging jets oscillate with high regularity for a much wider range of Reynolds numbers when an expansion of the exit channel is provided.
Figure 2.4 shows the evolution of the oscillation frequency for four oscillator designs having the same configuration ( $h=380 \mu m, w=100 \mu m, s=800 \mu m$ ) except that they present different widths in their output channels, as schematically presented in the figure insert. In all cases, the value of $R e$ at the threshold for which oscillations start is the same, but the larger the output channel, the larger the range of $R e$ for which regular operation can be maintained, i.e. $R e_{i r r}$ increases.

Moreover, the oscillation frequency at a given value of $R e$ is slightly smaller for designs presenting a smaller width in their output channel. If self-oscillations occur for low values of $R e$ in simple straight crossing channels of adequate dimensions, providing an expansion in the output channel allows to stabilize the oscillation mechanism and extends the oscillation regime over a wider range of $R e$, with only a minor effect on the frequency of oscillations. The extension of the oscillation regime between straight channels and channels presenting an expansion has been observed in all cases, regardless of the height $h$ of the oscillator structure.

### 2.2.2 Flow patterns created in the exit channels

Figure 2.5 shows the liquid flow close to the oscillator and further away laterally in one of the two output channels for $h=525 \mu m, w=100 \mu m, s=400 \mu m, L=2000 \mu m$. For low values of $R e$ (figure 2.5- $a$ and $b$ ), the amplitude of the oscillations is limited and smaller than the distance $s$ separating the two inlets. As the liquid is pushed in the expanding part of the exit channels, the pattern of the two fluids resulting from these oscillations is stretched along the channel width, resulting in temporal alternations of the fluids coming from the inlets. This appears as regularly spaced blue and red stripes of fluid in figure 2.5-a and $b$. When $R e$ increases, the oscillations become arrow-shaped jets of fluid, and the liquid issued from each nozzle is pushed towards the opposite side of the channel (figure 2.5-c). Further away in the exit channels, the fluid flow is rearranged but remains segmented in two parts because of the laminar flow conditions, each part showing a temporal alternation of both fluids with however unequal ratio with a prevalence in each branch of one fluid with respect to the other, but not with equal ratios (figure 2.5-d). If the temporal alternation of the fluid observed for low values of Re provides conditions of interest for microscale fluid mixing, it is not the case for the conditions created for larger values of $R e$, where the fluid flow remains segmented and only a limited mixing of the two fluids is expected in each of its parts. We have not further investigated the mixing efficiency from a quantitative point of view.

### 2.2.3 Evolution of the frequency with the oscillator geometry

The three main geometric parameters that may influence the self-oscillation phenomenon are the width of the jets $w$, the distance between the jets $s$ and the height of the device $h$. Figure 2.6$a$ shows the evolution of the frequency with $R e$, for different values of the width of the jet, all other geometric dimensions being identical across all devices ( $h=525 \mu m, s=500 \mu m$, $L=2000 \mu \mathrm{~m})$. For a given value of the jets width, the oscillation frequency increases with $R e$, and at a chosen value of $R e$, the oscillation frequency increases when the jet width decreases.
The threshold at which the oscillations appear is reached for smaller Re when the width of the jet is smaller. Colliding jets of identical design, having a width of $300 \mu \mathrm{~m}$ were also tested but oscillations of the fluid flows could not be observed. For the jets width of 150 and $200 \mu \mathrm{~m}$, the range of $R e$ where oscillations occur is limited: in both cases, the threshold where oscillations start is close to $R e=50$, and the flow stops to oscillate and gives way to a stable flow pattern similar to the one observed by Haward et al. Haward et al. (2016) when another threshold Re number is reached (in the order of $R e=250$ for $w=200 \mu m$ and $R e=290$ for $w=150 \mu m$ ). For smaller values of the width of the jet, regular and symmetric oscillations of the two impinging jets were observed until values of $R e$ close to 600 . Flow conditions for larger values of $R e$ could not be investigated as the syringe pumps used could not deliver larger flow rates. Oscillators having jets width smaller than $50 \mu \mathrm{~m}$ were not manufactured in the frame of this experiment, but other experiments we performed indicate that oscillations can be expected to occur for much smaller values of the width of the jet.
Figure 2.6 - $b$ shows the evolution of the frequency with $R e$, when the distance between the


Figure 2.5 - Images of the flow close to the oscillator (a and c) and further away laterally in one of the two output channels (b and d) for $h=525 \mu m, w=100 \mu m, s=400 \mu m, L=$ $2000 \mu \mathrm{~m}: a$ and $b$ ) correspond to $R e=35, F=120 \mathrm{~Hz}$, a temporal rearrangement of the fluid is observed. $c$ ) and $d$ ) correspond to $R e=47, F=165 \mathrm{~Hz}$, next to the oscillator a dead zone is visible where the fluid is stagnant. Further away, this dead zone gradually disappears but the fluid flow remains segmented into two parts, each part showing a temporal alternation of both inlet fluids, but not with equal ratios. See movie S3 in the Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevFluids.5.054202.
jets changes, all other geometric dimensions being identical across all devices ( $h=525 \mu \mathrm{~m}$, $w=100 \mu m, L=2000 \mu m$. For a chosen distance between the impinging jets, the oscillation frequency increases with $R e$, and at a chosen value of $R e$, the oscillation frequency increases when the distance between jets decreases. When the distance between the jets increases, the threshold at which the oscillations start, occurs for smaller values of Re. Oscillator geometries of identical design but having a distance of only $200 \mu \mathrm{~m}$ between the jets were also tested, but oscillations could not be observed with these devices. For a distance between the jets of $300 \mu m$, stable oscillations occur only in a limited range of $R e$, and stop when $R e$ is larger than 250 . When the distance between the jets is $2000 \mu m$, which corresponds to the full width of the exit channel, stable oscillations where the impinging jets alternate are also occurring in a limited range of $R e$. In this case, $s=L$, as it was the case in the experiments presented in figure 2.2 and 2.4, and the reduced range of oscillation frequencies observed is related to the absence of extension in the output channel as discussed previously.
Figure 2.6-c shows the evolution of the oscillation frequency with $R e$, when the overall dimension of the oscillator is changed while keeping constant the ratio $s / w$. The height of all oscillators studied here is $525 \mu \mathrm{~m}$. For all oscillators measured, the threshold where oscillations started was close to $R e=22$, and oscillations could be observed when increasing


Figure 2.6 - Evolution of the self-oscillation frequency with $R e$, when geometric parameters are changed. $a$ ) The width of the jets is changed, all other geometric dimensions being constant $(h=525 \mu m, s=500 \mu m, L=2000 \mu m) . b)$ The distance between of the jets is changed, all other geometric dimensions being constant ( $h=525 \mu m, w=100 \mu m, L=2000 \mu m$ ). $c$ ) The overall dimension of the oscillator is changed, with the ratio $s / w$ being constant. The height of the devices is $h=525 \mu m$. d) The height $h$ of the devices is changed, all other dimensions being equal ( $w=100 \mu m, s=800 \mu m, L=2000 \mu m$ ). The dotted lines are drawn only to guide the eye.
$R e$, until the maximal flow rate the syringe pumps could provide was reached. When the oscillator's dimensions are smaller, the frequency of the oscillations is higher for any given value of $R e$. Impinging jets having the same ratio s/w but an inlet channel of only $10 \mu \mathrm{~m}$ in width were also fabricated. These were very sensitive to the presence of dust particles in the water flows, but oscillations were observed when using filtered dye solutions. An accurate value of the oscillation frequency could not be measured, as the oscillations were very fast and a high-magnification microscope objective was used, which strongly limited the amount of light available to image the phenomenon with the high-speed camera.
Figure 2.6- $d$ shows the evolution of the oscillation frequency with the height of the fabricated structures. When performing measurements, oscillations were observed over a large range of $R e$ for all values of the height of the oscillators tested. However, for the oscillators of height smaller than $300 \mu \mathrm{~m}$, the impinging jets showed irregular oscillations frequencies, in particular for values of Re larger than 200. In this case, the jet oscillations superimposed with a large oscillation of the entire exit channel that occurs at a much lower frequency than the jet
oscillations.


Figure 2.7 - The frequency multiplied by the spacing between the jets $f \cdot s$ versus the average speed of the jets, $U$, for all the data presented in figure 2.6. The blue dotted line is a linear fit of all data.

Figure 2.7 shows the evolution of the parameter obtained by multiplying the frequency $f$ and the distance between the jets $s$ versus the average velocity $U$ of the liquid flow at the nozzles for all measurements previously presented in figure 2.6. A linear dependence is observed, indicating the importance of the spacing between the jets in the self-oscillation phenomenon. The linear fit of all data points presented in this figure has a slope of $1 / 6$, which is consistent with the measurements made by Denshikov et al. on large-scale facing jets in turbulent flow conditions (Denshikov presented an empirical formula that translates to $1 / f=6 s / U$, when using the notations of the present paper) Denshchikov et al. (1978). Without pretending more, as a matter of fact, the Strouhal number pertaining to many self-sustained oscillator flows (the wake of a cylinder for instance) is often found in the range 0.1-0.2.

### 2.2.4 Second oscillation mode

In the case of oscillator geometries based on large straight output channels (such as the oscillator of dimensions $w=100 \mu m, s=2000 \mu m, L=2000 \mu m, h=525 \mu m$ ), two oscillation modes can be observed. The first oscillation mode (figure 2.8-a) is similar to the oscillations presented previously, the jets first bifurcate in opposite directions and later come back towards
one another, collide and switch sides. This first oscillation mode occurs for low values of the Reynolds number (in the case of the oscillator presented in figure 2.8-a, for Re between 20 and 65). For large values of the Reynolds number, a second mode of regular oscillations was observed (figure 2.8-b), where the jets do not switch sides but bounce against each other at regular time intervals, each bounce resulting in a complex rotating flow motion at the center of the channel (in the case of the oscillator presented in figure $2.8-b$, this second mode is seen for Re between 65 and 160). This second oscillation mode has been observed for straightchannel oscillators where the ratio $s / w$ is larger than 20 , and seems to become dominant for straight-channel oscillators with even larger $s / w$ ratios.


Figure 2.8 - Evolution of the fluid flow during one oscillation. Experimental dyes concentration fields obtained for an oscillator of dimensions $w=100 \mu m, s=2000 \mu m, L=2000 \mu m, h=$ $525 \mu \mathrm{~m}$. Images are taken at regular time intervals (from left to right, top to bottom). a) $R e=47$, $F=23 \mathrm{~Hz}$, the liquid jets collide and switch sides at each oscillation. b) $R e=79, F=30 \mathrm{~Hz}$, the jets do not switch sides but bounce against each other regularly, each bounce resulting in a rotating flow motion in the center of the channel. See movie S4 in the Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevFluids.5.054202.


Figure 2.9 - Fully developed velocity profile, having unitary mean velocity, in a rectangular microchannel, imposed as a boundary condition at the inlets. Non-dimensional values of the x-coordinate between -0.5 and 0.5 correspond to a $100 \mu m$ inlet channel width $w$, while values of the $z$-coordinate between 0 to 5.25 corresponds to a microfluidic oscillator of $525 \mu \mathrm{~m}$ in height $h$.

### 2.3 Direct numerical simulations

### 2.3.1 Governing equations

The fluid motion inside the microfluidic oscillator domain, denoted by $\Omega$, is governed by the unsteady incompressible three-dimensional Navier-Stokes equations,

$$
\begin{gather*}
\nabla \cdot \mathbf{u}=0 \quad \text { on } \Omega  \tag{2.2}\\
\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u} \cdot \nabla \mathbf{u}=\nabla p+\frac{1}{R e} \nabla \cdot \boldsymbol{\tau} \quad \text { on } \Omega \tag{2.3}
\end{gather*}
$$

where $\mathbf{u}=\left\{u_{x}, u_{y}, u_{z}\right\}^{T}$ is the velocity flow field, $R e$ the Reynolds number and $\boldsymbol{\tau}=\left[\nabla \mathbf{u}+\nabla^{T} \mathbf{u}\right]$ the viscous stress tensor. Equations (2.2)-(2.3) are made non-dimensional by scaling lengths, velocity components and time respectively with the inlet channel width $w$, the average fluid velocity at the inlets $U$, and the convective time $w / U$, respectively. The Reynolds number is thus defined by equation (2.1), while the pressure is scaled by $\rho U^{2}$.
In addition to the fluid governing equations, we introduce a further advection-diffusion equation fully decoupled from equations (2.2)-(2.3) and describing the dynamics of a passive scalar, $\Phi$,

$$
\begin{equation*}
\frac{\partial \Phi}{\partial t}+\mathbf{u} \cdot \nabla \Phi=\frac{1}{P e} \Delta \Phi, \tag{2.4}
\end{equation*}
$$

(analogous to the temperature equation) which allows us to reproduce the two dyes injected during the experiments. The Péclet number, $P e$, appearing in equation (2.4) has been set to $P e=100$ in order to ensure good numerical stability and get a satisfactory flow visualization at the same time for all the particular geometries and control parameters, i.e. $R e$, considered. The oscillator cavity is assumed to be perfectly rigid, therefore a no-slip boundary condition for the velocity field, $\left.\mathbf{u}\right|_{\partial \Omega}=\mathbf{0}$, is enforced at the solid boundary domain, denoted by $\partial \Omega$. At the outlets, a traction-free boundary condition is imposed, $\mathbf{t}_{n}=\left[-p \mathbf{I}+\frac{1}{R e}\left(\nabla \mathbf{u}+\nabla^{T} \mathbf{u}\right)\right]$, where I denotes the identity matrix; in general, this boundary condition is used to model flow exits where details of the flow velocity and pressure are not known a priori; it is an appropriate boundary condition here, where the exit flow is close to be fully developed. At the inlets, the experimental constant flow rate is reproduced imposing the typical velocity profile present in rectangular micro-channels (see the analytical solution described in (Lee et al., 2006)), shown in figure 2.9. The length of the inlet ducts is such that assumed to be long enough for a fully developed flow is ensured.
Concerning the passive scalar equation, Dirichlet boundary conditions are imposed at the two inlets ( $\Phi=0,1$ ) to reproduce the injection of dyes, while outflow conditions are set at the outlets; no-flux is allowed through the solid walls.

### 2.3.2 Numerical procedure for DNS

The open-source code Nek5000 Lottes et al. has been used to perform the direct numerical simulation. The spatial discretization is based on the spectral element method (SEM). The three-dimensional geometry is divided into 7 macro boxes (as indicated in figure 2.10); each


Figure 2.10 - Domain's sub-division in macro boxes labelled by circled numbers: in the presence of an expansion channel, the mesh is stretched and remapped according to the prescribed radius of curvature.
macro box is then characterized by an imposed number of hexahedral elements, along the three Cartesian coordinates $x, y$ and $z$, within which, the solution is represented in terms of $N$-th order Lagrange polynomials interpolants, based on tensor product arrays of Gauss-Lobatto-Legendre (GLL) quadrature point in each spectral element; the common algebraic $P_{N} / P_{N-2}$ scheme is implemented, with $N$ fixed to 7 for velocity and 5 for pressure. In all cases
numerically examined, the overall length of the full oscillator structure in the $x$-direction, as well as the inlet ducts lengths in the $y$-direction (box 3 and 5), are kept constant and equal to $80 w$ and $6 w$, respectively. The inlet channel lengths are fixed to $10 w$ (value in the range where experimental tests showed insensitivity of the oscillation frequency with $R e$ to the inlet channel length). All the other characteristic sizes are changed in accordance with the definition of $w, h, s$ and $L$ associated with the considered microfluidic oscillator geometry. Macro boxes 2 and 6 , originally rectangular, are stretched or not depending on whether or not the expansion channel is present $(s \neq L$ or $s=L$ ). The domain is thus discretized with a structured multiblock grid consisting of, depending on the geometry analyzed, 32320 (if $s=8 w$ ) or 58880 (if $s=20 w$ ) spectral elements. The time-integration is handled with the semiimplicit method IM/EX, already implemented in Nek5000; the linear terms in equations (2.2)(2.3) are treated implicitly adopting a third order backward differentiation formula(BDF3), whereas the advective nonlinear term in equation (2.3) is estimated using a third order explicit extrapolation formula (EXT3). The semi-implicit scheme introduces a restriction on the time step (Karniadakis et al., 1991), therefore an adaptive time-step is set to guarantee the Courant-Friedrichs-Lewy (CFL) constraint.

### 2.4 Comparison between experiments and DNS

### 2.4.1 Dyes, concentration fields

Figure 2.11, 2.12 and 2.13 show the evolution of the dyes concentration field during one oscillation, with each figure corresponding to the case of an oscillator of specific geometry and a given flow condition. Figure 2.11 refers to an oscillator geometry made of a simple straight channel without expansion, similar to the one described in figure 2.2 ( $w=100 \mu \mathrm{~m}, \mathrm{~s}=800 \mu \mathrm{~m}$, $L=800 \mu m, h=525 \mu m$ and $R e=60$ ). Figure 2.12 shows an oscillator with an expansion in the output channel, similar to the one presented in figure 2.3 (oscillator dimensions are $w=100 \mu m, s=800 \mu m, L=2000 \mu m, h=525 \mu m$ and $R e=60$ ). Figure 2.13 corresponds to an oscillator geometry made of a simple straight channel having a large output width (oscillator dimensions are $w=100 \mu m, s=2000 \mu m, L=2000 \mu m, h=525 \mu m$ and $R e=50$ ). In all three figures, images are taken at regular time intervals during one oscillation and compare the measured and simulated concentration fields. The images obtained experimentally show a depth-averaged concentration, as they integrate the light passing through the full height of the microchannels, whereas in the case of the simulation, the images show the concentration field in the x -y plane of median height ( $212.5 \mu \mathrm{~m}$ from the bottom of the microchannel). All simulations have been run starting from zero initial conditions.

In all cases, there is a good agreement between the experimental and simulated dyes concentration fields, with the main flow features being similar for each chosen time step. The smaller features differ however between experiments and simulations, which may be related to the fact that the experimental images result from the integration of the light crossing the full height of the microstructure or to a non-optimal calibration of the Péclet number in the simulation.


Figure 2.11 - Evolution of the fluid flow with time during one oscillation. Comparison of experimental and simulated dye concentration fields in the case of an oscillator of dimensions $w=100 \mu m, s=800 \mu m, L=800 \mu m, h=525 \mu m$ at $R e=60$. The images are taken at regular time intervals (from top to bottom, left to right). See movie S5 in the Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevFluids.5.054202.


Figure 2.12-Comparison of experimental and simulated dye concentration fields in the case of an oscillator of dimensions $w=100 \mu m, s=800 \mu m, L=2000 \mu m, h=525 \mu m$ at $R e=60$. The images are taken at regular time intervals (from top to bottom, left to right). See movie S6 in the Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevFluids.5.054202.


Figure 2.13 - Comparison of experimental and simulated dye concentration fields in the case of an oscillator of dimensions $w=100 \mu m, s=2000 \mu m, L=2000 \mu m, h=525 \mu m$ at $R e=50$. The images are taken at regular time intervals (from top to bottom, left to right). See movie S7 in the Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevFluids.5.054202.

### 2.4.2 Non-dimensional frequency

In addition to the dyes concentration fields, simulations also provide the non-dimensional frequency of the self-oscillation phenomenon at the chosen value of the Reynolds number, expressed by the Strouhal number $S t=f \frac{w}{U}$. Figure 2.14- $a-b$ and $c$ compare the experimental and simulated values of $S t$, in the case of the three oscillator geometries presented in figure $2.11,2.12$ and 2.13 respectively. The DNS slightly overestimates the value of the oscillation frequency in all cases, however, the results of the simulation are generally close to the measurements. This little overestimation can be partially attributed to the numerical inlet velocity profile, which may not exactly represent the experimental profile. In the case of the oscillator with straight output channels (figure 2.14-a), a deviation between simulations and experiments can be seen at large values of $R e$. This is close to the conditions described in figure 2.2, where the jets stop to alternate regularly and which we linked to the absence of the expansion in the output channel. In such conditions, the liquid jets strongly interact with the walls, as the simulated pressure field shows in figure 2.15. This jet-wall interaction increases with increasing values of $R e$, and at some point, interferes with the increase of the self-oscillation frequency, inducing the stop of the alternating motion of the jets observed experimentally. Apparently, the DNS correctly predicts the interaction of the jets with the walls, but the ideal conditions described by the simulation do not correctly account for the


Figure 2.14 - Experimental and numerical non-dimensional oscillation frequency expressed by the Strouhal number $S t=f \frac{w}{U}$ versus the Reynolds number Re. a), b) and $c$ ) correspond to figure 2.11, figure 2.12 and figure 2.13, respectively.
change of frequency occurring experimentally close to this change of flow regime, probably induced by the small imperfections of the manufactured components and of the dust particles present in the liquid flows.
Note that this interaction of the jets with the walls does not occur in the case of oscillator geometries presenting an expansion in the output channel, as the jet's motion follows the wall curvature. This could explain the much wider range of stable oscillations, [ $R e_{c}, R e_{i r r}$ ], observed experimentally for such oscillator geometries.

### 2.5 Velocity field description

In section 2.4 we provided several comparisons between experimental results and numerical simulations in terms of dyes concentrations fields and oscillation frequency, showing a good agreement, which allows us to reasonably use the numerical results in order to investigate the various velocity fields more in depth.

### 2.5.1 Steady configuration

As mentioned, all the simulations were started from zero initial conditions, with the inlet velocity profile of figure 2.9 constantly enforced at the two inlets. After a first transient required for the flow to invade the whole cavity domain, a stationary configuration firstly manifests itself. This steady flow is always observed. If the Reynolds number is higher than the instability


Figure 2.15 - a) Simulated dye concentration and b) pressure fields (slice $x-y$ at $z=2.625$ ), showing the jet interaction with the walls occurring in the case of oscillators having straight output channels ( $w=100 \mu m, s=L=800 \mu m, h=525 \mu m$ at $R e=60$ ). The maximum pressure is always encountered at the domain's center, where the two jets face each other. Nevertheless, in $b$ ) we observe two regions of high pressure (highlighted in dashed black lines) occurring when the jets interact with the solid walls and whose intensity increases as $R e$ in increased.
threshold, it is observed for a certain time interval, after which the self-sustained oscillations start with the periodic flow configuration discussed in the previous sections. On the contrary, if $R e$ is set below this threshold, then the flow remains stationary indefinitely.
This stationary configuration is shown in figure 2.16 for the microfluidic oscillator based on straight output channels ( $w=100 \mu m, s=800 \mu m, L=800 \mu m, h=525 \mu m$ at $R e=32$ ).
From figure 2.16- $b$ we observe two large recirculation regions close to the channel inlets and resulting from the presence of walls, where a no-slip boundary condition is enforced. The three-dimensional shape of these recirculation regions can be clearly seen in figure 2.16$d$. Heading towards the channel outlets, the flow approaches a fully developed flow having substantially null $u_{y}$ and $u_{z}$ velocity components. From figure 2.16-c and $d$ we also note two regions of vortical motion in the collision region of the jets; this is due to the constant pressure that the two jets exert against each other; indeed, the pressure field (not represented in figure 2.16) is characterized by a high-pressure region spatially located where the jets collide. As the jets face each other, the fluid tends to escape in all directions, thus part of the fluid escaping along the $z$-direction meets the lateral solid walls at $z=0$ and $z=h$, which push back the fluid, leading to this vortical motion.
Figure 2.17 is the equivalent of figure 2.16, but in the case of the oscillator with an expansion in the output channel ( $w=100 \mu m, s=800 \mu m, L=2000 \mu m, h=525 \mu m$ and at $R e=30$ ).
This configuration is similar to the one presented in figure 2.16 with respect to the value of $R e$ and the spacing $s$. Now the presence of an expansion region in the output channel leads to


Figure $2.16-a)$ Dyes concentration and $b, c, d, e$ ) stationary velocity field numerically observed before the self-sustained oscillation start in the case of the microfluidic oscillator of figure 2.11, $w=100 \mu m, s=800 \mu m, L=800 \mu m, h=525 \mu m$ at $R e=32$. b) Filled 2D contour plot for $u_{x}$ and black arrow for the in-plane velocity vector, $\left\{u_{x}, u_{y}\right\} . c$ ) Filled 2D contour plot for $u_{z}$ and black arrow for the in-plane velocity vector, $\left\{u_{x}, u_{z}\right\}$. d) Filled 2D contour plot of the out of plane velocity $u_{x}$ and black arrow for the in-plane velocity vector, $\left\{u_{y}, u_{z}\right\}$. Slices' sizes are not to scale. Arrows provide a qualitative representation only. e) Filled 2D contour plot for $u_{y}$ and black arrow for the in-plane velocity vector, $\left\{u_{y}, u_{z}\right\}$. Slice represented in $b$ ), $c$ ) and $e$ ) correspond to the three main planes of symmetry (indicated in figure).
the formation of two much more elongated recirculation regions along the $x$-direction, which follow the curvature of the cavity (figure 2.17-b). Because of the cavity's curvature, the vertical velocity within these recirculation regions is larger when compared to figure2.16-b. Planes $x-z$ at $y=0$ for $u_{z}$ and $y-z$ at $x=0$ for $u_{y}$ are not shown here since they are qualitatively and quantitatively close to those of figure 2.16.

### 2.5.2 Self-oscillating configuration

When the Reynolds number is increased above the instability threshold, i.e. $R e>23$ for the microfluidic oscillator of figure 2.16 , the two jets start to oscillate regularly for a wide range of Re. As already mentioned in section 2.2, the jets regularly collide against each other and switch sides in a periodic motion. At each collision, a pair of three-dimensional vortices is emitted and advected towards the channel outlets, as can be observed in figure 2.18-b and $c$.
The two stable vortical regions represented in figure 2.16 and figure 2.17 are now alternately pushed up and down owing to the continuous switch of side of the oscillating jets, as show in figure $2.18-b$ and $d$. A qualitatively similar flow evolution in time is recognized for the microfluidic oscillator with the expansion channel, meaning the physical mechanism which


Figure $2.17-a$ ) Dyes concentration and $b, c, d, e$ ) stationary velocity field numerically observed before the self-sustained oscillation start in the case of the microfluidic oscillator of figure 2.12, $w=100 \mu m, s=800 \mu m, L=2000 \mu m, h=525 \mu m$ at $R e=30$. b) Filled 2D contour plot for $u_{x}$ and black arrow for the in-plane velocity vector, $\left\{u_{x}, u_{y}\right\} . c$. Filled 2D contour plot of the out of plane velocity $u_{x}$ and black arrow for the in-plane velocity vector, $\left\{u_{y}, u_{z}\right\}$. Slices' sizes are not to scale. Arrow provides a qualitative representation only.
breaks the symmetry of the stationary configuration and leads to the unsteady periodic motion is the same, while the expansion channel only contributes to stabilize the regular oscillations up to a much higher $R e$.

### 2.5.3 Perturbation fields

As mentioned in section 2.5.1, the steady configuration, which is linearly stable for $R e<R e_{c}$, is transiently observed even for $R e>R e_{c}$, before the amplitude of the oscillating perturbation, which grows exponentially, becomes large enough for the self-sustained oscillations to settle into a limit cycle. In the spirit of the linear global stability analysis, the total velocity and pressure fields in the vicinity of the threshold can be decomposed as the sum of a steady base flow and a time-dependent perturbation field:

$$
\begin{align*}
& \mathbf{u}(x, y, z, t)=\mathbf{u}_{b f}(x, y, z)+\mathbf{u}_{p}(x, y, z, t)  \tag{2.5}\\
& p(x, y, z, t)=p_{b f}(x, y, z)+p_{p}(x, y, z, t) \tag{2.6}
\end{align*}
$$

The total velocity field, $\mathbf{u}$, and pressure field, $p$, extracted from the DNS can thus be used to separate the corresponding perturbation fields, $\mathbf{u}_{p}$ and $p_{p}$, from the base-flow fields, $\mathbf{u}_{b f}$ and


Figure 2.18 - Snapshot of $a$ ) dyes concentration and $b, c, d, e$ ) unsteady velocity field numerically observed once the self-sustained oscillations reached the limit cycle in the case of the microfluidic oscillator of figure 2.11, $w=100 \mu \mathrm{~m}, \mathrm{~s}=800 \mu \mathrm{~m}, L=800 \mu \mathrm{~m}, \mathrm{~h}=525 \mu \mathrm{~m}$ at $R e=60 . b$ ) Filled 2D contour plot for $u_{x}$ and black arrow for the in-plane velocity vector, $\left\{u_{x}, u_{y}\right\} . c$ ) Filled 2D contour plot for $u_{z}$ and black arrow for the in-plane velocity vector, $\left\{u_{x}, u_{z}\right\}$. d) Filled 2D contour plot of the out of plane velocity $u_{x}$ and black arrow for the in-plane velocity vector, $\left\{u_{y}, u_{z}\right\}$. e) Filled 2D contour plot for $u_{y}$ and black arrow for the in-plane velocity vector, $\left\{u_{y}, u_{z}\right\}$. Slices represented in $b$ ), $c$ ) and $e$ ) correspond to the three main planes of symmetry (indicated in the figure). Slices' sizes are not to scale. Arrows provide a qualitative representation only.
$p_{b f}$, and highlight where the origin of the regular oscillations is located. Let us consider, i.e., the microfluidic geometry of figure 2.16 and 2.18. A series of numerical simulations, starting from zero initial conditions, were performed in the range $R e=18-25$ (the threshold, $R e_{c}$, for the case here considered is approximatively 23). Figure $2.19-a$ ) and $b$ ) show the value of the of the $x$ - an $y$-velocity components at the coordinate $(x, y, z)=(3,0,2.625)$. Since the oscillating flow configuration breaks the antisymmetry of the $y$-velocity component with respect to the $x-z$ plane in $y=0$, the $y$-component is then monitored (see figure $2.19-b$ )) in time to establish at which $R e$ and time-instant the oscillations start to be visible.
As shown in figure $2.19-b$ ), the flow does not exhibit any oscillations below $R e_{c}$, where only the linearly stable base-flow is observed. For $R e=25>R e_{c}$, oscillations start to grow from zero with a very small growth rate, given the vicinity of the marginal stability. In such conditions, the stationary base-flow velocity and pressure fields, $\mathbf{u}_{b f}$ and $p_{b f}$, can be identified where the perturbation is still very small, i.e., at $t=400$, where the order of magnitude of the perturbation is lower than $10^{-10}$. Subtracting this base flow from the total flow, i.e. at $t=1375$ in figure 2.19, allows to isolate the growing perturbation, as presented in figure 2.20.
The analysis of the perturbation velocity fields allows us to locate the origin of the oscillations


Figure $2.19-a$ ) Horizontal, $u_{x}$, and $b$ ) vertical, $u_{y}$, velocity components at $(x, y, z)=(3,0,2.625)$. The plane $x-y$ at $z=h / 2$ is a plane of antisymmetry for the perpendicular velocity component, $\left.u_{z}\right|_{z=h / 2}=0 . b$ ) The antisymmetry of $u_{y}$ with respect to the plane $x-z$ at $y=0$ is broken for $R e=25$, which is slightly higher than the threshold value, $R e_{c} \approx 23$. Note that the resulting Strouhal number agrees well with the experimental one presented in figure $2.14-a$ ), even if the limit cycle has not been reached yet.
in the central region, where the jets collide and curve towards the output channels. Welldefined counter-rotating vortical structures, whose extension in the $z$-direction covers the entire channel height $h$ and which are separated by a wavelength $\lambda$ suggesting a correlation with the distance separating the inlets, $s$, are generated and advected downstream (left and right) by the base-flow (see figure 2.20-a) and $b$ ). The $z$-velocity component is significantly smaller than the other two components in the central region and negligible in the rest of the domain, as shown in figure 2.20-c) and $d$ ).

### 2.5.4 Discussion

Despite the insight brought by the numerical simulations to visualize the total velocity and pressure fields, and the perturbation fields, no physical mechanism could be precisely identified, from which these self-sustained oscillations would originate. Several plausible candidates can be tentatively identified. Hyperbolic stagnation points and lines are well known to be unstable (Ortiz and Chomaz, 2011; Sipp et al., 1999), although they often lead to static bifurcations (Fani et al., 2013). The existence of recirculation regions is quite similar to sudden expansion flows which are also known to become statically unstable (Fani et al., 2012). But these recirculation regions also form an intense shear layer, which could possibly become the


Figure $2.20-a)$-b) Filled contours of the $x$ and $y$ perturbation velocity components extracted for $t=1375$ in figure 2.19-b) for the microfluidic oscillator with $w=100 \mu \mathrm{~m}, \mathrm{~s}=800 \mu \mathrm{~m}$, $L=800 \mu m, h=525 \mu m$ at $R e=25$. The black arrows represent the orientation of the in-plane velocity vector, $\left.\left\{u_{x}, u_{y}\right\} . c\right)$ Filled contours for $u_{z}$ in the $x-z$ slice at $y=0$. Slices' sizes are not to scale. Arrows provide a qualitative representation only. $d$ ) Filled contours of the out of plane velocity $u_{x}$ and black arrows for the in-plane velocity vector $\left\{u_{y}, u_{z}\right\}$ in the $y-z$ slice at $x=0$.
source of a Kelvin-Helmholtz instability. Indeed, the structure of the perturbation velocity field in the left and right channels shown in figure 2.20 is typical of sinuous shear instabilities. In order to translate into a global instability, this shear layer instability would either need to be of absolute nature, possibly because of the presence of nearby walls, known to enhance absolute instability in confined shear flows (Biancofiore and Gallaire, 2011; Healey, 2009; Juniper, 2006; Rees and Juniper, 2010). Even if this shear layer instability were to be convective, other feedback mechanisms, as the ones investigated in Villermaux (Villermaux et al., 1993; Villermaux and Hopfinger, 1994) could also ensure the global, sefl-sustained nature of the observed oscillations. In order to get further insight, an exhaustive stability analysis of the present flow needs to be conducted, which could locate the wavemaker region and clearly identify the governing instability mechanisms at stake.

### 2.6 Comments and conclusions

Pulsatile liquid flows showing a self-oscillatory behaviour were studied at the microscale. Experimentally, oscillating water jets were generated in microfabricated silicon cavities, from steady and equal inlet flows and without external stimuli. They were colored and imaged using a microscope and a high-speed camera. The oscillators we described here can be categorized
as based on jet interactions: Two facing jets first bifurcate in opposite directions and later come back towards one another, collide and switch sides, with a very regular temporal periodicity.
Direct numerical simulations were performed to solve the unsteady incompressible threedimensional Navier-Stokes equations in the studied geometries, using a spectral element method. The Nek5000 was used to perform the simulation. Experiments and simulations show a good agreement for all studied oscillators, for both the dye concentration fields and the non-dimensional oscillation frequency.
The self-oscillation phenomenon starts at a threshold, in terms of Reynolds number, that depends on the geometrical parameters of the oscillator cavity. Threshold values close to $R e=20$ were observed for many of the studied geometries.
When the oscillator is based on simple straight crossing channels, the self-oscillation phenomenon can be observed for a limited range of values of the Reynolds number, since when $R e$ exceeds a second threshold $R e_{i r r}$, the flows stop to switch sides regularly and periodicity is lost. The corresponding simulated pressure field evolution shows that the jets strongly interact with the output channel walls in this case, which induces this change of flow regime. When the output channel is no longer a simple straight channel but is supplemented by an expansion, this interaction with the walls is no longer occurring, as the jet's motion follows the wall curvature. This leads to a much wider range of stable oscillations. Experimentally, the impinging jets were observed to switch sides regularly until the pumps used could not deliver higher flow rates and stalled (for $R e=630$ ).
The evolution of the self-oscillation frequency was studied when the main geometric parameters of the oscillator cavity were changed. A linear dependence between the average flow velocity and the parameter obtained by multiplying the oscillation frequency and the distance between the jets was observed, which underlines the importance of the distance separating the jets and the jet velocity in the oscillation phenomenon.
The simulated velocity fields for the various studied oscillator cavities provide additional information on the flow behaviour, showing how vortices evolve in the flow at the onset of self-oscillations.
Finally, the oscillator cavities we studied can also be classified as "static mixers" as they provide a rearrangement of the inlet flows without moving parts or external stimuli. For values of $R e$ close to the onset of the self-oscillation phenomenon, a regular temporal rearrangement of the inlet flows was observed in the output channels, but for larger values of $R e$, the fluid flow in the output channel remains segmented, with only limited mixing. The studied microdevices cannot consequently be considered for efficient mixing at the microscale, however, cavities of adapted geometry can certainly be devised to take advantage of the self-oscillating phenomenon for the creation of efficient micromixers, these will additionally show a relatively low power dissipation as the output channels are of large dimensions compared to the input channels.
As a next step, a thorough linear stability analysis should enable the identification of the governing destabilization mechanism, and determine if this self-sustained oscillation results from the instability of the hyperbolic stagnation line, from the symmetry breaking of the recirculation regions or from the intense shear layers. Additionally, a subsequent weakly
nonlinear analysis, which we could not explore numerically in this work, so as to maintain a reasonable computational cost, could confirm the supercritical nature of the bifurcation.
Lastly, since flows in cross-slot geometries are typically known to show hysteretic behavior for certain combinations of the characteristic geometrical parameters (Burshtein et al., 2019), a weakly nonlinear analysis could also allow to numerically perform a parametric analysis and investigate possible interactions of the self-sustained regime with eventual non-oscillating symmetry breaking conditions, in particular when the gap separating the two facing inlets, $s$, and the height, $h$, approach the inlet width, $w$.

### 2.7 Appendix

### 2.7.1 Self-sustained oscillations in more complex jet networks

In this short Appendix we report the experimental observation of the self-oscillation phenomenon in other geometries involving more complex jet networks, as that presented in figure 2.21, with three colliding jets. Similarly to the case with two inlets, the three jets oscillate and switch sides regularly. The physical mechanism at play in the self-sustained oscillations observed here is still unclear and an in-depth stability analysis needs to be conducted to elucidate it.


Figure 2.21 - Evolution of the dye concentration fields with time in a micro-oscillator structure with 3 inlet channels, for $R e=32$. The images are taken at regular time intervals during one oscillation (from left to right, top to bottom). The jets width is $100 \mu \mathrm{~m}$, the three jets are placed at $120^{\circ}$ angle on a circle of $800 \mu \mathrm{~m}$ in diameter, the output channels width is $2000 \mu \mathrm{~m}$, the thickness of the device is $525 \mu \mathrm{~m}$. These images are associated with a video winner of the 2019 American Physical Society's Division of Fluid Dynamics (DFD) Gallery of Fluid Motion Award for work presented at the DFD Gallery of Fluid Motion. The original video is available online at the Gallery of Fluid Motion, https://doi.org/10.1103/APS.DFD.2019.GFM.V0036.

## 3 Impinging planar jets: hysteretic behaviour and origin of the selfsustained oscillations

Remark: this chapter is largely inspired by the publication of the same name.

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The experimental and numerical investigation presented in Chapter 2 (Bertsch et al., 2020a) describes the self-sustained oscillations induced by the interaction of two impinging jets in microfluidic devices. While the oscillatory regime induced by interacting jets has been studied in detail, the physical mechanism behind these oscillations remains still undetermined. In parallel, but for a different range of aspect ratios, Burshtein et al. (2019) experimentally found that hysteretic behaviours due to multiple symmetry-breakings can appear in cross-slot flows. The present work focuses on two-dimensional oscillators subjected to a fully developed inlet flow, as in Bertsch et al. (2020a) and in contradistinction with Pawlowski et al. (2006), who focused on plug inlet flow. The linear global stability analysis performed confirms the existence of an oscillating global mode, whose spatial structure qualitatively coincides with the one computed numerically by Bertsch et al. (2020a), suggesting that the physical mechanism from which the oscillations would originate is predominantly two-dimensional. The mode interaction of the oscillating mode with a steady symmetry-breaking mode is examined making use of the weakly nonlinear theory, which shows how the system exhibits hysteresis in a certain range of aspect ratios. Lastly, sensitivity analysis is exploited to identify the wavemaker associated with the global modes, whose examination allows us to spot the core of the symmetry-breaking instability at the stagnation point and to identify the Kelvin-Helmholtz instability, located in the jets interaction region, as the main candidate for the origin of the oscillations observed in both two-dimensional and three-dimensional fluidic devices.

## Chapter 3. Impinging planar jets: hysteretic behaviour and origin of the self-sustained oscillations

The Chapter is organized as follows. In section 3.1 the flow configuration and the governing equations describing the fluid motion inside a two-dimensional microfluidic cavity with an imposed fully developed inlet flow are introduced. In section 3.2 the numerical approaches adopted are described. In section 3.3 the steady symmetric base-flow is determined, while the tools of the linear global stability analysis are employed to derive the associated stability chart, where the two control parameters, Reynolds number and aspect ratio are varied in a wide range. The nonlinear global mode interaction emerging from the stability analysis is then discussed in section 3.4 making use of the weakly nonlinear theory and the multiple scale technique. The resulting bifurcation diagram is validated in section 3.5. Sensitivity analyses are carried out in section 3.6, which is devoted to the understanding of the physical mechanism behind the various of instability observed. We finally analyze the effect of a different inlet velocity profile by applying the weakly nonlinear model to the flow case of plug inlet profiles, revisiting the analysis of Pawlowski et al. (2006). Conclusions are presented in section 3.8.

### 3.1 Flow configuration and governing equations

Let us consider the two-dimensional X-junction (also called cross-junction) presented in figure 3.1. An incompressible fluid with density $\rho$ and dynamic viscosity $\mu$ enters the device through two facing inlets of width $w$, denoted by $\partial \Omega_{i}$, and it is allowed to flow out along the two symmetric arms of the main lateral channel. The two symmetric inlets mimic the action of two inlet channels separated by a distance $s$ to create two facing jets when they reach the lateral channel. Outlets, $\partial \Omega_{o}$, are provided at both ends of the channel, at a distance $L_{\text {out }}$, far away from the intersection. In figure $3.1, \Omega$ denotes the fluid domain, while $U$ is the average velocity of the fluid at the inlet channels. Taking advantage of the geometric symmetries of this microfluidic oscillator, the computational domain can be reduced to a quarter of the full domain, with y - and x-axes of symmetry $\partial \Omega_{\nu}$ and $\partial \Omega_{h}$ respectively. Proper boundary conditions for the fluid problem, listed in sections section 3.3 and section 3.4 , are then imposed at $\partial \Omega_{v}$ and $\partial \Omega_{h}$. As sketched in figure 3.1, a fully developed flow is imposed at the inlets at $y= \pm s / 2$. This assumption, removing the influence of the inlet channel length, allows us to reduce the number of geometrical parameters, simplifying the parametric analysis. The introduction of the following dimensionless variables (the star denotes the dimensional quantities),

$$
\begin{equation*}
x=\frac{x^{*}}{w}, \quad y=\frac{y^{*}}{s}, \quad u=\frac{u^{*}}{U}, \quad v=\frac{v^{*}}{U}, \quad p=\frac{p^{*}}{\rho U^{2}}, \quad t=\frac{t^{*}}{w / U} . \tag{3.1}
\end{equation*}
$$

leads to the definition of the aspect ratio $A R=s / w$ and of the nabla operator, $\nabla_{A R}=\left\{\frac{\partial}{\partial x}, \frac{1}{A R} \frac{\partial}{\partial y}\right\}^{T}$. The fluid motion within the microfluidic oscillator cavity, $\Omega$, is governed by the two-dimensional incompressible Navier-Stokes equations, whose non-dimensional form reads:

$$
\begin{equation*}
\frac{\partial \mathbf{u}}{\partial t}+\left(\mathbf{u} \cdot \nabla_{A R}\right) \mathbf{u}+\nabla_{A R} p-\frac{1}{R e} \Delta_{A R} \mathbf{u}=0 \tag{3.2}
\end{equation*}
$$



Figure 3.1 - Microfluidic oscillator cavity with straight output channels explored in this work. Notation: inlet width $w$, gap size $s$, overall length $2 L_{o u t}$, walls $\partial \Omega_{w}$, outlets $\partial \Omega_{o}$, x-axis of symmetry at $y=0, \partial \Omega_{h}$ and y-axis of symmetry at $x=0, \partial \Omega_{v}$. $U$ denotes the mean value of the velocity profile imposed at the inlets, $\partial \Omega_{i}$.

$$
\begin{equation*}
\nabla_{A R} \cdot \mathbf{u}=0 \tag{3.3}
\end{equation*}
$$

In (3.2)-(3.3), $\mathbf{u}=\{u, v\}^{T}$ is the velocity field, $p$ is the pressure field and $R e=\rho U w / \mu$ is the Reynolds number. The no-slip boundary condition is imposed at the rigid solid wall, $\partial \Omega_{w}$, $\left.\mathbf{u}\right|_{\partial \Omega_{w}}=0$, while an outflow boundary condition is imposed at the outlet, $\partial \Omega_{o},\left(-p \mathbf{I}+\frac{1}{R e} \nabla_{A R} \mathbf{u}\right)$. $\mathbf{n}=0$, where $\mathbf{n}$ is the unit normal to $\partial \Omega_{o}$ and $\mathbf{I}$ is the identity tensor. At the inlet, $\partial \Omega_{i}$, a fully developed parabolic velocity profile is imposed:

$$
\begin{equation*}
\left.\mathbf{u}\right|_{\partial \Omega_{i}}=\left\{0,-\frac{3}{2}\left(1-4 x^{2}\right)\right\}^{T} \tag{3.4}
\end{equation*}
$$

### 3.2 Numerical approach

Two different numerical approaches are adopted in the present work. The numerical scheme used to derive the global stability chart, section 3.3, to analyze the weakly nonlinear global mode interaction, section 3.4, and to perform sensitivity analysis, section 3.6, is a finite element method based on the FreeFem++ software (Hecht et al., 2011). The mesh refinement is controlled by the vertex densities on both external and internal boundaries. Regions, where the mesh density varies, are depicted in figure 3.2. The unknown velocity and pressure fields $\{\mathbf{u}, p\}^{T}$ are spatially discretized using a basis of Taylor-Hood elements $\left(P_{2}, P_{1}\right)$. The matrix inverses are computed using the UMFPACK package (Davis and Duff, 1997). The steady baseflow is obtained by the classic iterative Newton method, while eigenvalue calculations are performed using the ARPACK package (Lehoucq et al., 1998). For other details see Meliga et al. (2009a); Meliga and Gallaire (2011); Meliga et al. (2012a); Sipp and Lebedev (2007). With reference to figure 3.2, five different meshes, denoted $M 1-M 5$, exhibiting different boundary vertex densities, $n_{i}$, have been used to assess convergence in the numerical result. In the following, we will focus on the mesh $M 5$ to present all results. A detailed convergence analysis of meshes $M 1-M 5$ is given in Appendix 3.9.1.

## Chapter 3. Impinging planar jets: hysteretic behaviour and origin of the self-sustained oscillations

The results obtained from the weakly nonlinear investigation are then compared to direct numerical simulations (DNS) in section 3.5. The open-source code Nek5000 (Lottes et al.) has been used to perform the DNS. The spatial discretization is based on the spectral element method. The full two-dimensional geometry (without imposing any symmetry conditions) is divided into macro boxes; each macro box is then characterized by an imposed number of quadrilateral elements, along the two Cartesian coordinates $x$ and $y$, within which the solution is represented in terms of $N$-th order Lagrange polynomials interpolants, based on tensor product arrays of Gauss-Lobatto-Legendre (GLL) quadrature point in each spectral element; the common algebraic $P_{N} / P_{N-2}$ scheme is implemented, with $N$ fixed to 7 for velocity and 5 for pressure. The domain is thus discretized with a structured multiblock grid consisting of 4920 spectral elements, which largely guarantees convergence. The time-integration is handled with the semi-implicit method, already implemented in Nek5000; the linear terms in equations (3.3)-(3.2) are treated implicitly adopting a third order backward differentiation formula (BDF3), whereas the advective nonlinear term is estimated using a third order explicit extrapolation formula (EXT3). The semi-implicit scheme introduces a restriction on the time step (Karniadakis et al., 1991), therefore an adaptive time-step is set to guarantee the Courant-Friedrichs-Lewy (CFL) constraint. See Bertsch et al. (2020a) for more details.


Figure 3.2 - Computational domain considered in the global stability analysis, weakly nonlinear study and sensitivity analysis. $w=1, L_{2}=5 w, L_{3}=20 w, L_{\text {out }}=70 w$ and $s=1$. The number of elements per unit length used for the various line with different thicknesses: $n_{1}, n_{2}$, $n_{3}$ and $n_{4}$.

### 3.3 Steady base-flow and linear global stability analysis

The flow field $\mathbf{q}=\{\mathbf{u}, p\}^{T}$ is decomposed in a steady base-flow, $\mathbf{q}_{0}=\left\{\mathbf{u}_{0}, p_{0}\right\}^{T}$ and a small perturbation $\mathbf{q}=\left\{\mathbf{u}_{1}, p_{1}\right\}^{T}$, of infinitesimal amplitude $\epsilon$.

### 3.3.1 Steady base-flow

The base flow, $\mathbf{q}_{0}=\left\{\mathbf{u}_{0}, p_{0}\right\}^{T}$, is sought as a steady solution of the nonlinear Navier-Stokes equations,

$$
\begin{equation*}
\left(\mathbf{u}_{0} \cdot \nabla_{A R}\right) \mathbf{u}_{0}+\nabla_{A R} p_{0}-\frac{1}{R e} \Delta_{A R} \mathbf{u}_{0}=\mathbf{0}, \quad \nabla_{A R} \cdot \mathbf{u}_{0}=0 \tag{3.5}
\end{equation*}
$$

with the boundary conditions,

$$
\begin{equation*}
\left.\mathbf{u}_{0}\right|_{\partial \Omega_{w}}=\mathbf{0},\left.\quad\left(-p_{0} \mathbf{I}+\frac{1}{R e} \nabla_{A R} \mathbf{u}_{0}\right) \cdot \mathbf{n}\right|_{\partial \Omega_{o}}=0,\left.\quad \mathbf{u}_{0}\right|_{\partial \Omega_{i}}=\left\{0,-\frac{3}{2}\left(1-4 x^{2}\right)\right\}^{T} \tag{3.6}
\end{equation*}
$$

The steady base-flow velocity fields, $u_{0}(x, y)$ and $\nu_{0}(x, y)$, are characterized by the following symmetry and antisymmetry properties with respect to the y - and x -axes of symmetry, $\partial \Omega_{v}$ and $\partial \Omega_{h}$,

$$
\begin{align*}
& u_{0}(x, y)=u_{0}(x,-y)=-u_{0}(-x, y),  \tag{3.7}\\
& v_{0}(x, y)=-v_{0}(x,-y)=v_{0}(-x, y), \tag{3.8}
\end{align*}
$$

which translate in the following boundary conditions imposed at $\partial \Omega_{h}$ and $\partial \Omega_{\nu}$ :

$$
\begin{equation*}
\left.v_{0}\right|_{\partial \Omega_{h}}=0,\left.\frac{\partial u_{0}}{\partial y}\right|_{\partial \Omega_{h}}=0,\left.\quad u_{0}\right|_{\partial \Omega_{v}}=0,\left.\frac{\partial v_{0}}{\partial x}\right|_{\partial \Omega_{v}}=0 \tag{3.9}
\end{equation*}
$$

An approximate guess solution satisfying the required boundary conditions is first obtained by solving the associated Stokes problem, where the advective term is neglected. The solution of the steady nonlinear equation, $\mathbf{q}_{0}$, is then obtained using an iterative Newton method (Barkley, 2006; Barkley et al., 2002). Here the iterative process is carried out until the $L^{2}-$ norm of the residual of the governing equations for $\mathbf{q}_{0}$ becomes smaller than $1 \times 10^{-12}$.

Figure 3.3 shows the symmetric spatial structure of the magnitude of the steady velocity


Figure 3.3 - Steady base-flow for $R e=22.65$ and $A R=6.98$. Color map: magnitude of the velocity field. White lines: streamlines associated with the steady base-flow. Red dashed lines: boundaries of the four symmetric recirculation regions. The solution in the full flow domain is rebuilt using the symmetry properties. Only the central portion, $x \in[-25,25]$, is shown here.
field for $R e=22.65$ and $A R=6.98$. As observed in figure 3.3 , the $y$-velocity component is dominant in the central region, near the two inlets. The two facing jets collide and the fluid
is repulsed and advected downstream, towards the two outlets. A stagnation point is thus present at $x=y=0$ owing to the symmetry properties. We also observe the presence of four symmetric recirculation regions close to the channel inlets and resulting from the presence of walls, where a no-slip boundary conditions is enforced. Heading towards the channel outlets, the flow approaches a fully developed flow. The present base-flow configuration is qualitatively comparable to the one recently observed in the three-dimensional experimental and numerical investigations carried out by Bertsch et al. (2020a).

### 3.3.2 Global eigenmode analysis

At leading order in $\epsilon, \mathbf{q}_{1}=\left\{\mathbf{u}_{1}, p_{1}\right\}^{T}$ is an unsteady solution of the linearized Navier-Stokes equations around the $\epsilon^{0}$-order solution (steady base-flow):

$$
\begin{equation*}
\frac{\partial \mathbf{u}_{1}}{\partial t}+\left(\mathbf{u}_{0} \cdot \nabla_{A R}\right) \mathbf{u}_{1}+\left(\mathbf{u}_{1} \cdot \nabla_{A R}\right) \mathbf{u}_{0}+\nabla_{A R} p_{1}-\frac{1}{R e} \Delta_{A R} \mathbf{u}_{1}=\mathbf{0}, \quad \nabla_{A R} \mathbf{u}_{1}=0, \tag{3.10}
\end{equation*}
$$

with the boundary conditions,

$$
\begin{equation*}
\left.\mathbf{u}_{1} \cdot \mathbf{n}\right|_{\partial \Omega_{w}}=\mathbf{0},\left.\quad\left(-p_{1} \mathbf{I}+\frac{1}{R e} \nabla_{A R} \mathbf{u}_{1}\right) \cdot \mathbf{n}\right|_{\partial \Omega_{o}}=0,\left.\quad \mathbf{u}_{1}\right|_{\partial \Omega_{i}}=\mathbf{0} . \tag{3.11}
\end{equation*}
$$

The system can be written in a compact form as:

$$
\begin{equation*}
\left(\mathscr{B} \partial_{t}+\mathscr{A}\right) \mathbf{q}_{1}=\mathbf{0}, \tag{3.12}
\end{equation*}
$$

where the matrices $\mathscr{A}$ and $\mathscr{B}$ read:

$$
\mathscr{A}=\left(\begin{array}{cc}
\mathscr{C}_{A R}\left(\mathbf{u}_{0}, \cdot\right)-\frac{1}{R e} \Delta_{A R} & \nabla_{A R}  \tag{3.13}\\
\nabla_{A R}^{T} & 0
\end{array}\right), \quad \mathscr{B}=\left(\begin{array}{cc}
\mathscr{I} & 0 \\
0 & 0
\end{array}\right) .
$$

being $\mathscr{I}$ the identity matrix and $\mathscr{C}_{A R}$ the $\epsilon^{0}$-order symmetric advection operator, $\mathscr{C}_{A R}(\mathbf{a}, \mathbf{b})=$ $\left(\mathbf{a} \cdot \nabla_{A R}\right) \mathbf{b}+\left(\mathbf{b} \cdot \nabla_{A R}\right)$ a. We thus look for a first order solution which takes the normal mode form

$$
\begin{equation*}
\mathbf{q}_{1}=\hat{\mathbf{q}}_{1} e^{(\sigma+\mathrm{i}(\omega) t}+\text { c.c. }, \tag{3.14}
\end{equation*}
$$

where c.c. denotes the complex conjugate. Substituting (3.14) in (3.12) the $\epsilon$-order system reduces to the generalized eigenvalue problem:

$$
\begin{equation*}
[(\sigma+\mathrm{i} \omega) \mathscr{B}+\mathscr{A}] \hat{\mathbf{q}}_{1}=\mathbf{0} . \tag{3.15}
\end{equation*}
$$

In figure 3.4 the eigenvalues are displayed for different Reynolds numbers and aspect ratio values. In order to build the full eigenvalue spectrum using the reduced computational domain, we explored all the possible symmetries and antisymmetries of the perturbation velocity field $\mathbf{u}_{1}$ by imposing different axis boundary conditions analogous to (3.9). From the stability chart displayed in the ( $R e, A R$ ) plane of figure $3.4-(b)$ it emerges that the steady


Figure 3.4 - (a) Eigenvalues displayed in the $(\sigma, \omega)$ plane for $R e=R e_{C_{2}}=22.65$ and $A R=$ $A R_{C_{2}}=6.98$. A pair of complex eigenvalues, denoted by $B$ together with a pure real eigenvalue, $A$, are found to be simultaneously marginally stable for the present combination of parameters. Eigenvalues on the left side of the spectrum are not physical and correspond to spurious modes, whose presence is due to the influence of outlet boundary conditions. The position of eigenvalues $A, B$ and $C$ is not affected by $L_{\text {out }}$ in the range $L_{o u t} \in[30,100]$. (b) Marginal stability curves corresponding to the modes $A$ and $B$ and to a second steady mode $C$ as a function of $R e$ and $A R$. A codimension-2 point, $C_{2}$, is found for $R e=R e_{C_{2}}=22.65$ and $A R=A R_{C_{2}}=6.98$.
base-flow is stable below a critical aspect ratio, whose value is found to be approximately $A R \approx 1.75$ for a Reynolds number $R e=230$ (maximum value investigated in the present study). Analogously, the base-flow is stable below a Reynolds number $R e \approx 8$ for an aspect ratio $A R=70$ (maximum value considered here). As depicted in figure 3.4-(b) a codimension-2 point, $C_{2}$, is found for $R e=R e_{C_{2}}=22.65$ and $A R=A R_{C_{2}}=6.98$, where two different global modes, mode $A$, non-oscillating, and mode $B$, oscillating and characterized by a Strouhal number $S t_{C_{2}}=f w / U=\omega / 2 \pi=0.016$, are simultaneously marginally stable. This evidence motivates the weakly nonlinear analysis presented in section 3.4, which aims to investigate the interaction between modes $A$ and $B$. The presence of a second steady mode, denoted by $C$, is also observed. From the linear analysis, a second codimension-2 point appears between the oscillating mode $B$ and the second steady mode $C$, however at a parameter setting $(R e, A R)=(62,4)$, mode $A$ is far above its threshold, which jeopardizes the use of the linear and weakly nonlinear stability tools. Further considerations about the effect of the second steady mode $C$ are provided in Appendix 3.9.2, while hereinafter we will focus on global modes $A$ and $B$ and their global interactions.
As a side remark to figure 3.4-(b), an extrapolation of the marginal stability curve associated to mode $C$ suggests that it would cross the curve of mode $A$ for $R e>100$. Nevertheless, the eigenvalue calculation performed in the range $R e \in[100,230]$ (not visible in 3.4-(b)), showed that for $R e=230$ and $A R=1.75$ the stability boundary is still delimited by mode $A$ ( $C$ does not cross $A$ ). Indeed the two curves for modes $A$ and $C$ seem to approach two asymptotes (as well as the curve for mode $B$ ), whose actual existence could be confirmed by higher Reynolds calculations, which are however beyond the scope of this work.

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Figure 3.5 - Spatial structure of the $x$ - and $y$-velocity components associated with the direct global modes $A$ and $B$ at the codimension-2 point, $C_{2}=\left(R e_{C_{2}}, A R_{C_{2}}\right)=(22.65,6.98)$. (a), (c) xand $y$-velocity fields corresponding to the direct steady mode $A$. (b), (d) Real part of the $\mathrm{x}-$ and y -velocity fields corresponding to the direct oscillating mode $B$.

The symmetry properties which characterized the two global modes $A$ and $B$, reading

$$
\begin{array}{ll}
u_{1}^{A}(x, y)=-u_{1}^{A}(x,-y)=-u_{1}^{A}(-x, y), & v_{1}^{A}(x, y)=v_{1}^{A}(x,-y)=v_{1}^{A}(-x, y), \\
u_{1}^{B}(x, y)=-u_{1}^{B}(x,-y)=u_{1}^{B}(-x, y), & v_{1}^{B}(x, y)=v_{1}^{B}(x,-y)=-v_{1}^{B}(-x, y), \tag{3.17}
\end{array}
$$

lead to the following axis boundary conditions:

$$
\begin{array}{ll}
\left.u_{1}^{A}\right|_{\partial \Omega_{h}}=0,\left.\frac{\partial v_{1}^{A}}{\partial y}\right|_{\partial \Omega_{h}}=0, & \left.u_{1}^{A}\right|_{\partial \Omega_{v}}=0,\left.\frac{\partial v_{1}^{A}}{\partial x}\right|_{\partial \Omega_{v}}=0 \\
\left.u_{1}^{B}\right|_{\partial \Omega_{h}}=0,\left.\frac{\partial v_{1}^{B}}{\partial y}\right|_{\partial \Omega_{h}}=0, & \left.v_{1}^{B}\right|_{\partial \Omega_{v}}=0,\left.\frac{\partial u_{1}^{B}}{\partial x}\right|_{\partial \Omega_{v}}=0 . \tag{3.19}
\end{array}
$$

For a given global mode, $\hat{\mathbf{q}}_{1}$, we also compute the corresponding adjoint global mode, $\hat{\mathbf{q}}_{1}^{\dagger}$, which will be used in section 3.4 and which satisfies the adjoint eigenvalue problem,

$$
\begin{equation*}
\left[(\sigma-\mathrm{i} \omega) \mathscr{B}^{\dagger}+\mathscr{A}^{\dagger}\right] \hat{\mathbf{q}}_{1}^{\dagger}=\mathbf{0} . \tag{3.20}
\end{equation*}
$$

where $\mathscr{A}^{\dagger}$ and $\mathscr{B}^{\dagger}$ are the adjoint operators of the linear operator $\mathscr{A}$ and the mass matrix $\mathscr{B}$, obtained by integrating by parts system (3.10).

$$
\mathscr{A}^{\dagger}=\left(\begin{array}{cc}
\mathscr{C}_{A R}^{\dagger}\left(\mathbf{u}_{0}, \cdot\right)-\frac{1}{R e} \Delta_{A R} & -\nabla_{A R}  \tag{3.21}\\
\nabla_{A R}^{T} & 0
\end{array}\right), \quad \mathscr{B}^{\dagger}=\left(\begin{array}{cc}
\mathscr{I} & 0 \\
0 & 0
\end{array}\right) .
$$

Here $\mathscr{C}_{A R}^{\dagger}(\mathbf{a}, \mathbf{b})$ is the adjoint advection operator, which is not symmetric and which reads $\mathscr{C}_{A R}^{\dagger}(\mathbf{a}, \mathbf{b})=-\left(\mathbf{a} \cdot \nabla_{A R}\right)^{T} \mathbf{b}+\left(\mathbf{b} \cdot \nabla_{A R}\right) \mathbf{a}$. The adjoint boundary conditions are defined so that all boundary terms arising from the integration by parts are nil. Thus we obtain,

$$
\begin{equation*}
\left.\mathbf{u}_{1}^{\dagger}\right|_{\partial \Omega_{w}}=\mathbf{0}, \quad\left(\mathbf{u}_{0} \cdot \mathbf{n}\right) \hat{\mathbf{u}}_{1}^{\dagger}+\left.\left(p_{1}^{\dagger} \mathbf{I}+\frac{1}{R e} \nabla \mathbf{u}_{1}^{\dagger}\right) \cdot \mathbf{n}\right|_{\partial \Omega_{o}}=0,\left.\quad \mathbf{u}_{1}^{\dagger}\right|_{\partial \Omega_{i}}=\mathbf{0}, \tag{3.22}
\end{equation*}
$$



Figure 3.6 - Spatial structure of the $x$ - and $y$-velocity components associated with the direct and adjoint global modes at the codimension-2 point, $C_{2}=\left(\operatorname{Re}_{C_{2}}, A R_{C_{2}}\right)=(22.65,6.98)$. (a), (c) x - and y -velocity fields corresponding to the adjoint steady mode $A$. (b), (d) Real part of the x -and y -velocity fields corresponding to the adjoint oscillating mode $B$.

$$
\begin{array}{ll}
\left.u_{1}^{A \dagger}\right|_{\partial \Omega_{h}}=0,\left.\frac{\partial v_{1}^{A \dagger}}{\partial y}\right|_{\partial \Omega_{h}}=0, & \left.u_{1}^{A \dagger}\right|_{\partial \Omega_{v}}=0,\left.\frac{\partial v_{1}^{A \dagger}}{\partial x}\right|_{\partial \Omega_{v}}=0, \\
\left.u_{1}^{B \dagger}\right|_{\partial \Omega_{h}}=0,\left.\frac{\partial v_{1}^{B \dagger}}{\partial y}\right|_{\partial \Omega_{h}}=0, & \left.v_{1}^{B \dagger}\right|_{\partial \Omega_{v}}=0,\left.\frac{\partial u_{1}^{B \dagger}}{\partial x}\right|_{\partial \Omega_{v}}=0 . \tag{3.24}
\end{array}
$$

We checked a posteriori that both direct and adjoint problems have an identical spectrum and the direct and adjoint modes satisfy the bi-orthogonality property (see Meliga et al. (2009a)).
Figures 3.5 and 3.6 show the spatial structure of the velocity fields along the $x$ - and $y$-axis associated with the direct and adjoint global modes $A$ and $B$ respectively. While the direct modes are normalized using the value of the $y$-velocity field, $\hat{v}_{1}$, in a generic grid point, i.e. $(x, y)=(0.5,0)$, the adjoint modes are normalized such that $<\hat{\mathbf{q}}_{1}^{\dagger}, \mathscr{B} \hat{\mathbf{q}}_{1}>=1$, where $<,>$ is the inner product defined by $<\mathbf{a}, \mathbf{b}>=\int_{\Omega} \mathbf{a}^{*} \cdot \mathbf{b} d \Omega$, the star * denotes the complex conjugate and $\cdot$ indicates the canonical hermitian scalar product in $\mathbb{C}^{n}$. This normalization will simplify the expression of the various coefficients derived in section 3.4. In figure 3.5-(b) and (d) the real part velocity components of the oscillating mode along the $x$ - and $y$-axis are represented. Their spatial structure is qualitatively analogous to the one recently presented in the threedimensional study performed by Bertsch et al. (2020a), which confirms that this kind of instability arises in both the two-dimensional and three-dimensional problems for proper combinations of control parameters, $R e$ and $A R$, and which suggests that the same physical mechanism is behind the origin of the self-sustained oscillations regime. As mentioned by Bertsch et al. (2020a), the structure of the perturbation velocity fields of mode $B$ in the left and right output channels and their well defined wave-length is typical of sinuous shear instabilities, like the famous one characterizing the unsteady flow past a circular cylinder (Barkley, 2006; Ding and Kawahara, 1999; Sipp and Lebedev, 2007). From the analysis of the corresponding adjoint mode (see figure $3.6-(b)$ and $(d)$ ), we see that the spatial structure of the adjoint is localized in the central region, near the two inlets. In classic shear instabilities of open flow, a downstream localization of the global mode and an upstream localization of the adjoint global mode resulting from the convective non-normality of the linearized Navier-

Stokes operator (Chomaz, 2005) is observed. Identifying two downstream directions towards the outlets and two upstream directions corresponding to the inlets, a similar characteristic is found. This evidence motivates the detailed investigation, presented in section 3.6, of the nature of this instability, which, from the knowledge of the authors, remained undetermined so far.
Concerning the steady global mode $A$ (see figure 3.5-(a) and (c)), it represents a steady symmetry-breaking condition with respect to the x -axis of symmetry. Given the symmetries of mode $A$, this steady instability corresponds to two possible new steady configurations (bistability), symmetric with respect to the x -axis. It leads to a positive off-set of the stagnation point above the x -axis (respectively a negative off-set below the x -axis) in the y -direction (at $x=0$ ); the two recirculation regions above (respectively below) the axis become smaller than the two below (respectively above) the axis. The corresponding adjoint mode (see figure 3.6-(a) and (c)) maintains a structure similar to the one of the direct mode.
The existence of a steady symmetry-breaking global mode and an oscillating global mode, which can be unstable in different regions of a stability map is also qualitatively consistent with the numerical analysis proposed by Pawlowski et al. (2006), who examined the same 2D-configuration with the only difference that a plug inlet profile was considered (see section section 3.7 for further comments about the influence of a plug inlet velocity profile).

### 3.4 Weakly nonlinear formulation

### 3.4.1 Presentation

Since a codimension-2 point, $C_{2}=\left(\operatorname{Re}_{C_{2}}, A R_{C_{2}}\right)=(22.65,6.98)$, is found from the linear stability analysis, we present in this section a weakly nonlinear analysis in order to investigate the mode interaction between the steady mode $A$ and the oscillating mode $B$. In other words, we implement an asymptotic expansion where the two modes have the same order of magnitude. The departure from criticality, in terms of Reynolds number and aspect ratio, is assumed to be of order $\epsilon^{2}$. Hence, we introduce the two order one parameters, $\delta=\epsilon^{2} \tilde{\delta}$ and $\alpha=\epsilon^{2} \tilde{\alpha}$, such that:

$$
\begin{equation*}
\frac{1}{R e}=\frac{1}{R e_{C_{2}}}-\epsilon^{2} \tilde{\delta}, \quad \frac{1}{A R}=\frac{1}{A R_{C_{2}}}+\epsilon^{2} \tilde{\alpha} . \tag{3.25}
\end{equation*}
$$

In the spirit of the multiple scale technique, we introduce the slow time scale $T=\epsilon^{2} t$, being $t$ the fast time scale defined in (3.1). Hence, the entire flow field is expanded as:

$$
\begin{equation*}
\mathbf{q}=\{u, v, p\}^{T}=\mathbf{q}_{0}+\epsilon \mathbf{q}_{1}+\epsilon^{2} \mathbf{q}_{2}+\epsilon^{3} \mathbf{q}_{3}+\mathrm{O}\left(\epsilon^{4}\right), \tag{3.26}
\end{equation*}
$$

In order to easily write the equations at the various order in $\epsilon$ in a compact form, it is useful to introduce the following expansion for the nabla operator, $\nabla$ :

$$
\begin{equation*}
\nabla_{A R}=\left\{\frac{\partial}{\partial x}, \frac{1}{A R} \frac{\partial}{\partial y}\right\}^{T}=\left\{\frac{\partial}{\partial x}, \frac{1}{A R_{C_{2}}} \frac{\partial}{\partial y}\right\}^{T}+\epsilon^{2} \tilde{\alpha}\left\{0, \frac{\partial}{\partial y}\right\}^{T}=\nabla_{A C_{C_{2}}}+\epsilon^{2} \tilde{\alpha} \nabla_{\alpha}+\mathrm{O}\left(\epsilon^{3}\right) . \tag{3.27}
\end{equation*}
$$

The definition of the Laplacian follows:

$$
\begin{array}{r}
\Delta_{A R}=\nabla_{A R}^{T} \nabla_{A R}=\left(\nabla_{A R_{C_{2}}}+\epsilon^{2} \tilde{\alpha} \nabla_{\alpha}\right)^{T}\left(\nabla_{A R_{C_{2}}}+\epsilon^{2} \tilde{\alpha} \nabla_{\alpha}\right)=  \tag{3.28}\\
\left(\frac{\partial^{2}}{\partial x^{2}}+\frac{1}{A R_{C_{2}}^{2}} \frac{\partial^{2}}{\partial y^{2}}\right)+\epsilon^{2} \frac{2 \tilde{\alpha}}{A R_{C_{2}}} \frac{\partial^{2}}{\partial y^{2}}=\Delta_{A R_{C_{2}}}+\epsilon^{2} 2 \tilde{\alpha} \Delta_{\alpha A R_{C_{2}}}+\mathrm{O}\left(\epsilon^{3}\right) .
\end{array}
$$

Substituting the expansions defined above in the governing equations (3.3)-(3.2) with their boundary conditions, a series of problems at the different orders in $\epsilon$ are obtained.

### 3.4.2 Order $\epsilon^{0}$ : steady base-flow

At order $\epsilon^{0}$ the system is represented by the nonlinear equations for the steady symmetric base-flow (3.5) with boundary conditions (3.6)-(3.9). The solution, computed for $R e_{C_{2}}$ and $A R_{C_{2}}$ via iterative Newton's method, was described in section 3.3.1.

### 3.4.3 Order $\epsilon$ : linear global stability

At leading order in $\epsilon$, the system is represented by the unsteady Navier-Stokes equations linearized around the base-flow for $R e_{C_{2}}$ and $A R_{C_{2}}$, whose solution has been presented in section 3.3.2. In this framework, the solution of the leading order system is assumed to be composed by the sum of the two global modes, $A$ and $B$,

$$
\begin{equation*}
\mathbf{q}_{1}=A(T) \hat{\mathbf{q}}_{1}^{A}+\left(B(T) \hat{\mathbf{q}}_{1}^{B} e^{\mathrm{i} \omega t}+\text { c.c. }\right), \tag{3.29}
\end{equation*}
$$

that destabilized the steady state $\mathbf{q}_{0}$. In equation (3.29), the amplitude $A(T)$, which varies with the slow time scale $T$ and the associated normalized eigenfunction are purely real, while the amplitude $B(T)$ and eigenfunction for mode $B$ are complex. Introducing (3.29) in the $\epsilon-$ order system, a generalized eigenvalue problem for mode $A$ and $B$, whose general form reads $(\mathrm{i} \omega \mathscr{B}+\mathscr{A}) \hat{\mathbf{q}}_{1}=\mathbf{0}$, is retrieved. We remark that at the codimension-2 point, both modes are marginally stable, therefore their growth rate are nil, $\sigma_{A}=\sigma_{B}=0$ in $C_{2}$, while the oscillation frequency of mode $B$ is $\omega=0.10157$.

### 3.4.4 Order $\epsilon^{2}$ : base-flow modifications, mean-flow corrections, mode-interaction and second-harmonic response

At order $\epsilon^{2}$ we obtain the linearized Navier-Stokes equations applied to $\mathbf{q}_{2}=\left\{\mathbf{u}_{2}, p_{2}\right\}^{T}$ :

$$
\begin{equation*}
\left(\mathscr{B} \partial_{t}+\mathscr{A}\right) \mathbf{q}_{2}=\mathscr{F}_{2}, \tag{3.30}
\end{equation*}
$$

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with the boundary conditions

$$
\begin{equation*}
\left.\mathbf{u}_{2}\right|_{\partial \Omega_{w}}=\mathbf{0},\left.\quad\left(-p_{2} \mathbf{I}+\frac{1}{R e_{C_{2}}} \nabla_{A R_{C_{2}}} \mathbf{u}_{2}\right) \cdot \mathbf{n}\right|_{\partial \Omega_{o}}=0,\left.\quad \mathbf{u}_{2}\right|_{\partial \Omega_{i}}=\mathbf{0}, \tag{3.3}
\end{equation*}
$$

and forced by a term $\mathscr{F}_{2}$ depending only on zero and first-order solutions,

$$
\begin{equation*}
\mathscr{F}_{2}=\binom{-\tilde{\delta} \Delta_{A R C_{2}} \mathbf{u}_{0}+\frac{2 \tilde{\alpha}}{R e_{2}} \Delta_{\alpha A C_{C_{2}}} \mathbf{u}_{0}-\tilde{\alpha} \nabla_{\alpha} p_{0}-\frac{\tilde{\alpha}}{2} \mathscr{C}_{\alpha}\left(\mathbf{u}_{0}, \mathbf{u}_{0}\right)-\frac{1}{2} \mathscr{C}_{A R C_{2}}\left(\mathbf{u}_{1}, \mathbf{u}_{1}\right)}{-\tilde{\alpha} \nabla_{\alpha} \cdot \mathbf{u}_{0}}, \tag{3.32}
\end{equation*}
$$

where $\mathscr{C}_{\alpha}$ is the $\epsilon^{2}$-order symmetric advection operator, $\mathscr{C}_{\alpha}(\mathbf{a}, \mathbf{b})=\left(\mathbf{a} \cdot \nabla_{\alpha}\right) \mathbf{b}+\left(\mathbf{b} \cdot \nabla_{\alpha}\right) \mathbf{a}$, while $\mathscr{C}_{A R_{C_{2}}}(\mathbf{a}, \mathbf{b})=\left(\mathbf{a} \cdot \nabla_{A R_{C_{2}}}\right) \mathbf{b}+\left(\mathbf{b} \cdot \nabla_{A R_{C_{2}}}\right)$ a. Terms proportional to $\delta$ and $\alpha$ arise from the Reynolds number and aspect ratio variations with respect to the codimension-2 point definition and they act on the base-flow. The last term in the $y$-component of (3.32) is due to the transport of the first-order solution $\mathbf{q}_{1}$ by itself. Introducing the first-order normal form (3.29) in the forcing term expressed in (3.32), the different contributions can be individualized:

$$
\begin{equation*}
\mathscr{F}_{2}=\underbrace{\tilde{\delta} \hat{\mathscr{F}}_{2}^{\delta}+\tilde{\alpha} \hat{\mathscr{F}}_{2}^{\alpha}+A^{2} \hat{\mathscr{F}}_{2}^{A^{2}}+|B|^{2} \hat{\mathscr{F}}_{2}^{|B|^{2}}}_{\mathscr{F}_{2}^{j}=\left\{\mathscr{F}_{2 x}^{j} \mathscr{F}_{2 y}^{j}\right\}^{T}}+\left(B^{2} \hat{\mathscr{F}}_{2}^{B^{2}} e^{\mathrm{i} 2 \omega t}+A B \hat{\mathscr{F}}_{2}^{A B} e^{\mathrm{i} \omega t}+\text { c.c. }\right) \tag{3.33}
\end{equation*}
$$

Looking at (3.33), we recognize the second harmonic for mode $B$, which is pulsating at $2 \omega \neq \omega$ and thus it does not resonate and does not need the imposition of any compatibility condition. In principle, all the other terms could be classified as resonating terms in mode $A$ or $B$ for which the forced problem results to be singular and hence it is necessary to verify the solvability condition or Fredholm alternative. However, we can make use of the symmetry properties of the various forcing terms, as recently proposed in Camarri and Mengali (2019), to show that some of these conditions are implicitly satisfied. Indeed, the first four forcing terms, having $\omega=0$, are characterized by the following symmetries at the x - and y -axis,

$$
\begin{equation*}
\left.\mathscr{F}_{2 y}^{j}\right|_{\partial \Omega_{h}}=0,\left.\frac{\partial \mathscr{F}_{2 x}^{j}}{\partial y}\right|_{\partial \Omega_{h}}=0,\left.\quad \mathscr{F}_{2 x}^{j}\right|_{\partial \Omega_{v}}=0,\left.\frac{\partial \mathscr{F}_{2 y}^{j}}{\partial x}\right|_{\partial \Omega_{v}}=0, \tag{3.34}
\end{equation*}
$$

which does not coincide with the axis boundary conditions for mode $A$ given in (3.18). Consequently, the solvability condition for $A$ is naturally satisfied by symmetry properties. The same argument is applicable to the last terms oscillating in $\omega$, arising from the direct competition of modes $A$ and $B$, which is characterized by the symmetries,

$$
\begin{equation*}
\left.\hat{\mathscr{F}}_{2 y}^{A B}\right|_{\partial \Omega_{h}}=0,\left.\frac{\partial \hat{\mathscr{F}}_{2 x}^{A B}}{\partial y}\right|_{\partial \Omega_{h}}=0,\left.\quad \hat{\mathscr{F}}_{2 y}^{A B}\right|_{\partial \Omega_{v}}=0,\left.\frac{\partial \hat{\mathscr{F}}_{2 x}^{A B}}{\partial x}\right|_{\partial \Omega_{v}}=0, \tag{3.35}
\end{equation*}
$$

that differ from the boundary conditions for mode $B$ given in (3.19) and automatically satisfy the solvability condition. It follows that using the mentioned symmetry considerations, no solvability condition needs to be imposed at the $\epsilon^{2}$-order. We thus look for a second-order
solution having the expression:

$$
\begin{equation*}
\mathbf{q}_{2}=\tilde{\delta} \hat{\mathbf{q}}_{2}^{\delta}+\tilde{\alpha} \hat{\mathbf{q}}_{2}^{\alpha}+A^{2} \hat{\mathbf{q}}_{2}^{A^{2}}+|B|^{2} \hat{\mathbf{q}}_{2}^{|B|^{2}}+\left(B^{2} \hat{\mathbf{q}}_{2}^{B^{2}} e^{\mathrm{i} 2 \omega t}+A B \hat{\mathbf{q}}_{2}^{A B} e^{\mathrm{i} \omega t}+\text { c.c. }\right) \tag{3.36}
\end{equation*}
$$

where each single response is evaluated by means of a global resolvent technique (Garnaud et al., 2013; Viola et al., 2016a).
All the second-order responses are displayed in figure 3.7 in terms of their x-velocity component. As shown in figure 3.7-( $f$ ) the second harmonic response for the global mode $B$ is essentially periodic in space with a wavelength twice the one of the direct modes (see figure 3.5-(b) and (d)), while the interaction between $A$ and $B$ (see figure 3.7-(e)) is nearly periodic in space with a wavelength close to the one of the direct mode $B$ within a central region near the jets collision, where the direct mode $A$ mainly acts, and it vanishes far away as the mode $A$ vanishes too (see figure 3.5-(a) and (c)).


Figure 3.7 - Second-order responses corresponding respectively to $(a)$-(b) base-flow modifications due to Reynolds number and aspect ratio variations with respect to the codimension-2 point, $C_{2},(c)-(d)$ mean flow correction associated to mode $A$ and $B$ respectively, $(e)-(f)$ harmonic interaction between the steady mode $A$ and the oscillating mode $B$ and second harmonic for mode $B$.

### 3.4.5 Order $\epsilon^{3}$ : amplitude equations

At the $\epsilon^{3}$-order we derive the system of amplitude equations which describe the weakly nonlinear global mode interaction of $A$ and $B$. The problem at order $\epsilon^{3}$ is similar to the one obtained at order $\epsilon^{2}$, as it indeed appears as a linear system forced by the previous order solutions, englobed in $\mathscr{F}_{3}$,

$$
\begin{equation*}
\left(\mathscr{B} \partial_{t}+\mathscr{A}\right) \mathbf{q}_{3}=\mathscr{F}_{3}, \tag{3.37}
\end{equation*}
$$

and subjected to the boundary conditions,

$$
\begin{equation*}
\left.\mathbf{u}_{3}\right|_{\partial \Omega_{w}}=\mathbf{0},\left.\quad\left(-p_{3} \mathbf{I}+\frac{1}{R e_{C_{2}}} \nabla_{A R_{C_{2}}} \mathbf{u}_{3}\right) \cdot \mathbf{n}\right|_{\partial \Omega_{o}}=0,\left.\quad \mathbf{u}_{3}\right|_{\partial \Omega_{i}}=\mathbf{0} \tag{3.38}
\end{equation*}
$$

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The $\epsilon^{3}$-order forcing term, $\mathscr{F}_{3}$, by substituting the first- and second-order solutions, reads:

$$
\begin{gather*}
\left\{\begin{array}{c}
\mathscr{F}_{3}= \\
-\partial_{T} \mathbf{u}_{1}-\tilde{\delta} \Delta_{A R_{C_{2}}} \mathbf{u}_{1}+\frac{2 \tilde{\alpha}}{\operatorname{Re}_{C_{2}}} \Delta_{\alpha A R_{C_{2}}} \mathbf{u}_{1}-\tilde{\alpha} \nabla_{\alpha} p_{1}-\tilde{\alpha} \mathscr{C}_{\alpha}\left(\mathbf{u}_{0}, \mathbf{u}_{1}\right)-\mathscr{C}_{A R_{C_{2}}}\left(\mathbf{u}_{1}, \mathbf{u}_{2}\right) \\
-\tilde{\alpha} \nabla_{\alpha} \mathbf{u}_{1}
\end{array}\right)=  \tag{3.39}\\
=-\frac{\partial A}{\partial T} \mathscr{B} \hat{\mathbf{q}}_{1}^{A}+A\left(\tilde{\delta} \hat{\mathscr{F}}_{3}^{\delta A}+\tilde{\alpha} \hat{\mathscr{F}}_{3}^{\alpha A}\right)+A^{3} \hat{\mathscr{F}}_{3}^{A^{3}}+A|B|^{2} \hat{\mathscr{F}}_{3}^{A|B|^{2}}+ \\
+\left\{\left[-\frac{\partial B}{\partial T} \mathscr{B} \hat{\mathbf{q}}_{1}^{B}+B\left(\tilde{\delta} \hat{\mathscr{F}}_{3}^{\delta B}+\tilde{\alpha} \hat{\mathscr{F}}_{3}^{\alpha B}\right)+|B|^{2} B \hat{\mathscr{F}}_{3}^{|B|^{2} B}+A^{2} B \hat{\mathscr{F}}_{3}^{A^{2} B}\right] e^{\mathrm{i} \omega t}+\text { c.c. }\right\}+\text { N.R.T. } \tag{3.40}
\end{gather*}
$$

where N.R.T. gathers all the non-resonating terms, not relevant for the further analysis and omitted thereafter. The first term in the $y$-component of equation (3.39) corresponds to the slow time evolution of the amplitudes $A(T)$ and $B(T)$ with the slow time scale $T=\epsilon^{2} t$. The last term is due to the advection of the leading order solution by the second-order solution and vice versa. All the other terms arise from the Reynolds number and aspect ratio variation acting on the $\epsilon$-order solution. As standard in multiple scale analysis, in order to avoid secular terms and solve the expansion procedure at the third order, a compatibility condition must be enforced through the Fredholm alternative (Friedrichs, 2012).

The compatibility condition imposes the amplitudes $A(T)$ and $B(T)$ to obey the following relations:

$$
\begin{gather*}
\frac{d A}{d t}=\left(\delta \zeta_{A}+\alpha \eta_{A}\right) A-\mu_{A} A^{3}-\chi_{A} A|B|^{2},  \tag{3.41}\\
\frac{d B}{d t}=\left(\delta \zeta_{B}+\alpha \eta_{B}\right) B-\mu_{B}|B|^{2} B-\chi_{B} B A^{2} . \tag{3.42}
\end{gather*}
$$

where the physical time scale $t$ has been reintroduced, $\delta=\epsilon^{2} \tilde{\delta}=1 / R e_{C_{2}}-1 / R e, \alpha=\epsilon^{2} \tilde{\alpha}=$ $1 / A R-1 / A R_{C_{2}}$ and the various coefficients, whose values are reported in Appendix 3.9.1, are computed as scalar products between the adjoint global modes $\hat{\mathbf{q}}_{1}^{\dagger}$ and the resonant forcing terms $\hat{\mathscr{F}}_{3}^{i}$, i.e. for instance,

$$
\begin{equation*}
\zeta_{A}=\frac{\left\langle\hat{\mathbf{q}}_{1}^{A \dagger}, \hat{\mathscr{F}}_{3}^{\delta A}\right\rangle}{\left\langle\hat{\mathbf{q}}_{1}^{A \dagger}, \mathscr{B} \hat{\mathbf{q}}_{1}^{A}\right\rangle}=\left\langle\hat{\mathbf{q}}_{1}^{A \dagger}, \hat{\mathscr{F}}_{3}^{\delta A}\right\rangle, \quad \zeta_{B}=\frac{\left\langle\hat{\mathbf{q}}_{1}^{B \dagger}, \hat{\mathscr{F}}_{3}^{\delta B}\right\rangle}{\left\langle\hat{\mathbf{q}}_{1}^{B \dagger}, \mathscr{B} \hat{\mathbf{q}}_{1}^{B}\right\rangle}=\left\langle\hat{\mathbf{q}}_{1}^{B \dagger}, \hat{\mathscr{F}}_{3}^{\delta B}\right\rangle, \tag{3.43}
\end{equation*}
$$

since $\left\langle\hat{\mathbf{q}}_{1}^{A \dagger}, \mathscr{B} \hat{\mathbf{q}}_{1}^{A}\right\rangle=\left\langle\hat{\mathbf{q}}_{1}^{B \dagger}, \mathscr{B} \hat{\mathbf{q}}_{1}^{B}\right\rangle=1$ due to the normalization introduced in section 3.3.2. The detailed expression of each normal form coefficient is provided in Appendix A. Equations (3.41)-(3.42) differ from the classic Stuart-Landau equations, describing the pitchfork and Hopf bifurcations of single modes, by the two coupling terms, $\chi_{A} A|B|^{2}$ and $\chi_{B} B A^{2}$, coming from third-order nonlinearities. The structure of system (3.41)-(3.42) is well known in literature and is analogous to that derived by Meliga et al. (2012a), where a formally equivalent analysis is performed to investigate weakly nonlinear interactions for mode selection in swirling jets.

## Stability analysis of the amplitude equations

Here we perform the stability analysis of the amplitude equations (3.41)-(3.42). Recalling that the amplitude $A$ is purely real, as well as all the coefficients associated with its equation, while amplitude $B$ and the related amplitude equation coefficients are complex so that we can turn to polar coordinates, i.e. $B=|B| e^{i \Phi_{B}}$, and split the modulus and phase parts of equations (3.41)-(3.42):

$$
\begin{gather*}
\frac{d A}{d t}=\left(\delta \zeta_{A}+\alpha \eta_{A}\right) A-\mu_{A} A^{3}-\chi_{A} A|B|^{2}  \tag{3.44}\\
\frac{d|B|}{d t}=\left(\delta \zeta_{B r}+\alpha \eta_{B r}\right)|B|-\mu_{B r}|B|^{3}-\chi_{B r}|B| A^{2}  \tag{3.45}\\
\frac{d \Phi_{B}}{d t}=\left(\delta \zeta_{B i}+\alpha \eta_{B i}\right)-\mu_{B i}|B|^{2}-\chi_{B i} A^{2} \tag{3.46}
\end{gather*}
$$

System (3.44)-(3.45) presents different possible equilibria (Kuznetsov, 2013). Below the threshold the system is stable and the trivial equilibrium with $A=|B|=0$ is retrieved. Two other possible equilibria correspond to $(A \neq 0,|B|=0)$ (pitchfork bifurcation for mode $A$ ) or $(A=0,|B| \neq 0)$ (Hopf bifurcation for mode $B$ ). The single mode pitchfork and Hopf bifurcations are easily found, removing the coupling terms by setting $\chi_{A}=\chi_{B}=0$ and looking for a stationary solution of equations (3.44)-(3.45), $\frac{d A}{d t}=\frac{d|B|}{d t}=0$. This leads to the classic solutions,

$$
\begin{equation*}
A^{2}=\frac{\delta \zeta_{A}+\alpha \eta_{A}}{\mu_{A}}, \quad|B|^{2}=\frac{\delta \zeta_{B r}+\alpha \eta_{B r}}{\mu_{B r}} \tag{3.47}
\end{equation*}
$$

The non-trivial equilibrium with $(A \neq 0,|B| \neq 0)$ is obtained reintroducing the coupling terms and investigating the existence of a parameter region in which both mode coexist. Indeed, looking for a stationary solution $\frac{d A}{d t}=\frac{d|B|}{d t}=0$ we obtain the following system,

$$
\left[\begin{array}{cc}
\mu_{A} & \chi_{A}  \tag{3.48}\\
\chi_{B r} & \mu_{B r}
\end{array}\right]\left\{\begin{array}{c}
A^{2} \\
|B|^{2}
\end{array}\right\}=\left\{\begin{array}{c}
\delta \zeta_{A}+\alpha \eta_{A} \\
\delta \zeta_{B r}+\alpha \eta_{B r}
\end{array}\right\}
$$

which admits a physical solution only for $R e$ and $A R$ values for which $A^{2}>0$ and $|B|^{2}>0$. Solving (3.48), we get:

$$
\begin{align*}
A^{2} & =\frac{\left(\delta \zeta_{A}+\alpha \eta_{A}\right) \mu_{B_{r}}-\chi_{A}\left(\delta \zeta_{B_{r}}+\alpha \eta_{B_{r}}\right)}{\mu_{A} \mu_{B_{r}}-\chi_{A} \chi_{B_{r}}}  \tag{3.49}\\
|B|^{2} & =\frac{\left(\delta \zeta_{B_{r}}+\alpha \eta_{B_{r}}\right) \mu_{A}-\chi_{B_{r}}\left(\delta \zeta_{A}+\alpha \eta_{A}\right)}{\mu_{A} \mu_{B_{r}}-\chi_{A} \chi_{B_{r}}} \tag{3.50}
\end{align*}
$$

The general relation for the phase of mode $B$ at large time, which varies linearly in time, reads,

$$
\begin{equation*}
\left.\Phi_{B}\right|_{t \rightarrow+\infty}=\left[\left(\delta \zeta_{B i}+\alpha \eta_{B i}\right)-\mu_{B i}|B|^{2}-\chi_{B i} A^{2}\right] t \tag{3.51}
\end{equation*}
$$

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meaning that the frequency at large time will saturate to the following prescribed valued, function of the Reynolds number and aspect ratio variation with respect to the codimension-2 point, $C_{2}$ :

$$
\begin{equation*}
\left.\omega\right|_{t \rightarrow+\infty}=\omega_{C_{2}}+\left[\left(\delta \zeta_{B i}+\alpha \eta_{B i}\right)-\mu_{B i}|B|^{2}-\chi_{B i} A^{2}\right] . \tag{3.52}
\end{equation*}
$$



Figure 3.8 - (a) Weakly nonlinear map predicted by the normal form (3.41)-(3.42) in the ( $R e, A R$ )-plane. Green and blue dotted lines indicate the linear marginal stability curves for mode $A$ and $B$ respectively, as presented in section 3.3.2. In the black region the steady mode $A$ prevails, while the oscillating mode $B$ dominates in the wide grey region. A region of hysteresis, highlighted in light grey shade, is found for $A R$ smaller than $A R_{C_{2}}$. (b) Bifurcation diagram as a function of the Reynolds number for a fixed value of aspect ratio, $A R=6.5<A R_{C_{2}}$. Dashed and dot-dashed lines mean unstable branches, while solid lines denote stable branches. The vertical red dotted lines represents the thresholds for the pitchfork bifurcation of mode $A$ $(P A)$, the backward bifurcation of mode $A(B A)$ and the secondary Hopf bifurcation of mode $B$ $(S H B)$. The light gray shaded region corresponds to the hysteresis range of (a). (c) Bifurcation diagram as a function of the Reynolds number for a fixed value of aspect ratio, $A R=7.5>A R_{C_{2}}$. The vertical red dotted lines represents the thresholds for the Hopf bifurcation for mode $B$.

Combining all these ingredients, the bifurcation diagram proposed in figure 3.8-(a), (b) and (c) presents a complex series of bifurcations. The stability of the various branches was numerically assessed by time-marching equations (3.44)-(3.45) using the Matlab function ode23. As depicted in figure 3.8-(b) for a fixed value of aspect ratio, $A R=6.5<A R_{C_{2}}$, the steady mode $A$ bifurcates first at $R e_{P A}=23.85$ (pitchfork bifurcation PA) breaking the symmetry of the base-flow with respect to the x-axis, $\partial \Omega_{h}$. The oscillating mode $B$ then bifurcates from the x -symmetry-breaking pitchfork bifurcation at $R e_{S H B}=24.63$ through a secondary Hopf bifurcation $(S H B)$. Notwithstanding the subcritical nature of this bifurcation, which makes it unstable, such a bifurcated branch is fundamental for the emergence of the self-sustained oscillation regime, through a backward bifurcation of mode $A(B A)$ at $R e_{B A}=24.35$. The sub-criticality of the system in the range $R e_{B A}<R e<R e_{S P B}$ leads to an hysteretic behaviour where either the steady mode $A$ or the oscillating mode $B$ can dominate, depending on the initial conditions to which the system is subjected. Figure 3.8-(a) shows the full weakly nonlinear map predicted by the normal form (3.41)-(3.42) in the (Re, $A R$ )-plane around the codimension-2 point. Lastly, as shown in figure 3.8-(c), above $A R_{C_{2}}$ only the oscillating mode $B$, which settles into a limit cycle via classic Hopf bifurcation ( $H B$ ), exists. In this range, the self-sustained oscillations regime is observed above a certain Reynolds number.

### 3.5 Comparison with direct numerical simulations (DNS)

In this section, the results derived in section 3.4 via weakly nonlinear analysis are compared with direct numerical simulations (DNS). The full nonlinear unsteady dynamics represented by the system of governing equations (3.3)-(3.2) with its boundary conditions is solved using the open-source code Nek5000, as described in section 3.2.
In addition to the fluid governing equations, Nek5000 allows to easily introduce a further advection-diffusion equation describing the dynamics of a passive scalar, $\Phi$,

$$
\begin{equation*}
\frac{\partial \Phi}{\partial t}+\mathbf{u} \cdot \nabla \Phi=\frac{1}{P e} \Delta \Phi \tag{3.53}
\end{equation*}
$$

which enables us to reproduce the presence of two dyes continuously injected through the inlets, in order to visualize the instantaneous flow configuration (Bertsch et al., 2020a). The Péclet number, $P e$, appearing in (3.53) has been set to $P e=100$, a value which ensures good numerical stability and a satisfactory flow visualization at the same time for all the explored cases. Concerning this passive scalar equation, Dirichlet boundary conditions are imposed at the two inlets ( $\left.\Phi\right|_{y=-\frac{s}{2}}=0,\left.\Phi\right|_{y=\frac{s}{2}}=1$ ) to reproduce the injection of two different dyes, while outflow conditions are set at the outlets; no-flux is allowed through the solid walls.

### 3.5.1 Regime comparison

In figure 3.9 the nonlinear map of figure 3.8-(a) for a specific value of aspect ratio in the region characterized by the hysteretic behaviour, i.e. $A R=6.5$, is recalled. Different direct

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numerical simulations, covering the range of Reynolds numbers from the stable region ( $R e<$ $\left.R e_{P A}=23.85\right)$ to the region dominated by the oscillating mode $B\left(R e>R e_{S H B}=24.63\right)$ were performed. The investigated cases are indicated in figure 3.9 with symbols. The results extracted from the DNS are presented in figure 3.10 and 3.11.


Figure 3.9 - Nonlinear map of figure 3.8-(a) for a specific value of aspect ratio in the region characterized by the hysteretic behaviour, i.e. $A R=6.5$. Cases investigated by performing direct numerical simulations are indicated by symbols. The red diamond ( $R e=24.55$ ) corresponds to a case in which the existence of the hysteresis region has been checked using the same control parameters, $A R$ and $R e$, but different initial conditions, given by the final steady state or limit cycle of the two closest simulations, whose Reynolds number have been increased or decreased respectively, as sketched by the green arrows.


Figure 3.10 - Snapshots of the flow patterns in terms of dyes concentrations observed at large time, $(a)-(d)$ once the steady state (stable base-flow or symmetry breaking for mode $A$ ) is reached or, alternatively, $(e)-(f)$ once the limit cycle for mode $B$ is fully established, for the various Reynolds numbers indicated by white squares in figure 3.9. The white dashed lines represent the axes of symmetry characterizing the steady base-flow. The flow configuration for $R e=24.55$ is shown in figure 3.11.

All the numerical simulations displayed in figure 3.10 were started from zero initial conditions.


Figure 3.11 - Snapshots of the flow patterns in terms of dyes concentrations observed at large time for $A R=6.5$ and the same Reynolds number, $R e=24.55$ (hysteresis region), but with different initial conditions: $(a)$ the new steady state (symmetry-breaking condition) obtained for $R e=24.4$ is used as initial condition; (b) a time-instant of the unsteady solution corresponding to $R e=24.7$ (limit cycle for mode $B$ ) is imposed as initial condition. Streamlines and arrows are used to visualize the velocity fields associated with the steady and oscillating configurations respectively.

Figure 3.10-(a) shows the steady-state obtained for $R e=22$, which confirms that the steady base-flow is stable for $R e<R e_{P A}=23.85$, indeed no symmetry breaking can be observed. For $R e=24,24.2$ and 24.4 (which lies in the hysteresis range) we retrieved that the steady mode $A$ first bifurcates via a pitchfork bifurcation. The symmetry with respect to the $x$ axis is lost, the position of the stagnation point lies below the $x$-axis of symmetry and the size of the recirculation regions differs on either side of the x-axis of symmetry $\partial \Omega_{h}$. For $R e>R e_{S H B}=24.63$, i.e. $R e=24.7,24.84,25$ and 27, only the self-sustained oscillation regime is observed.

In order to confirm the existence of the hysteretic behaviour found in section 3.4.5, the solutions obtained at large time for $R e=24.4$ (asymmetric steady configuration) and $R e=24.7$ (limit cycle for the self-oscillations) are used as initial conditions for two more simulations, where the Reynolds number is fixed to $R e=24.55$ in both cases (filled red diamond and green arrows in figure 3.9). The results of this numerical procedure are given in figure 3.11. We clearly see in figure 3.11-(a) and (b) that depending on the initial conditions imposed to the system, for this fixed value of $R e=24.55$ within the hysteresis region, both modes can emerge. Other simulations (not shown) performed in the upper region of figure 3.8-(a), i.e. for $R e=25$ and $A R=7.5$ and 8 , confirm the supercritical nature of the Hopf bifurcation associated to mode $B(H B)$.
Next, global linear stability and weakly nonlinear analysis are both compared with the direct numerical simulations in terms of amplitude of modes $A$ and $B$ and oscillation frequency for the self-sustained oscillatory regime.

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Figure 3.12 - (a) Oscillation frequency, extracted from the y-velocity component at $(x, y)=$ $(0.5,0)$, versus Reynolds number for $A R=6.5$. The solid black line indicates the weakly nonlinear analysis (WNL). The dotted black line and plus signs represent the linear global stability analysis (LGS). The dashed black line and circles are associated with the direct numerical simulations (DNS) performed. (b) Amplitude of modes $A$ and $B$ extracted from DNS and compared with the bifurcation diagram obtained from the WNL analysis for $A R=6.5$. Symbols and green arrows in (b) correspond to the ones introduced in figure 3.9 and are associated to the DNS presented above.

### 3.5.2 Frequency comparison

Figure 3.12-(a) shows that, near the threshold, the linear global stability analysis (LGS), the weakly nonlinear analysis (WNL) and direct numerical simulation (DNS) agree well and prescribe the correct oscillation frequency. However, if the LGS soon diverges from the DNS, as extensively described in the literature (Barkley, 2006), the WNL theory, applied to the problem presented in this work provides a wider range of Reynolds numbers in which the model follows the DNS trend with a satisfactory agreement, showing an error of $3 \%$ for $R e=40$ against the $13.2 \%$ of the LGS. Additionally, it needs to be underlined that the results shown in this section refer to an aspect ratio of $A R=6.5$, hence a double offset (in terms of $R e$ and $A R$ ) with respect to the codimension-2 point, $C_{2}$, is considered in the WNL curve of figure 3.12. Indeed, the precision of the asymptotic expansion prediction increases as $\left|R e-R e_{C_{2}}\right|$ and $\left|A R-A R_{C_{2}}\right|$ decrease.

### 3.5.3 Amplitude comparison

In figure 3.12-(b) we compare the amplitude of mode $A$ and $B$ extracted from the DNS with the ones prescribed by the WNL model. The total flow solutions in the steady and oscillatory regimes evaluated via weakly nonlinear formulation read:

$$
\begin{gather*}
\mathbf{u}_{W N L}^{A}=\mathbf{u}_{0}+A \mathbf{u}_{1}^{A}+\delta \mathbf{u}_{2}^{\delta}+\alpha \mathbf{u}_{2}^{\alpha}+A^{2} \mathbf{u}_{2}^{A^{2}}  \tag{3.54}\\
\mathbf{u}_{W N L}^{B}=\mathbf{u}_{0}+\left(B \mathbf{u}_{1}^{B} e^{\mathrm{i} \omega t}+\text { c.c. }\right)+\delta \mathbf{u}_{2}^{\delta}+\alpha \mathbf{u}_{2}^{\alpha}+|B|^{2} \mathbf{u}_{2}^{|B|^{2}}+\left(B^{2} \mathbf{u}_{2}^{B^{2}} e^{\mathrm{i} 2 \omega t}+\text { c.c. }\right) \tag{3.55}
\end{gather*}
$$

Specifying equations (3.54) and (3.55) for the y-velocity components, $v_{W N L}^{A, B}$, at the x-axis of symmetry $(y=0)$, given the symmetries of the various terms, we have $v^{A}(x, 0)_{W N L}=A v_{1}^{A}(x, 0)$ and $\nu^{B}(x, 0)_{W N L}=\left(B v_{1}^{B}(x, 0) e^{\mathrm{i} \omega t}+c . c.\right)$. Selecting then the point $x=0.5$, used to normalize the global modes $\left(v_{1}^{A}(0,5,0)=1\right.$ and $\left.v_{1}^{B}(0.5,0)=1\right)$, we derive the following simple expressions,

$$
\begin{gather*}
v^{A}(0.5,0)_{W N L}=A  \tag{3.56}\\
v^{B}(0.5,0)_{W N L}=2|B| \cos \left(\omega t+\Phi_{B}\right) \tag{3.57}
\end{gather*}
$$

which allows us to easily compare the amplitudes $A$ and $|B|$ from the WNL model with the ones extracted from the DNS, $v(0.5,0)_{D N S}$. Figure $3.12-(b)$ shows not only a qualitative but also a quantitative agreement between DNS and WNL, which captures well the hysteretic behaviour of the flow for $A R<A R_{C_{2}}$ (sufficiently close to $A R_{C_{2}}$ ).

### 3.5.4 Evolution of the oscillation frequency with the aspect ratio

A linear dependence of the self-oscillations on the inverse of the spacing between the jets, $s$, which highlights the importance of the distance $s$ in the oscillatory phenomenon, was observed in Bertsch et al. (2020a), who proposed the following scaling law,

$$
\begin{equation*}
f \sim \frac{U}{s} \tag{3.58}
\end{equation*}
$$

where the slope, derived by fitting the experimental data, was seen to be approximatively $1 / 6$, which is also consistent with the measurements made by Denshchikov et al. (1978) on large scale facing jets in turbulent flow conditions. Given the definition of the Strouhal number introduced in section 3.3.2, $S t=f w / U$, and the aspect ratio $A R=s / w$, the non-dimensional form of the scaling law (3.58) reads

$$
\begin{equation*}
S t_{B} \sim \frac{1}{A R}, \tag{3.59}
\end{equation*}
$$

where the subscript ${ }_{B}$ is used to denote the frequency associated with the oscillating mode $B$. It follows that in our two-dimensional model, equation (3.58) translates in a linear dependence of the Strouhal number on $1 / A R$. Moreover, according to such a law, the variation of $S t$ with $R e$ is not predominant.
In order to verify whether the evolution of the oscillation frequency in our 2D flow behaves similarly to that of the 3D one studied in Bertsch et al. (2020a), we performed a series of direct numerical simulations fixing $R e$, i.e. $R e=R e_{C_{2}}=22.65$, and varying $A R\left(>A R_{C_{2}}\right)$. A quantitative comparison of DNS and the WNL model is shown in figure 3.13.
In this context, it is important to note that the parameter $1 / A R$ in (3.59) naturally appears in the weakly nonlinear formulation, which, indeed, prescribes a linear variation of the dimensionless frequency with $1 / A R$, as displayed in figure 3.13. DNS results agree well with the WNL model, which also provides a theoretical expression for the slope $m$ indicated in figure 3.13. In analogy with Bertsch et al. (2020a) and Denshchikov et al. (1978), we retrieved a factor close to

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Figure 3.13 - Variation of the Strouhal number, $S t_{B}=f w / U$, with the aspect ratio, $A R=s / w$, for a fixed Reynolds number, $R e=R e_{C_{2}}=22.65$. Black solid line: weakly nonlinear analysis (WNL). Black circles: direct numerical simulation (DNS). Inset: variation of $S t_{B}$ with $R e$ for different $A R$ according to the WNL model. For values of Re smaller than the WNL stability boundary, the instability does not occur and no oscillations can be observed.

1/6.
Moreover, as shown in the inset of figure 3.13 (see also figure 3.12-(a)), the dependence of the frequency on the Reynolds number is much weaker (at least in the first range of $R e$ ) than the dependence on $A R$, which is in agreement with the scaling law (3.59).

### 3.6 Instability mechanisms: sensitivity analysis

The presence of a stationary $(y,-y)$ symmetry-breaking bifurcation, as revealed by the existence of the $A$ mode analyzed in this study bears a certain similarity with sudden expansion flows, where the origin of the symmetry-breaking instability was found to lie in the recirculation regions (Fani et al., 2012; Lanzerstorfer and Kuhlmann, 2012; Lashgari et al., 2014). The physical mechanism associated to the symmetry breaking is often referred to as a Coandă effect, where the shear layers surrounding the recirculation regions are deflected towards one of the two confining walls.
The present flow however is characterized not only by two, but rather by four symmetric recirculation regions surrounding an hyperbolic stagnation point. Note that the existence of four recirculation regions invariant under two axial symmetries and one central symmetry suggests also the possibility for an $(x,-x)$ symmetry breaking, akin to the buckling of two colliding jets at their meeting point. The presence of a hyperbolic stagnation point is also known to give rise to the so-called hyperbolic instability (Friedlander and Vishik, 1991; Lifschitz and Hameiri, 1991), which was found to contribute in the destabilization of arrays of vortices (Godeferd et al., 2001; Ortiz and Chomaz, 2011; Sipp et al., 1999). This instability mechanism, which is
best understood in the short-wave and inviscid asymptotic limits, is however known to give rise to spanwise disturbances which cannot be active within the present 2D framework.
Turning our attention now to the self-sustained oscillatory global mode B, we note that the simple scaling of its intrinsic frequency with the physical parameters observed in the threedimensional microfluidic experiments and numerical simulations of Bertsch et al. (2020a) does not yet point clearly to a physical governing mechanism. As tentatively argued in Bertsch et al. (2020a), the oscillatory nature of this instability suggests the presence of a feedback mechanism, as the ones investigated in Villermaux et al. (1993); Villermaux and Hopfinger (1994). This suggests several candidates, such as the presence of a pocket of absolute instability, or a global pressure feedback. The perturbation field numerically extracted in Bertsch et al. (2020a) and qualitatively retrieved in the present study, shows a sinuous structure in the left and right outlet channels which is reminiscent of two synchronized sinuous shear instabilities. This suggests that the Kelvin-Helmholtz instability of the confined jet profiles prevailing in the outlet channels participates in the self-sustained oscillation process.
We have indeed determined the dispersion relation of the streamwise velocity profiles pertaining at different streamwise stations ( $x \in[1,10]$ ) in the side arm for $A R=A R_{C_{2}}=6.98$ and $R e=R e_{C_{2}}=22.65$ (see figure 3.15-(b)). We have found that the sinuous mode was indeed unstable in the region $1<x<5$ (see Appendix C for more details), while the varicose mode remained damped. This indicates that in this region, the shear is sufficiently intense for the Kelvin-Helmholtz instability to overcome the conjugate stabilizing effect of confinement and viscosity. Additionally, we found that the most unstable wavelength was close to 9 , in visual agreement with figure $3.5-(d)$, while the associated frequency was 0.1 , also in good agreement with the global mode frequency.
However, in order to translate into a self-sustained global instability, this shear layer instability would either need to be of absolute nature, possibly because of the presence of nearby walls, known to enhance absolute instability in confined shear flows (Biancofiore and Gallaire, 2011; Healey, 2009; Juniper, 2006; Rees and Juniper, 2010). As explained in Appendix C, our calculations however showed that the instability remains convective in the entire unstable region $x \in[1,5]$. Another source of strong shear is represented by the two facing $y$-velocity jets issuing from the inlets (see figure 3.15-(a)). Indeed, even if iso-thermal jets are usually known to be convectively unstable, the present geometry differs from a classical free jet. The two jets face each other and collide, slowing down while redirecting fluid towards the outlets. In this interaction region, the flow is however far from weakly non-parallel and the application of local stability analysis is therefore questionable.
Global instability of shear flows in open flows has indeed been historically studied under the parallel flow assumption, where the local linear stability theory is applied to determine whether the flow is absolutely unstable and hence a global instability is to be expected (Huerre and Monkewitz, 1985). Further progresses has been made by extending the analysis to spatially developing (Chomaz et al., 1988; Huerre and Monkewitz, 1990) with the introduction of the WKBJ approximation for weakly non-parallel flows, which extends the domain of validity of the local analysis and provides fair agreement when compared with the linear global stability analysis (Siconolfi et al., 2017; Viola et al., 2016a). Meanwhile, global stability analysis
(sometimes called bi-global (Theofilis, 2011)) has become increasingly popular in recent years, thanks to the large memory capabilities of modern computers.
As mentioned above, the flow is strongly nonparallel in both the x - and y -directions in the central interaction region of the X -junction, which jeopardizes the chances to apply successfully a weakly non-parallel approach to determine the physical mechanism governing this oscillatory instability. We thus propose to follow a different approach to investigate the nature of the instability.
The approach proposed in this section makes use of the properties of the adjoint eigenfunctions associated with the direct eigenmodes and it is formally known as sensitivity analysis. Following Giannetti and Luchini (2007), Chomaz (2005) popularized the definition of the wavemaker region as the region of the flow which is predominantly active in sustaining the global instability. He demonstrated that the wavemaker region can be identified as the overlapping region between the direct and adjoint global eigenvectors. Giannetti and Luchini (2007) indeed demonstrated that the concept of wavemaker identifies regions of the flow where the presence of a local instantaneous feedback produces the strongest drift of the leading eigenvalue. The wavemaker region has then been successfully used to analyze the canonical circular cylinder wake flow (Camarri and Iollo, 2010; Giannetti et al., 2019, 2010; Marquet et al., 2008). Meliga et al. (2009c) applied the theory to the wake of solid disks and spheres, while Ledda et al. (2018) made use of the wavemaker definition in the understanding of the suppression of von Kármán vortex streets past porous rectangular cylinders.
Here we apply the theory of sensitivity analysis in order to investigate both the nature of the steady symmetry-breaking mode and to identify the physical mechanism from which the self-sustained oscillations originate.

### 3.6.1 Core of the steady symmetry-breaking instability

As mentioned, the wavemaker region is defined by the overlapping region of the direct and adjoint global modes. Using the results from the global stability analysis presented in section 3.3.2, the direct and adjoint velocity fields for the steady mode $A$, here analyzed and shown in figure 3.5 and 3.6, are used to build the wavemaker, defined as the product of the direct and adjoint velocity magnitudes $\left\|\hat{\mathbf{u}}_{1}^{A}\right\| \cdot\left\|\hat{\mathbf{u}}_{1}^{A \dagger}\right\|$.
The resulting wavemaker region for $R e=22.65$ and $A R=6.98$, normalized by its maximum value, $\max \left(\left\|\hat{\mathbf{u}}_{1}^{A}\right\| \cdot\left\|\hat{\mathbf{u}}_{1}^{A+}\right\|\right)$ is displayed in figure 3.14 , together with streamlines extracted by the steady base-flow and the edge of the four symmetric recirculation bubbles. The stagnation point in $x=0$ and $y=0$ is clearly highlighted by the streamlines. The spatial distribution of the wavemaker is concentrated in the origin of the fluid domain, perfectly coincident with the stagnation point. As shown in section 3.5 , such instability leads to an offset of the stagnation point with respect to the x -axis of symmetry. While quite similar to symmetry-breaking bifurcations in expansion flows, the physical origin of the A-instability lies therefore probably here more in the structural instability of the stagnation point than in a Coandă effect where the side jets are attracted towards one wall. Broadly speaking, as one jet prevails over the other, the stagnation point is translated in either direction along the $y$-axis, and the streamlines are


Figure 3.14 - Structural sensitivity to a local feedback of the steady global mode $A$ for $R e=$ $R e_{C_{2}}=22.65$ and $A R=A R_{C_{2}}=6.98$. Color map: wavemaker region. Black lines: streamlines extracted from the steady base-flow, $\mathbf{u}_{0}$. Red dashed lines: recirculation bubble edges.
bent with the dominated jet that has less space to curve towards the outlet channels. The size of the recirculation regions is readapted to maintain a steady configuration.

Note that a buckling-like instability of the colliding symmetric jets, in analogy with the classic buckling typical of structural mechanics, would intuitively lead to a steady bending of the jets which would displace the stagnation point along the $y=0$ axis towards a positive or negative $x \neq 0$ offset. Whether the second steady mode $C$ can be reasonably interpreted as such is discussed in Appendix 3.9.2.

### 3.6.2 Physical mechanism behind the origin of the self-sustained oscillatory mode B

## Structural sensitivity: wavemaker region

Here we apply the same technique to the oscillatory instability. The wavemaker for mode $B$ is thus given by $\left\|\hat{\mathbf{u}}_{1}^{B}\right\| \cdot\left\|\hat{\mathbf{u}}_{1}^{B \dagger}\right\|$. Figure 3.15-(c) displays as a color map the wavemaker region for $R e=22.65$ and $A R=6.98$, normalized by its maximum value. From the observation of the wavemaker region, it can be deduced that, as expected, the origin of the oscillations is located in the central portion of the domain, where the two facing jets strongly interact with each other. Moreover, the structure of the wavemaker associated with the oscillating global mode $B$ coincides with all the aspect ratio values which have been checked, i.e. $A R \in[6,20]$. Further progress in understanding the physical mechanism of this instability can be made by analyzing the vorticity field and the local maximum shear. The local $x$ - and $y$-velocity profiles, independently considered as in the standard local and parallel linear theory and shown in figure 3.15-( $a$ ) and (b), have been analyzed and the corresponding loci of maximum shear, taken section by section, are plotted as red dashed lines in figure 3.15-(c). It is seen that the local maximum shear related to the local y-velocity profiles follows surprisingly well the region of maximum values of the wavemaker. Additionally, the wavemaker presents four symmetric maximum intensity points (white circles in figure 3.15-(c)) which approximately coincide with the intersections of the local maximum shear for the $y$ - and $x$-velocity profiles.
As a final comment, we can thus argue that, while the non-parallelism of the flow in the

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Figure 3.15-Structural sensitivity to a local feedback of the oscillating global mode for $\mathrm{Re}=$ $R e_{C_{2}}=22.65$ and $A R=A R_{C_{2}}=6.98$. (a) and (b) y-and x-base-flow velocity profiles, $\nu_{0}$ and $u_{0}$, independently considered as in the classic local parallel theory. (c) Color map: wavemaker region. White circles: maximum value of the normalized wavemaker. Black contours: baseflow vorticity field. Dashed red line: maximum shear of the y - and x - base-flow velocity profiles, $v_{0}$ and $u_{0}$ displayed in ( $a$ ) and ( $b$ ) respectively.
central region precludes the use of the classic local and parallel analysis to compare the present study and to firmly confirm the Kelvin-Helmholtz mechanism as the origin of the oscillatory instability, figure 3.15 suggests that the regions of maximum shear and the interaction of various shear layers play an important role in the physical mechanism engineering the global self-sustained oscillation.

## Sensitivity to base-flow modifications

Let us now consider the sensitivity analysis to arbitrary and small-amplitude base-flow modifications, $\delta \mathbf{u}_{\mathbf{0}}$. In the linear global stability framework, the parameter that defines if a mode is stable or unstable for a certain combination of control parameters, i.e. Reynolds number, is the growth rate, $\sigma$. We thus focus on the sensitivity of the growth rate associated with the global mode $B, \nabla_{\mathbf{u}_{0}} \sigma_{B}$, which is a real quantity expressed as,

$$
\begin{equation*}
\nabla_{\mathbf{u}_{0}} \sigma_{B}=\underbrace{-\operatorname{Re}\left(\left(\nabla_{\mathbf{u}_{0}} \cdot \hat{\mathbf{u}}_{1}^{B}\right)^{H} \cdot \hat{\mathbf{u}}_{1}^{B \dagger}\right)}_{\nabla_{\mathbf{u}_{0}, T} \sigma_{B}}+\underbrace{\operatorname{Re}\left(\nabla_{\mathbf{u}_{0}} \hat{\mathbf{u}}^{B \dagger} \cdot \mathbf{u}_{1}^{B *}\right)}_{\nabla_{\mathbf{u}_{0, P}} \sigma_{B}}, \tag{3.60}
\end{equation*}
$$

where here $\Re$ stands for the real part of the complex vector field, ${ }^{H}$ designates the transconjugate, while the star * denotes the complex conjugate. For a complete and detailed description


Figure 3.16 - Sensitivity of the growth rate $\sigma_{B}$ to base-flow modification for $R e=R e_{C_{2}}=22.65$ and $A R=A R_{C_{2}}=6.98$. (a) Sensitivity function to modification of the production, $\nabla_{\mathbf{u}_{0, P}} \sigma_{B}$. (b) Sensitivity function to modification of the transport, $\nabla_{\mathbf{u}_{0, T}} \sigma_{B}$. Filled contours: magnitude of the two real vector velocity fields. Red arrows: vector field orientation.
of the method see (Bottaro et al., 2003; Marquet et al., 2008). Two different physical interpretations are inherent in the two terms appearing on the right-hand side of (3.60) (Marquet et al., 2008). The first term, denoted by $\nabla_{\mathbf{u}_{0, T}} \sigma_{B}$, represents the sensitivity of the growth rate $\sigma_{B}$ to modifications of the transport since it originates from the transport of the perturbations by the base-flow, $\nabla \hat{\mathbf{u}}_{1}^{B} \cdot \mathbf{u}_{0}$. The second term, $\nabla_{\mathbf{u}_{0, P}} \sigma_{B}$, expresses the sensitivity to production, as it comes from the production of the perturbation by the base-flow, $\nabla \mathbf{u}_{0} \cdot \hat{\mathbf{u}}_{1}^{B}$ (see Marquet et al. (2008) for further details). An expression analogous to (3.60) can be derived for the sensitivity of the oscillation frequency, where the imaginary part of the complex vector field is considered. Marquet et al. (2008) argued that this distinction between transport and production mechanisms identified from the sensitivity analysis is directly connected to the concept of convective and absolute instability adopted in the local stability theory, where the competition of transport and production mechanism defines the global behaviour of the flow (Huerre and Monkewitz, 1990).

The sensitivity of the growth rate associated with the oscillating global mode $B, \sigma_{B}$, to modification of production and transport is shown in figure 3.16-(a) and (b) respectively. The magnitude of the two different sensitivity fields is similar, meaning that the two mechanisms are equally important. However, an interesting aspect that can be clearly observed in figure 3.16 is the decoupling of the directions in which the two mechanisms mainly act. Indeed, the production mechanism is essentially located in the facing jets and in the y-flow direction (pointing towards the center), while it vanishes moving away from the jets region. On the other hand, the transport mechanism, whose maximum intensity is also close to the jet's region, mainly acts on the x-direction of the output channels. In other terms, if an increase of the base-flow velocity in the jets region and oriented in the y-direction is considered, this modification will contribute to a destabilization via production mechanism (figure 3.16-(a)), but it will involve the transport mechanism only weakly, since the two directions of action are almost decoupled. In the same way, considering the $x$-direction, if one considers a decrease of the base-flow velocity in the central region (or alternatively an increase in the size of the recirculation regions), then such a modification will destabilize the flow via transport mecha-

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nism, but it will not play together with the production mechanism because of the mentioned decoupling.

### 3.7 Different inlet velocity profiles: plug flow

We conclude our analysis by examining the influence of a different inlet velocity profile, by specifically focusing on a plug flow, inspired by Pawlowski et al. (2006). In this study, the authors perform a detailed stability analysis, which provides a wide-ranged stability map in the ( $R e, A R$ )-parameter space. The very same steady symmetry-breaking and oscillatory instabilities, as well as the existence of a codimension-2 point, were found, suggesting that the inlet profile does not seem to qualitatively influence the nature of these instabilities. However, from a more quantitative viewpoint, the instability thresholds are significantly affected when a fully developed flow is replaced by a plug flow. Interestingly, despite the fact that the overall nature of the bifurcation nature does not change when varying the inlet profile, Pawlowski et al. (2006) did not report the presence of any hysteretic behaviour. In the following, we apply the weakly nonlinear theory outlined in section section 3.4 to the case of the plug inlet flow studied by Pawlowski et al. (2006) and we briefly discuss their results in relation with our analysis.

In figure 3.17, we propose a zoom of their stability map in the neighbourhood of the codimension-2 point. We extracted manually values, shown as white triangles and circles in figure 3.17 (in both the main figure and inset), from their stability curves (note that their aspect ratio is defined as $1 / A R=w / s$ ). Clearly, the wide range of $R e$ and $A R$ and the large thickness of the lines displayed in figure 10 of Pawlowski et al. (2006) make the extraction procedure only approximate. We note that the value of the codimension-2 point reported in the main text, $C_{2}=(11.2,13.33)$, by Pawlowski et al. (2006) does not seems to match the value extracted from their plot, which is instead in fairly good agreement with our calculations, for which $C_{2}=(20.9,10.53)$. If our marginal stability curve for mode $A$ (dark green dotted lines in the inset) matches very well their result, this is not the case for the curve associated to mode $B$ (blue dotted line in the inset) for $A R<A R_{C_{2}}$. Indeed, the white circles are obtained from the linear stability analysis of the bifurcated steady asymmetric state, while our blue dotted line is evaluated from the stability of the steady symmetric base-flow. We thus apply the WNL analysis around $C_{2}$ and the corresponding weakly nonlinear stability boundaries are displayed in the inset as black solid lines. First, we notice that the WNL analysis based on the symmetric base-flow captures their threshold for the unsteady mode B correctly. Furthermore, analogously to the fully developed flow case, the WNL approach detects a hysteresis region, not described in Pawlowski et al. (2006). We, therefore, performed DNS to confirm the WNL prediction.
In figure 3.18 the $y$-position of the stagnation point normalized by the aspect ratio, $y^{s p} / A R$ ( $A R=8$ ), is used to characterize the bifurcation diagram. As in section 3.4, four regions, denoted here by numbers, can be identified and the associated phase diagrams (amplitude of mode $|B|$ vs. $A$ ) are shown for completeness, following Kuznetsov (2013). The black solid


Figure 3.17 - Stability map taken from figure 10 of Pawlowski et al. (2006). White triangles and circles are the values that we extracted manually from their curves. Inset: our weakly nonlinear map for the case of a uniform inlet flow, where our definition of $A R$ is adopted in the $y$-axis. Dark green and blue dotted lines in the inset indicate the linear marginal stability curves obtained from our calculation for mode $A$ and $B$ respectively. Black lines in the inset represent our weakly nonlinear stability boundaries, as described in section 3.4. The light gray shaded region in the inset corresponds to the hysteresis. The green filled circle indicate the position of the codimension-2 point reported in Pawlowski et al. (2006), while the red filled circle is the codimension-2 point obtained from our calculation.
and dashed lines correspond to the stable and unstable branches described in figure 11 of Pawlowski et al. (2006), while the light gray shaded region is the hysteresis detected by our WNL model. Symbols correspond to our DNS. Pawlowski et al. (2006) started from $R e=1$ and increased $R e$ progressively. The reason they could not detect hysteresis is intrinsic to their continuation algorithm, which describes the transition from region 2 to region 4 (see figure 3.18) following the same branch. In other words, their initial conditions in region 3 (hysteresis) are taken from region 2 (steady asymmetric state) and therefore always lie in the lower right part of the phase diagram 3. Consequently, the final solution converges to the steady asymmetric configuration $A$. Indeed, the left upper part of the phase portrait 3 can be explored only by considering progressive decreases of $R e$ from region 4 (oscillating regime) to 3 , as our DNS, in good agreement with the WNL model prediction, could confirm. Hence, the WNL model adds new pieces of information, at least in the region of the parameter space close to the codimension-2 point, to the thorough stability analysis by Pawlowski et al. (2006).

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Figure 3.18 - Bifurcation diagram: $y$-position of the stagnation point versus $R e$. The black solid and dashed lines correspond to the stable and unstable branches of the bifurcation diagram shown in figure 11 of Pawlowski et al. (2006), calculated following the path of increasing $R e(A R$ is fixed to 8 ). The light gray shade region is the hysteresis predicted by our WNL model. Symbols indicate the DNS results (see figure 3.12-b) for the notation). A sketch of the phase portrait is given for each regime: 1 stable symmetric base-flow, 2 steady asymmetric configuration, 3 hysteresi and 4 oscillating regime.

### 3.8 Conclusion

In this Chapter we investigated different physical mechanisms arising in a two-dimensional fluidic oscillator with two impinging jets, in a so-called two-dimensional X-junction. The tools of the linear global stability analysis were used to identify different global modes, whose stability properties depend on the two main control parameters, the Reynolds number, $R e$, and the aspect ratio, $A R$. An oscillating mode that produces self-sustained oscillations qualitatively analogous to the ones observed in three-dimensional fluidic cavities (Bertsch et al., 2020a) was retrieved. The origin of such a phenomenon appears therefore as mainly twodimensional and due to the interaction of the two facing jets.

In a certain range of aspect ratios, when the gap length, $s$, separating the two inlets approaches the inlet width, $w$, the unsteady mode is seen to globally interact with a steady symmetry-breaking instability. A weakly nonlinear analysis (WNL), based on the multiple scale technique and showing how the system may present hysteretic behaviours depending on the initial conditions, was formalized. The predicted normal form describes the nonlinear interactions between global modes $A$ (steady) and $B$ (oscillating) and reduces the full dynamics to a low-dimensional model, as typical of WNL formulations. For codimensions larger than one, as in the present case, which displays a codimension-2 point, the normal form often predicts the complex system behaviours successfully (Crawford and Knobloch, 1991; Meliga et al., 2009a; Zhu and Gallaire, 2017). Indeed, a quantitative comparison of our WNL results against direct numerical simulation (DNS), in terms of oscillation frequency and mode
amplitudes, confirms the validity of the WNL analysis and, in particular, the existence of a narrow region of hysteresis for $A R<A R_{C_{2}}$ and $R e_{B A}<R e<R e_{S H B}$.
Furthermore, in analogy with the three-dimensional flow studied by Bertsch et al. (2020a), the oscillation frequency associated with unsteady instability was seen to be inversely proportional to the distance separating the two inlets, $s$, or, in non-dimensional terms, to the aspect ratio, AR.
In principle, a steady symmetry-breaking condition, as the one represented by the global mode $A$, and the associated hysteresis, similar to that here described in 2D, is expected to be retrieved in three-dimensional cavities for proper geometrical parameters, i.e. for a size of the perpendicular $z$-direction sufficiently larger than the distance $s$. Nevertheless, the eventual narrowness of the hysteresis region in the control parameter space could make it hard to be experimentally detected.
A linear sensitivity analysis and the definition of the wavemaker region were then systematically applied in order to explore the origin of the various instabilities observed. The core of the steady instability associated with mode $A$, which breaks the base-flow symmetry with respect to the x -axis, was shown to be spotted in the hyperbolic stagnation point. We showed how the self-sustained oscillatory regime, also observed in three-dimensional flow configurations (Bertsch et al., 2020a), was relying on shear instabilities. The structural sensitivity of the unsteady mode and its accurate examination allowed us to identify the Kelvin-Helmholtz shear instability, located in the jet's interaction region, as the heart of the physical mechanism behind the self-sustained oscillatory regime.
Lastly, we examined the effect of a different inlet velocity profile, e.g. a plug flow, in analogy with Pawlowski et al. (2006). Similarly to the case with a fully developed inlet flow, the weakly nonlinear analysis could detect hysteresis in a narrow region of the parameter space, whose existence was not discussed by Pawlowski et al. (2006). The physical nature of the instabilities remained the same, but their thresholds can differ significantly, calling for a sensitivity analysis of the inlet velocity profile. Indeed, in many practical situations, the inlet profile is neither fully developed, nor uniform, but rather lies in an intermediate case.

### 3.9 Appendix

### 3.9.1 Convergence analysis for the eigenvalue calculations and the amplitude equation coefficients

The convergence analysis for the eigenvalue calculations presented in section 3.3 is shown in table 3.1 for five different meshes $M 1-M 5$, which differ by the vertex densities $n_{i}$ in the various sub-domains displayed in figure 3.2. A similar convergence analysis for the nonlinear coefficients of the normal form (3.41)-(3.42) derived in section 3.4, is provided in table II. As shown in table 3.1, mesh $M 1$ is already excellent for the linear eigenvalue problem. Moreover, the structural sensitivity presented in section 3.6 highlights the fluid domain region in which all the physical mechanisms occur, suggesting that the length of the computational domain could be reduced from $L_{o u t}=70 w$ up to $\approx 30 \mathrm{w}$, without any influence on the eigenvalue

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| $R e$ | $A R$ | Mesh | $n_{1}$ | $n_{2}$ | $n_{3}$ | $n_{4}$ | $n_{\text {tot }}$ | $n_{\text {d.o.f. }}$ | $\operatorname{Re}\left(\lambda^{A}\right)$ | $\operatorname{Re}\left(\lambda^{B}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 22.65 | 6.98 | M1 | 145 | 115 | 75 | 35 | 240251 | 1098685 | $-2.6 e^{-5}$ | $3.3 e^{-6}$ |
|  |  | M2 | 150 | 120 | 80 | 35 | 257110 | 1175053 | $-2.5 e^{-5}$ | $-7.5 e^{-8}$ |
|  |  | M4 | 160 | 130 | 85 | 40 | 307080 | 1401813 | $-2.6 e^{-5}$ | $-8.9 e^{-7}$ |
|  |  | M5 | 175 | 145 | 95 | 45 | 383395 | 1747633 | $-2.6 e^{-5}$ | $-3.3 e^{-6}$ |
|  |  | M | 160 | 105 | 50 | 475963 | 2166624 | $-2.6 e^{-5}$ | $-3.3 e^{-6}$ |  |

Table 3.1 - Eigenvalue convergence associated with the computational domain presented in figure 3.2. Tolerance on the real part of the eigenvalues $\lambda^{A}$ and $\lambda^{B}$, associated to global modes $A$ and $B$, is set to tol $l_{\operatorname{Re}(\lambda)}=5 e^{-5}$. When $|\operatorname{Re}(\lambda)|<t o l_{\operatorname{Re}(\lambda)}$, the modes are considered marginally stable for such combination of Reynolds number, $R e$, and aspect ratio, $A R$, which will define a codimension-2 point $\left(R e_{C_{2}}, A R_{C_{2}}\right)$. Mesh M1 ensures the convergence of the eigenvalue computations in a range of $A R$ and $R e$ explored, however, mesh M5 must be adopted to guarantee an acceptable convergence in the weakly nonlinear analysis (see Table 2). $L_{\text {out }}$ is fixed to 70 . The imaginary part of $\lambda^{B}$ amounts to 0.10157 for all the meshes reported above.

| Mesh | $\zeta_{A}$ | $\eta_{A}$ | $\mu_{A}$ | $\chi_{A}$ |
| :---: | :---: | :---: | :---: | :---: |
| M1 | 1.22 | -0.257 | 0.157 | 1.01 |
| M2 | 1.22 | -0.256 | 0.157 | 1.01 |
| M3 | 1.22 | -0.256 | 0.157 | 1.01 |
| M4 | 1.22 | -0.257 | 0.157 | 1.01 |
| M5 | 1.22 | -0.257 | 0.157 | 1.01 |
|  | $\zeta_{B}$ | $\eta_{B}$ | $\mu_{B}$ | $\chi_{B}$ |
| M1 | $2.67+\mathrm{i} 0.0499$ | $-0.738+\mathrm{i} 1.00$ | $0.410+\mathrm{i} 0.0014$ | $0.164-\mathrm{i} 0.0963$ |
| M2 | $2.67+\mathrm{i} 0.0438$ | $-0.737+\mathrm{i} 1.00$ | $0.410+\mathrm{i} 0.0014$ | $0.164-\mathrm{i} 0.0963$ |
| M3 | $2.67+\mathrm{i} 0.0505$ | $-0.737+\mathrm{i} 1.00$ | $0.410+\mathrm{i} 0.0014$ | $0.164-\mathrm{i} 0.0963$ |
| M4 | $2.67+\mathrm{i} 0.0489$ | $-0.738+\mathrm{i} 1.00$ | $0.410+\mathrm{i} 0.0014$ | $0.164-\mathrm{i} 0.0963$ |
| M5 | $2.67+\mathrm{i} 0.0490$ | $-0.738+\mathrm{i} 1.00$ | $0.410+\mathrm{i} 0.0014$ | $0.164-\mathrm{i} 0.0963$ |

Table 3.2 - Values of the amplitude equation coefficients for global mode $A$ and $B$ corresponding to the case of a fully developed inlet velocity profile and calculated for different mesh M1-M5.
calculation (numerically verified). However, the weakly nonlinear problem and the calculation of the coefficient of the normal form requires a finer mesh and an adequate domain length in order to get an optimal convergence. Table 2 shows that refining from mesh $M 4$ to $M 5$ the major relative error (coefficient $\eta_{A}$ ) is less than $1 \%$. Note that this is the numerical precision of the calculation performed, which is not linked to the convergence of the asymptotic expansion, whose precision increases as $\left|R e-R e_{C_{2}}\right|$ and $\left|A R-A R_{C_{2}}\right|$ decrease.

The expression of the various normal form coefficients are provided in the following:

$$
\begin{align*}
& \zeta_{A}=-<\hat{\mathbf{q}}_{1}^{A \dagger},\left(\mathscr{D}_{A R_{C_{2}}} \hat{\mathbf{q}}_{1}^{A}+\mathscr{C}_{A C_{C_{2}}}\left[\hat{\mathbf{q}}_{1}^{A}, \hat{\mathbf{q}}_{2}^{\delta}\right]\right)>,  \tag{3.61}\\
& \zeta_{B}=-<\hat{\mathbf{q}}_{1}^{B \dagger},\left(\mathscr{D}_{A C_{2} C_{2}} \hat{\mathbf{q}}_{1}^{B}+\mathscr{C}_{A C_{C_{2}}}\left[\hat{\mathbf{q}}_{1}^{B}, \hat{\mathbf{q}}_{2}^{\delta}\right]\right)>, \tag{3.62}
\end{align*}
$$

$$
\begin{gather*}
\eta_{A}=-<\hat{\mathbf{q}}_{1}^{A \dagger},\left(-\frac{2}{\operatorname{Re}_{C_{2}}} \mathscr{D}_{\alpha A R_{C_{2}}} \hat{\mathbf{q}}_{1}^{A}+\mathscr{G}_{\alpha} \hat{\mathbf{q}}_{1}^{A}+\mathscr{C}_{\alpha}\left[\mathbf{q}_{0}, \hat{\mathbf{q}}_{1}^{A}\right]+\mathscr{C}_{A R_{C_{2}}}\left[\hat{\mathbf{q}}_{1}^{A}, \hat{\mathbf{q}}_{2}^{\alpha}\right]\right)>  \tag{3.63}\\
\eta_{B}=-<\hat{\mathbf{q}}_{1}^{B \dagger},\left(-\frac{2}{\operatorname{Re}_{C_{2}}} \mathscr{D}_{\alpha A R_{C_{2}}} \hat{\mathbf{q}}_{1}^{B}+\mathscr{G}_{\alpha} \hat{\mathbf{q}}_{1}^{B}+\mathscr{C}_{\alpha}\left[\mathbf{q}_{0}, \hat{\mathbf{q}}_{1}^{B}\right]+\mathscr{C}_{A R_{C_{2}}}\left[\hat{\mathbf{q}}_{1}^{B}, \hat{\mathbf{q}}_{2}^{\alpha}\right]\right)>  \tag{3.64}\\
\mu_{A}=<\hat{\mathbf{q}}_{1}^{A \dagger}, \mathscr{C}_{A R_{C_{2}}}\left[\hat{\mathbf{q}}_{1}^{A}, \hat{\mathbf{q}}_{2}^{A^{2}}\right]>  \tag{3.65}\\
\mu_{B}=<\hat{\mathbf{q}}_{1}^{B \dagger},\left(\mathscr{C}_{A R_{C_{2}}}\left[\hat{\mathbf{q}}_{1}^{B}, \hat{\mathbf{q}}_{2}^{|B|^{2}}\right]+\mathscr{C}_{A R_{C_{2}}}\left[\hat{\mathbf{q}}_{1}^{B^{*}}, \hat{\mathbf{q}}_{2}^{B^{2}}\right]\right)>  \tag{3.66}\\
\chi_{A}=<\hat{\mathbf{q}}_{1}^{A \dagger},\left(\mathscr{C}_{A R_{C_{2}}}\left[\hat{\mathbf{q}}_{1}^{A}, \hat{\mathbf{q}}_{2}^{|B|^{2}}\right]+\mathscr{C}_{A R_{C_{2}}}\left[\hat{\mathbf{q}}_{1}^{B^{*}}, \hat{\mathbf{q}}_{2}^{A B}\right]\right)>  \tag{3.67}\\
\chi_{B}=<\hat{\mathbf{q}}_{1}^{B \dagger},\left(\mathscr{C}_{A R_{C_{2}}}\left[\hat{\mathbf{q}}_{1}^{B}, \hat{\mathbf{q}}_{2}^{A^{2}}\right]+\mathscr{C}_{A R_{C_{2}}}\left[\hat{\mathbf{q}}_{1}^{A}, \hat{\mathbf{q}}_{2}^{A B}\right]\right)> \tag{3.68}
\end{gather*}
$$

where, given two vectors a and $\mathbf{b}, \mathscr{C}_{A R_{C_{2}}}[\mathbf{a}, \mathbf{b}]=\left\{\mathscr{C}_{A R_{C_{2}}}(\mathbf{a}, \mathbf{b}), 0\right\}^{T}, \mathscr{D}_{A R_{C_{2}}} \mathbf{a}=\left\{\Delta_{A R_{C_{2}}} \mathbf{a}, 0\right\}^{T}$, $\mathscr{C}_{\alpha}[\mathbf{a}, \mathbf{b}]=\left\{\mathscr{C}_{\alpha}(\mathbf{a}, \mathbf{b}), 0\right\}^{T}, \mathscr{D}_{\alpha A R_{C_{2}}} \mathbf{a}=\left\{\Delta_{\alpha A R_{C_{2}}} \mathbf{a}, 0\right\}^{T}$, while $\mathscr{G}_{\alpha} \hat{\mathbf{q}}_{1}^{A, B}=\left\{\nabla_{\alpha} \hat{p}_{1}^{A, B}, \nabla_{\alpha}^{T} \hat{\mathbf{u}}_{1}^{A, B}\right\}^{T}$. The star * denotes the complex conjugate.

### 3.9.2 Flow behaviour at higher Reynolds numbers

In section 3.3.2 the existence of a second steady global mode (denoted by $C$ ), which, from the global stability analysis, appears to be unstable for $R e=41.5$, for $A R=6.5$ (see figure 3.19(c)), was mentioned. When the threshold for mode $C$ is met, global modes $A$ and $B$ are both unstable. This evidence does not justify either the application of the linear stability tools or the weakly nonlinear analysis (the corresponding thresholds are too far from each other). Nevertheless, some information can still be extracted by looking at the DNS results for higher Reynolds numbers, i.e. $R e=50$, and, in particular, at the spatial structure of this mode. Figure 3.19-( $a$ ) shows the flow configuration for $R e=50$ and $A R=6.5$. As can be observed, in figure 3.19-(b), where the magnitude of the velocity field is plotted for two symmetric slices at coordinates $x=-5$ and $x=+5$, for such combination of control parameters, the symmetry of the flow (in terms of field magnitude) with respect to the $y$-axis is lost.

It can be argued that the cause of such behaviour is the second steady mode, $C$, which results to be unstable, with growth rate $\sigma_{C}=0.016$ (and frequency $\omega_{C}=0$ ), for the DNS parameters here presented. Figure 3.20-( $a$ ) and (b) displays the spatial structure of the x-and y-velocity components associated with the mode. The eigenfunctions for $\hat{u}_{1}^{C}$ and $\hat{v}_{1}^{C}$ exhibits the same symmetry properties characterizing the oscillating mode $B$ (see figure $3.5-(b)$ and (d)). However, the steady nature of this mode leads to a double steady symmetry-breaking condition (both axes of symmetry). The associated wavemaker region, computed as described in section 3.6, is shown in figure 3.20-(c). The overlapping region highlighted by the structural sensitivity appears to be approximately localized at the boundaries of the recirculation regions.

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Figure 3.19 - Snapshot of the unsteady flow configuration in terms of dyes concentrations for $R e=50$ and $A R=6.5$. Two slices, at $x=-5$ and $x=+5$ are extracted and used to plot the magnitude of the velocity field (b). (c) Marginal stability curves for global mode $A, B$, and $C$ as a function of $R e$ and $A R$ (as in figure 3.4). LS: Linearly Stable. LU: Linearly Unstable. Red circle: DNS parameters for the present case.

The nature of this instability could be reasonably classified as a buckling-like instability, where the symmetric configuration with two facing jets, above a certain critical Reynold number, which represents a measure of the jet intensities, becomes unstable and the jets tend to bend towards opposite directions, as clearly shown in figure 3.20-(d). The shape and size of the four recirculation regions are then readapted to the new steady configurations.
In the recent three-dimensional experimental and numerical investigation proposed by Bertsch et al. (2020a) for straight output channels, as the two-dimensional one analyzed in the present study, the self-sustained oscillatory regime, observed in a certain range of Reynolds numbers, is seen to be strongly altered as $R e$ is increased ( $R e \approx 100$ or higher). In particular, the two facing jets tend to suddenly switch left or right (and vice versa) and to keep that position steadily for a while. Fast oscillations are simultaneously present and sometimes the jets switch side. The existence of an analogous steady symmetry breaking condition in the threedimensional problem is in principle expected and its strong nonlinear interaction with the self-sustained oscillations for high Reynolds numbers could hypothetically and qualitatively justify the flow behaviour shown in Bertsch et al. (2020a) (see associated supplemental material at http://link.aps.org/supplemental/10.1103/PhysRevFluids.5.054202).


Figure 3.20 - Spatial structure of the $x$ - and $y$-velocity components associated with the direct global mode $C$ for $R e=50$ and $A R=6.5$, for which the base-flow is marginally stable. (a) xvelocity component. (b) y-velocity component. (c) Color map: structural sensitivity to a local feedback of the steady global mode $C$, expressed as $\left\|\hat{\mathbf{u}}_{1}^{C}\right\| \cdot\left\|\hat{\mathbf{u}}_{1}^{C \dagger}\right\|$ and normalized by its maximum value. Black contours: magnitude of the base-flow field. Red dashed lines: boundaries of the recirculation bubbles. (d) Streamline associated with the sum of the steady base-flow and the steady unstable mode $C$. A fictitious amplitude of 0.25 is imposed to the perturbation in order to get a good visualization of the streamlines modification. Red solid lines: axes of symmetry.

### 3.9.3 Temporal linear stability of the local velocity profiles in the lateral channel

The wavemaker analysis proposed in section section 3.6 suggests that the Kelvin-Helmholtz (KH) mechanism plays an important role in the oscillatory instability. Nevertheless, the KH instability has an inviscid origin, while the low Reynolds numbers encountered in this flow suggest that viscous effects could be dominant and consequently that they could inhibit the KH instability. In this appendix, we propose a temporal linear stability of the local velocity profiles in the lateral channel (see figure 3.15-(b)), which highlights that the KH mechanism is actually active in the underlying process.
If we assume that the steady base-flow in the right (or left, symmetric base-flow) output channel is locally parallel, i.e. we assume that the $y$-steady base-flow velocity component is zero and the $x$-component depends only on $y, \mathbf{u}_{0}=\left\{u_{0}(y), 0\right\}^{T}$, then we can tentatively apply the parallel stability theory. Linearizing the Navier-Stokes equation around the locally parallel base-flow and using the ansatz, $\mathbf{u}(x, y, t)=\hat{\mathbf{u}}(y) e^{\mathrm{i}(k x-\lambda t)}$ and $p(x, y, t)=\hat{p}(y) e^{\mathrm{i}(k x-\lambda t)}$, with $k$

Chapter 3. Impinging planar jets: hysteretic behaviour and origin of the self-sustained oscillations
spatial wavenumber, we obtain the following linear system,

$$
\begin{gather*}
0=\mathrm{i} k \hat{u}+\frac{\partial \hat{v}}{\partial y},  \tag{3.69}\\
-\mathrm{i} \lambda \hat{u}=-\mathrm{i} k u_{0} \hat{u}-\hat{v} \frac{\partial \hat{u}_{0}}{\partial y}-\mathrm{i} k \hat{p}+\frac{1}{R e}\left(-k^{2}+\frac{\partial^{2}}{\partial y^{2}}\right) \hat{u},  \tag{3.70}\\
-\mathrm{i} \lambda \hat{v}=-\mathrm{i} k u_{0} \hat{u}-\frac{\partial p}{\partial y}+\frac{1}{R e}\left(-k^{2}+\frac{\partial^{2}}{\partial y^{2}}\right) \hat{v}, \tag{3.71}
\end{gather*}
$$

subjected to no-slip boundary condition at the upper and lower walls. The system above, formally equivalent to the Orr-Sommerfeld equation expressed in primitive variables, reduces to a generalized eigenvalue problem in $\lambda$ (the real wavenumber $k$ is an input), whose temporal stability associated with the base-flow for each $x$-slice is studied numerically using a validated Chebyshev pseudo-spectral code. A one-dimensional grid in the $y$-direction made of 100 collocation points ensures convergence for the present case. The main results are shown in figure 3.21. We observe that there exists a spatial region, approximatively between $x=1$


Figure 3.21 - Temporal analysis of the $x$-velocity profiles shown in figure 3.15-(b) and corresponding to $R e=R e_{C_{2}}=22.65$ and $A R=A R_{C_{2}}=6.98$. Left plot: frequency $-\omega$ vs. wavenumber $k$. Right plot: growth rate $\sigma$ vs. wavenumber $k$. The maximum growth rate is found for $x=2$ and corresponds to $k \approx 0.71$ (wavelength $\approx 8.8$ ) and to an oscillation frequency $\omega=0.101$.
and $x=5$ in which the local profiles are temporally unstable. Interestingly, the maximum growth rate, obtained for $x=2$, is characterized by a spatial wavenumber $k \approx 0.7$, which corresponds to a wavelength $\approx 9$, in good agreement with the one observed in our oscillatory global mode (see figure 3.5-(d)). Furthermore, the associated oscillation frequency is $\omega=0.1$, a value which matches well the global frequency. Lastly, the local temporal analysis predicts a sinuous mode (not shown here), while varicose modes are always stable, in agreement with
global observations again.
A similar analysis can be repeated for the jet profiles selected along the $y$-axis (figure 15-(a)). These profiles are also found to be temporally unstable, but the interpretation of the results in terms of wavenumber and frequency is far from being trivial, since the features of the instability are clearly visible only in the lateral output channels.
We then performed a spatio-temporal instability analysis, where $\lambda$ and $k$ are both complex quantities, but we found that the pocket of temporal instability is associated with a convective instability (results not shown here).

As stated in section 3.6 and highlighted by the wavemaker (figure 15-(c)), the instability mechanism seems to be intrinsically global and due to the interaction of multiple shear layers (jets and horizontal flows), which communicate in the central region of the domain, where the flow is strongly non-parallel. For all these reasons, we believe that the employment of the classic local theory is not legit in our case. Nevertheless, the temporal analysis proposed in this appendix, together with consideration about the location of the maximum shear made in section 3.6, shows that the KH instability is active and that it could play a relevant role in the instability mechanism, despite the potentially high viscous effects.

# Harmonic and super-harmonic Part II sloshing dynamics of orbital-shaken cylindrical reservoirs 

## Introduction

Sloshing, i.e., the oscillations of a free liquid surface in partially filled containers, is an important issue in mechanical and aerospace engineering as well as in daily life. For instance, casually shaking a glass of water or a cup of coffee may lead to unpleasant liquid spilling (Mayer and Krechetnikov, 2012) (see figure II.1(a)). Closer to engineering, the total weight of launch vehicles and road or ship tankers is constituted in a large percentage by the liquids transported and by fuel. As the sloshing frequencies might be close to the control system frequencies, possible resonant sloshing dynamics can induce significant displacements of the vehicle's center of mass, thus endangering its dynamical stability (Ibrahim, 2005), with critical consequences on the transport safety and vehicle's performances (see figure II.1(b).
In some other applications however, enhancement of sloshing waves is seen as beneficial (see figure II.1(c)): in biology for example, cellular growth takes place in nutritive media placed into bioreactors (McDaniel and Bailey, 1969; Wurm, 2004). These containers are agitated so as to mix the liquid, prevent sedimentation and enhance gas transfer, which provides suitable oxygenation to the growing cell population (Klöckner and Büchs, 2012).
Therefore, a proper predictive understanding and modelling of the sloshing hydrodynamics at stake is essential in the design process of liquid tanks, so as to implement active control systems of vehicles and ensure efficient mixing processes.
For moderately large-size containers, sloshing is classically modelled by determining the oscillation modes compatible with a given tank shape using potential flow theory supplemented by viscous dissipation coming from bulk potential flow and Stokes boundary layers along walls (Faltinsen and Timokha, 2009). More precisely, gravity waves are restricted into modes with a discrete set of wavenumbers, owing to the action of the container walls. The values of the associated natural frequencies depend on the geometrical and fluid parameters through the well-known dispersion relation for capillary-gravity waves (Lamb 1932),

$$
\begin{equation*}
\omega_{m n}^{2}=g k_{m n}\left(1+\gamma k_{m n}^{2} / \rho g\right) \tanh \left(k_{m n} h\right) \tag{3.72}
\end{equation*}
$$

where $g$ is the gravitational acceleration, $h$ is the depth of the liquid layer, $\rho$ and $\gamma$ are the liquid's density and surface tension, while $k_{m n}$ is a wavenumber.
Since analytical solutions are limited to regular geometric tank shapes, the case of sloshing in partially filled cylindrical reservoirs has represented over the last 60 years one of the archetypal sloshing systems (Abramson, 1966). In this specific configuration, the wavenumbers

## Sloshing as an issue



Sloshing as beneficial


Figure II. 1 - (a) Example of daily-life liquid spilling. (b) Top, sloshing experiments in largescale tanks for LNG (liquefied natural gas) carrier (Pastoor et al., 2005). Bottom, a sloshing test carried out by ESA (European Space Agency) to test the response of a launcher's liquid propellants to the violence of take-off, so as to better understand the forces involved and enhance future launcher performances. (c) Left, stirred tank in operation: gas and chemicals are injected at the bottom, while the agitation is ensured by the propeller, which also breaks the largest bubbles. Gas exchange occurs at the interface of the bubbles. Right, orbital-shaken bioreactor: the motion is imposed at the whole vessel, and transmitted to the liquid by the walls, with the gas exchange occurring at the free surface (modified figure from Reclari (2013)).
$k_{m n}$ are given by the $n$ th-roots of the first derivative of the mth-order Bessel function satisfying $J_{m}^{\prime}\left(R k_{m n}\right)$, with $R$ the container's radius and the indices ( $m, n$ ) denoting, respectively, the number of nodal circles and nodal diameters of the associated eigenmode. The lowest or first system's natural frequency has typically $(m, n)=(1,1)$ and it is therefore denoted by $\omega_{11}$.

Among all the possible forcing conditions and container trajectories, orbital shaking is particularly interesting, despite its apparent simplicity. Previous experimental studies have carefully described the close-to-resonance dynamics for the two limiting cases, namely circular and purely longitudinal shaking (see figure II.2), casting light on a rich variety of wave regimes, i.e. planar waves, irregular motion or swirling waves, symmetry-breaking, etc., attracting interest to dynamicists over the last decades (Hutton, 1963; Miles, 1984c,d; Ockendon and Ockendon, 1973).


Figure II. 2 - Schematic illustration of possible operating parameters of the shaking configurations. Note that the container does not rotate around its own axis, but rather keeps its orientation fixed. The orbit aspect ratio is defined as $\alpha=a_{y} / a_{x}$, and it is $\alpha \neq 0$ for a generic elliptic orbit. The two limiting cases correspond to $\alpha=1$, rotary shaking, and $\alpha=0$, longitudinal shaking. The external driving is harmonic with angular frequency $\Omega=2 \pi / T$.

For circular orbits, the system responds with a swirling wave always co-directed with the container motion. This well-defined hydrodynamics, often simply modelled by a one-degree-of-freedom Duffing oscillator (Ockendon and Ockendon, 1973), is advantageously exploited in the design of bioreactors for bacterial and cellular cultures (McDaniel and Bailey, 1969; Wurm, 2004), where circular shaking is used as a method to gently mix the liquid content of a container by its displacement at fixed container orientation along a circular trajectory and at a constant angular velocity. Particularly, it constitutes an alternative to stirred tanks (see figure II.1 (c)), where the liquid agitation results from a rotating impeller or the rotation of a

|  | Sub-Harmonic | Harmonic | Super-Harmonic |
| :---: | :---: | :---: | :---: |
| $\omega / \Omega$ | $1 / 2$ | 1 | 2 |
| $\omega$ | $\Omega / 2$ | $\Omega$ | $2 \Omega$ |
| $\Omega / \omega$ | 2 | 1 | $1 / 2$ |

Table II. 1 - Definition of fundamental sub-harmonic, harmonic and super-harmonic resonances based on the relation between driving frequency, $\Omega=2 \pi / T$, and wave oscillation frequency, $\omega$. The case of sub-harmonic wave responses will be tackled in Part III in the context of the parametric Faraday instability.
magnetic rod. In these cultivation protocols, cells are in suspension in the extracellular liquid medium, which serves as buffer for consumables from which they feed and for their secretions. The motion of the liquid prevents sedimentation and homogenizes the concentration of dissolved oxygen and nutrients and of secreted proteins and carbon dioxide. Thanks to the possible gas exchanges at the free surface, oxygen supply from the container bottom can possibly be circumvented, avoiding the formation of bubbles and thereby the damages that their collapse can exert on cells (Handa-Corrigan et al., 1989; Kretzmer and Schügerl, 1991; Papoutsakis, 1991), sparking interest in the development of large-scale, in the hectoliter range, orbital-shaken bioreactors (Jesus et al., 2004; Liu and Hong, 2001; Muller et al., 2007). It is therefore not a surprise if a significant body of research on the gas exchange and mixing in these devices has emerged over the last two decades (Büchs, 2001; Büchs et al., 2000a,b; Maier et al., 2004; Micheletti et al., 2006; Muller et al., 2005; Tan et al., 2011; Tissot et al., 2010, 2011; Zhang et al., 2009).
Since the shear stresses and, therefore, the mixing are proportional to the velocity gradients in the liquid phase, most of the gas exchange phenomena listed above are directly linked to the liquid motion, with the optimal working conditions essentially dictated by the wave pattern (Reclari, 2013). For these reasons, at a more fundamental level, the hydrodynamics of these orbital shaking devices has received recent attention, from both experimental (Bouvard et al., 2017; Moisy et al., 2018; Reclari et al., 2014) and theoretical (Horstmann et al., 2020; Reclari et al., 2014) perspectives, predominantly using linear potential flow models. These models are often complemented with effective viscous damping rates to incorporate the energy dissipation responsible for the phase-shifts between wave and shaker, which was also seen to be sometimes responsible for damping-induced symmetry-breaking linear mechanisms resulting in linear spiral wave patterns (Horstmann et al., 2021, 2020). Previous studies, reviewed for instance in Ibrahim (2005) or Faltinsen and Timokha (2009), make mostly use of classical existing theories for general linear and weakly nonlinear sloshing dynamics in the vicinity of the fundamental harmonic resonance, i.e. when the system is harmonically driven at a frequency close to the lowest natural frequency, $\omega_{11}$.
However, the seminal work of Reclari (2013) cast light on the importance of super-harmonic resonances occurring for an excitation frequency far below $\omega_{11}$, which may possibly manifest with large amplitude responses and wave breaking, hence potentially raising an issue for the robustness of bioreactors if not accounted for. Among these super-harmonics, the doublecrest (DC) dynamics (as defined by Reclari (2013)) is particularly relevant, as it displays a


Figure II. 3 - Non-dimensional wave amplitude (re-scaled with the container's radius $R$ ) between the apparition of super-harmonic double-crested waves (at $\Omega \approx \omega_{21} / 2$, vertical red dashed line) and the first system's natural frequency ( $\omega_{11}$ ) (Reclari, 2013). $d_{s}$ is the orbit of the container's trajectory and represents the dimensional forcing amplitude. The black solid lines correspond to the theoretical predictions given by a linear potential model. The single-crest (SC) and double-crest (DC) wave shapes, visualized along the wall, i.e. $\theta \in[0,2 \pi]$ with $\theta$ the azimuthal coordinate, are shown in the two insets (modified figure from Reclari (2013)).
notably large amplitude response, that is strongly favoured by the spatial structure of the external forcing (see figure II.3). In the following we will refer to this resonance as fundamental super-harmonic. To avoid confusion with the contradictory terminology in literature, in table II. 1 we define what we mean for fundamental sub-harmonic, harmonic and superharmonic resonances.

In order to refine the linear potential model and, specifically, to predict the occurrence of the super-harmonic wave dynamics observed experimentally (we remark that by superharmonic, we mean here a wave of a certain wave frequency $\omega$ emerging from an excitation at $\Omega=\omega / 2$, with $\Omega$ the driving angular frequency), Reclari (2013) and Reclari et al. (2014) proposed an inviscid weakly nonlinear analysis based on a second order straightforward asymptotic expansion procedure, which was shown to be capable of capturing the emergence of the observed resonance. However, their analysis, as typical of straightforward asymptotic expansions, suffers from secular terms (Castaing, 2005; Nayfeh, 2008a) and, therefore, it still fails in describing the correct nonlinear behaviour close to both harmonic and super-harmonic resonances.
With regards to the experiments of Reclari (2013) and Reclari et al. (2014), Timokha and

Raynovskyy (2017) and Raynovskyy and Timokha (2018a,b) have applied the NarimanovMoiseev multimodal sloshing theory (Dodge et al., 1965; Faltinsen, 1974; Lukovsky, 1990; Moiseev, 1958; Narimanov, 1957; Narimanov et al., 1977). The theory is capable of accurately describing the nonlinear wave dynamics near the fundamental harmonic resonance when no secondary resonances occur (Faltinsen et al., 2016, 2005). Despite the fact that the experiments performed by Reclari (2013) and Reclari et al. (2014) were made for nondimensional fluid depths $H=h / R=1.04$ and 1 , which lie slightly beyond the applicability threshold of the multimodal theory ( $H_{t h}$ should be $\gtrsim 1.05$ as stated by Raynovskyy and Timokha (2020)) and imposed by the occurrence of secondary resonances, the authors found a quantitative good agreement with the experimental observations associated with the hardening-spring type single-crest swirling.
In the spirit of the aforementioned multimodal theory but with regards to square-base basins, the resonant amplification of higher order modes for forcing frequency in the vicinity of the primary resonance (secondary or internal resonances) was investigated by Faltinsen et al. (2005), who formalized a so-called adaptive asymptotic modal approach capable to improve the agreements with earlier experiments. A thorough discussion on this regard is also outlined in chapters 8 and 9 of Faltinsen and Timokha (2009), where the importance of the ratio of tank liquid depth to tank width on the occurrence of the internal resonance phenomenon is carefully discussed. Generally speaking, secondary resonance is a broader concept, and it may occur even far from the primary resonance zone, as in the case of the double-crest swirling observed in Reclari et al. (2014). To our knowledge, the adaptive modal approach was never extended to super-harmonic system responses of orbital-shaken circular cylindrical containers far from the primary resonance.
For these reasons, it appears that a quantitatively accurate model for the prediction of the super-harmonic double-crest (DC) dynamics observed during the thorough experimental campaign carried out by Reclari (2013) and Reclari et al. (2014) has not been provided yet. Chapter 4 is precisely dedicated to the development of a weakly nonlinear analysis based on the multiple timescale method, which will be seen to successfully capture nonlinear effects for the main additive harmonic resonances as well as the more subtle additive and multiplicative resonance governing the super-harmonic double-crest swirling. Amplitude equations are rigorously derived in an inviscid framework, which once amended with an ad-hoc damping term as the only tuning parameter, well match the experimental findings of Reclari (2013) and Reclari et al. (2014). Lastly, the obtained amplitude equations for harmonic single-crest and super-harmonic double-crest waves are found to be compatible with the two well-known one-degree-of-freedom (1dof) systems, the Duffing (already introduced in Chapter 1) and the Helmholtz-Duffing oscillators, respectively.

The study of the double-crest (DC) super-harmonic resonance is extended to longitudinal shaking in Chapter 5. The latter forcing condition has been analytically and experimentally studied for decades (Abramson, 1966; Chu, 1968; Hutton, 1963) and it is of interest from the perspective of hydrodynamic instabilities due to the occurrence of hysteretic symmetry-breaking conditions (Miles, 1984a,d). With regards to circular cylindrical containers, particularly rel-


Figure II. 4 - (a) Time evolution of the wave amplitude at a probed location in a circular container of radius $R=78 \mathrm{~mm}$ filled with water to a non-dimensional depth $h / R \approx 1.5$ and undergoing a longitudinal motion of non-dimensional amplitude $a_{x} / R=0.0033$ and driving frequency $\Omega / \omega_{11}=0.98$, with $\omega_{11}$ the lowest system's natural frequency. The time series shows a wave envelope modulation occurring on a much larger time scale than the forcing period $T=2 \pi / \Omega$. The absence of a steady wave amplitude regime is a clear sign of irregular motion. (b) Images from Royon-Lebeaud et al. (2007) of a swirling wave in a circular cylinder of radius $R=150 \mathrm{~mm}$ filled to a depth $h / R \approx 1.2$ and longitudinally forced with $a_{x} / R=0.023$ and $\Omega / \omega_{11} \approx 1.02$. Views are in the direction normal to the tank motion. The ten images represent slightly more than one wave period. (c) Theoretical (solid lines, from Faltinsen et al. (2016)) and experimental (circles, from Royon-Lebeaud et al. (2007)) estimates of bounds, in the forcing parameter space, $\left(\Omega / \omega_{11}, a_{x} / R\right)$, between the frequency ranges where planar, irregular and swirling waves occur for close-to-resonance longitudinal forcing conditions, i.e. $\Omega / \omega_{11} \approx 1$.
evant are the experimental studies by Abramson et al. (1966), Royon-Lebeaud et al. (2007) and Hopfinger and Baumbach (2009), who detected the stability bounds between harmonic planar, swirling and irregular waves and whose estimates were later used by Faltinsen et al. (2016) to validate their theoretical analysis (see figure II.4). However, these works were mostly focused on the investigation of system responses in the neighbourhood of harmonic resonances, whereas, with the exception of Reclari (2013); Reclari et al. (2014) in the context of circular sloshing, the literature seems to lack comprehensive experimental and theoretical
studies dealing with the fundamental secondary super-harmonic resonances (discussed in Chapter 4) under longitudinal or, more generally, elliptical container excitation.
In this Chapter, we take a first step in this direction by extending to longitudinal planar forcing the analysis formalized in Chapter 4 for circular container motions. In the spirit of the multiple timescale method, we develop a weakly nonlinear (WNL) model leading to a system of two amplitude equations, whose prediction anticipates that a planar wave symmetry-breaking via stable swirling may also occur under super-harmonic excitation. This finding is confirmed by our experimental observations, which indeed identify three possible super-harmonic regimes, i.e. (i) stable planar DC waves, (ii) irregular motion and (iii) stable swirling DC waves, whose corresponding stability boundaries in the forcing frequency-amplitude plane quantitatively match the present theoretical estimates.

Chapter 5 ends with a brief demonstration of how a straightforward extension of the present analysis to a generic container's elliptic orbit can be readily obtained without any further calculation. This paves the way for the analysis and experimental investigation of the next Chapter, which has been stimulated by the surprising fact that no experimental studies devoted to the more generic case of elliptic container orbits have been reported so far in the sloshing literature. Existing theoretical analyses of this forcing condition brought out interesting features of the resonant liquid response that depend on the orbit's ellipticity. In particular, the inviscid theory of Faltinsen et al. (2016) suggested the counter-intuitive existence, under resonant elliptic forcing, of stable swirling waves that propagate in the direction opposite to the forcing direction. Moreover, the theory anticipated that such counter-waves may exist even for quasi-circular orbits and travel with a smaller amplitude than co-directed waves. This, if confirmed, would further enrich the variety of observable dynamical sloshing regimes.
Therefore, Chapter 6 aims at providing a joint experimental and theoretical characterization of the free liquid surface response for a generic, elliptic periodic container trajectory, so as to bridge the gap between the two diametrically opposed shaking conditions previously discussed. Specifically, with the main focus here on harmonic resonances, we intend to experimentally identify the range of external control parameters, i.e. driving frequency, amplitude and orbit aspect ratio, for which stable counter-directed swirling waves occur. Our findings provide the first strong evidence of the existence of a frequency range where stable swirling can be either co- or counter-directed with respect to the container's direction of motion. Lastly, these results are successfully rationalized and predicted by the inviscid asymptotic model developed in Chapter 5, amended with heuristic damping by analogy with Chapter 4.

# 4 An amplitude equation modelling the double-crest swirling in orbital shaken cylindrical containers 

Remark: this chapter is largely inspired by the publication of the same name.

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The container motion along a planar circular trajectory at a constant angular velocity, i.e. orbital shaking, is of interest in several industrial applications, e.g. for fermentation processes or in cultivation of stem cells, where good mixing and efficient gas exchange are the main targets. Under these external forcing conditions, the free surface typically exhibits a primary steady state motion through a single-crest dynamics, whose wave amplitude, as a function of the external forcing parameters, shows a Duffing-like behaviour. However, previous experiments in lab-scale cylindrical containers have unveiled that, owing to the excitation of super-harmonics, diverse dynamics are observable in certain driving-frequency ranges. Among these superharmonics, the double-crest dynamics is particularly relevant, as it displays a notably large amplitude response, that is strongly favoured by the spatial structure of the external forcing. In the inviscid limit and with regards to circular cylindrical containers, we formalize here a weakly nonlinear analysis via multiple timescale method of the full hydrodynamic sloshing system, leading to an amplitude equation suitable to describe such a double-crest swirling motion. The weakly nonlinear prediction is shown to be in fairly good agreement with previous experiments described in the literature. Lastly, we discuss how an analogous amplitude equation can be derived by solving asymptotically for the first super-harmonic of the forced Helmholtz-Duffing equation with small nonlinearities.

## Chapter 4. An amplitude equation modelling the double-crest swirling in orbital shaken cylindrical containers



Figure 4.1 - Sketch of a cylindrical container of diameter $D=2 R$ and filled to a depth $h$. The gravity acceleration is denoted by $g . O^{\prime} \mathbf{e}_{x}^{\prime} \mathbf{e}_{y}^{\prime} \mathbf{e}_{z}^{\prime}$ is the Cartesian inertial reference frame, while $O \mathbf{e}_{x} \mathbf{e}_{y} \mathbf{e}_{z}$ is the Cartesian reference frame moving with the container. The origin of the moving cylindrical reference frame $(r, \theta, z)$ is placed at the container revolution axis and, specifically, at the unperturbed liquid height, $z=0$. The perturbed free surface and contact line elevation are denoted by $\eta$ and $\delta$, respectively. $d_{s}$ is the diameter of the circular shaking trajectory, characterized by a driving angular frequency $\Omega_{d}$.

The chapter is organized as follows. The flow configuration and governing equations are introduced in $\S 4.1$. $\S 4.2$ is briefly summarizes the salient points of the asymptotic model proposed by Reclari et al. (2014), whose limitations motivated the present work. After tackling the more common case of harmonic single-crest wave in $\S 4.3 .1$, the weakly nonlinear amplitude equation governing the super-harmonic double-crest wave dynamics is derived in §4.3.2. Final comments and conclusions are outlined in §4.4.

### 4.1 Flow configuration and governing equations: potential model

We consider a cylindrical container of diameter $D=2 R$ filled to a depth $h$ with a liquid of density $\rho$. The air-liquid surface tension is denoted by $\gamma$. The orbital (circular) shaking motion (see sketch in figure 4.1) can be represented as the combination of two sinusoidal translations with a $\pi / 2$ phase shift, thus leading to the following equations of motion for the container axis intersection with the $z=0$ plane, parametrized in cylindrical coordinates $(r, \theta)$

$$
\dot{\mathbf{X}}_{0}=\left\{\begin{array}{l}
-\frac{d_{s}}{2} \Omega_{d} \sin \left(\Omega_{d} t-\theta\right) \mathbf{e}_{r}  \tag{4.1}\\
\frac{d_{s}}{2} \Omega_{d} \cos \left(\Omega_{d} t-\theta\right) \mathbf{e}_{\theta}
\end{array} .\right.
$$

In the classical potential flow limit, i.e. the flow is assumed to be inviscid, irrotational and incompressible, the motion is described in terms of free surface deformation, $\eta$, and a potential velocity field, $\Phi_{t o t}$, which is typically separated into a container, $\Phi_{c}$, and a fluid component, $\Phi$. Hence, the liquid motion within the moving container is governed by the Laplace equation,

$$
\begin{equation*}
\Delta \Phi=\frac{1}{r} \frac{\partial \Phi}{\partial r}+\frac{\partial^{2} \Phi}{\partial r^{2}}+\frac{1}{r^{2}} \frac{\partial^{2} \Phi}{\partial \theta^{2}}+\frac{\partial^{2} \Phi}{\partial z^{2}}=0 \tag{4.2}
\end{equation*}
$$

subjected to the homogeneous no-penetration condition, $\nabla \Phi \cdot \mathbf{n}=\mathbf{0}$, at the solid sidewall and bottom, and by the dynamic and kinematic free surface boundary conditions at $z=\eta$ (see Ibrahim (2005)),

$$
\begin{gather*}
\frac{\partial \Phi}{\partial t}+\frac{1}{2} \nabla \Phi \cdot \nabla \Phi+\eta-\frac{\kappa(\eta)}{B o}=r f \cos (\Omega t-\theta)  \tag{4.3a}\\
\frac{\partial \eta}{\partial t}+\frac{\partial \Phi}{\partial r} \frac{\partial \eta}{\partial r}+\frac{1}{r^{2}} \frac{\partial \Phi}{\partial \theta} \frac{\partial \eta}{\partial \theta}-\frac{\partial \Phi}{\partial z}=0 \tag{4.3b}
\end{gather*}
$$

which have been made non-dimensional by using the container's characteristic length $R$, the characteristic velocity $\sqrt{g R}$ and the time scale $\sqrt{R / g}$. In (4.3a), $\kappa(\eta)$ denotes the fully nonlinear curvature, while $B o=\rho g R^{2} / \gamma$ is the Bond number. The non-dimensional driving amplitude and angular frequency read $f=d_{s} \Omega_{d}^{2} /(2 g)$ and, $\Omega=\Omega_{d} / \sqrt{g / R}$, respectively. When surface tension is accounted for, an additional contact line boundary condition is required at $z=\eta$ and $r=1$, typically written as $\partial \eta / \partial r=\cot \vartheta$, where $\vartheta$ is the macroscopic contact angle. Under the classic free-end edge contact line assumption with $\vartheta=\pi / 2$ adopted here, the latter dynamic equation simply reduces to $\partial \eta / \partial r=0$. This means that the free surface at rest is flat and that a $\pi / 2$ static contact angle is maintained when the contact line elevation changes dynamically.

### 4.2 Linear solution and second-order straightforward asymptotic expansion

In order to enlighten the limitations of the expansion procedure developed by Reclari et al. (2014), which motivates the formalization of the new theoretical framework proposed in the present work, we briefly recall the salient points. Let us consider the following asymptotic expansion for the flow quantities,

$$
\begin{gather*}
\Phi=\Phi_{0}+\epsilon \Phi_{1}+\epsilon^{2} \Phi_{2}+\mathrm{O}\left(\epsilon^{3}\right),  \tag{4.4a}\\
\eta=\eta_{0}+\epsilon \eta_{1}+\epsilon^{2} \eta_{2}+\mathrm{O}\left(\epsilon^{3}\right), \tag{4.4b}
\end{gather*}
$$

together with the further assumption of small driving forcing amplitudes of order $\mathrm{O}(\epsilon)$, i.e. $f=\epsilon F$, with $\epsilon$ a small parameter $\epsilon \ll 1$ and the auxiliary variable $F$ of order $O(1)$. Solution $\mathbf{q}_{0}=$ $\left\{\Phi_{0}, \eta_{0}\right\}^{T}$ represents the rest state, which has a potential velocity field null everywhere, $\Phi_{0}=0$, and a flat interface, $\eta_{0}=0$, as the contact angle is here assumed to be $\vartheta=\pi / 2$. Substituting

## Chapter 4. An amplitude equation modelling the double-crest swirling in orbital shaken cylindrical containers

the expansions above in equations (4.2)-(4.3b), a series of systems at the various order in $\epsilon$ is obtained. At leading order, equations (4.2)-(4.3b) reduce to a forced linear system, whose matrix compact form reads,

$$
\begin{equation*}
\left(\partial_{t} \mathscr{B}-\mathscr{A}\right) \mathbf{q}_{1}=\mathscr{F}_{1}, \tag{4.5}
\end{equation*}
$$

with $\mathbf{q}_{1}=\left\{\Phi_{1}, \eta_{1}\right\}^{T}, \mathscr{F}_{1}=F\{0, r / 2\}^{T} e^{\mathrm{i}(\Omega t-\theta)}+c . c .=F \hat{F}_{1}^{F} e^{\mathrm{i}(\Omega t-\theta)}+c . c$. and

$$
\mathscr{B}=\left(\begin{array}{ll}
0 & 0  \tag{4.6}\\
I_{\eta} & 0
\end{array}\right), \mathscr{A}=\left(\begin{array}{cc}
\Delta & 0 \\
0 & -I_{\eta}+\frac{1}{B o} \frac{\partial \alpha}{\partial \eta}
\end{array}\right),
$$

where $c . c$. stands for complex conjugate, $\partial \kappa / \partial \eta$ represents the first order variation of the curvature associated with the small perturbation $\epsilon \eta_{1}$ and $I_{\eta}$ is the identity matrix associated with the interface $\eta$. Note that the kinematic condition does not explicitly appear in (4.6), but it is enforced as a boundary condition at the interface (Viola et al., 2018). In the limit of zero external forcing, i.e. $F=0$, system (4.5) is a linear homogeneous problem which, by seeking for solutions having the following normal form

$$
\begin{equation*}
\hat{\mathbf{q}}_{m n}(r, z) e^{\mathrm{i}\left(\omega_{m n} t-m \theta\right)}+c . c ., \tag{4.7}
\end{equation*}
$$

reduces to the classic generalized eigenvalue problem for inviscid capillary-gravity waves

$$
\begin{equation*}
\left(\mathrm{i} \omega_{m n} \mathscr{B}-\mathscr{A}_{m}\right) \hat{\mathbf{q}}_{m n}=\mathbf{0}, \tag{4.8}
\end{equation*}
$$

where indices $(m, n)$ represent the number of nodal circles and nodal diameters, respectively, with $m$ also commonly known as azimuthal wavenumber. Owing to the normal mode expansion, we note that the operator $\mathscr{A}$ depends on the azimuthal wavenumber, $m$, and, therefore, we denote it by $\mathscr{A}_{m}$. An exact analytical solution to equation (4.8) can be readily obtained via a Bessel-Fourier-series representation leading to the well-known dispersion relation (Lamb, 1993)

$$
\begin{equation*}
\omega_{m n}^{2}=\left(k_{m n}+k_{m n}^{3} / B o\right) \tanh \left(k_{m n} H\right), \tag{4.9}
\end{equation*}
$$

with $H=h / R$ and where the wavenumbers $k_{m n}$ is given by the $n$ th-root of the first derivative of the $m$ th-order Bessel function of the first kind satisfying $J_{m}^{\prime}\left(k_{m n}\right)=0$.
Despite the existence of this analytical solution, in this work we opt for a numerical scheme based on a discretization technique, where linear operators $\mathscr{B}$ and $\mathscr{A}_{m}$ are discretized in space by means of a Chebyshev pseudospectral collocation method with a two-dimensional mapping implemented in Matlab, which is analogous to that described by Viola et al. (2018). This numerical technique will enable us to avoid straightforward, but cumbersome calculations, otherwise required in the development of the rest of this work and, particularly, of section §4.3.2. One must note that when (4.8) is solved numerically as in the present case, additional boundary conditions need to be made explicit in order to regularize the problem on the revolution axis ( $r=0$ ), i.e.

$$
\begin{equation*}
m=0: \quad \frac{\partial \hat{\eta}_{m n}}{\partial r}=\frac{\partial \hat{\Phi}_{m n}}{\partial r}=0 \tag{4.10a}
\end{equation*}
$$

$$
\begin{equation*}
|m| \geq 1: \quad \hat{\eta}_{m n}=\hat{\Phi}_{m n}=0 \tag{4.10b}
\end{equation*}
$$

It is also useful to note that owing to the symmetries of the problem, system (4.8) is invariant under the transformation

$$
\begin{equation*}
\left(\hat{\mathbf{q}}_{m n},+m, \mathrm{i} \omega_{m n}\right) \longrightarrow\left(\hat{\mathbf{q}}_{m n},-m, \mathrm{i} \omega_{m n}\right) \tag{4.11}
\end{equation*}
$$

Convergence of the numerical solution was checked by computing the first 16 modes ( $m=0,2,3,4$ with $n=1,2,3,4$ ), whose corresponding natural frequency values, $\omega_{m n}$, matched the analytical ones given by (4.9) up to the fourth digit for a computational grid $N_{r}=N_{z}=60$, with $N_{r}$ and $N_{z}$ the number of radial and axial grid points, respectively.
Let us now reintroduce the forcing term on the r.h.s. of equation (4.5). In contradistinction with the cases of unidirectional forcing (Miles, 1984a,d), for circular orbits, given the azimuthal periodicity of the associated forcing, the shaking at linear order is expected to excite nonaxisymmetric modes only and, specifically, those with $m=1$. Therefore, the linear response to the external forcing can be sought as

$$
\begin{equation*}
\mathbf{q}_{1}=F \hat{\mathbf{q}}_{1}^{F} e^{\mathbf{i}(\Omega t-\theta)}+c . c . \tag{4.12}
\end{equation*}
$$

with $\hat{\mathbf{q}}_{1}^{F}$ being the solution of the following forced problem

$$
\begin{equation*}
\left(\mathrm{i} \Omega \mathscr{B}-\mathscr{A}_{1}\right) \hat{\mathbf{q}}_{1}^{F}=\hat{\mathscr{F}}_{1}^{F} . \tag{4.13}
\end{equation*}
$$

The response structure $\hat{\mathbf{q}}_{1}^{F}$ is here computed numerically, but, in practice, it is formally equivalent to that obtained analytically by Reclari et al. (2014) by projecting the forcing term $\hat{\mathscr{F}}_{1}$ onto the basis formed by the first order Bessel functions of the first kind, except that surface tension is retained here because of the finite Bond number. Noting that $\epsilon F=f=d_{s} \Omega^{2} /(2 g)$, in figure 4.2 the linear solution $\epsilon \mathbf{q}_{1}^{F}$ from (4.12) is shown (black solid lines) and compared with experimental measurements reported by Reclari et al. (2014) in terms of maximum nondimensional crest-to-trough contact line amplitudes, $\tilde{\delta}=\delta R / D$, with $\delta(\theta, t)=\eta(r=1, \theta, t)$. Measurements for different values of the non-dimensional shaking diameters, $\tilde{d}_{s}=d_{s} / D$, are shown. Blue and green markers in figure 4.2 correspond to highly nonlinear scenarios manifesting a free surface breaking, which will be therefore ignored thereafter. As discussed by Reclari et al. (2014) and reproduced here, the linear solution describes well the single-crest (SC) wave dynamics for driving frequencies far enough from harmonic resonances and, particularly, for small $\tilde{d}_{s}$. However, as typical of undamped forced oscillators, the amplitude of the inviscid linear response to the external forcing is proportional to $\propto 1 /\left|\Omega^{2}-\omega_{1 n}^{2}\right|$ and therefore it naturally diverges close to $\omega_{1 n}$, thus failing in predicting the close-to-resonance behaviour, e.g. for $\tilde{d}_{s}=0.02$ at $\Omega \approx \omega_{11}$. Introduction of viscous dissipation would regularize the divergent behaviour at $\Omega=\omega_{11}$, however, in the absence of any nonlinear restoring term, the hardening nonlinearity displayed in figure 4.2 cannot be retrieved.
Furthermore, in experiments multiple-crested waves were observed at fractions of the natural frequencies (red markers in figure 4.2), i.e. the system responses with a frequency which is

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Figure 4.2 - Markers correspond to the experimentally measured maximum crest-to-trough contact line amplitude (non-dimensional), with $\tilde{\delta}=\delta R / D=\delta / 2$, reported by Reclari et al. (2014) for two container diameters, $D=0.144 \mathrm{~m}$ and $D=0.287 \mathrm{~m}$, a non-dimensional depth $\tilde{H}=h / D=0.52$ and five values of $\tilde{d}_{s}=d_{s} / D$, as a function of the non-dimensional shaking frequency $\Omega$ normalized by the natural frequency of the first non-axisymmetric mode, $\omega_{11}=1.3286(m=1)$ on the bottom-x-axis and by that of first non-axisymmetric mode with $m=2, \omega_{21}=1.7475$, on the top-x-axis (the frequency values correspond to $D=0.287 \mathrm{~m}$ ). Colors denote different wave conditions. Black solid lines: linear potential model solution, from (4.12), computed by solving numerically equation (4.13). Red solid lines: weakly nonlinear solution close to the $\Omega \approx \omega_{21} / 2$, obtained by computing (4.17). Note that in order to ease the comparison with experiments, the non-dimensional $\delta$ was rescaled by a factor $R / D=1 / 2$, as the container diameter $D$, rather than the container radius $R$, was used by Reclari et al. (2014) to make the equations non-dimensional.
$n$-times (with $n$ positive integer) that of the external forcing. Here we refer to such conditions as super-harmonic dynamics (note that the terminology subharmonic was used by Reclari et al. (2014) instead). Among these super-harmonics, the double-crest (DC) wave dynamics, occurring at a driving frequency $\Omega \approx \omega_{21} / 2$, was seen to be the most relevant (see figure 4.2), i.e. the most stable and the one displaying the largest deviation from the linear approximation. This specific multiple-crest dynamics, which is intrinsically nonlinear, is indeed favoured by the azimuthal symmetry of the external forcing. Reclari et al. (2014) tentatively described such double-crest dynamics by pursuing the asymptotic analysis up to the second order in $\epsilon$, as in equations (4.4a) and (4.4b), in order to account for second order system weak nonlinearities.
At the second order in $\epsilon$, one obtains the following forced linear system,

$$
\begin{equation*}
\left(\partial_{t} \mathscr{B}-\mathscr{A}\right) \mathbf{q}_{2}=\mathscr{F}_{2}=F^{2}\left(\hat{\mathscr{F}}_{2}^{F F} e^{\mathrm{i}(2 \Omega t-2 \theta)}+c . c .\right)+F^{2} \hat{\mathscr{F}}_{2}^{F \bar{F}}, \tag{4.14}
\end{equation*}
$$

where $\mathscr{F}_{2}$ gathers a series of terms produced by the first-order solution through the secondorder system nonlinearities. For the sake of brevity, the explicit expression of these forcing terms is here omitted (see Appendix 4.5.4 for details). The bar denotes the complex conjugate. Also, note that amplitude $F$ is actually a real quantity and in the following the superscript $\bar{F}$ will be used only to indicate forcing terms produced by the combination of the direct and complex conjugate contributions of the first order response to the external forcing. The r.h.s. of equation (4.14) clearly shows how second-order terms naturally induce a super-harmonic response, whose spatial periodicity is $m=2$, hence precisely corresponding to the double-crest dynamics experimentally observed. The second forcing term on the r.h.s. of (4.14) has $\omega=0$ and $m=0$, i.e. it is steady and axisymmetric. It originates in the leading order contribution in the time and azimuthally averaged flow, the so-called mean flow. Equation (4.14) was solved analytically by Reclari et al. (2014) by retaining for convenience only two modes, namely those with $(m, n)=(2,1)$ and $(0,1)$, expected to be the relevant ones. The numerical scheme employed in this work allows us to effortlessly account for all the $(2, n)$ and $(0, n)$ modes simultaneously, as their contribution will be directly encompassed in the spatial function $\hat{\mathbf{q}}_{2}^{F F}$ and $\hat{\mathbf{q}}_{2}{ }^{F} \bar{F}$, appearing in the second order solution,

$$
\begin{equation*}
\mathbf{q}_{2}=\left(F^{2} \hat{\mathbf{q}}_{2}^{F F} e^{\mathrm{i}(2 \Omega t-2 \theta)}+\text { c.c. }\right)+F^{2} \hat{\mathbf{q}}_{2}^{F \bar{F}}, \tag{4.15}
\end{equation*}
$$

whose contributions are computed by solving the following systems

$$
\begin{equation*}
\left(\mathrm{i} 2 \Omega \mathscr{B}-\mathscr{A}_{2}\right) \hat{\mathbf{q}}_{2}^{F F}=\hat{\mathscr{F}}_{2}^{F F}, \quad-\mathscr{A}_{0} \hat{\mathbf{q}}_{2}^{F \bar{F}}=\hat{\mathscr{F}}_{2}^{F \bar{F}} \tag{4.16}
\end{equation*}
$$

The total flow field, obtained through the asymptotic model is then given by the sum of the first and second-order solutions,

$$
\begin{equation*}
\mathbf{q}=\left(f \hat{\mathbf{q}}_{1}^{F} e^{\mathrm{i}(\Omega t-\theta)}+f^{2} \hat{\mathbf{q}}_{2}^{F F} e^{\mathrm{i}(2 \Omega t-2 \theta)}+c . c .\right)+f^{2} \hat{\mathbf{q}}_{2}^{F \bar{F}}, \tag{4.17}
\end{equation*}
$$

where, in order to eliminate the implicit small parameter $\epsilon$, the amplitudes $\epsilon F$ and $\epsilon^{2} F^{2}$ are recast in terms of the physical amplitudes, $f$ and $f^{2}$, respectively. The resulting prediction of the maximum crest-to-trough contact line amplitude, $\delta(\theta, t)=\eta(r=1, \theta, t)$ is shown in figure 4.2 for driving frequencies close to $\Omega / \omega_{21} \approx 0.5$ (see top-x-axis) as red solid lines. Although this straightforward asymptotic expansion detects the emergence of the super-harmonic doublecrest wave in that frequency window, it completely fails in capturing the correct nonlinear wave amplitude saturation displaying a hardening behaviour clearly visible in figure 4.2. Once again, the amplitude of the inviscid second harmonic response is proportional to $\propto 1 /\left|\Omega^{2}-\omega_{2 n}^{2} / 4\right|$ and the total solution tends to diverge close to the double-crest super-harmonic at $\omega_{21} / 2$.
Such a symmetric and, in the absence of dissipation, close-to-resonance divergent behaviour is actually expected when performing straightforward asymptotic expansions as they typically suffer from secular (or resonating) terms that must be properly treated (see Castaing (2005) and Nayfeh (2008a) among many other references).

### 4.3 Weakly nonlinear analysis via multiple timescale method

In order to overcome the aforementioned limitations of the straightforward asymptotic expansion procedure and thus to attempt to bridge the gap between theoretical predictions and experimental observations, we conduct in this section a weakly nonlinear analysis (WNL) based on the multiple timescale method. With the aim to derive a weakly nonlinear amplitude equation governing the double-crest dynamics (DC), we first tackle the simpler problem of single-crest waves (SC). In both cases, we look for a third-order asymptotic solution to the system

$$
\begin{equation*}
\mathbf{q}=\{\Phi, \eta\}^{T}=\epsilon \mathbf{q}_{1}+\epsilon^{2} \mathbf{q}_{2}+\epsilon^{3} \mathbf{q}_{3}+\mathrm{O}\left(\epsilon^{4}\right) \tag{4.18}
\end{equation*}
$$

where the zero order solution, $\mathbf{q}_{0}=\mathbf{0}$, is omitted.

### 4.3.1 Single-crest dynamics (SC)

In $\S 4.2$ the forcing amplitude $f$ was assumed of order $\epsilon$, thus leading to a linear first-order problem directly forced by the external shaking, which produces a divergent response close to harmonic resonances. With regards to single-crest waves and specifically to the harmonic response at a driving frequency close to that of one of the non-axisymmetric modes, $\omega_{1 n}$, we assume here a small forcing amplitude of order $\epsilon^{3}$. This assumption is justified by the fact that close-to-resonance, $\Omega \approx \omega_{1 n}$, and in the absence of dissipation, even a small forcing will induce a large system response. The following analysis is therefore expected to hold for $\Omega=\omega_{1 n}+\lambda$, where $\lambda$ is a small detuning parameter assumed of order $\epsilon^{2}$. Lastly, in the spirit of the multiple scale technique, we introduce the slow time scale $T_{2}=\epsilon^{2} t$, with $t$ being the fast time scale at which the free surface oscillates with angular frequency $\approx \omega_{1 n}$. Hence, the following scalings are assumed:

$$
\begin{equation*}
f=\epsilon^{3} F, \quad \lambda=\Omega-\omega_{1 n}=\epsilon^{2} \Lambda, \quad T_{2}=\epsilon^{2} t \tag{4.19}
\end{equation*}
$$

with $F$ and $\Lambda$ of order $\mathrm{O}(1)$. We note that the forcing amplitude could be assumed of order $\epsilon^{2}$ (as the other parameters), however, this complicates unnecessarily the second order problem without modifying the final amplitude equation, even if the values of its coefficients will end up being slightly different (up to a order $\epsilon$ ).
Although the asymptotic expansion is here pursued up to the third order in $\epsilon$, the procedure of the weakly nonlinear analysis is essentially equivalent to that of the straightforward asymptotic model discussed in $\S 4.2$. The major difference lies in the solution form of the leading order problem that is now a homogenous problem, as in equation (4.8). Given the azimuthal periodicity of the external forcing, among all possible natural eigenmodes, we assume a leading order solution as

$$
\begin{equation*}
\mathbf{q}_{1}=A_{1}\left(T_{2}\right) \hat{\mathbf{q}}_{1}^{A_{1}} e^{\mathrm{i}\left(\omega_{1 n} t-\theta\right)}+\text { c.c. } \tag{4.20}
\end{equation*}
$$

where $\hat{\mathbf{q}}_{1}^{A_{1}}$ is the eigenmode (computed by solving (4.8)) associated with ( $m, n$ ) $=(1, n$ ) and $\omega_{1 n}$ is the corresponding natural frequency (solution of (4.9)).

The complex amplitude $A_{1}$, function of the slow time scale $T_{2}$ and still unknown at this stage of the expansion, describes the slow time amplitude modulation of the oscillating wave $\hat{\mathbf{q}}_{1}^{A_{1}}$ and introduces a new arbitrariness in the problem, which must be fixed at a higher order. Eigen-surface, $\hat{\eta}_{1}^{A_{1}}$, and eigen-potential field, $\hat{\Phi}_{1}^{A_{1}}$, computed for $\omega_{1 n}=\omega_{11}$, are shown in figure 4.3(a) and (b), respectively.

By pursuing the expansion to the second order, a linear system forced by the first order solution and analogous to that of equation (4.14) is obtained (see Reclari (2013) for the full expansion of the original nonlinear governing equation up to the second order). Nevertheless, the forcing terms on the r.h.s. are here proportional to $A_{1}^{2}$ (super- or second-harmonic) and to $A_{1} \bar{A}_{1}$ (mean flow). Thus, we seek a second-order solution in the form

$$
\begin{equation*}
\mathbf{q}_{2}=A_{1} \bar{A}_{1} \hat{\mathbf{q}}_{2}^{A_{1} \bar{A}_{1}}+\left(A_{1}^{2} \hat{\mathbf{q}}_{2}^{A_{1} A_{1}} e^{\mathrm{i}\left(2 \omega_{1 n} t-2 \theta\right)}+c . c .\right) \tag{4.21}
\end{equation*}
$$



Figure 4.3 - (a), (b) and (c): real part of the first, $\hat{\eta}_{1}^{A_{1}}$, and second order, $\hat{\eta}_{2}^{A_{1} A_{1}}$ and $\hat{\eta}_{2}^{A_{1} \overline{A_{1}}}$, free surface deformations computed for $\omega_{1 n}=\omega_{11}$. (d), (e) and (f): imaginary part of the associated first order, $\hat{\Phi}_{1}^{A_{1}}$, and second order, $\hat{\Phi}_{2}^{A_{1} A_{1}}$ and $\hat{\Phi}_{2}^{A_{1} \overline{A_{1}}}$, potential velocity field. Each response is denoted by its amplitude dependence, $\epsilon A_{1}, \epsilon^{2} A_{1} \bar{A}_{1}$ and $\epsilon^{2} A_{1} A_{1}$. The first order eigenmode is normalized with the amplitude and phase of the contact line (at $r=1$ ), such that the free surface, $\hat{\eta}_{1}^{A_{1}}$ is purely real, whereas $\hat{\Phi}_{1}^{A_{1}}$ is purely imaginary. Note that the second order mean flow constantly induces an upside down bell-like axisymmetric interface deformation pushing the free surface downward at the center of the moving container. Calculations are performed for the case of figure 4.2, i.e. pure water with $\rho=1000 \mathrm{~m} / \mathrm{m}^{3}, \gamma=0.072 \mathrm{~N} / \mathrm{m}, D=0.287 \mathrm{~m}$ and $\tilde{H}=h / D=0.52$, for which $B o=2802.8$ and $\omega_{11}=1.3286$.

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with $\hat{\mathbf{q}}_{2}^{A_{1} \bar{A}_{1}}$ and $\hat{\mathbf{q}}_{2}^{A_{1} A_{1}}$ computed numerically and displayed in figure 4.3(b)-(d) and (c)-(f), respectively, in terms of second order free surface deformations and potential velocity fields evaluated for $\omega_{1 n}=\omega_{11}$. From a numerical perspective, we note that the second-order responses can be straightforwardly computed as long as the pairs $(\omega, m)=\left(2 \omega_{1 n}, 2\right)$ and $(0,0)$ do not correspond to eigenvalues of (4.8), i.e. the second order operators (i2 $\omega_{1 n} \mathscr{B}-\mathscr{A}_{2}$ ) and $-\mathscr{A}_{0}$ are non-singular and hence invertible.
With regards to figure 4.3, it is interesting to note how the second-order mean flow potential velocity field is null everywhere. This can be rigorously proven by first noticing that the mean flow corresponds to a time- and azimuthal-averaged flow, i.e. $\partial / \partial t=\partial / \partial \theta=0$. Moreover, in the inviscid limit, free surface elevation and potential field have a $\pi / 2$ phase shift, meaning that the first-order eigenmode can be normalized such that the eigen-surface is purely real, whereas the eigen-potential is purely imaginary. Under these conditions, the mean flow forcing term on the r.h.s. of the kinematic equation cancels out, so that the associated Laplace equation appears to be constrained by homogeneous Neumann conditions on all the domain boundaries, thus prescribing a trivial constant potential field and therefore a null velocity field. In other words, the second order mean flow system reduces to forced linear meniscus equation (resulting from (4.3a)) and its conditions at $r=0$ and $r=1$ (both $\partial \hat{\eta}_{2}^{A_{1} \bar{A}_{1}} / \partial r=0$ ), which prescribes a static mean interface deformation only. Such a result was expected since the second-order mean flow response represents the Eulerian mean flow, which, together with the so-called Stokes drift, contributes to the overall Lagrangian mean flow (see Bremer and Breivik (2018) for a thorough review).

While the Stokes drift is a pure kinematic concept, the Eulerian mean flow, often referred to as streaming flow (Bouvard et al., 2017), is generally believed to be of viscous origin, although another appealing interpretation has been recently proposed (Faltinsen and Timokha, 2019). Sticking to the well-accepted viscous Eulerian mean flow generation mechanism, it is not a surprise that the absence of viscous boundary layers results in a vanishing Eulerian mean flow.
We now move forward to the $\epsilon^{3}$-order problem, which is once again a linear problem forced by combinations of the first and second order solutions as well as by the slow time derivative of the leading order solution and by the external forcing, which was assumed of order $\epsilon^{3}$,

$$
\begin{array}{r}
\left(\partial_{t} \mathscr{B}-\mathscr{A}_{m}\right) \mathbf{q}_{3}=\mathscr{F}_{3}=  \tag{4.22}\\
=-\partial_{T_{2}} A_{1} \mathscr{B} \hat{\mathbf{q}}_{1}^{A_{1}} e^{\mathrm{i}\left(\omega_{1 n} t-\theta\right)}+\left|A_{1}\right|^{2} A_{1} \hat{\mathscr{F}}_{3}^{A_{1} \bar{A}_{1} A_{1}} e^{\mathrm{i}\left(\omega_{1 n} t-\theta\right)}+F \hat{\mathscr{F}}_{3}^{F} e^{\mathrm{i} \Lambda T_{2}} e^{\mathrm{i}\left(\omega_{1 n} t-\theta\right)} \\
+\mathrm{N} . \text { R.T. }+ \text { c.c. },
\end{array}
$$

with $\hat{\mathscr{F}}_{3}^{F}=\{0, r / 2\}^{T}$ and where N.R.T. stands for non-resonating terms, which are not relevant for further analysis. As standard in multiple scale analysis, the indeterminacy introduced by the unknown amplitude $A_{1}$ is resolved by requiring that secular terms do not appear in the solution to equation (4.22). Secularity results from all resonant forcing terms in $\mathscr{F}_{3}$, i.e. all terms sharing the same frequency and wavenumber ( $\omega_{1 n}, 1$ ) of $\mathbf{q}_{1}$, and in effect all terms explicitly written in (4.22). It follows that a compatibility condition must be enforced through the Fredholm alternative (Friedrichs, 2012). Such a compatibility condition imposes the
amplitude $B=\epsilon A_{1} e^{\mathrm{i} \lambda t}$ to obey the following normal form

$$
\begin{equation*}
\frac{d B}{d t}=-\mathrm{i} \lambda B+\mathrm{i} \mu_{S C} f+\mathrm{i} v_{S C}|B|^{2} B, \tag{4.23}
\end{equation*}
$$

where the physical time $t=T_{2} / \epsilon^{2}$ has been reintroduced and where forcing amplitude and detuning parameter are recast in terms of their corresponding physical value, $f=\epsilon^{3} F$ and $\lambda=\epsilon^{2} \Lambda$. Moreover, the small implicit parameter $\epsilon$ is eliminated by defining the total physical amplitude $A=\epsilon A_{1}$ (Bongarzone et al., 2021a). The subscript SC stands for single-crest (SC). The various normal form coefficients, which turn out to be real-valued quantities owing to the absence of dissipation, are computed as scalar products between the adjoint mode, $\hat{\mathbf{q}}_{1}^{A_{1} \dagger}$, associated with $\hat{\mathbf{q}}_{1}^{A_{1}}$, and the third order resonant forcing terms as follows

$$
\begin{gather*}
\mathrm{i} \mu_{S C}=\frac{\left\langle\hat{\mathbf{q}}_{1}^{A_{1} \dagger}, \mathscr{B} \hat{\mathscr{F}}_{3}^{F}>\right.}{\left\langle\hat{\mathbf{q}}_{1}^{A_{1} \dagger}, \mathscr{B} \hat{\mathbf{q}}_{1}^{A_{1}}>\right.}=\frac{\int_{z=0} r \overline{\hat{\eta}}_{1}^{A_{1} \dagger} / 2 r \mathrm{~d} r}{\int_{z=0}\left(\hat{\eta}_{1}^{A_{1} \dagger} \hat{\Phi}_{1}^{A_{1}}+\hat{\Phi}_{1}^{A_{1} \dagger} \hat{\eta}_{1}^{A_{1}}\right) r \mathrm{~d} r},  \tag{4.24a}\\
\mathrm{i} v_{S C}=\frac{<\hat{\mathbf{q}}_{1}^{A_{1} \dagger}, \mathscr{B} \hat{\mathscr{F}}_{3}^{A_{1} \bar{A}_{1} A_{1}}>}{\left\langle\hat{\mathbf{q}}_{1}^{A_{1} \dagger}, \mathscr{B} \hat{\mathbf{q}}_{1}^{A_{1}}>\right.}=\frac{\int_{z=0}\left(\hat{\eta}_{1}^{A_{1} \dagger} \hat{\mathscr{F}}_{3_{\mathrm{dyn}}}^{A_{1} \bar{A}_{1} A_{1}}+\hat{\Phi}_{1}^{A_{1} \dagger} \hat{\mathscr{F}}_{3_{\mathrm{kin}}}^{A_{1} \bar{A}_{1} A_{1}}\right) r \mathrm{~d} r}{\int_{z=0}\left(\hat{\eta}_{1}^{A_{1} \dagger} \hat{\Phi}_{1}^{A_{1}}+\hat{\Phi}_{1}^{A_{1} \dagger} \hat{\eta}_{1}^{A_{1}}\right) r \mathrm{~d} r} . \tag{4.24b}
\end{gather*}
$$

Here $\hat{\mathbf{q}}_{1}^{A_{1} \dagger}=\overline{\hat{\mathbf{q}}}_{1}^{A_{1}}$, since the inviscid problem is self-adjoint with respect to the Hermitian scalar product $\langle\mathbf{a}, \mathbf{b}\rangle=\int_{\Sigma} \overline{\mathbf{a}} \cdot \mathbf{b} \mathrm{d} V$, with $\mathbf{a}$ and $\mathbf{b}$ two generic vector (see Viola et al. (2018) for a thorough discussion and derivation of the adjoint problem). For the sake of brevity, the explicit expression of $\hat{\mathscr{F}}_{3}^{A_{1} \bar{A}_{1} A_{1}}$ is omitted, as it only involves straightforward calculations, i.e. a Taylor expansion of nonlinear governing equations and boundary conditions (4.2)-(4.3b) around the rest state $\mathbf{q}_{0}=\mathbf{0}$. Here we simply denote with the subscript ${ }_{d y n}$ and ${ }_{k i n}$ the forcing components appearing in the dynamic and kinematic boundary condition, respectively.
By turning to polar coordinates, $B=|B| e^{i \Theta}$, splitting the modulus and phase parts of (4.23) and looking for stationary solution, $d / d t=0$ with $|B| \neq 0$, the following implicit relation is obtained,

$$
\begin{equation*}
\tilde{d}_{s} \Omega^{2} \mp \frac{\left(\lambda-v_{S C}|B|^{2}\right)|B|}{\mu_{S C}}=0, \tag{4.25}
\end{equation*}
$$

or, in a more common polynomial form,

$$
\begin{equation*}
P(|B|)=|B|^{3}-\frac{\lambda}{v_{S C}}|B| \pm \frac{\mu_{S C} \tilde{d}_{s} \Omega^{2}}{v_{S C}}=0, \tag{4.26}
\end{equation*}
$$

where $f=\tilde{d}_{s} \Omega^{2}, \lambda=\left(\Omega-\omega_{1 n}\right)$ and the $\mp$ signs correspond to the phases $\Theta=0$ and $\pi$, respectively. The two branches prescribed by (4.25) for $|B|$ as a function of $\Omega$ at a fixed nondimensional shaking diameter $\tilde{d}_{s}$ can be easily computed using the Matlab function fimplicit. After evaluating the stable and unstable stationary solutions for $|B|$ and $\Theta$, the total single-crest wave solution is reconstructed as

$$
\begin{equation*}
\mathbf{q}_{S C}=\{\Phi, \eta\}^{T}=\mathbf{q}_{1}+\mathbf{q}_{2} \tag{4.27}
\end{equation*}
$$

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| Figure | $(m, n)$ | $\tilde{H}=h / D$ | $D[\mathrm{~m}]$ | $\omega_{m n}$ | $\mu_{S C}$ | $v_{S C}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | ---: |
| $4(a)$ | $(1,1)$ | 0.50 | 0.287 | 1.324 | 0.277 | 1.526 |
| $4(a)$ | $(1,2)$ | 0.50 | 0.287 | 2.321 | 0.042 | 17.025 |
| $4(b)$ | $(1,1)$ | 0.52 | 0.287 | 1.323 | 0.278 | 1.485 |

Table 4.1 - Value of the amplitude equation coefficients $\mu_{S C}$ and $v_{S C}$ used to produce figure 4.4.


Figure 4.4 - Comparison of the WNL prediction for single-crest waves (SC) with experiments in terms of maximum crest-to-trough contact line amplitude (non-dimensional), $\Delta \tilde{\delta}$. (a) Black pentagons correspond to the experimental measurements presented in figure 4.30 of Reclari (2013) (R13) for $\tilde{d}_{s}=d_{s} / D=0.01$, where the first two non-axisymmetric modes $(m, n)=(1,1)$ and $(1,2)$ are detected. Dotted black lines: solution of the linear potential model according to (4.12). Light blue and blue lines: WNL SC prediction (4.27). Unstable branches are represented as a dashed line. Violet solid lines: theoretical prediction by Raynovskyy and Timokha (2018a) (R\&T18) (see §4.3.1 for further comments). (b) Same as (a) with the black filled circles corresponding to the measurements of Reclari et al. (2014) (R14) reported in figure 4.2 for $\tilde{d}_{s}=0.02$. The values of normal form coefficient $\mu_{S C}$ and $v_{S C}$ computed for $\Omega \approx \omega_{11}$ and $\omega_{12}$ (see bottom and top x-axes) are given in table 4.1, together with the corresponding values of $\tilde{H}, D$ and natural frequencies $\omega_{m n}$ used in this calculation.

## Experiments vs weakly nonlinear prediction: wave amplitude

In figure 4.4(a) and (b) the weakly nonlinear (WNL) prediction in terms of maximum crest-totrough contact line amplitude, $\Delta \tilde{\delta}$, for SC waves is compared with two sets of experimental measurements and with the potential linear solution (4.12). In comparison to the linear theory presented in $\S 4.2$, the agreement with experiments improves for different shaking diameters and for different harmonic resonances, e.g. those associated with modes $(m, n)=(1,1)$ and $(1,2)$ of figure 4.4(a). The hardening nonlinearity is correctly captured and the amplitude prediction matches well the measurements until the free surface eventually breaks and the wave regime leaves the weakly nonlinear regime, hence suggesting the little relevance of
dissipative effects attributable to viscosity in this regime.
However, one must note that in this weakly nonlinear approach, the driving frequency is essentially fixed around that of a unique non-axisymmetric natural mode, $\Omega \approx \omega_{1 n}$. Consequently, when performing the analysis for a mode $(1, n)$, the influence of all other modes is completely overlooked. In consequence, the accuracy of the asymptotic solution rapidly deteriorates moving away from harmonic resonances, when compared to the linear solution (4.12), which turns out to be more accurate. This is visible looking at the bottom stable branch in the multi-solution range of figure $4.4(\mathrm{~b})$ or by looking at the driving frequency window $\Omega \in\left[0.7 \omega_{12}, 0.9 \omega_{12}\right]$ in figure 4.4(a). In other words, the detuning parameter should be small in order for the present weakly nonlinear analysis, based on a single-mode expansion, to hold. In this regard, as no other natural frequencies are encountered for $\Omega<\omega_{11}$, an exception is made for the left branch associated with the harmonic resonance of the first (or fundamental) non-axisymmetric mode, where an excellent agreement of the single mode prediction, comparable to that of the linear solution, lasts until $\Omega \approx 0$.

## Comparison with the multimodal theory by Raynovskyy and Timokha (2018a)

The violet solid curves reported in figure 4.4(a) and (b) correspond to the predictions associated with the $\omega_{11}$-single-crest swirling from the Narimanov-Moiseev multimodal sloshing theory employed by Raynovskyy and Timokha (2018a) (R\&T18) (only the stable branches are reported). Their curves have been here carefully reproduced by manually sampling those reported in figure 3 of R\&T18 in the range of frequency available, i.e. $\Omega / \omega_{11} \in[0.8,1.3]$. By looking at the upper stable branch, although an increasing discrepancy is observed for increasing wave amplitudes, one can see that both analyses are in fairly good agreement with experiments and with each other until wave braking eventually occurs. Such a discrepancy could be tentatively attributed to the different definition of the detuning parameter employed in R\&T18, i.e. $\epsilon^{2} \Lambda_{R \& T 18}=\omega_{1 n}^{2} / \Omega^{2}-1$. On the other hand, by looking at the lower stable branch, one sees that, whereas the jump-up frequency according to R\&T18 and to the present model essentially coincide, the discrepancy between the two predictions increases at a larger frequency, i.e. $\Omega / \omega_{11}>1$, with the one of R\&T18 that is closer to the linear potential model. One should also note that, in contradistinction with our analysis, that of R\&T18 accounts for damping and predicts the jump-down transition visible in figure $4.4(a)$. This damping was essentially fitted from the experimental measurements and, specifically, from the jump-down frequency occurring at a larger frequency, once the wave breaking regime is fully established, i.e. $\Omega / \omega_{11}=1.27$ for $\tilde{d}_{s}=0.01$ and $\Omega / \omega_{11}=1.45$ for $\tilde{d}_{s}=0.02$ (see figure 4.4). However, experiments suggest that the damping effect on the curves displayed in figure 4.4 would not be easy to observe, even for $\tilde{d}_{s}=0.01$. Indeed, the motion undergoes a single crest wave breaking, thus entering a fully nonlinear regime, where both our analysis and that of R\&T18 lose predictive power. We, therefore, decided to discard damping while comparing our results with the close-to-harmonic resonance experiments from Reclari (2013) and Reclari et al. (2014).

# Chapter 4. An amplitude equation modelling the double-crest swirling in orbital shaken cylindrical containers 

## The Duffing oscillator analogy

Mass-spring models are widely employed in several engineering fields, e.g. in aerospace engineering, for the description of close-to-resonance sloshing motions (Bauer, 1966; Dodge, 2000; Moiseev, 1958), where nonlinearities are of crucial importance. The most popular driven nonlinear mass-spring model is that developed by Duffing (1918), who added a cubic nonlinear spring deformation (cubic term) to the classically driven harmonic oscillator

$$
\begin{equation*}
\ddot{x}+2 \sigma \dot{x}+x+c_{3} x^{3}=p \cos \Omega t, \tag{4.28}
\end{equation*}
$$

where $\sigma$ is the damping coefficient and where, depending on the sign of $c_{3}$ the resonance curve bends and the nonlinear resonance frequency shifts, i.e. it decreases for softening spring ( $c_{3}<0$ ), whereas it increases for a hardening spring ( $c_{3}>0$ ), thus explaining the original observation of Duffing on vibration mechanism. Ockendon and Ockendon (1973) showed via asymptotic expansion of the potential flow solution in the neighbourhood of a harmonic resonance that for small external forcing amplitudes, sloshing in a two-dimensional rectangular container responds exactly as an undamped Duffing oscillator (with $\sigma=0$ ). In Appendix 4.5.2, we briefly show that, as expected, the same holds for close-to-harmonicresonance sloshing in orbital shaken cylindrical containers, whose formal amplitude equation, starting from the full inviscid hydrodynamic system, was derived in $\S 4.3 .1$ (see equation (4.23)). Typically, when the Duffing equation is employed to model close-to-resonance responses in sloshing dynamics and experimental measurements are available, the nonlinear coefficient is often computed by fitting the experimental measurements. Recently, with regards to quasi-two-dimensional rectangular containers laterally excited, Bäuerlein and Avila (2021) have carried out careful quantitative comparisons between experiments and theoretical predictions from the damped Duffing equation, showing that their actual sloshing system is remarkably well described by the forced-damped Duffing oscillator. Nevertheless, for increasing wave amplitude responses, experiments deviate from the Duffing solution, which is not capable to predict correctly the phase lag between driving and response, shown to be the key factor for an accurate estimation of the sloshing amplitude of the maximal nonlinear resonance (Bäuerlein and Avila, 2021; Cenedese and Haller, 2020). We note that, by analogy with the undamped Duffing equation, the weakly nonlinear analysis formalized in §4.3.1 exacerbates this aspect, since, owing to the lack of dissipation, it can only predict the classic phase lag bounds, 0 and $\pi$ (see Appendix 4.5.1 for further comments on this regard). Nevertheless, one should notice that this intrinsic limitation turns to be unimportant in cases as those of figure 4.4, where for increasing amplitude a wave breaking eventually occurs and the weakly nonlinear theory as well as the Duffing mechanical analogy no longer apply.

### 4.3.2 Double-crest dynamics (DC)

We now tackle the double-crest (DC) wave response. Its formalization is slightly more subtle, as it requires a new reordering of the small control parameter magnitudes as well as an unusual
form of the leading order problem, involving both a homogenous and a particular solution. We remind that the double-crest dynamics in figure 4.2 occurs at a driving frequency $\Omega \approx \omega_{21} / 2$. Results at the end of this section will be presented for mode $(2,1)$, for which experiments are available, however, for the sake of generality, we formalize the analysis for any mode $(2, n)$, i.e. $\Omega=\omega_{2 n} / 2+\lambda$, where $\lambda$ is the small detuning parameter.

## Formalism

To determine a suitable scaling for the forcing amplitude $f$ and detuning parameter $\lambda$ it is instructive to look at the experimental measurements shown in figure 4.2 for $\Omega$ close to $\omega_{21} / 2$. One can see that approaching $\Omega \approx \omega_{21} / 2$ from lower frequencies, the doublecrest wave emerges on the top of a single-crest dynamics, with the latter being correctly described by the linear solution, which still behaves well as $\omega_{21} / 2$ is far enough from the primary harmonic resonance occurring at $\omega_{11}$. It follows that the forcing amplitude $f$ and detuning $\lambda$ could be retained at order $\epsilon$ and the first order problem takes the form (4.5), with $\mathscr{F}_{1}=F\{0, r / 2\}^{T} e^{\mathrm{i}(\Omega t-\theta)}+c . c .=F \hat{\mathscr{F}}_{1}^{F} e^{\mathrm{i}(\Omega t-\theta)}+c . c$., with $f=\epsilon F$ and $\lambda=\epsilon \Lambda$.
Furthermore, in $\S 4.2$ we have shown how, close enough to the super-harmonic resonance, the divergent behaviour is produced by a second order resonating term, which breaks the straightforward expansion, as $\epsilon^{2}$-order terms should not become larger than the $\epsilon$-order ones. In the following, this asymptotic breakdown is overcome by assuming that the leading order solution is given by the sum of (i) a particular solution, given by the linear response to the external forcing computed by solving (4.13), and (ii) a homogeneous solution, represented by the natural mode $(m, n)=(2, n)$, obtained by solving the generalized eigenvalue problem (4.8), up to an amplitude to be determined at higher orders. The second-order resonating term will then require, in the spirit of multiple timescale analysis, an additional second-order solvability condition, complementing the third-order non-resonance condition already obtained in the single-crest wave weakly nonlinear model. This suggests that two slow time scales exist, namely $T_{1}$ and $T_{2}$, with $T_{1}$ one $\epsilon$-order faster than $T_{2}$, hence implying that quadratic nonlinearities are stronger than cubic ones. To summarize, the fundamental scalings underpinning the weakly nonlinear multiple scale expansion for double-crest waves are the following:

$$
\begin{equation*}
f=\epsilon F, \quad \lambda=\Omega-\omega_{2 n} / 2=\epsilon \Lambda, \quad T_{1}=\epsilon t, \quad T_{2}=\epsilon^{2} t, \tag{4.29}
\end{equation*}
$$

The first order solution reads

$$
\begin{equation*}
\mathbf{q}_{1}=A_{2}\left(T_{1}, T_{2}\right) \hat{\mathbf{q}}_{1}^{A_{2}} e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)}+F \hat{\mathbf{q}}_{1}^{F} e^{\mathrm{i}\left(\left(\omega_{2 n} / 2\right) t-\theta\right)} e^{\mathrm{i} \Lambda T_{1}}+c . c . \tag{4.30}
\end{equation*}
$$

where the unknown slow time amplitude modulation, $A_{2}$, is here a function of the two time scales $T_{1}$ and $T_{2}$, while the amplitude of the particular solution simply equals the forcing amplitude and $\hat{\mathbf{q}}_{1}^{F}$ is computed from (4.13) for $\Omega=\omega_{2 n} / 2$.

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|  | $\epsilon A_{2}$ | $\epsilon F$ | $\epsilon^{2} A_{2} A_{2}$ | $\epsilon^{2} \Lambda F$ | $\epsilon^{2} A_{2} \bar{A}_{2}$ | $\epsilon^{2} F \bar{F}$ | $\epsilon^{2} A_{2} F$ | $\epsilon^{2} A_{2} \bar{F}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\breve{m}$ | 2 | 1 | 4 | 1 | 0 | 0 | 3 | 1 |
| $\breve{\omega}$ | $\omega_{2 n}$ | $\omega_{2 n} / 2$ | $2 \omega_{2 n}$ | $\omega_{2 n} / 2$ | 0 | 0 | $3 \omega_{2 n} / 2$ | $\omega_{2 n} / 2$ |

Table 4.2 - First order linear solutions and second-order non-resonating forcing terms gathered by their amplitude dependency and corresponding azimuthal and temporal periodicity ( $\breve{m}, \breve{\omega}$ ). Six terms have been omitted as they are the complex conjugates of $\epsilon A_{2}, \epsilon F, \epsilon^{2} A_{2} A_{2}, \epsilon^{2} \Lambda F$, $\epsilon^{2} A_{2} F$ and $\epsilon^{2} A_{2} \bar{F}$.

The second-order linearized forced problem reads

$$
\begin{equation*}
\left(\partial_{t} \mathscr{B}-\mathscr{A}_{m}\right) \mathbf{q}_{2}=\mathscr{F}_{2}=\mathscr{F}_{2}^{i j}-\frac{\partial A_{2}}{\partial T_{1}} \mathscr{B} \hat{\mathbf{q}}_{1}^{A_{2}} e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)}-\mathrm{i} \Lambda F \mathscr{B} \hat{\mathbf{q}}_{1}^{F} e^{\mathrm{i}\left(\left(\omega_{2 n} / 2\right) t-\theta\right)} e^{\mathrm{i} \Lambda T_{1}}+c . c . \tag{4.31}
\end{equation*}
$$

The first order solution is made of four different contributions of amplitude $A_{2}, \bar{A}_{2}, F$ and $\bar{F}$, therefore it generates 10 different second-order forcing terms, here denoted by $\mathscr{F}_{2}^{i j}$, which exhibit a certain frequency and azimuthal periodicity, $(\breve{\omega}, \breve{m})$. The additional two forcing terms stem from the time-derivative of the first order solution (4.30) with respect to the first order slow time scale $T_{1}$. In order to interpret the last term in (4.31), it is worth first noting that, while the amplitude of the linear solution (4.12), computed at a generic driving frequency, grows with $\Omega$ as $F /\left|\Omega^{2}-\omega_{11}^{2}\right|=\tilde{d}_{s} \Omega^{2} /\left|\Omega^{2}-\omega_{11}^{2}\right| \propto \Omega^{2} /\left|\Omega^{2}-\omega_{11}^{2}\right|$, in the weakly nonlinear model for double-crest waves, the amplitude of the particular solution (4.30) is proportional to $F /\left|\omega_{21}^{2} / 4-\omega_{11}^{2}\right|=\tilde{d}_{s} \Omega^{2} /\left|\omega_{21}^{2} / 4-\omega_{11}^{2}\right| \propto \Omega^{2}$, since the driving frequency was frozen at $\Omega=\omega_{21} / 2+\lambda$, with the small detuning parameter, $\lambda$, contributing to modify its phase, but not its amplitude. This leads to an increasing discrepancy between (4.12) and the leading order particular solution (4.30) away from the super-harmonic resonance. The response to the forcing term proportional to $\Lambda F$ in (4.31) can be then interpreted as a second-order correction of the amplitude of the first-order particular solution accounting for a detuning from the exact resonance through $\Lambda F \propto \tilde{d}_{s} \Omega^{2}\left(\Omega-\omega_{2 n} / 2\right)$ and contributing to improve the asymptotic approximation in a wider range of driving frequency in the neighborhood of the super-harmonic frequency.

None of the forcing terms in (4.31) is resonant, as their oscillation frequency and azimuthal wavenumber differ from those of the leading order homogeneous solution, except the term produced by the second-harmonic of the leading order particular solution, i.e. $\mathscr{F}_{2}^{F F}=F^{2} \hat{F}_{2}^{F F} e^{\mathrm{i}\left(\omega_{2 n}-2 \theta\right)} e^{\mathrm{i} 2 \Lambda T_{1}}+$ c.c.. To avoid secular terms, a second-order compatibility condition is imposed, requiring that the following normal form is verified

$$
\begin{equation*}
\frac{\partial A_{2}}{\partial T_{1}}=\mathrm{i} \mu_{D C} F^{2} e^{\mathrm{i} 2 \Lambda T_{1}}, \tag{4.32}
\end{equation*}
$$

with $\mu_{D C}$ computed as before, i.e.

$$
\begin{equation*}
\mathrm{i} \mu_{D C}=\frac{\int_{z=0}\left(\hat{\eta}_{1}^{A_{2} \dagger} \hat{\mathscr{F}}_{2_{\mathrm{dyn}}}^{F F}+\hat{\Phi}_{1}^{A_{2} \dagger} \hat{\mathscr{F}}_{2_{\mathrm{kin}}}^{F F}\right) r \mathrm{~d} r}{\int_{z=0}\left(\hat{\eta}_{1}^{A_{2} \dagger} \hat{\Phi}_{1}^{A_{2}}+\hat{\Phi}_{1}^{A_{2} \dagger} \hat{\eta}_{1}^{A_{2}}\right) r \mathrm{~d} r}, \tag{4.33}
\end{equation*}
$$

Taken alone, the dynamics resulting from (4.32) is however of little relevance, since the solution, i.e. the frequency-response curve, would still diverge (symmetrically) to infinity for $\Delta=$ $\Omega-\omega_{2 n} / 2 \rightarrow 0$ in absence of any restoring term, i.e. the nonlinear mechanism responsible for the finite amplitude saturation, which only comes into play at order $\epsilon^{3}$. This means that the expansion must be pursued up to the following order, and thereby that we must solve for the second-order solution.
By substituting (4.32) in the forcing expression, equation (4.31) can be rearranged as follows

$$
\begin{array}{r}
\left(\partial_{t} \mathscr{B}-\mathscr{A}_{m}\right) \mathbf{q}_{2}=\mathscr{F}_{2_{N R T}}^{i, j}+\mathscr{F}_{2_{R T}}^{i, j}=  \tag{4.34}\\
=\mathscr{F}_{2_{N R T}}^{i, j}+F^{2}\left(\hat{\mathscr{F}}_{2}^{F F}-\mathrm{i} \mu_{D C} \mathscr{B} \hat{\mathbf{q}}_{1}^{A_{2}}\right) e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)} e^{\mathrm{i} 2 \Lambda T_{1}}+c . c .,
\end{array}
$$

where the subscripts ${ }_{N R T}$ and ${ }_{R T}$ denote non-resonating (whose frequencies and azimuthal periodicities are gathered in table 6.1) and resonating terms, respectively. Note that the term proportional to $\Lambda F$ has been included in the non-resonating forcing terms, whereas the resonant term is written explicitly. The compatibility condition is now trivially satisfied, meaning that the new forcing term is orthogonal to the adjoint mode, $\hat{\mathbf{q}}_{1}^{A_{2} \dagger}=\overline{\hat{\mathbf{q}}}_{1}^{A_{2}}$, by construction and therefore, according to the Fredholm alternative, a non-trivial solution exists. Hence, we seek a second-order solution having the following form

$$
\begin{array}{r}
\mathbf{q}_{2}=A_{2} \bar{A}_{2} \hat{\mathbf{q}}_{2}^{A_{2} \bar{A}_{2}}+F^{2} \hat{\mathbf{q}}_{2}^{F \bar{F}}+ \\
+A_{2}^{2} \hat{\mathbf{q}}_{2}^{A_{2} A_{2}} e^{\mathrm{i}\left(2 \omega_{2 n} t-4 \theta\right)}+\Lambda F \hat{\mathbf{q}}_{2}^{\Lambda F} e^{\mathrm{i}\left(\left(\omega_{2 n} / 2\right) t-\theta\right)} e^{\mathrm{i} \Lambda T_{1}}+ \\
+A_{2} F \hat{\mathbf{q}}_{2}^{A_{2} F} e^{\mathrm{i}\left(\left(3 \omega_{2 n} / 2\right) t-3 \theta\right)} e^{\mathrm{i} \Lambda T_{1}}+A_{2} \bar{F} \hat{\mathbf{q}}_{2}^{A_{2} \bar{F}} e^{\mathrm{i}\left(\left(\omega_{2 n} / 2\right) t-\theta\right)} e^{-\mathrm{i} \Lambda T_{1}}+ \\
+F^{2} \hat{\mathbf{q}}_{2}^{F F} e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)} e^{\mathrm{i} 2 \Lambda T_{1}}+\text { c.c.. } \tag{4.35}
\end{array}
$$

All non-resonant responses in (4.35) are handled similarly, i.e. they are computed in Matlab by performing a simple matrix inversion using standard LU solvers (as in $\S \S 4.2$ and 4.3.1). As anticipated above, although the operator associated with the resonant forcing term, i.e. (i $\omega_{2 n} \mathscr{B}-\mathscr{A}_{2}$ ), is singular, the value of the normal form coefficient (4.33) ensures that a nontrivial solution for $\hat{\mathbf{q}}_{2}^{F F}$ exists. Diverse approaches can be followed to compute this response. Here such a response is computed by using the pseudoinverse matrix of the singular operator (Orchini et al., 2016). Another possible approach is given in Appendix A of Meliga et al. (2012b), where a two-step regularization procedure, involving an intermediate factious solution for $\hat{\mathbf{q}}_{2}^{F F}$ is employed. We also note that in (4.35), exactly as in (4.21), a second-order homogeneous solution has not been accounted for as its introduction would be irrelevant to the final solution.
The first order solutions together with all the non-resonating second-order responses are shown in the various panels of figure 4.5 , where the two leading order contributions, $\epsilon A_{2}$ and $\epsilon F$, corresponding to the double- and single-crest wave, respectively, can be identified. Moreover, we note that the second order response proportional to $\epsilon^{2} \Lambda F$ has a spatial structure similar to that of the leading order response $\epsilon F$, as it represents the second order correction to the latter caused by small frequency shifts of order $\epsilon$.

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Figure 4.5 - Real part of the first, (a) $\hat{\eta}_{1}^{A_{2}}$ and (e) $\hat{\eta}_{1}^{F}$, and non-resonating second order, (b) $\hat{\eta}_{2}^{A_{1} \bar{A}_{2}}$, (c) $\hat{\eta}_{2}^{A_{2} A_{2}}$, (d) $\hat{\xi}_{2}^{\Lambda_{1} F}$, (f) $\hat{\eta}_{2}^{F \bar{F}}$, (g) $\hat{\eta}_{2}^{A_{2} F}$ and (h) $\hat{\eta}_{2}^{A_{2} \bar{F}}$, free surface deformations computed for $\omega_{2 n}=\omega_{21}$. The first order eigenmode is normalized with the amplitude and phase of the contact line (at $r=1$ ), such that the free surface, $\hat{\eta}_{1}^{A_{2}}$ is purely real, whereas $\hat{\Phi}_{1}^{A_{2}}$ is purely imaginary. Note that the second order mean flow $\hat{\eta}_{2}^{F \bar{F}}$ constantly induces an upside down bell-like an axisymmetric interface deformation pushing the free surface downward at the center of the moving container, by analogy with the effect produced by $\hat{\eta}_{2}^{A_{1} \bar{A}_{1}}$ for SC waves, as the two responses are essentially equivalent up to a prefactor. Here the mean flow $\hat{\eta}_{2}^{A_{2}} \overline{A_{2}}$ for DC pushes the interface upward at the wall (same as $\hat{\eta}_{2}^{F \bar{F}}$ ) and, at the same time, downward in an annular region at intermediate radial coordinates, without altering the free surface elevation at the container revolutions axis.

Lastly, at third order in $\epsilon$, the problem reads

$$
\begin{array}{r}
\left(\partial_{t} \mathscr{B}-\mathscr{A}_{m}\right) \mathbf{q}_{3}=\mathscr{F}_{3}=  \tag{4.36}\\
=-\frac{\partial A_{2}}{\partial T_{2}} \mathscr{B} \hat{\mathbf{q}}_{1}^{A_{2}} e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)}-\mathrm{i} 2 \Lambda F^{2} \mathscr{B} \hat{\mathbf{q}}_{2}^{F F} e^{\mathrm{i}\left(\omega_{2 n}-2 \theta\right)} e^{\mathrm{i} 2 \Lambda T_{1}}+ \\
+\left|A_{2}\right|^{2} A_{2} \hat{\mathscr{F}}_{3}^{A_{2} \bar{A}_{2} A_{2}} e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)}+A_{2} F^{2} \hat{\mathscr{F}}_{3}^{A_{2} F \bar{F}} e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)}+ \\
+\Lambda F^{2} \hat{\mathscr{F}}_{3}^{\Lambda F F} e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)} e^{\mathrm{i} 2 \Lambda T_{1}}+\text { N.R.T. }+ \text { c.c. }
\end{array}
$$

where the first two forcing terms arise from the time-derivative of the first order solution with respect to the second order slow time scale $T_{2}$ and from that of the second order solution with respect to the first order slow time scale $T_{1}$, respectively (see Appendix 4.5.4 for the full expression of $\mathscr{F}_{2}$ and $\mathscr{F}_{3}$ ). By noticing that the second and last forcing terms share the same amplitude dependence, i.e. $\Lambda F^{2}$, they can be recast into a single forcing term, say $\Lambda F^{2} \hat{\mathscr{F}}_{3}^{\Lambda F F} e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)} e^{\mathrm{i} 2 \Lambda T_{1}}+c . c .$.
Once again, all terms explicitly written in (4.36) are resonant, as they share the same pair
$\left(\omega_{2 n}, 2\right)$ than the first order homogeneous solution, hence a third order compatibility condition, leading to the following normal form, must be enforced

$$
\begin{equation*}
\frac{\partial A_{2}}{\partial T_{2}}=\mathrm{i} \zeta_{D C} \Lambda F^{2} e^{\mathrm{i} 2 \Lambda T_{1}}+\mathrm{i} \chi_{D C} A_{2} F^{2}+\mathrm{i} v_{D C}\left|A_{2}\right|^{2} A_{2} \tag{4.37}
\end{equation*}
$$

with

$$
\begin{gather*}
\mathrm{i} \zeta_{D C}=\frac{\int_{z=0}\left(\hat{\eta}_{1}^{A_{2} \dagger} \hat{\mathscr{F}}_{3_{\mathrm{dyn}}}^{\Lambda F F}+\hat{\Phi}_{1}^{A_{2} \dagger} \hat{\mathscr{F}}_{3_{\mathrm{kin}}}^{\Lambda F F}\right) r \mathrm{~d} r}{\int_{z=0}\left(\hat{\eta}_{1}^{A_{2} \dagger} \hat{\Phi}_{1}^{A_{2}}+\hat{\Phi}_{1}^{A_{2} \dagger} \hat{\eta}_{1}^{A_{2}}\right) r \mathrm{~d} r},  \tag{4.38a}\\
\mathrm{i} \chi_{D C}=\frac{\int_{z=0}\left(\hat{\eta}_{1}^{A_{2} \dagger} \hat{\mathscr{F}}_{3_{\mathrm{dyn}}}^{A_{2} F \bar{F}}+\hat{\Phi}_{1}^{A_{2} \dagger} \hat{\mathscr{F}}_{3_{\mathrm{kin}}}^{A_{2} \bar{F}}\right) r \mathrm{~d} r}{\int_{z=0}\left(\hat{\eta}_{1}^{A_{2} \dagger} \hat{\Phi}_{1}^{A_{2}}+\hat{\Phi}_{1}^{A_{2} \dagger} \hat{\eta}_{1}^{A_{2}}\right) r \mathrm{~d} r},  \tag{4.38b}\\
\mathrm{i} v_{D C}=\frac{\int_{z=0}\left(\hat{\eta}_{1}^{A_{2} \dagger} \hat{F}_{3_{\mathrm{dyn}}}^{A_{2} \bar{A}_{2} A_{2}}+\hat{\Phi}_{1}^{A_{2} \dagger} \hat{\mathscr{F}}_{3_{\mathrm{k}}}^{A_{2} \bar{A}_{2} A_{2}}\right) r \mathrm{~d} r}{\int_{z=0}\left(\hat{\eta}_{1}^{A_{2} \dagger} \hat{\Phi}_{1}^{A_{2}}+\hat{\Phi}_{1}^{A_{2} \dagger} \hat{\eta}_{1}^{A_{2}}\right) r \mathrm{~d} r} \tag{4.38c}
\end{gather*}
$$

As a last step in the derivation of the final amplitude equation for the double-crest (DC) waves and in order to eliminate the implicit small parameter $\epsilon$, we unify (4.32) and (4.37) into a single equation recast in terms of the physical time $t=T_{1} / \epsilon=T_{2} / \epsilon^{2}$, physical forcing control parameters, $f=\epsilon F, \lambda=\epsilon \Lambda$ and total amplitude, $A=\epsilon A_{2}$. This is achieved by summing (4.32) and (4.37) along with their respective weights $\epsilon^{2}$ and $\epsilon^{3}$, thus obtaining

$$
\begin{equation*}
\frac{d B}{d t}=-\mathrm{i}\left(2 \lambda-\chi_{D C} f^{2}\right) B+\mathrm{i}\left(\zeta_{D C} \lambda+\mu_{D C}\right) f^{2}+\mathrm{i} v_{D C}|B|^{2} B \tag{4.39}
\end{equation*}
$$

where the change of variable $A=B e^{\mathrm{i} 2 \lambda t}$ has been introduced for convenience. As in $\S 4.3 .1$, by turning to polar coordinates, $B=|B| e^{i \Theta}$, splitting the modulus and phase parts of (4.39) and looking for stationary solution, $d / d t=0$ with $|B| \neq 0$, the following implicit relation is obtained,

$$
\begin{equation*}
\tilde{d}_{S} \Omega^{2}-\sqrt{\left(2 \lambda-v_{D C}|B|^{2}\right)|B| /\left[\chi_{D C}|B| \pm\left(\zeta_{D C} \lambda+\mu_{D C}\right)\right]}=0 \tag{4.40}
\end{equation*}
$$

where only the real solutions corresponding to $f=\tilde{d}_{s} \Omega^{2}>0$ are retained, as the combinations $\tilde{d}_{s} \Omega^{2}<0$ are not physically meaningful.
Although two more terms appear in (4.39) and the dependence on the forcing amplitude is different with respect to the SC case, i.e. $f^{2}$ instead of $f$, thus leading to the square root in (4.40), amplitude equation (4.39) is reminiscent of that given in (4.23). Indeed, equation (4.39) contains essentially three contributions,

$$
\begin{equation*}
\lambda \leftrightarrow\left(2 \lambda-\chi_{D C} f^{2}\right), \quad \mu_{S C} f \leftrightarrow\left(\zeta_{D C} \lambda+\mu_{D C}\right) f^{2}, \quad v_{S C} \leftrightarrow v_{D C}, \tag{4.41}
\end{equation*}
$$

in order, a detuning term (forcing amplitude-dependent), an additive (quadratic) forcing term (forcing frequency dependent) and the classic cubic restoring term, respectively. Hence, the same qualitative hardening or softening nonlinear behaviours as well as hysteresis, typical features of the Duffing-equation, are expected under the hypotheses of the present analysis.

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The total flow solution predicted by the WNL for DC waves and reconstructed as

$$
\begin{equation*}
\mathbf{q}_{D C}=\{\Phi, \eta\}^{T}=\mathbf{q}_{1}+\mathbf{q}_{2}, \tag{4.42}
\end{equation*}
$$

is compared in figures 4.6 and 4.7 with experiments from Reclari (2013) and Reclari et al. (2014) (see also figure 4.2).

## Experiments vs weakly nonlinear prediction: wave amplitude

In figures 4.6 and 4.7, the weakly nonlinear (WNL) prediction of double-crest (DC) waves according to (4.42) (light blue solid and dashed lines) is quantitatively compared with the experimental measurements from Reclari (2013) and Reclari et al. (2014) in terms of maximum non-dimensional crest-to-trough contact line amplitude, $\Delta \tilde{\delta}$, for different values of the shaking diameters, $\tilde{d}_{s}$ corresponding to those of figure 4.2 in the frequency window close to $\omega_{21} / 2$.
The improvement gained through the formal WNL analysis, when compared with the linear and straightforward asymptotic models, is striking. The amplitude equation model correctly predicts the surge of the double-crest swirling and the resulting finite amplitude saturation via hardening nonlinear mechanism, thus remarkably narrowing the gap with experiments for all the values of $\tilde{d}_{s}$ considered and for different container configurations.
Notwithstanding such an improvement, figure 4.6 highlights the main limitation of the present amplitude equation model for DC waves. Indeed, one notices that, while at larger shaking diameters, i.e. $\tilde{d}_{s}=0.13$ and 0.20 , a DC wave first emerges on the top of a single-crest (SC) dynamics and eventually a double-crest wave breaking occurs at larger frequencies, a jump-down transition from $D C$ to SC takes place by increasing $\Omega$ at lower shaking diameters, i.e. $\tilde{d}_{s}=0.07$ and 0.10 for $D=0.144 \mathrm{~m}$. This well-known hysteretic behaviour can be reasonably ascribed to the viscous dissipation of the system. For instance, at sufficiently small shaking diameters, e.g. $\tilde{d}_{s} \approx 0.02$ (see figure 4.2 ), the DC dynamics does not manifest at all, as the energy pumped into the system by the external forcing is likely not sufficient to dominate over the system viscous dissipation, whose effect also depends on the container diameter, $D$. Indeed, figure 4.7 clearly shows that larger diameters, i.e. $D=0.287 \mathrm{~m}$, generate less dissipation. It follows that for larger $D$, by increasing the driving frequency at a fixed shaking diameter, e.g. $\tilde{d}_{s}=0.10$, the free surface is more likely to undergo a wave breaking, rather than a jump-down transition (see figures 4.6(b) and 4.7(c)). Obviously, the inviscid model employed here is not capable to predict the so-called jump-down frequency. In Appendix 4.5.1, a heuristic viscous damping model is introduced to tentatively overcome the beforehand mentioned limitations.
Finally, we note that for frequency moderately far from the super-harmonic resonance, the agreement of the WNL model with experiments and with the linear solution, which behaves well away from $\Omega \approx \omega_{11}$, progressively deteriorates. This is particularly evident on the lower stable branch and can be ascribed to the fact that the asymptotic model is essentially formalized for a fixed driving frequency, i.e. $\Omega \approx \omega_{21} / 2$. The second order correction (proportional to $\Lambda F$ and discussed in $\S 4.3 .2$ ) to the leading order particular solution seems to be sufficient to guarantee a fairly good agreement of the upper stable branch in a relatively wide range of
frequency $\Omega<\omega_{21} / 2$. However, for the lower stable branch, i.e. $\Omega>\omega_{21} / 2$, the agreement with non-breaking-wave experiments sufficiently far from resonance is still relatively poor, despite the fact that these measurements essentially follow the linear prediction.


Figure $4.6-(a)-(d)$ Weakly nonlinear (WNL) prediction for double-crest (DC) waves versus experiments by Reclari et al. (2014) (R14) (reported in figure 4.2) in terms $\Delta \tilde{\delta}$ for a container diameter $D=0.144 \mathrm{~m}, \tilde{H}=0.52$ and for various $\tilde{d}_{s}$. Dotted black lines: linear model (4.12). Red lines: straightforward asymptotic (4.17). Light blue solid and dashed lines: stable and unstable branches, respectively, predicted by the WNL solution (4.42). The normal form coefficient values for this configurations are $\chi_{D C}=2.755, \zeta_{D C}=0.150, \mu_{D C}=0.129$ and $v_{D C}=10.018$. Violet lines: WNL solution (4.45) including the $\epsilon^{3}$-order correction discussed in $\S 4.3 .2$.

## $\epsilon^{3}$-order correction to the leading order single-crest particular solution

The last consideration in $\$ 4.3 .2$ suggests that the leading order single-crest solution with its second order correction is only accurate in a limited range of frequency and higher order terms should be accounted for in order to better retrieve the linear prediction far from resonance. On this regard, as the present asymptotic expansion is pursued up to the $\epsilon^{3}$-order, we note that the third order forcing contains a term, namely

$$
\begin{equation*}
\mathscr{F}_{3}^{\Lambda \Lambda F}=-\mathrm{i} \Lambda^{2} F \mathscr{B} \hat{\mathbf{q}}_{2}^{\Lambda F} e^{\mathrm{i}\left(\left(\omega_{2 n} / 2\right) t-\theta\right)} e^{\mathrm{i} \Lambda T_{1}}+c . c . \tag{4.43}
\end{equation*}
$$

generated by the derivative of the second order solution with respect to the slow time scale $T_{1}$. This term is not resonating in $e^{\mathrm{i}\left(\omega_{2 n}-2 \theta\right)}$ and therefore it can be gathered together with the other third-order non-resonating terms (N.R.T.) in equation (4.36), which in asymptotic

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Figure 4.7 - Same as figure 4.6 (same color convention) but for a container diameter $D=$ 0.287 m and $\tilde{H}=0.50$ (experiments are reported in figure 4.19 of Reclari (2013) (R13)). Normal form coefficients: $\chi_{D C}=2.709, \zeta_{D C}=0.169, \mu_{D C}=0.134$ and $v_{D C}=9.885$.
models are typically ignored unless one aims to proceed to higher orders. Nevertheless, such a forcing term produces a system response,

$$
\begin{equation*}
\mathbf{q}_{3}^{\Lambda \Lambda F}=\Lambda^{2} F \hat{\mathbf{q}}_{3}^{\Lambda \Lambda F} e^{\mathrm{i}\left(\left(\omega_{2 n} / 2\right) t-\theta\right)} e^{\mathrm{i} \Lambda T_{1}}+c . c . \tag{4.44}
\end{equation*}
$$

which precisely represents the third order frequency correction of the leading order particular solution and acts similarly to the second order correction $\mathbf{q}_{2}^{\Lambda F}=\Lambda F \hat{\mathbf{q}}_{2}^{\Lambda F} e^{\mathrm{i}\left(\left(\omega_{2 n} / 2\right) t-\theta\right)} e^{\mathrm{i} \Lambda T_{1}}+$ c.c. discussed in §4.3.2.
A better intuition about why this third order correction should improve the prediction farther away from the super-harmonic resonance is given in the following. As anticipated in $\S 4.3 .2$, the first order particular solution simply represents the linear system response to the external forcing $\epsilon F \cos (\Omega t-\theta)$. In general, the resulting amplitude is $\alpha \epsilon F /\left(\Omega^{2}-\omega_{1 n}^{2}\right)$, but in our asymptotic analysis, the leading order forcing frequency is frozen to $\omega_{2 n} / 2$, which implies that the first order particular solution has an amplitude fixed to $\epsilon F /\left(\omega_{2 n}^{2} / 4-\omega_{1 n}^{2}\right)$, which does not account for the frequency detuning. If one replaces $\Omega=\omega_{2 n} / 2+\epsilon \Lambda$ in $\epsilon F /\left(\Omega^{2}-\omega_{1 n}^{2}\right)$ and takes its Taylor-expansion, the $\epsilon^{2}$-order term is $\propto \epsilon^{2} F \Lambda$, while the $\epsilon^{3}$-order term is $\propto \epsilon^{3} F \Lambda^{2}$. It naturally follows that accounting for this third order correction (4.44) leads to a more accurate description of the linear response to the external forcing and, therefore, it should give a better model prediction farther away from resonance, where the double-crest amplitude $|B| \approx 0$ and the non-breaking-wave experiments follow the linear theory.

To conclude, taking the total solution as

$$
\begin{equation*}
\mathbf{q}_{D C}=\{\Phi, \eta\}^{T}=\mathbf{q}_{1}+\mathbf{q}_{2}+\mathbf{q}_{3}^{\Lambda \Lambda F} \tag{4.45}
\end{equation*}
$$

is expected to leave the amplitude saturation prediction for the double-crest wave, $|B|$, unaltered, as it does not contribute to the amplitude equation solution, and, simultaneously, to better describe the single-crest swirling farther away from $\Omega \approx \omega_{2 n} / 2$ (at least where no breaking waves occur).
The influence of the aforementioned $\epsilon^{3}$-order corrections to the prediction for DC waves is shown as violet lines in figure 4.6 and 4.7 , where it can be seen that accounting for the additional term allows one to eventually close the gap with the experiments even farther away from resonance, hence ensuring to the WNL model a wider frequency range of validity in all cases examined.

## Experiments vs weakly nonlinear prediction: free surface reconstruction

In figure 4.8, the weakly nonlinear (WNL) model prediction for the double-crest waves (DC) is compared versus the straightforward asymptotic prediction discussed in $\$ 4.2$ and the experimental measurements for DC waves from Reclari (2013) and Reclari et al. (2014). The direct quantitative comparison is here outlined in terms of dimensionless and phase-averaged wave height measured at the sidewall, $\tilde{\delta}(\theta)$.

We observe that, if at $\Omega / \omega_{21}=0.490$ both models match satisfactorily the experimental points, as soon as $\Omega / \omega_{21}=0.5$ is approached, the straightforward asymptotic solution diverges due to the resonant (second order) super-harmonic term, while the WNL solution predicts correctly the finite amplitude saturation and the emergence of a DC wave on the top of a singlecrest (SC) one. The WNL model for DC waves remains in fairly good agreement even at larger driving frequency, although the increasing phase-asymmetry between the two local peaks at $\theta=\pi / 2$ and $3 \pi / 2$ is not retrieved by the present inviscid asymptotic analysis, where secondary effects, e.g. the phase shift induced by viscous dissipation and influence of other higher modes, as well as stronger nonlinear effects for increasing wave amplitudes are overlooked.
For completeness, the three-dimensional free surface, $\eta(r, \theta, \pi / \Omega)$, is reconstructed through (4.42) and shown in the right-panels of figure 4.8, where, for increasing shaking frequencies, the nonlinear transition from a nearly single-crest to a double-crest swirling is enlightened.

## The Helmholtz-Duffing oscillator analogy

While the Duffing equation is known to capture period-3 and period-1/3 dynamics arising from the cubic nonlinearity (Jordan and Smith, 1999; Kalmár-Nagy and Balachandran, 2011), as those observed by Bäuerlein and Avila (2021) and occasionally by Reclari et al. (2014), it cannot predict the period-halving (the system responds at a frequency which is twice that of the external forcing) dynamics associated with the super-harmonic resonance investigated in this work. Therefore, in connection with $\S 4.3$.1, here we aim to identify the simplest possible

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Figure 4.8 - Left-panels: comparison of the dimensionless and phase-averaged wave height measured at the wall, $\tilde{\delta}(\theta, \pi / \Omega)$, (black circles) with the straightforward asymptotic solution rebuilt via (3.14) (grey solid line) and the weakly nonlinear (WNL) solution for the double-crest (DC) wave (4.25). Panels correspond to $\tilde{H}=0.52, \tilde{d}_{s}=0.11$ and $D=0.144 \mathrm{~m}$. The experimental measurements, here shown as black circles, are available in Reclari (2013), except for panel (c), which is provided in Reclari et al. (2014). Note that (b) the nonlinear prediction has a very large amplitude. Right-panels: corresponding three-dimensional free surface deformation, $\eta(r, \theta, \pi / \Omega)$, reconstruct via (4.25). The transition from a single-crest to a double-crest swirling via hardening nonlinearity is visible moving from top to bottom, i.e. for increasing frequency.
mechanical oscillator that could mimic, at least from a qualitative perspective, the period-1/2 dynamics studied in this work.

The weakly nonlinear analysis (WNL) as well as the straightforward asymptotic model highlighted the crucial role of quadratic nonlinearities emerging at second order and from which the double-crest (DC) dynamics stems. At the same time, the WNL model enlightened that second-order terms only are not sufficient to capture all the dynamics features owing to the lack of restoring terms and, therefore, cubic nonlinearities must be retained. These considerations suggest that the DC dynamics could be tentatively described by a driven oscillator with both quadratic (asymmetric) and cubic (symmetric) nonlinear terms, i.e.

$$
\begin{equation*}
\ddot{x}+2 \sigma \dot{x}+x+c_{2} x^{2}+c_{3} x^{3}=p \cos \Omega t . \tag{4.46}
\end{equation*}
$$

Equation (4.46), also commonly known as Helmholtz-Duffing equation, has wide applications in engineering problems as those related to beams, plates and shells subjected to an initial static curvature (Askari et al., 2011; Mirzabeigy et al., 2014), whose governing equations are reconduced to a second-order nonlinear ordinary equation with quadratic and cubic nonlinear terms (Alijani et al., 2011; Fallah and Aghdam, 2011; Ke et al., 2010).
Among the diverse asymptotic solutions of (4.46) in different limits (Benedettini and Rega, 1989; Kovacic and Brennan, 2011; Rega, 1995), the most relevant to our work is that of Benedettini and Rega (1989). Within the context of planar nonlinear response of suspended elastic cables to external excitation, they derived an amplitude equation which concerns with the first or fundamental super-harmonic excitation, i.e. $\Omega \approx 1 / 2$, of (4.46). Their weakly nonlinear approach is detailed in Appendix 4.5.3, with the additional assumption of vanishing damping $\sigma=0$. Assuming $2 \Omega=1+\lambda=1+\epsilon \hat{\lambda}$, small nonlinearities, $c_{2}=\epsilon \hat{c_{2}}$ and $c_{3}=\epsilon^{2} \hat{c}_{3}$, and introducing two slow time scales, one obtains

$$
\begin{equation*}
d D / d t=-\mathrm{i}\left(2 \lambda+c_{5} f^{2}\right) D+\mathrm{i}(1-\lambda) c_{2} f^{2} / 2-\mathrm{i} 4 c_{4}|D|^{2} D \tag{4.47}
\end{equation*}
$$

with $C=D e^{\mathrm{i} 2 \lambda t}$ and with the auxiliary coefficients $c_{4}$ and $c_{5}$ (both functions of $c_{2}$ and $c_{3}$ ) defined in Benedettini and Rega (1989). By comparing term by term, the analogy with equation (4.39) is evident.
To conclude, although the DC dynamics examined in this study is intrinsically related to the simultaneous interplay of multiple waves, thus making particularly challenging an accurate $v i s$ - $\grave{a}$-vis quantitative comparison with a single-degree-of-freedom mechanical model, equation (4.47) seems to suggest that the actual inviscid sloshing dynamics in the DC regime may be, at least qualitatively, described by the undamped Helmholtz-Duffing equation (4.46) driven super-harmonically.

### 4.4 Conclusion

With regards to orbital shaken cylindrical containers and, specifically, to the careful experimental campaign reported in Reclari (2013) and Reclari et al. (2014), a weakly nonlinear

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analysis (WNL) via multiple timescale method was formalized in $\S 4.3$ in order to investigate diverse features of the steady state free surface dynamics and, particularly, the double-crest (DC) wave dynamics pertaining at half the frequency of the first $m=2$ natural mode.

After having discussed the substantial limitations of the straightforward expansion procedure proposed by Reclari et al. (2014) and summarized in $\S 4.2$, the WNL analysis was first formulated under the most common condition of pure harmonic resonance. Despite the inviscid assumption, the WNL analysis developed for the single-crest (SC) wave dynamics was shown to be in fairly good agreement with all the experimental measurements. In fact, the present model correctly describes the close-to-resonance hardening nonlinear behaviour experimentally observed. The agreement remains sufficiently accurate until the free surface eventually breaks and a transition to a fully nonlinear regime occurs.
It is well-assessed in the literature that the close-to-harmonic-resonance sloshing dynamics can be modelled (from both qualitative and quantitative perspectives (Bäuerlein and Avila, 2021)) by a single degree of freedom (1dof) system with a cubic nonlinearity and driven harmonically, i.e. by the famous Duffing oscillator, as rigorously proved for a two-dimensional rectangular container laterally excited (Ockendon and Ockendon, 1973). Without surprise, this was shown to hold for the case of orbital-shaken (circular) cylindrical containers as well.
The WNL analysis was then extended to the more complex case of a double-crest swirling. The overall agreement with experiments and, especially, the improvements with respect to the simple straightforward asymptotic model are remarkable in all cases considered, although the slight asymmetry observed in the reconstruction of the periodic free surface dynamics at the sidewall was not retrieved in the present model.
To the knowledge of the authors, a formal amplitude equation describing the super-harmonic DC sloshing dynamics in orbitally shaken containers and coupled with a thorough experimental validation, has not been reported in the literature yet, hence representing the most significant finding of this work.
Lastly, by analogy with the close-to-harmonic-resonance dynamics for SC waves, for which the Duffing oscillator represents the suitable mechanical analogy, a one-degree-of-freedom (1dof) mechanical oscillator having both quadratic and cubic nonlinear terms, commonly referred to as Helmholtz-Duffing (HD) oscillator, driven super-harmonically, was tentatively identified as the simplest possible mechanical system that could mimic, at least qualitatively, the super-harmonic DC sloshing dynamics investigated in this study. The HD equation was largely adopted in the last few decades within the context of structural analysis, i.e. beams, plates and shells subjected to an initial static curvature as well as suspended elastic cables (Benedettini and Rega, 1989; Nayfeh, 1984), and it was here proposed as a direct mechanical analogy with the present orbital sloshing system.

The main limitation of the models derived in this work is intrinsic to the fundamental assumption of an inviscid fluid. This precludes one to correctly accounting for the jump-down transition experimentally observed for DC waves at low shaking amplitudes and, therefore, for an accurate estimation of the maximum amplitude response when such a transition occurs. Furthermore, in the absence of viscous boundary layers, the weakly nonlinear time- and azimuthal-averaged mean flow reduces to a free surface deformation only. This is in stark
contrast with the existence of the so-called Eulerian mean flow (Bremer and Breivik, 2018), also known as viscous streaming flow, typically observed in experiments (Bouvard et al., 2017). Therefore, the present work overlooks one of the essential points of interest in applications of orbital shaking. The mean flow, which contributes to efficient mixing, is not captured.
The extension of the asymptotic models developed in this work to a viscous analysis is desirable, as it would enable one to predict quantitatively these secondary but fundamental effects for both cases of harmonic and super-harmonic resonances. However, it presently hinges on the subtle modelling of the moving contact line condition.

### 4.5 Appendix

### 4.5.1 Heuristic damping model: jump-down frequency and DC dynamics suppression at low driving amplitudes

In §4.3.2 the weakly nonlinear (WNL) model for double-crest (DC) waves was compared with experimental measurements from Reclari (2013) and Reclari et al. (2014) in terms nondimensional maximum crest-to-trough contact line amplitude, $\Delta \tilde{\delta}$, for different non-dimensional shaking diameters, $\tilde{d}_{s}$, and container diameters, $D$ (see figures 4.6 and 4.7). We have observed that at larger shaking amplitudes, $\tilde{d}_{s}$, and for larger container diameter, $D$, a DC wave first emerges on the top of a single-crest (SC) wave at $\Omega \approx \omega_{21} / 2$ and eventually wave breaking occurs at larger frequencies. On the contrary, a jump-down transition from DC to SC then takes place by increasing $\Omega$ at lower values of $\tilde{d}_{s}$ and/or for smaller $D$. The latter well-known hysteretic behaviour can be ascribed to the viscous dissipation of the system, obviously overlooked by the present inviscid analysis. In this Appendix, viscous dissipation is tentatively reintroduced by employing a simple heuristic viscous damping model, as described in the following.
The viscous dissipation essentially arises at three locations, (i) at the solid tank boundary layers, i.e. bottom and sidewall, (ii) in the fluid bulk and (iii) at the free surface, the latter being typically negligible for ideal surface waves (in absence of any form of contamination). A well-known formula for the prediction of the viscous damping coefficient of capillary-gravity waves in upright cylindrical containers was provided by Case and Parkinson (1957) and Miles (1967). Such an estimation is computed according to the following formula

$$
\begin{equation*}
\sigma=\frac{2 k_{m n}^{2}}{R e}+\left(\frac{\omega_{m n}}{2 R e}\right)^{1 / 2} \frac{k_{m n}}{\sinh \left(2 k_{m n} H\right)}+\left(\frac{\omega_{m n}}{2 R e}\right)^{1 / 2}\left[\frac{1}{2} \frac{1+\left(m / k_{m n}\right)}{1-\left(m / k_{m n}\right)}-\frac{k_{m n} H}{\sinh \left(2 k_{m n} H\right)}\right], \tag{4.48}
\end{equation*}
$$

where the first term represents the bulk dissipation, whereas the second and third terms are related to the dissipation occurring at the solid bottom and sidewall, respectively. In equation (4.48), $H=h / R$ is the non-dimensional fluid depth, $k_{m n}$ is the non-dimensional wavenumber associated with mode $(m, n), \omega_{m n}$ is the corresponding natural frequency obeying to the dispersion relation (4.9) and $R e=g^{1 / 2} R^{3 / 2} / v$ is the Reynolds number ( $v$ denotes the kinematic viscosity of the fluid). In $\S 4.3 .2$ an amplitude equation, governing the dynamics of

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Figure 4.9 - (a) Same case of figure 4.6 -(a) with $\tilde{d}_{s}=0.07$. (b) same as (a), but for $\tilde{d}_{s}=0.02$ (from figure 4.2), value at which the double-crest dynamics does not manifest. Solid and dashed lines correspond to stable and unstable branches, respectively, computed via the weakly nonlinear analysis in the inviscid case and for different values of damping coefficient, with $\sigma$ given by (4.48). Markers correspond to the experimental points shown in figure 4.2 and are extracted from Reclari et al. (2014).
a natural mode ( $2, n$ ) (which leads the DC wave dynamics observed close to $\Omega \approx \omega_{21} / 2$ ), was derived. For mode $(2,1)$ in the conditions of figure 4.6, i.e. pure water with $\rho=1000 \mathrm{~kg} / \mathrm{m}^{3}$, $\gamma=0.072 \mathrm{~N} / \mathrm{m}, v=1 \times 10^{-6} \mathrm{~m}^{2} / \mathrm{s}, D=0.144 \mathrm{~m}$ (for which the Bond number is $B o=705.6$ ) and $H=1.04=2 \tilde{H}$, the values $R e=60480, k_{21}=3.0542$ and $\omega_{21}=1.7561$ give a non-dimensional viscous damping coefficient $\sigma=0.0051$, mostly produced by the sidewall boundary layer. Typically, as in the present case and as supported by experimental (Cocciaro et al., 1993) and numerical (Viola et al., 2018) evidence, the viscous damping rate can be interpreted as a slow damping process over a faster time scale represented by the wave oscillation. Under this hypothesis, which translates in the assumption of a viscous damping coefficient of order $\epsilon^{2}$ within the present WNL framework, the damping coefficient can be added a posteriori, i.e. in a phenomenological way, to the final inviscid amplitude equation from (4.39), leading to

$$
\begin{equation*}
\frac{d B}{d t}=-\left[\sigma+\mathrm{i}\left(2 \lambda-\chi_{D C} f^{2}\right)\right] B+\mathrm{i}\left(\zeta_{D C} \lambda+\mu_{D C}\right) f^{2}+\mathrm{i} v_{D C}|B|^{2} B . \tag{4.49}
\end{equation*}
$$

The stationary form of (4.49) can be rearranged in the following implicit form

$$
\begin{equation*}
\left(2 \lambda-v_{D C}|B|^{2}-\chi_{D C} f^{2}\right)|B| \pm \sqrt{f^{4}\left(\zeta_{D C} \lambda+\mu_{D C}\right)^{2}-(\sigma|B|)^{2}}=0 \tag{4.50}
\end{equation*}
$$

which can be solved using the Matlab function fimplicit. The effect of viscous dissipation on the DC regime is investigated in figure 4.9 for two representative values of the shaking diameter.
The case of figure 4.9 (a) shows that the so-called jump-down frequency is somewhere in
between $\Omega \in[0.675,0.685]$. The damping value produced by (4.48) appears to be too small to match the experimental jump-down frequency, hence we tentatively added a prefactor in order to fit the measurements. It turns out that a prefactor of 1.35 is sufficient to provide a fairly good prediction of the jump-down frequency. We note that prediction (4.48) does not involve any dissipation mechanism associated with the contact line, i.e. contact line hysteresis (Bongarzone et al., 2021c; Cocciaro et al., 1993; Dussan, 1979; Hocking, 1987; Keulegan, 1959; Kidambi, 2009a; Miles, 1967; Viola et al., 2018; Viola and Gallaire, 2018) or possible surface contamination (Henderson and Miles, 1990, 1994). Indeed, depending on the configuration, contact line dynamics may rule the overall dissipation, with a measured damping coefficient up to 10-20 times larger (Benjamin and Ursell, 1954; Hocking, 1987; Kidambi, 2009a) than that predicted by (4.48). Comparison of the theoretical damping coefficient value with that measured in moving contact line experiments, due to unavoidable sources of uncertainty in the meniscus dynamics, have always been mostly qualitative, rather than quantitative, requiring often the use of fitting parameters. For instance, in their predictive theory for single-mode Faraday experiments, Henderson and Miles (1990) used an effective fluid viscosity 3 times larger than the actual one. Recently, Bäuerlein and Avila (2021) have measured the damping coefficient of the first anti-symmetric sloshing mode in a quasi-two-dimensional rectangular container, which was seen to be approximately 1.5 larger than that predicted by the theory (Faltinsen and Timokha, 2009).
Even in absence of strong contact line dissipation, free surface contamination may strongly contribute to the overall damping coefficient. Henderson and Miles (1990) have considered a fully contaminated free surface, which can be modelled by a surface film that is free to move vertically but cannot stretch horizontally. They have also derived an analytical expression for the associated damping, which reads $\left(\omega_{m n} / 2 R e\right)^{1 / 2} k_{m n} \cosh ^{2} k_{m n} H / \sinh 2 k_{m n} H$. If such a contribution was accounted for in (4.48), it would produce a value of 0.0109 , which is approximately twice the damping obtained without contamination (and would correspond to the green lines in figure $4.9(a)$ for $2 \sigma$ ). The need for a prefactor of 1.35 in figure $4.9(a)$, which approximately corresponds to a fictitious fluid with a dynamic viscosity 1.8 times larger, is therefore not surprising when the damping is computed via (4.48) and contact line dissipation as well as free surface contamination are neglected.
We remark that the reasonings outlined in this Appendix in order to elucidate the effect of viscosity are in fact only qualitative. Many aspects are ignored in the present inviscid analysis with phenomenological damping, two of which are commented in the following.

Prediction (4.48) is only valid for free capillary-gravity waves, whereas dissipation rates of forced wave motions are generally more complex. In particular, the double-crest wave evolves out of a non-resonant forced single-crest swirling motion. A proper viscous WNL analysis would produce complex eigenmodes and responses (due to the phase shift owing to viscosity) and hence complex-valued normal form coefficients. For instance, among these coefficients, the imaginary part of $v_{D C}$ (or $v_{S C}$ ) multiplied by $|B|^{2}$ in (4.49), could be interpreted as a sort of nonlinear damping (Douady, 1990), $\left(\sigma-\operatorname{Im}[v]|B|^{2}\right)$, whose contribution to the overall dissipation mechanisms is expected to increase at larger wave amplitudes, hence influencing the location of the jump-down frequency (note that $-\operatorname{Im}[v]$ can be $>0$ ). Moreover, the imaginary

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part of $\chi f^{2}$ in DC-waves would contribute to the overall damping, $\left(\sigma+\operatorname{Im}[\chi] f^{2}-\operatorname{Im}[v]|B|^{2}\right)$, by introducing a further effect, proportional to $f^{2}$, that finds its origin in the fact that the double-crest wave emerges at second order owing to nonlinear square terms in the first order non-resonant SC-motion.
In contradistinction with the case of a pinned (or fixed) contact line, a formal viscous analysis undertaking the case of a moving contact line would require the introduction of a slip length model in order to regularize the well-known contact line stress-singularity (Davis, 1974; Huh and Scriven, 1971; Lauga et al., 2007; Navier, 1823; Viola and Gallaire, 2018).
Most importantly, the inviscid WNL model is not capable to describe the continuous modulation of the phase lag between the external forcing and the wave amplitude response, which has been recently demonstrated by Bäuerlein and Avila (2021) (for uni-directional sloshing waves in three-dimensional rectangular container) to be of crucial importance in the correct prediction of the jump-down frequency, otherwise often inaccurate, even when the considered damping coefficient is that measured experimentally. In principle, a formal viscous analysis, as briefly introduced above, is expected to correctly capture such a phase lag.
Another interesting case, that is worth to be commented, is that shown in figure 4.9(b). At a shaking diameter $\tilde{d}_{s}=0.02$ (the lowest reported in figure 4.2), the DC dynamics was not observed at all. This is in conflict with the inviscid straightforward asymptotic analysis, which always prescribes a divergent behaviour close to the dominant super-harmonic, $\Omega \approx \omega_{21} / 2$, even for vanishing $\tilde{d}_{s}$. However, as soon as viscous dissipation is introduced, the energy pumped into the system is not sufficient to overcome dissipative effects and DC waves are essentially suppressed, with a system response that follows satisfactorily the linear solution (see figure 4.2) showing a single-crest dynamics ranging over the whole frequency window, $\Omega / \omega_{11} \in[0,1]$, in agreement with experimental evidences.

### 4.5.2 Asymptotic harmonic solution of the undamped Duffing equation

By analogy with the weakly nonlinear analysis for harmonic single-crest wave dynamics presented in $\S 4.3 .1$, we look for an asymptotic solution of the undamped Duffing equation

$$
\begin{equation*}
\ddot{x}+x+c_{3} x^{3}=p \cos \Omega t, \tag{4.51}
\end{equation*}
$$

having the form $x=x_{0}+\epsilon x_{1}$. Additionally, as standard in asymptotic solutions of the Duffing equation, we assume a small external forcing amplitude, $p=\epsilon \hat{p}$ and detuning from the exact resonance, i.e. $\Omega=1+\lambda=1+\epsilon \hat{\lambda}$, small nonlinearities through $c_{3}=\epsilon \hat{c}_{3}$ and the existence of a characteristic slow time scale $\hat{t}_{1}=\epsilon t$. Under these assumptions, the $\epsilon^{0}$-order homogeneous solutions simply reads

$$
\begin{equation*}
x_{0}=C\left(\hat{t}_{1}\right) e^{\mathrm{i} t}+c . c . . \tag{4.52}
\end{equation*}
$$

with $C\left(\hat{t}_{1}\right)$ to be determined at next order. At order $\epsilon$ one can readily verify that, in order to avoid secular terms, a solvability condition must be satisfied. Such a condition leads to the
very classical amplitude equation

$$
\begin{equation*}
d D / d t=-\mathrm{i} \lambda D+\mathrm{i}(-1 / 4) p+\mathrm{i}\left(3 c_{3} / 2\right)|D|^{2} D \tag{4.53}
\end{equation*}
$$

where the change of variable $C=D e^{\mathrm{i} \lambda t}$ was introduced and each quantity was recast in terms of the corresponding physical value (to eliminate the implicit small parameter $\epsilon$ ).

By noticing that

$$
\begin{equation*}
-1 / 4 \leftrightarrow \mu_{S C}, \quad 3 c_{3} / 2 \leftrightarrow v_{S C}, \tag{4.54}
\end{equation*}
$$

one immediately recognizes that equation (4.23) has indeed the same structure of the formal amplitude equation (4.53), thus suggesting that the continuous sloshing system and the one-degree-of-freedom (1dof) Duffing system, under the specific conditions listed above, behave essentially in the same way.

### 4.5.3 Asymptotic super-harmonic solution of the undamped Helmholtz-Duffing equation

In this Appendix, although with the additional assumption of vanishing damping, we briefly summarize the super-harmonic weakly nonlinear solution of the Helmholtz-Duffing equation,

$$
\begin{equation*}
\ddot{x}+x+c_{2} x^{2}+c_{3} x^{3}=p \cos \Omega t \tag{4.55}
\end{equation*}
$$

derived by Benedettini and Rega (1989) and introduced in §4.3.1.
We look for an asymptotic solution of the form $x=x_{0}+\epsilon x_{1}+\epsilon^{2} x_{2}$, to equation (4.55) with $\sigma=0$ (undamped oscillator), $2 \Omega=1+\lambda=1+\epsilon \hat{\lambda}$ and with small nonlinearities through $c_{2}=\epsilon \hat{c}_{2}$ and $c_{3}=\epsilon^{2} \hat{c}_{3}$ (with the cubic term one order smaller than the quadratic one). The existence of two slow time scales is hypothesized, $\hat{t}_{1}=\epsilon t$ and $\hat{t}_{2}=\epsilon^{2} t$. Under these assumptions, the solution of the $\epsilon^{0}$-order forced linear problem reads

$$
\begin{equation*}
x_{1}=C\left(\hat{t}_{1}, \hat{t}_{2}\right) e^{\mathrm{i} t}+f e^{\mathrm{i}(1 / 2) t} e^{\mathrm{i}(\hat{\lambda} / 2) \hat{t}_{1}}+c . c . \tag{4.56}
\end{equation*}
$$

with $f=(2 / 3) p$ and $C\left(\hat{t}_{1}, \hat{t}_{2}\right)$ to be determined at next order. At orders $\epsilon$ and $\epsilon^{2}$, resonating terms produced by the weak quadratic and cubic nonlinearities, respectively, arise, thus requiring the imposition of two solvability conditions prescribing that amplitude $C(\hat{t})$ must obey to the following normal forms

$$
\begin{gather*}
\boxed{\epsilon^{1}}: \quad d C / d \hat{t}_{1}=\mathrm{i}\left(c_{2} / 2\right) f^{2} e^{\mathrm{i} \hat{\lambda} \hat{t}_{1}},  \tag{4.57a}\\
\epsilon^{2}: \quad d C / d \hat{t}_{2}=-\mathrm{i} \hat{\lambda}\left(c_{2} / 4\right) e^{\mathrm{i} \hat{\lambda} \hat{t}_{1}}-\mathrm{i} c_{5} f^{2} A-\mathrm{i} 4 c_{4}|C|^{2} C \tag{4.57b}
\end{gather*}
$$

with the full expression of the auxiliary coefficients $c_{4}$ and $c_{5}$ (both functions of $c_{2}$ and $c_{3}$ ), given in Benedettini and Rega (1989). Combining (4.57a) and (4.57b) into a single amplitude

## Chapter 4. An amplitude equation modelling the double-crest swirling in orbital shaken cylindrical containers

equation (by summing the two expressions by their respective weights, i.e. $\epsilon$ and $\epsilon^{2}$, and reintroducing the physical quantities in order to eliminate the dependence on the implicit small parameter $\epsilon$ ), one obtains

$$
\begin{equation*}
d D / d t=-\mathrm{i}\left(\lambda+c_{5} f^{2}\right) D+\mathrm{i}(1-\lambda / 2) c_{2} f^{2} / 2-\mathrm{i} 4 c_{4}|D|^{2} D, \tag{4.58}
\end{equation*}
$$

with $C=D e^{\mathrm{i} \lambda t}$. Note that the procedure used in the perturbation analysis above and outlined in Benedettini and Rega (1989) is in fact equivalent to that followed in Nayfeh (1984) for treating the same second-order super-harmonic resonance in a more general case of a twoterm excitation. By comparing the various terms of (4.58) with those of (4.39), the analogy is evident, thus suggesting that the actual inviscid sloshing dynamics in the double-crest wave regime may be, at least qualitatively, described by the undamped Helmholtz-Duffing equation (4.46) driven super-harmonically.

### 4.5.4 $\epsilon^{2}$ and $\epsilon^{3}$-order dynamic and kinematic equations

For completeness, in this Appendix we provide the second and third-order asymptotic expansions of the free surface boundary conditions with regards to the most complex formulation presented in this Chapter, i.e. that related to the double-crest (DC) swirling. At second-order, the dynamic and kinematic equations evaluated at $z=\eta_{0}=0$ read, respectively

$$
\begin{gather*}
\frac{\partial \Phi_{2}}{\partial t}+\eta_{2}-\frac{1}{B o} \frac{\partial \kappa}{\partial \eta}\left(\eta_{2}\right)=-\frac{\partial \Phi_{1}}{\partial T_{1}}-\eta_{1} \frac{\partial}{\partial z} \frac{\partial \Phi_{1}}{\partial t}-\frac{1}{2} \nabla \Phi_{1} \cdot \nabla \Phi_{1}+\frac{1}{B o} \frac{1}{2} \frac{\partial^{2} \kappa}{\partial \eta^{2}}\left(\eta_{1}\right)^{2}=\mathscr{F}_{2 \mathrm{dyn}}  \tag{4.59}\\
\frac{\partial \eta_{2}}{\partial t}-\frac{\partial \Phi_{2}}{\partial z}=-\frac{\partial \eta_{1}}{\partial T_{1}}-\nabla \Phi_{1} \cdot \nabla \eta_{1}+\eta_{1} \frac{\partial}{\partial z} \frac{\partial \Phi_{1}}{\partial z}=\mathscr{F}_{2_{\mathrm{kin}}} \tag{4.60}
\end{gather*}
$$

where $\nabla \eta=\left\{\partial \eta / \partial r, r^{-1} \partial \eta / \partial \theta, 0\right\}^{T}, \partial \kappa / \partial \eta$ represents the first order variation of the curvature,

$$
\begin{equation*}
\frac{\partial \kappa}{\partial \eta}=\frac{\partial^{2}}{\partial r^{2}}+\frac{1}{r} \frac{\partial}{\partial r}+\frac{1}{r^{2}} \frac{\partial^{2}}{\partial \theta^{2}} \tag{4.61}
\end{equation*}
$$

and it is applied to $\eta_{2}$, while $\partial^{2} \kappa / \partial \eta^{2}$ is its second order variation applied to $\left(\eta_{1}\right)^{2}$. However, as the system is expanded around $z=\eta_{0}=0$, it turns out that $\partial^{2} \kappa / \partial \eta^{2}=0$.

By pursuing the expansion up to the third order in $\epsilon$, one obtains

$$
\begin{array}{r}
\frac{\partial \Phi_{3}}{\partial t}+\eta_{3}-\frac{1}{B o} \frac{\partial \kappa}{\partial \eta}\left(\eta_{3}\right)=-\frac{\partial \Phi_{1}}{\partial T_{2}}-\frac{\partial \Phi_{2}}{\partial T_{1}}-\eta_{1} \frac{\partial}{\partial z} \frac{\partial \Phi_{1}}{\partial T_{1}}-  \tag{4.62}\\
-\eta_{2} \frac{\partial}{\partial z} \frac{\partial \Phi_{1}}{\partial t}-\eta_{1} \frac{\partial}{\partial z} \frac{\partial \Phi_{2}}{\partial t}-\frac{1}{2}\left(\eta_{1}\right)^{2} \frac{\partial^{2}}{\partial z^{2}} \frac{\partial \Phi_{1}}{\partial t}-\nabla \Phi_{1} \cdot \nabla \Phi_{2}-\eta_{1} \nabla \Phi_{1} \cdot \frac{\partial}{\partial z} \nabla \Phi_{1}+ \\
+\frac{1}{B o} \frac{1}{2} \frac{\partial^{2} \kappa}{\partial \eta^{2}}\left(2 \eta_{1} \eta_{2}\right)+\frac{1}{B o} \frac{1}{6} \frac{\partial^{3} \kappa}{\partial \eta^{3}}\left(\eta_{1}\right)^{3}=\mathscr{F}_{3_{\mathrm{dyn}}}
\end{array}
$$

### 4.5. Appendix

$$
\begin{array}{r}
\frac{\partial \eta_{3}}{\partial t}-\frac{\partial \Phi_{3}}{\partial z}=-\frac{\partial \eta_{1}}{\partial T_{2}}-\frac{\partial \eta_{2}}{\partial T_{1}}-\nabla \Phi_{1} \cdot \nabla \eta_{2}-\nabla \Phi_{2} \cdot \nabla \eta_{1}-\eta_{1} \nabla \eta_{1} \cdot \frac{\partial}{\partial z} \nabla \Phi_{1}+  \tag{4.63}\\
+\eta_{2} \frac{\partial}{\partial z} \frac{\partial \Phi_{1}}{\partial z}+\eta_{1} \frac{\partial}{\partial z} \frac{\partial \Phi_{2}}{\partial z}+\frac{1}{2}\left(\eta_{1}\right)^{2} \frac{\partial^{2}}{\partial z^{2}} \frac{\partial \Phi_{1}}{\partial z}=\mathscr{F}_{3_{\mathrm{kin}}}
\end{array}
$$

with $\partial^{3} \kappa / \partial \eta^{3}(\neq 0)$ the third order variation of the curvature, whose explicit expression is here omitted for the sake of brevity, applied to $\left(\eta_{1}\right)^{3}$.
Note that the second order in the straightforward asymptotic model is retrieved by retaining the $\epsilon^{2}$-order system only and imposing $\partial / \partial T_{1}=\partial / \partial T_{2}=0$. On the other hand, the equations above reduce to the second and third-order problem in the single-crest (SC) swirling formulation when $\partial / \partial T_{1}=0$ and the external forcing term, $r F \cos (\Omega t-\theta)$, appears on the right-hand-side of equation (4.62). At each order in $\epsilon$, by substituting the previous order solutions, it is possible to explicitly separate the various forcing contributions by their temporal and azimuthal periodicities, thus leading to expressions (4.14) for the straightforward model, (4.22) for the single-crest model and (4.34), (4.36) for the double-crest one.
For the calculation of the amplitude equation coefficients at $\epsilon^{3}$-order, only resonant terms matter. These terms, with their corresponding amplitudes, are proportional to $e^{\mathrm{i}\left(\omega_{1 n} t-\theta\right)}$ for SC waves and to $e^{\mathrm{i}\left(\mathrm{i} \omega_{2 n} t-2 \theta\right)}$ for DC waves (or, more generally, proportional to $e^{\mathrm{i}\left(\omega_{m n} t-m \theta\right)}$ ). As an example, in the following, we provide the expression of $\hat{\mathscr{F}}_{3, \text { kin }}^{A \bar{A} A}$ used in (4.24b) (with $A=A_{1}$ ) and in (4.38c) (with $A=A_{2}$ ) and which, together with $\hat{\mathscr{F}} \hat{F}_{3, \text { dyn }}^{A \bar{A} A}$, represents the most involved third order resonant forcing term:

$$
\begin{align*}
& \hat{\mathscr{F}}_{3, \mathrm{kin}}^{A \bar{A} A}=-\nabla \hat{\Phi}_{1}^{A} \cdot \nabla \hat{\eta}_{2}^{A \bar{A}}-\nabla \overline{\hat{\Phi}_{1}^{A}} \cdot \nabla \hat{\eta}_{2}^{A A}-\nabla \hat{\Phi}_{2}^{A A} \cdot \nabla \overline{\hat{\eta}_{1}^{A}}-\nabla \hat{\Phi}_{2}^{A \bar{A}} \cdot \nabla \hat{\eta}_{1}^{A}-  \tag{4.64}\\
&-\hat{\eta}_{1}^{A} \nabla \hat{\eta}_{1}^{A} \cdot \frac{\partial}{\partial z} \nabla \overline{\hat{\Phi}_{1}^{A}}-\hat{\eta}_{1}^{A} \nabla \overline{\hat{\eta}_{1}^{A}} \cdot \frac{\partial}{\partial z} \nabla \hat{\Phi}_{1}^{A}-\overline{\hat{\eta}_{1}^{A}} \nabla \hat{\eta}_{1}^{A} \cdot \frac{\partial}{\partial z} \nabla \hat{\Phi}_{1}^{A}+ \\
&+ \hat{\eta}_{2}^{A \bar{A}} \frac{\partial}{\partial z} \frac{\partial \hat{\Phi}_{1}^{A}}{\partial z}+\hat{\eta}_{2}^{A A} \frac{\partial}{\partial z} \frac{\partial \overline{\hat{\Phi}_{1}^{A}}}{\partial z}+\hat{\eta}_{1}^{A} \frac{\partial}{\partial z} \frac{\partial \hat{\Phi}_{2}^{A \bar{A}}}{\partial z}+\hat{\eta}_{1}^{A} \\
& \frac{\partial}{\partial z} \frac{\partial \hat{\Phi}_{2}^{A A}}{\partial z}+ \\
&+\left(\hat{\eta}_{1}^{A} \hat{\eta}_{1}^{A}\right) \frac{1}{2} \frac{\partial^{2}}{\partial z^{2}} \frac{\partial \hat{\Phi}_{1}^{A}}{\partial z}+\left(\hat{\eta}_{1}^{A} \overline{\hat{\eta}_{1}^{A}}\right) \frac{1}{2} \frac{\partial^{2}}{\partial z^{2}} \frac{\partial \hat{\Phi}_{1}^{A}}{\partial z}+\left(\overline{\hat{\eta}}_{1}^{A}\right.\left.\hat{\eta}_{1}^{A}\right) \frac{1}{2} \frac{\partial^{2}}{\partial z^{2}} \frac{\partial \hat{\Phi}_{1}^{A}}{\partial z}
\end{align*}
$$

The expression of $\hat{F}_{3, \text { dyn }}^{A \bar{A} A}$ (not reported here for the sake of brevity) is computed analogously. The extraction of resonant terms, especially those appearing at third-order, involves tedious calculations due to several possible combinations of the previous-order solutions. Nevertheless, the procedure is straightforward and systematic, so that tools of symbolic calculus, e.g. the software Wolfram Mathematica, which was indeed used in this work, can be employed to ease such a procedure.

## 5 Super-harmonically resonant swirling waves in longitudinally forced circular cylinders

Remark: this chapter is largely inspired by the publication of the same name.

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#### Abstract

Author's contributions: A.B. and F.G. created the research plan. A.B. formulated analytical and numerical models. A.M. and A.B. led model solutions. A.M. designed and performed all the experiments with inputs from A.B. and F.G.. A.M. and A.B. wrote the manuscript with inputs from F.G..


Resonant sloshing in circular cylinders was studied by Faltinsen et al. (2016), whose theory was used to describe steady-state resonant waves due to time-harmonic container's elliptic orbits. In the limit of longitudinal container motions, a symmetry-breaking of the planar wave solution occurs, with clockwise and anti-clockwise swirling equally likely. In addition to this primary harmonic dynamics, previous experiments have unveiled that diverse superharmonic dynamics are observable far from primary resonances. Among these, the so-called double-crest (DC) dynamics, first observed by Reclari et al. (2014) for rotary sloshing, is particularly relevant, as its manifestation is the most favoured by the spatial structure of the external driving. Following Bongarzone et al. (2022a), in this work, we develop a weakly nonlinear (WNL) analysis to describe the system response to super-harmonic longitudinal forcing. The resulting system of amplitude equations predicts that a planar wave symmetry-breaking via stable swirling may also occur under super-harmonic excitation. This finding is confirmed by our experimental observations, which identify three possible super-harmonic regimes, i.e. (i) stable planar DC waves, (ii) irregular motion and (iii) stable swirling DC waves, whose corresponding stability boundaries in the forcing frequency-amplitude plane quantitatively match the present theoretical estimates.

## Chapter 5. Super-harmonically resonant swirling waves in longitudinally forced circular cylinders



Figure 5.1 - Sketch of a cylindrical container of diameter $D=2 R$ and filled to a depth $h$. The gravity acceleration is denoted by $g . O^{\prime} \mathbf{e}_{x}^{\prime} \mathbf{e}_{y}^{\prime} \mathbf{e}_{z}^{\prime}$ is the Cartesian inertial reference frame, while $O \mathbf{e}_{x} \mathbf{e}_{y} \mathbf{e}_{z}$ is the Cartesian reference frame moving with the container. The origin of the moving cylindrical reference frame ( $r, \theta, z$ ) is placed at the container revolution axis and, specifically, at the unperturbed liquid height, $z=0$. The perturbed free surface and contact line elevation are denoted by $\eta$ and $\delta$, respectively. $\bar{a}_{x}$ is the amplitudes of the longitudinal periodic forcing of driving angular frequency $\bar{\Omega}$.

The Chapter is organized as follows. The flow configuration and governing equations are given in $\S 5.1$. In $\S 5.2$ we briefly introduce the classical linear potential model together with a short description of the numerical method employed in this work. By analogy with Bongarzone et al. (2022a), in §5.3, we first tackle the simpler case of harmonic single-crest (SC) wave. The WNL system of amplitude equations governing the double-crest (DC) wave dynamics under super-harmonic longitudinal forcing, which represents the core of this study, is then formalized in $\S 5.4$. The experimental apparatus, procedure and findings are described in $\S 5.5$, where a thorough quantitative comparison with the present theoretical estimates is carried out. Final comments and conclusions are outlined in §5.6. Lastly, Appendix 5.7.2 complements the theoretical model by briefly showing how a straightforward extension of the present analysis to generic container's elliptic orbits can be readily obtained without any further calculation, hence paving the way for further analyses and experimental investigations.

### 5.1 Flow configuration and governing equations

We consider a cylindrical container of diameter $D=2 R$ filled to a depth $h$ with a liquid of density $\rho$. The air-liquid surface tension is denoted by $\gamma$, whereas the gravity acceleration is denoted by $g$. $O^{\prime} \mathbf{e}_{x}^{\prime} \mathbf{e}_{y}^{\prime} \mathbf{e}_{z}^{\prime}$ is the Cartesian inertial reference frame, while $O \mathbf{e}_{x} \mathbf{e}_{y} \mathbf{e}_{z}$ is the Cartesian reference frame moving with the container. The origin of the moving cylindrical reference frame $(r, \theta, z)$ is placed at the container revolution axis and, specifically, at the unperturbed liquid height, $z=0$ (see figure 1). A longitudinal shaking in the horizontal plane, e.g. along the
$x$-axis, can be represented by the following equations describing the motion velocity of the container axis intersection with the $z=0$ plane, parametrised in polar coordinates $(r, \theta)$,

$$
\dot{\mathbf{X}}_{0}=\left\{\begin{array}{c}
-\bar{a}_{x} \bar{\Omega} \sin (\bar{\Omega} t) \cos \theta \mathbf{e}_{r}  \tag{5.1}\\
\bar{a}_{x} \bar{\Omega} \sin (\bar{\Omega} t) \sin \theta \mathbf{e}_{\theta}
\end{array}\right.
$$

with $\bar{a}_{x}$ the dimensional forcing amplitude and $\bar{\Omega}$ the dimensional driving angular frequency. In the potential flow limit, the liquid motion within the moving container is governed by the Laplace equation, subjected to the homogeneous no-penetration condition at the solid lateral wall and bottom,

$$
\begin{equation*}
\Delta \Phi=0, \quad \nabla \Phi \cdot \mathbf{n}=\mathbf{0} \tag{5.2}
\end{equation*}
$$

and by the dynamic and kinematic boundary conditions at the free surface $z=\eta(r, \theta)$ (Faltinsen and Timokha, 2009; Ibrahim, 2005),

$$
\begin{gather*}
\frac{\partial \Phi}{\partial t}+\frac{1}{2} \nabla \Phi \cdot \nabla \Phi+\eta-\frac{\kappa(\eta)}{B o}=r f \cos (\Omega t) \cos \theta  \tag{5.3a}\\
\frac{\partial \eta}{\partial t}+\nabla \Phi \cdot \nabla \eta-\frac{\partial \Phi}{\partial z}=0 \tag{5.3b}
\end{gather*}
$$

which have been made non-dimensional by using the container's characteristic length $R$, the velocity $\sqrt{g R}$ and the time scale $\sqrt{R / g}$. In (5.3a), $\kappa(\eta)$ denotes the fully nonlinear curvature, while $B o=\rho g R^{2} / \gamma$ is the Bond number. As soon as the Bond number is sufficiently large, i.e. $B o \sim 10^{3}$ (Bouvard et al., 2017), surface tension effects are almost negligible (fully negligible for $B o \gtrsim 10^{4}$, except in the neighbourhood of the contact line (Faltinsen et al., 2016)). In the following, we assume large Bond numbers and accordingly, the curvature term in (5.3a) is neglected. The non-dimensional driving acceleration along the $x$-axis reads $f=a_{x} \Omega^{2}$, with $a_{x}=\bar{a}_{x} / R$ and $\Omega=\bar{\Omega} / \sqrt{g / R}$. Lastly, the non-dimensional fluid depth is $H=h / R$.

### 5.2 Linear potential model

Far from resonances and in the limit of small forcing amplitudes, the linear theory is expected to provide a good approximation of the harmonic system response. Let us consider small perturbations of a base-state $\mathbf{q}_{0}$,

$$
\begin{equation*}
\mathbf{q}(r, \theta, z, t)=\{\Phi(r, \theta, z, t), \eta(r, \theta, t)\}^{T}=\mathbf{q}_{0}+\epsilon \mathbf{q}^{\prime}=\epsilon\left\{\Phi^{\prime}, \eta^{\prime}\right\}^{T}+\mathrm{O}\left(\epsilon^{2}\right) \tag{5.4}
\end{equation*}
$$

together with the assumption of small driving forcing amplitudes of order $\mathrm{O}(\epsilon)$, i.e. $f=\epsilon F$, with $\epsilon$ a small parameter $\epsilon \ll 1$ and with the auxiliary variable $F$ of order $\mathrm{O}(1)$.

In the following, we assume that the dynamic contact line freely slides along the lateral wall with a constant slope while keeping a contact angle equal to a static value of $90^{\circ}$. The latter hypothesis implicitly assumes the absence of a static meniscus so that the base-state configuration is $\mathbf{q}_{0}=\left\{\Phi_{0}, \eta_{0}\right\}^{T}=\mathbf{0}$, i.e. the fluid is at rest with a flat static interface.

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At order $\epsilon$, equations (5.2)-(5.3b) reduce to a forced linear system, whose matrix compact form reads,

$$
\begin{equation*}
\left(\partial_{t} \mathscr{B}-\mathscr{A}\right) \mathbf{q}^{\prime}=\mathscr{F}^{\prime}, \tag{5.5}
\end{equation*}
$$

with $\mathscr{F}^{\prime}=F \hat{\mathscr{F}}\left(\frac{1}{2} e^{\mathrm{i}(\Omega t-\theta)}+\frac{1}{2} e^{\mathrm{i}(\Omega t+\theta)}\right)+c . c ., \hat{\mathscr{F}}=\left\{0, \frac{r}{2}\right\}^{T}$ and

$$
\mathscr{B}=\left(\begin{array}{cc}
0 & 0  \tag{5.6}\\
I_{\eta} & 0
\end{array}\right), \mathscr{A}=\left(\begin{array}{cc}
\Delta & 0 \\
0 & -I_{\eta}
\end{array}\right),
$$

where $c . c$. stands for complex conjugate and $I_{\eta}$ is the identity matrix associated with the interface $\eta$. Note that the kinematic condition does not explicitly appear in (5.6), but it is enforced as a boundary condition at the interface (Viola et al., 2018). We then seek a standing wave solution in the form

$$
\begin{equation*}
\mathbf{q}^{\prime}(r, \theta, z, t)=F \hat{\mathbf{q}}(r, z)\left(\frac{1}{2} e^{\mathrm{i}(\Omega t-\theta)}+\frac{1}{2} e^{\mathrm{i}(\Omega t+\theta)}\right)+c . c ., \tag{5.7}
\end{equation*}
$$

where $\hat{\mathbf{q}}$ is straightforwardly computed by solving the system

$$
\begin{equation*}
\left(\mathrm{i} \Omega \mathscr{B}-\mathscr{A}_{m=1}\right) \hat{\mathbf{q}}=\hat{\mathscr{F}}, \tag{5.8}
\end{equation*}
$$

Note that, due to the normal mode ansatz (5.7), the linear operator $\mathscr{A}_{m}$ depends on the azimuthal wavenumber $m$, here $m=1$. Even though an exact analytical solution to equation (5.8) can be readily obtained via a Bessel-Fourier-series representation, in this work, as in Bongarzone et al. (2022a), we opt for a numerical scheme based on a discretization technique, where linear operators $\mathscr{B}$ and $\mathscr{A}_{m}$ are discretized in space by means of a Chebyshev pseudo-spectral collocation method with a two-dimensional mapping implemented in Matlab, which is analogous to that described by Viola et al. (2018) and Bongarzone et al. (2021c). The numerical scheme requires explicit boundary conditions at $r=0$ in order to regularize the problem on the revolution axis ( $r=0$ ), i.e.

$$
\begin{align*}
& m=0: \frac{\partial \hat{\eta}}{\partial r}=\frac{\partial \hat{\Phi}}{\partial r}=0,  \tag{5.9a}\\
& m \geq 1: \hat{\eta}=\hat{\Phi}=0 . \tag{5.9b}
\end{align*}
$$

The numerical convergence of the results presented throughout the work is achieved using a computational grid $N_{r}=N_{z} \leq 40$, with $N_{r}$ and $N_{z}$ the number of radial and axial collocation grid points, respectively. Due to the low computational cost, we used $N_{r}=N_{z}=60$.
We recall the well-known dispersion relation for inviscid gravity waves (Lamb, 1993),

$$
\begin{equation*}
\omega_{m n}^{2}=k_{m n} \tanh \left(k_{m n} H\right), \tag{5.10}
\end{equation*}
$$

where the wavenumber $k_{m n}$ is given by the $n$ th-root of the first derivative of the $m$ th-order Bessel function of the first kind satisfying $J_{m}^{\prime}\left(k_{m n}\right)=0$. By denoting the eigenvector associated with the natural frequency $\omega_{m n}$ as $\hat{\mathbf{q}}_{m n}$, solution of the homogeneous version of equation (5.8) for $\Omega=\omega_{m n}$, it is useful for the rest of the analysis to note that owing to the symmetries of the
problem, the system admits the following invariant transformation

$$
\begin{equation*}
\left(\hat{\mathbf{q}}_{m n},+m, \mathrm{i} \omega_{m n}\right) \longrightarrow\left(\hat{\mathbf{q}}_{m n},-m, \mathrm{i} \omega_{m n}\right) \tag{5.11}
\end{equation*}
$$

Such an invariance suggests that the spatial structure, $\hat{\mathbf{q}}(r, z)$, of the system response to an external forcing with temporal and azimuthal periodicity $(\Omega, m)$ is the same of that computed for $(\Omega,-m)$, so that the linear solution form (5.7) holds true.

### 5.3 Harmonic single-crest (SC) resonance

With the aim to derive a weakly nonlinear (WNL) system of amplitude equations governing the super-harmonic double-crest dynamics (DC) under longitudinal excitation, we first tackle the simpler problem of harmonic single-crest waves (SC). We look for a third-order asymptotic solution for the system

$$
\begin{equation*}
\mathbf{q}=\{\Phi, \eta\}^{T}=\epsilon \mathbf{q}_{1}+\epsilon^{2} \mathbf{q}_{2}+\epsilon^{3} \mathbf{q}_{3}+\mathrm{O}\left(\epsilon^{4}\right) \tag{5.12}
\end{equation*}
$$

where the zero order solution, $\mathbf{q}_{0}=\mathbf{0}$, associated with the rest state, is omitted.
With regards to SC waves and, specifically, to the harmonic response at a driving frequency close to the natural frequency of one of the non-axisymmetric $m= \pm 1$ modes, $\omega_{1 n}$, we assume here a small forcing amplitude of order $\epsilon^{3}$. This assumption is justified by the fact that close to resonance, $\Omega \approx \omega_{1 n}$, and in the absence of dissipation, even a small forcing will induce a large system response. Hence, the analysis is expected to hold for $\Omega=\omega_{1 n}+\lambda$, where $\lambda$ is a small detuning parameter assumed of order $\epsilon^{2}$. In the spirit of the multiple scale technique, we introduce the slow time scale $T_{2}=\epsilon^{2} t$, with $t$ being the fast time scale. Hence, the following scalings are assumed:

$$
\begin{equation*}
f=\epsilon^{3} F, \quad \Omega=\omega_{1 n}+\epsilon^{2} \Lambda, \quad T_{2}=\epsilon^{2} t \tag{5.13}
\end{equation*}
$$

with the auxiliary parameters, $F$ and $\Lambda$, of order $O(1)$.
Given the azimuthal periodicity of the external forcing, i.e. $m= \pm 1$, we assume a leading order solution as the sum of two counter-propagating travelling waves,

$$
\begin{equation*}
\mathbf{q}_{1}=A_{1}\left(T_{2}\right) \hat{\mathbf{q}}_{1}^{A_{1}} e^{\mathrm{i}\left(\omega_{1 n} t-\theta\right)}+B_{1}\left(T_{2}\right) \hat{\mathbf{q}}_{1}^{B_{1}} e^{\mathrm{i}\left(\omega_{1 n} t+\theta\right)}+c . c . \tag{5.14}
\end{equation*}
$$

where $\hat{\mathbf{q}}_{1}^{A_{1}}=\hat{\mathbf{q}}_{1}^{B_{1}}$ (owing to (5.11)) is the eigenmode computed by solving (5.8) for its homogeneous solution at $\Omega=\omega_{1 n}$, where $\omega_{1 n}$ is given by (5.10). The complex amplitudes $A_{1}$ and $B_{1}$, functions of the slow time scale $T_{2}$ and still undetermined at this stage of the expansion, describe the slow time amplitude modulation of the two oscillating waves and must be determined at a higher order.
By pursuing the expansion to the second order, one obtains a linear system forced by combinations of the first-order solutions. These forcing terms are proportional to $A_{1}^{2}$ and $B_{1}^{2}$ (second harmonics), to $\left|A_{1}\right|^{2}$ and $\left|B_{1}\right|^{2}$ (steady and axisymmetric mean flow corrections) and to $A_{1} B_{1}$

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and $A_{1} \bar{B}_{1}$ (cross-quadratic interactions),

$$
\begin{align*}
& \left(\partial_{t} \mathscr{B}-\mathscr{A}_{m}\right) \mathbf{q}_{2}=\mathscr{F}_{2}=\left(\left|A_{1}\right|^{2} \hat{\mathscr{F}}_{2}^{A_{1} \bar{A}_{1}}+\left|B_{1}\right|_{1}^{2} \hat{\mathscr{F}}_{2}^{B_{1} \bar{B}_{1}}\right) \\
& +\left(A_{1}^{2} \hat{\mathscr{F}}_{2}^{A_{1} A_{1}} e^{\mathrm{i} 2\left(\omega_{1 n} t-\theta\right)}+B_{1}^{2} \hat{\mathscr{F}}_{2}^{B_{1} B_{1}} e^{\mathrm{i} 2\left(\omega_{1 n} t+\theta\right)}+\text { c.c. }\right) \\
& \quad+\left(A_{1} B_{1} \hat{\mathscr{F}}_{2}^{A_{1} B_{1}} e^{\mathrm{i} 2 \omega_{1 n} t}+A_{1}{\left.\overline{B_{1}} \hat{\mathscr{F}}_{2}^{A_{1} \bar{B}_{1}} e^{-\mathrm{i} 2 \theta}+\text { c.c. }\right) .}^{\text {. }}\right. \text {. } \tag{5.15}
\end{align*}
$$

Thus, we seek a second-order solution of the form

$$
\begin{align*}
\mathbf{q}_{2}=\left|A_{1}\right|^{2} \hat{\mathbf{q}}_{2}^{A_{1} \bar{A}_{1}}+\left|B_{1}\right|^{2} \hat{\mathbf{q}}_{2}^{B_{1} \bar{B}_{1}} & +\left(A_{1}^{2} \hat{\mathbf{q}}_{2}^{A_{1} A_{1}} e^{\mathrm{i} 2\left(\omega_{1 n} t-\theta\right)}+B_{1}^{2} \hat{\mathbf{q}}_{2}^{B_{1} B_{1}} e^{\mathrm{i} 2\left(\omega_{1 n} t+\theta\right)}+\text { c.c. }\right) \\
& +\left(A_{1} B_{1} \hat{\mathbf{q}}_{2}^{A_{1} B_{1}} e^{\mathrm{i} 2 \omega_{1 n} t}+A_{1} \bar{B}_{1} \hat{\mathbf{q}}_{2}^{A_{1} \bar{B}_{1}} e^{-\mathrm{i} 2 \theta}+\text { c.c. }\right) . \tag{5.16}
\end{align*}
$$

Given the invariant transformation (5.11), only some of these second-order responses need to be computed explicitly, as, e.g., $\hat{\mathbf{q}}_{2}^{A_{1} \bar{A}_{1}}=\hat{\mathbf{q}}_{2}^{B_{1} \bar{B}_{1}}$ and $\hat{\mathbf{q}}_{2}^{A_{1} A_{1}}=\hat{\mathbf{q}}_{2}^{B_{1} B_{1}}$.
We now move forward to the $\epsilon^{3}$-order problem, which is once again a linear problem forced by combinations of the first (5.14) and second order solutions (5.16), produced by third order non-linearities such as $\left(\nabla \Phi_{1} \cdot \nabla \Phi_{2}+\nabla \Phi_{2} \cdot \nabla \Phi_{1}\right) / 2$ in the dynamic condition or $\nabla \Phi_{1} \cdot \nabla \eta_{2}+\nabla \Phi_{2} \cdot \nabla \eta_{1}$ in the kinematic equation, as well as by the slow time- $T_{2}$ derivative of the leading order solution and by the external forcing, which was assumed of order $\epsilon^{3}$,

$$
\begin{array}{r}
\left(\partial_{t} \mathscr{B}-\mathscr{A}_{m}\right) \mathbf{q}_{3}=\mathscr{F}_{3}=-\frac{\partial A_{1}}{\partial T_{2}} \mathscr{B} \hat{\mathbf{q}}_{1}^{A_{1}} e^{\mathrm{i}\left(\omega_{1 n} t-\theta\right)}-\frac{\partial B_{1}}{\partial T_{2}} \mathscr{B} \hat{\mathbf{q}}_{1}^{B_{1}} e^{\mathrm{i}\left(\omega_{1 n} t+\theta\right)}  \tag{5.17}\\
+\left|A_{1}\right|^{2} A_{1} \hat{\mathscr{F}}_{3}^{\left|A_{1}\right|^{2} A_{1}} e^{\mathrm{i}\left(\omega_{1 n} t-\theta\right)}+\left|B_{1}\right|^{2} B_{1} \hat{\mathscr{F}}_{3}^{\left|B_{1}\right|^{2} B_{1}} e^{\mathrm{i}\left(\omega_{1 n} t+\theta\right)} \\
+\left|B_{1}\right|^{2} A_{1} \hat{\mathscr{F}}_{3}^{\left|B_{1}\right|^{2} A_{1}} e^{\mathrm{i}\left(\omega_{1 n} t-\theta\right)}+\left|A_{1}\right|^{2} B_{1} \hat{\mathscr{F}}_{3}^{\left|A_{1}\right|^{2} B_{1}} e^{\mathrm{i}\left(\omega_{1 n} t+\theta\right)} \\
+\frac{1}{2} F \hat{\mathscr{F}}_{3}^{F} e^{\mathrm{i}\left(\omega_{1 n} t-\theta\right)} e^{\mathrm{i} \Lambda T_{2}}+\frac{1}{2} F \hat{\mathscr{F}}_{3}^{F} e^{\mathrm{i}\left(\omega_{1 n} t+\theta\right)} e^{\mathrm{i} \Lambda T_{2}} \\
+ \text { N.R.T. }+ \text { c.c., }
\end{array}
$$

with $\hat{\mathscr{F}}_{3}^{F}=\{0, r / 2\}^{T}$ and where N.R.T. stands for non-resonating terms. These terms are not strictly relevant for further analysis and can therefore be neglected. Amplitudes equations for $A_{1}$ and $B_{1}$ are obtained by requiring that secular terms do not appear in the solution to equation (5.17), where secularity results from all resonant forcing terms in $\mathscr{F}_{3}$ (see Appendix D of Bongarzone et al. (2022a) for its explicit expression), i.e. all terms sharing the same frequency and wavenumber of $\mathbf{q}_{1}$, e.g. $(\omega, m)=\left(\omega_{1 n}, \pm 1\right)$, and in effect all terms explicitly written in (5.17). It follows that a compatibility condition must be enforced through the Fredholm alternative (Friedrichs, 2012; Olver, 2014b), which imposes the amplitudes $A=\epsilon A_{1} e^{-\mathrm{i} \lambda t}$ and $B=\epsilon B_{1} e^{-\mathrm{i} \lambda t}$ to obey the following normal form

$$
\begin{align*}
& \frac{d A}{d t}=-\mathrm{i} \lambda A+\mathrm{i} \frac{\mu_{S C}}{2} f+\mathrm{i} v_{S C}|A|^{2} A+\mathrm{i} \xi_{S C}|B|^{2} A,  \tag{5.18a}\\
& \frac{d B}{d t}=-\mathrm{i} \lambda B+\mathrm{i} \frac{\mu_{S C}}{2} f+\mathrm{i} v_{S C}|B|^{2} B+\mathrm{i} \xi_{S C}|A|^{2} B, \tag{5.18b}
\end{align*}
$$

where the physical time $t=T_{2} / \epsilon^{2}$ has been reintroduced and where forcing amplitude and detuning parameter are recast in terms of their corresponding physical values, $f=\epsilon^{3} F$ and $\lambda=\epsilon^{2} \Lambda=\Omega-\omega_{1 n}$, so as to eliminate the small implicit parameter $\epsilon$ (Bongarzone et al., 2021a, 2022b). The subscript $S C$ stands for single-crest (SC). The various normal form coefficients, which turn out to be real-valued quantities due to the absence of dissipation, are computed as scalar products between the adjoint mode, $\hat{\mathbf{q}}_{1}^{A_{1} \dagger}=\hat{\mathbf{q}}_{1}^{B_{1} \dagger}$, associated with $\hat{\mathbf{q}}_{1}^{A_{1}}=\hat{\mathbf{q}}_{1}^{B_{1}}$, and the third order resonant forcing terms (see Appendix 5.7.1 and Bongarzone et al. (2022a) for further details).

Once stable stationary solutions are computed, $A$ and $B$ are replaced in expressions (5.14) and (5.16) and the total harmonic SC wave solution is reconstructed as

$$
\begin{equation*}
\mathbf{q}_{S C}=\{\Phi, \eta\}^{T}=\epsilon \mathbf{q}_{1}+\epsilon^{2} \mathbf{q}_{2} \tag{5.19}
\end{equation*}
$$

To this end, it is first convenient to express equations (5.18a)-(5.18b) in polar coordinates, i.e. by defining $A=|A| e^{i \Phi_{A}}$ and $B=|B| e^{i \Phi_{B}}$, and then to introduce the following change of variables, $|a|=|A|+|B|$ and $|b|=|A|-|B|$. By looking for periodic solutions with stationary amplitudes $|A|,|B| \neq 0$, one can sum and subtract equations (5.18a) and (5.18b), hence obtaining,

$$
\begin{gather*}
f=a_{x} \Omega^{2}= \pm|a|\left(\lambda-\left(\frac{v_{S C}+\xi_{S C}}{4}\right)|a|^{2}-\left(\frac{3 v_{S C}-\xi_{S C}}{4}\right)|b|^{2}\right) \frac{1}{\mu_{S C}}  \tag{5.20a}\\
0=|b|\left(\lambda-\left(\frac{v_{S C}+\xi_{S C}}{4}\right)|b|^{2}-\left(\frac{3 v_{S C}-\xi_{S C}}{4}\right)|a|^{2}\right) \tag{5.20b}
\end{gather*}
$$

As expected, equation (5.20b) suggests that two possible solutions exist. The planar (or standing) wave solution is retrieved for

$$
\begin{gather*}
|b|=|A|-|B|=0 \rightarrow|A|=|B|  \tag{5.21a}\\
a_{x} \Omega^{2}= \pm|a|\left(\lambda-\left(\frac{v_{S C}+\xi_{S C}}{4}\right)|a|^{2}\right) \frac{1}{\mu_{S C}} \tag{5.21b}
\end{gather*}
$$

whereas the swirling wave solution is found when $|b| \neq 0$ and

$$
\begin{gather*}
|b|^{2}=\left(\lambda-\left(\frac{3 v_{S C}-\xi_{S C}}{4}\right)|a|^{2}\right)\left(\frac{4}{v_{S C}+\xi_{S C}}\right),  \tag{5.22a}\\
a_{x} \Omega^{2}= \pm 2|a|\left(\frac{\xi_{S C}-v_{S C}}{v_{S C}+\xi_{S C}}\right)\left(\lambda-v_{S C}|a|^{2}\right) \frac{1}{\mu_{S C}} \tag{5.22b}
\end{gather*}
$$

The various branches prescribed by (5.21b) and (5.22a)-(5.22b) for $|a|$ and $|b|$ as a function of $\tau=\Omega / \omega_{1 n}$ and at a fixed non-dimensional shaking amplitude $a_{x}$ are here computed by means of the Matlab function fimplicit.
From (5.18a)-(5.18b) expressed in polar coordinates, one finds that the stationary module equations read $\mu_{S C} f \sin \Phi_{A} / 2=0$ and $\mu_{S C} f \sin \Phi_{B} / 2=0$, hence implying $\sin \Phi_{A}=\sin \Phi_{B}=0$. We therefore note that four possible combinations of stationary phases, $\Phi_{A}$ and $\Phi_{B} \in[0,2 \pi]$,

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Figure 5.2 - Estimates of bounds, in the $\left(\Omega / \omega_{11}, a_{x}\right)$-plane, between the frequency ranges where planar, irregular and swirling waves occur when the container undergoes a longitudinal and harmonic motion. Filled markers: experiments by Royon-Lebeaud et al. (2007). Black dashed lines: theoretical prediction by Faltinsen et al. (2016), whose theoretical curves have been here reproduced by manually sampling those reported in their original figure 8(a).
are in principle admitted, i.e. (i) $\Phi_{A}=\Phi_{B}=0$, (ii) $\Phi_{A}=\Phi_{B}=\pi$, (iii) $\Phi_{A}=0, \Phi_{B}=\pi$ and (iv) $\Phi_{A}=\pi, \Phi_{B}=0$. However, (iii) and (iv) are totally equivalent to (i) and (ii), respectively, with amplitudes $|a| \rightarrow|b|$ and $|b| \rightarrow|a|$. Therefore, only combinations (i) $\Phi_{A}=\Phi_{B}=\Phi=0$ and (ii) $\Phi=\pi$, which produce the $\pm$ sign in (5.20a), are retained.

### 5.3.1 Comparison with existing experiments and theoretical predictions

In figure 5.2 we reproduce figure 8 of Faltinsen et al. (2016), which shows the estimates of bounds between the frequency ranges where harmonic planar, irregular and swirling waves occur. The outcomes of the present analysis are consistent with those of Faltinsen et al. (2016) and with the experimental measurements by Royon-Lebeaud et al. (2007). The values of the normal form coefficients $\mu_{S C}, v_{S C}$ and $\xi_{S C}$ reported in table 5.1 of Appendix 5.7.1 confirm that the stability boundaries vary weakly with the liquid depth, as stated by Faltinsen et al. (2016) for non-dimensional fluid depths $H \gtrsim 1.05$, but strongly depend on the forcing amplitude, with the frequency range for irregular and swirling waves widening for increasing forcing amplitudes. In this context, irregular means that both the planar and the swirling wave solutions are unstable, hence one could expect irregular and chaotic patterns with a switching between planar and swirling motion. The green-shaded region corresponds to stable single-crest (SC) swirling waves, while the light purple-shaded region corresponds to the multi-solution regime,


Figure 5.3 - Non-dimensional maximum steady-state wave elevation, $\max _{t, \theta=0, \pi / 2} \eta$, (the maximum is taken from values at two probes located at $(x, y)=(0.875,0)$ and $(0,0.875))$ versus the forcing frequency $\Omega / \omega_{11}$ and for different $x$-longitudinal shaking amplitudes, $a_{x}$ : (a) $0.0033,0.0066,0.0133$ and 0.0266 ; (b) 0.023 and 0.045 . Markers are associated with two experimental series by Royon-Lebeaud et al. (2007) (experimental data from their original figures 2 (now (a)) and 7 (now (b)). Filled circles correspond to measurements done for the planar regime, whereas filled squares indicate swirling. The black dashed lines represent the stable branches predicted by Faltinsen et al. (2016). Their curves have been here carefully reproduced by manually sampling those reported in their original figure 10 in the range of frequency available, i.e. $\Omega / \omega_{11} \in[0.7,1.2]$. Colored solid lines correspond to the present theoretical predictions for stable branches.
where both stable swirling SC and planar SC wave motions are possible depending on the initial transient, i.e. on the initial conditions, as typical of hysteretic systems.
In figure 5.3(a) and (b) the non-dimensional maximum steady-state wave elevation, computed by reconstructing the total flow solution in accordance with (5.19), is compared with the theoretical estimations by Faltinsen et al. (2016) (black dashed lines) from their figure 8 and with the corresponding experimental measurements by Royon-Lebeaud et al. (2007) (colored filled markers). The agreement between the present model and experiments is fairly good and consistent with predictions by Faltinsen et al. (2016). The larger disagreement between theory and experiments at smaller forcing amplitudes was tentatively attributed by Faltinsen et al. (2016) to the fact that the actual elevation of these wave amplitudes was approximately 1 mm and may therefore be more difficult to measure with sufficient accuracy.
A comparable mismatch is here retrieved. As a side comment, we note that, within the present inviscid framework, the lower left stable planar branch, $\Omega / \omega_{11}<1$, is obtained for a phase $\Phi=0$, which implies a fluid motion in phase with the container motion, whereas the lower right planar branch, $\Omega / \omega_{11}>1$, has a phase $\Phi=\pi$, hence implying a phase opposition. The stable swirling branch is characterized by $\Phi=0$. This is consistent with previous studies (Royon-Lebeaud et al., 2007).

### 5.3.2 Discussion: present analysis vs. the Narimanov-Moiseev multimodal theory

In this section, the present model for harmonic resonances has been compared with the Narimanov-Moiseev multimodal theory employed by Faltinsen et al. (2016) and has been shown to provide very consistent and close predictions. Before moving to the next section, it is therefore worth pointing out the methodological analogies and differences as well as the pros and cons of the two approaches.

Adopting a variational formulation (Miles, 1976) and assuming an incompressible and irrotational flow, the multimodal method reduces the hydrodynamic sloshing system to a modal system of nonlinearly coupled ordinary differential equations written in terms of the so-called generalised coordinates (Faltinsen and Timokha, 2009). This projection step uses a Fourier-type representation of the time-dependent surface elevation and potential velocity field. Because the resulting coefficients in the equation system are derived analytically and only ODEs must be numerically time-integrated, numerical errors are negligible with small CPU time relative to that for CFD methods based on the governing equations of the full hydrodynamic sloshing system.
For theoretical modelling purposes, postulating proper asymptotic relations between these generalised coordinates simplifies the system to a weakly nonlinear form. Specifically, in the case of harmonic resonances, the Narimanov-Moiseev asymptotic relations assume a leading dynamics of order $\sim \epsilon$, a frequency detuning $\sim \epsilon^{2}$, a forcing amplitude $\sim \epsilon^{3}$ and a slow time scale $\sim \epsilon^{2}$. As surface tension effects are neglected and the contact line is assumed to freely sleep along the sidewall with a constant and zero slope, the Fourier basis (Bessel functions for circular cylinders) also coincides with the actual natural sloshing modes.
Under these assumptions, although we do not go through this initial projection step, the present model and the Narimanov-Moiseev multimodal theory are essentially equivalent. The $\epsilon$-order leading dynamics (5.14) is indeed written in terms of natural sloshing modes (here computed numerically) and the same asymptotic scaling is adopted (see (5.13)).
Nevertheless, the reintroduction of surface tension in the multimodal theory can be challenging. In particular, it is not clear yet how to account for static meniscus and contact line dynamics (Raynovskyy and Timokha, 2020). Moreover, in these cases, the element of the Fourier basis (Bessel-)functions no more coincide with the actual natural sloshing modes.
The numerical nature of our approach, based on primitive equations, not only allows us to reintroduce straightforwardly surface tension (see Bongarzone et al. (2022a)) but also to possibly account (asymptotically) for static meniscus effects and contact angle dynamics while keeping the leading order dynamics expressed in terms of exact (up to a numerical convergence error) linear natural modes computed numerically. This has been shown possible in a series of works by some of the authors (Bongarzone et al., 2022b, 2021c; Viola et al., 2018; Viola and Gallaire, 2018) and makes the present approach in this sense more versatile to study the sloshing problem under non-ideal sidewall conditions.
Most important for the following analysis, is the case of the resonant amplification of higherorder modes. Below a critical liquid depth, typically $H_{c r} \approx 1.05$ for circular cylinders, such
amplification (secondary or internal resonances) can happen in the vicinity of the primary resonance. These cases, which require a reordering of the asymptotic scaling, can still be tackled in the framework of the multimodal theory by employing the so-called adaptive modal approach (see chapters 7-9 of Faltinsen and Timokha (2009) and Raynovskyy and Timokha (2020)). However, secondary resonances may also occur more broadly even far from the primary resonance zone and for $H>H_{c r}$, as in the case of the DC super-harmonic resonance. To the authors' knowledge, even though its formalization appears possible, no variant of the abovementioned adaptive modal approach capable of dealing with super-harmonic resonances far from the primary one has been reported yet. The WNL analysis of the next section proposes an asymptotic reordering allowing one to deal with the specific case of super-harmonic DC resonances (Bongarzone et al., 2022a).

### 5.4 Super-harmonic double-crest (DC) resonance

We now tackle the double-crest (DC) wave response to longitudinal shaking, whose investigation represents the core of the present work. We remind that the double-crest dynamics occurs at a driving frequency $\Omega \approx \omega_{2 n} / 2$ (see figure 4 of Reclari et al. (2014)). For the sake of generality, the following analysis is therefore formalized for any mode ( $2, n$ ), i.e. $\Omega=\omega_{2 n} / 2+\lambda$, where $\lambda$ is the small detuning parameter.
By analogy with Bongarzone et al. (2022a), the leading order solution is here assumed to be given by the sum of a particular solution, given by the linear response to the external forcing, computed by solving (5.8) with $\Omega=\omega_{2 n} / 2$ and $m= \pm 1$, and a homogeneous solution, represented by the two natural modes for $(m, n)=( \pm 2, n)$ associated with $\omega_{2 n}$, up to their amplitudes to be determined at higher orders. At second order, quadratic terms in $(\Omega, m)=\left(\omega_{2 n} / 2, \pm 1\right)$ will produce resonant terms in $\left(\omega_{2 n}, \pm 2\right)$. These $\epsilon^{2}$-order resonating terms will then require, in the spirit of multiple timescale analysis, an additional second-order solvability condition, hence suggesting that two slow time scales exist, namely $T_{1}$ and $T_{2}$. Thus, the asymptotic scalings of the weakly nonlinear expansion for double-crest (DC) waves are the following:

$$
\begin{equation*}
f=\epsilon F, \quad \Omega=\omega_{2 n} / 2+\epsilon \Lambda, \quad T_{1}=\epsilon t, \quad T_{2}=\epsilon^{2} t \tag{5.23}
\end{equation*}
$$

with a first-order solution reading

$$
\begin{gather*}
\mathbf{q}_{1}=A_{2}\left(T_{1}, T_{2}\right) \hat{\mathbf{q}}_{1}^{A_{2}} e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)}+B_{2}\left(T_{1}, T_{2}\right) \hat{\mathbf{q}}_{1}^{B_{2}} e^{\mathrm{i}\left(\omega_{2 n} t+2 \theta\right)} \\
+\frac{1}{2} F \hat{\mathbf{q}}_{1}^{F} e^{\mathrm{i}\left(\left(\omega_{2 n} / 2\right) t-\theta\right)} e^{\mathrm{i} \Lambda T_{1}}+\frac{1}{2} F \hat{\mathbf{q}}_{1}^{F} e^{\mathrm{i}\left(\left(\omega_{2 n} / 2\right) t+\theta\right)} e^{\mathrm{i} \Lambda T_{1}}+c . c . . \tag{5.24}
\end{gather*}
$$

In (5.24), $\hat{\mathbf{q}}_{1}^{A_{2}}=\hat{\mathbf{q}}_{1}^{B_{2}}$, whereas $A_{2}$ and $B_{2}$ are the unknown slow time amplitude modulations, here functions of the two time scales $T_{1}$ and $T_{2}$. The second-order linearized forced problem

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(a)

(b)

(c)


Figure 5.4 - Spatial structures of the first order contributions (a) $\mathbf{q}_{1}^{F}(r, z) e^{i} \cos \theta$ (single-crest SC) and (b) $\hat{\mathbf{q}}_{1}^{A_{2}} \cos 2 \theta=\hat{\mathbf{q}}_{1}^{B_{2}} \cos 2 \theta$ (double-crest DC) appearing in (5.24) and computed for $t=0$ and $T_{1}=0$. (c) Superposition of (a) and (b). Here the corresponding amplitudes have been arbitrarily chosen for visualization purposes, but we note that, while amplitude $A_{2}$ and $B_{2}$ still need to be determined, the amplitude of the single-crest solution (a) is univocally defined once the amplitude, $F$, and the oscillation frequency, $\Omega$, of the external driving are prescribed.
reads

$$
\begin{align*}
\left(\partial_{t} \mathscr{B}-\mathscr{A}_{m}\right) \mathbf{q}_{2} & =\mathscr{F}_{2}=\mathscr{F}_{2}^{i j}-\left(\frac{\partial A_{2}}{\partial T_{1}} \mathscr{B} \hat{\mathbf{q}}_{1}^{A_{2}} e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)}+\frac{\partial B_{2}}{\partial T_{1}} \mathscr{B} \hat{\mathbf{q}}_{1}^{B_{2}} e^{\mathrm{i}\left(\omega_{2 n} t+2 \theta\right)}+c . c .\right) \\
& -\mathrm{i} \Lambda F\left(\frac{1}{2} \mathscr{B} \hat{\mathbf{q}}_{1}^{F} e^{\mathrm{i}\left(\left(\omega_{2 n} / 2\right) t-\theta\right)} e^{\mathrm{i} \Lambda T_{1}}+\frac{1}{2} \mathscr{B} \hat{\mathbf{q}}_{1}^{F} e^{\mathrm{i}\left(\left(\omega_{2 n} / 2\right) t+\theta\right)} e^{\mathrm{i} \Lambda T_{1}}+c . c .\right) . \tag{5.25}
\end{align*}
$$

The first order solution is indeed made of 8 different contributions (including the complex conjugates) and it generates, in total, 36 different second-order forcing terms, here implicitly gathered in $\mathscr{F}_{2}^{i j}$, each characterized by a certain oscillation frequency and azimuthal periodicity. For the sake of brevity, indices $(i, j)$ are used to remind one that each forcing is proportional to a quadratic combination of leading order amplitudes. These indices can assume the following values: $i, j=A_{2}, B_{2}, F, \bar{A}_{2}, \bar{B}_{2}, \bar{F}$. For instance, the quadratic interaction of $A_{2}\left(T_{1}, T_{2}\right) \hat{\mathbf{q}}_{1}^{A_{2}} e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)}$ with itself will have indices $\left(i=A_{2}, j=A_{2}\right)$ and will produce a forcing term proportional to $A_{2}^{2}$, i.e $\mathscr{F}_{2}^{A_{2} A_{2}}$. The additional eight forcing terms, with their complex conjugates, appearing in (5.25) stem from the time derivative of the first-order solution (5.24) with respect to the first-order slow time scale $T_{1}$. None of the forcing terms in (5.25) is resonant, as their oscillation frequency or azimuthal wavenumber differ from those of the leading order homogeneous solution, except the two terms produced by the second-harmonic of the leading order particular solution, i.e. $\mathscr{F}_{2}^{F F}=\frac{1}{4} F^{2} \hat{\mathscr{F}}_{2}^{F F} e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)} e^{\mathrm{i} 2 \Lambda T_{1}}+\frac{1}{4} F^{2} \hat{\mathscr{F}}_{2}^{F F} e^{\mathrm{i}\left(\omega_{2 n} t+2 \theta\right)} e^{\mathrm{i} 2 \Lambda T_{1}}+$ c.c. To avoid secular terms, a second-order compatibility condition is thus imposed, requiring that the following normal form equations are verified

$$
\begin{equation*}
\frac{\partial A_{2}}{\partial T_{1}}=\mathrm{i} \frac{\mu_{D C}}{4} F^{2} e^{\mathrm{i} 2 \Lambda T_{1}}, \quad \frac{\partial B_{2}}{\partial T_{1}}=\mathrm{i} \frac{\mu_{D C}}{4} F^{2} e^{\mathrm{i} 2 \Lambda T_{1}} . \tag{5.26}
\end{equation*}
$$

Taken alone, the dynamics resulting from system (5.26) is still of little relevance, since it can be shown that the wave amplitudes $A_{2}$ and $B_{2}$ scale like $\sim \frac{1}{\Lambda}$, hence diverging symmetrically to
infinity for $\Lambda \rightarrow 0\left(\Omega \rightarrow \omega_{2 n} / 2\right)$ in absence of any restoring term, i.e. the nonlinear mechanism responsible for the finite amplitude saturation, which only comes into play at order $\epsilon^{3}$. The expansion must be therefore pursued up to the next order, and thereby one must solve for the second-order solution (Fujimura, 1989, 1991).
By substituting (5.24) and (5.26) in the forcing expression, equation (5.25) can be rewritten as

$$
\begin{align*}
& \left(\partial_{t} \mathscr{B}-\mathscr{A}_{m}\right) \mathbf{q}_{2}=\mathscr{F}_{2_{N R T}}^{i j}+\mathscr{F}_{2_{R T}}^{i j}=\mathscr{F}_{2_{N R T}}^{i j}+c . c .+  \tag{5.27}\\
& \frac{1}{4} F^{2}\left(\hat{\mathscr{F}}_{2}^{F F}-\mathrm{i} \mu_{D C} \mathscr{B} \hat{\mathbf{q}}_{1}^{A_{2}}\right) e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)} e^{\mathrm{i} 2 \Lambda T_{1}}+c . c .+ \\
& \frac{1}{4} F^{2}\left(\hat{\mathscr{F}}_{2}^{F F}-\mathrm{i} \mu_{D C} \mathscr{B} \hat{\mathbf{q}}_{1}^{B_{2}}\right) e^{\mathrm{i}\left(\omega_{2 n} t+2 \theta\right)} e^{\mathrm{i} 2 \Lambda T_{1}}+c . c .,
\end{align*}
$$

where the subscripts ${ }_{N R T}$ and ${ }_{R T}$ denote non-resonating and resonating terms, respectively. Note that the term proportional to $\Lambda F$ in (5.25) has been included in the non-resonating forcing terms, while resonant terms are written explicitly. The compatibility condition is now satisfied, meaning that the new resonant forcing term is orthogonal to the adjoint mode, $\hat{\mathbf{q}}_{1}^{A_{2} \dagger}=\overline{\hat{\mathbf{q}}}_{1}^{A_{2}}$, by construction so that, according to the Fredholm alternative, a non-trivial unique solution can be computed. Hence, we can write the second order solution as

$$
\begin{align*}
& \mathbf{q}_{2}=\left(\left|A_{2}\right|^{2} \hat{\mathbf{q}}_{2}^{A_{2} \bar{A}_{2}}+\frac{1}{4}|F|^{2} \hat{\mathbf{q}}_{2}^{F \bar{F}}\right)+  \tag{5.28}\\
& \left(A_{2}^{2} \hat{\mathbf{q}}_{2}^{A_{2} A_{2}} e^{\mathrm{i}\left(2 \omega_{2 n} t-4 \theta\right)}+\frac{1}{2} \Lambda F \hat{\mathbf{q}}_{2}^{\Lambda F} e^{\mathrm{i}\left(\left(\omega_{2 n} / 2\right) t-\theta\right)} e^{\mathrm{i} \Lambda T_{1}}+c . c .\right)+ \\
& \left(\frac{1}{2} A_{2} F \hat{\mathbf{q}}_{2}^{A_{2} F} e^{\mathrm{i}\left(\left(3 \omega_{2 n} / 2\right) t-3 \theta\right)} e^{\mathrm{i} \Lambda T_{1}}+\frac{1}{2} A_{2} \bar{F} \hat{\mathbf{q}}_{2}^{A_{2} \bar{F}} e^{\mathrm{i}\left(\left(\omega_{2 n} / 2\right) t-\theta\right)} e^{-\mathrm{i} \Lambda T_{1}}+c . c .\right)+ \\
& \left(\left|B_{2}\right|^{2} \hat{\mathbf{q}}_{2}^{B_{2} \bar{B}_{2}}+\frac{1}{4}|F|^{2} \hat{\mathbf{q}}_{2}^{F \bar{F}}\right)+ \\
& \left(B_{2}^{2} \hat{\mathbf{q}}_{2}^{B_{2} B_{2}} e^{\mathrm{i}\left(2 \omega_{2 n} t+4 \theta\right)}+\frac{1}{2} \Lambda F \hat{\mathbf{q}}_{2}^{\Lambda F} e^{\mathrm{i}\left(\left(\omega_{2 n} / 2\right) t+\theta\right)} e^{\mathrm{i} \Lambda T_{1}}+c . c .\right)+ \\
& \left(\frac{1}{2} B_{2} F \hat{\mathbf{q}}_{2}^{B_{2} F} e^{\mathrm{i}\left(\left(3 \omega_{2 n} / 2\right) t+3 \theta\right)} e^{\mathrm{i} \Lambda T_{1}}+\frac{1}{2} B_{2} \bar{F} \hat{\mathbf{q}}_{2}^{B_{2}} \bar{F} e^{\mathrm{i}\left(\left(\omega_{2 n} / 2\right) t+\theta\right)} e^{-\mathrm{i} \Lambda T_{1}}+c . c .\right)+ \\
& \left(A_{2} B_{2} \hat{\mathbf{q}}_{2}^{A_{2} B_{2}} e^{\mathrm{i} 2 \omega_{2 n} t}+A_{2} \bar{B}_{2} \hat{\mathbf{q}}_{2}^{A_{2} \bar{B}_{2}} e^{-\mathrm{i} 4 \theta}+c . c .\right)+ \\
& \left(\frac{1}{4} F^{2} \hat{\mathbf{q}}_{2}^{F F} e^{\mathrm{i} \omega_{2 n} t} e^{\mathrm{i} 2 \Lambda T_{1}}+\frac{1}{4} F \bar{F} \hat{\mathbf{q}}_{2}^{F \bar{F}} e^{-\mathrm{i} 2 \theta}+c . c .\right)+ \\
& \left(\frac{1}{2} A_{2} F \hat{\mathbf{q}}_{2}^{A_{2} F} e^{\mathrm{i}\left(\left(3 \omega_{2 n} / 2\right) t-\theta\right)} e^{\mathrm{i} \Lambda T_{1}}+\frac{1}{2} A_{2} \bar{F} \hat{\mathbf{q}}_{2}^{A_{2}} \bar{F} e^{\mathrm{i}\left(\left(\omega_{2 n} / 2\right) t-3 \theta\right)} e^{-\mathrm{i} \Lambda T_{1}}+c . c .\right)+ \\
& \left(\frac{1}{2} B_{2} F \hat{\mathbf{q}}_{2}^{B_{2} F} e^{\mathrm{i}\left(\left(3 \omega_{2 n} / 2\right) t+\theta\right)} e^{\mathrm{i} \Lambda T_{1}}+\frac{1}{2} B_{2} \bar{F} \hat{\mathbf{q}}_{2}^{B_{2} \bar{F}} e^{\mathrm{i}\left(\left(\omega_{2 n} / 2\right) t+3 \theta\right)} e^{-\mathrm{i} \Lambda T_{1}}+c . c .\right)+ \\
& \left(\frac{1}{4} F^{2} \hat{\mathbf{q}}_{2}^{F F} e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)} e^{\mathrm{i} 2 \Lambda T_{1}}+\frac{1}{4} F^{2} \hat{\mathbf{q}}_{2}^{F F} e^{\mathrm{i}\left(\omega_{2 n} t+2 \theta\right)} e^{\mathrm{i} 2 \Lambda T_{1}}+c . c .\right) .
\end{align*}
$$

All non-resonant responses in (5.28) are handled similarly, i.e. they are computed in Matlab by performing a simple matrix inversion using standard LU solvers. Although the operator associated with the resonant forcing term, i.e. (i $\omega_{2 n} \mathscr{B}-\mathscr{A}_{2}$ ), is singular, the value of the normal

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form coefficient $\mu_{D C}$ ensures that a non-trivial solution for $\hat{\mathbf{q}}_{2}^{F^{2}}$ exists. Diverse approaches can be followed to compute this response, which was here computed by using the pseudo-inverse matrix of the singular operator (Orchini et al., 2016) obtained via the built-in Matlab function pinv. We also recall that due to the invariant transformation (5.11) only some of the spatial structures appearing in (5.28) need to be computed. Lastly, at third order in $\epsilon$, the problem reads

$$
\begin{array}{r}
\left(\partial_{t} \mathscr{B}-\mathscr{A}_{m}\right) \mathbf{q}_{3}=\mathscr{F} 3  \tag{5.29}\\
=-\frac{\partial A_{2}}{\partial T_{2}} \mathscr{B} \hat{\mathbf{q}}_{1}^{A_{2}} e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)}-\frac{\partial B_{2}}{\partial T_{2}} \mathscr{B} \hat{\mathbf{q}}_{1}^{B_{2}} e^{\mathrm{i}\left(\omega_{2 n} t+2 \theta\right)} \\
-\mathrm{i} \frac{1}{4} 2 \Lambda F^{2} \mathscr{B} \hat{\mathbf{q}}_{2}^{F^{2}} e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)} e^{\mathrm{i} 2 \Lambda T_{1}}-\mathrm{i} \frac{1}{4} 2 \Lambda F^{2} \mathscr{B} \hat{\mathbf{q}}_{2}^{F^{2}} e^{\mathrm{i}\left(\omega_{2 n} t+2 \theta\right)} e^{\mathrm{i} 2 \Lambda T_{1}} \\
+\left|A_{2}\right|^{2} A_{2} \hat{\mathscr{F}}_{3}^{\left|A_{2}\right|^{2} A_{2}} e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)}+\left|B_{2}\right|^{2} B_{2} \hat{\mathscr{F}}_{3}^{\left|B_{2}\right|^{2} B_{2}} e^{\mathrm{i}\left(\omega_{2 n} t+2 \theta\right)} \\
+\left|B_{2}\right|^{2} A_{2} \hat{\mathscr{F}}_{3}^{\left|B_{2}\right|^{2} A_{2}} e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)}+\left|A_{2}\right|^{2} B_{2} \hat{\mathscr{F}}_{3}^{\left|A_{2}\right|^{2} B_{2}} e^{\mathrm{i}\left(\omega_{2 n} t+2 \theta\right)} \\
+\frac{1}{4} F^{2} A_{2} \hat{\mathscr{F}}{ }_{3}^{|F|^{2} A_{2}} e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)}+\frac{1}{4} F^{2} B_{2} \hat{\mathscr{F}}_{3}^{|F|^{2} B_{2}} e^{\mathrm{i}\left(\omega_{2 n} t+2 \theta\right)} \\
+\frac{1}{4} \Lambda F^{2} \hat{\mathscr{F}}_{3}^{\Lambda F^{2}} e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)} e^{\mathrm{i} 2 \Lambda T_{1}}+\frac{1}{4} \Lambda F^{2} \hat{\mathscr{F}}_{3}^{\Lambda F^{2}} e^{\mathrm{i}\left(\omega_{2 n} t+2 \theta\right)} e^{\mathrm{i} 2 \Lambda T_{1}}+\mathrm{N} . \text { R.T. }+c . c .,
\end{array}
$$

where the first two forcing terms arise from the time-derivative of the first-order solution with respect to the second-order slow time scale $T_{2}$ and from that of the second-order solution with respect to the first-order slow time scale $T_{1}$, respectively (see Appendix D of Bongarzone et al. (2022a) for the full expression of $\mathscr{F}_{2}$ and $\mathscr{F}_{3}$ ). Once again, all terms explicitly written in (5.29) are resonant, as they share the same pair ( $\omega_{2 n}, \pm 2$ ) than the first order homogeneous solutions, hence a third order compatibility condition, leading to the following normal form, must be enforced

$$
\begin{align*}
& \frac{\partial A_{2}}{\partial T_{2}}=\mathrm{i} \frac{\zeta_{D C}}{4} \Lambda F^{2} e^{\mathrm{i} \Lambda \Lambda T_{1}}+\mathrm{i} \frac{\chi_{D C}}{4} A_{2} F^{2}+\mathrm{i} v_{D C}\left|A_{2}\right|^{2} A_{2}+\mathrm{i} \xi_{D C}\left|B_{2}\right|^{2} A_{2},  \tag{5.30a}\\
& \frac{\partial B_{2}}{\partial T_{2}}=\mathrm{i} \frac{\zeta_{D C}}{4} \Lambda F^{2} e^{\mathrm{i} 2 \Lambda T_{1}}+\mathrm{i} \frac{\chi_{D C}}{4} B_{2} F^{2}+\mathrm{i} v_{D C}\left|B_{2}\right|^{2} B_{2}+\mathrm{i} \xi_{D C}\left|A_{2}\right|^{2} B_{2} . \tag{5.30b}
\end{align*}
$$

where the coefficients are defined in Appendix 5.7.1.
As a last step in the derivation of the final amplitude equation for the double-crest (DC) waves and in order to eliminate the implicit small parameter $\epsilon$, we unify systems (5.26) and (5.30a)(5.30b) into a single system of equations recast in terms of the physical time $t=T_{1} / \epsilon=T_{2} / \epsilon^{2}$, physical forcing control parameters, $f=\epsilon F, \lambda=\epsilon \Lambda$ and total amplitudes, $A=\epsilon A_{2} e^{-\mathrm{i} 2 \lambda t}$ and $B=\epsilon B_{2} e^{-\mathrm{i} 2 \lambda t}$. This is achieved by summing (5.26) to (5.30a) and (5.30b) along with their respective weights $\epsilon^{2}$ and $\epsilon^{3}$, thus obtaining

$$
\begin{equation*}
\frac{d A}{d t}=-\mathrm{i}\left(2 \lambda-\frac{\chi_{D C}}{4} f^{2}\right) A+\mathrm{i} \frac{\left(\zeta_{D C} \lambda+\mu_{D C}\right)}{4} f^{2}+\mathrm{i} v_{D C}|A|^{2} A+\mathrm{i} \xi_{D C}|B|^{2} A, \tag{5.31a}
\end{equation*}
$$

$$
\begin{equation*}
\frac{d B}{d t}=-\mathrm{i}\left(2 \lambda-\frac{\chi_{D C}}{4} f^{2}\right) B+\mathrm{i} \frac{\left(\zeta_{D C} \lambda+\mu_{D C}\right)}{4} f^{2}+\mathrm{i} v_{D C}|B|^{2} B+\mathrm{i} \xi_{D C}|A|^{2} B \tag{5.31b}
\end{equation*}
$$

We note that no second order homogeneous solutions, e.g. proportional to amplitudes $C_{2}\left(T_{1}, T_{2}\right)$ and $D_{2}\left(T_{1}, T_{2}\right)$, have been accounted for in (5.28), as their presence will produce two resonant third order terms, $\frac{\partial C_{2}}{\partial T_{1}} \mathscr{B} \hat{\mathbf{q}}_{2}^{C_{2}} e^{\mathrm{i}\left(\omega_{2 n} t-2 \theta\right)}\left(\hat{\mathbf{q}}_{2}^{C_{2}}=\hat{\mathbf{q}}_{2}^{A_{2}}\right)$ and $\frac{\partial D_{2}}{\partial T_{1}} \mathscr{B} \hat{\mathbf{q}}_{2}^{D_{2}} e^{\mathrm{i}\left(\omega_{2 n} t+2 \theta\right)}\left(\hat{\mathbf{q}}_{2}^{C_{2}}=\hat{\mathbf{q}}_{2}^{A_{2}}\right)$, that can be incorporated in the final amplitude equations (5.31a)-(5.31b) by simply defining $A=\epsilon\left(A_{2}+\epsilon C_{2}\right) e^{-\mathrm{i} 2 \lambda t}$ and $B=\epsilon\left(B_{2}+\epsilon D_{2}\right) e^{-\mathrm{i} 2 \lambda t}$.
As in $\S 5.3$, we first turn to polar coordinates, $A=|A| e^{\mathrm{i} \Phi_{A}}$ and $B=|B| e^{\mathrm{i} \Phi_{A}}$, and we split the modulus and phase parts of (5.31a)-(5.31b). We then look for stationary solutions, $d / d t=$ 0 with $|A|,|B| \neq 0\left(\Phi_{A}=\Phi_{B}=\Phi=0\right.$, $\pi$, see $\left.\S 5.3\right)$. By summing and subtracting (5.31a) and (5.31b), after introducing the auxiliary amplitudes $|a|=|A|+|B|$ and $|b|=|A|-|B|$, the following implicit relations are obtained,

$$
\begin{gather*}
f^{2}=|a|\left(2 \lambda-\frac{v_{D C}+\xi_{D C}}{4}|a|^{2}-\frac{3 v_{D C}-\xi_{D C}}{4}|b|^{2}\right) \frac{4}{\left(|a| \chi_{D C} \pm 2\left(\zeta_{D C} \lambda+\mu_{D C}\right)\right)}  \tag{5.32a}\\
0=|b|\left(\frac{\chi_{D C}}{4} f^{2}-\left(2 \lambda-\frac{v_{D C}+\xi_{D C}}{4}|b|^{2}-\frac{3 v_{D C}-\xi_{D C}}{4}|a|^{2}\right)\right), \tag{5.32b}
\end{gather*}
$$

with $f=a_{x} \Omega^{2}$ and $\lambda=\Omega-\omega_{2 n} / 2$. By analogy with harmonic forcing conditions, two possible (super-harmonic) solutions exist, i.e. a planar wave solution for $|b|=0$,

$$
\begin{equation*}
f=\sqrt{|a|\left(2 \lambda-\frac{v_{D C}+\xi_{D C}}{4}|a|^{2}\right) \frac{4}{\left(|a| \chi_{D C} \pm 2\left(\zeta_{D C} \lambda+\mu_{D C}\right)\right)}}, \tag{5.33}
\end{equation*}
$$

and a swirling solution for $|b| \neq 0$ defined by,

$$
\begin{gather*}
|b|^{2}=\left(2 \lambda-\frac{\chi_{D C}}{4} f^{2}-\frac{3 v_{D C}-\xi_{D C}}{4}|a|^{2}\right)\left(\frac{4}{v_{D C}+\xi_{D C}}\right)  \tag{5.34a}\\
f=\sqrt{2|a|\left(\frac{\xi_{D C}-v_{D C}}{v_{D C}+\xi_{D C}}\right)\left(2 \lambda-v_{D C}|a|^{2}\right) \frac{4}{\left(2|a| \frac{\left(\xi_{D C}-v_{D C}\right)}{\left(v_{D C}+\xi_{D C}\right)} \chi_{D C} \pm 2\left(\zeta_{D C} \lambda+\mu_{D C}\right)\right)}}, \tag{5.34b}
\end{gather*}
$$

where only real solutions corresponding to $f=a_{x} \Omega^{2}>0$ are retained, as the combinations $a_{x} \Omega^{2}<0$ are not physically meaningful.

The stability of such stationary solutions $\mathbf{y}_{s}=\left(|A|, \Phi_{A},|B|, \Phi_{B}\right)$ is computed by introducing small amplitude and phase perturbations ( $\ll 1$ ) with the ansatz $\mathbf{y}_{p}(t)=\left(\left|A_{p}\right|, \Phi_{A, p},\left|B_{p}\right|, \Phi_{B, p}\right) e^{s t}$ in (5.31a)-(5.31b), which are then linearized around $\mathbf{y}_{0}$, hence obtaining at first order an eigenvalue problem in the complex eigenvalue $s=s_{R}+\mathrm{i} s_{I}$. For each $\left(|A|, \Phi_{A},|B|, \Phi_{B}\right)$ one obtains four eigenvalues $s$ and if the real part $s_{R}$ of at least one of these eigenvalues is positive, then that configuration is deemed as unstable. An analogous procedure has been followed for the case of harmonic resonances discussed in §5.3.

Once the various branches for $|a|$ and $|b|$ as a function of $\tau=\Omega / \omega_{2 n}$ and at a fixed nondimensional shaking amplitude $a_{x}$ are computed and their stability is determined, amplitudes $A$ and $B$ are substituted in (5.24) and (5.28), so that the total flow solution predicted by the WNL for DC waves is reconstructed as

$$
\begin{equation*}
\mathbf{q}_{D C}=\{\Phi, \eta\}^{T}=\epsilon \mathbf{q}_{1}+\epsilon^{2} \mathbf{q}_{2} . \tag{5.35}
\end{equation*}
$$

As discussed in Bongarzone et al. (2022a) for circular sloshing, although the quantitative dependence on the external control parameters, i.e. driving amplitude and frequency, is different with respect to the SC case, e.g. $f^{2}$ instead of $f$, system (5.31a)-(5.31b) is essentially analogous to that given in (5.18a)-(5.18b). Indeed, equations (5.31a)-(5.31b) contain four main contributions,

$$
\begin{equation*}
\lambda \leftrightarrow\left(2 \lambda-\frac{\chi_{D C}}{4} f^{2}\right), \quad \mu_{S C} f \leftrightarrow \frac{\zeta_{D C} \lambda+\mu_{D C}}{4} f^{2}, \quad v_{S C} \leftrightarrow v_{D C}, \quad \xi_{S C} \leftrightarrow \xi_{D C}, \tag{5.36}
\end{equation*}
$$

corresponding respectively to a detuning term (forcing amplitude-dependent), an additive (quadratic) forcing term (driving frequency dependent), the classic cubic restoring term and, lastly, the cubic term dictating the nonlinear interaction between the two counter-propagating travelling waves. For these reasons, figure 5.5 shows the nonlinear amplitude saturation for $|a|=|A|+|B|$ and $|b|=|A|-|B|$ which are reminiscent of those commented and displayed by Faltinsen et al. (2016) in their figure 7 with regard to harmonic system responses, although the phases associated to each super-harmonic branch are $\pi$-shifted with respect to their harmonic analogous.
A more detailed description of the bifurcation diagrams shown in figure 5.5(a) and (b) is given in Faltinsen et al. (2016). Here we limit to note that the branching diagrams contain three bifurcation points, namely U (turning point), H (Hopf bifurcation) and P (Poincaré bifurcation, (Miles, 1984d)), whose positions determine the frequency ranges where stable planar (standing), swirling or irregular waves are theoretically expected. By keeping track of the position of these three bifurcations points in the $\left(\Omega / \omega_{21},|a|\right)$-plane as the forcing amplitude, $a_{x}$, is varied, one can draw a super-harmonic stability chart in the $\left(\Omega / \omega_{21}, a_{x}\right)$-plane similar to that of figure 5.3 for harmonic resonances and which is shown in figure 5.6.
The first striking difference with respect to the harmonic stability chart of figure 5.2 is the opposite curvature of the stability boundaries between the various super-harmonic regimes. As mentioned above, this is due to the quantitative dependence of the additive forcing term in system (5.31a)-(5.31b) on the driving amplitude, which is here quadratic in $f$, thus leading to the square root in equations (5.33) (planar DC) and (5.34b) (swirling DC).
Furthermore, there is a substantial difference in terms of free surface patterns. As suggested by the form of the first order solution (5.24), the leading order dynamics, governing the super-harmonic system response to longitudinal forcing, results from a superposition of a stable planar (or standing) single-crest (SC) wave, oscillating harmonically at a frequency $\omega_{S C}=\Omega \approx \omega_{2 n} / 2$ and generated by the two $m= \pm 1$ counter-rotating travelling waves of equal amplitudes, and a super-harmonic double-crest (DC) wave dynamics oscillating at a frequency
of approximately $\omega_{D C}=2 \Omega \approx \omega_{2 n}$ (period-halving). We can also relate the SC wave frequency to its own natural frequency by writing $\omega_{S C}=\Omega \approx \frac{1}{2} \sqrt{\frac{k_{2 n} \tanh k_{2 n} H}{k_{1 n} \tanh k_{1 n} H}} \omega_{1 n}$, which is $\approx 0.657 \omega_{1 n}$ (in deep water) for $n=1$ and approaches $0.5 \omega_{1 n}$ for large $n$, hence showing that the DC resonance always occurs far from the primary harmonic resonance.
When the amplitudes of the two travelling waves with $m= \pm 2$ are equal, i.e. $|A|=|B|$ (or $|b|=0$ ), the DC dynamics manifests itself via planar motion and the global solution takes the form of a planar wave (planar SC+DC, light blue shaded region in figure 5.6). On the contrary, when $|A| \neq|B| \neq 0$, one of the two $m= \pm 2$ waves dominates over the other and a stable swirling motion, responsible of the system symmetry-breaking, is established. In this case, the total solution is given by the sum of a harmonic planar SC wave and a super-harmonic swirling DC wave (swirling DC+planar SC, green shaded region in figure 5.6). The white-dotted region and the light red shaded regions in figure 5.6 correspond, respectively, to the super-harmonic irregular motion regime (see $\$ 5.5$ for further details) and to the multi-solution range where both types of motion are possible depending on the initial conditions, i.e. to the region of hysteresis.


Figure 5.5 - Typical response curve for $a$ and $b$ for a fluid depth $H=1.5$ with longitudinal superharmonic forcing of amplitude $a_{x}=0.2$. Panel (a) shows a projection of the three-dimensional branch structure $\left(\Omega / \omega_{21},|a|,|b|\right)$ in the $\left(\Omega / \omega_{21},|a|\right)$-plane, whereas panel (b) shows the same projection, but on the $\left(\Omega / \omega_{21},|b|\right)$-plane. Black solid lines mark stable steady-state planar waves, whereas light blue solid lines indicate stable steady-state swirling waves. Dashed lines denote the corresponding unstable branches. U: turning point. H: Hopf bifurcation. P: Poincaré bifurcation. For completeness, the phase values $\Phi_{A}=\Phi_{B}=\Phi=0$ or $\pi$ associated to each branch are reported in panel (a).

### 5.5 Experiments

In this section, we present our experimental set-up dedicated to the generation and characterization of sloshing waves under longitudinal super-harmonic forcing with driving (dimen-

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Figure 5.6 - Estimates of bounds, in the $\left(\Omega / \omega_{21}, a_{x}\right)$-plane, between the frequency ranges where planar, irregular and swirling waves occur when the container undergoes a longitudinal and super-harmonic motion at a forcing frequency $\Omega \approx \omega_{21} / 2$. In this range of frequency, the theory predicts the superposition of an unconditionally stable planar single-crest (SC) wave ( $m= \pm 1$ ) oscillating harmonically with the driving frequency and a super-harmonic doublecrest ( DC ) dynamics ( $m= \pm 2$ ), which can manifest itself via planar, swirling or irregular wave motions. The stability boundaries (black solid lines) were computed for a fluid depth $H=1.5$, as in Royon-Lebeaud et al. (2007). The corresponding values of the normal form coefficients appearing in (5.31a)-(5.31b) are given in table 5.1.
sionless) frequency $\Omega \approx \omega_{21} / 2$. The bounds between the different regimes for the resulting super-harmonic wave are experimentally retrieved as a function of the driving amplitude and frequency, and compared to the theoretical estimates. Finally, we measure the wave amplitude saturation in the vicinity of the super-harmonic resonance and compare it with the theoretical weakly nonlinear prediction (5.35).

### 5.5.1 Experimental set-up

The experimental set-up used to generate the sloshing waves in the cylindrical container and to observe the resulting free-surface motion is shown in figure 5.7. A Plexiglas cylindrical container of height 50 cm and inner diameter $D=2 R=17.2 \mathrm{~cm}$, partially filled with a column of distilled water of height $h=11 \mathrm{~cm}$, is fixed on a single-axis linear motion actuator (AEROTECH PRO165LM). Sloshing waves are generated by imposing to the container a longitudinal sinu-


Figure 5.7 - Experimental apparatus.
soidal forcing of angular frequency $\bar{\Omega}$ and amplitude $\bar{a}_{x}$.
The motion of the fluid free surface is recorded with a digital camera (NIKON D850) coupled with a Nikon $60 \mathrm{~mm} \mathrm{f} / 2.8 \mathrm{D}$ lens and operated in slow motion mode, allowing for an acquisition frequency of 120 frames per second. The optical axis of the camera is aligned with the container motion axis. A LED panel (not depicted in Figure 5.7) placed behind the tank provides back illumination of the fluid-free surface for a better optical contrast.
The actuation of the moving stage as well as the camera triggering for movie recording are set and controlled via a home-made Labview program. In a typical experiment, the container undergoes a harmonic motion of fixed amplitude in the range $4 \mathrm{~mm} \leq \bar{a}_{x} \leq 34 \mathrm{~mm}$ (i.e. $\left.a_{x}=\bar{a}_{x} / R \in[0.05,0.40]\right)$, while a sweep in forcing frequency is implemented within the inter$\operatorname{val} \bar{\Omega} / 2 \pi \in[1.35 \mathrm{~Hz}, 1.58 \mathrm{~Hz}]$ corresponding to the dimensionless range $\Omega / \omega_{21} \in[0.45,0.53]$. Each frequency step lasts 100 oscillation periods while the frequency increment between two consecutive steps is typically of 10 mHz . Along the sweeping, a movie is recorded for each ( $\bar{a}_{x}, \bar{\Omega}$ ) set of parameters. To ensure that the steady-state amplitude regime is established at each step in the recorded free-surface dynamics, the camera is triggered only after a certain number of cycles, typically 50, see Appendix 5.7.3.

### 5.5.2 Analysis of the free-surface dynamics

## Qualitative observations

While operating a sweep in forcing frequency at fixed forcing amplitude, we observe in the vicinity of the super-harmonic resonance three different kinds of motion, namely planar, irregular and swirling ones, whose occurrence depends on the forcing amplitude and frequency,

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Figure 5.8 - Images of the fluid-free surface while the container is subjected to a longitudinal harmonic forcing of amplitude $a_{x}=\bar{a}_{x} / R \approx 0.23$ at various driving angular frequencies $\Omega$ close to $\omega_{21} / 2$. The fluid-free surface is observed in the direction aligned with the container motion. For each driving frequency (a), (b) and (c), the time interval between two snapshots is about $T / 4$, with $T=2 \pi / \Omega$ the corresponding oscillation period. On each snapshot, the vertical middle axis is represented by a red dotted line. For a forcing frequency $\Omega \approx 0.48 \omega_{21}$ (a) and $\Omega \approx 0.52 \omega_{21}$ (c) the free-surface image at each time $t$ is mirror-symmetric with respect to the middle vertical axis, a signature of a planar wave regime, while the symmetry is broken for $\Omega \approx 0.50 \omega_{21}$ (c) revealing a swirling wave.
see for instance the snapshots displayed on figure 5.8.
For a given (and large enough) amplitude and starting from a frequency higher than a certain amplitude-dependent threshold $\Omega_{P}\left(a_{x}\right)$, the free surface responds to the longitudinal harmonic forcing by displaying a planar dynamics such as shown in figure 5.8(c). When the critical frequency $\Omega=\Omega_{P}\left(a_{x}\right)$ is reached, the motion bifurcates to a swirling wave, which propagates along the container wall with a stationary amplitude, see figure 5.8(b). The wave can rotate either clockwise or anti-clockwise (both rotation directions were observed during the experiments). When the forcing frequency is further decreased below a critical frequency $\Omega=\Omega_{H}\left(a_{x}\right) \approx \omega_{21} / 2<\Omega_{P}\left(a_{x}\right)$, the free surface exhibits an irregular dynamics, characterized by a switching between planar and swirling motion (not shown in figure 5.8). For forcing frequencies lower than a certain threshold $\Omega<\Omega_{U}\left(a_{x}\right)$, the free surface motion stabilizes into a steady planar wave such as shown on figure 5.8(a).
All together, these observations are qualitatively consistent with the outcomes of the weakly nonlinear analysis of Section 5.4, that predicts the existence of three different dynamical
regimes -namely planar, irregular and swirling motion-, for a longitudinal forcing frequency in the vicinity of $\omega_{21} / 2$. One of the main purposes of the present experimental investigation is to determine the amplitude-dependent frequency bounds of these different regimes and to compare them to our theoretical prediction of the positions of the bifurcation points $U$ (turning point), H (Hopf bifurcation) and P (Poincaré bifurcation) (see figures 5.5 and 5.6).


Figure 5.9 - General procedure for the analysis of the free surface dynamics. (a) On each frame, the edges of the container are detected (black dotted lines) and the vertical $Z(t)$ axis is set as the middle line between these two edges, while the scale of the horizontal direction $Y(t)$ is fixed by the distance between both edges. (b) Schematic of the container illustrating the link between the Cartesian coordinate system $(Y(t), Z(t))$ attached to each frame, and the cylindrical coordinates in the referential frame of the container. (c) Left, the intensity profile along a vertical line of coordinate $(Y(t)=y)$ with $y \in[-R, R]$ is then measured on each frame $t$ and plot as a function of time (here for $y=0$ ). The position of the front contact line at the azimuthal coordinate $\theta=\arcsin (y / R)=0$ as a function of time is highlighted in red. Right, frames from which the intensity profiles at times $t_{i}$ and $t_{j}$ on the left-hand side image, along the line $(Y(t)=0)$ (represented by a red dotted line), are extracted. At $t_{i}$, the wave is climbing the front wall of the container (with respect to the camera position) whereas at $t_{j}$, it reaches the back of the tank.

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## Procedure

Since the camera optical axis is aligned with the direction of the container motion, we note that a planar wave is characterized by its symmetry with respect to the vertical middle axis of the container image, whereas a swirling wave breaks this symmetry while travelling clockwise or anti-clockwise along the container walls, see figure 5.8.
We take benefit of these observations to build a more quantitative description of the freesurface dynamics, with the aim of identifying the various types of sloshing waves in the vicinity of the super-harmonic resonance. This can be done by exploiting the symmetry properties of the image of the free surface response with respect to the vertical middle axis of the container image, and by characterizing the regularity of these waves as a function of the forcing parameters, so as to identify the irregular regime.
To do so, the time evolution of the free surface dynamics is extracted from the movies along vertical directions that are mirror-symmetric with respect to the vertical middle axis of the container image. Comparing the resulting temporal signals with each other allows one to discriminate between planar and swirling motions and to study the wave regularity.
The first step is to attach to each frame $t$ of a given movie, a Cartesian reference frame ( $Y(t), Z(t)$ ), such that $Y(t)=0$ corresponds to the vertical middle axis of the container image, and that $Y(t)=R$ represents the right-hand-side edge of the container image. To this end, the edges of the container are automatically detected in a dedicated Matlab program. The vertical $Z(t)$ axis $(Y(t)=0)$ on the frame corresponding to time $t$ is then set as the middle line between these two edges, while the distance between both edges sets the scale of the horizontal direction $Y$. Note that we neglect the 4 mm thickness of the container wall.
A direction $y \in[-R, R]$ is then chosen to extract from each frame corresponding to time $t_{i}$, the intensity profile $I_{t_{i}}(y)$ along the vertical line $Y\left(t_{i}\right)=y$. The resulting intensity profiles are then plotted as a function of time to build an image $I(y)$ composed as $I(y)=\left[I_{t_{1}}(y), I_{t_{2}}(y), \ldots\right]$ such as displayed in figure 5.9 (c).

We note that at each time $t$, the intensity profile $I_{t}(y)$ contains the intersection of the front contact line image with the vertical axis $(Y(t)=y)$, that corresponds to the point of coordinates ( $R, \theta, \eta(R, \theta, t)$ ) in the moving cylindrical frame of reference of the container, where $\theta=\arcsin (y / R)$ (see figure $5.9(\mathrm{~b})$ ). As a consequence, the final image $I(y)$ also contains the dynamics of the front contact line in the azimuthal direction $\theta$.
The resulting image $I(y)$ exhibits a periodic dark pattern that represents the free surface response to the harmonic forcing, see an example in figure 5.9 (c) in which $y=0$. Indeed, on each frame of the movie, the free surface appears as the darkest feature, so that the intensity profile along a given line $(Y(t)=y)$ actually represents the vertical extension of the free surface at time $t$ along this direction, which is maximal whenever the sloshing wave reaches its maximal elevation $\max _{t} \eta(R, \theta, t)$ along the azimuthal direction $\theta=\arcsin (y / R)$ (in the front of the container with respect to the camera position, corresponding to $\theta \in]-\pi / 2, \pi / 2$ [) or along $\theta=\pi-\arcsin (y / R)$ (in the back of the container). Furthermore, when the contact line reaches its maximal elevation in the front of the container, the free surface is imaged from below, so that it appears darker than when the maximal elevation is reached in the back, where the free
surface is imaged from above, see the snapshots in figure 5.9(c). These observations allow us to identify in the image $I(y)$ the position as a function of time of the front contact line $\eta(R, \theta, t)$, with $\theta=\arcsin (y / R)$, as highlighted in red in figure $5.9(\mathrm{c})$, and following the method detailed in Appendix 5.7.4.
Note that this procedure does not give a quantitative access to the actual amplitude of the front contact line oscillations, since the intensity profiles $I_{t_{i}}(y)$ constituting the image $I(y)$ are simply juxtaposed with each other without rescaling the pixel width along the vertical direction. However, the position extracted from $I(y)$ of the image of the points of coordinates $\eta(R, \pm \theta, t)$ as a function of time still encloses the symmetry properties of the free surface response, its regularity as well as its frequency content, which are the only quantities needed in order to identify the wave regimes.

### 5.5.3 Regularity and frequency content of the free surface response

The resulting image $I(y)$ is then revealing of the free surface dynamics $\eta(r, \theta, t)$ and in particular of its dynamics at the front wall $\eta(r=R, \theta=\arcsin (y / R), t)$. Figure 5.10(a)-(d) displays $I(0)$ for various forcing frequencies close to the super-harmonic resonance, at the same forcing amplitude. These images reveal that depending on the forcing frequency, the free surface oscillations (dark periodic pattern) can be either regular (a), (c) and (d) -i.e. the oscillations are enclosed into an envelope of constant amplitude- or irregular (b) with a temporal modulation of the amplitude envelope. Therefore, the profiles $I(0)$ allow us to characterize the regularity of the sloshing wave, and in particular to identify the irregular regime. The latter will be described in more detail in Section 5.5.6, but such details are not needed for the identification of the irregular regime bounds, for which the analysis of the regularity property of the $I(0)$-pattern is sufficient. Therefore in the following, we will focus on the regular planar and swirling motions, that cannot be unambiguously distinguished from each other on the basis of the profiles $I(0)$.
Figure 5.10 (e)-(h) displays the (normalized) power spectral densities of the front contact line dynamics $\eta(R, \theta=0, t)$ extracted from the profiles $I(0)$ (a)-(d). It appears that in all cases, the energy of the sloshing wave is massively distributed to its first (harmonic) and second (super-harmonic) components, while the contribution of higher modes is fairly negligible. This incidentally implies that the symmetry properties of a regular wave are directly linked to the symmetry properties of these two first oscillation modes.
In other words, a planar dynamics should necessarily consist of the superposition of two planar waves: a planar single-crest (SC) wave harmonically oscillating with the driving frequency $\Omega$ and one super-harmonic planar double-crest (DC) wave oscillating at $\omega_{D C}=2 \Omega \approx \omega_{21}$. On the other hand, a swirling dynamics must contain at least one symmetry-breaking (swirling) component that, as predicted by the present weakly nonlinear analysis, should correspond to the super-harmonic $\omega_{21}$ component.

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Figure 5.10 - Panels (a) to (d): intensity profiles as a function of time along the vertical middle axis $(Y=0)$-denoted $I(0)$ - for various forcing frequencies $\Omega$ in the vicinity of the superharmonic resonance $\Omega \approx \omega_{21} / 2$ at same forcing amplitude $\bar{a}_{x} / R \approx 0.23$. In each case, the profile $I(0)$ is extracted from a movie whose recording has been started after about 50 oscillation cycles following each change in forcing frequency, thus ensuring that initial transients are filtered out (see also Appendix 5.7.3). Panels (e) to (h) Power spectral densities (PSD) -normalized by the maximal peak amplitude- corresponding to the front contact line dynamics as extracted from the profiles $I(0)$ displayed in panels (a) to (d).

### 5.5.4 Symmetry properties of the regular regimes: planar versus swirling waves

We now focus on the regular regimes, namely the steady planar and swirling motions. As stated before, the profiles $I(0)$ cannot discriminate between a planar and a swirling dynamics and instead only contain information on their regularity and their frequency content. To distinguish a planar from a swirling motion, we then compare the profiles along two ( $Y \neq 0$ )-


Figure 5.11 - Symmetry properties of the stationary waves. (a), (c) and (e): images of the fluidfree surface while the container is subjected to a longitudinal harmonic forcing of amplitude $\bar{a}_{x} / R \approx 0.23$ at various driving angular frequencies $\Omega$ close to $\omega_{21} / 2$ : (a) $\Omega \approx 0.48 \omega_{21}$, (c) $\Omega \approx 0.50 \omega_{21}$ and (e) $\Omega \approx 0.52 \omega_{21}$ (same forcing parameters as in figure 5.10(a), (c) and (d)). For each driving frequency ( $\mathrm{a}, \mathrm{c}, \mathrm{e}$ ), the time interval between two snapshots is about $T / 4$, with $T=2 \pi / \Omega$ the corresponding oscillation period. On each snapshot, the vertical axes $(Y=R / 2)$ and $(Y=-R / 2)$ are represented by a blue and red dotted line, respectively. For a forcing frequency $\Omega=0.48 \omega_{21}$ (a) and $\Omega=0.52 \omega_{21}$ (e) the free-surface image at each time $t$ is mirror-symmetric with respect to the middle vertical axis, while the symmetry is broken for $\Omega=0.50 \omega_{21}$. (b), (d) and (f): superposition of the intensity profiles as a function of time along the vertical axis $(Y=R / 2)$ and $(Y=-R / 2)$-denoted $I(R / 2)$ (in blue) and $I(-R / 2)$ (in red) respectively-, for the same forcing parameters as in figure 5.10(a), (c), and (e). The grey regions show where $I(R / 2)$ and $I(-R / 2)$ have the same intensities.

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directions that are symmetric with respect to the vertical middle axis of the container image.
Figure 5.11 (b), (d) and (f) show composite images, each produced using the Matlab function imshowpair applied to the pair $I(R / 2)$ and $I(-R / 2)$, for three different forcing frequencies that both result in a regular motion (same forcing parameters as in figure 5.10(a), (c), and (d)). Briefly, imshowpair $\left(I_{\alpha}, I_{\beta}\right)$ creates from a pair of grayscale images $I_{\alpha}$ and $I_{\beta}$, a RGB image where each pixel is represented by a RGB triplet, the R-intensity being the intensity of the corresponding pixel in $I_{\alpha}$, and the G- and B-intensities being equal to the intensity of the corresponding pixel in $I_{\beta}$. A pixel where $I_{\alpha}$ and $I_{\beta}$ have the same intensity will be represented by a RGB triplet of the forme $[a, a, a]$, where $a \in[0,255]$, i.e. will appear as grey. On the contrary, if this pixel has a much larger intensity on $I_{\alpha}$ (resp. on $I_{\beta}$ ) than it has on $I_{\beta}$ (resp. on $I_{\alpha}$ ), it will appear in red (resp. in cyan) on the resulting composite image. The composite images displayed in Figure 5.11(b), (d) and (f) thus highlight in each case the differences between $I(R / 2)$ and $I(-R / 2)$. They are then a direct signature of the symmetry of the freesurface dynamics with respect to the vertical middle axis ( $Y=0$ ), and reveal two different kinds of motion, (i) a planar motion, for which $I(R / 2)$ and $I(-R / 2)$ perfectly overlap with each other due to the mirror-symmetry of the wave, and (ii) a circular motion, characterized by a symmetry-breaking between the right and left hand-side free-surface dynamics: the maximum of the wave along $\theta=\arcsin (1 / 2)=\pi / 6$ is indeed phase-shifted with respect to the maximum of the wave along $\theta=-\pi / 6$, thus revealing a travelling wave propagating along the wall of the container.
To determine which $\omega$-component is responsible for the symmetry-breaking induced by the swirling motion, we extract from $I(R / 2)$ and $I(-R / 2)$ the position of the front contact line as a function of time $\eta(R, \theta, t)$ where $\theta= \pm \pi / 6$, see figure 5.12(b), (e). This makes it possible to compute the power spectrum of both signals, as well as the phase difference between the phase angle of their components that oscillate at the frequencies corresponding to their spectrum's first and second peaks (see figure $5.12(\mathrm{c}-\mathrm{f})$ ). A planar wave oscillating at a frequency $\omega$ is then characterized by the $\omega$-components of $\eta(R, \pi / 6, t)$ and of $\eta(R,-\pi / 6, t)$ being in phase with each other, while a swirling wave is characterized by a $m \pi / 3$-phase shift between the $\omega$-components of these signals, where $m$ denotes the azimuthal wavenumber of the swirling wave ( $m=1$ for a harmonically oscillating single-crest wave, $m=2$ for a super-harmonic double-crest wave).
The Fourier analysis of the signals $\eta(R, \pi / 6, t)$ and $\eta(R,-\pi / 6, t)$ reveals that for forcing frequencies $\Omega$ close to $\omega_{21} / 2$, the free surface motion mostly results from the combination of a single-crest wave harmonically oscillating at the forcing frequency $\omega_{S C}=\Omega \approx \omega_{21} / 2$, and of a super-harmonic double-crest wave oscillating at a frequency $\omega_{D C}=2 \Omega \approx \omega_{21}$. From figure 5.12 (c) and ( f ), it is clear that the single-crest wave is a planar wave for both planar (figure $5.12(\mathrm{c})$ ) and swirling (figure 5.12(f)) dynamics, as revealed by the vanishing phase-shift between the harmonic components of $\eta(R, \pi / 6, t)$ and $\eta(R,-\pi / 6, t)$ in both cases. On the other hand, the phase shift between the super-harmonic components is zero in the case of the planar dynamics, and close to $2 \pi / 3$ in the case of the swirling dynamics.


Figure 5.12 - Analysis of the steady-state free-surface dynamics under a harmonic forcing of amplitude $a_{x}=0.23$ and frequency (a)-(c) $\Omega / \omega_{21} \approx 0.48$ and (d)-(f) $\Omega / \omega_{21} \approx 0.50$. (a) and (d) Image of the free surface with vertical lines intersecting the image of the front contact line at the points of coordinates $(R, \pi / 6, \eta(R, \pi / 6, t)$ ) (blue arrows) and ( $R,-\pi / 6, \eta(R,-\pi / 6, t)$ ) (red arrows) in the moving reference frame of the container. (b) and (e) Normalized elevation of the front contact line $\tilde{\eta}(R, \pi / 6, t)$ (blue dots) and $\tilde{\eta}(R,-\pi / 6, t)$ (red dots) extracted from the corresponding profiles $I(R / 2)$ and $I(-R / 2)$ (not shown here). The $\tilde{\eta}$-functions are defined according to $\tilde{\eta}(R, \theta, t)=(\eta(R, \theta, t)-\sigma) / \delta$, where $\sigma=\left(\min _{t}\left(\eta(R, \theta, t)+\max _{t}(\eta(R, \theta, t)) / 2\right.\right.$ and $\delta=\max _{t}(\eta(R, \theta, t))-\min _{t}(\eta(R, \theta, t))$. (c) and (f) Left, Power spectral densities of $\tilde{\eta}(R, \pi / 6, t)$ (blue dots) and of $\tilde{\eta}(R,-\pi / 6, t)$ (red dots). Right, Absolute value of the phase shift between the components of $\tilde{\eta}(R, \pi / 6, t)$ and of $\tilde{\eta}(R,-\pi / 6, t)$ oscillating at the frequencies corresponding to the first peak $(\omega=\Omega)$ and to the second peak $(\omega=2 \Omega)$ of the power spectra.

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These observations are general to the whole range of forcing frequencies and amplitudes investigated along this study: in the vicinity of the super-harmonic resonance, the single-crest wave is always a planar wave, as revealed by the vanishing phase-shift between the harmonic components of $\eta(R, \pi / 6, t)$ and $\eta(R,-\pi / 6, t)$ for both planar and swirling dynamics (this is also true in the case of the irregular regime, see later Section 5.5.6). In the case of a regular dynamics, the double-crest wave is either a planar (for vanishing phase-shift between the corresponding components) or a swirling wave (characterized by a $2 \pi / 3$ phase-shift between the $\omega_{21}$-component of the right and left-hand-side signals), depending on the exact ratio between $\Omega$ and $\omega_{21}$, as well as on the forcing amplitude $\bar{a}_{x} / R$.

### 5.5.5 Experimental estimate of regime bounds

From the above analysis, it appears that consistently with the predictions provided by our theoretical weakly nonlinear analysis, the sloshing waves resulting from the longitudinal super-harmonic forcing of the container at a frequency $\Omega \approx \omega_{21}$, consist in the superposition of a planar single-crest wave, harmonically oscillating with the forcing at $\omega=\Omega$, and of a double-crest wave, that can exhibit either a planar, irregular or swirling dynamics.
Having identified the three different regimes for the free-surface dynamics in the vicinity of the super-harmonic resonance, we can now experimentally determine their stability regions in the $\left(\Omega / \omega_{21}, a_{x}\right)$ space. To do so, we fix the forcing amplitude while operating a frequency sweep from high to low frequencies, within the range $\Omega / \omega_{21} \in[0.45,0.53]$, by frequency decrements of 10 mHz . Note that a downward frequency sweep ensures to recover the stability bound between the super-harmonic planar and swirling regimes, as the transition in this direction occurs exactly at the threshold frequency $\Omega_{P}\left(a_{x}\right)$ below which the super-harmonic planar motion becomes unstable. On the contrary, since the super-harmonic swirling wave is still stable for frequencies larger than $\Omega_{P}\left(a_{x}\right)$ (hysteresis), an upward frequency sweep will maintain the system's response on the swirling branch, thus it is not suitable to experimentally detect the bifurcation point $P$.
The downward frequency sweep also enables one to detect the bounds that separate the irregular regime from steady planar $\left(\Omega=\Omega_{U}\left(a_{x}\right)\right)$ and swirling motions $\left(\Omega=\Omega_{H}\left(a_{x}\right)\right)$.

This procedure is applied for various forcing amplitudes $a_{x} \in[0.05,0.4]$, enabling us to build the stability regions diagram displayed on figure 5.13. All together, the experimental measurements are in very good quantitative agreement with the theoretical regime bounds for $a_{x}>0.15$, below which the super-harmonic irregular and swirling regimes appear to be suppressed by dissipative mechanisms, e.g. viscous dissipation occurring in the fluid bulk, sidewall and free surface boundary layers (Bongarzone et al., 2022b; Case and Parkinson, 1957; Miles, 1967; Raynovskyy and Timokha, 2020) as well as in the neighbourhood of the moving contact line (Cocciaro et al., 1993; Dussan, 1979; Hocking, 1987; Keulegan, 1959; Viola and Gallaire, 2018). This last contribution is likely to be important since no particular precautions, such as wall treatment or pre-wetting, have been taken in order to minimize contact angle hysteresis. Below this threshold amplitude, experiments have shown a vanishing super-harmonic contribution to the dynamics, with a harmonic planar motion produced by


Figure 5.13 - Estimates of regime bounds in the $\left(\Omega / \omega_{21}, a_{x}\right)$-plane for a container of diameter $D=0.172 \mathrm{~m}$, filled to a depth $H=1.3$, driven longitudinally and super-harmonically at a frequency $\Omega \approx \omega_{21} / 2$ : comparison between the theoretical predictions (solid lines) and experimental measurements (markers). Grey thick solid lines: present theoretical predictions. Black empty squares: super-harmonic planar motion. Black crosses: irregular regime. Black-filled circles: super-harmonic swirling motion.
the single-crest wave only and well described by the potential linear model, thus suggesting that the double-crest wave has been entirely killed by dissipative mechanism. Note that such a suppression of the DC dynamics at low forcing amplitude is reminiscent of what was observed by Reclari et al. (2014) for circular shaking.

### 5.5.6 Irregular regime

In this section, we provide a more thorough description of the irregular regime. When fixing the forcing frequency slightly below $\omega_{21} / 2$ and progressively increasing the forcing amplitude, the free-surface response is first very regular and displays a planar dynamics for low enough forcing amplitudes. Above a threshold amplitude, the dynamics becomes irregular and at large enough amplitudes, the response is again regular but consists of a swirling motion. Figure 5.14(a) displays the free surface response along the vertical middle axis $Y=0$ for increasing forcing amplitudes at a fixed forcing frequency $\Omega \approx 0.496 \omega_{21}$. The regular regimes (top and bottom panels) are characterized by a constant amplitude of the free surface oscillations. In contrast, the oscillations of the free surface for intermediate forcing amplitudes (second and

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Figure 5.14 - (a) Intensity profiles $I(0)$ for various forcing amplitudes and same forcing frequency $\Omega \approx 0.496 \omega_{21}$. The surface oscillations are enclosed into an envelope, plotted in black on top of the images. (b) Left, Frequency of the main peak in the envelope's power spectrum (non-dimensionalized by $\bar{\Omega} / 2 \pi$ ) as a function $a_{x}$, for same $\Omega \approx 0.496 \omega_{21}$, and for upwards (white markers) and downwards (black markers) amplitude sweeps. When the envelope is a straight line (as here for $a_{x}<0.15$, which corresponds to regular planar dynamics), the power spectrum is flat and we set the corresponding frequency equal to zero. For $a_{x} \approx 0.40$, the power spectrum of the envelope is dominated by a low amplitude and small frequency noise, causing a brutal decrease of the "burst" frequency, thus indicating a transition from irregular to regular swirling motion. Right, Power spectra of the envelope for $a_{x} \approx 0.12, a_{x} \approx 0.23$ and $a_{x} \approx 0.40$. (c) Correlation between $I(R / 2)$ and $I(-R / 2)$ versus time, for the same set of forcing parameters as in (a). (d) Left, Superposition of $I(R / 2)$ (blue) and $I(-R / 2)$ (red) and right, front contact line position $\eta(R, \pm \pi / 6, t)$ as a function of time extracted from $I(-R / 2)$ (red curve) and from $I(R / 2)$ (blue curve). The signals presented in (d) are taken from the full signals used to compute their correlation in (c) for $a_{x} \approx 0.23$, over the time ranges highlighted in blue and denoted as (1) (maximum of correlation) and as (2) (minimum of correlation). (e)-(f) Normalized power spectra of $\eta(R,-\pi / 6, t)$ (red curve) and of $\eta(R, \pi / 6, t)$ (blue curve), and absolute value of the phase-shift between their harmonic and super-harmonic components, where the dynamics of $\eta(R, \pm \pi / 6, t)$ is considered over (e): time range (1) and (f): time range (2).
third panels) are enclosed into a quasi-periodic envelope, whose frequency linearly increases with the forcing amplitude (see figure 5.14(b), and Appendix 5.7.4 for the methodology used to compute the envelope). This is very reminiscent of the observations by Royon-Lebeaud et al. (2007) of the irregular regime present in the vicinity of the harmonic resonance under longitudinal forcing. Note that these features are also quantitatively recovered by proceeding with a downward amplitude sweep. In particular, upward and downward forcing amplitude sweeps provide the same threshold amplitudes between planar and irregular dynamics, and between irregular and swirling regimes. Furthermore, the frequency of the main peak in the power spectrum of the amplitude envelope seems to be a robust feature that does not depend on the sweep direction, see Figure 5.14(b).
To gain more insight into this irregular dynamics, we compute at each time $t_{i}$ the spatial correlation between $I_{t_{i}}(R / 2)$ and $I_{t_{i}}(-R / 2)$, which we refer to as $\operatorname{corr}\left(t_{i}\right)$

$$
\begin{equation*}
\operatorname{corr}\left(t_{i}\right)=\frac{\sum_{n}\left(I_{t_{i}, n}(R / 2)-\bar{I}_{t_{i}}(R / 2)\right)\left(I_{t_{i}, n}(-R / 2)-\bar{I}_{t_{i}}(-R / 2)\right)}{\sqrt{\sum_{n}\left(I_{t_{i}, n}(R / 2)-\bar{I}_{t_{i}}(R / 2)\right)^{2} \sum_{n}\left(I_{t_{i}, n}(-R / 2)-\bar{I}_{t_{i}}(-R / 2)\right)^{2}}}, \tag{5.37}
\end{equation*}
$$

where $n \in[1, N]$, with $N$ the number of pixels in the vertical direction, and $\bar{I}_{t_{i}}(y)$ represents the mean of the N-element vector $I_{t_{i}}(y)$. A high and constant correlation is a sign of a steady planar motion, while a low but still constant correlation is characteristic of the steady swirling regime. At intermediary forcing amplitudes -i.e. in the irregular regime- the correlation is a quasi-periodic function of time, with the same quasi-period as the envelope, see figure 5.14(c). A comparison between $I(R / 2)$ and $I(-R / 2)$ on time ranges corresponding to the maximum and minimum of the correlation function reveals that in the time interval where the signals are highly correlated, the motion is planar-like (although irregular), while in the time range where they are poorly correlated, the maxima of the right and left-hand-side signals are phase-shifted with respect to each other, thus reflecting the presence of a swirling wave, see figure $5.14(\mathrm{~d})$.

This is further confirmed by the power spectra of the front contact line dynamics along the azimuthal directions $\theta= \pm \pi / 6$, extracted from $I(R / 2)$ and $I(-R / 2)$, on time ranges where these signals are highly correlated and where they are poorly correlated, see figure 5.14(e-f). In both cases, the sloshing wave contains a planar single-crest wave, as revealed by the vanishing phase-shift between the harmonic components of $\eta(R, \pi / 6, t)$ and $\eta(R,-\pi / 6, t)$. The wave also contains a super-harmonic component, that is responsible for the switching between a planar-like motion (vanishing phase-shift between the $\omega_{21}$-components of the $\eta(R, \pi / 6, t)$ and $\eta(R,-\pi / 6, t)$ signals, figure $5.14(\mathrm{e})$ ) and a swirling dynamics (rotating, symmetry-breaking wave that is super-harmonically oscillating at $\omega \approx \omega_{21}$, figure $5.14(\mathrm{f})$ ).
This is again very similar to the features of the irregular regime in the vicinity of the harmonic resonance described by Royon-Lebeaud et al. (2007) that relate the "bursts" in the free-surface oscillation amplitude to the quasi-periodic occurrence of a swirling wave. However, in the case of super-harmonic resonance, the irregular regime consists here of the superposition of a stable planar single-crest wave and of super-harmonic double-crest dynamics. The latter is responsible for the irregularity of the total dynamics, by quasi-periodically switching between

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$a_{x}=0.2093$

$a_{x}=0.2558$



$$
a_{x}=0.2325
$$


$a_{x}=0.2791$


Figure 5.15 - Quantitative comparison with experimental measurements in terms of finite amplitude saturation for various non-dimensional forcing amplitudes, $a_{x}$. Black-dotted lines: linear potential solution according to (5.7) and (5.8). Light blue solid lines: stable planar and swirling branches predicted by the present weakly nonlinear (WNL) model according to (5.35). Markers: experimental measurements. Black empty squares correspond to planar motion whereas black-filled circles refer to swirling dynamics.
super-harmonic planar and swirling motion.

### 5.5.7 Wave amplitude saturation: theoretical predictions versus experiments

In this last section, we provide a more quantitative comparison in terms of wave amplitude saturation between the theoretical predictions according to (5.35) and the experimental measurements. On this point, the dimensional wave amplitude, $\Delta \bar{\delta}=\max _{\theta, t} \eta(r=R, \theta, t)-$ $\min _{\theta, t} \eta(r=R, \theta, t)$, is experimentally measured by fixing the forcing amplitude while operating a frequency sweep in two directions. A backward sweep is used so as to follow the right lower planar branch until the sub-critical jump-up transition to swirling (P: Poincaré bifurcation) occurs ( $\Omega=\Omega_{P}\left(a_{x}\right)$ ). On the other hand, an upward sweep is performed in order to maintain a stable super-harmonic swirling response from bifurcation point $H\left(\Omega=\Omega_{H}\left(a_{x}\right)\right)$ and beyond the threshold frequency $\Omega=\Omega_{P}\left(a_{x}\right)$, above which the super-harmonic planar and swirling motions are both stable solutions (right region in the stability chart of figure 5.13).
For each set of forcing parameters ( $a_{x}, \Omega / \omega_{21}$ ), the height in pixel of the wave crest (resp.
trough) on the front wall is manually extracted from the corresponding movies and is compared, in the same frame it is extracted from, to the height of the fluid at rest (flagged by a black mark on the container, also used as a scale) to obtain the maximal (resp. minimal) front contact line elevation $\max _{\theta, t} \eta(r=R, \theta, t)\left(\right.$ resp. $\left.\min _{\theta, t} \eta(r=R, \theta, t)\right)$. This value is converted into meters using the conversion factor provided by the black scale. The resulting amplitude $\Delta \bar{\delta}$ is then averaged over 3 to 5 cycles of oscillations and finally normalized by the container radius $R$.
The experimental dimensionless wave amplitude $\Delta \delta=\Delta \bar{\delta} / R$ as a function of the forcing frequency for various forcing amplitudes is displayed in figure 5.15 together with the theoretical weakly nonlinear prediction (5.35) (light blue solid lines) and with the linear potential solution (5.7) for comparison (black dashed line).
The experimental data associated with the two planar branches compare generally well with the present weakly nonlinear prediction, although the WNL theory slightly underestimates the wave amplitude in the swirling regime. We recall from $\S 5.4$ that, at leading order, the wave solution in these two branches is made by the superposition of two planar waves, i.e. a harmonic planar single-crest component, oscillating in space and time as $\cos (\Omega t) \cos \theta$, and a super-harmonic planar double-crest component, characterized by $\cos (2 \Omega t+\Phi) \cos 2 \theta$, with a phase $\Phi=\pi$ in the left branch and $\Phi=0$ in the right one. The information on the phase $\Phi$ is not directly discernible from the amplitude plot of figure 5.15 , but it is contained in the snapshots sequence reported in figure 5.11 (a) for $\Omega / \omega_{21}<0.5$ and (e) for $\Omega / \omega_{21}>0.5$. Due to the temporal periodicity of the single-crest wave, snapshots taken at $t=T / 4=\pi / 2 \Omega$ and $t=3 T / 4=3 \pi / 2 \Omega$ represent temporal nodes for the harmonic component, so that, as a firstorder approximation, only the double-crest component, whose azimuthal spatial structure reads $\cos (\pi+\Phi) \cos 2 \theta$, is instantaneously left. It is then clear that for $\Omega / \omega_{21}<0.5$ and $\Phi=\pi$, the free surface maximum is reached at the azimuthal coordinates $\theta=0$ and $\pi$, whereas the minimum is at $\theta= \pm \pi / 2$ (vice versa for $\Omega / \omega_{21}>0.5$ and $\Phi=0$ ). This produces the concave and convex shapes in the instantaneous free surface displayed in figure 5.11 (a) and (e), respectively.
Consistently with the stability chart in figure 5.13 obtained through a backward frequency sweep, the threshold frequency $\Omega_{H}\left(a_{x}\right)$, at which the swirling branch becomes stable from lower driving frequency $\Omega$, is correctly detected. Furthermore, the upward sweep allows us to detect also the jump-down transition from the swirling to the lower right planar branch.
The occurrence of the jump-down transition was to be expected as it is produced by dissipative mechanisms (see also §5.5.5), which are overlooked by the present inviscid analysis. The associated damping, which is a function of the wave amplitude and of the forcing acceleration amplitude (see Raynovskyy and Timokha (2018b),Raynovskyy and Timokha (2020) and the discussion in Appendix A of Bongarzone et al. (2022a)), is responsible for the modulation in the phase lag between the external driving and the wave response, which was shown by Bäuerlein and Avila (2021) (for unidirectional sloshing waves in a rectangular container) to be of crucial importance for a correct prediction of the jump-down frequency.
The damping coefficient could be tentatively fitted from experiments and phenomenologically introduced a posteriori in amplitude equations (5.31a)-(5.31b) as done in Appendix A
of Bongarzone et al. (2022a). Nevertheless, the jump-down transition in the cases examined in this section (see figure 5.15) was seen to be extremely sensitive to the frequency sweeping rate. A decrease in the frequency step increment from 5 mHz to 1 mHz (used to produce the swirling branch in figure 5.15) was observed to give different jump-down frequencies. This is also expected as it is known from the literature that in the multi-solution range, the characteristic of the response mainly depends on the sweep rate (Bourquard and Noiray, 2019; Park et al., 2011; Yu et al., 2020). Since we did not try frequency increments smaller than 1 mHz , the jump-down frequency predictions as shown in figure 5.15 are not entirely reliable for fitting the damping at stake in the experiments.
In spite of such limitations, the weakly nonlinear model is seen to describe fairly well the experimental swirling branch until the measured jump-down frequency. A relatively small departure of the swirling response from the theoretical prediction is typically observed at larger driving amplitude for increasing wave frequency. In agreement with previous studies (Bäuerlein and Avila, 2021; Dodge et al., 1965; Ibrahim, 2005), our experiments reveal that this is due to the progressive steepening and broadening of the wave crest and troughs, respectively, in the vicinity of the container wall. This nonlinear mechanism eventually becomes strong enough for the weakly nonlinear model to lose accuracy.

### 5.6 Conclusion

In this work, the behaviour of sloshing waves in a cylindrical container submitted to longitudinal periodic forcing with driving amplitude $a_{x}$ and angular frequency $\Omega$ was investigated. While previous studies of this forcing condition and geometry mostly focused on the investigation of the free surface response in the vicinity of harmonic resonance, i.e. $\Omega / \omega_{1 n} \approx 1$, the core of the present work was dedicated to the most relevant secondary super-harmonic resonances $\Omega / \omega_{2 n} \approx 1 / 2$, characterized by the occurrence of a double-crest (DC) dynamics oscillating at a frequency $\omega=2 \Omega \approx \omega_{2 n}$.
Such a super-harmonic resonance was first experimentally observed by Reclari (2013) and Reclari et al. (2014) for circular container motions, but its investigation under different forcing conditions, e.g. longitudinal forcing, seemed to be still unreported.
With the aim to take a further step in this direction, a weakly nonlinear analysis (WNL) via multiple timescale method together with a dedicated experimental campaign were implemented in order to account for the steady-state free surface dynamics, and for the symmetrybreaking due to the emergence of a double crest swirling wave in the vicinity of the superharmonic resonance.
In a similar fashion to Bongarzone et al. (2022a), the WNL analysis was first formalized to tackle the simpler case of harmonic resonances. The outcomes of the model were compared to previous experimental measurements and to former theoretical predictions based on the Narimanov-Moiseev multimodal sloshing theory (Faltinsen et al., 2016; Raynovskyy and Timokha, 2020). All together, our analysis addressing the single-crest (SC) wave dynamics was shown to be consistent with the previously reported experimental and theoretical results. In
particular, the WNL model successfully captured the regime bounds between single-crest planar, swirling and irregular waves, and correctly described the close-to-resonance nonlinear behaviour, thus validating the relevance of this theoretical approach.
The WNL analysis was then extended to the more complex case of the super-harmonic resonance. A dedicated lab-scale experiment was set up to observe and characterise the superharmonic response to longitudinal forcing. In remarkable agreement with the outcomes of the WNL model, the experimental investigation showed that the free surface dynamics in the vicinity of the super-harmonic resonance results from the superposition of a permanent, first-order forced harmonic planar single-crest wave, and of a super-harmonic double-crest wave that can exhibit either a planar, irregular or swirling dynamics, the latter being responsible for a symmetry-breaking in the system's response through equally probable clockwise or anti-clockwise swirling waves. The bounds in the ( $a_{x}, \Omega / \omega_{21}$ ) plane between the three different regimes were experimentally retrieved and were shown to be in very good quantitative agreement with the WNL predictions, at least above a threshold forcing amplitude, below which the swirling and irregular dynamics appear to be suppressed by dissipative mechanisms, which are not accounted for by the present inviscid analysis. Finally, the predicted wave amplitude saturation, computed by reconstructing the total flow solution, was compared to the experimentally measured steady-state wave amplitude and was shown to correctly describe the stable planar and swirling branches in the neighbourhood of the super-harmonic resonance.
The fairly good agreement between the theoretical predictions and the experimental findings validates the relevance of the WNL approach to successfully describe the sloshing wave dynamics resulting from nonlinear harmonic and super-harmonic interactions. As discussed in Appendix 5.7.2, this analysis is not restricted to longitudinal forcing, but can be straightforwardly generalized without any further calculation to any elliptic trajectory, hence recovering the limit of circular sloshing investigated in Bongarzone et al. (2022a). In this respect, the theory of Faltinsen et al. (2016) for elliptical container motions interestingly predicts the occurrence of counter-rotating swirling waves, i.e. propagating in the direction opposed to that of the container motion. The qualitative analogy between the harmonic and super-harmonic system behaviour outlined in this Chapter would suggest that such counter-propagating swirling waves could also be triggered by exciting the system in the vicinity of the super-harmonic double-crest resonance, thus calling for new experimental campaigns.

### 5.7 Appendix

### 5.7.1 Computation of the normal form coefficients

The normal form coefficients appearing in (5.18a)-(5.18b) for the harmonic single-crest (SC) dynamics are computed as follows

$$
\begin{equation*}
\mathrm{i} \mathscr{I}_{S C} \mu_{S C}=<\hat{\mathbf{q}}_{1}^{A_{1} \dagger}, \hat{\mathscr{F}}_{3}^{F}>=\int_{0}^{1}(r / 2) \overline{\hat{\eta}}_{1}^{A_{1} \dagger} r \mathrm{~d} r, \tag{5.38a}
\end{equation*}
$$

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Table 5.1 - Value of the normal form coefficients appearing in (5.18a)-(5.18b) (SC) and in (5.31a)-(5.31b) (DC) computed at different fluid depths $H=h / R$ and associated with mode $(m, n)=(1,1)$. Note that in (5.31a)-(5.31b), $\chi_{D C}=\chi_{-}+\chi_{+}$.

$$
\begin{align*}
& \mathrm{i} \mathscr{I}_{S C} v_{S C}=\left\langle\hat{\mathbf{q}}_{1}^{A_{1} \dagger}, \hat{\mathscr{F}}_{3}^{\left|A_{1}\right|^{2} A_{1}}>=\int_{0}^{1}\left(\overline{\hat{\eta}}_{1}^{A_{1} \dagger} \hat{\mathscr{F}}_{3_{\mathrm{dyn}}\left|A_{1}\right|^{2} A_{1}}+\overline{\hat{\Phi}}_{1}^{A_{1} \dagger} \hat{\mathscr{F}}_{3_{\mathrm{kin}}}^{\left|A_{1}\right|^{2} A_{1}}\right) r \mathrm{~d} r,\right.  \tag{5.38b}\\
& \mathrm{i} \mathscr{I}_{S C} \xi_{S C}=<\hat{\mathbf{q}}_{1}^{A_{1} \dagger}, \hat{\mathscr{F}}_{3}^{\left|B_{1}\right|^{2} A_{1}}>=\int_{0}^{1}\left(\overline{\hat{\eta}}_{1}^{A_{1} \dagger} \hat{\mathscr{F}}_{3_{\mathrm{dyn}}\left|B_{1}\right|^{2} A_{1}}+\overline{\hat{\Phi}}_{1}^{A_{1} \dagger} \hat{\mathscr{F}}_{3_{\mathrm{kin}}}^{\left|B_{1}\right|^{2} A_{1}}\right) r \mathrm{~d} r . \tag{5.38c}
\end{align*}
$$

where $\mathscr{I}_{S C}=<\hat{\mathbf{q}}_{1}^{A_{1} \dagger}, \mathscr{B} \hat{\mathbf{q}}_{1}^{A_{1}}>=\int_{0}^{1}\left(\overline{\hat{\eta}}_{1}^{A_{1} \dagger} \hat{\Phi}_{1}^{A_{1}}+\overline{\hat{\Phi}}_{1}^{A_{1} \dagger} \hat{\eta}_{1}^{A_{1}}\right) r \mathrm{~d} r$. Here $\left(\hat{\mathbf{q}}_{1}^{A_{1} \dagger}, \hat{\mathbf{q}}_{1}^{B_{1} \dagger}\right)=\left(\overline{\hat{\mathbf{q}}}_{1}^{A_{1}}, \overline{\hat{\mathbf{q}}}_{1}^{B_{1}}\right)$, since the inviscid problem is self-adjoint with respect to the Hermitian scalar product $<\mathbf{a}, \mathbf{b}>=$ $\int_{\Sigma} \overline{\mathbf{a}} \cdot \mathbf{b} \mathrm{d} V$, with $\mathbf{a}$ and $\mathbf{b}$ two generic vector (see Viola et al. (2018) for a thorough discussion and derivation of the adjoint problem).
Expressions (5.38a) and (5.38b) were already given in Bongarzone et al. (2022a). The lefthand side of those expressions was typed mistakenly, as the mass matrix $\mathscr{B}$ should not appear in their numerators. The present version is instead written down correctly.
For the calculation of the amplitude equation coefficients at $\epsilon^{3}$ order, only resonant terms matter. These terms, with their corresponding amplitudes, are proportional to $e^{\mathrm{i}\left(\left(\omega_{1 n} t \pm \theta\right)\right.}$ for SC waves and to $e^{\mathrm{i}\left(\omega_{2 n} t \pm 2 \theta\right)}$ for DC waves. As an example, the expression of $\hat{F}_{3_{k i n}}^{|A|^{2} A}$, with $A=A_{1}$ for SC waves and $A=A_{2}$ for DC waves, is given in Appendix D of Bongarzone et al. (2022a).

The extraction of resonant terms was performed by using tools of symbolic calculus, e.g. the software Wolfram Mathematica.
Analogously, the normal form coefficients appearing in (5.31a)-(5.31b) for the super-harmonic double-crest (DC) dynamics are calculated as

$$
\begin{align*}
& \mathrm{i} \mathscr{I}_{D C} \mu_{D C}=\int_{0}^{1}\left(\overline{\hat{\eta}}_{1}^{A_{2} \dagger} \hat{\mathscr{F}}_{2 \mathrm{dyn}}^{F^{2}}+\overline{\hat{\Phi}}_{1}^{A_{2} \dagger} \hat{\mathscr{F}}_{2 \mathrm{kn}}^{F^{2}}\right) r \mathrm{~d} r,  \tag{5.39a}\\
& \mathrm{i} \mathscr{I}_{D C} \zeta_{D C}=\int_{0}^{1}\left(\overline{\hat{\eta}}_{1}^{A_{2} \dagger} \hat{\mathscr{F}}_{3 \mathrm{dyn}}^{\Lambda F^{2}}+\overline{\hat{\Phi}}_{1}^{A_{2} \dagger} \hat{\mathscr{F}}_{3_{\mathrm{kin}}} \mathrm{~F}^{2}\right) r \mathrm{~d} r,  \tag{5.39b}\\
& \mathrm{i} \mathscr{I}_{D C} \chi_{D C}=\int_{0}^{1}\left(\overline{\hat{\eta}}_{1}^{A_{2} \dagger} \hat{\mathscr{F}}_{3 \mathrm{dyn}}^{A_{2}|F|^{2}}+\overline{\hat{\Phi}}_{1}^{A_{2} \dagger} \hat{\mathscr{F}}_{\mathrm{H}_{\mathrm{kin}}}^{A_{2}|F|^{2}}\right) r \mathrm{~d} r,  \tag{5.39c}\\
& \mathrm{i} \mathscr{I}_{D C} v_{D C}=\int_{0}^{1}\left(\overline{\hat{\eta}}_{1}^{A_{2} \dagger} \hat{\mathscr{F}}_{3_{\mathrm{dyn}}}^{\left|A_{2}\right|^{2} A_{2}}+\overline{\hat{\Phi}}_{1}^{A_{2} \dagger} \hat{\mathscr{F}}_{3 \mathrm{kin}}^{\left|A_{\mathrm{i}}\right|^{2} A_{2}}\right) r \mathrm{~d} r,  \tag{5.39d}\\
& \mathrm{i} \mathscr{I}_{D C} \xi_{D C}=\int_{0}^{1}\left(\left.\overline{\hat{\eta}}_{1}^{A_{2} \dagger} \hat{\mathscr{F}}_{3_{\mathrm{dyn}}} \mathrm{~B}_{2}\right|^{2} A_{2}+\left.\overline{\hat{\Phi}}_{1}^{A_{2} \dagger} \hat{\mathscr{F}}_{3_{\mathrm{kin}}} \mathrm{~B}_{2}\right|^{2} A_{2}\right) r \mathrm{~d} r, \tag{5.39e}
\end{align*}
$$

with $\mathscr{I}_{D C}=<\hat{\mathbf{q}}_{1}^{A_{2} \dagger}, \mathscr{B} \hat{\mathbf{q}}_{1}^{A_{2}}>=\int_{0}^{1}\left(\overline{\hat{\eta}}_{1}^{A_{2} \dagger} \hat{\Phi}_{1}^{A_{2}}+\overline{\hat{\Phi}}_{1}^{A_{2} \dagger} \hat{\eta}_{1}^{A_{2}}\right) r \mathrm{~d} r$. The integrals are all evaluated at the free surface $z=0$.
We note that the value of the normal form coefficient $\chi_{D C}$ contains two different contributions. Indeed, it could be conveniently rewritten as $\chi_{D C}=\chi_{-}+\chi_{+}$, with the value of $\chi_{-}$and $\chi_{+}$given in table 5.1. $\chi_{-}$precisely corresponds to the coefficient $\chi_{D C}$ computed in Bongarzone et al. (2022a) and, by adopting the present formalism, e.g. for mode $A_{2}$ (same for mode $B_{2}$ ), it is produced by the interaction of the second order responses

$$
\begin{equation*}
(1 / 2) A_{2} F \hat{\mathbf{q}}_{2}^{A_{2} F} e^{\mathrm{i}\left(\left(3 \omega_{2 n} / 2\right) t-3 \theta\right)} e^{\mathrm{i} \Lambda T_{1}}+(1 / 2) A_{2} \bar{F} \hat{\mathbf{q}}_{2}^{A_{2} \bar{F}} e^{\mathrm{i}\left(\left(\omega_{2 n} / 2\right) t-\theta\right)} e^{-\mathrm{i} \Lambda T_{1}}, \tag{5.40}
\end{equation*}
$$

in equation (5.28) with the complex conjugate of the leading order particular solution characterized by $m=-1$ in (5.24). On the contrary, the contribution $\chi_{+}$is the result of the interaction between the second-order responses

$$
\begin{equation*}
(1 / 2) A_{2} F \hat{\mathbf{q}}_{2}^{A_{2} F} e^{\mathrm{i}\left(\left(3 \omega_{2 n} / 2\right) t-\theta\right)} e^{\mathrm{i} \Lambda T_{1}}+(1 / 2) A_{2} \bar{F} \hat{\mathbf{q}}_{2}^{A_{2} \bar{F}} e^{\mathrm{i}\left(\left(\omega_{2 n} / 2\right) t-3 \theta\right)} e^{-\mathrm{i} \Lambda T_{1}} \tag{5.41}
\end{equation*}
$$

in equation (5.28) and the complex conjugate of the leading order particular solution for $m=+1$ in (5.24).

### 5.7.2 Generalization to elliptic orbits

In this appendix, we show how the analysis outlined in this Chapter for longitudinal container motions can be straightforwardly generalized to any elliptic-like shaking. For elliptical orbits

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in the horizontal $(x, y)$-plane, equations (5.1) are modified as follows

$$
\dot{\mathbf{X}}_{0}=\left\{\begin{array}{c}
\left(-a_{x} \Omega \sin (\Omega t) \cos \theta+a_{y} \Omega \cos (\Omega t) \sin \theta\right) \mathbf{e}_{r}  \tag{5.42}\\
\left(a_{x} \Omega \sin (\Omega t) \sin \theta+a_{y} \Omega \cos (\Omega t) \sin \theta\right) \mathbf{e}_{\theta}
\end{array},\right.
$$

with $a_{x}$ and $a_{y}$ the non-dimensional major- and minor-axis forcing amplitude components, respectively, and $\Omega$ the non-dimensional driving angular frequency. Under these forcing conditions, the unsteady and forced Bernoulli's equation at $z=\eta$ reads

$$
\begin{equation*}
\frac{\partial \Phi}{\partial t}+\frac{1}{2} \nabla \Phi \cdot \nabla \Phi+\eta=r\left(f_{x} \cos (\Omega t) \cos \theta+f_{y} \sin (\Omega t) \sin \theta\right) \tag{5.43}
\end{equation*}
$$

where $f_{x}=a_{x} \Omega^{2}$ and $f_{y}=a_{y} \Omega^{2}$. By introducing the aspect ratio $\alpha=a_{y} / a_{x}=f_{y} / f_{x}$, so that $f_{x}=f$ and $f_{y}=\alpha f$, equation (5.43) can be conveniently rewritten as

$$
\begin{equation*}
\frac{\partial \Phi}{\partial t}+\frac{1}{2} \nabla \Phi \cdot \nabla \Phi+\eta=r \frac{f}{2}\left(\left(\frac{1+\alpha}{2}\right) e^{\mathrm{i}(\Omega t-\theta)}+\left(\frac{1-\alpha}{2}\right) e^{\mathrm{i}(\Omega t+\theta)}\right)+c . c . . \tag{5.44}
\end{equation*}
$$

A value $0<\alpha<1$ implies elliptic orbits, whereas the two limit cases with $\alpha=0\left(a_{x} \neq 0, a_{y}=0\right)$ and $\alpha=1\left(a_{x}=a_{y} \neq 0\right)$ correspond, respectively, to longitudinal, as in the present work, and circular (Bongarzone et al., 2022a), shaking conditions. For the convenience of notation, we also introduce the auxiliary variables

$$
\begin{equation*}
\alpha_{-}=\frac{1+\alpha}{2}, \quad \alpha_{+}=\frac{1-\alpha}{2}, \tag{5.45}
\end{equation*}
$$

with $1 / 2 \leq \alpha_{-} \leq 1$ and $0 \leq \alpha_{+} \leq 1 / 2$. By accounting for the two auxiliary aspect ratios, $\alpha_{-}$ and $\alpha_{+}$in the expression of the forcing term, the whole derivation can be repeated, hence leading, without any further computation, to the following system of amplitude equations for harmonic single-crest (SC) waves

$$
\begin{align*}
& \frac{d A}{d t}=-\mathrm{i} \lambda A+\mathrm{i} \mu_{S C} \alpha_{-} f+\mathrm{i} v_{S C}|A|^{2} A+\mathrm{i} \xi_{S C}|B|^{2} A,  \tag{5.46a}\\
& \frac{d B}{d t}=-\mathrm{i} \lambda B+\mathrm{i} \mu_{S C} \alpha_{+} f+\mathrm{i} v_{S C}|B|^{2} B+\mathrm{i} \xi_{S C}|A|^{2} B . \tag{5.46b}
\end{align*}
$$

and for super-harmonic double-crest (DC) waves

$$
\begin{array}{r}
\frac{d A}{d t}=-\mathrm{i}\left(2 \lambda-\left(\alpha_{-}^{2} \chi_{-}+\alpha_{+}^{2} \chi_{+}\right) f^{2}\right) A+\mathrm{i}\left(\zeta_{D C} \lambda+\mu_{D C}\right) \alpha_{-}^{2} f^{2} \\
+\mathrm{i} v_{D C}|A|^{2} A+\mathrm{i} \xi_{D C}|B|^{2} A, \\
\frac{d B}{d t}=-\mathrm{i}\left(2 \lambda-\left(\alpha_{+}^{2} \chi_{+}+\alpha_{-}^{2} \chi_{-}\right) f^{2}\right) B+\mathrm{i}\left(\zeta_{D C} \lambda+\mu_{D C}\right) \alpha_{+}^{2} f^{2} \\
+\mathrm{i} v_{D C}|B|^{2} B+\mathrm{i} \xi_{D C}|A|^{2} B, \tag{5.47b}
\end{array}
$$



Figure 5.16 - Intensity profile along the middle axis of the container as a function of time. The free surface, initially at rest $(t<0)$ is submitted to forced harmonic oscillations from $t=0$.
with the values of the normal form coefficients still given in table 5.1.
We note that in the limit of $\alpha=0$ (longitudinal), $\alpha_{-}=\alpha_{+}=1 / 2$ and equations (5.18a)-(5.18b) and (5.31a)-(5.31b) are retrieved. On the contrary, in the limit of $\alpha=1$ (circular), one has $\alpha_{-}=1$ and $\alpha_{+}=0$, so that equations (5.46a) and (5.47a) corresponds to equations (4.6) and (4.22) of Bongarzone et al. (2022a), with $A \neq 0$ and $B=0$ the only possible stable stationary solution for (5.47a) and (5.47b).

### 5.7.3 Estimation of the duration of the transient regime

In this study, we only consider the permanent response of the free surface to forced oscillations. To ensure we discard the transient regime in our analysis of the free surface dynamics, we first obtained an estimation of the transient time by recording for various forcing amplitudes $\bar{a}_{x}$ and angular frequencies $\bar{\Omega}$, the full dynamics of the free-surface, initially at rest and then put into oscillations. The temporal evolution of the intensity profile along the middle axis of the container extracted from our movies, is a direct signature of the variation in time of the sloshing wave amplitude, and reveals that for all ( $\bar{a}_{x}, \bar{\Omega}$ ) set of parameters investigated, the free-surface dynamics can be safely considered as having reached a steady-state after typically 50 cycles of oscillations, see figure 5.16.

### 5.7.4 Extracting the contact line dynamics from the intensity profiles

Here we present the general methodology to extract, from the intensity profiles, the front contact line position as a function of time. First, we binarize the intensity profiles, so as to isolate the periodic pattern due to the free surface motion from the rest of the image (see figure 5.17 (a)). Then the vertical positions of the first and last non-zero pixel of each column in the binarized intensity profile are retrieved at each time step, thus providing the top and bottom envelopes of the intensity profile, see figure 5.17 (b). The front contact line position periodically coincides with the elevation of the lower or of the upper envelope of the pattern. By detecting the local minima of the distance between the top and bottom envelopes as a

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Figure 5.17 - (a) Binarized intensity profile from figure 5.9(c). (b) Upper and lower envelopes of the binarized profile. (c) Distance (in pixel) between the upper and lower envelope as a function of time. The local minima (highlighted by blue dots) correspond to the time steps where the back and front contact line have similar elevations. (d) Upper and lower envelopes of the binarized profile (black) and front contact line position as a function of time (red dotted line).
function of time (see figure 5.17 (c)), we identify the time instants at which the front and back contact lines have similar elevations, which corresponds for the front contact line position to a switch from the lower to the upper envelope, or vice-versa. The front contact line dynamics is then obtained by extracting from the bottom and top envelopes the time series corresponding to the front contact line position, see figure 5.17 (d). The exact position of the front contact line at intermediary time steps where the distance between the lower and upper envelopes is minimal cannot be precisely detected, so that the successive time series are connected through a simple linear interpolation. Note that this procedure does not affect the frequency content of the dynamics.
Lastly, to compute the amplitude envelope of the front contact line oscillations, such as displayed in figure 5.14(a), we use the Matlab function islocalmax (resp. islocalmin), that extracts the local maxima (resp. minima) from the front contact line profile, thus providing the position of the top (resp. bottom) amplitude envelope.

### 5.7.5 Can the asymptotic WNL model predict the irregular regime?

This Appendix is devoted to verifying whether the asymptotic weakly nonlinear model could be used to better grasp the features of the irregular regimes. In the latter regime, no stable


Figure 5.18 - Left: frequency of the main peak in the power spectrum of the envelope (nondimensionalized by the forcing frequency $\bar{\Omega} / 2 \pi$ ) as a function of the non-dimensional forcing amplitude $a_{x}$ and for same forcing angular frequency $\Omega=0.496 \omega_{21}$. The white squares and black circles correspond to the experimental points reported in figure 5.14 and are obtained, respectively, for an upward and downward forcing-amplitude sweep. Right: prediction from the WNL model for a damping coefficient $\sigma=0.01$ (red crosses) and $\sigma=0.015$ (blue crosses).


Figure 5.19 - Predicted WNL time-series corresponding to the contact line elevation at an azimuthal coordinate $\theta=0$ and constructed according to equation 5.2 of the manuscript for three different forcing amplitudes $a_{x}$. Amplitudes $A(t)=|A(t)| e^{\mathrm{i} \Phi_{A}(t)}$ and $B(t)=|B(t)| e^{\mathrm{i} \Phi_{B}(t)}$ are computed by time-integrating the complex system of equations (5.31a)-(5.31b) with an additional damping $\sigma=0.015$. The time integration is performed via the built-in Matlab function ode23.
stationary amplitude solutions are found. Nevertheless, one can still solve the system of amplitude equations in time assigning some initial conditions so as to observe the transient and, more interestingly, the large time dynamics.

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When time-integrating the inviscid system of amplitude equations (5.31a)-(5.31b), a subtle technical issue arises. In the absence of any form of dissipation, what we define as stable stationary-amplitude solutions (computed by cancelling out the terms $d A / d t$ and $d B / d t$ in equations (5.31a)-(5.31b)) are actually only marginally stable, meaning that a small perturbation of these states will neither grow nor decay in time. Since the real system is dissipative, the large-time solutions always tend to the marginally stable states, as external small perturbations are damped down. Hence, to obtain, for instance, the stable stationary states of figure 5.5(a) by time-marching the coupled amplitude equations, one needs to introduce a small damping factor so as to damp down perturbations (due to initial conditions for instance). This can be done by adding a small dissipative term to equations (5.31a)-(5.31b), i.e. $d A / d t=-\sigma A+\ldots$ and $d B / d t=-\sigma B+\ldots$, and waiting for a sufficiently long time. We can then run in time the modified system of equations in the range of parameters corresponding to the irregular regime and monitor the statistical properties of the resulting signal.

In figure 5.19 we report three time-series obtained at three different non-dimensional forcing amplitudes $a_{x}$ and at a driving frequency $\Omega=0.496 \omega_{21}$ (as in figure 5.14 ). We can clearly spot an envelope modulation enclosing the fast wave oscillations. The frequency of the main peak in the PSD associated with these envelope signals can be then compared with the experimental values reported in figure 5.18 of this document.
The agreement is qualitatively good and a similar linear trend is retrieved using the WNL model, meaning that the simple system of two complex amplitude equations can reproduce the main features of the full system dynamics even in the irregular regime.
However, this analysis requires the time-marching of the equations (5.31a)-(5.31b), the $a d$ hoc introduction of a damping coefficient and it provides outputs which depend on the value of the latter coefficient (see Right panel of figure 5.18). Further investigations in this direction could require alone a dedicated theoretical study

# 6 Swirling against the forcing: evidence of stable counter-directed sloshing waves in orbital-shaken reservoirs 

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#### Abstract

Author's contributions: A.B. and F.G. created the research plan. A.B. formulated analytical and numerical models. A.M. and A.B. led model solutions. A.M. designed and performed all the experiments with inputs from A.B. and F.G.. A.M. and A.B. wrote the manuscript with inputs from F.G..


We study the free surface response in a cylindrical container undergoing an elliptic periodic orbit. For small forcing amplitudes and deep liquid layers, we quantify the effect of the orbit's aspect ratio on the surface dynamics in the vicinity of the fluid system's lowest natural frequency $\omega_{0}$. We provide experimental evidence of the existence of a frequency range where stable swirling can be either co- or counter-directed with respect to the container's direction of motion. Our findings are successfully predicted by an inviscid asymptotic model, amended with heuristic damping.

In the general introduction to Part II, we have mentioned how the problem of liquid sloshing pertains to many aspects of daily life, ranging from mundane wine tasting to more pragmatic issues such as liquid spilling (Mayer and Krechetnikov, 2012) and transport safety (Faltinsen and Timokha, 2009). With regards to orbital sloshing in partially filled circular cylinders, we have described how previous experimental studies have investigated the close-to-resonance dynamics for either circular (in Chapter 4) or purely longitudinal (in Chapter 5) shaking, casting light on a rich variety of wave regimes attracting interest to dynamicists over the last decades (Hutton, 1963; Miles, 1984c,d; Ockendon and Ockendon, 1973). These fascinating and complex features are briefly recalled below. For circular orbits, the system responds with

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a swirling wave always co-directed with the container motion (Reclari et al., 2014). This welldefined hydrodynamics, often simply modelled by a one-degree-of-freedom Duffing oscillator (Bongarzone et al., 2022a; Ockendon and Ockendon, 1973), is advantageously exploited, for example in biology, in the design of bioreactors, where the container is shaken so as to mix the liquid, prevent sedimentation and enhance gas transfer, hence providing suitable oxygenation to the growing cell population (Klöckner and Büchs, 2012). In the case of longitudinal forcing, the standing wave solution may undergo close-to-resonance a symmetry-breaking, with clockand anti-clockwise swirling waves equally probable, or completely lose regularity showing an irregular and chaotic alternation of planar and swirling motions (Hutton, 1963; RoyonLebeaud et al., 2007). Such a configuration finds a close mechanical analogy in the resonant motion of a forced spherical pendulum (Miles, 1984d), a four degrees-of-freedom system that has been widely studied in the context of order-to-chaos transitions (Miles, 1984c; Tritton, 1986) for its similarities with the Lorentz's problem (Lorenz, 1963).

Surprisingly however, no experimental studies devoted to the more generic case of elliptic orbits have been reported so far in the sloshing literature. Yet, existing theoretical analyses of this forcing condition brought out interesting features of the resonant liquid response that depend on the orbit's ellipticity. In particular, a recent inviscid theory (Faltinsen et al., 2016) suggested the counter-intuitive existence, under resonant elliptic forcing, of stable swirling waves that propagate in the direction opposite to the forcing direction. Moreover, the theory anticipated that such counter-waves may exist even for quasi-circular orbits and travel with a smaller amplitude than co-directed waves. This, if confirmed, would further enrich the variety of observable dynamical sloshing regimes and possibly open a new room in this rich dynamic system field.

For moderately large-size containers, the use of inviscid hydrodynamic models is well accepted (Ibrahim, 2005), but in real sloshing problems, waves are always subjected to a non-vanishing viscous dissipation. Hence, the counter-swirling wave predicted by the inviscid model (Faltinsen et al., 2016), being intrinsically disfavored by the forcing direction, is likely to be more sensitive to damping than co-swirling solutions, and it is currently unclear whether such a solution can actually arise in a real-life lab-experiment.

In this Chapter, we aim to provide a joint experimental and theoretical characterization of the free liquid surface response for a generic, elliptic periodic container trajectory, so as to bridge the gap between the two diametrically opposed shaking conditions previously discussed. Specifically, we intend to identify the range of external control parameters, i.e. driving frequency, amplitude and orbit aspect ratio, for which stable counter-directed swirling waves do occur, and assess the extent of the forcing regime where inviscid models break down.


Figure 6.1 - (a) Experimental setup. Sloshing waves are generated by the container elliptic trajectory, achieved by imposing along the x and y axes two sinusoidal forcing of driving angular frequency $\bar{\Omega}$ and amplitudes $\bar{a}_{x}$ and $\bar{a}_{y} . \delta(\theta, t)$ denotes the free surface elevation measured at the sidewall, $r=R$. (b) Sketch illustrating the extraction from the frame corresponding to time-instant $t_{i}$ of the intensity profiles along the vertical middle axis of the container image (labelled as $I_{t_{i}}(0)$ ) and along the vertical axes located at coordinates $\mp R / 2$.

### 6.1 Experimental setup and procedure

In our experimental campaign, we used a Plexiglas circular cylindrical container of total height 50 cm and internal radius $R=0.086 \mathrm{~m}$, filled to a depth $h=0.15 \mathrm{~m}$ with water: density $\rho=1000 \mathrm{~kg} \mathrm{~m}^{-3}$, surface tension $\gamma=0.0725 \mathrm{Nm}^{-1}$ and dynamic viscosity $\mu=0.001 \mathrm{~kg} \mathrm{~m}^{-1} \mathrm{~s}^{-1}$. The gravity acceleration is denoted by $g$ (see figure 6.1(a)). The container is fixed on a doubleaxes linear motion actuator (Aerotech pro165LM + pro225LM), which imposes along the x and y axes two sinusoidal forcings of angular frequency $\bar{\Omega}$ and amplitudes $\bar{a}_{x}$ and $\bar{a}_{y}$, that are $\pi / 2$-phase shifted with respect to each other. The fluid motion is recorded with a digital camera (Nikon D850) coupled with a Nikon $60 \mathrm{~mm} \mathrm{f} / 2.8 \mathrm{D}$ lens and operated in slow mode with an acquisition frequency of 120 fps . The camera's optical axis is aligned with the x -axis. A LED panel is placed behind the container so as to provide a back illumination for better optical contrast.
In the moving reference frame, any planar elliptic-like shaking can be represented by the following equations describing the motion acceleration of the container axis parametrized in polar coordinates $(r, \theta)$,

$$
\frac{\mathrm{d}^{2} \mathbf{X}_{0}}{\mathrm{~d} t^{2}}=\left\{\begin{array}{l}
\left(-f_{x} \cos \Omega t \cos \theta-f_{y} \sin \Omega t \sin \theta\right) \mathbf{e}_{r}  \tag{6.1}\\
\left(f_{x} \cos \Omega t \sin \theta-f_{y} \sin \Omega t \sin \theta\right) \mathbf{e}_{\theta}
\end{array}\right.
$$

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where $f_{x}=f=\bar{a}_{x} \Omega^{2} / R$ and $f_{y}=\alpha f=\bar{a}_{y} \Omega^{2} / R$ are the non-dimensional major- and minoraxis driving acceleration components, respectively, and $\Omega=\bar{\Omega} / \sqrt{g / R}$ the non-dimensional driving angular frequency. The bar symbol refers to the dimensional quantities. Note that the minor-to-major-axis component ratio, $\alpha=\bar{a}_{y} / \bar{a}_{x}=f_{y} / f_{x}$, has been introduced. A value $0<\alpha<1$ refers to elliptic orbits, whereas the two limiting cases $\alpha=0$ and $\alpha=1$ correspond, respectively, to longitudinal and circular shaking conditions.
With this experimental campaign we intend to study the free surface response in the vicinity of the lowest natural frequency $\omega_{0}=\bar{\omega}_{0} / \sqrt{g / R}=\sqrt{k \tanh (k h / R)}=1.3547$ (with wavenumber $k=1.8412$ ) (Lamb, 1993), for varying orbit's aspect ratios $\alpha$ and forcing amplitudes $\bar{a}_{x}$. In particular, we aim at recovering the whole set of stationary wave amplitude solutions, i.e. co- and counter-swirling waves, and at studying how their stability depends on the forcing parameters. If co- and counter-swirling waves happen to be coexisting stable solutions for a certain combination of control parameters, then a co-directed swirling motion will very likely be naturally favoured by the forcing direction and will therefore spontaneously arise from the time-harmonic forcing. On the other hand, triggering counter-swirling would require escaping the basin of attraction of the co-swirling wave solution, which is only possible by introducing a sufficient flow perturbation. The experimental procedure described in the following is thus suitably designed so as to reveal steady-state counter-directed waves, whenever this dynamics is a stable admissible solution.
In a typical experiment, the amplitude $\bar{a}_{x} \in[1,3] \mathrm{mm}$ and ellipticity $\alpha \in[0.1,0.95]$ are fixed, while frequencies are swept up- and downward within the (dimensionless) range $\Omega / \omega_{0} \in$ [ $0.82,1.21]$. The increment between two consecutive steps in the frequency sweep is 0.0217 . Each frequency step consists of two parts: the container undergoes first a harmonic elliptic forcing that is in the anti-clockwise direction for 150 oscillation periods and then in the clockwise direction for another 150 oscillation periods. Two movies are then recorded at each step so as to monitor the free surface response to both clockwise and anti-clockwise forcing. Switching the direction of the tank's trajectory in the second phase of the experimental procedure induces a flow perturbation that is enough to produce a counter-directed wave if the latter is an admissible stable configuration for the system. For each frequency step and container direction, the camera is triggered only after 100 cycles so that it only records the last 50 oscillation periods. Preliminary longer measurements performed for a few forcing parameters sampled within our experimental range showed that the transient regime typically lasts less than 100 cycles. Successively, we made sure that every movie recorded after 100 oscillation periods indeed corresponds to stationary wave amplitude regimes, except when the system exhibits the irregular dynamics described later in the section.

### 6.1.1 Analysis of the free-surface response

The procedure to analyze the free surface response is extensively described in Chapter 5 and illustrated here in figure 6.1(b). Briefly, we build from each movie an image $I(y)=$ [ $\left.I_{t_{1}}(y), I_{t_{2}}(y), \ldots\right]$ where $I_{t_{i}}(y)$ is the intensity profile along the vertical axis $Y\left(t_{i}\right)=y$ on the frame $i$ corresponding to time $t_{i}$, with $Y\left(t_{i}\right)=0$ being the vertical middle axis between the


Figure 6.2 - (a)-(d) Intensity profiles as a function of time along the middle vertical axis $I(0)$ for ellipticity $\alpha=0.5$, amplitude $\bar{a}_{x}=1.5 \mathrm{~mm}$ and frequency (a)-(b) $\Omega / \omega_{0} \approx 0.95$ or (c)-(d) $\Omega / \omega_{0} \approx 1.04$. The intensity profiles (b) and (d) are obtained from the binarization of (a) and (c) so as to filter out the signal of weaker intensity coming from the back contact line whenever the elevation of the front contact line is minimal. The oscillations of the front contact line are then enclosed into a top-bottom envelope, plotted in red in panel (d).
edges of the container image (represented by $Y\left(t_{i}\right)=R$ and $Y\left(t_{i}\right)=-R$ ). The resulting image, as illustrated in figure 6.2, displays a periodic dark pattern that represents the free surface response to the harmonic forcing. The free surface appears as the darkest feature on each frame so that the intensity profile along a vertical line at a given time $t_{i}$ represents the vertical extension of the free surface in this direction.
The usefulness of the resulting image $I(y)$ is threefold: (i) it allows the detection of irregular dynamics. This corresponds to the absence of any stable wave amplitude for a given set of forcing parameters and is easily identified by the time-varying envelope modulating the free surface oscillations, see figure 6.2(a). (ii) For a regular response, $I(0)$ enables one to measure the amplitude of the front contact line in the azimuthal direction $\theta=0$. (iii) The comparison of the profiles along two vertical directions that are mirror-symmetric with respect to the vertical middle axis, e.g. $I(-R / 2)$ and $I(R / 2)$, makes it possible to determine the propagation direction of the wave and to compare it with the container's motion direction.

### 6.1.2 Detecting the irregular regime

Figure 6.2 displays two intensity profiles as a function of time along the vertical middle axis ( $Y=0$ ) for the same forcing amplitude $\bar{a}_{x}$ and ellipticity $\alpha$ but for two different forcing frequencies $\Omega / \omega_{0}$. Those images show that depending on the forcing parameters, the amplitude of the free surface oscillations can be either irregular, figure 6.2(a)-(b), or stationary, figure 6.2(c)(d). In the analysis of the close-to-resonance dynamics, we, therefore, use the profile $I(0)$ to identify the irregular regime.

### 6.1.3 Measuring the wave amplitude

The intensity profile $I(0)$ also provides the amplitude $\delta(\theta=0, t)$ of the swirling wave at the front wall of the container, i.e. at the azimuthal coordinate $\theta=0$, such as defined in figure 6.1 (b). Indeed, due to the backlighting, the intensity signal corresponding to the front contact line appears darker than the one due to the back contact line, so that pieces of information associated with the latter can be filtered out by a proper thresholding of profile $I(0)$. On the resulting binarized image, the maximal and minimal heights of the final periodic pattern correspond then to the peaks and troughs of the swirling wave at the front wall along $\theta=0$. The amplitude of the wave (in pixel) is thus experimentally retrieved as half the difference between the height of the top and of the bottom envelopes enclosing its oscillations, displayed in figure $6.2(\mathrm{~d})$ as red lines, and converted into millimeters by using a scale put on the front wall of the container. Note that in this procedure, we neglect the variation of the pixel size that can occur along the container motion, the camera being fixed. This is justified by the very small forcing amplitude ( $1 \mathrm{~mm} \leq \bar{a}_{x} \leq 3 \mathrm{~mm}$ ) with respect to the distance between the camera and the front wall of the container ( 1 m ). The error related to the variation of the pixel size is therefore of the order of $0.1 \%$, i.e. negligible compared to the typical dispersion of our measurements.


Figure 6.3 - Superposition of the intensity profiles as a function of time along the vertical axis $(Y=R / 2)$ and $(Y=-R / 2)$, denoted $I(R / 2)$ (in blue) and $I(-R / 2)$ (in red) respectively, for a harmonic forcing of frequency $\Omega / \omega_{0} \approx 1.04$, amplitude $\bar{a}_{x}=1.5 \mathrm{~mm}$, and (a)-(b) ellipticity $\alpha=0.50$ and (c)-(d) $\alpha=0.95$. The container moves either in the anti-clockwise direction ((a) and (c)) or in the clockwise direction ((b) and (d)).

### 6.1.4 Identifying the swirling direction

To detect the direction of propagation of the wave, we compare for each movie the intensity profiles along two vertical directions that are mirror-symmetric with respect to the vertical middle axis of the container. figure 6.3 shows composite images that each consists in the


Figure 6.4 - Free surface snapshots corresponding to the case of figure 6.3(a)-(b) with $\alpha=0.50$. Direction of the container motion: left, anti-clockwise; right, clockwise (follow the black arrows). The white arrows indicate the direction of the wave rotation. A visual indication of the different wave amplitudes is provided by the black double-sided arrows.
superposition of $I(R / 2)$ and $I(-R / 2)$ into a composite RGB image, where grey areas correspond to pixels where $I(R / 2)$ and $I(-R / 2)$ have the same intensity, while red (resp. blue) areas correspond to the part of $I(-R / 2)$ (resp. $I(R / 2)$ ) that do not overlap with $I(R / 2)$ (resp. $I(-R / 2)$ ). Thus, a red (resp. blue) peak preceding a blue (resp. red) peak corresponds to a wave travelling from the left (resp. right) to the right-hand (resp. left-hand) side of the front wall of the container, i.e. in the anti-clockwise (resp. clockwise) direction. The propagation direction of the wave can then be determined and compared to the direction of the container motion. In figure 6.3, the dynamics associated with two different aspect ratios $\alpha=0.5$ and $\alpha=0.95$ (quasi-circular orbit) are compared for the same forcing frequency and amplitude. For each $\alpha$, the right and left-hand-side signals $I(R / 2)$ and $I(-R / 2)$ are superposed to each other for two motion configurations, namely an anti-clockwise followed by a clockwise container trajectory. In the case of the anti-clockwise tank's motion, figure 6.3(a)-(b), the swirling wave travels in the

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same direction as the container, but the change of container's motion direction induces a flow perturbation sufficient to produce a robust counter-directed wave, if the latter corresponds to a system's stable solution. We indeed observe in the case of $\alpha=0.5$ that the wave, though of smaller amplitude, is still travelling from the left to the right-hand-side of the container's front wall despite the reverse of direction in the tank trajectory. This appears glaringly in figure 6.4, where the two series of free surface snapshots show how the wave's direction of rotation remains unchanged despite the reversal of the container's direction of motion. On the contrary, figure 6.3 (c)-(d), the wave switches direction for the large ellipticity $\alpha=0.95$ and is therefore co-directed with the forcing for both container motion directions. These results provide the first experimental evidence for the existence of counter-swirling solutions and validate this procedure as suitable to trigger and identify stable counter-directed waves.

### 6.2 Inviscid asymptotic model

To assess the extent of the validity of an inviscid hydrodynamic model to predict resonant counter-swirling in a lab-scale experiment, in this section we compare our experimental results with the theoretical estimates provided by the asymptotic model formalized in Chapter 5 and recalled in the following. This weakly nonlinear model has been extensively compared Faltinsen et al. (2016) for both purely longitudinal (Marcotte et al., 2023a) and circular (Bongarzone et al., 2022a) shaking conditions, and it has been shown to provide consistent results. See Chapter 5 for a discussion on the methodological analogies and differences as well as on the pros and cons of the present approach versus the Narimanov-Moiseev multimodal theory employed in Faltinsen et al. (2016).

### 6.2.1 Governing equations

In the potential flow limit, i.e. the flow is assumed inviscid, irrotational and incompressible, the liquid motion is governed by the Laplace equation, subjected to the homogeneous nopenetration condition at the solid lateral wall, $r=R$, and bottom $z=-h$,

$$
\begin{equation*}
\Delta \Phi=0, \quad \nabla \Phi \cdot \mathbf{n}=\mathbf{0}, \tag{6.2}
\end{equation*}
$$

and by the kinematic and dynamic boundary conditions at the free surface $z=\eta(r, \theta, t)$ (Faltinsen and Timokha, 2009; Ibrahim, 2005),

$$
\begin{gather*}
\frac{\partial \eta}{\partial t}+\nabla \Phi \cdot \nabla \eta-\frac{\partial \Phi}{\partial z}=0  \tag{6.3a}\\
\frac{\partial \Phi}{\partial t}+\frac{1}{2} \nabla \Phi \cdot \nabla \Phi+\eta=r\left(f_{x} \cos (\Omega t) \cos \theta+f_{y} \sin (\Omega t) \sin \theta\right) \tag{6.3b}
\end{gather*}
$$

made non-dimensional using the characteristic length $R$ and velocity $\sqrt{g R} . \Phi(r, \theta, z, t)$ and $\eta(r, \theta, t)$ denote potential velocity field and free surface elevation, respectively. Note that, as in

Faltinsen et al. (2016) and Marcotte et al. (2023a), surface tension effects have been neglected. By recalling the definition of the orbit aspect ratio, $\alpha=\bar{a}_{y} / \bar{a}_{x}=f_{y} / f_{x}$, so that $f_{x}=f$ and $f_{y}=\alpha f$, equation (6.3b) can be conveniently rewritten as

$$
\begin{equation*}
\frac{\partial \Phi}{\partial t}+\frac{1}{2} \nabla \Phi \cdot \nabla \Phi+\eta=r \frac{f}{2}\left(\alpha_{A} e^{\mathrm{i}(\Omega t-\theta)}+\alpha_{B} e^{\mathrm{i}(\Omega t+\theta)}\right)+c . c . \tag{6.4}
\end{equation*}
$$

with $c . c$. denoting the complex conjugate and with $\alpha_{A}=(1+\alpha) / 2$ and $\alpha_{B}=(1-\alpha) / 2$ two auxiliary orbit parameter.

### 6.2.2 Multiple time-scales weakly nonlinear analysis

The weakly nonlinear multiple timescale analysis formalized in Chapter 5 is based on the following asymptotic expansion for the flow quantities,

$$
\begin{equation*}
\mathbf{q}(r, \theta, z, t)=\{\Phi, \eta\}^{T}=\mathbf{q}_{0}+\epsilon \mathbf{q}_{1}+\epsilon^{2} \mathbf{q}_{2}+\epsilon^{3} \mathbf{q}_{3}+\mathrm{O}\left(\epsilon^{4}\right) \tag{6.5}
\end{equation*}
$$

and on the assumption of a small forcing amplitude of order $f=\epsilon^{3} F$, which is justified by the fact that close to the resonance $\Omega \approx \omega_{0}$, even a small forcing will induce a large system response. We then allow for a small frequency detuning with respect to the first system's natural frequency, $\omega_{0}$, such that $\Omega=\omega_{0}+\lambda$, with $\lambda=\epsilon^{2} \Lambda, \epsilon$ a small parameter $\ll 1$ and the new auxiliary parameters $F$ and $\Lambda$ assumed of order $O$ (1). Note that the $\epsilon^{0}$-order solution, $\mathbf{q}_{0}$ represents the rest state, for which $\Phi_{0}$ and $\eta_{0}$ are simply zero.
Given the azimuthal periodicity of the forcing term on the right-hand-side of (6.4), i.e. $m= \pm 1$ (with $m$ a so-called azimuthal wavenumber), we postulate a leading order solution as the sum of two counter-propagating travelling waves

$$
\begin{equation*}
\mathbf{q}_{1}(r, \theta, z, t)=A_{1}\left(T_{2}\right) \hat{\mathbf{q}}_{1}^{A_{1}}(r, z) e^{\mathrm{i}\left(\omega_{0} t-\theta\right)}+B_{1}\left(T_{2}\right) \hat{\mathbf{q}}_{1}^{B_{1}}(r, z) e^{\mathrm{i}\left(\omega_{0} t+\theta\right)}+c . c . \tag{6.6}
\end{equation*}
$$

with c.c. denoting the complex conjugate. As typical of multiple timescale analyses (Nayfeh, 2008b; Whitham, 1974), the complex amplitudes $A_{1}$ and $B_{1}$, functions of the slow time scale $T_{2}=\epsilon^{2} t$ and still undetermined at this stage of the expansion, describe the slow time amplitude modulation of the two oscillating waves and must be determined at a higher order of the asymptotic expansion.
The natural frequency $\omega_{0}$ and structure $\hat{\mathbf{q}}_{1}^{A_{1}}$ (and $\hat{\mathbf{q}}_{1}^{B_{1}}$ ) assume the meaning of eigenvalue and associated eigenmode of the leading order linearized sloshing operator, whose matrix compact form can be written as $\left(\mathrm{i} \omega_{0} \mathscr{B}-\mathscr{A}_{m= \pm 1}\right) \hat{\mathbf{q}}_{1}^{A_{1}, B_{1}}=0$ (see Viola and Gallaire (2018), Bongarzone et al. (2022a) and Marcotte et al. (2023a) for the expression of $\mathscr{B}$ and $\mathscr{A}_{m}$ ). As in Bongarzone et al. (2022a), those matrices are numerically discretized in space by means of a Gauss-Lobatto-Chebyshev pseudo-spectral collocation method with a two-dimensional mapping implemented in Matlab, which is analogous to the method described in Viola and Gallaire (2018) and Bongarzone et al. (2021c).
By pursuing the expansion to the second order in $\epsilon$, one obtains a linear system forced by second-order non-linear terms produced by combinations of the two leading order waves

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Figure 6.5 - (a) First-order and (b)-(e) second-order free surface deformations. Top: top-view of the full surface deformations, reconstructed according to the corresponding azimuthal periodicity and shown for $t=0$, i.e. $\hat{\eta}_{1}^{A_{1}, B_{1}}(r) \cos m \theta$. Bottom: interface as a function of the radial coordinate only and at $\theta=0$, e.g. $\hat{\eta}_{1}^{A 1, B_{1}}(r)$. The first-order solution is normalized with the amplitude and phase of the contact line elevation (at $r=1$ ), such that the free surface $\eta_{1}^{A_{1}, B_{1}}$ is purely real, whereas the potential velocity field $\hat{\Phi}_{1}^{A_{1}, B_{1}}$ is purely imaginary. Note that, owing to the symmetries of the problem, the system admits the following invariant transformation: $\left(\hat{\mathbf{q}},+m, \mathrm{i} \omega_{0}\right) \longrightarrow\left(\hat{\mathbf{q}},-m, \mathrm{i} \omega_{0}\right)$, so that $\hat{\eta}_{1}^{A_{1}}=\hat{\eta}_{1}^{B_{1}}, \hat{\eta}_{2}^{\left|A_{1}\right|^{2}}=\hat{\eta}_{2}^{\left|B_{1}\right|^{2}}$ and $\hat{\eta}_{2}^{A_{1}^{2}}=\hat{\eta}_{2}^{B_{1}^{2}}$. In other words, only part of the first and second-order responses need to be computed explicitly.
through, e.g., $\nabla \Phi_{1} \cdot \nabla \Phi_{1} / 2$ in the dynamic condition and $\nabla \Phi_{1} \cdot \nabla \eta_{1}$ in the kinematic equation. These forcing terms, $\hat{\mathscr{F}}_{2}^{i j}$, are proportional to $A_{1}^{2}$ and $B_{1}^{2}$ (second harmonics), to $\left|A_{1}\right|^{2}$ and $\left|B_{1}\right|^{2}$ (steady and axisymmetric mean flow corrections) and to $A_{1} B_{1}$ and $A_{1} \bar{B}_{1}$ (cross-quadratic interactions), and therefore they call for a second-order solution in the form

$$
\begin{align*}
\mathbf{q}_{2}=\left|A_{1}\right|^{2} \hat{\mathbf{q}}_{2}^{A_{1} \bar{A}_{1}}+\left|B_{1}\right|^{2} \hat{\mathbf{q}}_{2}^{B_{1} \bar{B}_{1}}+ & \left(A_{1}^{2} \hat{\mathbf{q}}_{2}^{A_{1} A_{1}} e^{\mathrm{i} 2\left(\omega_{0} t-\theta\right)}+B_{1}^{2} \hat{\mathbf{q}}_{2}^{B_{1} B_{1}} e^{\mathrm{i} 2\left(\omega_{0} t+\theta\right)}+c . c .\right)  \tag{6.7}\\
& +\left(A_{1} B_{1} \hat{\mathbf{q}}_{2}^{A_{1} B_{1}} e^{\mathrm{i} 2 \omega_{0} t}+A_{1} \bar{B}_{1} \hat{\mathbf{q}}_{2}^{A_{1} \bar{B}_{1}} e^{-\mathrm{i} 2 \theta}+c . c .\right) .
\end{align*}
$$

None of the associated forcing terms being resonant, each spatial structure, $\hat{\mathbf{q}}_{2}^{i j}(r, z)$ can be computed numerically as described in Bongarzone et al. (2022a) by simply inverting the corresponding linear operator, e.g.

$$
\begin{gather*}
\hat{\mathbf{q}}_{2}^{A_{1} \bar{A}_{1}}=\left(-\mathscr{A}_{0}\right)^{-1} \hat{F}_{2}^{A_{1} \bar{A}_{1}}, \quad \hat{\mathbf{q}}_{2}^{A_{1} A_{1}}=\left(\mathrm{i} 2 \omega_{0} \mathscr{B}-\mathscr{A}_{-2}\right)^{-1} \hat{\mathscr{F}}_{2}^{A_{1} A_{1}},  \tag{6.8}\\
\hat{\mathbf{q}}_{2}^{A_{1} B_{1}}=\left(\mathrm{i} 2 \omega_{0}-\mathscr{A}_{0}\right)^{-1} \hat{\mathscr{F}}_{2}^{A_{1} B_{1}}, \quad \hat{\mathbf{q}}_{2}^{A_{1} \bar{B}_{1}}=\left(-\mathscr{A}_{-2}\right)^{-1} \hat{\mathscr{F}}_{2}^{A_{1} \bar{B}_{1}} .
\end{gather*}
$$

The resulting structures are shown in figure 6.5 in terms of free surface deformations.
We now move forward to the $\epsilon^{3}$-order problem, which is once again a linear problem forced by combinations of the first (6.6) and second order (6.7) solutions, produced by third order non-linearities through, e.g., $\left(\nabla \Phi_{1} \cdot \nabla \Phi_{2}+\nabla \Phi_{2} \cdot \nabla \Phi_{1}\right) / 2$ in the dynamic condition or $\nabla \Phi_{1}$. $\nabla \eta_{2}+\nabla \Phi_{2} \cdot \nabla \eta_{1}$ in the kinematic equation, as well as by the slow time- $T_{2}$ derivative of the
leading order solution and by the external forcing, which was assumed of order $\epsilon^{3}$ :

$$
\begin{array}{r}
\left(\partial_{t} \mathscr{B}-\mathscr{A}_{m}\right) \mathbf{q}_{3}=\mathscr{F}_{3}=-\frac{\partial A_{1}}{\partial T_{2}} \mathbf{B} \hat{\mathbf{q}}_{1}^{A_{1}} e^{\mathrm{i}\left(\omega_{0} t-\theta\right)}-\frac{\partial B_{1}}{\partial T_{2}} \mathbf{B} \hat{\mathbf{q}}_{1}^{B_{1}} e^{\mathrm{i}\left(\omega_{0} t+\theta\right)}  \tag{6.9}\\
+\left|A_{1}\right|^{2} A_{1} \hat{\mathscr{F}}_{3}^{\left|A_{1}\right|^{2} A_{1}} e^{\mathrm{i}\left(\omega_{0} t-\theta\right)}+\left|B_{1}\right|^{2} B_{1} \hat{\mathscr{F}}_{3}^{\left|B_{1}\right|^{2} B_{1}} e^{\mathrm{i}\left(\omega_{0} t+\theta\right)} \\
+\left|B_{1}\right|^{2} A_{1} \hat{\mathscr{F}}_{3}^{\left|B_{1}\right|^{2} A_{1}} e^{\mathrm{i}\left(\omega_{0} t-\theta\right)}+\left|A_{1}\right|^{2} B_{1} \hat{\mathscr{F}}_{3}^{\left|A_{1}\right|^{2} B_{1}} e^{\mathrm{i}\left(\omega_{0} t+\theta\right)} \\
+\alpha_{A} F \hat{\mathscr{F}}_{3}^{F} e^{\mathrm{i}\left(\omega_{0} t-\theta\right)} e^{\mathrm{i} \Lambda T_{2}}+\alpha_{B} F \hat{\mathscr{F}}_{3}^{F} e^{\mathrm{i}\left(\omega_{0} t+\theta\right)} e^{\mathrm{i} \Lambda T_{2}} \\
+ \text { N.R.T. }+ \text { c.c. }
\end{array}
$$

with $\hat{\mathscr{F}}_{3}^{F}=\{0, r / 2\}^{T}$ and where N.R.T. stands for non-resonating terms. The latter terms are not strictly relevant for further analysis and can therefore be neglected. The arbitrariness on amplitudes $A_{1}$ and $B_{1}$ is fixed by requiring that secular terms do not appear in the solution to equation (6.9), where secularity results from all resonant forcing terms in $\mathscr{F}_{3}$ (see Chapter 4 for its explicit expression), i.e. all terms sharing the same frequency and wavenumber of $\mathbf{q}_{1}$, e.g. $\left(\omega_{0}, m= \pm 1\right)$, and in effect, all terms explicitly written in (6.9). It follows that a compatibility condition must be enforced through the Fredholm alternative (Friedrichs, 2012), which imposes the amplitudes $A=\epsilon A_{1} e^{-\mathrm{i} \lambda t}$ and $B=\epsilon B_{1} e^{-\mathrm{i} \lambda t}$ to obey the following normal form

$$
\begin{align*}
& \frac{\mathrm{d} A}{\mathrm{~d} t}=-\mathrm{i} \lambda A+\mathrm{i} \alpha_{A} \mu f+\mathrm{i} v|A|^{2} A+\mathrm{i} \xi|B|^{2} A  \tag{6.10a}\\
& \frac{\mathrm{~d} B}{\mathrm{~d} t}=-\mathrm{i} \lambda B+\mathrm{i} \alpha_{B} \mu f+\mathrm{i} v|B|^{2} B+\mathrm{i} \xi|A|^{2} B \tag{6.10b}
\end{align*}
$$

where the physical time $t=T_{2} / \epsilon^{2}$ has been reintroduced and where forcing amplitude and detuning parameter are recast in terms of their corresponding physical values, $f=\epsilon^{3} F$ and $\lambda=\epsilon^{2} \Lambda=\Omega-\omega_{0}$, so as to eliminate the small implicit parameter $\epsilon$ (Bongarzone et al., 2021a, 2022b).
The values of the normal form coefficients $\mu, v$ and $\xi$ as a function of the non-dimensional fluid depth, $H=h / R$, are reported in Appendix 6.5.1. These coefficients, which turn out to be real-valued quantities due to the absence of dissipation, are computed as scalar products between the adjoint mode, $\left(\hat{\mathbf{q}}_{1}^{A_{1} \dagger}, \hat{\mathbf{q}}_{1}^{B_{1} \dagger}\right)$, associated with $\left(\hat{\mathbf{q}}_{1}^{A_{1}}, \hat{\mathbf{q}}_{1}^{B_{1}}\right)$, and the third order resonant forcing terms:

$$
\begin{gather*}
\mathrm{i} \mathscr{I} \mu=<\hat{\mathbf{q}}_{1}^{A_{1} \dagger}, \hat{\mathscr{F}}_{3}^{F}>=\int_{0}^{1}(r / 2) \overline{\hat{\eta}}_{1}^{A_{1} \dagger} r \mathrm{~d} r,  \tag{6.11a}\\
\mathrm{i} \mathscr{I} v=<\hat{\mathbf{q}}_{1}^{A_{1} \dagger}, \hat{\mathscr{F}}_{3}^{\left|A_{1}\right|^{2} A_{1}}>=\int_{0}^{1}\left(\overline{\hat{\eta}}_{1}^{A_{1} \dagger} \hat{\mathscr{F}}_{3_{\mathrm{dyn}}}^{\left|A_{1}\right|^{2} A_{1}}+\overline{\hat{\Phi}}_{1}^{A_{1} \dagger} \hat{\mathscr{F}}_{3_{\mathrm{kin}}}^{\left|A_{1}\right|^{2} A_{1}}\right) r \mathrm{~d} r  \tag{6.11b}\\
\mathrm{i} \mathscr{I} \xi=<\hat{\mathbf{q}}_{1}^{A_{1} \dagger}, \hat{\mathscr{F}}_{3}^{\left|B_{1}\right|^{2} A_{1}}>=\int_{0}^{1}\left(\overline{\hat{\eta}}_{1}^{A_{1} \dagger} \hat{\mathscr{F}}_{3_{\mathrm{dyn}}\left|B_{\mathrm{d}}\right|^{2} A_{1}}+\overline{\hat{\Phi}}_{1}^{A_{1} \dagger} \hat{\mathscr{F}}_{3_{\mathrm{kin}}}^{\left|B_{1}\right|^{2} A_{1}}\right) r \mathrm{~d} r \tag{6.11c}
\end{gather*}
$$

where

$$
\begin{equation*}
\mathscr{I}=<\hat{\mathbf{q}}_{1}^{A_{1} \dagger}, \mathbf{B} \hat{\mathbf{q}}_{1}^{A_{1}}>=\int_{0}^{1}\left(\overline{\hat{\eta}}_{1}^{A_{1} \dagger} \hat{\Phi}_{1}^{A_{1}}+\overline{\hat{\Phi}}_{1}^{A_{1} \dagger} \hat{\eta}_{1}^{A_{1}}\right) r \mathrm{~d} r \tag{6.12}
\end{equation*}
$$

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Here $\left(\hat{\mathbf{q}}_{1}^{A_{1} \dagger}, \hat{\mathbf{q}}_{1}^{B_{1} \dagger}\right)=\left(\overline{\hat{\mathbf{q}}}_{1}^{A_{1}}, \overline{\hat{\mathbf{q}}}_{1}^{B_{1}}\right)$, since the inviscid problem is self-adjoint with respect to the Hermitian scalar product $\left\langle\mathbf{a}, \mathbf{b}>=\int_{V} \overline{\mathbf{a}} \cdot \mathbf{b} \mathrm{~d} V\right.$, with $\mathbf{a}$ and $\mathbf{b}$ two generic vectors (see Viola et al. (2018) for a thorough discussion and derivation of the adjoint problem).

For the sake of brevity, we do not report the expression of the various forcing terms. As an example, the full expression of $\hat{\mathscr{F}}_{3 k i n}^{\left|A_{1}\right|^{2}} A_{1}$ is given in Chapter 4. The other forcing terms are calculated analogously.

### 6.2.3 Phenomenological damping coefficient

Consistently with the inviscid analysis of Faltinsen et al. (2016), the system of amplitude equations (6.10a)-(6.10b) unrealistically predicts counter-waves for $\alpha \rightarrow 1$ (Faltinsen et al., 2016; Raynovskyy and Timokha, 2020), while the condition $\alpha=1$ gives only co-directed waves (Raynovskyy and Timokha, 2018b; Reclari et al., 2014) (see Appendix 6.5.2 for further details) This implies that the response curve branching is not a continuous function of $\alpha$, which is in contradiction with our experimental evidence reported in the next section. By analogy with Raynovskyy and Timokha (2020), we, therefore, introduce in equations (6.10a)-(6.10b) a heuristic damping coefficient, $\sigma$, that serves to regularize the limit for $\alpha \rightarrow 1$.
The value of $\sigma$ is estimated according to the well-known expression (Case and Parkinson, 1957; Henderson and Miles, 1990; Miles, 1967)

$$
\begin{equation*}
\sigma=\underbrace{\frac{2 k^{2}}{R e}}_{\text {bulk }}+\sqrt{\frac{\omega_{0}}{2 R e}} \underbrace{\left(\frac{k \cosh ^{2} k H}{\sinh 2 k H}\right)}_{\text {surf. contamination }}+\sqrt{\frac{\omega_{0}}{2 R e}}(\underbrace{\frac{k}{\sinh 2 k H}}_{\text {bottom }}+\underbrace{\frac{1}{2} \frac{1+(1 / k)^{2}}{1-(1 / k)^{2}}-\frac{k H}{\sinh 2 k H}}_{\text {sidewall }}) . \tag{6.13}
\end{equation*}
$$

The damping associated with lowest natural frequency, $\omega_{0}=\bar{\omega}_{0} / \sqrt{g / R}=\sqrt{k \tanh (k H)}=$ 1.3547 (with wavenumber $k=1.8412$ ) (Lamb, 1993), in a container of diameter $D=2 R=$ 0.172 m filled to a depth $H=h / R=1.744$ with distilled water, i.e. $\rho=1000 \mathrm{~kg} / \mathrm{m}^{3}, \mu=$ $0.001 \mathrm{~kg} / \mathrm{ms}$ and $\gamma=0.072 \mathrm{~N} / \mathrm{m}$, for which $R e=\rho \sqrt{g R^{3}} / \mu=78952$ (Reynolds number), amounts to $\sigma=0.0055$. Typically the viscous damping rate can be interpreted as a slow damping process (Cocciaro et al., 1993; Viola and Gallaire, 2018), i.e. $1 / \sigma \approx 180$, over a faster time scale represented by the wave oscillation, i.e. $1 / \omega_{0} \approx 0.5$. When this hypothesis holds, as in the present experimental study, the damping coefficient is assumed to be small of order $\epsilon^{2}$, such that damping terms as $-\sigma A$ and $-\sigma B$, both of order $\sim \mathrm{O}\left(\epsilon^{3}\right)(A, B \sim \mathrm{O}(\epsilon))$, can be phenomenologically added $a$ posteriori to the final inviscid amplitude equations.
Before moving forward, it is worth noticing that expression (6.13) englobes different effects, i.e. viscous dissipation occurring in the Stokes boundary layers (at the solid lateral and bottom wall), bulk dissipation and possible sources of dissipation associated with free surface contamination effects (Case and Parkinson, 1957; Henderson and Miles, 1994), but it does not account for any form of dissipation induced by contact angle dynamics (Bongarzone et al., 2021c; Dollet et al., 2020; Dussan, 1979; Hocking, 1987; Keulegan, 1959), by wave breaking and overturning (Raynovskyy and Timokha, 2020) or by Prandtl mass-transport phenomena
(Faltinsen and Timokha, 2019; Hutton, 1964).
Moreover, as pointed out in Chapter 4, prediction (6.13) is only valid for small-amplitude capillary-gravity waves, whereas the dissipation rates of forced wave motions are generally more complex, i.e. it is typically a function of the wave amplitude (Raynovskyy and Timokha, 2020). A more rigorous viscous analysis would indeed produce complex eigenfunctions and, therefore, complex-valued normal form coefficients (Bongarzone et al., 2022b), e.g. $v=\operatorname{Re}[v]+\mathrm{i} \operatorname{Im}[v]$ (same for $\xi$ ), so that the effective damping will be asymptotically proportional to the square of the wave amplitude through the cubic term in the amplitude equation, i.e. $\left(\sigma+\operatorname{Im}[v]|A|^{2}+\operatorname{Im}[\xi]|B|^{2}\right)$ for amplitude $A$ and $\left(\sigma+\operatorname{Im}[v]|B|^{2}+\operatorname{Im}[\xi]|A|^{2}\right)$ for amplitude $B$.
For these reasons, we do not expect the heuristic damping model to provide an accurate estimation of the actual amplitude-dependent dissipation of the system, crucial for a correct prediction of the phase-lag between forcing and the system response (Bäuerlein and Avila, 2021). However, accounting for a damping coefficient $\sigma$ in equations (6.14a)-(6.14b) is essential in order to regularize the weakly nonlinear model prediction as the orbit aspect ratio $\alpha$ approaches 1, i.g. for circular orbits.

### 6.2.4 Lowest order asymptotic solution

In conclusion, after accounting for the small damping terms $-\sigma A$ and $-\sigma B$, the lowest order asymptotic solution governing the close-to-resonance interaction of the two $m= \pm 1$ counterpropagating waves is ruled by the following system of complex amplitude equations

$$
\begin{align*}
& \frac{\mathrm{d} A}{\mathrm{~d} t}=-(\sigma+\mathrm{i} \lambda) A+\mathrm{i} \mu \alpha_{A} f+\mathrm{i} v|A|^{2} A+\mathrm{i} \xi|B|^{2} A  \tag{6.14a}\\
& \frac{\mathrm{~d} B}{\mathrm{~d} t}=-(\sigma+\mathrm{i} \lambda) B+\mathrm{i} \mu \alpha_{B} f+\mathrm{i} v|B|^{2} B+\mathrm{i} \xi|A|^{2} B \tag{6.14b}
\end{align*}
$$

The leading order free surface deformation writes

$$
\begin{equation*}
\eta(r, \theta, t)=\hat{\eta}_{1}^{A_{1}, B_{1}}(r)\left(A e^{\mathrm{i}(\Omega t-\theta)}+B e^{\mathrm{i}(\Omega t+\theta)}\right)+c . c . . \tag{6.15}
\end{equation*}
$$

Given the choice of the mode normalization, for which $\hat{\eta}_{1}^{A_{1}, B_{1}}(r=1)=1$, we can express the dimensionless contact line elevation, $\delta(\theta, t) / R$, at any azimuthal coordinate, e.g. at $\theta=0$, as

$$
\begin{equation*}
\delta(0, t) / R=(A+B) e^{\mathrm{i} \Omega t}+c . c . \tag{6.16}
\end{equation*}
$$

This quantity will be used in the next section for comparison with the experimental measurements of the stable stationary wave amplitudes. The stationary solutions and their stability can be computed and predicted from (6.14a)-(6.14b) as explained in Appendix 6.5.2.

# Chapter 6. Swirling against the forcing: evidence of stable counter-directed sloshing waves in orbital-shaken reservoirs 

### 6.3 Comparison with experiments

We now compare, in figure 6.6, our measurements to the asymptotic model (6.14a)-(6.14b). It is important to note that the comparison is outlined only in terms of steady-state wave amplitude. In other words, the experimental transient dynamics following the reverse of the container's direction of motion is ignored and, more generally, the specific structure of such an initial perturbation does not enter the theoretical model, as we only look for large time stationary solutions of equations (6.14a)-(6.14b) with $d / d t=0$.
Figure 6.6 shows that at small ellipticity, e.g. $\alpha$ close to 0.10 , the amplitude response curve is similar to that induced by a purely longitudinal forcing (Marcotte et al., 2023a; Royon-Lebeaud et al., 2007) except that the planar wave solution no longer exists, owing to the preferential direction of motion, and that the co- and counter-rotating waves are no more equally probable, with the counter-wave exhibiting a slightly lower amplitude. By increasing the value of $\alpha$, the counter-wave displays a decreasing amplitude and the range of frequency for which irregular motion occurs shrinks down and ultimately vanishes (Faltinsen et al., 2016). For longitudinal sloshing, irregular motions are the result of an irregular alternation of planar and swirling dynamics (Royon-Lebeaud et al., 2007). In the context discussed here, irregular means that both the co- and counter-swirling solutions are unstable and the system exhibits irregular and chaotic patterns switching between co- and, at a small ellipticity, counterswirling dynamics alternating transient intervals of nearly-planar motions. As $\alpha$ approaches 1 , the admissible frequency range associated with counter-waves reduces and it is eventually suppressed, whereas the frequency range associated with co-directed swirling widens and covers all of the frequency range at $\alpha=0.95$, i.e. approaching the limiting case of a circular trajectory ( $\alpha=1$ ) (Bongarzone et al., 2022a; Reclari et al., 2014). We also observe a decrease in the wave amplitude at $\bar{a}_{x}=3 \mathrm{~mm}$ for $\alpha \geq 0.5$, occurring just before the jump-down frequency (see grey boxes in figure 6.6) and which can be tentatively attributed to highly nonlinear effects, e.g wave breaking leading to the atomization of the wave crests, overlooked by the weakly nonlinear model.
The experimental steady-state wave amplitudes are in good quantitative agreement with the theoretical predictions for all $\bar{a}_{x}$ and $\alpha$ values explored, hence proving the validity of the inviscid analysis in our regime of operation. The only major limitation of the asymptotic analysis is intrinsic to the use of a simple phenomenological damping. As the latter does not depend on the wave amplitude, it cannot accurately predict the phase-lag between forcing and the system response (Bäuerlein and Avila, 2021). This translates into an imprecise estimation of the jump-down frequency occurring above resonance and of the frequency range associated with the counter-swirling, which appears slightly overestimated.

### 6.4 Conclusion

In this work, we have investigated the sloshing dynamics in the vicinity of the first harmonic resonance for container elliptic orbits. The amplitude-response curves at different forcing amplitudes were examined versus the orbit's aspect ratio. We have reported for the first time


Figure 6.6 - Non-dimensional wave amplitude, $\Delta \delta=\left(\max _{t} \delta(0, t)-\min _{t} \delta(0, t)\right) / 2 R$ versus $\Omega / \omega_{0}$ for different values of $\bar{a}_{x}$ (rows) and $\alpha$ (columns). Markers: experiments (black for coand red for counter-waves). The typical dispersion in the measurements is well represented by the size of the markers. Curves: stable branches predicted by the present WNL theory (solid for co- and dashed for counter-waves). Vertical dotted lines indicate frequency values at which experiments have shown irregular motion. Unstable branches are not displayed for the sake of clarity.

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experimental evidence of the existence of a frequency range where stable swirling can be counter-directed with respect to the container's direction of motion. Particularly, our experiments demonstrated the existence of a significant frequency range associated with stable counter-swirling up to surprisingly high orbit aspect ratios.
Our findings have been rationalised by the asymptotic model formalized in Chapter 5 supplemented with a heuristic damping coefficient, which shows how the close-to-resonant sloshing dynamics for any container's elliptic-like orbit is well represented by four degrees of freedom only. This suggests that generalising the resonantly forced spherical pendulum (Miles, 1984c) could provide a suitable mechanical analogy for this entire family of sloshing dynamics, thus offering additional room in this archetypical low degrees-of-freedom class of dynamical systems.
We have discussed how the phenomenological damping is sufficient to resolve the singular limiting behaviour for $\alpha \rightarrow 1$, but its simplistic estimation does not allow for an accurate prediction of the jump-down frequency and of the frequency range associated with counterswirling. The adequate embedding of dissipative viscous effects is a long-standing problem in the hydrodynamics community and still represents a current key challenge in modelling sloshing dynamics. The use of machine learning algorithms has been recently suggested as a pursuable approach (Miliaiev and Timokha, 2023), but their use obviously requires the $a$ priori knowledge of an experimental dataset for training. Therefore, future perspectives of this work could include the extension of the weakly nonlinear model to a viscous framework in the same spirit as Bongarzone et al. (2022b). Although the latter presently hinges on the subtle modelling of the moving contact line dynamics, such an extension is desirable, as it would enable one to better quantify the overall system dissipation and also to predict the viscous streaming experimentally observed in orbitally shaken containers (Bouvard et al., 2017).

### 6.5 Appendix

### 6.5.1 Values of the normal form coefficients

In table 6.1 we report the values of the normal form coefficients, $\mu, v$ and $\xi$ appearing in (6.14a)(6.14b) as a function of the non-dimensional fluid depth $H=h / R$. Note that our experiments have been performed at a fluid depth $H=1.744$.
For completeness we also report the value of the system's lowest natural frequency $\omega_{0}$, which satisfies the well-known dispersion relation for gravity waves $\omega_{0}=\bar{\omega}_{0} / \sqrt{g / R}=\sqrt{k \tanh (k H)}$ (with $k=1.8412$ ) Lamb (1993). We do not report the value of the damping coefficient $\sigma$ as a function of $H$, since for $H \geq 1$ the fluid depth does not significantly affect its value estimated according to (6.13).

| $H=h / R$ | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 | 1.6 | 1.7 | $\mathbf{1 . 7 4 4}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mu$ | -0.279 | -0.280 | -0.281 | -0.282 | -0.283 | -0.283 | -0.283 | $\mathbf{- 0 . 2 8 3}$ |
| $\nu$ | 1.414 | 1.407 | 1.406 | 1.407 | 1.409 | 1.410 | 1.411 | $\mathbf{1 . 4 1 2}$ |
| $\xi$ | -7.487 | -7.914 | -8.101 | -8.211 | -8.281 | -8.328 | -8.359 | $\mathbf{- 8 . 3 6 9}$ |
| $\omega_{0}$ | 1.334 | 1.341 | 1.346 | 1.349 | 1.352 | 1.353 | 1.354 | $\mathbf{1 . 3 5 5}$ |

Table 6.1 - Value of the normal form coefficients appearing in (6.14a)-(6.14b) computed at different non-dimensional fluid depths $H=h / R$ (as reported in table 1 of Marcotte et al. (2023a)) and associated with the lowest natural frequency mode. The subscript $S C$ was used in Bongarzone et al. (2022a) and Marcotte et al. (2023a) to indicate the shape of the associated free surface response close to harmonic resonance, initially denominated single-crest (SC) by Reclari et al. (2014). Here the subscript SC has been omitted, but in practice, $\mu, v$ and $\xi$ coincide with $\mu_{S C}, v_{S C}$ and $\xi_{S C}$ in Marcotte et al. (2023a). For completeness, we also report the value of the system's lowest natural frequency $\omega_{0}$. The bold values correspond to those used in the main document for comparison with experiments.

### 6.5.2 Stationary wave amplitude solutions and their stability

By turning (6.14a)-(6.14b) into polar coordinates, i.e. $A=|A| e^{\mathrm{i} \Phi_{A}}$ and $B=|B| e^{\mathrm{i} \Phi_{B}}$, we can split real and imaginary parts, hence obtaining

$$
\begin{gather*}
\frac{\mathrm{d}|A|}{\mathrm{d} t}=-\sigma|A|+\alpha_{A} \mu f \sin \Phi_{A}  \tag{6.17a}\\
|A| \frac{\mathrm{d} \Phi_{A}}{\mathrm{~d} t}=-\lambda|A|+\alpha_{A} \mu f \cos \Phi_{A}+v|A|^{3}+\xi|B|^{2}|A|,  \tag{6.17b}\\
\frac{\mathrm{d}|B|}{\mathrm{d} t}=-\sigma|B|+\alpha_{B} \mu f \sin \Phi_{B}  \tag{6.17c}\\
|B| \frac{\mathrm{d} \Phi_{B}}{\mathrm{~d} t}=-\lambda|B|+\alpha_{B} \mu f \cos \Phi_{B}+v|B|^{3}+\xi|A|^{2}|B| \tag{6.17d}
\end{gather*}
$$

Let us then decompose amplitudes and phases as the sum of stationary values plus timedependent small perturbations of order $\epsilon \ll 1$.

$$
\mathbf{y}(t)=\left(\begin{array}{c}
|A|(t)  \tag{6.18}\\
\Phi_{A}(t) \\
|B|(t) \\
\Phi_{B}(t)
\end{array}\right)=\left(\begin{array}{c}
a_{0} \\
\phi_{A, 0} \\
b_{0} \\
\phi_{B, 0}
\end{array}\right)+\epsilon\left(\begin{array}{c}
a_{1}(t) \\
\phi_{A, 1}(t) \\
b_{1}(t) \\
\phi_{B, 1}(t)
\end{array}\right)=\mathbf{y}_{0}+\epsilon \mathbf{y}_{1}(t)=\mathbf{y}_{0}+\epsilon\left(\hat{\mathbf{y}}_{1} e^{s t}+\text { c.c. }\right)
$$

with $s=s_{R}+\mathrm{i} s_{I} \in \mathbb{C}$ an eigenvalue and c.c. denoting the complex conjugate part of the small linear perturbation. The substitution of (6.18) in (6.17a)-(6.17d) and the linearization around $\mathbf{y}_{0}$, lead to two problems at order $\epsilon^{0}$ and $\epsilon$, respectively. As the nonlinear system of equations at order $\epsilon^{0}$ does not admit an analytical solution, we apply a numerical procedure after rewriting
the problem in the form:

$$
\mathscr{F}=0=\left\{\begin{array}{l}
\alpha_{A} \mu f \sin \phi_{A, 0}-\sigma a_{0}  \tag{6.19}\\
\alpha_{A} \mu f \cos \phi_{A, 0}-a_{0}\left(\lambda-v a_{0}^{2}-\xi b_{0}^{2}\right) \\
\alpha_{B} \mu f \sin \phi_{B, 0}-\sigma b_{0} \\
\alpha_{B} \mu f \cos \phi_{B, 0}-b_{0}\left(\lambda-v b_{0}^{2}-\xi a_{0}^{2}\right)
\end{array}\right.
$$

System (6.19) is then solved in Matlab function using the built-in function fsolve and prescribing some initial guesses (ig) for $\left(a_{0}^{i g}, \phi_{A, 0}^{i g}, b_{0}^{i g}, \phi_{B, 0}^{i g}\right)$. In practice, we provide in input the external control parameters, $\Omega, a_{x}=\bar{a}_{x} / R$ and $\alpha$, whereas the associated combination of stationary amplitudes and phases, $\left(a_{0}, \phi_{A, 0}, b_{0}, \phi_{B, 0}\right)$ are computed as outputs.
In the following we study the stability properties of these steady-state amplitude and phase solutions. Given the ansatz $\mathbf{y}_{1}(t)=\hat{\mathbf{y}}_{1} e^{s t}+c . c$. , at order $\epsilon$ the linearized and unsteady system, describing the evolution of small amplitude perturbations around the stationary states can be written in a matrix form as

$$
\begin{equation*}
s \mathbf{M} \hat{\mathbf{y}}_{1}=\mathbf{K} \hat{\mathbf{y}}_{1} \tag{6.20}
\end{equation*}
$$

with matrices $\mathbf{M}$ and $\mathbf{K}$ reading

$$
\mathbf{M}=\left[\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{6.21}\\
0 & a_{0} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & b_{0}
\end{array}\right], \quad \mathbf{K}=\left[\begin{array}{cccc}
K_{11} & K_{12} & 0 & 0 \\
K_{21} & K_{22} & K_{24} & 0 \\
0 & 0 & K_{33} & K_{34} \\
K_{41} & 0 & K_{43} & K_{44}
\end{array}\right],
$$

$\hat{\mathbf{y}}_{1}=\left(\hat{a}_{1}, \hat{\phi}_{A, 1}, \hat{b}_{1}, \hat{\phi}_{B, 1}\right)^{T}$ and

$$
\begin{gather*}
K_{11}=-\sigma, K_{33}=-\sigma  \tag{6.22a}\\
K_{12}=\alpha_{A} \mu f \cos \phi_{A, 0}, K_{34}=\alpha_{B} \mu f \cos \phi_{B, 0}  \tag{6.22b}\\
K_{21}=-\lambda+3 v a_{0}^{2}+\xi b_{0}^{2}, K_{43}=-\lambda+3 v b_{0}^{2}+\xi a_{0}^{2}  \tag{6.22c}\\
K_{22}=-\alpha_{A} \mu f \sin \phi_{A, 0}, K_{44}=-\alpha_{B} \mu f \sin \phi_{B, 0}  \tag{6.22d}\\
K_{24}=2 \xi a_{0} b_{0}, K_{41}=2 \xi a_{0} b_{0} \tag{6.22e}
\end{gather*}
$$

We proceed as follows. For each $\left(a_{0}, \phi_{A, 0}, b_{0}, \phi_{B, 0}\right)$, solution of (6.19), we obtain four eigenvalues $s$. If the real part of at least one of these eigenvalues is positive, then that configuration, associated with the set of external parameters $\left(\Omega, a_{x}, \alpha\right)$, is labelled as unstable.


Figure 6.7 - Close-to-resonance branching diagram illustrated in terms of dimensionless wave amplitude, $\Delta \delta$, versus the rescaled forcing frequency $\Omega / \omega_{0}$ and for $\bar{a}_{x}=3 \mathrm{~mm}$. (a) $\alpha=0.99$ and $\sigma=0$; (b) $\alpha=1$ and $\sigma=0$; (c) $\alpha=0.99$ as in (a), but $\sigma=0.0055$ as prescribed by equation (6.13). Panel (c) shows how accounting for a small damping coefficient is sufficient to suppress the counter-directed swirling branch for $\alpha=0.99$, hence regularizing the branching diagram in the limit of $\alpha \rightarrow 1$ clearly highlighted by panels (a) and (b) for $\sigma=0$.

### 6.5.3 Bifurcation diagram for $\alpha \rightarrow 1$

In this Appendix, we illustrate the role of the phenomenological damping coefficient on the branching diagram in the limit of $\alpha \rightarrow 1$. Indeed, we have observed in our experiments that for increasing $\alpha$, the frequency range associated with the existence of a stable counterswirling wave progressively shrinks until it eventually disappears (for $\bar{a}_{x}=3 \mathrm{~mm}$, this occurs between $\alpha=0.85$ and $\alpha=0.95$ ). However, as discussed in $\$ 6.2 .3$, the inviscid model predicts an extended branch associated with stable counter-directed waves for any $\alpha<1$, e.g. $\alpha=0.99$ (see figure 6.7(a) of this document), and no branch at all for $\alpha$ exactly equal to 1 (figure 6.7(b)), thus indicating that the response curves branching is not a continuous function of $\alpha$. Instead, accounting for a damping coefficient, $\sigma$, allows for a continuous shrinking of the counterdirected wave branch, that eventually disappears (figure 6.7 (c)), in qualitative agreement with our experimental observations.

Sub-harmonic Faraday waves in Part III circular cylinders and thin annuli

## Introduction

Orderly and intricate structures often emerge from basic building blocks in nature, such as the crystallization of water molecules into snowflakes or the self-assembly of nucleotides into complex DNA structures. Sand also piles into patterns of ripples or stripes in the desert, showcasing a more diligent and efficient creation process than many human approaches that require piece-by-piece construction.
The assembling of microscale materials has been receiving increasing attention due to high demands in engineering architectures and systems across various fields such as tissue engineering (Athanasiou et al., 2013; Gurkan et al., 2012), microelectromechanical systems (Knuesel and Jacobs, 2010; Stauth and Parviz, 2006), and micro-photonics (Lu et al., 2001). For instance, tissue engineering is particularly interesting as it involves organizing cells into repeating units with well-defined 3D architectures to achieve tissue-specific functions necessary for various applications.
In the context of microscale technologies, several methods are nowadays available for creating various structures using microscale materials. Among those, Chen et al. (2014) presented a highly adaptable and biocompatible method for generating a wide range of structures using microscale materials (see figure III.1). By leveraging the topography of liquid surfaces created by standing waves, they could direct the assembly of a large number of microscale materials into various ordered and symmetric structures. This liquid-based template can be dynamically reconfigured in a very short time (in the order of a few seconds) and allows for scalable and parallel assembly. Moreover, they demonstrated that the assembled structures can be immobilized through chemical- and photo-crosslinking for subsequent use.
In this technique, standing wave patterns are generated by imposing to a partially filled container a vertical harmonic forcing, with an amplitude above a critical threshold, so as to trigger parametric Faraday waves (see also Chapter 1). It is therefore crucial to characterize and predict the hydrodynamics at stake and, particularly, the instability onset of these waves.
In the following, we give an overview of the origin of the Faraday instability. Specifically, we discuss the classical theoretical frameworks typically employed in the prediction and characterization of such standing wave patterns, with a particular focus on some important limitations and oversimplifications of these models. The latter will indeed motivate the studies carried out in Chapters 7 and 8.


Figure III. 1 - Dynamical reconfigurability of liquid-based templated assembly (figure modified from (Chen et al., 2014)). (a) Chamber shape effect on the assembly: circular (top) versus squared (bottom) vessels. (b) A schematic of dynamic reconfiguration of the assembled structures: ( $\mathrm{fA}, \mathrm{aA}$ ) and ( $\mathrm{fB}, \mathrm{aB}$ ) are vibrational frequencies and accelerations for the formation of structures A and B, respectively. (c) Photo crosslinking of the assembled structure. Once the hydrogels were assembled, crosslinking was performed to immobilize the assembled pattern. Scale bars: 4 mm .

When a vessel containing liquid undergoes periodic vertical oscillations, the free liquid surface may be parametrically destabilized with the excitation of standing waves depending on the combination of forcing amplitude and frequency. The threshold at which the instability appears is a function of the corresponding mode dissipation and the excited wavelength is generally specified by the wave whose natural frequency is half that of the parametric excitation, as first noticed by Faraday (1831), who observed experimentally that the resonance was typically of sub-harmonic nature. This observation was later confirmed by Rayleigh (1883a,b), in contrast with Matthiessen (1868, 1870), who observed synchronous vibrations of the free surface with the vertical shaking. The pioneering work of Benjamin and Ursell (1954) gave momentum to the theoretical investigations of the Faraday instability. Using first principles, Benjamin and Ursell (1954) determined the linear stability of the flat free surface of an ideal fluid within a vertically vibrating container displaying a sliding contact line which intersects orthogonally the container sidewalls. The stability is governed by a system of uncoupled Mathieu equations (see Chapter 1), which predict that standing capillary-gravity waves appear
inside the so-called Faraday tongues in the driving frequency-amplitude space, with the wave response that can be sub-harmonic, harmonic or super-harmonic, hence reconciling previous observations.

## Dissipation in absence of walls

The effect of viscous dissipation, taken to be linear and sufficiently small, was initially introduced heuristically (Lamb, 1993; Landau and Lifshitz, 1959) in the inviscid solution, resulting in a semi-phenomenological damped Mathieu equation, which was later proven by the viscous linear Floquet theory of Kumar and Tuckerman (1994b) to be inaccurate, even at small viscosities. An improved version of the damped Mathieu equation, accounting in a more rigorous manner for the dissipation taking place in the free surface and bottom boundary layer, was proposed by Müller et al. (1997), who also noticed in their experiments that the fluid depth can affect the Faraday threshold, with harmonic responses most likely to be triggered for thin fluid layers. The viscous theory of Kumar and Tuckerman (1994b), formulated for a horizontally infinite domain, was found to give good agreement with the small-depth large-aspect-ratio experiments of Edwards and Fauve (1994), where the influence of lateral walls was negligible. If indeed, at large excitation frequencies, where the excited wavelength is much smaller than the container characteristic length, the accessible range of spatial wavenumber is nearly continuous, in the low-frequency regime of single-mode excitation the mode quantization owing to the container sidewall becomes a dominant factor, leading to a discrete spectrum of resonances.

## Mobile contact lines

A generalization of the viscous Floquet theory to spatially finite systems can be readily obtained by analogy with the inviscid formulation of Benjamin and Ursell (1954), as Batson et al. (2013) recently proposed (see figure III.2(a)). It has however intrinsic limitations as it relies on ideal lateral wall conditions, i.e. the unperturbed free surface is assumed to be flat, the contact line is ideally free to slip with a constant zero slope and the stress-free sidewall boundary condition is required for mathematical tractability, since it allows for convenient Bessel-eigenfunctions representation. With the noticeable exception of the sophisticated experiments by Batson et al. (2013) and Ward et al. (2019) using a gliding liquid coating, these assumptions, by overlooking the contact line dynamics, lead in most experimental cases to a considerable underestimation of the actual overall dissipation, resulting in many cases in an inaccurate prediction of the linear Faraday thresholds in small-container experiments (Benjamin and Ursell, 1954; Ciliberto and Gollub, 1985; Das and Hopfinger, 2008; Dodge et al., 1965; Henderson and Miles, 1990; Tipton and Mullin, 2004). The complexity lies primarily in the region of the moving contact line, where molecular, boundary layer and macroscopic scales are intrinsically connected. Despite the significant efforts devoted by several authors to its theoretical understanding (Case and Parkinson, 1957; Cocciaro et al., 1993, 1991; Davis, 1974; Hocking, 1987; Jiang et al., 2004; Keulegan, 1959; Miles, 1967, 1990, 1991; Perlin and Schultz, 2000; Ting and Perlin, 1995), the

## (a) <br> Ideal Side-wall conditions


(b)


Figure III. 2 - (a) Stability map in the driving parameter space computed via Floquet analysis by Batson et al. (2013) for a bi-layer fluid system in a small cylindrical container and assuming ideal sidewall conditions: flat static surface and stress-free sidewall, i.e. the viscous boundary layer at the lateral wall is neglected and the static contact angle is ideally assumed $\theta_{s}=90^{\circ}$. The insets show few free surface shapes. (b) A way to eliminate the static meniscus is to fill the container up to the rim (brimful). This configuration also allows one to control the shape and size of the meniscus by slightly underfilling or overfilling the container (nearlybrimful condition, $\theta_{s} \neq 90^{\circ}$ ). An oscillating meniscus emits meniscus waves, which have a zero threshold, oscillate harmonically with the forcing and appears as concentric ripples. For forcing amplitudes $f$ above the Faraday threshold, $f_{t h}$, those waves interact with the parametric waves and produce new patterns.
comparison with moving-contact-line experiments, due to unavoidable sources of uncertainty in the meniscus dynamics, remained mostly qualitative, rather than quantitative, requiring often the use of fitting parameters, e.g. a larger effective fluid viscosity (Henderson and Miles, 1990).

## Pinned contact lines

A natural means to get rid of the extra dissipation produced by the contact line dynamics is to simply pin the free surface at the edge of the sidewall, i.e. the container is filled to the brim (brimful condition), as shown in figure III.2(b)). In such a condition, the overall dissipation is ruled by that occurring in the fluid bulk and in the Stokes boundary layers at the bottom and at the solid lateral walls, where the fluid obeys the classic no-slip boundary condition, relaxing the stress-singularity at the contact line (Davis, 1974; Huh and Scriven, 1971; Lauga et al., 2007; Miles, 1990; Navier, 1823; Ting and Perlin, 1995). Even in the inviscid context, the problem of a pinned contact line boundary condition is well-posed, as shown by the seminal works of Benjamin and Scott (1979) and Graham-Eagle (1983), who first solved the resulting dispersion relation for inviscid capillary-gravity waves with a free surface pinned at the container brim using a variational approach and a suitable Lagrange multiplier. Since then, several semi-analytical techniques, often combining an inviscid solution with boundary layer approximations and asymptotic expansions accounting for viscous dissipation, have been therefore developed to solve the pinned contact line problem, for example in cylindrical containers (Henderson and Miles, 1994; Kidambi, 2009b; Martel et al., 1998; Miles and Henderson, 1998; Nicolás, 2002, 2005). The resulting predictions of natural frequencies and damping coefficients of these capillary-gravity waves, in contradistinction with the case of a moving contact line, showed a remarkable agreement with experimental measurements (Henderson and Miles, 1994; Howell et al., 2000).

## Ubiquity of Meniscus waves

Within the framework of the Faraday instabilities, this pinned contact line condition can be reached by carefully filling up the vessel to the brimful condition, as done by Douady (1990) and Edwards and Fauve (1994), among others. Nevertheless, as noticed by Bechhoefer et al. (1995), these delicate experimental conditions are not always perfectly achieved, leading to the presence of a minute meniscus. As mentioned for instance by Douady (1990), the meniscus cannot remain steady upon the oscillating vertical motion of the vessel, which results in the emission of travelling waves from the sidewall to the interior. Irrespective of the pinned or free-edge nature of the contact line, these so-called meniscus waves are synchronized with the excitation frequency. They are not generated by the parametric resonance, but rather by the modulation of the gravitational acceleration resulting in an oscillating capillary length. They do not need to overcome a minimal threshold in forcing amplitude to appear, are therefore observable in the whole driving frequency-amplitude space and are well described by a purely linear response, i.e. at sufficiently small forcing amplitude, the meniscus-wave amplitude is


Figure III. 3 - (a) A meniscus, the typical length of which is the capillary length $l_{c}=(\gamma / \rho g)^{1 / 2}$, with $\gamma$ the liquid surface tension, $\rho$ the liquid density and $g$ the gravity acceleration, is always excited by a vertical oscillation. When the cell goes up, the effective gravity is increased and the meniscus length decreases. So it emits a surface wave in order to preserve the mass. For a vertical oscillation of the vessel, the meniscus thus produces an isochronous wave. (b) Photograph of a wave emitted by the meniscus of an oil layer of depth $h=2 \mathrm{~mm}$, in a square cell $80 \times 80 \times 5 \mathrm{~mm}^{2}$, at a forcing frequency of 20 Hz , visualized by the vertical reflection of a light beam. The waves are clearly generated from the boundaries and quickly damped so that the center of the cell is still flat. Without any meniscus, the surface remains flat even during vertical oscillation (Douady, 1990).
proportional to the external forcing amplitude.
As stated by Douady (1990), edge waves constitute a new time-dependent base state on which the instability of parametric waves may develop, possibly blurring the experimental detection of the true Faraday thresholds (see figure III.3). This has led researchers to attempt to suppress edge waves by selecting large-aspect-ratio containers where sidewall effects are negligible, using sloping sides or shelf conditions to mitigate edge waves by impedance matching (Bechhoefer et al., 1995), or employing highly viscous fluid which damps out these waves (Bechhoefer et al., 1995; Douady, 1990).
With interests in pattern formation, pure meniscus-waves-patterns were investigated for themselves by Torres et al. (1995), while complex patterns originated by the coupling of meniscus and Faraday waves were recently described by Shao et al. (2021a,b) for small circularcylinder experiments. A discussion about harmonic Faraday waves disturbed by harmonic meniscus waves is also outlined in Batson et al. (2013), where the presence of edge waves in a small circular-cylinder-bilayer experiment leads to an imperfect bifurcation diagram, also referred to as a tailing effect by Virnig et al. (1988), who analyzed sub-harmonic responses only. Interestingly, in some cases, e.g. liquid-based biosensors for DNA detection (Picard and Davoust, 2007), tunable small-amplitude stationary waves as meniscus waves are actually desired and preferred to saturated larger-amplitude Faraday waves. In such applications, a starting brimful condition, having a contact line fixed at the brim, is ideal since the effective static contact angle at the wall and hence the size and shape of the static meniscus, which will emit edge waves under vertical excitations, can be adjusted simply by increasing or decreasing
the bulk volume (nearly-brimful condition, see figure III.2(b)).
Although the non-conventional eigenvalue problem for natural frequencies and damping coefficients of pinned-contact-line capillary-gravity waves was tackled by several authors mentioned above and in spite of the vastness of literature focused on Faraday waves, there is a lack of a comprehensive theoretical framework for the investigation of such a configuration within the context of Faraday instability.
An important exception is the work of Kidambi (2013). Assuming inviscid Faraday waves in a brimful cylinder with an ideally flat static free surface, he represented the problem using appropriate modal solutions followed by a projection on a test function space and showed that pinned contact line condition resulted in an infinite system of coupled Mathieu equations, unlike the classic case of an ideal moving contact line (Benjamin and Ursell, 1954). Nevertheless, viscosity, crucial for an accurate prediction of the Faraday threshold and the associated emergence of the standing wave pattern, was not included in the analysis, nor was the presence of a static meniscus and its consequent emission of meniscus waves. Some attempts to include meniscus modifications to the Faraday thresholds have been made by several authors by including periodic inhomogeneities (Ito et al., 1999; Tipton, 2003) and phenomenological terms (Lam and Caps, 2011) to an ad hoc damped Mathieu equation.

Following this literature survey, the purpose of Chapter 7 is to take one more step in the direction undertaken by Kidambi (2013), by rigorously accounting for (i) viscous damping, (ii) a pinned contact line and (iii) the presence of a static meniscus at rest. As mentioned above, a contact angle different from 90 degrees not only results in a static meniscus but also induces the emission of meniscus waves as the static meniscus shape is no longer a solution to the forced problem, even below the Faraday threshold. A Floquet-inspired linear theory $\grave{a}$ la Kumar and Tuckerman (1994b) cannot be pursued, as perturbations develop around an oscillating base flow. In contrast, we propose to use the weakly nonlinear approach (WNL) to approximate the linear Faraday bifurcations, although it is expected to involve cumbersome calculations.

## Weakly nonlinear analysis

Weakly nonlinear analyses (Chen and Vinals, 1999; Douady, 1990; Henderson and Miles, 1990; Jian and Xuequan, 2005; Meron and Procaccia, 1986; Miles, 1984b; Milner, 1991; Nagata, 1989; Nayfeh, 1987; Rajchenbach and Clamond, 2015a; Skeldon and Guidoboni, 2007; Zhang and Vinals, 1997) have indeed been widely used in the context of Faraday instabilities to study the wave amplitude saturation via super and subcritical bifurcations, as well as to investigate pattern and quasi-pattern formation (Edwards and Fauve, 1994; Stuart and Fauve, 1993) or spatiotemporal chaos (Ciliberto and Gollub, 1985; Gluckman et al., 1993), arising when two modes with nearly the same frequency share the same unstable region in the parameter space and strongly interact. In contradistinction to these previous studies, the presence of a static meniscus calls for a WNL approach not only to estimate the wave amplitude saturation in
the weakly nonlinear regime but also to predict the Faraday threshold. Hence, with regard to cylindrical straight-sidewalls and sharp-edged containers, as the one considered by Shao et al. (2021b), we derive a WNL model capable of simultaneously accounting for viscous dissipation, static meniscus and meniscus waves, thus allowing us to predict their influence on the linear Faraday threshold for standing capillary-gravity waves with pinned contact line as well as their saturation to finite amplitude. Following the recent experimental evidence of Shao et al. (2021b), we focus on single-mode sub-harmonic resonances. To this end, the full system of equations governing the fluid motion is solved asymptotically by means of the method of multiple timescales, involving a series of linear problems, which are solved numerically. The theoretical model results in a final amplitude equation for the wave amplitude, $B$, whose form corresponds to that derived by Douady (1990) using symmetry arguments solely and keeping low order terms only,

$$
\frac{d B}{d t}=-(\sigma+\mathrm{i} \Lambda / 2) B+\zeta F B^{*}+v|B|^{2} B
$$

where $\sigma$ is the damping coefficient, $\Lambda$ is the frequency-detuning parameter and the star symbol denotes the complex conjugate. This equation correctly predicts the existence of a so-called sub-harmonic Faraday tongue in the driving frequency-amplitude (i.e. the $\Omega_{d}-F_{d}$ ) plane. Within the tongue, the forced response driven at $\Omega_{d}$ is linearly unstable and a solution oscillating $\omega$ (which is sufficiently close to $\Omega_{d} / 2$ ) emerges. The equation above is indeed valid whatever the shape of the static surface, mode structure and the boundary condition are, but the normal form coefficients $\zeta$ and $v$, which account for the effect of the static contact angle and which are complex values owing to the presence of viscosity, are here formally determined in closed form from first principles and computed numerically.

## Faraday waves in Hele-Shaw cells

When it comes to Faraday waves in Hele-Shaw cells, it is even more crucial to pay close attention to the treatment of the sidewall and contact line conditions, as these factors play a dominant role in this configuration.
Recent Hele-Shaw cell experiments have enriched the knowledge of Faraday waves (Faraday, 1831). Researchers have uncovered a new type of highly localized standing waves, referred to as oscillons, that are both steep and solitary-like in nature (Rajchenbach et al., 2011) (see figure III.4(a,b)). These findings have spurred further experimentations with Hele-Shaw cells filled with one or more liquid layers, using a variety of fluids, ranging from silicone oil, and water-ethanol mixtures to pure ethanol (Li et al., 2018b) (figure III.4(c)). Through these experiments, new combined structures produced by triadic interactions of oscillons were discovered by Li et al. (2014) (figure III.4(d,e)). Additionally, another new family of waves was observed in a cell filled solely with pure ethanol and at extremely shallow liquid depths (Li et al., 2016, 2015) (figure III.4(f)).


Figure III. 4 - (a) Even and (b) odd standing solitary waves. Driving frequency, 11 Hz ; vibration amplitude, 4.1 mm ; the wave amplitudes are of the order of 1.2 cm (Rajchenbach et al., 2011). (c) The wave profile of two coupled Faraday waves observed in a two layers system of pure ethanol (depth $d_{1}=4 \mathrm{~mm}$ ) and silicone oil (depth $d_{2}=8 \mathrm{~mm}$ ) in the case of a forcing frequency of 18 Hz and acceleration $16 \mathrm{~m} / \mathrm{s}^{2}$ (Li et al., 2018b). (c,d) High localization of oscillons. Experiments were performed in $15 \%$ ethanol-water solution at a frequency of 18 Hz , an acceleration amplitude of $20.503 \mathrm{~m} / \mathrm{s}^{2}$, and a fluid depth of 2 cm . The right oscillon preserves the same structure in the (d) and (e). The left oscillon is single-peaked in the (d) but becomes doublepeaked in the (e) by additional disturbance of the free surface (Li et al., 2014). (a)-(e) are experimental time snapshots. (f) Snapshots of the Faraday wave profiles in extreme shallow depth ( 2 mm ) and observed for absolute ethanol at a forcing frequency of 18 Hz and forcing acceleration $19.80 \mathrm{~m} / \mathrm{s}^{2}$ (Li et al., 2015). In the various sub-panels, $T$ denotes the wave period.

These findings represent a new contribution to the understanding of the wave behaviour in Hele-Shaw configurations. In this regard, it becomes therefore essential to have a reliable stability theory that can explain and predict the instability onset for the emergence of initial wave patterns.

Notwithstanding two-dimensional direct numerical simulations (Périnet et al., 2016; Ubal et al., 2003) have qualitatively reproduced standing wave patterns reminiscent of those observed experimentally (Li et al., 2014), ignoring the effect of internal wall attenuation leads to an oversimplified model that is not capable of quantitatively predicting the instability regions (Benjamin and Ursell, 1954; Kumar and Tuckerman, 1994a) and is not suitable for modelling Hele-Shaw flows. On the other hand, when attempting to perform three-dimensional simulations of fluid motions within a Hele-Shaw cell, one of the primary challenges that arises is the high computational cost associated with this task. Due to the small dimension in the narrow direction, the grid cell size must be set even smaller in order to accurately capture the
shear dissipation that occurs within the boundary layer. As a result, the computational cost of performing such simulations rapidly increases.
To overcome these challenges and arrive at a more accurate yet efficient approach for resolving the fluid dynamics within this system, researchers have largely invoked the use of Darcy's law to treat the confined fluid between two vertical walls as though it were flowing through a porous medium. When gap-averaging the linearized Navier-Stokes equation, this approximation, which assumes a steady parabolic flow in the short dimension, translates into a real-valued damping coefficient $\sigma \in \mathrm{R}$ that scales as $12 v / b^{2}$, with $v$ the fluid kinematic viscosity and $b$ the cell's gap-size, and which represents the boundary layer dissipation at the lateral walls. However, Darcy's model is known to be inaccurate when unsteady and convective inertias, e.g. through the advection of momentum, are not negligible, such as in waves (Kalogirou et al., 2016). It is not mathematically straightforward to consistently reintroduce convective terms in the gap-averaged Hele-Shaw equations (Plouraboué and Hinch, 2002; Ruyer-Quil, 2001).
Li et al. (2019) applied a Kelvin-Helmholtz-Darcy theory proposed by Gondret and Rabaud (1997) to reintroduce advection and obtain the nonlinear gap-averaged Navier-Stokes equations, which have been then implemented in the open-source code developed by (Popinet, 2003, 2009) to simulate Faraday waves in a Hele-Shaw cell. Although this gap-averaged model has been compared to several experiments showing fairly good agreements, the surface tension term is still two-dimensional, as the out-of-plane interface shape is not directly taken into account. This simplified treatment overlooks the contact line dynamics and may sometimes lead to miscalculations. Advances in this direction were made by Li et al. (2018a), who found that the out-of-plane capillary forces or curvature should be retained in order to improve the description of the wave dynamics, as experimental evidence suggests. By employing a more sophisticated model, coming from molecular kinetics theory (Blake, 1993, 2006; Hamraoui et al., 2000), to include the capillary contact line motion arising from the small scale of the gap-size between the two walls of a Hele-Shaw cell, they derived a novel dispersion relation, which indeed better predicts the observed instability onset.
Unfortunately, they couldn't exactly predict the exact instability thresholds as some discrepancies were still found. This mismatch was tentatively attributed to factors that are not accounted for in the gap-averaged model, such as the extra dissipation on the lateral walls in the elongated direction. Of course, a lab-scale experiment using a rectangular cell cannot entirely replace an infinite-length model, but if the container is sufficiently long, then this extra dissipation should be negligible. Other candidates were identified in the phenomenological contact line model or free surface contaminations.

If these factors can certainly be sources of discrepancies, our guess is that something more profound could be at the origin of the discordance between theory and experiments in the first place.
Despite the use of the Darcy approximation is well-assessed in the literature, the choice of a steady Poiseuille flow to build up the gap-averaged model appears in fundamental contrast with the unsteady nature of oscillatory Hele-Shaw flows, such as Faraday waves. At low enough oscillation frequency $\omega$ or for sufficiently viscous fluids, the thickness of the oscillating Stokes
boundary layer, $\delta^{\prime}=\sqrt{2 v / \omega}$, becomes comparable to the cell gap, $b$, i.e. $2 \delta^{\prime} / b \approx 1$ : the Stokes layers over the lateral solid faces of the cell merge and eventually invade the entire fluid bulk. In such scenarios, the Poiseuille profile gives an adequate flow description, but this requisite is rarely met in the above-cited experimental campaigns. It appears, thus, very natural to ask oneself whether a more appropriate description of the oscillating boundary layer impacts the prediction of stability boundaries.


Figure III. 5 - Womersley velocity profiles (modified figure from San and Staples (2012)) in a cell of width $b$ and for different Womersley number $W o=b \sqrt{2} / \delta^{\prime} . \delta^{\prime}=\sqrt{2 v / \omega}$ denotes the Stokes boundary layer thickness and is a function of the fluid kinematic viscosity $v$ and the characteristic oscillation frequency of the flow $\omega$. For $W o \leq 2$, viscous forces dominate the flow, and the pulse is considered quasi-static with a parabolic profile. For $W o \geq 2$ the inertial forces are dominant in the central core, whereas viscous forces dominate near the boundary layer. Thus, the velocity profile gets flattened, and the phase between the pressure and velocity gets shifted towards the core, with a complete phase opposition in the limit of a plug flow.

The study reported in Chapter 8 is precisely devoted to answering this question by proposing a revised gap-averaged Floquet analysis, based on the classical Womersley-like solution for the pulsating flow in a channel (Womersley, 1955) (see figure III.5).
Following the approach taken by Viola et al. (2017), we examine the impact of inertial effects on the instability threshold of Faraday waves in Hele-Shaw cells, with a focus on the unsteady term of the Navier-Stokes equations. This scenario corresponds to a pulsatile flow where the fluid's motion reduces to a two-dimensional oscillating Poiseuille flow and it seems better suited than the steady Poiseuille profile to investigate the stability properties of the system. When gap-averaging the linearized Navier-Stokes equation, this results in a modified damping coefficient, a function of the ratio between the Stokes boundary layer thickness and the cell's gap, and whose complex value will depend on the frequency of the wave response specific to each unstable parametric region.

First, we consider the case of horizontally infinite rectangular Hele-Shaw cells by also accounting for the same dynamic contact angle model employed by Li et al. (2019), so as to quantify the predictive improvement brought by the present theory. A vis-à-vis comparison with experiments by Li et al. (2019) points out how the standard Darcy model often underestimates the Faraday threshold, whereas the present theory can explain and close the gap with these experiments.
The analysis is then extended to the case of thin annuli. This less common configuration has already been used to investigate oscillatory phase modulation of parametrically forced surface waves (Douady et al., 1989) and drift instability of cellular patterns (Fauve et al., 1991). For our interest, an annular cell is convenient as it naturally filters out the extra dissipation that could take place on the lateral boundary layer in the elongated direction, hence allowing us to reduce the sources of extra uncontrolled dissipation and perform a cleaner comparison with experiments. Our homemade experiments for this configuration highlight how Darcy's theory overlooks a frequency detuning that is essential to correctly predict the locations of the Faraday's tongues in the frequency spectrum. These findings are well rationalized and captured by the present model.

## 7 Sub-harmonic parametric instability in nearly-brimful circular-cylinders: a weakly nonlinear analysis

Remark: this chapter is largely inspired by the publication of the same name.

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In lab-scale Faraday experiments, meniscus waves respond harmonically to small-amplitude forcing without threshold, hence potentially cloaking the instability onset of parametric waves. Their suppression can be achieved by imposing a contact line pinned at the container brim with static contact angle $\theta_{s}=90^{\circ}$ (brimful condition). However, tunable meniscus waves are desired in some applications as those of liquid-based biosensors, where they can be controlled adjusting the shape of the static meniscus by slightly under/over-filling the vessel $\left(\theta_{s} \neq 90^{\circ}\right)$ while keeping the contact line fixed at the brim. Here, we refer to this wetting condition as nearly-brimful. Although classic inviscid theories based on Floquet analysis have been reformulated for the case of a pinned contact line (Kidambi, 2013), accounting for (i) viscous dissipation and (ii) static contact angle effects, including meniscus waves, makes such analyses practically intractable and a comprehensive theoretical framework is still lacking. Aiming at filling this gap, in this work we formalize a weakly nonlinear analysis via multiple timescale method capable to predict the impact of (i) and (ii) on the instability onset of viscous sub-harmonic standing waves in both brimful and nearly-brimful circular-cylinders. Notwithstanding that the form of the resulting amplitude equation is in fact analogous to that obtained by symmetry arguments (Douady, 1990), the normal form coefficients are here computed numerically from first principles, thus allowing us to rationalize and systematically quantify the modifications on the Faraday tongues and on the associated bifurcation diagrams induced by the interaction of meniscus and sub-harmonic parametric waves.

The Chapter is organized as follows. In $\S 7.1$ the flow configuration and governing equations

## Chapter 7. Sub-harmonic parametric instability in nearly-brimful circular-cylinders: a weakly nonlinear analysis



Figure 7.1 - Sketch of a straight-sidewalls sharp-edged cylindrical container of radius $R$ and filled to a depth $h$ with a liquid of density $\rho$ and dynamic viscosity $\mu$. The air-liquid surface tension is denoted by $\gamma$. (a) The free surface, $\eta$, is represented in a generic static configuration characterized by a static contact angle $\theta_{s}$. (b) Generic dynamic configuration under the external vertical periodic forcing of amplitude $F_{d}$ and angular frequency $\Omega_{d}$. The contact line is pinned and the dynamic angle, oscillating around its static value, $\theta_{s}$, is denoted by $\theta$. $(r z)$-plane: reference working plane.
are introduced, while the numerical methods and tools employed in the work are presented in §7.2. In $\S 7.3$ we formulate a linear eigenvalue problem for the damping and natural frequency of viscous capillary-gravity waves with pinned contact line, whose numerical solution is compared with several previous experiments and theories in Appendix 7.7.1. The WNL model for sub-harmonic Faraday resonances is formalized in §7.4. A vis- $\grave{a}$-vis comparison with recent experiments by Shao et al. (2021b) with a pure brimful configuration are discussed before moving to a systematic investigation of meniscus effects. Lastly, for validation purposes, in §7.5 the modified bifurcation diagram presented in $\S 7.4$ is compared for a specific case, i.e. pure axisymmetric dynamics, with fully nonlinear direct numerical simulation (DNS). Final comments and conclusions are outlined in §7.6.

### 7.1 Flow Configuration and governing equations

We consider a cylindrical vessel of radius $R$ and filled to a depth $h$ with a liquid of density $\rho$ and dynamic viscosity $\mu$ (see figure 7.1). The vessel undergoes a vertical periodic acceleration $F_{d}=$ $A_{d} \Omega_{d}^{2}$, where $A_{d}$ and $\Omega_{d}=2 \pi f_{d}$ are the driving amplitude and angular frequency, respectively. In a non-inertial reference frame, the fluid experiences a vertical acceleration due to the unsteady apparent gravitational acceleration $g_{\text {app }}(t)=g\left[1-\left(F_{d} / g\right) \cos \Omega_{d} t\right]$. The viscous fluid motion is thus governed by the incompressible Navier-Stokes equations,

$$
\begin{equation*}
\nabla \cdot \mathbf{u}=0 \quad, \frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u} \cdot \nabla) \mathbf{u}+\nabla p-\frac{1}{R e} \Delta \mathbf{u}=-\left(1-\frac{F_{d}}{g} \cos \Omega_{d} t\right) \hat{\mathbf{e}}_{z} \tag{7.1}
\end{equation*}
$$

where $\mathbf{u}(r, \phi, z, t)=\left\{u_{r}(r, \phi, z, t), u_{\phi}(r, \phi, z, t), u_{z}(r, \phi, z, t)\right\}^{T}$ is the velocity field and $p(r, \phi, z, t)$ is the pressure field. Equations (7.1) are made non-dimensional by using the container's characteristic length $R$, the characteristic velocity $\sqrt{g R}$ and the time scale $\sqrt{R / g}$. The pressure
gauge is set to $\rho g R$. Consequently, the Reynolds number is defined as $R e=\rho g^{1 / 2} R^{3 / 2} / \mu$ and the term on the r.h.s. represents the time-modulation of the non-dimensional gravity acceleration. The domains of validity for $r, \phi$ and $z$ are, respectively, $r \in[0,1], \phi \in[0,2 \pi]$ and $z \in[-h / R, \eta]$, with $\eta(r, \phi, t)$ the interface coordinate. Then, at $z=\eta$ we impose the kinematic and dynamic boundary conditions (b.c.),

$$
\begin{gather*}
\frac{\partial \eta}{\partial t}+\left.u_{r}\right|_{\eta} \frac{\partial \eta}{\partial r}+\frac{\left.u_{\phi}\right|_{\eta}}{r} \frac{\partial \eta}{\partial \phi}-\left.u_{z}\right|_{\eta}=0,  \tag{7.2a}\\
-\left.p\right|_{\eta} \mathbf{n}(\eta)+\left.\frac{1}{R e}\left(\nabla \mathbf{u}+\nabla^{T} \mathbf{u}\right)\right|_{\eta} \cdot \mathbf{n}(\eta)=\frac{1}{B o} \kappa(\eta) \mathbf{n}(\eta), \tag{7.2b}
\end{gather*}
$$

where $\kappa(\eta)$ is the free surface curvature, $\mathbf{n}(\eta)$ is unit vector locally normal to the interface and $B o$ is the Bond number defined as $B o=\rho g R^{2} / \gamma$, with $\gamma$ air-liquid surface tension. At the solid bottom, $z=-h / R=-H$ and sidewall, $r=1$, we impose the no-slip b.c., $\mathbf{u}=\mathbf{0}$. Lastly, the dynamic pinned (or fixed) contact line condition is enforced as

$$
\begin{equation*}
\left.\frac{\partial \eta}{\partial t}\right|_{r=1}=0 . \tag{7.3}
\end{equation*}
$$

### 7.2 Numerical methods and tools

Different numerical approaches are adopted in the present Chapter. The numerical scheme used in the eigenvalue calculation, $\$ 7.3$, and in the weakly nonlinear analysis, $\$ 7.4$, is a staggered Chebyshev-Chebyshev collocation method implemented in Matlab. The three velocity components are discretized using a Gauss-Lobatto-Chebyshev (GLC) grid, whereas the pressure is staggered on Gauss-Chebyshev (GC) grid. Accordingly, the momentum equation is collocated at the GLC nodes and the pressure is interpolated from the GC to the GLC grid, while the continuity equation is collocated at the GC nodes and the velocity components are interpolated from the GLC to the GC grid. This results in the classical $P_{N}-P_{N-2}$ formulation, which automatically suppresses spurious pressure modes in the discretized problem (Viola et al., 2016b; Viola and Gallaire, 2018). A two-dimensional mapping is then used to map the computational space onto the physical space, which has, in general, a curved boundary due to the presence of concave or convex static meniscus. Lastly, the partial derivatives in the computational space are mapped onto the derivatives in the physical space, which depend on the mapping function. For other details see Heinrichs (2004), Canuto et al. (2007), Sommariva (2013) and Viola et al. (2018).

Mesh convergence was tested for different refinements, starting from a grid size of $N_{r}=$ $N_{z}=20$ up to $N_{r}=N_{z}=90$ with a progressive increment of 10 GLC nodes in both directions. $N_{r}$ and $N_{z}$ denotes here the number of radial and axial nodes, respectively. A mesh size of $N_{r}=N_{z}=40$ was seen to be sufficient to ensure a convergence of the natural frequencies and damping coefficients, $\S 7.7 .1$, up to the third digit. However, a mesh $N_{r}=N_{z}=80$ was required to ensure the same convergence for the normal form coefficients in the weakly nonlinear

Chapter 7. Sub-harmonic parametric instability in nearly-brimful circular-cylinders: a weakly nonlinear analysis
model, §7.4.
The weakly nonlinear model presented in $\S 7.4$ involves a third-order asymptotic expansion of the full hydrodynamic system introduced in $\S 7.1$, that turns out to be very tedious to derive analytically. Therefore, the linearization and expansion procedures have been fully automated using the software Wolfram Mathematica, a powerful tool for symbolic calculus, which has been then integrated within the main Matlab code. The Mathematica codes are provided in the supplementary material available at link https://doi.org/10.1017/jfm.2022.600 as a support to the reader.
In $\S 7.5$, the results obtained from the weakly nonlinear analysis are compared and validated for a specific case, i.e. axisymmetric dynamics, with axisymmetric and fully nonlinear direct numerical simulations (DNS), which have been performed using the finite-element software COMSOL Multiphysics v5.6. Further details about the specific DNS setting will be given in §7.5.

### 7.3 Natural oscillations with pinned contact line and static meniscus

Assuming at first the case with zero external forcing, in this section we provide the framework for the numerical study of the damping coefficients and natural frequencies of viscous capillary-gravity waves with fixed contact line and in the presence of a static meniscus. The flow field $\mathbf{q}(r, \phi, z, t)=\{\mathbf{u}(r, \phi, z, t), p(r, \phi, z, t)\}^{T}$ and the interface $\eta(r, \phi, t)$ are decomposed in a static axisymmetric base flow, $\mathbf{q}_{0}(z)=\left\{\mathbf{0}, p_{0}(z)\right\}^{T}$ and $\eta_{0}(r)$, and a small perturbation, $\mathbf{q}_{1}(r, \phi, z, t)=\left\{\mathbf{u}_{1}(r, \phi, z, t), p_{1}(r, \phi, z, t)\right\}^{T}$ and $\eta_{1}(r, \phi, z, t)$, of infinitesimal amplitude $\epsilon$, i.e. $\mathbf{q}=\mathbf{q}_{0}+\epsilon \mathbf{q}_{1}$ and $\eta=\eta_{0}+\epsilon \eta_{1}$.

### 7.3.1 Static meniscus

At rest, the velocity field $\mathbf{u}_{0}$ is null everywhere and the pressure is hydrostatic, i.e. $p_{0}=-z$. Therefore, the static configuration is obtained by solving the nonlinear equation associated with the shape of the axisymmetric static meniscus, $\eta_{0}(r)$,

$$
\begin{equation*}
\eta_{0}-\frac{\kappa\left(\eta_{0}\right)}{B o}=0 \tag{7.4}
\end{equation*}
$$

with $\kappa\left(\eta_{0}\right)=\left(\eta_{0, r r}+\eta_{0, r}\left(1+\eta_{0, r}^{2}\right) / r\right)\left(1+\eta_{0, r}^{2}\right)^{-3 / 2}$. At the centerline, $r=0$, the regularity condition $\eta_{0, r}=0$ holds owing to axisymmetry. The shape of the meniscus is obtained by imposing the geometric relation at the contact line, $r=1$,

$$
\begin{equation*}
\left.\frac{\partial \eta_{0}}{\partial r}\right|_{r=1}=\cot \theta_{s} \tag{7.5}
\end{equation*}
$$

where $\theta_{s}$ is a prescribed static contact angle (see also figure $7.1(\mathrm{a})$ ). When $\theta_{s}$ is set to $\pi / 2$, then the static interface appears flat.

### 7.3.2 Linear eigenvalue problem

Governing equations (7.1) and their boundary conditions (7.2) are then linearized around the static base flow. It follows that at order $\epsilon$ the velocity and pressure fields satisfy the Stokes equations

$$
\begin{equation*}
\nabla \cdot \mathbf{u}_{1}=0, \quad \frac{\partial \mathbf{u}_{1}}{\partial t}+\nabla p_{1}-\frac{1}{R e} \Delta \mathbf{u}_{1}=\mathbf{0} \tag{7.6}
\end{equation*}
$$

with the linearized kinematic and dynamic free surface boundary conditions (at $z=\eta_{0}$ )

$$
\begin{gather*}
\frac{\partial \eta_{1}}{\partial t}+\left.u_{r, 1}\right|_{\eta_{0}} \frac{\partial \eta_{0}}{\partial r}-\left.u_{z, 1}\right|_{\eta_{0}}=0  \tag{7.7}\\
-\left.p_{1}\right|_{\eta_{0}} \mathbf{n}\left(\eta_{0}\right)+\eta_{1} \mathbf{n}\left(\eta_{0}\right)+\left.\frac{1}{R e}\left(\nabla \mathbf{u}_{1}+\nabla^{T} \mathbf{u}_{1}\right)\right|_{\eta_{0}} \cdot \mathbf{n}\left(\eta_{0}\right)=\left.\frac{1}{B o} \frac{\partial \kappa(\eta)}{\partial \eta}\right|_{\eta_{0}} \eta_{1} \mathbf{n}\left(\eta_{0}\right), \tag{7.8}
\end{gather*}
$$

where $\mathbf{n}\left(\eta_{0}\right)=\left\{-\eta_{0, r}, 0,1\right\}^{T}\left(1+\eta_{0, r}^{2}\right)^{-1 / 2}$ and

$$
\begin{equation*}
\left.\frac{\partial \kappa(\eta)}{\partial \eta}\right|_{\eta_{0}} \eta_{1}=\frac{\left(1+\eta_{0, r}^{2}\right)-3 r \eta_{0, r} \eta_{0, r}}{\left(1+\eta_{0, r}^{2}\right)^{5 / 2}} \frac{1}{r} \frac{\partial \eta_{1}}{\partial r}+\frac{1}{\left(1+\eta_{0, r}^{2}\right)^{3 / 2}} \frac{\partial^{2} \eta_{1}}{\partial r^{2}}+\frac{1}{\left(1+\eta_{0, r}^{2}\right)^{1 / 2}} \frac{1}{r^{2}} \frac{\partial^{2} \eta_{1}}{\partial \phi^{2}} \tag{7.9}
\end{equation*}
$$

is the first order variation of the curvature associated with the small perturbation $\epsilon \eta_{1}$. The azimuthal coordinate is denoted by $\phi$. The no-slip boundary condition is imposed at the solid walls, $\mathbf{u}_{1}=\mathbf{0}$, while the pinned contact line condition is enforced at the contact line, $z=\eta_{0}$ and $r=1$,

$$
\begin{equation*}
\left.\frac{\partial \eta_{1}}{\partial t}\right|_{r=1}=0 \tag{7.10}
\end{equation*}
$$

Hence, the linear system can be written in compact form as

$$
\left(\mathscr{B} \partial_{t}-\mathscr{A}\right) \mathbf{q}_{1}=\mathbf{0} \text {, with } \mathscr{B}=\left(\begin{array}{ll}
I & 0  \tag{7.11}\\
0 & 0
\end{array}\right), \mathscr{A}=\left(\begin{array}{cc}
R e^{-1} \Delta & -\nabla \\
\nabla^{T} & 0
\end{array}\right) .
$$

We note that the kinematic and the dynamic b.c.s (7.7) and (7.8) do not explicitly appear in (7.11), but they are imposed as conditions at the interface (Viola and Gallaire, 2018). More precisely, in the numerical scheme, the kinematic condition governing the state variable $\eta$ is implemented as an additional equation dynamically coupled with $\mathbf{u}_{1}$ and $p_{1}$ in (7.11) (this is better clarified in Appendix B of Bongarzone et al. (2021c)), whereas the three stress components of the dynamic condition are enforced as standard boundary conditions in the corresponding components of the momentum equation. The solution can be then expanded

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in terms of normal modes in time and in the azimuthal direction as

$$
\begin{align*}
& \mathbf{q}_{1}(r, \phi, z, t)=\hat{\mathbf{q}}_{1}(r, z) e^{\lambda t} e^{\mathrm{i} m \phi}+\text { c.c. }  \tag{7.12}\\
& =\left\{\hat{u}_{1 r}(r, z), \hat{u}_{1 \phi}(r, z), \hat{u}_{1 z}(r, z), \hat{p}_{1}(r, z)\right\}^{T} e^{\lambda t} e^{\mathrm{i} m \phi}+c . c ., \\
& \eta_{1}(r, \phi, t)=\hat{\eta}_{1}(r) e^{\lambda t} e^{\mathrm{i} m \phi}+c . c . . \tag{7.13}
\end{align*}
$$

Substituting the normal form (7.12)-(7.13) in system (7.11), we obtain a generalized linear eigenvalue problem,

$$
\begin{equation*}
\left(\lambda \mathscr{B}-\mathscr{A}_{m}\right) \hat{\mathbf{q}}_{1}=\mathbf{0}, \tag{7.14}
\end{equation*}
$$

where the linear operator $\mathscr{A}_{m}$ depends on the azimuthal wavenumber $m$ and $\hat{\mathbf{q}}_{1}$ is the global mode associated with the eigenvalue $\lambda=-\sigma+\mathrm{i} \omega$, with $\sigma$ and $\omega$ the damping coefficient and the natural frequency, respectively, of the ( $m, n$ ) global mode. Here the indices ( $m, n$ ) represent the number of nodal circles and nodal diameters, respectively. Owing to the normal mode expansion (7.12), we notice that the operator $\mathscr{A}_{m}$ is complex since $\phi$ derivatives produce i $m$ terms. A complete expansion of the complex operator can be found in Meliga et al. (2009b) and Viola and Gallaire (2018).
In order to regularize the problem at the axis, depending on the selected azimuthal wavenumber $m$, different regularity conditions must be imposed at $r=0$ (Liu and Liu, 2012; Viola and Gallaire, 2018),

$$
\begin{align*}
& |m|=0: \quad \hat{u}_{1 r}=\hat{u}_{1 \phi}=\frac{\partial \hat{u}_{1 z}}{\partial r}=\frac{\partial \hat{p}_{1}}{\partial r}=0,  \tag{7.15a}\\
& |m|=1: \quad \frac{\partial \hat{u}_{1 r}}{\partial r}=\frac{\partial \hat{u}_{1 \phi}}{\partial r}=\hat{u}_{1 z}=\hat{p}_{1}=0,  \tag{7.15b}\\
& |m|>0: \quad \hat{u}_{1 r}=\hat{u}_{1 \phi}=\hat{u}_{1 z}=\hat{p}_{1}=0 . \tag{7.15c}
\end{align*}
$$

Lastly, owing to the symmetries of the problem, system (7.14) with its boundary conditions is invariant under the

$$
\begin{equation*}
\left(\hat{u}_{1 r}, \hat{u}_{1 \phi}, \hat{u}_{1 z}, \hat{p}_{1}, \hat{\eta}_{1},+m,-\sigma+\mathrm{i} \omega\right) \rightarrow\left(\hat{u}_{1 r},-\hat{u}_{1 \phi}, \hat{u}_{1 z}, \hat{p}_{1}, \hat{\eta}_{1},-m,-\sigma+\mathrm{i} \omega\right) \tag{7.16}
\end{equation*}
$$

transformation, so in this section, $\S 7.3$, we consider only the case with $m \geqslant 0$. Furthermore, the following relations hold

$$
\begin{align*}
& \left(\hat{\mathbf{q}}_{1}, \hat{\eta}_{1},+m,-\sigma+\mathrm{i} \omega\right) \rightarrow\left(\hat{\mathbf{q}}_{1}^{*}, \hat{\eta}_{1}^{*},-m,-\sigma-\mathrm{i} \omega\right),  \tag{7.17}\\
& \left(\hat{\mathbf{q}}_{1}, \hat{\eta}_{1},-m,-\sigma+\mathrm{i} \omega\right) \rightarrow\left(\hat{\mathbf{q}}_{1}^{*}, \hat{\eta}_{1}^{*},+m,-\sigma-\mathrm{i} \omega\right), \tag{7.18}
\end{align*}
$$

(where the star designates the complex conjugate), i.e. the eigenvalues are complex conjugates and all spectra $( \pm m)$ in the ( $\sigma, \omega$ )-plane are symmetric with respect to the real axis $(\omega=0)$, but the complex conjugates of the corresponding eigenvectors, except for the axisymmetric
7.4. Weakly-nonlinear model for sub-harmonic Faraday thresholds with contact angle
effects

| Literature survey | meniscus-free $\left(\theta_{s}=90^{\circ}\right)$ | Acr. | with-meniscus $\left(\theta_{s} \neq 90^{\circ}\right)$ | Acr. |
| :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |
| Experimental | Henderson \& Miles (1994) | HM94 | Cocciaro et al. (1993) | C93 |
| campaigns | Howell et al. (2000) | H2000 | Picard \& Davoust (2007) | PD07 |
| Viscous <br> analyses | Henderson \& Miles (1994) | HM94 |  |  |
|  | Marte et al. (1998) | M98 |  |  |
|  | Miles \& Henderson (1998) | MH98 | Kidambi (2009b) | K09 |
|  | Nicolás (2002) | N02 |  |  |
|  | Kidambi (2009b) | K09 |  |  |
| Inviscid | Graham-Eagle (1983) | GE83 |  | N05 |
| analyses | Henderson \& Miles (1994) | HM94 |  |  |
|  | Kidambi (2013) | K13 | Nicolás (2005) |  |

Table 7.1 - Literature survey on the natural frequencies and damping coefficients of smallamplitude capillary-gravity waves in lab-scale upright cylindrical containers with pinned contact line and in both meniscus-free and with-meniscus configurations. The present work lies within the conditions highlighted by the shaded frames. The case examined by K13 and S21 will be discussed afterwards in $\S 7.4$ within the context of sub-harmonic Faraday waves.
dynamics ( $m=0$ ), are not eigenmodes of the same spectrum. The damping coefficients and natural frequencies of viscous capillary-gravity waves with fixed contact line in both the meniscus-free and with-meniscus configuration are thus computed by solving numerically the generalized eigenvalue problem (7.14), as described in §7.2.
With regard to the literature survey outlined in table 7.1, in Appendix 7.7.1 we propose a thorough validation of our numerical tools via comparison with several pre-existing experiments and theoretical/semi-analytical predictions focusing on both brimful and nearly-brimful circular-cylinders.

### 7.4 Weakly-nonlinear model for sub-harmonic Faraday thresholds with contact angle effects

In this section, the numerical tools presented and validated in $\$ 7.3$ are employed to formalize a weakly nonlinear model accounting for contact angle effects, i.e. static meniscus and harmonic meniscus capillary waves, on the sub-harmonic Faraday instability with pinned contact line.

### 7.4.1 Presentation

Here, the full system (7.1)-(7.3) is solved through a weakly nonlinear (WNL) analysis based on the multiple scale method that is valid in the regime of small perturbations of the static configuration and small external control parameters, namely the driving amplitude and detuning from the parametric resonance. Let us thus introduce the following asymptotic

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expansion for the flow quantities,

$$
\begin{gather*}
\mathbf{q}=\{\mathbf{u}, p\}^{T}=\mathbf{q}_{0}+\epsilon \mathbf{q}_{1}+\epsilon^{2} \mathbf{q}_{2}+\epsilon^{3} \mathbf{q}_{3}+\mathrm{O}\left(\epsilon^{4}\right),  \tag{7.19}\\
\eta=\eta_{0}+\epsilon \eta_{1}+\epsilon^{2} \eta_{2}+\epsilon^{3} \eta_{3}+\mathrm{O}\left(\epsilon^{4}\right) . \tag{7.20}
\end{gather*}
$$

In the spirit of the multiple scale technique, we introduce the slow time scale $T=\epsilon^{2} t$, with $t$ being the fast time scale at which the free surface oscillates. For sub-harmonic resonances the system is expected to respond with a frequency equal to half the driving frequency. In order to determine the boundaries of the instability tongues, we assume the external forcing angular frequency to be $\Omega_{d}=2 \omega+\Lambda$, where $\omega$ is the natural frequency associated with the generic ( $m, n$ ) capillary-gravity wave considered and $\Lambda$ is the detuning parameter. As, by construction, the WNL analysis is valid close to the instability threshold only, we assume a departure from criticality to be of order $\epsilon^{2}$. In terms of control parameters, this assumption translates to the following scalings for the external forcing amplitude, $F_{d} / g$, and detuning $\Lambda$,

$$
\begin{equation*}
F_{d} / g=F=\epsilon^{2} \hat{F}, \quad \Lambda=\epsilon^{2} \hat{\Lambda} . \tag{7.21}
\end{equation*}
$$

It should be noted that the presence of viscosity leads to a damped $\epsilon$-order solution $\mathbf{q}_{11}$ (as discussed in $\S 7.3$ ), whereas standard multiple scale methods apply to marginally stable systems (Nayfeh, 2008a). Nevertheless, as the Reynolds number is typically high enough, the damping coefficient results in a slow damping process over fast wave oscillations (see $\$ 7.3$ ). In such a regime, a multiple scale analysis can still be applied by postulating that the damping coefficient of the ( $m, n$ ) wave is of order $\epsilon^{2}$, i.e. $\sigma=\epsilon^{2} \hat{\sigma}$, therefore the $(m, n$ ) eigenvalue reads $\lambda=-\epsilon^{2} \hat{\sigma}+\mathrm{i} \omega$. A simple way to account for this second order departure from neutrality consists in replacing the leading order operator $\mathscr{A}_{m}=\mathscr{A}_{m}(R e)$ defined in (7.14), for which $\hat{\mathbf{q}}_{1}$ is not neutral, but rather stable, by the shifted operator (Meliga et al., 2009b), $\tilde{\mathcal{A}}_{m}=\mathscr{A}_{m}+\epsilon^{2} \mathscr{S}_{m}$, where $\mathscr{S}_{m}$ is the shift operator defined as $\mathscr{S}_{m} \hat{\mathbf{q}}_{1}=-\hat{\sigma} \hat{\mathbf{q}}_{1}$. The shifted operator $\tilde{\mathcal{A}}_{m}$ is characterized by the same spectra of $\mathscr{A}_{m}$, except that the ( $m, n$ ) eigenmode $\hat{\mathbf{q}}_{1}$ associated with $\hat{\sigma}$ is now marginally stable, and hence the WNL formalism can be applied. For a thorough discussion about the formalism of the shift operator see Meliga et al. (2009b, 2012b). Although a different approach to account for a damped first-order solution was followed by Viola and Gallaire (2018), leading to a different (but equivalent) asymptotic expansion, we use in this Chapter the shift operator approach.
Finally, substituting the asymptotic expansions and scalings above in the governing equations (7.1)-(7.3) with their boundary conditions, a series of problems at the different orders in $\epsilon$ are obtained.
As anticipated in $\$ 7.2$, when contact angle effects are included in the analysis, i.e. the initial static interface is not flat, the third order asymptotic expansion of the full viscous hydrodynamic system introduced in $\$ 7.1$ turns out to be very complex to be derived analytically. Particularly tedious is the dynamic boundary condition, as it involves free surface boundary terms, which, within the linearization process, must be flattened at the static interface, $\eta_{0}$,
as well as the full nonlinear curvature. In order to overcome these practical difficulties, the linearization and expansion procedures have been fully automated using symbolic calculus within the software Wolfram Mathematica, which has been then integrated within the main code implemented in Matlab. The corresponding Mathematica codes are provided as supplementary material available at the link: https://doi.org/10.1017/jfm.2022.600.

### 7.4.2 Order $\epsilon^{0}$ : static meniscus

At order $\epsilon^{0}$ the system reduces to the nonlinear equation associated with the shape of the axisymmetric static meniscus. The velocity field is null, $\mathbf{u}_{0}=\mathbf{0}$ and the pressure is hydrostatic, $p_{0}=-z$. As described in $\S 7.3 .1$, the static interface, $\eta_{0}(r)$, is obtained by prescribing a static contact angle, $\theta_{s}$, which enters through the geometrical relation (7.5) imposed at the contact line.

| $\epsilon^{2}$ | $A_{1}^{+} A_{1}^{+*}$ | $A_{1}^{-} A_{1}^{-*}$ | $\hat{F}$ | $A_{1}^{+} A_{1}^{+}$ | $A_{1}^{-} A_{1}^{-}$ | $A_{1}^{+} A_{1}^{-}$ | $A_{1}^{+} A_{1}^{-*}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m^{i j}$ | 0 | 0 | 0 | $2 m$ | $-2 m$ | 0 | $2 m$ |
| $\omega^{i j}$ | 0 | 0 | $2 \omega$ | $2 \omega$ | $2 \omega$ | $2 \omega$ | 0 |

Table 7.2 - Second order nonlinear forcing terms gathered by their amplitude dependency, and corresponding azimuthal and temporal periodicity $\left(m^{i j}, \omega^{i j}\right)$. Seven terms have been omitted as they are the complex conjugates.

### 7.4.3 Order $\epsilon$ : capillary-gravity waves

At leading order in $\epsilon$ the system is represented by the unsteady Stokes equations (7.6), together with the kinematic and dynamic boundary conditions (7.7)-(7.8), linearized around the static base flow $\mathbf{q}_{0}=\left\{\mathbf{u}_{0}, p_{0}\right\}^{T}=\{\mathbf{0},-z\}^{T}$ and $\eta_{0}$, and subjected to the no-slip b.c. at the solid walls, regularity conditions at the axis (7.15a)-(7.15c), and to the pinned contact line condition (7.10):

$$
\begin{equation*}
\left(\mathscr{B} \partial_{t}-\tilde{\mathscr{A}}_{m}\right) \mathbf{q}_{1}=\mathbf{0} . \tag{7.22}
\end{equation*}
$$

Within the framework of the Faraday instability, we are interested in a standing waveform of the solution, which can be seen as a result of the balance of two counter-rotating waves. Hence, we seek a first-order solution in the form

$$
\begin{align*}
& \mathbf{q}_{1}=A_{1}^{+}(T) \hat{\mathbf{q}}_{1}^{A^{+}} e^{\mathrm{i}(\omega t+m \phi)}+A_{1}^{-}(T) \hat{\mathbf{q}}_{1}^{A^{-}} e^{\mathrm{i}(\omega t-m \phi)}+c . c .  \tag{7.23}\\
& \eta_{1}=A_{1}^{+}(T) \hat{\eta}_{1}^{A^{+}} e^{\mathrm{i}(\omega t+m \phi)}+A_{1}^{-}(T) \hat{\eta}_{1}^{A^{-}} e^{\mathrm{i}(\omega t-m \phi)}+c . c . \tag{7.24}
\end{align*}
$$

( $\eta_{1}$ takes the same form) that destabilizes the static configuration. A single azimuthal wavenumber $m$ is considered at a time. In (7.23) $A_{1}^{+}$and $A_{1}^{-}$, unknown at this stage of the expansion, are the complex amplitudes of the oscillating mode $\hat{\mathbf{q}}_{1}^{A^{+}}$and $\hat{\mathbf{q}}_{1}^{A^{-}}$respectively and they are functions of the slow time scale $T$. The eigensolution of (7.22) has been widely discussed in $\S 7.3$ for $m \geqslant 0$. We note in addition that the eigenmode for the $-m$ perturbation is similar to

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that of the $+m$ perturbation, more precisely, it oscillates with the same frequency $\omega$, but it has the opposite pitch and it rotates in the opposite direction.

### 7.4.4 Order $\epsilon^{2}$ : meniscus waves, second-harmonics and mean-flow corrections

At order $\epsilon^{2}$ we obtain the linearized Stokes equations and boundary conditions applied to $\mathbf{q}_{2}=\left\{\mathbf{u}_{2}, p_{2}\right\}^{T}$ and $\eta_{2}$,

$$
\begin{equation*}
\left(\mathscr{B} \partial_{t}-\tilde{\mathscr{A}}_{m}\right) \mathbf{q}_{2}=\mathscr{F}_{2}, \tag{7.25}
\end{equation*}
$$

and forced by a term $\mathscr{F}_{2}$ depending only on zero-, first-order solutions and on the external forcing

$$
\begin{align*}
& \mathscr{F}_{2}=\left|A_{1}^{+}\right|^{2} \hat{\mathscr{F}}_{2}^{A^{+} A^{+^{*}}}+\left|A_{1}^{-}\right|^{2} \hat{\mathscr{F}}_{2}^{A^{-} A^{-*}}+\left(\hat{F} \hat{\mathscr{F}}_{2}^{\hat{F}} e^{\mathrm{i}(2 \omega t+\hat{\Lambda} T)}+\text { c.c. }\right)+  \tag{7.26}\\
&+\left(A_{1}^{+} \hat{\mathscr{F}}_{2}^{A^{+} A^{+}} e^{\mathrm{i}(2 \omega t+2 m \phi)}+A_{1}^{-2} \hat{\mathscr{F}}_{2}^{A^{-} A^{-}} e^{\mathrm{i}(2 \omega t-2 m \phi)}+\text { c.c. }\right)+ \\
&+\left(A_{1}^{+} A_{1}^{-} \hat{\mathscr{F}}_{2}^{A^{+} A^{-}} e^{\mathrm{i} 2 \omega t}+A_{1}^{+} A_{1}^{-*} \hat{\mathscr{F}}_{2}^{A^{+} A^{-*}} e^{\mathrm{i} 2 m \phi}+\text { c.c. }\right)
\end{align*}
$$

All terms contributing to the forcing vector $\mathscr{F}_{2}$ were extracted using symbolic calculus in Wolfram Mathematica (see supplementary material). The first order solution is made of four different contributions of amplitude $A_{1}^{+}, A_{1}^{+^{*}}, A_{1}^{-}$and $A_{1}^{-^{*}}$, therefore it generates 10 different second-order forcing terms, $\hat{\mathscr{F}}_{2}^{i j} e^{\mathrm{i}\left(\omega^{i j} t+m^{i j} \phi\right)}$, which exhibits a certain frequency and spatial periodicity, gathered in table 7.2. The two additional terms, $\hat{F}_{2} \hat{F}$, appearing in the forcing expression (7.26), come from the spatially uniform axisymmetric external forcing typical of Faraday waves, whose amplitude was assumed to be of order $\epsilon^{2}$. All these forcing terms are non-resonant, as their oscillation frequencies and their spatial symmetries, through the azimuthal wavenumber, differ from those of the leading order solution (see table 7.2). Hence no solvability conditions are required at the present order (Meliga et al., 2009b). We can thus seek for a second-order solution as the superposition of the second-order response to the external forcing, $\hat{\mathbf{q}}_{2}^{\hat{F}}$, and 10 responses $\hat{\mathbf{q}}_{2}^{i j}$ to each single forcing terms,

$$
\begin{align*}
\boldsymbol{q}_{2}= & \left|A_{1}^{+}\right|^{2} \hat{\mathbf{q}}_{2}^{A^{+} A^{+^{*}}}+\left|A_{1}^{-}\right|^{2} \hat{\mathbf{q}}_{2}^{A^{-} A^{-^{*}}}+\left(\hat{F} \hat{\mathbf{q}}_{2}^{\hat{F}} e^{\mathrm{i}(2 \omega t+\hat{\Lambda} T)}+\text { c.c. }\right)+  \tag{7.27}\\
& +\left(A_{1}^{+^{2}} \hat{\mathbf{q}}_{2}^{A^{+^{2}}} e^{\mathrm{i}(2 \omega t+2 m \phi)}+A_{1}^{-2} \hat{\mathbf{q}}_{2}^{A^{-2}} e^{\mathrm{i}(2 \omega t-2 m \phi)}+\text { c.c. }\right)+ \\
& +\left(A_{1}^{+} A_{1}^{-} \hat{\mathbf{q}}_{2}^{A^{+} A^{-}} e^{\mathrm{i} 2 \omega t}+A_{1}^{+} A_{1}^{-^{*}} \hat{\mathbf{q}}_{2}^{A^{+} A^{-*}} e^{\mathrm{i} 2 m \phi}+\text { c.c. }\right)
\end{align*}
$$

(the same form is assumed for $\eta_{2}$ ) each of which is computed as a solution of a linear forced problem

$$
\begin{equation*}
\left(\mathrm{i} \omega^{i j} \mathscr{B}-\tilde{\mathscr{A}}_{m^{i j}}\right) \hat{\mathbf{q}}_{2}^{i j}=\hat{\mathscr{F}}_{2}^{i j}, \tag{7.28}
\end{equation*}
$$

with $m^{i j}$ and $\omega^{i j}$ for $(i, j)$ from table 7.2 and which can be inverted (non-singular operator) as long as any of the combinations ( $m^{i j}, \omega^{i j}$ ) is not an eigenvalue (none of them has $m^{i j}= \pm m$ ). As an example, the $\epsilon$-order eigensurface and some of the various second-order surfaces are


Figure 7.2 - (a)-(f) Top: real part of the free surface elevation, $\operatorname{Re}(\hat{\eta})$ associated with (a) mode $(1,2)$ and with (b)-(f) some of the corresponding second-order responses for different values of the static contact angle, $\theta_{s}$. The $\epsilon$-order solution is normalized such that the phase of the interface at the contact line in $\phi=0$ is zero and the corresponding slope is one, i.e. $\hat{\mathbf{q}}_{1} \rightarrow \hat{\mathbf{q}}_{1} e^{-\mathrm{iarctan}\left[\hat{\eta}_{1}(r=1,0)\right]} /\left(\partial \hat{\eta}_{1}(r, 0) /\left.\partial r\right|_{r=1}\right)$. Bottom: free surface visualization in terms of the absolute value of the real part of the interface slope at $\theta_{s}=45^{\circ}$. The colormaps were individually saturated for visualization purposes only. (g)-(m) Same as (a)-(f), but for mode $(3,2)$. (n)-(s) Same as (a)-(f), but for the axisymmetric mode ( 0,2 ). Parameter setting: $R=$ $0.035 \mathrm{~m}, h=0.022 \mathrm{~m}, \rho=997 \mathrm{~kg} \mathrm{~m}^{-3}, \mu=0.001 \mathrm{~kg} \mathrm{~m}^{-1} \mathrm{~s}^{-1}, \gamma=0.072 \mathrm{Nm}^{-1}$, for which $B o=$ 166.2 and $R e=20437$, and a static contact angle $\theta_{s}=45^{\circ}$. The light red boxes highlight the second-order response to the external forcing, i.e. second-order harmonic meniscus waves.
shown in figure 7.2 for three different waves, i.e. $(m, n)=(1,2),(3,2)$ and $(0,2)$. Owing to the symmetries of the system (given in equation (7.16)), some of the second-order responses corresponding to the generic ( $m, n$ ) wave have the same solution with opposite azimuthal velocity, therefore in figure 7.2 we show only the solutions with different surface shapes. Furthermore, as can be deduced from figure $7.2(\mathrm{n})$-(s), in the axisymmetric case $(0, n)$ all the responses are axisymmetric with zero azimuthal velocity, thus some of the second order responses share exactly the same solution. In this case, indeed, the second order solution

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could be formulated a priori as the sum of three terms only, whose amplitudes are proportional to $\hat{F}, A_{1}^{2}$ (second harmonic) and $\left|A_{1}\right|^{2}$ (mean flow correction), respectively.
Of particular interest is the second-order response to the external forcing, whose interface shape is highlighted by the red boxes in figure 7.2. With the present scaling, the forcing enters at second order in the $z$-component of the momentum equation (see equation (7.1)). If the initial static interface is assumed to be flat $\left(\theta_{s}=90^{\circ}\right)$, then the response $\left(\hat{\mathbf{q}}_{2}^{\hat{F}}, \hat{\eta}_{2}^{\hat{F}}\right)$, translates into a harmonic hydrostatic pressure modulation only, with a free surface remaining flat, i.e. $\hat{\mathbf{u}}_{2}^{\hat{F}}=\mathbf{0}$ and $\hat{\eta}_{2}^{\hat{F}}=0$, a case classically analyzed in the literature. On the other hand, as shown in figure $7.2(\mathrm{c})$, (i) and (p), if a static contact angle $\theta_{s} \neq 90^{\circ}$ is considered, then the $\epsilon^{0}$-order static meniscus induces at order $\epsilon^{2}$ axisymmetric meniscus capillary waves travelling from the sidewall to the interior and reflected back, which oscillates harmonically with the external forcing and with an amplitude proportional to the external forcing amplitude. In the present WNL analysis, these meniscus waves, which appear as concentric ripples (see figure 7.2(c), (i) and (p)), as typically observed in experiments (Batson et al., 2013; Shao et al., 2021a,b), will couple at third order with the first order solution and will contribute to modify both the linear stability boundaries associated with the sub-harmonic Faraday tongues as well as the bifurcation diagram, i.e. wave amplitude saturation to finite amplitude. Furthermore, figure 7.2 clearly shows that a static contact angle $\theta_{s} \neq 90$, depending on its value (here only values of $\theta_{s}<90^{\circ}$ have been considered), modifies not only the damping coefficients and frequencies of the leading order wave (see also figure 7.11 and 7.12), but also its spatial shape and, as consequence, all the associated second-order responses, whose modifications may have a significant influence on the corresponding saturation to a finite amplitude.

### 7.4.5 Order $\epsilon^{3}$ : amplitude equation for standing waves

Lastly, at the $\epsilon^{3}$-order we derive an amplitude equation for standing waves with a pinned contact line accounting for weakly nonlinear modifications of the sub-harmonic Faraday threshold owing to contact angle effects. The problem at order $\epsilon^{3}$ is similar to the one obtained at order $\epsilon^{2}$, as it appears as a linear system,

$$
\begin{equation*}
\left(\mathscr{B} \partial_{t}-\tilde{\mathscr{A}}_{m}\right) \mathbf{q}_{3}=\mathscr{F}_{3}, \tag{7.29}
\end{equation*}
$$

forced by combinations of the previous order solutions encompassed in $\mathscr{F}_{3}$, that contains several nonlinear terms of various space and time periodicities and which we denote as $\hat{\mathscr{F}}_{3}^{i j} e^{\mathrm{i}(\omega t+m \phi)}$. Since many of these terms are resonant, as standard in multiple scale analysis, in order to avoid secular terms and solve the expansion procedure at the third order, a compatibility condition must be enforced through the Fredholm alternative (Friedrichs, 2012). Such a compatibility condition imposes the amplitudes $A_{1}^{+}$and $A_{1}^{-}$to obey the following relation

$$
\begin{equation*}
\frac{d A^{ \pm}}{d t}=-\sigma A^{ \pm}+\zeta F A^{\mp^{*}} e^{\mathrm{i} \Lambda t / 2}+v_{1}\left|A^{ \pm}\right|^{2} A^{ \pm}+v_{2}\left|A^{\mp}\right|^{2} A^{ \pm} \tag{7.30}
\end{equation*}
$$

where the physical time $t=T / \epsilon^{2}$ has been reintroduced and where $\sigma=\epsilon^{2} \hat{\sigma}, F=F_{d} / g=$ $\epsilon^{2} \hat{F}$ and $\Lambda=\epsilon^{2} \hat{\Lambda}$. By considering the expansion $\mathbf{q}=\mathbf{q}_{0}+\epsilon A_{1} \hat{\mathbf{q}}_{1} \ldots$, the small parameter $\epsilon$ is eliminated by defining the amplitude $A=\epsilon A_{1}$, so that everything is recast in terms of actual physical quantities (Bongarzone et al., 2021a, 2022a). The various coefficients are computed as scalar products between the adjoint global modes and the resonant forcing terms $\hat{\mathscr{F}}_{3}^{i j}$, whose analytically complex expressions have been extracted from the third order forcing using the symbolic calculus tools of Wolfram Mathematica. For instance, the complex coefficient $\zeta$ is evaluated as

$$
\begin{equation*}
\zeta=\frac{\int_{\mathrm{u}_{1}}^{\hat{\mathbf{u}}_{1}^{*} A^{+}} \cdot \hat{\mathscr{F}}_{3, N S}^{\hat{F} A^{-*}} r \mathrm{~d} r \mathrm{~d} z+\int_{\eta_{0}} \hat{\mathbf{u}}_{1}^{+^{+} A^{+}} \cdot \hat{\mathscr{F}}_{3, D}^{\hat{F} A^{-*}} r \mathrm{~d} r+\int_{\eta_{0}} \xi^{\psi^{*} A^{+}} \hat{\mathscr{F}}_{3, K}^{\hat{F} A^{*}} r \mathrm{~d} r}{\int_{\mathrm{u}} \hat{\mathbf{u}}_{1}^{*^{*} A^{+}} \cdot \hat{\mathbf{u}}_{1}^{A^{+}} r \mathrm{~d} r \mathrm{~d} z+\int_{\eta_{0}} \xi^{\xi^{*} A^{+}} \hat{\eta}_{1}^{A^{+}} r \mathrm{~d} r} \tag{7.31}
\end{equation*}
$$

where V denotes the fluid bulk domain, the dagger symbol refers to the adjoint eigenmode, $\xi=$ $\left[-\frac{\eta_{0, r}}{R e}\left(\frac{\partial \hat{u}_{1 z}}{\partial r}+\frac{\partial \hat{u}_{r r}}{\partial z}\right)+\left(-\hat{p}_{1}+\frac{2}{R e} \frac{\partial \hat{u}_{1 z}}{\partial z}\right)\right]$ (see also Viola and Gallaire (2018)) and the subscripts ${ }_{N S}$, $D$ and ${ }_{K}$ designate the forcing components of $\hat{\mathscr{F}}_{3}^{\hat{F} A^{-*}}$ appearing in the $\epsilon^{3}$-order Navier-Stokes equations, dynamic boundary condition and kinematic boundary condition, respectively. Analogous expressions hold for $v_{1}$ and $v_{2}$ by replacing $\hat{\mathscr{F}}{ }_{3} A^{-^{*}}$ with $\hat{\mathscr{F}}_{3}^{A^{+} A^{+*} A^{+}}$and $\hat{\mathscr{F}}_{3}^{A^{-} A^{A^{*}} A^{+}}$, respectively. We notice that the adjoint eigenvector appearing in (7.31) does not need to be independently calculated. Indeed, Viola and Gallaire (2018) demonstrated that the linear operator $\mathscr{B}$ and $\mathscr{A}_{m}\left(\right.$ the same applies to the shifted operator $\left.\tilde{\mathscr{A}}_{m}\right)$ are self-adjoint, i.e. $\mathscr{B}^{\dagger}=\mathscr{B}$ and $\mathscr{A}_{m}^{\dagger}=\mathscr{A}_{m}$, with the adjoint eigenvalue being the complex conjugate of the direct one, $\lambda^{\dagger}=$ $\lambda^{*}$. Then, from (7.16), (7.17) and (7.18), it follows that for the couple ( $m,-\sigma+\mathrm{i} \omega$ ) associated with a direct mode, we have the relation

$$
\begin{equation*}
\left(\hat{u}_{1 r}^{*},-\hat{u}_{1 \phi}^{*}, \hat{u}_{1 z}^{*}, \hat{p}_{1}^{*}, \hat{\eta}_{1}^{*}\right) \rightarrow\left(\hat{u}_{1 r}^{\dagger}, \hat{u}_{1 \phi}^{\dagger}, \hat{u}_{1 z}^{\dagger}, \hat{p}_{1}^{\dagger}, \hat{\eta}_{1}^{\dagger}\right), \tag{7.32}
\end{equation*}
$$

which directly provides the desired adjoint mode without any further calculation. We also underline that due to the symmetry of the solution, the same value of $\zeta$ is obtained if one makes use of the scalar product between the adjoint mode for $A_{1}^{-}$and the forcing term $\hat{\mathscr{F}}{ }_{3}^{\hat{F} A^{+}}$ (same for $v_{1}$ and $v_{2}$ ).
As anticipated before, the standing wave solution corresponds to the superposition of two balanced counter-rotating waves of the same amplitude $A^{+}=A^{-}=A$. It follows that system (7.30) reduces to the single amplitude equation

$$
\begin{equation*}
\frac{d B}{d t}=-(\sigma+\mathrm{i} \Lambda / 2) B+\zeta F B^{*}+v|B|^{2} B \tag{7.33}
\end{equation*}
$$

where the change of variable $A=B^{i \Lambda / 2}$ has been introduced and where the complex coefficient $v$ is taken as the sum of $v_{1}$ and $v_{2}$. The form of (7.33) is totally equivalent to the normal form postulated by Douady (1990), using symmetry arguments only, and reported in this thesis in the introduction of Part III. Its structure indeed does not depend on the boundary conditions and on the mode shape, nevertheless, its coefficients do. In the present work these complex coefficients, $\zeta$ and $v$, as well as the frequency and damping of the wave, $\omega$ and $\sigma$, are formally computed by taking into account the full hydrodynamic system, whose solution is exact

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at numerical convergence. The damping coefficient must be small enough, but its value is numerically computed, rather than estimated heuristically. Most importantly, $\zeta$ and $v$, through the WNL formulation presented above, encompass in a formal manner, although within the assumptions of validity of a single-mode WNL theory, the effect of the static contact angle and of the coupling with harmonic meniscus waves (MW) on the sub-harmonic Faraday threshold of standing viscous capillary-gravity waves with pinned contact line.

### 7.4.6 Linear stability of the amplitude equation: sub-harmonic Faraday tongues

Here we perform the stability analysis of the amplitude equation (7.33), which prescribes the marginal stability boundaries, typically known as Faraday's tongues. By turning to polar coordinates

$$
\begin{equation*}
B=|B| e^{\mathrm{i} \varphi}, \quad-(\sigma+\mathrm{i} \Lambda / 2)=c_{1} e^{\mathrm{i} \varphi_{1}}, \quad \zeta=c_{2} e^{\mathrm{i} \varphi_{2}}, \quad v=c_{3} e^{\mathrm{i} \varphi_{3}} \tag{7.34}
\end{equation*}
$$

splitting the modulus and phase parts of (7.33) and introducing the change of variable $\Theta=$ $\Phi-\varphi_{2} / 2$, we obtain the following system

$$
\begin{gather*}
\frac{d|B|}{d t}=c_{1} \cos \left(\varphi_{1}\right)|B|+c_{2} \cos (2 \Theta) F|B|+c_{3} \cos \left(\varphi_{3}\right)|B|^{3}  \tag{7.35}\\
\frac{d \Theta}{d t}=c_{1} \sin \left(\varphi_{1}\right)-c_{2} \sin (2 \Theta) F+c_{3} \sin \left(\varphi_{3}\right)|B|^{2} \tag{7.36}
\end{gather*}
$$

Equation (7.35) admits two possible equilibria ( $d / d t=0$ ), having $|B|=0$ and $|B| \neq 0$, respectively. We first focus on the stability of the trivial stationary solution, $|B|=0$. By eliminating $\Theta$ from (7.35)-(7.36), the linear threshold or marginal stability boundaries (sub-harmonic Faraday tongues) are readily obtained (Douady, 1990; Rajchenbach and Clamond, 2015a),

$$
\begin{equation*}
F_{t h}^{L}=\left(F_{d} / g\right)_{t h}^{L}=c_{1} / c_{2} \rightarrow F_{t h}^{L}= \pm|\zeta|^{-1} \sqrt{\sigma^{2}+\left(\Omega_{d} / 2-\omega\right)^{2}}, \tag{7.37}
\end{equation*}
$$

where the relation $\Lambda=\Omega_{d}-2 \omega$ has been reintroduced and which predicts the lowest threshold, $F_{t h, \text { min }}^{L}=\sigma /|\zeta|$, at $\Omega_{d}=2 \omega$. The forcing amplitude at which the instability appears is therefore proportional to its dissipation, $\sim \sigma$ (note that this is true only for sub-harmonic resonances, e.g. the threshold for harmonic tongues is expected to scale as $\sim \sigma^{1 / 2}$, see Rajchenbach and Clamond (2015a)). Moreover, $F_{t h}^{L}$ depends on the coefficient $\zeta$, which is produced by the interaction of the first order response, proportional to the amplitude $A^{ \pm}$, with the second order response to the external forcing, proportional to $\hat{F}$. Therefore, contact angle modifications of the leading order solution and harmonic meniscus waves (see figure 7.2) enter directly in the calculation of $\zeta$, whose value contributes to the definition of the marginal stability boundaries. Presence of a static meniscus, as widely discussed in $\$ 7.3$, also modifies the natural frequency $\omega$ and the damping $\sigma$. Lastly, it is important to note that within the sub-harmonic tongue (linearly unstable) the signal is periodic with a frequency equal to half the driving frequency, $\Omega_{d} / 2$, whereas out of the tongue (linearly stable) it is periodic with frequency $\Omega_{d}$, owing to the


Figure 7.3 - Inviscid stability plots associated with modes $(1,1)$ and $(1,2)$ for two different Bond numbers, i.e. $B o=1000$ and 100 , and for a depth $h / R=H=1$. The gray-shaded regions have not been reproduced in this work, but rather they have been simply taken from figure 10 of K13. Sub-harmonic tongues are denoted by the subscript $s h$. For computational reasons, the instability regions (grey shaded) were obtained in K13 by truncating the number of basis function $N_{K 13}$ to 2, although convergence of the natural frequencies was achieved by taking $N_{K 13}=30$, as stated by K13 in his table 1 (with a systematic underestimation of approximately $5 \%$ ). The vertical black dash-dot lines correspond to the converged results reported in table 1 of K 13 . The blue solid lines correspond to the present numerical prediction computed through (7.37) for $R e=10^{6}$, while the colored dash-dot lines denote the present Faraday tongues shifted by $5 \%$. Black and colored lines have been added on top of the original figure from K13.
presence of harmonic meniscus waves.

## Brimful condition: validation with the inviscid analysis by K 13 for $\theta_{s}=90^{\circ}$

The most comprehensive investigation of Faraday thresholds with pinned contact line that the authors are aware of is that of K13 (see table 7.1), who considered the case of a perfect brimful condition (meniscus-free) in the inviscid limit. Unlike the classic case of an ideal moving contact line, K13 showed that the pinned contact line problem can be recast into an infinite system of coupled Mathieu equations taking the following form

$$
\begin{equation*}
\frac{d^{2} \mathbf{y}}{d \tau^{2}}+(\mathscr{P}-2 \mathscr{Q} \cos 2 \tau) \mathbf{y}=\mathbf{0} \tag{7.38}
\end{equation*}
$$

where matrices $\mathscr{P}$ and $\mathscr{Q}$, obtained via projection onto the test function space, are in general not diagonal (for a free contact line $\mathscr{P}$ and $\mathscr{Q}$ are diagonal, so that (7.38) reduces to (2.14) of Benjamin and Ursell (1954), i.e. uncoupled Mathieu equations). Three different methods

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(Nayfeh and Mook, 1995) can be used to solve (7.38), namely, (i) the mapping at a period (given by the Floquet theory), (ii) the Hill's infinite determinant method (used by Kumar and Tuckerman (1994b)) and (iii) the multiple scale method. The first two techniques were used in K13 and, particularly, the first one was employed in order to describe the so-called combination resonance tongues (indicated by the black arrows in figure 7.3), which are not studied in the present work focused on sub-harmonic tongues only (see K13 for a thorough discussion). The disadvantage of the multiple-scale method is that generally, it is not suitable for the exploration of a large part of the parameter space, however, as anticipated in the introduction, the application of the first two techniques is challenging when the initial free surface is not assumed to be flat. Here we use the inviscid results provided by K13 to validate the present WNL model for the prediction of sub-harmonic instability onset in the limit of high Reynolds numbers (e.g. Re is assumed to be $\sim 10^{6}$ in the present viscous analysis).
A quantitative comparison of the prediction of sub-harmonic Faraday thresholds with results by K13 is shown in figure 7.3 for $\theta_{s}=90^{\circ}, h / R=H=1$, for two different Bond numbers and for two non-axisymmetric modes, i.e. $(1,1)$ and $(1,2)$. For computational reasons, the instability regions (grey shaded) computed by K13 were obtained by truncating the number of basis function $N_{K 13}$ to 2, although convergence of the natural frequencies was achieved by taking $N_{K 13}=30$, as stated by K13 in his table 1, causing a systematic underestimation of approximately $5 \%$. The vertical black dash-dot lines, corresponding to the converged natural frequencies reported in table 1 for $N_{K 13}=30$, agree perfectly with the present prediction, which prescribes the correct slope of the right and left marginal stability boundaries (blue solid lines). If the present prediction is shifted by $-5 \%$ (orange dash-dot lines), the results match. We can hence conclude that the present model is congruent with the analysis by K13 and it prescribes correctly the sub-harmonic Faraday tongues for a pinned contact line case in the limit of validity of the WNL model, i.e. small external forcing amplitude and small detuning.

## Brimful condition: comparison with recent experiments by $\mathbf{S 2 1}$ for $\theta_{s}=90^{\circ}$

From the knowledge of the authors, no systematic calculations of the linear sub-harmonic Faraday tongues for a pinned contact line and including viscous dissipation are reported in the literature. With regard to small circular-cylinder experiments, this configuration was recently studied by Shao et al. (2021b) (S21). By properly filling the container they could reproduce an initially flat static free surface, which remains stable and flat below the Faraday threshold and thereby they could derive experimentally the boundaries of the unstable regions. Their experimental measurements (extracted from figure 4 of S21) are illustrated in figure 7.4(a), as colored filled circles, together with our numerical prediction from (7.37) (colored solid lines). Shao et al. (2021b) also employed a Rayleigh-Ritz approach (Bostwick and Steen, 2009) to estimate numerically the natural frequency in the inviscid limit and this result, which showed a good agreement with their experiments, is reported for completeness in figure 7.4(a) as vertical black dash-dot lines.

The present numerical analysis for $\theta_{s}=90^{\circ}$ predicts the occurrence of the same sub-
7.4. Weakly-nonlinear model for sub-harmonic Faraday thresholds with contact angle

| $(m, n)$ | $\lambda^{90^{\circ}}$ | $\zeta^{90^{\circ}}$ | $\sigma /\|\zeta\|^{90^{\circ}}$ | $\lambda^{45^{\circ}}$ | $\zeta^{45^{\circ}}$ | $\sigma /\|\zeta\|^{45^{\circ}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $(2,1)$ | $-0.007+\mathrm{i} 1.98$ | $-0.005-\mathrm{i} 0.41$ | 0.016 | $-0.009+\mathrm{i} 1.91$ | $-0.006-\mathrm{i} 0.39$ | 0.022 |
| $(0,1)$ | $-0.003+\mathrm{i} 2.17$ | $-0.003-\mathrm{i} 0.46$ | 0.008 | $-0.004+\mathrm{i} 2.12$ | $-0.107-\mathrm{i} 0.33$ | 0.012 |
| $(3,1)$ | $-0.008+\mathrm{i} 2.44$ | $-0.007-\mathrm{i} 0.47$ | 0.017 | $-0.011+\mathrm{i} 2.37$ | $-0.010-\mathrm{i} 0.47$ | 0.023 |
| $(1,2)$ | $-0.005+\mathrm{i} 2.69$ | $-0.003-\mathrm{i} 0.52$ | 0.010 | $-0.006+\mathrm{i} 2.62$ | $-0.006-\mathrm{i} 0.52$ | 0.012 |
| $(4,1)$ | $-0.010+\mathrm{i} 2.86$ | $-0.008-\mathrm{i} 0.52$ | 0.019 | $-0.014+\mathrm{i} 2.78$ | $-0.020-\mathrm{i} 0.65$ | 0.021 |
| $(2,2)$ | $-0.008+\mathrm{i} 3.16$ | $-0.003-\mathrm{i} 0.56$ | 0.014 | $-0.009+\mathrm{i} 3.08$ | $-0.006-\mathrm{i} 0.59$ | 0.015 |
| $(0,2)$ | $-0.007+\mathrm{i} 3.24$ | $-0.003-\mathrm{i} 0.56$ | 0.012 | $-0.008+\mathrm{i} 3.16$ | $-0.008-\mathrm{i} 0.61$ | 0.013 |
| $(5,1)$ | $-0.012+\mathrm{i} 3.28$ | $-0.010-\mathrm{i} 0.55$ | 0.022 | $-0.017+\mathrm{i} 3.18$ | $-0.014-\mathrm{i} 0.52$ | 0.032 |
| $(3,2)$ | $-0.010+\mathrm{i} 3.63$ | $-0.003-\mathrm{i} 0.58$ | 0.018 | $-0.012+\mathrm{i} 3.53$ | $-0.003-\mathrm{i} 0.66$ | 0.018 |
| $(6,1)$ | $-0.014+\mathrm{i} 3.69$ | $-0.010-\mathrm{i} 0.57$ | 0.025 | $-0.020+\mathrm{i} 3.58$ | $-0.015-\mathrm{i} 0.60$ | 0.034 |

Table 7.3 - Nondimensional natural frequencies, damping coefficients ( $\lambda$ is the eigenvalue $\lambda=-\sigma+\mathrm{i} \omega$ ) and complex normal form coefficient $\zeta=\zeta_{\mathrm{R}}+\mathrm{i} \zeta_{\mathrm{I}}$ for both $\theta_{s}=90^{\circ}$ and $\theta_{s}=45^{\circ}$, associated with the modes shown in figure 7.4 and computed for $R=0.034925 \mathrm{~m}, h=0.022 \mathrm{~m}$, $\rho=1000 \mathrm{~kg} \mathrm{~m}^{-3}, \mu=0.001 \mathrm{~kg} \mathrm{~m}^{-1} \mathrm{~s}^{-1}$ and $\gamma=0.072 \mathrm{Nm}^{-1}$, for which $B o=165.5$ and $R e=$ 20371. The number of points in the radial and axial directions for the GLC grid used is this calculation is $N_{r}=N_{z}=80$, for which convergence up to the third digit is achieved.
harmonic single-mode instabilities in the selected frequency window. In agreement with experimental observations, the viscous WNL analysis prescribes a minimum onset acceleration that is nearly constant for all $(m, n)$-modes in the range $f_{d} \in[10,20] \mathrm{Hz}$ with a discrete spectrum of sub-harmonic resonances. Moreover, the WNL model predicts correctly the coefficient $\zeta$, which prescribes the slope of the transition curves for all tongues.
All the experimental frequencies are slightly larger than the ones predicted here and this shift is roughly of the order of $+1 \%$ for all measurements. It is difficult to attribute a positive $+1 \%$ shift to a specific cause, especially because the pinned contact line configuration is known to produce the largest frequencies among the possible contact line boundary conditions, e.g. a free contact line. The presence of free surface contamination (surface film) is expected to slightly increase the rigidity of the free surface, leading to higher resonance frequencies, but also to larger damping coefficients, which reduce the frequencies (Henderson and Miles, 1990, 1994; Miles, 1967). However, any evidence of surface contamination is reported in S21. In the present case, such a slight systematic mismatch is more likely to be caused by little incongruities between numerics and experiments. For instance, in this case, the Bond number is relatively low, $B o=165.5$, so that little variations in the value of the surface tension or, alternatively, geometrical tolerances on the container radius could contribute to shift the tongues slightly.
In S21 the authors report the nominal container radius $R=0.035$. We have written to the authors, who have kindly provided us with the technical drawing of their cylindrical container. The actual nominal (inner) radius is $R=0.034925 \mathrm{~m}$. Unfortunately, the tolerance on the inner radius is not specified. Nevertheless, given the tolerances specified by the manufacturer on the outer radius, i.e. $\pm 0.000254 \mathrm{~m}$, it is natural to assume at least the same value for the inner one. In figure $7.4(a)$ the sub-harmonic tongues computed for the nominal radius are shown as

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Figure 7.4 - (a) Colored solid lines: boundaries of the sub-harmonic Faraday tongues predicted by (7.37) in the forcing acceleration amplitude-forcing frequency dimensional space, $\left(f_{d}, F_{d}\right)$. Here the static contact angle was set to $\theta_{s}=90^{\circ}$. The colored filled circles correspond to the original experimental values extracted from figure 4 of S21 for different waves ( $m, n$ ). The black dash-dot lines correspond to their inviscid numerical calculation. Parameters: $R=0.034925 \mathrm{~m}$, $h=0.022 \mathrm{~m}, \rho=1000 \mathrm{~kg} \mathrm{~m}^{-3}, \mu=0.001 \mathrm{~kg} \mathrm{~m}^{-1} \mathrm{~s}^{-1}$ and $\gamma=0.072 \mathrm{Nm}^{-1}$, for which $B o=165.5$ and $R e=20371$. Colored bands: marginal stability boundaries computed for a container radius $R=(0.034925-0.000254) \mathrm{m}$ (right boundary) and $R=(0.034925+0.000254) \mathrm{m}$ (left boundary). (b) Modification of the linearly unstable regions due to contact angle effects, where the results for three values of $\theta_{s}$, including $90^{\circ}$ (black dotted lines) as in (a), are compared for a nominal radius $R=0.035 \mathrm{~m}$.
colored solid lines, whereas the colored bands are associated with the geometrical tolerance on the container radius. One can see that a value of $R=0.034925-0.000254=0.034671 \mathrm{~m}$ (right boundaries) is sufficient to produce a $+1 \%$ frequency shift so to achieve a fairly good agreement with the experimental measurements. For completeness, the values of the damping coefficients, natural frequencies and of the normal form coefficient $\zeta$ for two different static contact angles used in figure 7.4 are given in table 7.3.

## Nearly-brimful condition: static contact angle effects and meniscus waves modifications

When the value of the prescribed static contact angle is $\theta_{s} \neq 90^{\circ}$, then the initial static free surface is not flat, but rather concave $\left(\theta_{s}<90^{\circ}\right)$ or convex $\left(\theta_{s}>90^{\circ}\right)$, and its effects on Faraday waves can be studied by exploiting the present WNL analysis.

In $\S 7.3$ we discussed how the static meniscus modifies the natural frequencies and damping coefficients in a non-trivial way depending on the wavenumber of the mode, on the Bond and Reynolds number and on the fluid depth (Kidambi, 2009b) (K09). Moreover, under vertical oscillations, the meniscus emits axisymmetric travelling waves (see figure 7.2(c), (i) and (p)), which, with the WNL scaling adopted in this work, are coupled at third order with the subharmonic parametric waves and hence contribute to alter the instability regions.

With regard to the same configuration of figure 7.4(a) (Shao et al., 2021b), in figure 7.4(b) we examine the influence of these capillary effects on the linear Faraday thresholds. For this configuration the natural frequencies are found to have a maximum for $\theta_{s} \approx 90^{\circ}$ (similarly to figure 7.11 (b) (Picard and Davoust, 2007) (PD07). This suggests that the little shift ( $+1 \%$ ) in the experimental measurements reported in figure 7.4(a) is not due to an uncontrolled nearly-brimful condition. When the static contact angle $\theta_{s}$ is decreased the meniscus introduces a negative shift in all resonances. This translates into a negative shift of all Faraday's tongues in the $\left(f_{d}, F_{d}\right)$-plane, which also shows a slightly higher onset acceleration owing to an increase of the dissipation occurring in the meniscus region (in spite of the fact that the natural frequencies are lower). For $\theta_{s}>90^{\circ}$, e.g. $100^{\circ}$, the onset is slightly lowered (slight decrease of the dissipation occurring in the meniscus region, in agreement with experimental observation by Henderson et al. (1992)). As a result of the mode shape modification by contact angle effects (see figure 7.2(a), (g) and (n)) and of the third order coupling with harmonic meniscus waves, the slope of the transition curves is also altered, but only slightly. In other words, harmonic meniscus waves do not affect significantly the linear instability onsets of these sub-harmonic resonances. This observation is in agreement with Batson et al. (2013), who noticed that a significant meniscus modification is more likely to occur for harmonic Faraday waves and particularly for axisymmetric $(0, n)$ modes. This is somewhat intuitive as meniscus waves, being axisymmetric and having zero thresholds, are essentially indistinguishable from harmonic axisymmetric parametric waves when the driving angular frequency is $\Omega_{d}=\omega_{0 n}$.
Notwithstanding that the coupling between meniscus and sub-harmonic-parametric waves is only weak, the shift in frequency may lead to a reorganization of the discrete spectrum. This is observable in figure $7.4(\mathrm{~b})$ for modes $(0,2)$ and $(5,1)$. Decreasing $\theta_{s}$, the region associated with mode $(5,1)$ progressively lies within that of mode $(0,2)$ and possibly disappears. Having a higher onset acceleration is less likely to be detected. This reorganization is expected to be more pronounced for higher frequency modes, where, for a fixed Bond number, the characteristic mode wavelength becomes comparable and eventually smaller than the characteristic meniscus length, i.e. the (static) capillary length $l_{c} \sim 1 / \sqrt{B o}$, thus enhancing contact angle effects.
Lastly, it should be noted that although parametric waves are linearly stable for all $\theta_{s}$ outside the Faraday's tongues, the free surface (which is maintained flat when $\theta_{s}=90^{\circ}$ ) appears as the superposition of the static meniscus and harmonic meniscus waves, whose amplitude (for a fixed frequency) is proportional to the forcing amplitude, giving rise to an imperfect bifurcation diagram that shows a tailing effect and that will be examined in the following.

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### 7.4.7 Weakly nonlinear threshold and bifurcation diagram

In this paragraph, we focus on the stability of the non-trivial equilibrium, $|B| \neq 0$, of system (7.35)-(7.36). Again, for stationary solutions, we find by eliminating $\Theta$ that

$$
\begin{equation*}
c_{3}|B|^{2}=-c_{1} \cos \left(\varphi_{1}-\varphi_{3}\right) \pm \sqrt{c_{2}^{2} F^{2}-c_{1}^{2} \sin ^{2}\left(\varphi_{1}-\varphi_{3}\right)} \tag{7.39}
\end{equation*}
$$

with physical real solutions for $F \geq \frac{c_{1}}{c_{2}}\left|\sin \left(\varphi_{1}-\varphi_{3}\right)\right|$. This well-known result prescribes either a supercritical or a subcritical transition when the marginal stability boundaries are crossed, i.e. by changing forcing frequency and amplitude. The location of the hysteresis depends on the sign of the nonlinear coefficient $v$ (Kovacic et al., 2018b), which assumes the meaning of a nonlinear detuning, while the boundary of the hysteresis region in the parameter space is defined by the nonlinear threshold

$$
\begin{equation*}
F_{t h}^{N L}=c_{1} / c_{2}\left|\sin \left(\varphi_{1}-\varphi_{2}\right)\right| \tag{7.40}
\end{equation*}
$$

In figure 7.5 the nonlinear wave amplitude saturation, for a fixed external acceleration amplitude, $F_{d}$, and for a varying excitation frequency, $\Omega_{d}$, is shown for two different modes, $(0,2)$ and $(3,2)$, and for different static contact angle values. The linear acceleration threshold (Faraday's tongue) is plotted versus a normalized driving frequency in order to better compare the difference between the two cases with $\theta_{s}=90^{\circ}$ (flat static surface, brimful condition) and $45^{\circ}$ (static meniscus and meniscus waves, nearly-brimful condition). As previously discussed, contact angle modifications on the linear thresholds are only weak. When a concave ( $\theta_{s}<90^{\circ}$ ) static meniscus is considered, the damping is generally higher, the shape of the mode is, however, modified, leading to a slightly different value of the complex linear coefficient $\zeta$ (see table 7.3), which also encompasses the second order coupling between parametric and meniscus waves. As a consequence, the minimum onset acceleration, given by the ratio $\sigma /|\zeta|$, is often comparable.
Supercritical and subcritical bifurcations of Faraday waves have been widely discussed in the literature (see for instance Douady (1990); Rajchenbach and Clamond (2015a) among other references), hence we limit here to recall that if $\cos \left(\varphi_{1}-\varphi_{3}\right)>0$, or alternatively $\Lambda=$ $\Omega_{d}-2 \omega>-2 \sigma v_{R} / v_{I}$, then the bifurcation is supercritical, while if $\cos \left(\varphi_{1}-\varphi_{3}\right)<0$, or $\Lambda=$ $\Omega_{d}-2 \omega<-2 \sigma v_{R} / v_{I}$, the transition is subcritical, the sign of $v_{I}$ determines whether hysteresis occurs on the left-side or on the right-side. The inferior boundary of the hysteresis region in the $\left(\Omega_{d}, F_{d}\right)$-plane is defined by equation (7.40). In other words, the ratio $v_{R} / v_{I}$, through the relation $\varphi_{3}=\tan ^{-1}\left(v_{I} / v_{R}\right)$, determines the importance of the subcritical region in the parameter space (Douady, 1990; Gu and Sethna, 1987; Hsu, 1977; Meron, 1987; Nayfeh and Mook, 1995).
We underline that the amplitude equation coefficients setting the nonlinear threshold and the bifurcation diagram are not calibrated from experimental data, but their values are here computed numerically from first principles through our WNL analysis.

## Wave amplitude increase and sub-criticality suppression




Figure 7.5 - Linear acceleration threshold (Faraday tongue) (left-y-axis, thin solid lines) and saturated wave amplitude, $|B|$, (right-y-axis, thick solid lines) for a fixed acceleration amplitude $F_{d}=0.5 \mathrm{~m} \mathrm{~s}^{-2}$, while the driving frequency is varied. Stable branches for $|B|$ are shown as solid lines, while unstable branches are shown as dashed lines. Two different modes corresponding, namely (a) $(m, n)=(3,2)$ and (b) $(0,2)$, are shown. Different static contact angles are considered. The frequency is normalized with twice the natural frequency of the corresponding excited mode so that the lowest linear threshold occurs for $\Omega_{d} / 2 \omega=1$ for all $\theta_{s}$. At convergence (GLC grid $N_{r}=N_{z}=80$ ), the complex nonlinear amplitude equation coefficient, $v=v_{R}+\mathrm{i} v_{I}$, for mode ( 0,2 ) (subplot (b)), assumed the values, $v^{90^{\circ}}=-0.0909-$ i 1.9094 and $v^{45^{\circ}}=-0.0184-\mathrm{i} 0.5617$. Geometrical and physical parameters are set as in figure 7.2.

We now discuss contact angle modifications on the nonlinear wave amplitude saturation in comparison with the results for the classic case with $\theta_{s}=90^{\circ}$ (flat static interface). A first striking result is shown in figure 7.5(a) for the second axisymmetric mode ( 0,2 ), which displays the bifurcation diagram (in the right y -axis) computed by sweeping the external forcing frequency at a fixed forcing amplitude, i.e. $F_{d}=0.5 \mathrm{~m} \mathrm{~s}^{-2}$ (left y-axis). Figure 7.5(a) shows that, despite contact angle effects do not alter substantially the sub-harmonic Faraday tongue (the unstable region is slightly wider), presence of the meniscus waves, from which the parametric wave bifurcates, can strongly increase the wave amplitude response (up to three times in this case). The magnitude of such an increase is found to be maximum for axisymmetric waves. Again, this can be intuitively explained by considering that axisymmetric parametric and meniscus waves share the same spatial symmetries, despite their different nature, i.e. sub-harmonic versus harmonic responses. Therefore, axisymmetric parametric waves, which emerge on top of meniscus waves, appear to be nonlinearly more destabilized by the latter when compared to other modes.
The second interesting result is shown in figure 7.5(b). In some cases, as for example for mode (3,2), we observe an inversion of the bifurcation diagram, caused by the change of sign of the nonlinear coefficient, $v$, as the static contact angle is varied from $90^{\circ}$ to $45^{\circ}$ (same extrema of figure 7.5(a)). This is mathematically not paradoxical as one more independent

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parameter, i.e. the contact angle $\theta_{s}$, is added to the overall parameter space. The increase of the wave amplitude response with a decrease of $\theta_{s}$ is accompanied by a progressive reduction of the region of hysteresis, until a threshold value, $\theta_{s}^{t h}\left(=56^{\circ}\right.$ for the case of figure $\left.7.5(\mathrm{~b})\right)$, is reached. Eventually, the direction of the bifurcation reverses and the size of the hysteresis region starts to increase again. At the threshold value, $\theta_{s}^{\text {th }}$, corresponding to figure 7.5 (b), the nonlinear coefficient $v$ takes the value $v=-0.0729+\mathrm{i} 0.0083$, yielding a large ratio $v_{R} / v_{I}$ in absolute value, for which the phase $\varphi_{3}$ is nearly $-\pi$, thus meaning that the sub-criticality is totally suppressed and the bifurcation is always supercritical for each combination of external control parameters in $\left(\Omega_{d}, F_{d}\right)$-plane (Douady, 1990). From the knowledge of the authors, such a contact-angle-related behaviour has not been reported in the literature yet, thus suggesting a pursuable direction that future lab-scale and controlled experiments could undertake.

## The imperfect bifurcation diagram: tailing effect



Figure 7.6 - Bifurcation diagram associated with $(m, n)=(0,2)$ (see also figure 7.5(a)) and for a static contact angle $\theta_{s}=45^{\circ}$. Here the dimensional centerline amplitude (axisymmetric dynamic) is reconstructed by summing the various order solutions, i.e. $\eta=\eta_{0}+\eta_{1}+\eta_{2}$ and it is plotted versus the external forcing acceleration for a fixed excitation angular frequency, while different colors correspond to different forcing frequencies. The tailing effect (imperfect bifurcation diagram) produced by the presence of harmonic meniscus waves and indicated by the black thin solid line (the amplitude of meniscus waves grows linearly with $F_{d}$, independently of the parameter combination $\left(\Omega_{d}, F_{d}\right)$ ), is well visible in the right-inset. Colored solid lines are used for stable branches, while colored dashed lines for the unstable ones. The hysteretic loop is indicated by the green arrows. The centerline amplitude is simply computed as $\max _{t} \eta(r=0, t) / 2-\min _{t} \eta(r=0, t) / 2$.

As shown in figure 7.5, the linear threshold given by (7.37) prescribes a stable solution outside the sub-harmonic Faraday tongues (see figure 7.4) with a stationary mode amplitude $|B|=0$. Nevertheless, we remind the reader that the total solution, e.g. in terms of free surface elevation, is given by the sum of the solutions at the various orders in $\epsilon$, i.e. $\eta=\eta_{0}+\eta_{1}(B)+$ $\eta_{2}\left(F_{d}, B^{2},|B|^{2}\right)$. In particular, meniscus waves, whose amplitude is proportional to the external acceleration amplitude, $F_{d}$, are contained in the second-order response $\eta_{2}$. If one considers an axisymmetric dynamics, e.g. $(0,2)$, the amplitude of the centerline elevation is a suitable quantity to monitor the free surface stability and thus to depict a comprehensive bifurcation diagram. This is done in figure 7.6 , where such a bifurcation diagram for $(0,2)$ is reported
for different excitation angular frequencies in a range which gathers both supercritical and subcritical bifurcations. Figure 7.6 clearly shows that, when a nearly-brimful condition is considered, e.g. $\theta_{s}<90^{\circ}$, the sub-harmonic parametric waves, stable outside the Faraday tongues, do not bifurcate from the rest state (as for $\theta_{s}=90^{\circ}$ ), but rather from the meniscus waves solution ( $\propto F_{d}$ ), oscillating harmonically with the driving frequency. This produces a socalled imperfect bifurcation diagram, which displays a tailing effect (highlighted by the black thin solid line) (Virnig et al., 1988). The bifurcation diagram of figure 7.6 is also reminiscent of that presented by Batson et al. (2013), although they focus on harmonic parametric waves.

### 7.5 Validation with axisymmetric direct numerical simulations



Figure 7.7 - (a) Faraday tongue (black solid line) for the axisymmetric mode $(0,2)$ and for a static contact angle $\theta_{s}=45^{\circ}$. Forcing frequency and amplitude in the ( $\Omega_{d}, F_{d}$ )-space, corresponding to the DNS points in (b), are indicated by colored filled markers. Note that the frequency in the x -axis is normalized using the natural frequency $\omega=3.16$ computed for $\theta_{s}=45^{\circ}$. The grey arrows denote the direction followed in the continuation procedure for DNS. For completeness, the Faraday tongue for $\theta_{s}=90^{\circ}$ is reported as grey dashed line. (b) Associated bifurcation digram: WNL prediction (lines) versus DNS (markers). The unstable branch is displayed as colored dashed lines. The black solid line indicating the slope of the meniscus wave response is also given to guide the eyes. The centerline amplitude is computed as $\max _{t} \eta(r=0, t) / 2-$ $\min _{t} \eta(r=0, t) / 2$.

In this section, with the purpose of partially validating the weakly nonlinear analysis, we perform nonlinear direct numerical simulations (DNS) associated with the system of equations (7.1)-(7.3) and, specifically, with the axisymmetric dynamics $(m, n)=(0,2)$, already discussed in $\S 7.4$. Indeed, differently from non-axisymmetric modes $(m, n)$ that would require computationally demanding full three-dimensional DNS, axisymmetric ( $0, n$ ) modes can be solved through axisymmetric DNS, thus reducing the computational burden. To this end, the built-in package for laminar flow with a moving interface and automatic remeshing implemented in the finite-element software COMSOL Multiphysics v5.6. were employed. In the underlying problem, we adopted a hybrid quadrilateral-triangular mesh. Specifically, triangular elements were used in the interior, where little deformations occur, while quadrilateral

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elements were adopted in the neighbourhood of the free surface (larger mesh deformation), sidewalls and bottom, where, in addition, boundary layer refinements were used to properly account for the viscous dissipation taking place in the oscillating Stokes boundary layers (see also figure 7.12). Globally, the grid is made of approximately 60000 mesh elements. $P_{1}-P_{1}$ elements (default), stabilized with a streamline diffusion scheme (SUPG, Streamline Upwind Petrov-Galerkin), were used, leading to roughly 230000 degrees of freedom, for which convergence was tested. Time integration is handled with a mixed-order backward differentiation formula (BDF1/BDF2) with adaptive time-step and the system at each time-step is solved via robust direct method MUMPS (MUltifrontal Massively Parallel sparse direct Solver) coupled with an inner iterative Newton solver.
By simulating an axisymmetric dynamics only, all the other non-axisymmetric instabilities are artificially filtered out, i.e. the Faraday tongues for $(0, n)$ are isolated, enabling a direct comparison of DNS with the single standing-wave expansion adopted in §7.4. Although such a simplification is not realistic, as often multiple tongues may share nearly the same region of instability and the associated parametric waves may therefore interact nonlinearly, it is extremely convenient for validation purposes and it enables us to easily highlight the various effects, i.e. contact angle and meniscus waves modifications of the Faraday threshold, tackled in in §7.4.

### 7.5.1 Procedure

To start, the shape of the static meniscus, computed in Matlab by solving Eq. (7.4) with its boundary conditions (prescribing a static contact angle value, e.g. $\theta_{s}=45^{\circ}$ ) was loaded in COMSOL Multiphysics and the static domain was meshed. First, simulations were initialized for time $t=0$ with a BDF1 scheme giving a zero velocity field and hydrostatic pressure $p=-z$ as initial conditions. A body forcing, corresponding to the non-dimensional time-dependent gravity acceleration, $-1+\left(F_{d} / g\right) \cos \Omega_{d} t$, was assigned. The starting point of the grey arrows in figure 7.7 (b) indicates the combination of external control parameter ( $\Omega_{d}, F_{d}$ ) (colored markers), chosen to initiate the simulations, as described above. Once the stationary state for these initial DNS was established, a continuation procedure (directions of the arrows), by slightly adjusting the external amplitude acceleration and angular frequency, was adopted in order to speed up the computations for all the other combinations of parameters here considered (see figure 7.7).

### 7.5.2 Amplitude saturation and free surface reconstruction: WNL vs. DNS

The WNL prediction (7.39) for the finite amplitude saturation is compared with DNS in figure 7.7. The selected combinations of control parameters, i.e. ( $\Omega_{d}, F_{d}$ ), for DNS calculations are indicated by colored markers in figure 7.7(a), where the grey arrows display the direction followed in the continuation procedure. Once the stationary state is established, i.e. the wave amplitude saturates, and being the underlying dynamics axisymmetric, the centerline free


Figure 7.8 - WNL (black) versus DNS (red) below Faraday threshold (outside the Faraday's tongue) for $\Omega_{d} / \omega^{45^{\circ}}=0.9804$ and $F_{d}=0.85 \mathrm{~m} \mathrm{~s}^{-2}$ (see figure 7.7). (a) Free surface shape computed when the centerline elevation is maximum. For completeness, the shape of the static meniscus for $\theta_{s}=45^{\circ}$ is reported as a black dotted line. (b) Corresponding frequency spectrum: power spectral density (PSD) versus the dimensional oscillation frequency of the system response.
surface elevation is used as a reference measure of the free surface destabilization and of its saturation to finite amplitude. The DNS results are therefore compared with the WNL prediction, where the centerline dynamics is reconstructed by evaluating $\eta=\eta_{0}+\eta_{1}+\eta_{2}$ in $r=0$ for any time. The resulting amplitude comparison is shown in figure 7.7(b). At small forcing amplitude below the Faraday threshold (see also figure 7.7(b)), only harmonic travelling meniscus waves, whose amplitude is proportional to $F_{d}$, are observed in the DNS, consistently with the WNL model (straight line in figure 7.7(b)). In this small amplitude regime, the WNL model and DNS are in fairly good agreement in terms of free surface dynamics (see figures 7.8(a) and (b)). The frequency spectrum in figure 7.8(b) clearly highlights the harmonic nature of these zero-threshold meniscus waves, directly forced by the container sidewalls as soon as the vertical excitation starts. By increasing the external acceleration amplitude $F_{d}$, the stability boundary (Faraday tongue in figure 7.7(a)) is eventually crossed and the parametric wave emerges on the top of edge waves, i.e. it bifurcates from the new stable and harmonically oscillating configuration. Employing a continuation technique by progressively increasing/decreasing the forcing amplitude at different driving frequencies, several DNS were performed in both the supercritical and subcritical regime (respectively filled colored circles and triangles in figure 7.7). The agreement between DNS and WNL prediction in terms of amplitude saturation is found to be fairly good. Moreover, as figure 7.7(a) shows, DNS are consistent with the frequency shift caused by the presence of the static meniscus for $\theta_{s}=45^{\circ}$. As an example, the fully nonlinear free surface dynamics obtained from DNS for $\Omega_{d} / \omega^{45^{\circ}}=1.0054$ and $F_{d}=0.675 \mathrm{~m} \mathrm{~s}^{-2}$ is compared with the WNL reconstruction in figure 7.9(a)-(c) for three different time-instants, while the corresponding centerline elevation and frequency spectrum are provided in figure $7.9(\mathrm{~g})$ and (h), respectively. The WNL model is in agreement with the DNS, which consistently predicts the excitation of a dominant sub-harmonic parametric wave $(0,2)$, coupled with smaller amplitude harmonic meniscus waves as well as with higher order

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Figure 7.9 - WNL (black) versus DNS (red) above Faraday threshold (within the Faraday tongue) for $\Omega_{d} / \omega^{45^{\circ}}=1.0054$ and $F_{d}=0.675 \mathrm{~m} \mathrm{~s}^{-2}$ (see figure 7.7). (a)-(c) Comparison in terms of free surface reconstruction for three different time instants: (a) when the centerline elevation is maximum, (b) when it is zero and equal to the static meniscus position and (c) when it is minimum. For completeness, the shape of the static meniscus for $\theta_{s}=45^{\circ}$ is reported as a black dotted line. (d)-(f) Full three-dimensional visualization extracted from the DNS. (g) Centerline elevation versus time associated with (a), (b) and (c). $t_{0}$ is an arbitrary time-instant. The constant value of the static meniscus elevation at $r=0$ is shown as a black dotted line. (h) Frequency spectrum computed from the time series shown in (g): power spectral density (PSD) versus the dimensional oscillation frequency of the system response.
harmonics (only second harmonics are included in the asymptotic expansion up to the third order in $\epsilon$ ).
As a final comment to this section, while not the purpose of the present analysis, few DNS were performed at higher external acceleration amplitudes, in the parameter region far from the hypotheses of validity of the WNL theory. For the case of figure 7.7(b), preliminary observations revealed that DNS tends to diverge when the centerline elevation approaches a value of approximately 5 mm , suggesting a potential transition to a highly nonlinear wave-breaking condition and eventually to a finite-time singularity with intense jet formation (Basak et al., 2021). See also Das and Hopfinger (2008) for a detailed investigation of the occurrence of such a phenomenon in Faraday experiments.

### 7.6 Conclusion

In this Chapter, we considered sub-harmonic parametric resonances of standing viscous capillary-gravity waves in straight-wall circular-cylindrical containers with brimful (flat static interface) or nearly-brimful (curved meniscus) conditions. We formalized a numerically-based weakly nonlinear expansion (in the spirit of the multiple timescale method) that provides an amplitude equation for the prediction of sub-harmonic Faraday thresholds, which corresponds to the classic one widely discussed by Douady (1990) and other authors using symmetry arguments solely. However, in this work amplitude equation (7.33) has been derived from first principles and the values of the complex normal form coefficients have not a heuristic (or fitting-based) nature, but rather they are obtained in closed form and evaluated numerically.
While a simplified version of the underlying fluid problem, i.e. ideal inviscid fluid and perfect brimful conditions ( $\theta_{s}=90^{\circ}$, meniscus-free), was investigated by Kidambi (2013), the present work accounts for (i) viscous dissipation and (ii) static contact angle effects, including harmonic travelling meniscus waves, i.e. nearly-brimful conditions, realistic features which are typically encountered in real Faraday experiments.
The numerical inviscid analysis by Kidambi (2013) and the recent experimental study by Shao et al. (2021b) were used to validate the WNL model in the simpler case of an initially flat static surface, i.e. meniscus-free configuration with a static contact angle $\theta_{s}=90^{\circ}$ (see figure 7.4(a)). The agreement with experiments by Shao et al. (2021b) was found to be fairly good in the whole frequency window examined. Starting from the reference brimful condition, we progressively introduced in the analysis contact angle effects, simulating the under-filling (or over-filling) of the container. The presence of a static meniscus was shown to determine a negative (at least in the cases examined) frequency shift of all the sub-harmonic Faraday tongues and to slightly increase (or decrease) the minimum onset forcing amplitude, as a consequence of a slightly higher (lower) dissipation in the meniscus region, as expected from previous studies. In addition, contact angle modifications, altering the position of the resonances, can induce a reorganization of the frequency spectrum, with some instabilities overlapping with other unstable regions, hence making them less likely to be detected, (see figure 7.4(b)).
The salient point of the present work is the introduction of harmonic meniscus or edge waves emitted by the oscillating static meniscus under the vertical external excitation, widely discussed in the literature, but mostly from an experimental perspective only. In the adopted asymptotic scaling, these directly forced waves appear at $\epsilon^{2}$ and they are coupled at order $\epsilon^{3}$ with the parametric waves, thus influencing not only the wave amplitude saturation, but also the marginal stability boundaries (through a modification of the slope of transition curves) as well as the solution outside the instability regions. If, indeed, for $\theta_{s}=90^{\circ}$ no meniscus is present and the sub-harmonic parametric waves bifurcate from the flat surface state, when $\theta_{s} \neq 90^{\circ}$, the instability emerges on the top of a still stable, but stationary oscillating free surface, i.e. edge or meniscus waves. This translates in the so-called imperfect bifurcation diagram shown in figure 7.6 , which displays a tailing effect owing to meniscus waves, whose amplitude is proportional to the external acceleration amplitude. In this regard, we note the

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analogy with previous experimental observations (Batson et al., 2013), although for different fluid systems and contact line conditions. One of the major influences of contact angle effects on the wave amplitude response was found to occur for axisymmetric sub-harmonic waves. Intuitively, this was explained by considering that harmonic meniscus waves, being directly forced by the spatially uniform forcing, are axisymmetric by construction, therefore axisymmetric parametric waves, although of different nature, are more likely to be destabilized by edge waves, as they share the same spatial symmetries. This effect is expected to be dominant for harmonic axisymmetric parametric waves, as proven experimentally by Batson et al. (2013).
Furthermore, the existence of a harmonic meniscus wave state, from which the parametric waves bifurcate (rather than the flat interface rest state), has been observed in some cases (see figure 7.5(b)) to induce a change of sign of the direction in the bifurcation diagram as the contact angle is varied. Specifically, in some cases the present analysis predicts the existence of a static contact angle for which the bifurcation is always supercritical no matter what the combination of external forcing amplitude and frequency be, thus leading to a suppression of the sub-criticality of the system. This does not seem to have been reported in the literature and it could be checked in future experiments.

Lastly in $\$ 7.5$, with the purpose of validation only, the single-mode WNL model, in the specific case of an axisymmetric dynamics, was compared with fully nonlinear axisymmetric direct numerical simulations, which revealed a good agreement, proving (at least partially) the correctness of the WNL prediction when contact angle effects were introduced.
To conclude, we add that the numerical tools developed in this work could enable us to explore different geometries, to revisit previous experiments with different contact line boundary conditions, e.g. the more involved sliding contact line condition, although those require the regularization of the well-known contact line stress-singularity, most likely via phenomenological slip length models (Miles, 1990; Ting and Perlin, 1995)), to introduce in the latter dynamical contact angle effects (Viola et al., 2018; Viola and Gallaire, 2018) and to explore different fluid systems of interest, e.g. multilayer configurations as those investigated by Batson et al. (2013). Moreover, with the aim of quantifying contact angle effects on the Faraday thresholds, the ad hoc asymptotic scaling for sub-harmonic parametric resonances defined in the present weakly nonlinear analysis could be modified so as to tackle other types of resonances, such as harmonic and super-harmonic parametric waves, combination resonances (see Kidambi (2013)), internal resonances (Faltinsen et al., 2016; Miles, 1984b; Miles and Henderson, 1990; Nayfeh, 1987) as well as secondary-drift instabilities triggered by pure viscous modes, which may break the symmetry of non-axisymmetric standing waves (Fauve et al., 1991; Knobloch et al., 2002; Martel and Knobloch, 1997; Martel et al., 2000; Périnet et al., 2017; Vega et al., 2001). Some of these directions are being pursued and will be reported elsewhere.

### 7.7 Appendix

### 7.7.1 Damping and frequency of capillary-gravity waves in brimful and nearlybrimful circular-cylinders

With regard to the literature survey outlined in table 7.1, in this appendix, we study the damping and natural frequencies of viscous capillary-gravity waves with fixed contact line and we compare our numerical results with existing experiments and with previous theoretical and numerical predictions.

Flat static free surface: $\theta_{s}=90^{\circ}$
Let us start by considering the case of a flat static interface, i.e. the static contact angle is set to $\theta_{s}=90^{\circ}$, for which $\eta_{0}(r)=0$, i.e. perfect brimful condition.

## Experiments and theories by HM94, MH98 and M98

We consider here the experimental measurements by HM94 for the first six modes in a brimful, sharp-edged cylinder in the absence of free surface contamination. The corresponding geometrical and fluid properties are reported in the caption of table 7.4 , while the eigen-surfaces associated with the first six modes, computed by solving numerically the eigenvalue problem (7.14), are shown in figure 7.10.

In table 7.4, the experimental damping coefficients and angular frequencies measured by HM94 are compared with their own viscous theoretical predictions, with the prediction of M98 for the very same case and with our numerical results. If the frequency prediction of HM94 is in good agreement with their own experiments, a significant mismatch is found in terms of the damping coefficient. However, this discrepancy is strongly reduced in the prediction of M98, which is in agreement with our numerical results. By analogy with M98, the theory proposed in HM94 was supplemented in MH98 by a calculation of the interior damping (based on Lamb's dissipation integral for an irrotational flow (Lamb, 1993)), which yields results (here omitted for the sake of brevity) of comparable accuracy with M98 and with the present predictions. We note that the predicted frequencies in both M98 and the present study are always within $0.3 \%$ of the experimental values.



Figure 7.10 - Shape of the eigen-surfaces associated with the six global modes considered in table 7.4 and denoted by the indices $(m, n)$. The magnitude of the eigen-surface slope is plotted. The eigenmodes are normalized such that the phase of the interface at the contact line in $\phi=0$ is zero and the corresponding slope is one, i.e. $\hat{\mathbf{q}}_{1} \rightarrow \hat{\mathbf{q}}_{1} e^{-\mathrm{iarctan}\left[\hat{\eta}_{1}(r=1,0)\right]} /\left(\partial \hat{\eta}_{1}(r, 0) /\left.\partial r\right|_{r=1}\right)$.

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|  | Exp. HM94 |  | Theory HM94 |  |  | Theory M98 |  |  |  | Present Num. |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $(m, n)$ | $f_{E}(\mathrm{~Hz})$ | $\Delta_{E}(-)$ | $f_{T}$ | $\Delta_{T}$ | $\Delta_{E} / \Delta_{T}$ | $f_{T}$ | $\Delta_{T}$ | $\Delta_{E} / \Delta_{T}$ | $f$ | $\Delta$ | $\Delta_{E} / \Delta$ |  |
| $(1,1)$ | 4.65 | 1.4 | 4.66 | 1.13 | 1.2 | 4.67 | 1.37 | 1.02 | 4.66 | 1.36 | 1.03 |  |
| $(2,1)$ | 6.32 | 1.8 | 6.32 | 1.24 | 1.4 | 6.34 | 1.75 | 1.03 | 6.34 | 1.74 | 1.03 |  |
| $(0,1)$ | 6.84 | 1.2 | 6.73 | 0.44 | 2.7 | 6.85 | 0.95 | 1.26 | 6.85 | 0.93 | 1.29 |  |
| $(3,1)$ | 7.80 | 2.2 | 7.79 | 1.29 | 1.7 | 7.82 | 2.11 | 1.04 | 7.82 | 2.08 | 1.06 |  |
| $(4,1)$ | 9.26 | 2.4 | 9.24 | 1.32 | 1.8 | 9.27 | 2.47 | 0.97 | 9.27 | 2.42 | 0.99 |  |
| $(1,2)$ | 8.57 | 1.5 | 8.57 | 0.48 | 3.1 | 8.59 | 1.45 | 1.03 | 8.59 | 1.43 | 1.05 |  |

Table 7.4 - Experimental frequency and damping by HM94, their theoretical prediction and the theoretical prediction by M98 are compared with the present numerical results. Geometrical and fluid properties: $R=0.02766 \mathrm{~m}, h=0.038 \mathrm{~m}, \rho=1000 \mathrm{~kg} \mathrm{~m}^{-3}, \mu=0.001 \mathrm{~kg} \mathrm{~m}^{-1} \mathrm{~s}^{-1}$, $\gamma=0.0724 \mathrm{Nm}^{-1}$, for which $R e=14401$ and $B o=103.6$, and a static angle $\theta_{s}=90^{\circ}$. The dimensionless damping coefficient $\sigma$ is rescaled according to HM94, i.e. $\Delta=4 \sqrt{R e / 2 \omega} \sigma$, where $\sigma$ and $\omega$ for the present numerical results (last three columns) are those computed by solving (7.14). The dimensional frequency is readily obtained as $f=(\omega / 2 \pi) \sqrt{g / R}$. The number of points in the radial and axial directions for the GLC grid used in this calculation is $N_{r}=N_{z}=40$, for which convergence is achieved.

## Experiments and theories by H2000, M98, N02 and K09

Table 7.5 provides a comparison of the present results with the experimental measurements of H2000, the asymptotic calculations of M98, the theoretical predictions of N02 and the calculations of K09.
All the theoretical methods accurately predict the natural frequencies, even at low $R e$, as the viscous correction is very small. However, in terms of damping, it is seen that the asymptotic model of M98 is increasingly inaccurate for decreasing Re. For the present case, our numerical calculations place in between N02 and K09, with frequency predictions within $0.7 \%$ of the experimental values.

|  | Exp. H2000 |  | Theory M98 |  | Theory N02 |  | Num. K09 |  | Present Num. |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $R e$ | $f_{E}(-)$ | $\Delta_{E}(-)$ | $f_{T} / f_{E}$ | $\Delta_{T} / \Delta_{E}$ | $f_{T} / f_{E}$ | $\Delta_{T} / \Delta_{E}$ | $f_{N} / f_{E}$ | $\Delta_{N} / \Delta_{E}$ | $f / f_{E}$ | $\Delta / \Delta_{E}$ |
| 13077 | 2.079 | 0.005 | 1.004 | 0.98 | 1.005 | 0.91 | 1.005 | 0.92 | 1.005 | 0.91 |
| 6423 | 2.075 | 0.009 | 1.005 | 0.98 | 1.007 | 0.94 | 1.007 | 0.95 | 1.007 | 0.95 |
| 2621 | 2.075 | 0.018 | 1.005 | 1.04 | 1.006 | 0.97 | 1.006 | 0.97 | 1.006 | 0.97 |
| 1317 | 2.072 | 0.033 | 1.006 | 1.06 | 1.006 | 0.94 | 1.006 | 0.95 | 1.006 | 0.95 |
| 575 | 2.066 | 0.066 | 1.008 | 1.13 | 1.005 | 0.98 | 1.006 | 0.98 | 1.005 | 0.98 |
| 270 | 2.059 | 0.127 | 1.010 | 1.19 | 1.001 | 0.98 | 1.001 | 0.98 | 1.001 | 0.98 |

Table 7.5 - Dimensionless damping and frequency of the first axisymmetric mode $(0,1)$ for different $R e$. Nondimensional parameters: $R=1, h / R=1.379, B o=365$ and $\theta_{s}=90^{\circ}$. Here the dimensionless natural frequency and damping correspond to $f=\omega$ and $\Delta=\sigma$ in our notation. The number of points in the radial and axial directions for the GLC grid used in this calculation is $N_{r}=N_{z}=40$, for which convergence is achieved. Comparisons outlined in this table (except for the last column) are provided in table 2 of K09.

## Presence of static meniscus: $\theta_{s} \neq 90^{\circ}$

We now analyze the case of an initially non-flat static interface, i.e. $\theta_{s} \neq 90^{\circ}$, for which $\eta_{0}(r) \neq 0$ (nearly-brimful condition), and its effect on the natural frequencies and damping coefficients of viscous capillary-gravity waves with a pinned contact line.

## Experiments by C93 and calculations by N05 and K09

C93 measured the frequency and damping rate of the first non-axisymmetric mode $(m, n)=$ $(1,1)$ in a cylindrical container where the static free surface had an effective static contact angle $\theta_{s}=62^{\circ}$. They identified two different regimes, namely, a higher and a smaller amplitude regime. In the latter, the contact line was observed to remain pinned. N05 and K09 have computed the damping and frequency for this case and a comparison with our numerical analysis is reported in table 7.6. We note that the prediction of N05 is close to the experimental values, however, such a prediction is based on an asymptotic representation of the static meniscus, while in the present calculation, as well as the one proposed by K09, it is computed numerically. Moreover, the damping prediction by N05 relies on HM94 and M98 theories, since its starting point is an inviscid analysis. Our result seems to be slightly closer to the experimental values than the one of K09, although both are in fairly good agreement.

| Exp. C93 |  | Theory N05 |  |  |  | Num. K09 |  |  |  | Present Num. |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $f_{E}(\mathrm{~Hz})$ | $\Delta_{E}(\mathrm{mHz})$ | $f_{T}$ | $\Delta_{T}$ | $\Delta_{E} / \Delta_{T}$ | $f_{N}$ | $\Delta_{N}$ | $\Delta_{E} / \Delta_{N}$ | $f$ | $\Delta$ | $\Delta_{E} / \Delta$ |  |  |
| 3.22 | $15 \pm 2$ | 3.22 | 14.6 | 0.977 | 3.23 | 16.3 | 1.085 | 3.23 | 15.4 | 1.027 |  |  |

Table 7.6 - Dimensional frequency and damping of the first non-axisymmetric mode ( 1,1 ). Parameter setting: $R=0.05025 \mathrm{~m}, h=0.13 \mathrm{~m}, \rho=1000 \mathrm{~kg} \mathrm{~m}^{-3}, \mu=0.00099 \mathrm{~kg} \mathrm{~m}^{-1} \mathrm{~s}^{-1}, \gamma=$ $0.0724 \mathrm{Nm}^{-1}$ and $\theta_{s}=62^{\circ}$, for which $R e=35628.103$ and $B o=346.363$. Here $f=(\omega / 2 \pi) \sqrt{g / R}$ and $\Delta=\sigma \sqrt{g / R}$. The number of points in the radial and axial directions for the GLC grid used is this calculation is $N_{r}=N_{z}=40$, for which convergence is achieved.

## Experiments by PD07 and theory by N05

PD07 presented a liquid surface biosensor for DNA detection based on resonant meniscus capillary waves. In their experimental setting, the contact line is pinned at the brim, so that the static contact angle can be modified by controlling the bulk volume. As their setup was developed to make use exclusively of axisymmetric stationary meniscus waves, by exciting the container below the Faraday threshold they could measure the amplitude spectra for a series of effective contact angles in a frequency window centered around one particular natural frequency (that of mode $(m, n)=(0,10)$ ), highlighting two main phenomena attributable to contact angle effects, namely a decrease of the resonance frequency and a strong increase of the wave amplitude with the curvature of the meniscus, the latter being typical of a meniscus waves response. The experimental values were found to be in qualitative agreement with the inviscid prediction of N05, according to which the frequency has a maximum for $\theta_{s}=90^{\circ}$ (the maximum experimental frequency is found for $\theta_{s} \in[90,100]$ ). The frequency shift as a function of the static contact angle measured by PD07 is shown in figure 7.11 together with

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our numerical prediction for this specific case. Even in this case, our frequency prediction lies within $0.3 \%$ of the experimental values.


Figure 7.11 - Comparison of the experimentally measured natural frequency for mode $(0,10)$ (filled white circles, extracted from figure 5 of PD07) versus static contact angle with the inviscid estimation of N05 (black solid line) and our numerical results (black crosses). The black dashed line indicates the flat case with $\theta_{s}=90^{\circ}$. Parameter setting: pure water, clean surface, $h=0.045 \mathrm{~m}$ and $R=0.025 \mathrm{~m}$, for which $h / R=1.8, B o=86.3$ and $R e=10855$. The number of points in the radial and axial directions for the GLC grid used in this calculation is $N_{r}=N_{z}=40$, for which convergence is achieved.

## Numerical study by K09

As mentioned in the introduction, an important combined theoretical and numerical work accounting for contact angle effects on the damping and frequency of viscous capillary-gravity waves is that of K09. In figure 7.12 our predictions are compared with Kidambi's results for the first non-axisymmetric mode $(1,1)$ and for two different combinations of nondimensional physical parameters. Our solution is found to be in good agreement with that of K09 for a wide range of static contact angles. In particular, the predicted frequencies are within $0.4 \%$ of each other. Different peculiar behaviours are observed as the contact angle and the other physical parameters are varied. K09 found that at shallow depths the presence of a static meniscus leads to an increase of the natural frequency irrespective of the static contact angle, while at large depths the frequency shows a maximum in the neighbourhood of $\theta_{s}=90^{\circ}$, in agreement with N05, with the experimental observations pointed out by PD07, and with the present study.

## Comments

Although the frequency predictions are in excellent agreement with experimental measurements (usually well within $1 \%$ ), we observe that the estimation of the damping coefficient is more sensitive to the various methods of calculation proposed in the literature. This is due to the fact that most of the existing theories are based on semi-analytical asymptotic expressions and boundary layer approximations with a leading order solution formulated in the inviscid framework (HM94,M98,MH98,N02,N05), as originally introduced by Benjamin and Scott (1979) and Graham-Eagle (1983). However, despite the sources of dissipation being several and hard to accurately quantify, especially with asymptotic approaches, the pinned contact line problem allows one to drastically reduce uncertainties related to contact line


Figure 7.12 - (a) Damping and (b) frequency of the first asymmetric mode ( 1,1 ) as a function of the static contact angle. White filled squares and circles: numerical results of K09. Black crosses: present numerical results. The Bond number is fixed to $B o=365$. The number of points in the radial and axial directions for the GLC grid used is this calculation is $N_{r}=N_{z}=40$, for which convergence is achieved. (c) Eigen-velocity field for $h / R=H=0.231, R e=13077.02$ and $\theta_{s}=45^{\circ}$ at $t=$ and $\phi=0$.
dynamics, thus leading in general to better agreements with experiments. Little uncertainties can still be present in experiments, where free surface contamination is not fully controlled.
A wide majority of studies, both experimental and numerical (or semi-analytic), have been focused on the classic case of a flat static free surface, with the exceptions of those by N05 and K09. Particularly K09, in the spirit of N02, projected the governing equations onto an appropriate basis and formulated a nonlinear eigenvalue problem (solved numerically with an iterative method) for the damping and frequency of viscous capillary-gravity waves with fixed contact line, which formally includes both static meniscus effects and viscous dissipation.

We underline that, unlike the previous analyses by K09, in the present work, through a fully numerical discretization technique, the problem of viscous capillary-gravity waves with pinned contact line is formulated as a classic generalized linear eigenvalue problem, which can be solved numerically with standard techniques. Hence, the numerical method used in this work allows one to directly solve capillary-gravity waves in brimful and nearly-brimful conditions accounting for contact angle effects and viscous dissipation without any simplification or assumption, i.e. the numerical solution at convergence is supposed to be accurate.

### 7.7.2 Values of the nonlinear normal form coefficient $v$

For completeness, the value of the nonlinear normal form coefficient $v$ associated with all modes in figure 7.4 is reported in table 7.7 for different static contact angle, $\theta_{s}$, i.e. $90^{\circ}, 75^{\circ}, 60^{\circ}$ and $45^{\circ}$. By looking at the imaginary part, $v_{I}$, one can see that an inversion of the bifurcation direction occurs for modes $(4,1),(3,2)$ and $(6,1)$ for a static contact angle between $\theta_{s}=60^{\circ}$ and $45^{\circ}$.

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| $(m, n)$ | $v^{90^{\circ}}$ | $v^{75^{\circ}}$ | $v^{60^{\circ}}$ | $v^{45^{\circ}}$ |
| :---: | :---: | :---: | :---: | :---: |
| $(2,1)$ | $-0.004-\mathrm{i} 0.438$ | $-0.004-\mathrm{i} 0.318$ | $-0.004-\mathrm{i} 0.182$ | $-0.005-\mathrm{i} 0.081$ |
| $(0,1)$ | $-0.029-\mathrm{i} 0.948$ | $-0.0230-\mathrm{i} 0.708$ | $-0.039-\mathrm{i} 0.415$ | $-0.049-\mathrm{i} 0.297$ |
| $(3,1)$ | $-0.002-\mathrm{i} 0.760$ | $-0.001-\mathrm{i} 0.512$ | $-0.005-\mathrm{i} 0.246$ | $-0.007-\mathrm{i} 0.086$ |
| $(1,2)$ | $-0.017-\mathrm{i} 1.156$ | $-0.014-\mathrm{i} 0.798$ | $-0.011-\mathrm{i} 0.394$ | $-0.008-\mathrm{i} 0.139$ |
| $(4,1)$ | $-0.023-\mathrm{i} 0.994$ | $-0.037-\mathrm{i} 0.560$ | $-0.046-\mathrm{i} 0.115$ | $-0.056+\mathrm{i} 0.118$ |
| $(2,2)$ | $-0.033-\mathrm{i} 1.378$ | $-0.031-\mathrm{i} 0.888$ | $-0.021-\mathrm{i} 0.370$ | $-0.010-\mathrm{i} 0.084$ |
| $(0,2)$ | $-0.091-\mathrm{i} 1.909$ | $-0.077-\mathrm{i} 1.458$ | $-0.046-\mathrm{i} 1.116$ | $-0.018-\mathrm{i} 0.562$ |
| $(5,1)$ | $-0.001-\mathrm{i} 1.548$ | $-0.009-\mathrm{i} 1.070$ | $-0.013-\mathrm{i} 0.497$ | $-0.010-\mathrm{i} 0.144$ |
| $(3,2)$ | $-0.091-\mathrm{i} 1.304$ | $-0.099-\mathrm{i} 0.647$ | $-0.080-\mathrm{i} 0.074$ | $-0.046+\mathrm{i} 0.128$ |
| $(6,1)$ | $-0.016-\mathrm{i} 1.765$ | $-0.025-\mathrm{i} 1.084$ | $-0.024-\mathrm{i} 0.360$ | $-0.016+\mathrm{i} 0.009$ |

Table 7.7-Nonlinear coefficient, $v=v_{R}+\mathrm{i} v_{I}$, associated with the modes shown in figure 7.4 and computed for different values of the static contact angle, i.e. $\theta_{s}=90^{\circ}, 75^{\circ}, 60^{\circ}$ and $45^{\circ}$. These coefficients were computed using a grid with $N_{r}=N_{z}=80$ GLC nodes, for which convergence up to the third digit was achieved.

# 8 A revised gap-averaged Floquet analysis of Faraday waves in Hele-Shaw cells 

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Existing theoretical analyses of Faraday waves in Hele-Shaw cells rely on the Darcy approximation and assume a parabolic flow profile in the narrow direction. However, Darcy's model is known to be inaccurate when convective and unsteady inertias are important. In this work, we propose a novel gap-averaged Floquet theory accounting for inertial effects induced by the unsteady terms in the Navier-Stokes equations, a scenario that corresponds to a pulsatile flow where the fluid motion reduces to a two-dimensional oscillating Poiseuille flow, similarly to the Womersley flow in arteries. When gap-averaging the linearized Navier-Stokes equation, this results in a modified damping coefficient, which is a function of the ratio between the Stokes boundary layer thickness and the cell's gap, and whose complex value depends on the frequency of the wave response specific to each unstable parametric region. We first revisit the standard case of horizontally infinite rectangular Hele-Shaw cells by also accounting for a dynamic contact angle model. A comparison with existing experiments shows the predictive improvement brought by the present theory and points out how the standard gap-averaged model often underestimates the Faraday threshold. The analysis is then extended to the less conventional case of thin annuli. A series of dedicated experiments for this configuration highlights how Darcy's thin-gap approximation overlooks a frequency detuning that is essential to correctly predict the locations of the Faraday tongues in the frequency-amplitude parameter plane. These findings are well rationalized and captured by the present model.

The Chapter is organized as follows. In $\$ 8.1$ we revisit the classical case of horizontally infinite rectangular Hele-Shaw cells. The present model is compared with theoretical predictions from the standard Darcy theory and with existing experiments. The case of thin annuli is then considered. The model for the latter unusual configuration is formulated in $\S 8.2$ and


Figure 8.1 - (a) Sketch of Faraday waves in a rectangular Hele-Shaw cell of width $b$ and length $l$ filled to a depth $h$ with a liquid. Here $b$ denotes the gap size of the Hele-Shaw cell, $l$ is the wavelength of a certain wave, such that $b / l \ll 1$, and $\theta$ is the dynamic contact angle of the liquid on the lateral walls. The vessel undergoes a vertical sinusoidal oscillation of amplitude $a$ and angular frequency $\Omega$. The free surface elevation is denoted by $\eta^{\prime}\left(x^{\prime}\right)$. (b) Same as (a), but in an annular Hele-Shaw cell with internal and external radii, respectively, $R-b / 2$ and $R+b / 2$. Here, $b / R \ll 1$ and the free surface elevation is a function of the azimuthal coordinate $\varphi^{\prime}$, i.e. $\eta^{\prime}\left(\varphi^{\prime}\right)$.
compared with homemade experiments in §8.3. Conclusions are outlined in §8.4.

### 8.1 Horizontally infinite Hele-Shaw cells

Let us begin by considering the case of a horizontally infinite Hele-Shaw cell of width $b$ filled to a depth $h$ with an incompressible fluid of density $\rho$, dynamic viscosity $\mu$ (kinematic viscosity $v=\mu / \rho$ ) and liquid-air surface tension $\gamma$ (see also sketch in figure 8.1(a)). The vessel undergoes a vertical sinusoidal oscillation of amplitude $a$ and angular frequency $\Omega$. In a frame of reference which moves with the oscillating container, the free liquid interface is flat and stationary for small forcing amplitudes, and the oscillation is equivalent to a temporally modulated gravitational acceleration, $G\left(t^{\prime}\right)=g-a \Omega^{2} \cos \Omega t^{\prime}$. The equation of motion for the fluid bulk are

$$
\begin{equation*}
\rho\left(\frac{\partial \mathbf{U}^{\prime}}{\partial t^{\prime}}+\mathbf{U}^{\prime} \cdot \nabla^{\prime} \mathbf{U}^{\prime}\right)=-\nabla^{\prime} P^{\prime}+\mu \nabla^{\prime 2} \mathbf{U}^{\prime}-\rho G(t) \mathbf{e}_{z}, \quad \nabla^{\prime} \cdot \mathbf{U}^{\prime}=0 \tag{8.1}
\end{equation*}
$$

Linearizing about the rest state $\mathbf{U}^{\prime}=\mathbf{0}$ and $P^{\prime}\left(z^{\prime}, t^{\prime}\right)=-\rho G\left(t^{\prime}\right) z^{\prime}$, the equations for the perturbation velocity, $\mathbf{u}^{\prime}\left(x^{\prime}, y^{\prime}, z^{\prime}, t^{\prime}\right)=\left\{u^{\prime}, v^{\prime}, w^{\prime}\right\}^{T}$, and pressure, $p^{\prime}\left(x^{\prime}, y^{\prime}, z^{\prime}, t^{\prime}\right)$, fields, associated with a certain perturbation's wavelength $l=2 \pi / k$ ( $k$, wavenumber), read

$$
\begin{equation*}
\rho \frac{\partial \mathbf{u}^{\prime}}{\partial t^{\prime}}=-\nabla^{\prime} p^{\prime}+\mu \nabla^{2} \mathbf{u}^{\prime}, \quad \nabla^{\prime} \cdot \mathbf{u}^{\prime}=0 \tag{8.2}
\end{equation*}
$$

Assuming that $b k \ll 1$, then the velocity along the narrow $y^{\prime}$-dimension $v^{\prime} \ll u^{\prime}, w^{\prime}$ and, by employing the Hele-Shaw approximation as in, for instance, Viola et al. (2017), one can simplify the linearized Navier-Stokes equations as follows:

$$
\begin{gather*}
\frac{\partial u^{\prime}}{\partial x^{\prime}}+\frac{\partial v^{\prime}}{\partial y^{\prime}}+\frac{\partial w^{\prime}}{\partial z^{\prime}}=0  \tag{8.3a}\\
\rho \frac{\partial u^{\prime}}{\partial t^{\prime}}=-\frac{\partial p^{\prime}}{\partial x^{\prime}}+\mu \frac{\partial^{2} u^{\prime}}{\partial y^{\prime 2}}, \quad \rho \frac{\partial w^{\prime}}{\partial t^{\prime}}=-\frac{\partial p^{\prime}}{\partial z^{\prime}}+\mu \frac{\partial^{2} w^{\prime}}{\partial y^{\prime 2}}, \quad \frac{\partial p^{\prime}}{\partial y^{\prime}}=0 \tag{8.3b}
\end{gather*}
$$

Equations (8.3a)-(8.3b) are made dimensionless using $k^{-1}$ for the directions $x^{\prime}$ and $z^{\prime}$, and $b$ for $y^{\prime}$. The forcing amplitude and frequency provide a scale $a \Omega$ for the in-plane $x z$-velocity components, whereas the continuity equation imposes the transverse component $v^{\prime}$ to scale as $v^{\prime} \sim b k a \Omega \ll a \Omega \sim u^{\prime}$, due to the strong confinement in the $y$-direction ( $b k \ll 1$ ). With these choices, dimensionless spatial scales, velocity components and pressure write:

$$
\begin{equation*}
x=x^{\prime} k, y=\frac{y^{\prime}}{b}, \quad z=z^{\prime} k, \quad u=\frac{u^{\prime}}{a \Omega}, \quad v=\frac{v^{\prime}}{b k a \Omega}, \quad w=\frac{w^{\prime}}{a \Omega}, \quad p=\frac{k p^{\prime}}{\rho a \Omega^{2}}, \quad t=\Omega t^{\prime} . \tag{8.4}
\end{equation*}
$$

The first two equations in (8.3b) in non-dimensional form are

$$
\begin{equation*}
\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}+\frac{\delta_{S t}^{2}}{2} \frac{\partial^{2} u}{\partial y^{2}}, \quad \frac{\partial w}{\partial t}=-\frac{\partial p}{\partial z}+\frac{\delta_{S t}^{2}}{2} \frac{\partial^{2} w}{\partial y^{2}} \tag{8.5}
\end{equation*}
$$

where $\delta_{S t}=\delta_{S t}^{\prime} / b$ and with $\delta_{S t}^{\prime}=\sqrt{2 v / \Omega}$ denoting the thickness of the oscillating Stokes boundary layer. The ratio $\sqrt{2} / \delta_{S t}$ is also commonly referred to as the Womersley number, $W o=b \sqrt{\Omega / v}$ (San and Staples, 2012; Womersley, 1955).

### 8.1.1 Floquet analysis of the gap-averaged equations

Given its periodic nature, the stability of the base flow, represented by a time-periodic modulation of the hydrostatic pressure, can be investigated via Floquet analysis. We therefore introduce the following Floquet ansatz (Kumar and Tuckerman, 1994a)

$$
\begin{array}{r}
\mathbf{u}(x, y, z, t)=e^{\mu_{F} t} \sum_{n=-\infty}^{+\infty} \tilde{\mathbf{u}}_{n}(x, y, z) e^{\mathrm{i}(n+\alpha / \Omega) t}=e^{\mu_{F} t} \sum_{n=-\infty}^{+\infty} \tilde{\mathbf{u}}_{n}(x, y, z) e^{\mathrm{i} \xi_{n} t}, \\
p(x, z, t)=e^{\mu_{F} t} \sum_{n=-\infty}^{+\infty} \tilde{p}_{n}(x, z) e^{\mathrm{i}(n+\alpha / \Omega) t}=e^{\mu_{F} t} \sum_{n=-\infty}^{+\infty} \tilde{p}_{n}(x, z) e^{\mathrm{i} \xi_{n} t}, \tag{8.6b}
\end{array}
$$

where $\mu_{F}$ is the real part of the non-dimensional Floquet exponent and represents the growth rate of the perturbation. We have rewritten $(n+\alpha / \Omega)=\xi_{n}$ to better explicit the parametric nature of the oscillation frequency of the wave response. In the following, we will focus on the condition for marginal stability (boundaries of the Faraday's tongues), which require the growth rate $\mu_{F}=0$. In addition, values of $\alpha=0$ and $\Omega / 2$ correspond, respectively, to harmonic and sub-harmonic parametric resonances (Kumar and Tuckerman, 1994a). This implies that $\xi_{n}$ is a parameter whose value is either $n$, for harmonics, or $n+1 / 2$, for sub-harmonics, with $n$ an integer $n=0,1,2, \ldots$ specific to each Fourier component in (8.6a)-(8.6b).
By injecting the ansatzs (8.6a)-(8.6b) in (8.5), we find that each component of the Fourier series must satisfy

$$
\begin{equation*}
\forall n: \quad \mathrm{i} \xi_{n} \tilde{u}_{n}=-\frac{\partial \tilde{p}_{n}}{\partial x}+\frac{\delta_{S t}^{2}}{2} \frac{\partial^{2} \tilde{u}_{n}}{\partial y^{2}}, \quad \mathrm{i} \xi_{n} \tilde{w}_{n}=-\frac{\partial \tilde{p}_{n}}{\partial z}+\frac{\delta_{S t}^{2}}{2} \frac{\partial^{2} \tilde{w}_{n}}{\partial y^{2}}, \tag{8.7}
\end{equation*}
$$

which, along with the no-slip condition at $y= \pm 1 / 2$, correspond to a two-dimensional pulsatile Poiseuille flow with solution

$$
\begin{equation*}
\tilde{u}_{n}=\frac{\mathrm{i}}{\xi_{n}} \frac{\partial \tilde{p}_{n}}{\partial x} F_{n}(y), \quad \tilde{w}_{n}=\frac{\mathrm{i}}{\xi_{n}} \frac{\partial \tilde{p}_{n}}{\partial z} F_{n}(y), \quad F_{n}(y)=\left(1-\frac{\cosh (1+\mathrm{i}) y / \delta_{n}}{\cosh (1+\mathrm{i}) / 2 \delta_{n}}\right), \tag{8.8}
\end{equation*}
$$

and where $\delta_{n}=\delta_{S t} / \sqrt{\xi_{n}}$, is a rescaled Stokes boundary layer thickness specific to the $n$th Fourier component. The function $F_{n}(y)$ is displayed in figure 8.2(b), which depicts how a decrease in the value of $\delta_{n}$ starting from large values corresponds to a progressive transition from a fully developed flow profile to a plug flow connected to thin boundary layers. The gap-averaged velocity along the $y$-direction satisfies a Darcy-like equation,

$$
\begin{equation*}
<\tilde{\mathbf{u}}_{n}>=\int_{-1 / 2}^{1 / 2} \tilde{\mathbf{u}}_{n} \mathrm{~d} y=\frac{\mathrm{i} \beta_{n}}{\xi_{n}} \nabla \tilde{p}_{n}, \quad \beta_{n}=1-\frac{2 \delta_{n}}{1+\mathrm{i}} \tanh \frac{1+\mathrm{i}}{2 \delta_{n}} \tag{8.9}
\end{equation*}
$$

In order to obtain a governing equation for the pressure $\tilde{p}_{n}$, we average the continuity equation and we impose the impermeability condition for the spanwise velocity, $v=0$ at $y= \pm 1 / 2$,

$$
\begin{equation*}
\frac{\left.\partial<\tilde{u}_{n}\right\rangle}{\partial x}+\underbrace{\int_{-1 / 2}^{1 / 2} \frac{\partial \tilde{v}_{n}}{\partial y} \mathrm{~d} y}_{\tilde{v}_{n}(1 / 2)-\tilde{v}_{n}(-1 / 2)=0}+\frac{\left.\partial<\tilde{w}_{n}\right\rangle}{\partial z}=\nabla \cdot<\tilde{\mathbf{u}}_{n}>=0, \tag{8.10}
\end{equation*}
$$

Since $<\tilde{\mathbf{u}}_{n}>=\mathrm{i}\left(\beta_{n} / \xi_{n}\right) \nabla \tilde{p}_{n}$, the pressure field $\tilde{p}_{n}$ must obey the Laplace equation

$$
\begin{equation*}
\nabla^{2} \tilde{p}_{n}=\frac{\partial^{2} \tilde{p}_{n}}{\partial x^{2}}+\frac{\partial^{2} \tilde{p}_{n}}{\partial z^{2}}=0 . \tag{8.11}
\end{equation*}
$$

It is now useful to expand each Fourier component $\tilde{p}_{n}(x, z)$ in the infinite $x$-direction as $\sin x$ such that the $y$-average implies,

$$
\begin{equation*}
\tilde{p}_{n}(x, z)=\hat{p}_{n}(z) \sin x, \tag{8.12a}
\end{equation*}
$$

$$
\begin{equation*}
<\tilde{u}_{n}>=\hat{u}_{n}=\frac{\mathrm{i} \beta_{n}}{\xi_{n}} \hat{p}_{n} \cos x, \quad<\tilde{w}_{n}>=\hat{w}_{n}=\frac{\mathrm{i} \beta_{n}}{\xi_{n}} \frac{\partial \hat{p}_{n}}{\partial z} \sin x . \tag{8.12b}
\end{equation*}
$$

Replacing (8.12a) in (8.11) leads to

$$
\begin{equation*}
\left(\frac{\partial^{2}}{\partial z^{2}}-1\right) \hat{p}_{n}=0 \tag{8.13}
\end{equation*}
$$

which admits the solution form

$$
\begin{equation*}
\hat{p}_{n}=c_{1} \cosh z+c_{2} \sinh z \tag{8.14}
\end{equation*}
$$

The presence of a solid bottom imposes that $\hat{w}_{n}=0$ and, therefore, that $\partial \hat{p}_{n} / \partial z=0$, at a non-dimensional fluid depth $z=-h k$, hence giving

$$
\begin{equation*}
\hat{p}_{n}=c_{1}[\cosh z+\tanh k h \sinh z] \tag{8.15}
\end{equation*}
$$

Let us now invoke the linearized kinematic boundary condition

$$
\begin{equation*}
\frac{\partial \eta}{\partial t}=w \tag{8.16}
\end{equation*}
$$

Note that free surface elevation, $\eta^{\prime}\left(x^{\prime}, y^{\prime}, t^{\prime}\right)$, has been rescaled by the forcing amplitude $a$, i.e. $\eta^{\prime} / a=\eta$, and represents the projection of the bottom of the transverse concave meniscus on the $x z$-plane of figure 8.1(a). Moreover, by recalling the Floquet ansatzs (8.6a)-(8.6b) (with $\mu_{F}=0$ ), here specified for the interface, we get an equation for each Fourier component $n$,

$$
\begin{equation*}
\eta=\sum_{n=-\infty}^{+\infty} \tilde{\eta}_{n} e^{\mathrm{i} \xi_{n} t} \quad \longrightarrow \quad \forall n: \quad \mathrm{i} \xi_{n} \tilde{\eta}_{n}=\tilde{w}_{n} \tag{8.17}
\end{equation*}
$$

Expanding $\tilde{\eta}_{n}$ in the $x$-direction as $\sin x$ and averaging in $y$, i.e. $<\tilde{\eta}_{n}>=\hat{\eta}_{n} \sin x$, leads to

$$
\begin{equation*}
\mathrm{i} \xi_{n} \hat{\eta}_{n}=\hat{w}_{n}=\left.\frac{\mathrm{i} \beta_{n}}{\xi_{n}} \frac{\partial \hat{p}_{n}}{\partial z}\right|_{z=0}=\frac{\mathrm{i} \beta_{n}}{\xi_{n}} c_{1} \tanh k h \longrightarrow c_{1}=\frac{\xi_{n}^{2}}{\beta_{n}} \frac{\hat{\eta}_{n}}{\tanh k h} \tag{8.18}
\end{equation*}
$$

Lastly, we consider the linearized dynamic condition (or linearized normal stress), evaluated at $z^{\prime}=\eta^{\prime}$ and where the term associated with the curvature of the free surface appears,

$$
\begin{equation*}
\left.-p^{\prime}+\rho G\left(t^{\prime}\right) \eta^{\prime}+2 \mu \frac{\partial w^{\prime}}{\partial z^{\prime}}-\gamma \frac{\partial \kappa^{\prime}}{\partial \eta^{\prime}} \right\rvert\, \eta^{\prime}=0 \tag{8.19}
\end{equation*}
$$

In (8.19), $\partial \kappa^{\prime} / \partial \eta^{\prime}$ represents the first-order variation of the curvature associated with the small perturbation $\eta^{\prime}$. Capillary force in the $x$-direction is only important at large enough wavenumbers, although the associated term can be retained in the analysis in order to retrieve the dispersion relation for capillary-gravity waves (Li et al., 2019). On the other hand, the small gap of Hele-Shaw cells is such that surface tension effects in the narrow $y$-direction are strongly exacerbated. In general, the curvature can be divided into two parts (Chuoke et al.,

1959; Saffman and Taylor, 1958):

$$
\begin{equation*}
\kappa^{\prime}\left(\eta^{\prime}\right)=\frac{\partial}{\partial x^{\prime}}\left(\frac{\partial_{x^{\prime}} \eta^{\prime}}{\sqrt{1+\left(\partial_{x^{\prime}} \eta^{\prime}\right)^{2}}}\right)+\frac{2}{b} \cos \theta \tag{8.20}
\end{equation*}
$$

where the first term indicates the principal radius of curvature and the second term represents the out-of-plane curvature of the meniscus (see figure 8.1(a)). A common treatment of Hele-Shaw cells assumes the out-of-plane interface shape to be semicircular (Afkhami and Renardy, 2013; McLean and Saffman, 1981; Park and Homsy, 1984; Saffman and Taylor, 1958). Nevertheless, laboratory observations have unveiled that liquid oscillations in Hele-Shaw cells experience an up-and-down driving force with $\theta$ constantly changing (Jiang et al., 2004), hence giving rise to a dynamic contact angle. Here, as in Li et al. (2019), we use the following model (Hamraoui et al., 2000) to evaluate the cosine of the dynamic contact angle $\theta$ as

$$
\begin{equation*}
\cos \theta=1-\frac{M}{\mu} C a=1-\frac{M w^{\prime}}{\gamma} \tag{8.21}
\end{equation*}
$$

where $C a=\mu w^{\prime} / \gamma$ is the Capillary number defined using the vertical contact line velocity $w^{\prime}=\partial \eta^{\prime} / \partial t^{\prime}$. The friction coefficient $M$, sometimes referred to as mobility parameter $M$ (Xia and Steen, 2018), can be interpreted in the framework of molecular kinetics theory (Blake, 1993, 2006; Hocking, 1987; Johansson and Hess, 2018; Voinov, 1976). Here, we simply view this coefficient as a constant phenomenological parameter that defines the energy dissipation rate per unit length of the contact line and, as in Li et al. (2019), we directly use the values employed by Hamraoui et al. (2000).
By combining equations (8.20)-(8.21) and taking their first-order curvature variation applied to the small perturbation, one can express

$$
\begin{equation*}
-\gamma \frac{\partial \kappa^{\prime}}{\partial \eta^{\prime}} \left\lvert\, \eta^{\prime}=-\gamma \frac{\partial^{2} \eta^{\prime}}{\partial x^{\prime 2}}+\frac{2 M}{b} \frac{\partial \eta^{\prime}}{\partial t^{\prime}}\right. \tag{8.22}
\end{equation*}
$$

After turning to non-dimensional quantities using the scaling in (8.4), equations (8.19) reads

$$
\begin{equation*}
-\Omega^{2} p+g \eta-\frac{\gamma}{\rho} k^{2} \frac{\partial^{2} \eta}{\partial x^{2}}+\frac{2 M}{\rho b} \Omega \frac{\partial \eta}{\partial t}=a \Omega^{2} \cos t \eta \tag{8.23}
\end{equation*}
$$

where the viscous stress term has been eliminated, as it is negligible compared to the others. With introduction of the Floquet ansatz (8.6b)-(8.17) and by recalling the $x$-expansion of the interface and pressure as $\sin x$, the averaged normal stress equations become

$$
\begin{equation*}
\forall n: \quad-\Omega^{2} \hat{p}_{n}+\mathrm{i}\left(\xi_{n} \Omega\right) \frac{2 M}{\rho b} \hat{\eta}_{n}+\left(1+\frac{\gamma}{\rho g} k^{2}\right) g \hat{\eta}_{n}=\frac{a \Omega^{2}}{2 g} g\left(\hat{\eta}_{n-1}+\hat{\eta}_{n+1}\right) . \tag{8.24}
\end{equation*}
$$

where the decomposition $\cos \Omega t^{\prime}=\left(e^{\mathrm{i} \Omega t^{\prime}}+e^{-\mathrm{i} \Omega t^{\prime}}\right) / 2=\left(e^{\mathrm{i} t}+e^{-\mathrm{i} t}\right) / 2$ has also been used. Equations (8.15) and (8.18) are finally used to express the dynamic equation as a function of the
non-dimensional averaged interface only,

$$
\begin{equation*}
-\frac{\left(\xi_{n} \Omega\right)^{2}}{\beta_{n}} \hat{\eta}_{n}+\mathrm{i}\left(\xi_{n} \Omega\right) \frac{2 M}{\rho b} k \tanh k h \hat{\eta}_{n}+(1+\Gamma) g k \tanh k h \hat{\eta}_{n}=\frac{g k \tanh k h}{2} f\left(\hat{\eta}_{n-1}+\hat{\eta}_{n+1}\right), \tag{8.25}
\end{equation*}
$$

with the auxiliary variables $f=a \Omega^{2} / g$ and $\Gamma=\gamma k^{2} / \rho g$, such that $(1+\Gamma) g k \tanh k h=\omega_{0}^{2}$, the well-known dispersion relation for capillary-gravity waves (Lamb, 1993).
As in the present form the interpretation of coefficient $\beta_{n}$ does not appear straightforward, it is useful to define the damping coefficients

$$
\begin{equation*}
\sigma_{n}=\sigma_{B L}+\sigma_{C L}, \quad \sigma_{B L}=\chi_{n} \frac{v}{b^{2}}, \quad \sigma_{C L}=\frac{2 M}{\rho b} k \tanh k h, \tag{8.26a}
\end{equation*}
$$

where $\chi_{n}$ is used to help rewriting $\frac{1}{\beta_{n}}=1-\mathrm{i} \frac{\delta_{n}^{2}}{2} \chi_{n}$,

$$
\begin{equation*}
\chi_{n}=\mathrm{i} \frac{2}{\delta_{n}^{2}}\left(\frac{1-\beta_{n}}{\beta_{n}}\right)=12\left[\frac{\mathrm{i}}{6 \delta_{n}^{2}}\left(\frac{\frac{2 \delta_{n}}{1+\mathrm{i}} \tanh \frac{1+\mathrm{i}}{2 \delta_{n}}}{1-\frac{2 \delta_{n}}{1+\mathrm{i}} \tanh \frac{1+\mathrm{i}}{2 \delta_{n}}}\right)\right] . \tag{8.26b}
\end{equation*}
$$

These auxiliary definitions allows one to express (8.25) as

$$
\begin{equation*}
\left.-\left(\xi_{n} \Omega\right)^{2} \hat{\eta}_{n}+\mathrm{i}\left(\xi_{n} \Omega\right) \sigma_{n} \hat{\eta}_{n}+\omega_{0}^{2} \hat{\eta}_{n}=\frac{\omega_{0}^{2}}{2(1+\Gamma)} f\left[\hat{\eta}_{n+1}+\hat{\eta}_{n-1}\right)\right] \tag{8.27}
\end{equation*}
$$

or, equivalently,

$$
\begin{equation*}
\frac{2(1+\Gamma)}{\omega_{0}^{2}}\left[-(n \Omega+\alpha)^{2}+\mathrm{i}(n \Omega+\alpha) \sigma_{n}+\omega_{0}^{2}\right] \hat{\eta}_{n}=f\left[\hat{\eta}_{n+1}+\hat{\eta}_{n-1}\right] . \tag{8.28}
\end{equation*}
$$

Subscripts $B L$ and $C L$ in (8.26a) denote, respectively, the boundary layers and contact line contributions to the total damping coefficient $\sigma_{n}$.
At the end of this mathematical derivation, a useful result is the modified damping coefficient $\sigma_{n}$. Since the boundary layer contribution, $\sigma_{B L}$ depends on the $n$th Fourier component, the overall damping, $\sigma_{n}$, is mode dependent and its value is different for each specific $n$th parametric resonant tongue considered. This is in stark contrast with the standard Darcy approximation, where $\sigma_{B L}$ is the same for each resonance and amounts to $12 \mathrm{v} / \mathrm{b}^{2}$. In our model, the case of $\alpha=0$ with $n=0$ constitutes a peculiar case, as $\xi_{n}=\xi_{0}=0$ and $\delta_{0} \rightarrow+\infty$. In such a situation, $F_{0}(y)$ tends to the steady Poiseuille profile, so that we take $\chi_{0}=12$.
Similarly to Kumar and Tuckerman (1994a), equation (8.28) is rewritten as

$$
\begin{equation*}
A_{n} \hat{\eta}_{n}=f\left[\hat{\eta}_{n+1}+\hat{\eta}_{n-1}\right] \tag{8.29}
\end{equation*}
$$

with

$$
\begin{equation*}
A_{n}=\frac{2(1+\Gamma)}{\omega_{0}^{2}}\left(-(n \Omega+\alpha)^{2}+\mathrm{i}(n \Omega+\alpha) \sigma_{n}+\omega_{0}^{2}\right)=A_{n}^{r}+\mathrm{i} A_{n}^{i} \in \mathbb{C} \tag{8.30}
\end{equation*}
$$



Figure 8.2 - (a) Real and imaginary parts of the complex auxiliary coefficient $\chi=\chi_{r}+\mathrm{i} \chi_{i}$ versus twice the non-dimensional Stokes boundary layer thickness $\delta$. The horizontal black dotted line indicates the constant value 12 given by the Darcy approximation. (b) Normalized profile $F(y)$ (Womersley profile) for different $\delta=b^{-1} \sqrt{2 v / \xi \Omega}$, whose values are specified by the filled circles in (a) with matching colors. The Poiseuille profile is also reported for completeness. In drawing these figures we let the oscillation frequency of the wave, $\xi \Omega$, free to assume any value, but we recall that the parameter $\xi$ can only assume discrete values, and so do $\chi$ and $F(y)$.

The non-dimensional amplitude of the external forcing, $f=a \Omega^{2} / g$ appears linearly, therefore (8.29) can be considered to be a generalized eigenvalue problem

$$
\begin{equation*}
\mathbf{A} \hat{\eta}=f \mathbf{B} \hat{\eta} \tag{8.31}
\end{equation*}
$$

with eigenvalues $f$ and eigenvectors whose components are the real and imaginary parts of $\hat{\eta}_{n}$. See Kumar and Tuckerman (1994a) for the structure of matrices A and B.
For one frequency forcing we use a truncation number $N=10$, which produces $2(N+1) \times$ $2(N+1)=22 \times 22$ matrices. Eigen-problem (8.31) is then solved in Matlab using the built-in function eigs. For a fixed forcing frequency $\Omega$ and wavenumber $k$, the eigenvalue with the smallest real part will define the instability threshold.
Figure 8.3 shows the results of this procedure for one of the configurations considered by Li et al. (2019) and neglecting the dissipation associated with the contact line motion, i.e. $M=0$. In each panel, associated with a fixed forcing frequency, the black regions correspond to the unstable Faraday tongues computed using $\sigma_{B L}=12 v / b^{2}$ as given by Darcy's approximation, whereas the red regions are the unstable tongues computed with the modified $\sigma_{B L}=\chi_{n} v / b^{2}$. At a forcing frequency 4 Hz the first sub-harmonic tongues computed using the two models essentially overlap. Yet, successive resonances display an increasing departure from Darcy's model due to the newly introduced complex coefficient $\sigma_{n}$. Particularly, the real part of $\chi_{n}$ is responsible for the higher onset acceleration, while the imaginary part is expected to act as a detuning term, which shifts the resonant wavenumbers $k$.


Figure 8.3 - Faraday tongues computed via Floquet analysis at different fixed driving frequencies, $\Omega / 2 \pi$ (reported on the top of each panel). Black regions correspond to the unstable Faraday tongues computed using $\sigma_{B L}=12 v / b^{2}$ as in the standard Darcy approximation, whereas red regions are the unstable tongues computed with the present modified $\sigma_{B L}=\chi_{n} v / b^{2}$. For this example, we consider ethanol $99.7 \%$ (see table 8.1) in a Hele-Shaw cell of gap size $b=2 \mathrm{~mm}$ filled to a depth $h=60 \mathrm{~mm} . f$ denotes the non-dimensional forcing acceleration, $f=a \Omega^{2} / g$, with dimensional forcing amplitude $a$ and angular frequency $\Omega$. For plotting, we define a small scale-separation parameter $\epsilon=k b / 2 \pi$ and we arbitrarily set its maximum acceptable value to 0.2 . Contact line dissipation is not included, i.e. $M=\sigma_{C L}=0$. SH stands for sub-harmonic, whereas $H$ stands for harmonic.

### 8.1.2 Asymptotic approximations

The main result of this analysis consists in the derivation of the modified damping coefficient $\sigma_{n}=\sigma_{n, r}+\mathrm{i} \sigma_{n, i}$ associated with each parametric resonance. Aiming at better elucidating how this modified complex damping influences the stability properties of the system, we would like to derive in this section an asymptotic approximation, valid in the limit of small forcing amplitudes, damping and detuning, of the first sub-harmonic (SH1) and harmonic (H1) Faraday tongues.
Unfortunately, the dependence of $\sigma_{n}$ on the parametric resonance considered and, more specifically, on the $n$th Fourier component, does not allow one to directly convert the governing equations (8.27), expressed in a discrete frequency domain, back into the continuous temporal domain. By keeping this in mind, we can still imagine fixing the value of $\sigma_{n}$ to that
corresponding to the parametric resonance of interest, e.g. $\sigma_{0}$ (with $n=0$ and $\xi_{0} \Omega=\Omega / 2$ ) for SH1 or $\sigma_{1}$ (with $n=1$ and $\xi_{1} \Omega=\Omega$ ) for H1. By considering then that for the SH1 and H1 tongues, the system responds in time as $\exp (i \Omega t / 2)$ and $\exp (i \Omega t)$, respectively, we can recast, for these two specific cases, equations (8.27) into a damped Mathieu equation (Benjamin and Ursell, 1954; Kumar and Tuckerman, 1994a; Müller et al., 1997)

$$
\begin{equation*}
\frac{\partial^{2} \hat{\eta}}{\partial t^{\prime 2}}+\hat{\sigma}_{n} \frac{\partial \hat{\eta}}{\partial t^{\prime}}+\omega_{0}^{2}\left(1-\frac{f}{1+\Gamma} \cos \Omega t^{\prime}\right) \hat{\eta}=0 \tag{8.32}
\end{equation*}
$$

with either $\hat{\sigma}_{n}=\sigma_{0}(\mathrm{SH1})$ or $\hat{\sigma}_{n}=\sigma_{1}(\mathrm{H} 1)$ and where one can recognize that $-\left(\xi_{n} \Omega\right)^{2} \hat{\eta} \leftrightarrow$ $\partial^{2} \hat{\eta} / \partial t^{\prime 2}$ and $\mathrm{i}\left(\xi_{n} \Omega\right) \hat{\eta} \leftrightarrow \partial \hat{\eta} / \partial t^{\prime}$. Asymptotic approximations can be then computed by expanding asymptotically the interface as $\hat{\eta}=\hat{\eta}_{0}+\epsilon \hat{\eta}_{1}+\epsilon^{2} \hat{\eta}_{2}+\ldots$, with $\epsilon$ a small parameter $\ll 1$.

## First sub-harmonic tongue

As anticipated above, when looking at the first or fundamental sub-harmonic tongue (SH1), one should take $\hat{\sigma}_{n} \rightarrow \sigma_{0}$ (with $\xi_{0} \Omega=\Omega / 2$ ), which is assumed small of order $\epsilon$. The forcing amplitude $f$ is assumed of order $\epsilon$ as well. Furthermore, a small detuning $\sim \epsilon$, such that $\Omega=2 \omega_{0}+\epsilon \lambda$, is also considered, and, in the spirit of the multiple timescale analysis, a slow time scale $T=\epsilon t^{\prime}$ (Nayfeh, 2008a) is introduced. At leading order, the solution reads $\hat{\eta}_{0}=$ $A(T) e^{\mathrm{i} \omega_{0} t^{\prime}}+c . c$. , with c.c. denoting the complex conjugate part. At the second order in $\epsilon$, the imposition of a solvability condition necessary to avoid secular terms prescribes the amplitude $B(T)=A(T) e^{-\mathrm{i} \lambda T / 2}$ to obey the following amplitude equation

$$
\begin{equation*}
\frac{d B}{d T}=-\frac{\sigma_{0}}{2} B-\mathrm{i} \frac{\lambda}{2} B-\mathrm{i} \frac{\omega_{0}}{4(1+\Gamma)} f \bar{B} \tag{8.33}
\end{equation*}
$$

Turning to polar coordinates, i.e. $B=|B| e^{\mathrm{i} \Phi}$, keeping in mind that $\sigma_{0}=\sigma_{0, r}+\mathrm{i} \sigma_{0, i}$ and looking for stationary solutions with $|B| \neq 0$ (we skip the straightforward mathematical steps), one ends up with the following approximation for the marginal stability boundaries associated with the first sub-harmonic Faraday tongue

$$
\begin{equation*}
\left(\frac{\Omega+\sigma_{0, i}}{2 \omega_{0}}-1\right)= \pm \frac{1}{4(1+\Gamma)} \sqrt{f^{2}-\frac{4 \sigma_{0, r}^{2}(1+\Gamma)^{2}}{\omega_{0}^{2}}} \tag{8.34}
\end{equation*}
$$

whose onset acceleration value, $\min f_{1_{S H}}$, for a fixed driving frequency $\Omega / 2 \pi$, amounts to

$$
\begin{equation*}
\min f_{S H 1}=2 \sigma_{0, r} \sqrt{\frac{1+\Gamma}{g k \tanh k h}} \approx 2 \sigma_{0, r} \sqrt{\frac{1}{g}\left(\frac{1}{k}+\frac{\gamma}{\rho g} k\right)}, \tag{8.35}
\end{equation*}
$$

Note that the final approximation on the right-hand-side of (8.35) only holds if $k h \gg 1$, so that $\tanh k h \approx 1$ (deep water regime). Given that $\chi_{0, r}>12$ and $\chi_{0, i}>0$ always, the asymptotic approximation (8.35), in its range of validity, suggests that Darcy's model underestimates
the sub-harmonic stability threshold. Moreover, from (8.34), the critical wavenumber $k$, associated with $\min f_{S H 1}$, would correspond to that prescribed by the Darcy approximation but at an effective forcing frequency $\Omega+\sigma_{0, i}=2 \omega_{0}$ instead of at $\Omega=2 \omega_{0}$. This explains why the modified tongues appear shifted towards higher wavenumbers. These observations are well visible in figure 8.4.


Figure 8.4 - First sub-harmonic and harmonic Faraday tongues at a driving frquency $1 / T=18 \mathrm{~Hz}$ for the same configuration of figure 8.3. Black and red regions show unstable tongues computed via Floquet analysis by using, respectively, $\sigma_{B L}=12 \mathrm{v} / \mathrm{b}^{2}$ and the modified $\sigma_{B L}=\chi_{1} v / b^{2}$ from the present model. Dashed and solid light-blue lines correspond to the asymptotic approximations according to (8.34) and (8.37).

## First harmonic tongue

By analogy with §8.1.2, an analytical approximation of the first harmonic tongue (H1) can be provided. In the same spirit of Rajchenbach and Clamond (2015b), we adapt the asymptotic scaling such that $f$ is still of order $\epsilon$, but $T=\epsilon^{2}, \hat{\sigma}_{n}=\sigma_{1} \sim \epsilon^{2}$ (with $\xi_{1} \Omega=\Omega$ ) and $\Omega=\omega_{0}+\epsilon^{2} \lambda$. Pursuing the expansion up to $\epsilon^{2}$-order, with $\hat{\eta}_{0}=A(T) e^{\mathrm{i} \omega_{0} t^{\prime}}+$ c.c. and $B(T)=A(T) e^{-\mathrm{i} \lambda T}$, will provide the amplitude equation

$$
\begin{equation*}
\frac{d B}{d T}=-\frac{\sigma_{1}}{2} B-\mathrm{i} \lambda B-\mathrm{i} \frac{\omega_{0}}{8(1+\Gamma)^{2}} f^{2} \bar{B}+\mathrm{i} \frac{\omega_{0}}{12(1+\Gamma)^{2}} f^{2} B \tag{8.36}
\end{equation*}
$$

The approximation for the marginal stability boundaries derived from (8.36) takes the form

$$
\begin{equation*}
\left(\frac{\Omega+\sigma_{1, i} / 2}{\omega_{0}}-1\right)=\frac{f^{2}}{12(1+\Gamma)^{2}} \pm \frac{1}{8(1+\Gamma)^{2}} \sqrt{f^{4}-\left(\frac{4 \sigma_{1, r}(1+\Gamma)^{2}}{\omega_{0}}\right)^{2}} \tag{8.37}
\end{equation*}
$$

with a minimum onset acceleration, $\min f_{1_{H}}$

$$
\begin{equation*}
\min f_{H}=2 \sqrt{\sigma_{1, r}}\left(\frac{(1+\Gamma)^{3}}{g k \tanh k h}\right)^{1 / 4} \approx 2 \sqrt{\sigma_{1, r}} \frac{1}{g^{1 / 4}}\left(\frac{1}{k^{1 / 3}}+\frac{\gamma}{\rho g} k^{5 / 3}\right)^{3 / 4} \tag{8.38}
\end{equation*}
$$

and where, as before, the final approximation on the right-hand side is only valid in the deep water regime. Similarly to the sub-harmonic case, the critical wavenumber $k$ corresponds to that prescribed by the Darcy approximation but at an effective forcing frequency $\Omega+\sigma_{1, i} / 2=$ $\omega_{0}$ instead of at $\Omega=\omega_{0}$ and the onset acceleration is larger than that predicted from the Darcy approximation (as $\chi_{1, r}>12$ ).

### 8.1.3 Comparison with experiments by Li et al. (2019)

| Liquid | $\mu[\mathrm{mPa} \mathrm{s}]$ | $\rho\left[\mathrm{kg} / \mathrm{m}^{3}\right]$ | $\gamma[\mathrm{N} / \mathrm{m}]$ | $M[\mathrm{~Pa} \mathrm{~s}]$ |
| :---: | :---: | :---: | :---: | :---: |
| ethanol $99.7 \%$ | 1.096 | 785 | 0.0218 | 0.04 |
| ethanol $70.0 \%$ | 2.159 | 835 | 0.0234 | 0.0485 |
| ethanol $50.0 \%$ | 2.362 | 926 | 0.0296 | 0.07 |

Table 8.1 - Characteristic fluid parameters for the three ethanol-water mixtures considered in this study. Data for the pure ethanol and ethanol-water mixture ( $50 \%$ ) are taken from Li et al. (2019). The value of the friction parameter $M$ for ethanol-70\% is fitted from the experimental measurements reported in $\S 8.3$, but lies well within the range of values used by Li et al. (2019) and agrees with the linear trend displayed in figure 5 of Hamraoui et al. (2000).


Figure 8.5 - Sub-harmonic instability onset, $\min f$, versus driving frequency, $\Omega / 2 \pi$. Comparison between theoretical data (empty squares: standard Darcy model, $\sigma_{B L}=12 \mathrm{v} / \mathrm{b}^{2}$; colored triangles: present model, $\sigma_{B L}=\chi_{n} v / b^{2}$ ) and experimental measurements by Li et al. (2019). The values of the mobility parameter $M$ here employed are reported in the figure.

Results presented so far were produced by assuming the absence of contact line dissipation, i.e. coefficient $M$ was set to $M=0$, so that $\sigma_{C L}=0$. In this section, we reintroduce such a dissipative contribution and we compare our theoretical predictions with a set of experimental measurements reported by Li et al. (2019), using the values they have proposed for M. This comparison, shown in figure 8.5, is outlined in terms of non-dimensional minimum onset acceleration, $\min f=\min f_{S H 1}$, versus driving frequency. These authors performed experiments in two different Hele-Shaw cells of length $l=300 \mathrm{~mm}$, fluid depth $h=60 \mathrm{~mm}$ and gap-size $b=2 \mathrm{~mm}$ or $b=5 \mathrm{~mm}$. Two fluids, whose properties are reported in table 8.1, were used: ethanol $99.7 \%$ and ethanol $50 \%$. The empty squares in figure 8.5 are computed
via Floquet stability analysis (8.31) using the Darcy approximation for $\sigma_{B L}=12 \mathrm{v} / \mathrm{b}^{2}$ and correspond to the theoretical prediction by Li et al. (2019), while the colored triangles are computed using the present theory, with the corrected $\sigma_{B L}=\chi_{n} v / b^{2}$. Although the trend is approximately the same, the Darcy approximation underestimates the onset acceleration with respect to the present model, which overall compares better with the experimental measurements (black-filled circles). Some disagreement still exists, especially at smaller cell gaps, i.e. $b=2 \mathrm{~mm}$, where surface tension effects are even larger. This is likely attributable to an imperfect phenomenological contact line model (Bongarzone et al., 2022b, 2021c), whose definition falls beyond the scope of this work. Yet, this comparison shows how the modifications introduced by the present model contribute to closing the gap between theoretical Faraday onset estimates and these experiments.

### 8.2 The case of thin annuli

We now consider the case of a thin annular container, whose nominal radius is $R$ and the actual inner and outer radii are $R-b / 2$ and $R+b / 2$, respectively (see the sketch in figure 8.1(b)). In the limit of $b / R \ll 1$, the wall curvature is negligible and the annular container can be considered a Hele-Shaw cell. The following change of variable for the radial coordinate, $r^{\prime}=R+y^{\prime}=R\left(1+y^{\prime} / R\right)$ with $y^{\prime} \in[-b / 2, b / 2]$, will be useful in the rest of the analysis. As in §8.1, we first linearize around the rest state. Successively, we introduce the following non-dimensional quantities,

$$
\begin{equation*}
r=\frac{r^{\prime}}{R}, \quad y=\frac{y^{\prime}}{b}, \quad z=\frac{z^{\prime}}{R}, \quad u=\frac{u_{\varphi}^{\prime}}{a \Omega}, \quad v=\frac{u_{r}^{\prime}}{a \Omega(b / R)}, \quad w=\frac{u_{z}^{\prime}}{a \Omega}, \quad p=\frac{p^{\prime}}{\rho R a \Omega^{2}} . \tag{8.39}
\end{equation*}
$$

It follows that, at leading order, $r=1+y b / R \sim 1 \longrightarrow 1 / r=1 /(1+y b / R) \sim 1$ but $\partial / \partial_{r}=$ $(R / b) \partial / \partial_{y} \sim(b / R)^{-1} \gg 1$. With this scaling and introducing the Floquet ansatzs (8.6a)-(8.6b), one obtains the following simplified governing equations,

$$
\begin{gather*}
\frac{\partial \tilde{u}_{n}}{\partial \varphi}+\frac{\partial \tilde{v}_{n}}{\partial y}+\frac{\partial \tilde{w}_{n}}{\partial z}=0,  \tag{8.40a}\\
\mathrm{i} \tilde{u}_{n}=-\frac{1}{\xi_{n}} \frac{\partial \tilde{p}_{n}}{\partial \varphi}+\frac{\delta_{n}^{2}}{2} \frac{\partial^{2} \tilde{u}_{n}}{\partial y^{2}}, \quad \mathrm{i} \tilde{w}_{n}=-\frac{1}{\xi_{n}} \frac{\partial \tilde{p}_{n}}{\partial z}+\frac{\delta_{n}^{2}}{2} \frac{\partial^{2} \tilde{w}_{n}}{\partial y^{2}} \quad \text { or } \quad \tilde{\mathbf{u}}_{n}=\frac{\mathrm{i}}{\xi_{n}} \nabla \tilde{p}_{n} F_{n}(y), \tag{8.40b}
\end{gather*}
$$

which are fully equivalent to those for the case of conventional rectangular cells if the transformation $\varphi \rightarrow x$ is introduced. Averaging the continuity equation with the imposition of the no-penetration condition at $y=\mp 1 / 2, v(\mp 1 / 2)$, eventually leads to

$$
\begin{equation*}
\nabla^{2} \tilde{p}_{n}=\frac{\partial^{2} \tilde{p}_{n}}{\partial z^{2}}+\frac{\partial^{2} \tilde{p}_{n}}{\partial \varphi^{2}}, \tag{8.41}
\end{equation*}
$$

identically to (8.11). Expanding $\tilde{p}_{n}$ in the azimuthal direction as $\tilde{p}_{n}=\hat{p}_{n} \sin m \varphi$, with $m$ the azimuthal wavenumber, provides

$$
\begin{equation*}
\left(\frac{\partial^{2}}{\partial z^{2}}-m^{2}\right) \hat{p}_{n}=0 \quad \rightarrow \quad \hat{p}_{n}=c_{1} \cosh m z+c_{2} \sinh m z \tag{8.42}
\end{equation*}
$$

and the no-penetration condition at the solid bottom located at $z=-h / R, \hat{w}_{n}=\partial_{z} \hat{p}_{n}=0$, prescribes

$$
\begin{equation*}
\hat{p}_{n}=c_{1}(\cosh m z+\tanh m h / R \sinh m z) \tag{8.43}
\end{equation*}
$$

Although so far the theory for the rectangular and the annular cases is basically the same, here it is crucial to observe that the axisymmetric container geometry translates into a periodicity condition:

$$
\begin{equation*}
\sin (-m \pi)=\sin (m \pi) \quad \longrightarrow \quad \sin m \pi=0 \tag{8.44}
\end{equation*}
$$

which always imposes the azimuthal wavenumber to be an integer. In other words, in contradistinction with the case of $\S 8.1$, where the absence of lateral wall ideally allows for any wavenumber $k$, here we have $m=0,1,2,3, \ldots \in \mathbb{N}$.
By repeating the calculations outlined in $\S 8.1$, one ends up with the same equation (8.28) (and subsequent (8.29)-(8.31)), but where $\omega_{0}$ obeys to the quantized dispersion relation

$$
\begin{equation*}
\omega_{0}^{2}=\left(\frac{g}{R} m+\frac{\gamma}{\rho R^{3}} m^{3}\right) \tanh m \frac{h}{R}=(1+\Gamma) \frac{g}{R} m \tanh m \frac{h}{R} \tag{8.45}
\end{equation*}
$$

with $\Gamma=\gamma m^{2} / \rho g R^{2}$. In this context, a representation of Faraday's tongues in the forcing frequency-amplitude plane appears most natural, as each parametric tongue will correspond to a fixed wavenumber $m$. Consequently, instead of fixing $\Omega$ and varying the wavenumber, here we solve (8.31) by fixing $m$ and varying $\Omega$.

### 8.2.1 Floquet analysis and asymptotic approximation

The results from this procedure are reported in figure 8.6, where, as in figure 8.3, the black regions correspond to the unstable tongues obtained according to the standard gap-averaged Darcy model, while the red ones are computed using the present theory with the corrected gapaveraged $\sigma_{B L}=\chi_{n} v / b^{2}$. The Faraday threshold is represented in terms of forcing acceleration (panels (a) and (b)) and forcing amplitude (panels (c) and (d)). Note the prediction reported in panels (c) and (d) are equivalent to those reported in panels (a) and (b) with the ordinate rescaled by a factor $\Omega^{2} / g$. In figure 8.6(a)-(c) no contact line model is included, whereas in (b)-(d) a mobility parameter $M=0.0485$ is accounted for. The use of this specific value for $M$ will be clarified in the next section when comparing the theory with dedicated experiments. The regions with the lowest thresholds in each panel are sub-harmonic tongues associated with modes from $m=1$ to 14 .
In general, the present model gives a higher instability threshold, consistent with the results reported in the previous section. However, the tongues are here shifted to the left.


Figure 8.6 - Faraday tongues computed via Floquet analysis (8.31) at different fixed azimuthal wavenumber $m$ and varying the driving frequency, $\Omega / 2 / \pi$. (a)-(b) Faraday thresholds in terms of forcing acceleration $f=a \Omega^{2} / g$; (c)-(d) Threshold in terms of forcing amplitude $a$. Black regions correspond to the unstable Faraday tongues computed using $\sigma_{B L}=12 v / b^{2}$, whereas red regions are the unstable tongues computed with the present modified $\sigma_{B L}=\chi_{n} v / b^{2}$. The fluid parameters used here correspond to those given in table 8.1 for ethanol $70 \%$. The gap-size is set to $b=7 \mathrm{~mm}$, the fluid depth to $h=65 \mathrm{~mm}$ and the nominal radius to $R=44 \mathrm{~mm}$. Contact line dissipation is included in (b) and (d) by accounting for a mobility coefficient $M=0.0485$. The regions with the lowest thresholds in each panel are sub-harmonic tongues associated with modes from $m=1$ to 14 .

The asymptotic approximation for the sub-harmonic onset acceleration, adapted to this case from (8.34) yields:

$$
\begin{equation*}
f_{S H 1}=2 \sqrt{(1+\Gamma) \frac{\sigma_{0, r}^{2}}{(g / R) m \tanh m h / R}+4(1+\Gamma)^{2}\left(\frac{\Omega+\sigma_{0, i}}{2 \omega_{0}}-1\right)^{2}} \tag{8.46}
\end{equation*}
$$

with

$$
\begin{equation*}
\min f_{S H 1}=2 \sigma_{0, r} \frac{1+\Gamma}{\omega_{0}}=2 \sigma_{0, r} \sqrt{\frac{1+\Gamma}{(g / R) m \tanh m h / R}} \approx 2 \sigma_{0, r} \sqrt{\frac{R}{g}\left(\frac{1}{m}+\frac{\gamma}{\rho g R^{2}} m\right)}, \tag{8.47}
\end{equation*}
$$

helps us indeed in rationalizing the influence of the modified complex damping coefficient. This apparent opposite correction is a natural consequence of the different representations: varying wavenumber at a fixed forcing frequency (as in figure 8.3) versus varying forcing frequency at a fixed wavenumber (figure 8.6). Such a behaviour is clarified by the asymptotic relation (8.46) and, particularly by the term $\left(\frac{\Omega+\sigma_{0, i}}{2 \omega_{0}}-1\right)$. In $\S 8.1$, the analysis is based on a fixed forcing frequency, while the wavenumber $k$ and, hence, the natural frequency $\omega_{0}$, are let free to vary. The first sub-harmonic Faraday tongue occurs when $\Omega+\sigma_{0, i} \approx 2 \omega_{0}$. Since $\Omega$ is fixed and $\sigma_{0, i}>0, \Omega+\sigma_{0, i}>\Omega$ such that $\omega_{0}$ and therefore $k$ have to increase in order to satisfy the
relation. On the other hand, if the wavenumber $m$ and, hence, $\omega_{0}$ are fixed as in this section, then $2 \omega_{0}-\sigma_{0, i}<2 \omega_{0}$ and the forcing frequency around which the sub-harmonic resonance is centered, decreases of a contribution $\sigma_{0, i}$, which introduces a frequency detuning responsible for the negative frequency shift displayed in figure 8.6.

### 8.2.2 Discussion on the system's spatial quantization

A first aspect that needs to be better discussed is the frequency-dependence of the damping coefficient $\sigma_{n}$ associated with each Faraday's tongue. In the case of horizontally infinite cells, the most natural description for investigating the system's stability properties is in the $(k, f)$ plane for a fixed forcing angular frequency $\Omega$ (Kumar and Tuckerman, 1994a). According to our model, the oscillating system's response occurring within each tongue is characterized by a Stokes boundary layer thickness $\delta_{n}=\sqrt{2 v /(n \Omega+\alpha)} / b$. For instance, let us consider sub-harmonic resonances with $\alpha=\Omega / 2$. As $\Omega$ is fixed (see any sub-panel of figure 8.3), each unstable region sees a constant $\delta_{n}$ (with $n=0,1,2, \ldots$ ) and hence a constant damping $\sigma_{n}$. On the other hand, in the case of quantized wavenumber as for the annular cell of $\S 8.2$, the most suitable description is in the driving frequency-driving amplitude plane at fixed wavenumber $m$ (see figure 8.6) (Batson et al., 2013). In this description, each sub-harmonic ( $\alpha=\Omega / 2$ ) or harmonic $(\alpha=\Omega) n$th tongue associated with a wavenumber $m$, sees a $\delta_{n}$, and thus a $\sigma_{n}$, changing with $\Omega$ along the tongue itself.

### 8.3 Experiments

In a real lab-scale experiment, the horizontal size of rectangular cells is never actually infinite due to the presence of lateral walls in the elongated direction. In such a case however, the solution form (8.9) prevents the no-slip condition for the in-plane $x z$-velocity components to be imposed (Viola et al., 2017). This always translates into a theoretical underestimation of the overall damping of the system in rectangular Hele-Shaw cells, although the sidewall contribution is expected to be negligible for sufficiently long cells.
On the other hand, the case of a thin annulus, by naturally filtering out this extra dissipation owing to the periodicity condition, offers a prototype configuration that can potentially allow one to better quantify the correction introduced by the present gap-averaged model when compared to dedicated experiments.

### 8.3.1 Setup

The experimental apparatus, shown in figure 8.7, consists in a Plexiglas annular container of height 100 mm , nominal radius $R=44 \mathrm{~mm}$ and gap-size $b=7 \mathrm{~mm}$, which is then filled to a depth $h=65 \mathrm{~mm}$ with ethanol $70 \%$ (see table 8.1 for the fluid properties). An air conditioning


Figure 8.7 - Photo of the experimental setup
system helps in maintaining the temperature of the room at around $22^{\circ}$. The container is mounted on a loudspeaker VISATON TIW $3608 \Omega$ placed on a flat table and connected to a wave generator TEKTRONIX AFG 1022, whose output signal is amplified using a wideband amplifier THURKBY THANDER WA301. The motion of the free surface is recorded with a digital camera NIKON D850 coupled with a $60 \mathrm{~mm} \mathrm{f} / 2.8 \mathrm{D}$ lens and operated in slow motion mode, allowing for an acquisition frequency of 120 frames per second. A LED panel placed behind the apparatus provides back illumination of the fluid interface for better optimal contrast. The wave generator imposes a sinusoidal alternating voltage, $v=\left(\mathrm{V}_{p p} / 2\right) \cos \left(\Omega t^{\prime}\right)$, with $\Omega$ the angular frequency and $V p p$ the full peak-to-peak voltage. The response of the loudspeaker to this input translates into a vertical harmonic motion of the container, $a \cos \left(\Omega t^{\prime}\right)$, whose amplitude, $a$ [mm], is measured with a chromatic confocal displacement sensor STI CCS PRIMA/CLS-MG20. This optical pen, which is placed around 2 cm (within the admissible working range of 2.5 cm ) above the container and points at the top flat surface of the outer container's wall, can detect the time-varying distance between the fixed sensor and the oscillating container's surface with a sampling rate in the order of kHz and a precision of $\pm 1 \mu \mathrm{~m}$. Therefore, the pen can be used to obtain a very precise real-time value of $a$ as the voltage amplitude $V p p$ and the frequency $\Omega$ are adjusted.

### 8.3.2 Identification of the accessible experimental range

Such a simple setup, however, put some constraints on the explorable experimental frequency range.
(i) First, we need to ensure that the loudspeaker's output translates into a vertical container's


Figure 8.8 - Top: vertical container displacement $a$ versus time at different forcing frequencies. The black curves are the measured signal, while the green dash-dotted curves are sinusoidal fitting. Below a forcing frequency of 8 Hz , the loudspeaker's output begins to depart from a sinusoidal signal. Bottom: same as in figure 8.6(d): sub-harmonic Faraday tongues computed by accounting for contact line dissipation with a mobility parameter $M=0.0485$. The light blue curve here superposed corresponds to the maximal vertical displacement $a$ achievable with our setup. With this constraint, Faraday waves are expected to be observable only in the frequency range highlighted in blue.
displacement following a sinusoidal time signal. To this end, the optical sensor is used to measure the container motion at different driving frequencies. These time signals are then fitted with a sinusoidal law. Figure 8.8 shows how below a forcing frequency of 8 Hz , the loudspeaker's output begins to depart from a sinusoidal signal. This check imposes a first lower bound on the explorable frequency range.
(ii) In addition, as Faraday waves only appear above a threshold amplitude, it is convenient to measure a priori the maximal vertical displacement $a$ achievable. The loudspeaker response curve is reported in the bottom part of figure 8.8. A superposition of this curve with the predicted Faraday's tongues immediately identifies the experimental frequency range within which the maximal achievable $a$ is larger than the predicted Faraday threshold so that standing waves are expected to emerge in our experiments. Assuming the herein proposed gap-averaged model (red regions) to give a good prediction of the actual instability onset, the experimental range explored in the next section is limited to approximately $\in[10.2,15.6] \mathrm{Hz}$.

### 8.3.3 Procedure



Figure 8.9 - Free surface shape at a forcing frequency $1 / T=11.7 \mathrm{~Hz}$ and corresponding to: (a) the lowest forcing amplitude value, $a=0.4693 \mathrm{~mm}$, for which the $m=6$ standing wave is present (the figure shows a temporal snapshot); (b) the largest forcing amplitude value, $a=0.4158 \mathrm{~mm}$, for which the surface becomes flat and stable again. Despite the small forcing amplitude variation, the change in amplitude is large enough to allow for a visual inspection of the instability threshold with sufficient accuracy.

Given the constraints discussed in $\S 8.3 .2$, experiments have been carried out in a frequency range between 10.2 Hz and 15.6 Hz with a frequency step of 0.1 Hz . For each fixed forcing frequency, the Faraday threshold is determined as follows: the forcing amplitude $a$ is set to the maximal value achievable by the loudspeaker, so as to quickly trigger the emergence of the unstable Faraday wave. The amplitude is then progressively decreased until the wave disappears and the surface becomes flat again.
More precisely, a first quick pass across the threshold is made to determine an estimate of the sought amplitude. A second pass is then made by starting again from the maximum amplitude and decreasing it. When we approach the value determined during the first pass, we perform finer amplitude decrements, and we wait several minutes between each amplitude change to ensure that the wave stably persists. We eventually identify two values: the last amplitude where the instabilities were present (see figure 8.9(a)) and the first one where the surface becomes flat again (see figure 8.9 (b)). Two more runs following an identical procedure are then performed to verify the values previously found. Lastly, an average between the smallest unstable amplitude and the largest stable one gives us the desired threshold.
Once the threshold amplitude value is found for the considered frequency, the output of the wave generator is switched off, the frequency is changed, and the steps presented above are repeated again for the new frequency. In this way we always start from a stable configuration, hence limiting the possibility of nonlinear interaction between different modes.
For each forcing frequency, the two limiting amplitude values, identified as described above, are used to define the error bars reported in figure 8.10. Those error bars must also account for the optical pen's measurement error $(0.1 \mu \mathrm{~m})$, as well as the non-uniformity of the output signal. By looking at the measured average, minimum, and maximum amplitude values in the temporal output signal, it is noteworthy that the average value typically deviates from the minimum and maximum by around $10 \mu \mathrm{~m}$. Consequently, we incorporate in the error bars this additional $10 \mu \mathrm{~m}$ of uncertainty in the value of $a$. The uncertainty in the frequency of the output signal is not included in the definition of the error bars, as it is extremely small, on the order of 0.001 Hz .


Figure 8.10 - Experiments (empty circles) are compared to the theoretically predicted subharmonic Faraday threshold computed via Floquet analysis (8.31) for different fixed azimuthal wavenumber $m$ and according to the standard (black solid lines) and revised (red regions) gap-averaged models. The tongues are computed by including contact line dissipation with a value of $M$ equal to 0.0485 as in figures 8.6 (b)-(d) and 8.8. As explained in $\S 8.3 .3$, the vertical error bars indicate the amplitude range between the smallest measured forcing amplitude at which the instability was detected and the largest one at which the surface remains stable and flat. These two limiting values are successively corrected by accounting for the optical pen's measurement error and the non-uniformity of the output signal of the loudspeaker.

### 8.3.4 Instability onset and wave patterns

The experimentally detected threshold at each measured frequency is reported in figure 8.10 in terms of forcing acceleration $f$ and amplitude $a$. Once again, the black unstable regions are calculated according to the standard gap-averaged model with $\sigma_{B L}=12 v / b^{2}$, whereas red regions are the unstable tongues computed using the modified damping $\sigma_{B L}=\chi_{n} v / b^{2}$. Both scenarios include contact line dissipation $\sigma_{C L}=(2 M / \rho b)(m / R) \tanh (m h / R)$, with a value of $M$ equal to 0.0485 for ethanol $70 \%$. Although, at first, this value has been simply selected in order to fit well our experimental measurements, it is in perfect agreement with the linear relation linking $M$ to the liquid's surface tension reported in figure 5 of Hamraoui et al. (2000) and used by Li et al. (2019) (see table 8.1).
As figure 8.10 strikingly shows, the present theoretical thresholds match well our experimental measurements. On the contrary, the poor description of the oscillating boundary layer in


Figure 8.11 - Snapshots of the wave patterns experimentally observed within the sub-harmonic Faraday tongues associated with the azimuthal wavenumbers $m=5,6,7,8$ and $9 . T$ is the forcing period, which is approximately half the oscillation period of the wave response. These patterns appear for: $(m=5) 1 / T=10.6 \mathrm{~Hz}, a=0.8 \mathrm{~mm} ;(m=6) 1 / T=11.6 \mathrm{~Hz}, a=1.1 \mathrm{~mm}$; $(m=7) 1 / T=12.7 \mathrm{~Hz}, a=0.9 \mathrm{~mm} ;(m=8), 1 / T=13.7 \mathrm{~Hz}, a=0.6 \mathrm{~mm} ;(m=9) 1 / T=14.8 \mathrm{~Hz}$, $a=0.4 \mathrm{~mm}$. These forcing amplitudes are the maximal achievable at their corresponding frequencies (see figure 8.8 for the associated operating points). The number of peaks is easily countable by visual inspection of two time snapshots of the oscillating pattern extracted at $t=0, T$ and $t=T / 2$. This provides a simple criterion for the identification of the resonant wavenumber $m$. See also supplementary movies 1-5 at: LINK.
the classical Darcy model translates into a lack of viscous dissipation. The arbitrary choice of a higher fitting parameter $M$ value, e.g. $M \approx 0.09$ would increase contact line dissipation and compensate for the underestimated Stokes boundary layer one, hence bringing these predictions much closer to experiments; however, such a value would lie well beyond the typical values reported in the literature. Furthermore, the real damping coefficient $\sigma_{B L}=12 \mathrm{v} / \mathrm{b}^{2}$ given by the Darcy theory does not account for the frequency detuning displayed by experiments. This frequency shift is instead well captured by the imaginary part of the new damping $\sigma_{B L}=\chi_{n} v / b^{2}$ (with $\chi_{n}=\chi_{n, r}+\mathrm{i} \chi_{n, i}$ ).
Within the experimental frequency range considered, five different standing waves, corresponding to $m=5,6,7,8$ and 9 , have emerged. The identification of the wavenumber $m$ has been simply performed by visual inspection of the free surface patterns reported in figure 8.11. Indeed, by looking at a time snapshot, it is possible to count the various wave peaks along the azimuthal direction.
When looking at figure 8.10, it is worth commenting that on the left sides of the marginal stability boundaries associated with modes $m=5$ and 6 we still have a little discrepancy between experiments and the model. Particularly, the experimental thresholds are slightly lower than the predicted ones. A possible explanation can be given by noticing that our experimental protocol is agnostic to the possibility of subcritical bifurcations and hysteresis, while such behaviour has been predicted by Douady (1990).
As a last comment, one has to keep in mind that the Hele-Shaw approximation remains good only if the wavelength, $2 \pi R / m$ does not become too small, i.e. comparable to the cell's gap, $b$. In other words, one must check that the ratio $m b / 2 \pi R$ is of the order of the small separation-of-scale parameter, $\epsilon$. For the largest wavenumber observed in our experiments, $m=9$, the ratio $m b / 2 \pi R$ amounts to 0.23 , which is not exactly small. Yet, the Hele-Shaw approximation is seen to remain fairly good.

### 8.3.5 Contact angle variation and thin film deposition

Before concluding, it is worth commenting on why the use of dynamic contact angle model (8.21) is justifiable and seen to give good estimates of the Faraday thresholds.
Existing lab experiments have revealed that liquid oscillations in Hele-Shaw cells constantly experience an up-and-down driving force with an apparent contact angle $\theta$ constantly changing (Jiang et al., 2004). Our experiments are consistent with such evidence. In figure 8.12 we report seven snapshots, (i)-(vii), covering one oscillation period, $T$, for the container motion. These snapshots illustrate a zoom of the dynamic meniscus profile and show how the macroscopic contact angle changes in time during the second half of the advancing cycle (i)-(v) and the first half of the receding cycle (vi)-(x), hence highlighting the importance of the out-of-plane meniscus curvature variations. Thus, on the basis of our observations, it seemed appropriate to introduce in the theory a contact angle model so as to justify this associated additional dissipation, which would be neglected by assuming $M=0$. The model used in this study, and already implemented by Li et al. (2019), is very simple; it assumes the cosine of the dynamic contact angle to linearly depend on the contact line speed through the capillary number Ca


Figure 8.12 - Zoom of the meniscus dynamics recorded at a driving frequency 11.6 Hz and amplitude $a=1.2 \mathrm{~mm}$ for $m=6$. Seven snapshots, (i)-(vii), covering one oscillation period, $T$, for the container motion are illustrated. These snapshots show how the meniscus profile and the macroscopic contact angle change in time during the second half of the advancing cycle and the first half of the receding cycle, hence highlighting the importance of the out-of-plane curvature or capillary effects. See also supplementary movie 6 at: LINK.
(Hamraoui et al., 2000). Accounting for such a model is shown, both in Li et al. (2019) and in this study, to supplement the theoretical predictions by a sufficient extra dissipation suitable to match experimental measurements.
This dissipation eventually reduces to a simple damping coefficient $\sigma_{C L}$ as it is of linear nature. A unique constant value of the mobility parameter $M$ is sufficient to fit all our experimental measurements at once, suggesting that the meniscus dynamics is not significantly affected by the evolution of the wave in the azimuthal direction, i.e. by the wavenumber, and $M$ can be seen as an intrinsic property of the liquid-substrate interface.
Several studies have discussed the dependence of the system's dissipation on the substrate material (Cocciaro et al., 1993; Dussan, 1979; Eral et al., 2013; Huh and Scriven, 1971; Ting and Perlin, 1995; Viola et al., 2018; Viola and Gallaire, 2018; Xia and Steen, 2018). These authors, among others, have unveiled and rationalized interesting features such as solid-like friction induced by contact angle hysteresis. This strongly nonlinear contact line behaviour does not seem to be present in our experiments. This can be tentatively explained by looking at figure 8.13. These snapshots illustrate how the contact line constantly flows over a wetted substrate, due to the presence of a stable thin film deposited and alimented at each oscillation cycle. This feature has been also recently described by Dollet et al. (2020), who showed that the relaxation dynamics of liquid oscillation in a U-shaped tube filled with ethanol, due to the presence of a similar thin film, obey an exponential law that can be well-fitted by introducing a simple linear damping, as done in this work.


Figure 8.13 - These three snapshots correspond to snapshots (ii), (iii) and (iv) of figure 8.12 and show, using a different light contrast, how the contact line constantly moves over a wetted substrate due to the presence of a stable thin film deposited and alimented at each cycle.

### 8.4 Conclusions

Previous theoretical analyses for Faraday waves in Hele-Shaw cells have so far relied on the Darcy approximation, which is based on the parabolic flow profile assumption in the narrow direction and that translates into a real-valued damping coefficient $\sigma_{B L}=12 v / b^{2}$, with $v$ the fluid kinematic viscosity and $b$ the cell's gap-size, that englobes the dissipation originated from the Stokes boundary layers over the two lateral walls. However, Darcy's model is known to be inaccurate whenever inertia is not negligible, e.g. in unsteady flows such as oscillating standing or travelling waves.
In this work, we have proposed a gap-averaged linear model that accounts for inertial effects induced by the unsteady terms in the Navier-Stokes equations, amounting to a pulsatile flow where the fluid motion reduces to a two-dimensional oscillating flow, reminiscent of the Womersley flow in cylindrical pipes. When gap-averaging the linearized Navier-Stokes equation, this results in a modified damping coefficient, $\sigma_{B L}=\chi_{n} v / b^{2}$, with $\chi_{n}=\chi_{n, r}+i \chi_{n, i}$ complex-valued, which is a function of the ratio between the Stokes boundary layer thickness and the cell's gap-size, and whose value depends on the frequency of the system's response specific to each unstable parametric Faraday tongue.
After having revisited the ideal case of infinitely long rectangular Hele-Shaw cells, for which we have found a good agreement against the experiments by Li et al. (2019), we have considered the case of Faraday waves in thin annuli. This annular geometry, owing to the periodicity condition, naturally filters out the additional, although small, dissipation coming from the lateral wall in the elongated direction of finite-size lab-scale Hele-Shaw cells. Hence, a thin annulus offers a prototype configuration that can allow one to better quantify the correction introduced by the present gap-averaged theory when compared to dedicated experiments and to the standard gap-averaged Darcy model.
A series of homemade experiments for the latter configuration has proven that Darcy's model typically underestimates the Faraday threshold, as $\chi_{n, r}>12$, and overlooks a frequency detuning introduced by $\chi_{n, i}>0$, which appears essential to correctly predict the location of the Faraday's tongue in the frequency spectrum. The frequency-dependent gap-averaged model proposed here successfully predicts these features and brings the Faraday thresholds estimated theoretically closer to the ones measured.

Furthermore, a close look at the experimentally observed meniscus and contact angle dynamics clearly highlighted the importance of the out-of-plane curvature, whose contribution has been neglected so far in the literature, with the exception of Li et al. (2019). This evidence justifies the employment of a dynamical contact angle model to recover the extra contact line dissipation and close the gap with experimental measurements.
A natural extension of this work is to examine the existence of a drift instability at higher forcing amplitudes.

Nonlinear relaxation dynamics of free Part IV surface oscillations due to contact angle hysteresis

## Introduction

In Part II, we have tackled several aspects of sloshing, an archetypal resonator system in fluid mechanics which sometimes represents a critical issue in mechanical engineering and daily life. It is therefore crucial to understand its associated damping. Indeed, the latter plays a fundamental role in the mitigation of the wave amplitude response in resonant conditions.
We have mentioned how originally the eigenfrequencies of standing capillary-gravity waves in closed basins were derived in the potential flow limit (Lamb, 1993), while the linear viscous dissipation at the free surface, at the solid walls and in the bulk for low-viscosity fluids was typically accounted for by a boundary layer approximation (Case and Parkinson, 1957; Miles, 1967; Ursell, 1952). This classic theoretical approach, which has been used in Part II, is built on the simplifying assumption that the free liquid surface, $\eta$, intersects the lateral wall orthogonally and the contact line can freely slip at a velocity $\partial \eta / \partial t(\sim U)$ and with a constant zero slope,

$$
\frac{\partial \eta}{\partial n}=0 \quad \text { free-end edge condition, }
$$

where $\partial / \partial n$ is the spatial derivative in the direction normal to the lateral wall. Chapters 4,5 and 6 , have proven these hypotheses reasonable for the modelling of gravity-dominated waves in moderately large-size containers, although some mismatch between theory and experiments is still present. Such a mismatch was partially attributed to additional dissipations sources acting at the moving contact line, whose dynamics is the central topic of this Part IV.
The classical assumption of a free-end edge condition has been relaxed in Part III, where two other scenarios have been studied within the framework of the Faraday instability in small-size partially filled containers. In Chapter 7, we have considered a diametrically opposed boundary condition, namely a pinned-end edge, according to which the contact line is fixed,

$$
\frac{\partial \eta}{\partial t}=0 \quad \text { pinned-end edge condition, }
$$

while the slope, $\partial \eta / \partial n$, is let free to vary (Benjamin and Scott, 1979; Graham-Eagle, 1983). In this case, theoretical predictions have provided an estimation of the sub-harmonic Faraday threshold in good agreement with experimental measurements. Indeed, with the contact line being fixed, the system's dissipation can be estimated accurately, since no extra and undetermined dissipation is generated by the contact line.
In Chapter 8, the instability onset of Faraday waves in Hele-Shaw cells and with a moving

## Partial Wetting in Uni-Directional Flows



Partial Wetting in Oscillatory Flows


Figure IV. 1 - (a) Advancing, $\theta_{a}$, and receding, $\theta_{r}$, contact angles in a droplet sliding down with velocity $U$ over a dry substrate (partial wetting). (d) Contact angles in an expanding and contracting liquid droplet. Both (a) and (d) are examples of uni-directional flows. The dynamic contact angle is seen experimentally to depend on the capillary number, $C a=\mu U / \gamma$, as reported by (b) Snoeijer and Andreotti (2013) and (c) Rio et al. (2005). The dependence of the contact angle, $\theta$, on the capillary number, $C a$, is modelled in the literature by the (e) de Gennes (Gennes, 1985) and (f) Cox-Voinov (Cox, 1986; Voinov, 1976) models. (g) Contact angle dynamics in a vertically vibrating droplet and in ( j ) sloshing waves (snapshots over a period) (Viola, 2016). For these oscillatory flows, experiments by (h) Xia and Steen (2018) and (l) Cocciaro et al. (1993) suggest as suitable phenomenological contact angle laws the (i) nonlinear Dussan model (Dussan, 1979; Jiang et al., 2004), sometimes simply approximated by the (m) Hocking linear law (Hocking, 1987) supplemented with hysteresis.
contact line has been estimated by introducing, in the same spirit of Li et al. (2019), an intermediate boundary condition that assumes a linear relation between the contact line speed and slope, $\partial \eta / \partial n \propto \partial \eta / \partial t$ (Hocking, 1987), with a proportionality constant, sometimes referred to as mobility parameter $M$ (Xia and Steen, 2018), that in our study has been kept constant in time. We note that, according to the linear relation,

$$
\frac{\partial \eta}{\partial n}=M \frac{\partial \eta}{\partial t} \quad \text { mixed condition, }
$$

the limiting values $M \rightarrow 0$ and $M \rightarrow \infty$ would correspond, respectively, to free-end and pinnedend edge contact line conditions. The agreement with experiments was found to be fairly good, although this proportionality constant was used as a fitting parameter.
With these simple contact line models, the damping of the system is assumed to have linear origins. Nevertheless, these assumptions altogether, by overlooking the actual nonlinear contact line dynamics, have led to a considerable underestimation of the actual overall dissipation in most of the small-size lab-scale experiments (Benjamin and Ursell, 1954; Henderson and Miles, 1990), for which the complexity of the region in the neighbourhood of the moving contact line, where molecular, boundary layer and macroscopic scales are intrinsically connected, is of extreme importance.
In order to understand and quantify better, at least from a macroscopic perspective, this extra dissipation, it is necessary to look more carefully at the dynamics of the oscillating contact line and at its wetting conditions, a long-standing problem in fluid mechanics that dates back to Navier Navier (1823) (see also Davis (1974); Eggers (2005); Eral et al. (2013); Huh and Scriven (1971); Keulegan (1959); Lauga et al. (2007); Miles (1990); Ting and Perlin (1995)).

When a liquid meniscus flows over a dry solid substrate, there is a triple-phase interface (air-liquid-solid), which experiences a complex nonlinear dynamics. For instance, let us consider two scenarios of uni-directional flows: a droplet sliding down with velocity $U$ on an inclined dry plate in partial wetting conditions (see figure IV.1(a)); an expanding or contracting (at velocity $U$ ) liquid droplet (see figure IV.1(d)). Experimental observations (Dussan, 1979; Grand et al., 2005; Rio et al., 2005) have shown that the dynamic advancing, $\theta_{a}$, and receding, $\theta_{r}$, contact angles deviate from their static values depending on the velocity of displacement of the advancing or receding meniscus. Moreover, there exists a range $\theta \in\left[\theta_{r}, \theta_{a}\right]$ within which the contact line seems to remain stationary. The existence of such a static range, defined as contact angle hysteresis, plays a critical role in the nonlinear damping and dynamics of capillary-gravity waves.
Several models have been suggested to explain the relation between the dynamic contact angles, $\theta$, and the capillary number defined by the drop velocity, $U$, i.e. $C a=\mu U / \gamma$, with $\gamma$ and $\mu$, the air-liquid surface tension and dynamic viscosity, respectively. One such model for these uni-directional flows has been established by Gennes (1985), who extended to partial wetting conditions the Tanner law, originally derived in total wetting. This law connects the dynamic contact angles $\theta$ and the static (equilibrium) angle $\theta_{s}$ with the capillary number $C a$. More precisely, the force required to draw the liquid is represented by $\gamma\left(\cos \theta_{s}-\cos \theta\right)$, while the viscous force is proportional to $\mu U \theta^{-1} \log \left(l_{\text {macro }} / l_{\text {micro }}\right)$. Here, $l_{\text {macro }}$ denotes a macroscopic
characteristic length and $l_{\text {micro }}$ is a microscopic cut-off length, which is necessary to prevent stress singularity, as pointed out by Snoeijer and Andreotti (2013). For small values of static and dynamic contact angles, the equation $\theta\left(\theta^{2}-\theta_{s}^{2}\right)= \pm 6 C a \log \left(l_{\text {macro }} / l_{\text {micro }}\right)$ holds true, with the $\pm$ signs that distinguish between the advancing and receding motion of the contact line.
Cox (1986) and Voinov (1976) arrived at a similar but different relation by solving lubrication equations for slightly curved air-liquid interfaces. Like the approach of de Gennes, their solution is truncated at both molecular and macroscopic scales, giving the law $\theta^{3}-\theta_{s}^{3}=$ $\pm 9 C a \log \left(l_{\text {macro }} / l_{\text {micro }}\right)$.
In the study by Grand et al. (2005), it was noted that while certain models accurately depict the contact line dynamics observed in experiments, they fail to account for wetting hysteresis. As a result, when comparing these models to experimental data, the static contact angle $\theta_{s}$ is substituted with the limit static angle $\theta_{a}$ for the advancing branch and $\theta_{r}$ for the receding branch. Figure IV.1(e,f) displays the resulting $\theta(C a)$ dependence for the de Gennes (e) and Cox-Voinov models (f), both of which incorporate a static hysteresis range $\Delta$.
For oscillatory flows, the contact angle laws proposed in the literature share the same qualitative features as those derived for uni-directional flows, such as the de Gennes or the Cox-Voinov ones, but are described by quantitatively different relations. As this thesis focuses on oscillatory flows, the bottom part of figure IV. 1 gives a brief overview of famous contact line models which have been used in this context. For instance, the contact angle dynamics observed for vertical vibrating sessile drops (figure $\operatorname{IV} .1(\mathrm{~g})$ ) or during the relaxation of sloshing waves (figure IV.1(j)) are seen to obey the nonlinear (cubic) Dussan model, $\left(\theta-\theta_{s}\right)^{3} \sim C a$ (see figure IV.1(h,i)), and are sometimes well approximated by a modified Hocking's law (supplemented with hysteresis, see figure IV. $1(1, m)$ ).
Furthermore, the rich dynamics of an oscillatory meniscus shows some interesting features that the next two Chapters of this thesis aim at reproducing and predicting. Those features are described in detail in figure IV.2. In a study conducted by Noblin et al. (2004), they investigated the behaviour of a water droplet on a solid surface with a finite contact angle hysteresis under vertical vibration (see figure IV.1(g)). The results showed two distinct types of oscillations. At low forcing amplitude, the contact line remains pinned (see figure IV.2(a)) and the drop displays eigenmodes at different resonance frequencies. At higher amplitudes, the contact line starts to move, remaining circular but with a radius oscillating at the excitation frequency. This transition between the two regimes occurs when the variations of the contact angle exceed the hysteresis range. They also observed a decrease in the resonance frequencies at larger vibration amplitudes for which the contact line is mobile. These features were attributed to the hysteresis acting as solid-like friction on the oscillations, leading to a stick-slip regime at intermediate amplitude (Dollet et al., 2020).
In his seminal works, Cocciaro et al. $(1993,1991)$ thoroughly characterized the contact angle dynamics during the natural (free-of-forcing) relaxation phase of the fundamental asymmetric sloshing mode in a small circular cylindrical container. Two different damping regimes were observed, corresponding to higher and smaller wave amplitude oscillations (see figure IV.2(b,c)). First, the contact line slides over the solid substrate experiencing progressive stick-slip transitions under the effect of the dynamic wall friction. In this phase, the damping


Figure IV. 2 - (a) Transition between stick and stick-slip motions in a water sessile drop deposited on a vertically vibrating substrate characterized by a finite contact angle hysteresis ( $\Delta \approx 10-15$ degrees) (Noblin et al., 2004). Lower curves are contact angle variations versus time, the dashed line represents $\theta_{s}$. Higher curves are the contact line position around the starting position before vibrations. The six curves for different non-dimensional acceleration amplitudes $f / g$ are joined together in the same plot for comparison. The driving frequency is $1 / T=9 \mathrm{~Hz}$. (b) Experimental contact angle dependence on the capillary number as measured by Cocciaro et al. (1993) during the natural relaxation dynamics of water oscillations in a cylindrical container initially perturbed using a loudspeaker, so as to induce the liquid motion. (c) Associated damping rate versus the amplitude of the angle measured at the container axis. The vertical dashed line indicates the value for which the contact line irreversibly pins.
increases considerably as the wave amplitude decreases, until it reaches a maximum value, after which it starts to decrease, and the small amplitude regime is established. A finite time of arrest for the contact line is found: the interface irreversibly pins and the following pure bulk motion is seen to decay exponentially owing to the linear viscous dissipation acting in the fluid bulk and in the Stokes boundary layers. The natural oscillations frequency initially matches the value associated with a free-end eigenmode, it increases during the decay, and it eventually tends to the value associated with a pinned-end eigenmode.

As an alternative to computationally expensive fully nonlinear direct numerical simulations (see (Amberg, 2022; Ludwicki et al., 2022) among others), different theoretical frameworks, attempting to rationalize the nonlinear dependence of the damping rate on the oscillation amplitude, have been recently proposed (Viola et al., 2018; Viola and Gallaire, 2018). These
works are based on an asymptotic formulation of the full hydrodynamic problem, which is tackled in the spirit of the weakly nonlinear and multiple timescale approach Stuart (1960), under precise assumptions and range of validity. The asymptotic analysis is found to be able to quantitatively predict the nonlinear trend of the damping in the higher amplitudes regime and the existence of a finite-time of arrest for the contact line, in agreement with experiments (Cocciaro et al., 1993; Dollet et al., 2020). However, it fails in capturing the transient stickslip motion and, most importantly, the transition to the small amplitude regime, when the interface pins but the fluid bulk keeps oscillating with a smaller amplitude motion following a purely pinned dynamics.

The purpose of Chapter 9 is to provide a different theoretical approach, which overcomes the limitations of these asymptotic analyses, thus successfully solving the overall flow dynamics and enabling us to extract and highlight realistic flow features, yet keeping a low computational cost. To this end, we consider viscous liquid oscillations in an idealized two-dimensional container and subjected to an experimentally inspired nonlinear contact line model, to which the contact line is forced to obey. Using a piecewise time splitting of the nonlinear contact line law, we formalize a mathematical model based on successive projections between different sets of linear eigenmodes pertaining to each linear split-piece composing the contact line law.
This procedure allows us to formally account for all the nonlinear features of small-amplitude capillary-gravity waves induced by a nonlinear contact line law acting at the lateral wall of a rectangular basin and, in particular, to correctly solve the transition from a contact line stick-slip (or nearly stick-slip) regime to the pinned (or nearly pinned) one. Indeed, each projection, corresponding to each stick-slip transition, eventually induces a rapid loss of total energy in the liquid motion and contributes to its nonlinear damping.

The projection method formalized in Chapter 9 for an idealized two-dimensional flow configuration with triple contact points (rather than lines), is extended in Chapter 10 to describe the more realistic situation of liquid oscillations in a U-shaped tube, as experimentally investigated by Dollet et al. (2020). A thorough quantitative comparison with these experiments shows that the projection method correctly captures the final stick-slip-to-stick transition, as well as the secondary fluid bulk motion following the arrest of the contact line, overlooked by previous asymptotic analyses.

## 9 Relaxation of capillary-gravity waves due to contact line nonlinearity: a projection method

Remark: this Chapter is largely inspired by the publication of the same name.

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The comprehension of the role of wetting properties in the damping of liquid oscillations in confined basins is a long-standing problem in the hydrodynamics field and for which renewed interest has emerged in recent years. A series of careful lab-scale experiments have revealed that the damping of liquid natural small oscillations varies nonlinearly with the oscillation amplitude, in contrast with previous theoretical predictions (Case and Parkinson, 1957; Lamb, 1993; Miles, 1967; Ursell, 1952), which prescribe a constant and unique value for the damping rate, thus indicating a dependence on the contact line behaviour and hence on the solid substrate material. This effect has been tentatively attributed to a source of dissipation localized in the proximity of the air-liquid-solid triple line, which, during the dynamics, may exhibit a complex hysteretic behaviour under the effect of solid-like wall friction. In this Chapter, assuming that the contact line behaves according to experimentallyinspired phenomenological laws, we formalize a mathematical method based on successive linear eigenmode projections for solving numerically the nonlinear fluid motion in the limit of small oscillation amplitudes. We show that each projection eventually induces a rapid loss of total energy in the liquid motion and contributes to its nonlinear damping. Particularly, this approach captures the transition from a contact line stick-slip (or nearly stick-slip) motion to a pinned (or nearly pinned) configuration, as well as the secondary fluid bulk motion following the arrest of the contact line, overlooked by previous asymptotic analyses.

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Figure 9.1 - Sketch of a two-dimensional rectangular container of width $2 l$ and filled to a depth $h$ (e.g. $h / l=3$, nearly deep water regime, without loss of generality) with a liquid of density $\rho$ and dynamic viscosity $\mu$. The air-liquid surface tension is $\gamma$. The origin of the Cartesian coordinate system is fixed at the center of the free liquid surface at rest, while the bottom is placed at $z=-h . \theta$ is the contact angle. The dashed-dotted line is the geometrical axis of symmetry. $\Omega$ denotes the bulk domain, $\partial \Omega$ its solid boundaries and $\eta$ denotes here the moving interface.

The Chapter is organized as follows. The flow configuration analyzed in this work and the physical model governing the problem are introduced in $\S 9.1$. For completeness, the key points of the weakly nonlinear formulation applied to the present case are synthetically re-proposed and commented in $\S 9.2$. The novel projection method is introduced and carefully described in $\$ 9.3$, where quantitative and qualitative comparisons with the weakly nonlinear model and previous experiments are made. Lastly, the extension of the method to more sophisticated contact line dynamics is discussed in $\$ 9.4$. Final conclusions and comments are outlined in $\$ 9.5$.

### 9.1 Flow configuration and governing equations

The viscous fluid motion within the two-dimensional vessel is governed by the incompressible Navier-Stokes equations,

$$
\begin{equation*}
\nabla \cdot \mathbf{u}=0, \quad \frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u} \cdot \nabla) \mathbf{u}+\nabla p-\frac{1}{R e} \Delta \mathbf{u}=-1 \hat{\mathbf{e}}_{z} \tag{9.1}
\end{equation*}
$$

which are made nondimensional by using the container's characteristic length $l$ and the velocity $\sqrt{g l}$ (see figure. 9.1). Consequently, the Reynolds number is defined as $R e=\rho g^{1 / 2} l^{3 / 2} / \mu$ and the term $-1 \hat{\mathbf{e}}_{z}$ is the nondimensional gravity acceleration. At the free surface, $z=\eta$
kinematic and dynamic boundary conditions hold

$$
\begin{gather*}
\frac{D(\eta-z)}{D t}=\frac{\partial \eta}{\partial t}+u \frac{\partial \eta}{\partial x}-w=0  \tag{9.2a}\\
{\left[-p \mathbf{I}+\frac{1}{R e}\left(\nabla \mathbf{u}+\nabla^{T} \mathbf{u}\right)-\frac{1}{B o} \kappa(\eta) \mathbf{I}\right] \cdot \mathbf{n}=\mathbf{0}} \tag{9.2b}
\end{gather*}
$$

where $D / D t$ is the material derivative, $\mathbf{u}=\{u, w\}^{T}$ is the velocity vector, $\kappa$ is the free surface curvature, $\kappa(\eta)=\partial_{x x} \eta\left(1+\partial_{x} \eta^{2}\right)^{-3 / 2}$, and $\mathbf{n}=\left(1+\partial_{x} \eta^{2}\right)^{-1 / 2}\left\{-\partial_{x} \eta, 1\right\}^{T}$ is unit vector normal to the interface. The Bond number is defined as $B o=\rho g l^{2} / \gamma$, with $\gamma$ designating the air-liquid surface tension. At the bottom wall the no-slip condition applies

$$
\begin{equation*}
\mathbf{u}=\{u, w\}^{T}=\mathbf{0} \quad \text { at } z=-\frac{h}{l} \tag{9.3}
\end{equation*}
$$

At the lateral walls a slip length model is adopted, thus assuming that the fluid speed relative to the solid wall is proportional to the viscous stress (Lauga et al., 2007; Navier, 1823) and that, together with no-penetration condition, provides the boundary conditions

$$
\begin{equation*}
u=0, \quad w+l_{s} \frac{\partial w}{\partial x}=0 \quad \text { at } x= \pm 1 \tag{9.4}
\end{equation*}
$$

Such a condition is indeed needed in order to regularize the stress singularity at the moving contact line (Davis, 1974; Huh and Scriven, 1971).

Lastly, at the contact line, $z=\eta$ and $x= \pm 1$, we include a phenomenological contact line law, which describes the nonlinear contact angle dynamic as a function of the contact line speed,

$$
\begin{equation*}
\frac{\partial \eta}{\partial x}= \pm \cot \theta, \quad \theta-\theta_{s}=\mathscr{F}\left(\frac{\partial \eta}{\partial t}\right) \tag{9.5}
\end{equation*}
$$

Relevant nonlinear laws, $\mathscr{F}\left(\frac{\partial \eta}{\partial t}\right)$, will be introduced in the next section $\S 9.2$.
It was hypothesized that a phenomenological macroscopic slip length appearing in equation (9.4) is not constant in space, but rather a function of the position along the lateral wall and that it vanishes at a certain distance away from the contact line, where the flow obeys the no-slip condition (Miles, 1990; Ting and Perlin, 1995). However, in order to avoid the resulting dynamical coupling of the slip length $l_{s}$ and the contact line motion, and since the dissipation at the wall in the contact line region (accounted for by the contact line model) dominates over that taking place at the lateral walls (Hocking, 1987), we assume here a slip length $l_{s} \gg 1$, constant in time and space along the lateral wall, so that equation (9.4) for $w$ reduces to a stress-free wall boundary condition. This simplistic assumption, which neglects the viscous boundary layer at the lateral walls, will result in an underestimation of the overall damping rate, but it will significantly simplify the mathematical treatment of the lateral boundaries.
Finally, it is important to note that the simultaneous application of stress-free wall conditions and a pinned contact line does not result in any inconsistency (Benjamin and Scott, 1979; Graham-Eagle, 1983), in marked contrast to the combination of no-slip wall conditions and a

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free-edge contact line.
Although a perfect slip condition is assumed at the lateral wall and albeit the overall dissipation is mainly ascribed to the nonlinear contact line dynamics, the role of the small linear viscous dissipation occurring in the fluid bulk and solid bottom is of crucial importance in avoiding the accumulation of energy in high frequencies in the forthcoming projection method, thus precluding the use of a simpler potential model.
Further details are given in Appendix 9.6.3.

### 9.2 Asymptotic formulation

The system of governing equations and boundary conditions introduced in $\S 9.1$ is nonlinear owing to the nonlinear contact angle law. Henceforth, we consider the following two experimentally based contact line laws from the literature (Cocciaro et al., 1993; Dussan, 1979; Jiang et al., 2004) (see figure 9.2),


Figure 9.2 - (a) Hocking's linear law (Hocking, 1987) plus hysteresis range (Dussan, 1979; Young and Davis, 1987). (b) Jiang et al. cubic model, obtained in the framework of unidirectional flow over a flat plate at low Reynolds number (Jiang et al., 2004).

$$
\theta-\theta_{s}= \begin{cases}\alpha C a \frac{\partial \eta}{\partial t}+\frac{\Delta}{2} \operatorname{sgn}\left(\frac{\partial \eta}{\partial t}\right) & \text { Hocking+hyst. }  \tag{9.6}\\ \beta C a^{1 / 3}\left(\frac{\partial \eta}{\partial t}\right)^{1 / 3} & \text { Jiang et al. }\end{cases}
$$

with the capillary number defined as $C a=\mu \sqrt{g l} / \gamma$ and $\partial \eta / \partial t$ is the non-dimensional contact line speed.
To avoid misleading interpretations, we specify that in the following the terminology "Hocking's law plus hysteresis" (or Hocking+hyst.) refers to a combination of the original Hocking's linear law (without static hysteresis) supplemented with a static hysteresis range, $\Delta$, as in Dus-
san (1979).
Given the contact line nonlinearity, one possible approach to tackle the problem is asymptotic theory. In this section, we briefly repropose a weakly nonlinear (WNL) formulation, based on a multiple timescale expansion and valid in the limit of small contact line parameters, which aims to incorporate as much as possible, the features of the contact line dynamics (Viola et al., 2018; Viola and Gallaire, 2018). We draw the reader's attention to the fact that the nomenclature adopted hereinafter relies on that used in the authors' earlier works (Viola et al., 2018; Viola and Gallaire, 2018), the limitations of which motivated the present study.

### 9.2.1 Presentation

Let us introduce the following asymptotic expansion for the flow quantities,

$$
\begin{equation*}
\mathbf{q}=\{\mathbf{u}, p, \eta\}^{T}=\mathbf{q}_{0}+\epsilon \mathbf{q}_{1}+\epsilon^{2} \mathbf{q}_{2}+\mathrm{O}\left(\epsilon^{3}\right) \tag{9.7a}
\end{equation*}
$$

(the same expansion holds for the contact angle $\theta$ ). Under the assumption of small viscous effects ( $R e \gg 1$ ) and introducing the following scalings for the slow time, damping coefficient and contact line parameters,

$$
\begin{equation*}
\underbrace{T=\epsilon t}_{\text {slow time scale }}, \underbrace{\sigma_{V}=\epsilon \hat{\sigma}_{V}}_{\text {viscous damping }}, \underbrace{\Delta=\epsilon^{2} \hat{\Delta}}_{\text {nonlinear range }}, \underbrace{\alpha C a=\epsilon \hat{\alpha}}_{\text {Hocking's linear variation }}, \underbrace{\beta C a^{1 / 3}=\epsilon^{5 / 3} \hat{\beta}}_{\text {Jiang } \text { et al. cubic law }} \tag{9.7b}
\end{equation*}
$$

with $\epsilon \ll 1$ small parameter, the contact line nonlinearities are retained only as a weakly nonlinear correction (order $\epsilon^{2}$ ) of an $\epsilon$-order dynamics representing an oscillatory small perturbation of the static flow configuration. Substituting the expansions above in the governing equations and boundary conditions, a series of systems are obtained at the various order in $\epsilon$. We note that the small parameter $\epsilon$ is not explicitly defined here, but rather it only serves to separate different order of magnitudes of the problem and it will be eliminated afterwards by recasting all quantities in terms of their corresponding physical values (Bongarzone et al., 2021a; Meliga et al., 2009b; Viola et al., 2018; Viola and Gallaire, 2018).

At order $\epsilon^{0}$, the nonlinear problem associated with the static shape of the interface is obtained. Although the procedure in principle applies to any static contact angle $\theta_{s}$, we consider the simplest case $\theta_{0}=\theta_{s}=\pi / 2$, so that the fluid at rest ( $\mathbf{u}_{0}=\mathbf{0}, p_{0}=-z$ ) has a flat static interface $\eta_{0}=0$. Note that the static contact angle $\theta_{s}$ represents the macroscopic contact angle measured with the fluid at rest before imposing any initial perturbation. At order $\epsilon$, the linear eigenvalue problem for viscous capillary-gravity waves is retrieved. The contact line boundary condition reads $\partial_{x} \eta_{1}=0\left(\theta_{1}=0\right)$, thus retrieving the classic free-end edge condition. Assuming a single mode expansion, the marginally stable first-order dynamics is described by equation (9.8),

$$
\begin{equation*}
\mathbf{q}_{1}(x, z, t)=A_{1}(T) \hat{\mathbf{q}}_{1}(x, z) e^{\mathrm{i} \omega t}+\text { c.c. } \tag{9.8}
\end{equation*}
$$

where $\hat{\mathbf{q}}_{1}(x, z)$ a viscous free-end edge eigenmode (set by the initial condition) and $\omega$ its corresponding eigenfrequency. We note that in a two-dimensional framework, the eigenmodes

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are represented by symmetric or anti-symmetric waves with respect to the container axis placed at $x=0$. In the following, we arbitrarily consider the fundamental anti-symmetric (first or lowest oscillation frequency) mode, but the very same single mode analysis applies to any eigenfunction. We also note that the viscous damping coefficient of the eigenmode $\sigma_{V}$ has been assumed of order $\epsilon$ and therefore only enters at the next order. This assumption only serves to apply the formalism of the multiple scale analysis, which classically applies to marginally stable system (Nayfeh, 2008a). As the Reynolds is typically high enough, the damping coefficient results in a slow damping process over fast wave oscillations and hence it can be directly englobed in the final amplitude equation. A possible way to account for such a first-order departure from marginal stability is discussed in Viola and Gallaire (2018). Another option, which leads to the same final results for this specific problem, but that is formally more general, is represented by the shift operator technique proposed in Meliga et al. (2009b). Here we opt for the second option. See Meliga et al. (2009b) for a thorough discussion in this regard. $A(T)$ is the mode amplitude, unknown at this stage, and allowed to vary on the slow time scale $T=\epsilon t$. In the spirit of multiple scale expansion, the resonating effect of secular terms on the asymptotic solution is avoided at order $\epsilon^{2}$ by imposing a compatibility condition, which prescribes an amplitude equation for $A$, slow amplitude modulation of the first order motion. The contact line dissipation enters at this order and, together with the viscous dissipation in the bulk, is incorporated in the structure of the amplitude equations (9.9) (according to the two contact line models introduced in equation (9.6)),

$$
\frac{d A_{1}}{d T}+\hat{\sigma}_{V} A_{1}= \begin{cases}-\hat{\zeta}_{H} A_{1}-\hat{\chi}_{H} \frac{A_{1}}{\left|A_{1}\right|} & \text { Hocking+hyst. }  \tag{9.9}\\ -\hat{\chi}_{J} \frac{A_{1}}{\mid A_{1} 2^{2 / 3}} & \text { Jiang et al. }\end{cases}
$$

with $\sigma_{V}=\epsilon \hat{\sigma}_{V}$ the linear viscous damping coefficient, computed numerically as a solution of the $\epsilon$-order eigenvalue problem. The r.h.s. of equation 9.9 clearly highlights the contact line terms contributing to the overall dissipation, namely the ones proportional to $\hat{\zeta}_{H}$ and $\hat{\chi}_{J}$, which represent the dissipation induced by the linear (Hocking) or cubic (Jiang et al.) variation of the angle with the contact line speed, while the second term in the Hocking's model plus hysteresis range, $\hat{\chi}_{H}$, reproduces the dissipation associated with the contact angle hysteresis. Coefficients $\zeta_{H}=\epsilon \hat{\zeta}_{H}, \chi_{H}=\epsilon^{2} \hat{\chi}_{H}$ read

$$
\begin{equation*}
\zeta_{H}=\frac{\lambda^{2} \sin \theta_{s} \alpha C a \kappa}{B o}, \quad \chi_{H}=\frac{\lambda^{2} \sin \theta_{s} \Delta \kappa}{|\lambda| \pi B o\left|\hat{\eta}_{1}\right|_{x=1}} . \tag{9.10}
\end{equation*}
$$

with $\kappa$ defined in Viola and Gallaire (2018) and with the complex eigenvalue $\lambda \approx \mathrm{i} \omega$ as $\sigma_{V}$ is of order $\epsilon$. The expression of $\chi_{J}=\epsilon^{5 / 3} \hat{\chi}_{J}$ is given in the Appendix B of Viola and Gallaire (2018). For a thorough description of the weakly nonlinear formulation, the derivation of the amplitude equation coefficients and the numerical scheme used in this work (based on a Chebyshev collocation method solved in Matlab) see Viola et al. (2018); Viola and Gallaire (2018). Henceforward, we will focus on the simpler Hocking's model plus hysteresis. The more sophisticated extension of the projection approach proposed in $\$ 9.3$ to any nonlinear contact
line function, including the Jiang et al. cubic model, will be discussed in §9.4.

### 9.2.2 Time of arrest and nonlinear damping

The amplitude $A$, obtained for the Hocking's law plus hysteresis, is first calculated solving (9.9) (see Viola and Gallaire (2018)) yielding (after eliminating $\epsilon$ by recasting each variable in terms of the corresponding total physical quantity, e.g. the physical time $t=T / \epsilon$, the total amplitude $A=\epsilon A_{1}$ and the initial condition $A_{0}=\epsilon a_{0}$ )

$$
\begin{equation*}
|A(t)|=\left[A_{0}+\frac{\chi_{H}}{\left(\zeta_{H}+\sigma_{V}\right)}\right] e^{-\left(\zeta_{H}+\sigma_{V}\right) t}-\frac{\chi_{H}}{\left(\zeta_{H}+\sigma_{V}\right)}, \tag{9.11}
\end{equation*}
$$

and then substituted in equation (9.8), which describes the time evolution of the first order dynamics, as displayed in figure 9.3 in terms of contact line elevation.
The associated damping rate can be then obtained as the logarithmic decrement of the


Figure 9.3 - Contact line elevation (black solid line), $e(t)=\left.\eta\right|_{x=1}$, modulated by the slow time amplitude (red solid line) versus time and corresponding to the dominant (first) free-end edge anti-symmetric natural mode. We assumed pure water in a container of width $l=5 \mathrm{~cm}$ for which $B o=336, R e=30717$ and $C a=0.011$. The chosen contact line parameters for the Hocking law are $\theta_{s}=\pi / 2, \alpha=77 \mathrm{rad}$ with $\Delta=20^{\circ}$. The initial contact line elevation and speed are set to 0 and 0.1 , respectively. The initial absolute value and phase of the complex amplitude $A, A_{0}$ and $\Phi_{0}$, respectively, follow. $t^{*}$ denotes the time of arrest (vertical red dashed line) prescribed by the WNL model.
amplitude $|A(t)|$ in time. However, we introduce here a different measure based on the gravitational potential energy density of the system, $E_{p g}$, which is in general a more suitable quantity to describe the overall system dynamics (and it will be used in the next section $\S 9.3$ ).

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Since $E_{p g}=\int_{z=\eta_{0}} \eta^{2} \mathrm{~d} x \sim|A|^{2}$, the damping rate is expressed as

$$
\begin{equation*}
D R_{E_{p g}}=-\frac{d}{d t}\left[\log \left(\frac{|A(t)|^{2}}{A_{0}^{2}}\right)\right]=-\frac{2}{|A(t)|} \frac{d|A(t)|}{d t}=2\left(\sigma_{V}+\zeta_{H_{r}}\right)+2 \frac{\chi_{H_{r}}}{|A(t)|} \tag{9.12}
\end{equation*}
$$

(the subscript ${ }_{r}$ stands for real part) and is presented in figure 9.4. The value of the nondi-


Figure 9.4 - Non-dimensional damping rate versus non-dimensional time, obtained for the same parameter setting of figure 9.3. The vertical lines indicate the time of arrest, $t^{*}$, for the contact line and, in this context, the total motion.
mensional oscillation frequency, $\omega$, and the viscous damping coefficient, $\sigma_{V}$, corresponding to the dominant (first) antisymmetric wave with free-end edge contact line condition, plotted in figure 9.3, are respectively 1.258 and $1.16 \times 10^{-4}$, while the damping coefficient associated with the Hocking's linear law is $\zeta_{H_{r}}=4 \times 10^{-3}$, hence $\left(\sigma_{V}+\zeta_{H_{r}}\right) \approx \zeta_{H_{r}}$. The value of $\chi_{H_{r}}$ is $8.3 \times 10^{-4}$.

### 9.2.3 Comments

The weakly nonlinear analysis from Viola et al. (2018); Viola and Gallaire (2018), and reproposed in this section, partially accounts for dissipative effects acting at the sliding contact line using asymptotic theory and reveals the strong influence of the nonlinear contact line dynamics on the damping of capillary-gravity waves in confined basins. Given that the linear viscous dissipation taking place in the fluid bulk and at the solid bottom, $\sigma_{V}$, is negligible compared to $\zeta_{H_{r}}$, equation 9.12 suggests that there exist two main dissipation sources, a first one due to the linear contact angle variation with the contact line speed, $\zeta_{H_{r}}$, and another due to hysteresis occurring at a zero speed, $\chi_{H_{r}} /|A(t)|$. The latter contribution depends on the sign of the contact line speed only and, therefore, it may be interpreted as a Coulomb-like friction force, responsible for the predominantly linear decay and thus for the arrest of the motion at a finite time. In fact, the damping rate is found to depend on the initial condition and on the wave amplitude variation in time (Cocciaro et al., 1993; Dollet et al., 2020; Keulegan, 1959; Viola et al., 2018; Viola and Gallaire, 2018). More precisely, as the amplitude decreases, the damping rate increases and eventually diverges in a finite time leading to the finite-time arrest of the contact line (see figure 9.4 ) for $\left|A\left(t=t^{*}\right)\right|=0$. Nevertheless, the weakly nonlinear theory
fails in capturing the transition to the smaller amplitude regime observed in Cocciaro et al. (1993); Dollet et al. (2020), when the contact line is pinned and the fluid bulk keeps oscillating with pinned-end edge dynamics. This limitation is intrinsic to the asymptotic formulation, which only predicts the evolution of the amplitude of a leading order free-end edge mode. Consequently, it prescribes the arrest of the total fluid motion once the contact line is pinned at $t=t^{*}$, thus overlooking the remaining bulk oscillations submitted to a finite, and small, damping rate. Furthermore, even during the initial higher amplitude sloshing motion, the contact line exhibits periodic stick-slip transitions, which are not captured with a weakly nonlinear approach. In other words, the asymptotic model neglects the slowing increasing fraction of the time period during which the contact angle changes while the contact line is at rest (pinned fraction of liquid motion).

### 9.3 Projection Method

In $\S 9.2$, we brought to light the main limitations of the weakly nonlinear analysis, which, by construction, introduces the contact line dissipation only at higher orders and is unable to capture the slip-sticking regime observed in the experiments. In this section, we propose and formalize a different mathematical approach based on a sequential eigenfunction projection aiming at overcoming these limitations and at providing a more complete and realistic characterization of the liquid motion. To this end, we first introduce here the projection method in its simplest formulation considering as contact line boundary condition the Hocking's model plus hysteresis of figure 9.2-(a), with the angle varying linearly for a non-zero contact line speed. Subsequently, the projection approach is compared with the asymptotic model. The results of the analysis are then discussed in the light of previous laboratory experiments (Cocciaro et al., 1993) (in a cylindrical container) before generalizing the method to solve the nonlinear Jiang et al. law in §9.4.

### 9.3.1 Application to the Hocking's law plus hysteresis

## Formalism

In contradistinction with the weakly nonlinear approach where the nonlinearities enter at second order, the contact line model is accounted for at first order, thus solving a nonlinear problem. In practice, the total flow field is expanded as

$$
\begin{equation*}
\mathbf{q}_{t o t}=\mathbf{q}_{0}+\epsilon \mathbf{q}+\mathrm{O}\left(\epsilon^{2}\right) \tag{9.13}
\end{equation*}
$$

with the rest state $\mathbf{q}_{0}=\left\{\mathbf{u}_{0}=\mathbf{0}, p_{0}=-z, \eta_{0}=0\right\}^{T}, \theta_{0}=\theta_{s}=\pi / 2$, and with the only assumption of small hysteresis of order $\epsilon$, i.e. $\Delta=\epsilon \hat{\Delta}$. Note that the viscous damping coefficient is not required to be small. In this limit of small perturbation, the only nonlinearity appearing in the system is attributed to the contact line dynamics through the geometrical relation (9.40e). When the contact line motion is schematized using the Hocking's law plus hysteresis range, we

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can identify two well-distinct phases of the dynamics, one in which the angle varies linearly with a slope $\alpha$ as a function of the contact line speed (Hocking's linear law) and one in which the contact line is pinned at a certain elevation and with zero velocity (static hysteresis). We denote these two phases as free and pinned phases, respectively. Hence, we can write two different boundary conditions associated with the two phases (e.g. at the contact line, $z=0$, $x=1$ ),

$$
\begin{gather*}
\underbrace{\left.\frac{\partial \eta_{f}}{\partial x}\right|_{x=1}+\left.\alpha C a \frac{\partial \eta_{f}}{\partial t}\right|_{x=1}}_{\text {free-phase l.h.s. }}=\underbrace{-\theta^{ \pm}}_{\text {free-phase r.h.s. }},  \tag{9.14}\\
\left.\frac{\partial \eta_{p}}{\partial t}\right|_{x=1}=0 \longrightarrow \underbrace{\left.\eta_{p}\right|_{x=1}}_{\text {pinned-phase l.h.s. }}=\underbrace{\text { const. }}_{\text {pinned-phase r.h.s. }} \tag{9.15}
\end{gather*}
$$

where subscripts ${ }_{f}$ and ${ }_{p}$ stand for free-phase and pinned-phase respectively and $\theta^{ \pm}= \pm \hat{\Delta} / 2$ for a symmetric hysteresis range centered around $\theta_{s}$. In other words, the free-phase has a non-homogenous Robin boundary condition at the contact line, while the pinned-phase has a non-homogeneous Dirichlet condition. In both phases, the solution is thus rewritten as the sum of a homogeneous solution (generalized eigenvalue problem) and a static ( $\partial_{t} \eta=0, \mathbf{u}=\mathbf{0}$ ) particular solution, which must satisfy the following static equation (linearized meniscus equation) and boundary conditions,

$$
\begin{gather*}
\eta_{f_{s}}-\frac{1}{B o} \frac{\partial^{2} \eta_{f_{s}}}{\partial x^{2}}=p_{f_{s}}=\text { const. with }\left.\frac{\partial \eta_{f_{s}}}{\partial x}\right|_{x=1}=-1,  \tag{9.16}\\
\eta_{p_{s}}=\frac{\eta_{f_{s}}}{F_{0}} \text { with } F_{0}=\left.\eta_{f_{s}}\right|_{x=1}, \tag{9.17}
\end{gather*}
$$

where ${ }_{s}$ denotes the static particular solution, $\partial_{x x} \eta$ represents the $\epsilon$-order curvature operator, linearized around $\eta_{0}=0$ for the present case with $\theta_{s}=\pi / 2$ (owing to the expansion (9.13)) and the minus sign in equations (9.14) and (9.16) comes from the linearization of $\cot \theta$ in equation (9.40e) at $x=1$. The constant on the r.h.s. of equation (9.16) (different from that of equation (9.15)), which is trivially zero for an anti-symmetric wave dynamics, is instead computed by imposing the mass conservation constraint, $\int_{z=\eta_{0}} \eta_{f_{s}} \mathrm{~d} x=0$, when the symmetric dynamics is considered. For the convenience of notation, instead of imposing $-\theta^{ \pm}$in the r.h.s. of the boundary condition to equation (9.16) at $x=1$, we impose -1 , and we kept $\theta^{ \pm}$ explicit in front of $\mathbf{q}_{f_{s}}$, state vector of the particular solution associated with the free-phase. $\mathbf{q}_{p_{s}}$ is its analogous in the pinned-phase. The two particular solutions, $\mathbf{q}_{f_{s}}$ and $\mathbf{q}_{p_{s}}$, up to their associated constant, $\theta^{ \pm}$and $e_{f p}$, are displayed in figure 9.5)-(a) and (b). The solution in the two phases is thus expressed as follows:

$$
\begin{equation*}
\mathbf{q}_{f}=\theta^{ \pm} \mathbf{q}_{f_{s}}+\left(\sum_{n=1}^{N_{f}} A_{f_{n}} \hat{\mathbf{q}}_{f_{n}} e^{\lambda_{f_{n}}\left(t-T_{f}\right)}+c . c .\right), \tag{9.18a}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{q}_{p}=e_{f p} \mathbf{q}_{p_{s}}+\left(\sum_{m=1}^{N_{p}} B_{p_{m}} \hat{\mathbf{q}}_{p_{m}} e^{\lambda_{p_{m}}\left(t-T_{p}\right)}+c . c .\right) \tag{9.18b}
\end{equation*}
$$

where $N_{f}$ and $N_{p}$ are the truncation numbers of the series representing the homogeneous solutions, $\hat{\mathbf{q}}_{f_{n}}, \hat{\mathbf{q}}_{p_{m}}$ and $\lambda_{f_{n}}, \lambda_{p_{m}}$ are the eigenmodes and eigenvalues obtained solving numerically the corresponding generalized eigenvalue problem (see figure 9.5-(c)-(j)), $A_{f_{n}}$ and $B_{p_{m}}$ are the complex amplitude coefficients associated with each mode. Without loss of generality, we consider $N_{f}=N_{p}=N$ in the following. Importantly, during this pinned-phase, the non-zero static solution (see equation (9.17)) allows the interface to transiently oscillate with a contact line elevation fixed to the value assumed at the transition from the previous free-phase to the next pinned-phase, $e_{f p}$. Figure 9.5 gathers all the ingredients needed to formalize the projection scheme and we can now proceed in the description of its practical application to temporal flow evolution. A convergence analysis for a test case of eigenvalue calculation is given in Appendix 9.6.1.

## Temporal evolution description

Let us initialize the dynamics starting from the free-phase (advancing, $\theta^{+}$or receding, $\theta^{-}$, path) assigning arbitrary values to the complex amplitudes $A_{f_{n}}=A_{f_{n}}^{0}$ for $t=T_{f}=0$. The system will then evolve following the dynamics described by equation (9.18a). By virtue of the mode normalization introduced in figure 9.5-(c)-(f), for which $\left.\eta_{f_{n}}\right|_{x=1}=1$ and $\left.\eta_{f_{s}}\right|_{x=1}=F_{0}$, the contact line elevation, $\left.\eta\right|_{x=1}=e$, and speed, $\left.\partial_{t} \eta\right|_{x=1}=\dot{e}$, in this phase read

$$
\begin{gather*}
e=\theta^{ \pm} F_{0}+\left(\sum_{n=1}^{N} A_{f_{n}} e^{\lambda_{f_{n}}\left(t-T_{f}\right)}+c . c .\right),  \tag{9.19a}\\
\dot{e}=\left(\sum_{n=1}^{N} \lambda_{f_{n}} A_{f_{n}} e^{\lambda_{f_{n}}\left(t-T_{f}\right)}+c . c .\right) . \tag{9.19b}
\end{gather*}
$$

The second equation (9.19b) provides a criterion for the transition to the pinned-phase. Indeed, letting evolving in time equation (9.19b) until $\dot{e}=0$, we can compute the time $T_{p}$ at which the transition from the free-phase to the pinned-phase occurs. At the transition, $t=T_{p}$, the physical condition is the continuity of the whole set of flow quantities, $\mathbf{q}_{f}\left(t=T_{p}\right)=\mathbf{q}_{p}\left(t=T_{p}\right)$, which translates into the following equivalence

$$
e_{f p} \mathbf{q}_{p_{s}}+\left(\sum_{m=1}^{N} B_{p_{m}} \hat{\mathbf{q}}_{p_{m}}+c . c .\right)=\theta^{ \pm} \mathbf{q}_{f_{s}}+\left(\sum_{n=1}^{N} A_{f_{n}} \hat{\mathbf{q}}_{f_{n}} e^{\lambda_{f_{n}} \Delta T_{f p}}+\text { c.c. }\right)
$$

with $\Delta T_{f p}=T_{p}-T_{f}$. Noting that the contact line elevation at the transition time is,

$$
\begin{equation*}
e_{f p}=\theta^{ \pm} F_{0}+\left(\sum_{n=1}^{N} A_{f_{n}} e^{\lambda_{f_{n}} \Delta T_{f p}}+c . c .\right) \tag{9.20}
\end{equation*}
$$

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Figure 9.5 - (a)-(b) Static particular solutions for the anti-symmetric free and pinned dynamics respectively. Note that the radial values of the two particular solutions are plotted up to their corresponding constants, $\theta^{ \pm}$and $e_{f p}$, respectively, which will be determined by the time-marching of the projection algorithm. (c)-(f) Eigensurface associated with the first four anti-symmetric modes corresponding to the free-dynamics. Only half domain, $x \in[0,1]$, is shown. The modes are normalized with the phase of the interface at the contact line and its absolute value so that the contact line elevation is $1 .(\mathrm{g})-(\mathrm{j})$ Eigensurface associated with the first four anti-symmetric modes corresponding to the pinned-dynamics. The modes are normalized with the phase of the interface at the contact line and its slope so that the slope in $x=1$ is 1 . As in figure 9.3, we assumed pure water in a container of width $l=5 \mathrm{~cm}$ for which $B o=336, R e=30717$ and $C a=0.011$. The static angle is $\theta_{s}=\pi / 2$ and the slope $\alpha$ is set to 77 rad. The linear eigenfrequency and damping associated with each mode are denoted by $\omega$ and $\sigma$, respectively. Note that, despite the lower frequencies, the damping associated with the free-modes, which englobe part of the contact line dissipation due to the linear variation of the angle (Hocking's linear law), is higher than the one for pinned modes, where only the viscous bulk dissipation is present (we recall that the viscous boundary layers at the lateral walls are neglected by imposing a stress-free condition, see also supplementary notes at the end of this Chapter).
and $\mathbf{q}_{f_{s}}=F_{0} \mathbf{q}_{p_{s}}$, equation (9.20) is rewritten as

$$
\begin{equation*}
\sum_{m=1}^{N} B_{p_{m}} \hat{\mathbf{q}}_{p_{m}}+c . c=\underbrace{\sum_{n=1}^{N} A_{f_{n}}\left(\hat{\mathbf{q}}_{f_{n}}-\mathbf{q}_{p_{s}}\right) e^{\lambda_{f_{n}} \Delta T_{f p}}+c . c . .}_{\mathbf{f}_{f p}} \tag{9.21}
\end{equation*}
$$

where the pinned static particular solution is subtracted from the r.h.s. In other words, once the particular solution is subtracted from the r.h.s., the field $\mathbf{f}_{f p}$ satisfies the boundary


Figure 9.6 - (a) Contact line elevation versus time. The blue and green colors indicate the free and pinned phases, respectively. Parameters are set as in figure 9.3, with a static hysteresis range $\Delta=20^{\circ}$. We initialize the problem setting the complex amplitude of the first antisymmetric mode in order to have an initial contact line elevation and speed equal to 0 and 0.1 , respectively. $t_{W N L}^{*}$ and $t_{P R O J}^{*}$ denote the final time of arrest for the contact line resulting from the weakly nonlinear calculation (WNL) and from the projection scheme (PROJ). (b) Different contributions to the instantaneous total energy of the system (log scale) versus time (linear scale). Only the total energy is displayed (red solid line) for the WNL model. The black dashed-dotted line represents the final exponential decaying following the pinning of the contact line. The series associated with each phase are truncated to $N=30$. The time step used to advance the algorithm in time was set to $\Delta t=0.005$. The filled circles in ( $a$ ) correspond to a sampling period of 0.05 . See also Integral Multimedia Movie 1 for a full free-surface dynamic representation. Multimedia view: https://doi.org/10.1063/5.0055898.1.
conditions of the linear problem for the pinned-phase, therefore $\mathbf{f}_{f p}$ can be represented as a linear combination of modes pertaining to the pinned-phase. The unknown amplitudes $B_{p_{m}}$ are thus obtained by applying a projection step. To this end, let us introduce the following weighted scalar product

$$
\begin{equation*}
<\mathbf{w}, \mathbf{v}>_{E}=\int_{\Omega} \overline{\mathbf{u}}_{\mathbf{w}} \mathbf{u}_{\mathbf{v}} \mathrm{d} \Omega+\int_{z=0}\left(\bar{\eta}_{\mathbf{w}} \eta_{\mathbf{v}}+\frac{1}{B o} \frac{\partial \bar{\eta}_{\mathbf{w}}}{\partial x} \frac{\partial \eta_{\mathbf{v}}}{\partial x}\right) \mathrm{dx}, \tag{9.22}
\end{equation*}
$$

where $\mathbf{v}=\left\{\mathbf{u}_{\mathbf{v}}, p_{\mathbf{v}}, \eta_{\mathbf{v}}\right\}^{T}$ and $\mathbf{w}=\left\{\mathbf{u}_{\mathbf{w}}, p_{\mathbf{w}}, \eta_{\mathbf{w}}\right\}^{T}$ are two generic vectors, the bar designates the complex conjugate, $\Omega$ denotes the fluid bulk domain ( $\mathrm{d} \Omega=\mathrm{dxdz}$ ) and the subscript ${ }_{E}$ stands for

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energy. It follows that the energy norm of a generic vector $\mathbf{v}$ is given by

$$
\begin{equation*}
<\mathbf{v}, \mathbf{v}>_{E}=\|\mathbf{v}\|_{E}^{2}=\underbrace{\int_{\Omega} \overline{\mathbf{u}}_{\mathbf{v}} \mathbf{u}_{\mathbf{v}} \mathrm{d} \Omega}_{\sim E_{k}^{\mathbf{v}}}+\underbrace{\int_{z=0} \bar{\eta}_{\mathbf{v}} \eta_{\mathbf{v}} \mathrm{dx}}_{\sim E_{p g}^{\mathbf{v}}}+\underbrace{\frac{1}{B o} \int_{z=0} \frac{\partial \bar{\eta}_{\mathbf{v}}}{\partial x} \frac{\partial \eta_{\mathbf{v}}}{\partial x} \mathrm{dx}}_{\sim E_{p s}^{\mathbf{v}}} \tag{9.23}
\end{equation*}
$$

One can recognize that the three integrals in equation (9.34) represent a measure of the total energy density, given by the sum of kinetic, gravitational potential and surface potential energy densities, respectively, stored in $\mathbf{v}$. Invoking the concept of adjoint modes, solutions of the adjoint linearized problem, it can be demonstrated that the direct modes, $\hat{\mathbf{q}}_{i}$, and the adjoint modes, $\hat{\mathbf{q}}_{j}^{\dagger}$, form a bi-orthogonal basis with respect to the scalar product (9.34). Moreover, the adjoint modes can be normalized such that $\left\langle\hat{\mathbf{q}}_{j}^{\dagger}, \hat{\mathbf{q}}_{i}>_{E}=\delta_{i j}\right.$, with $\delta_{i j}$ the Kronecker delta. Further insights about the derivation of the adjoint problem and the demonstration of the bi-orthogonality condition with respect to (9.34) are given in the supplementary notes.

Hence, we can project the known vector $\mathbf{f}_{f p}$ onto the pinned basis and the amplitude coefficients $B_{p_{m}}$ are obtained as

$$
\begin{equation*}
B_{p_{m}}=<\hat{\mathbf{q}}_{p_{m}}^{\dagger}, \mathbf{f}_{f p}>_{E} . \tag{9.24}
\end{equation*}
$$

where $\hat{\mathbf{q}}_{p_{m}}^{\dagger}$ is the adjoint mode that is bi-orthogonal to the direct mode basis, which is made of pairs of complex conjugate eigenvectors, $\hat{\mathbf{q}}_{p_{m}}$ and $\overline{\hat{\mathbf{q}}}_{p_{m}}$, associated with pairs of complex conjugate eigenvalues, $\lambda_{p_{m}}$ and $\bar{\lambda}_{p_{m}}$ (see also supplementary notes).

At this point the system is in the pinned-phase and its time evolution is described by equation (9.18b). The contact line elevation is fixed to $e_{f p}$ and contact angle varies within the prescribed hysteresis range. Given the eigenmode normalization used for $\eta_{p_{m}}$ (see figure 9.5-(g)-(j)), the contact angle variation can be expressed as,

$$
\begin{equation*}
\theta=e_{f p} / F_{0}+\left(\sum_{m=1}^{N} B_{p_{m}} e^{\lambda_{p_{m}}\left(t-T_{p}\right)}+c . c .\right) \tag{9.25}
\end{equation*}
$$

The transition to the next free-phase will occur at $t=T_{f}$, when $\theta=\theta^{\mp}$ (the sign depends on the semi-cycle considered), so that

$$
\begin{equation*}
\sum_{n=1}^{N} A_{f_{n}} \hat{\mathbf{q}}_{f_{n}}+c . c .=\underbrace{\sum_{m=1}^{N} B_{p_{m}}\left(\hat{\mathbf{q}}_{p_{m}}-\mathbf{q}_{f_{s}}\right) e^{\lambda_{p_{m}} \Delta T_{p f}}+\text { c.c.. }}_{\mathbf{f}_{p f}} \tag{9.26}
\end{equation*}
$$

Analogously to (9.24), we now project the known vector $\mathbf{f}_{p f}$ onto the free-basis, obtaining

$$
\begin{equation*}
A_{f_{n}}=<\hat{\mathbf{q}}_{f_{n}}^{\dagger}, \mathbf{f}_{p f}>_{E} \tag{9.27}
\end{equation*}
$$

Essentially, we let the system evolve in time applying a projection step at each transition. During the evolution, the system dissipates energy and eventually, after a certain number of cycles, it gets trapped in the pinned-phase, the contact line arrests and the motion decays
exponentially to its equilibrium state under the effect of the viscous bulk dissipation only.

### 9.3.2 Results and discussion

In this paragraph, the relevant results are discussed. In figure 9.6-(a) (Multimedia view) the contact line elevation is plotted versus time. First, we observe that the actual final time of arrest for the contact line is higher than the one predicted by the weakly nonlinear model. Furthermore, the projection scheme is able to describe the transient stick-slip contact line motion. Indeed, the blue and green colors correspond to the time spent by the contact line in the free-phase and pinned-phase, respectively. During the dynamics, the contact line slides over the solid substrate, is subjected to dynamic friction (linear contact angle variation in the free-phase), and transiently sticks when it reaches zero speed (hysteresis range), until the bulk inertia dominates again over the static friction and the triple line de-pins.
As shown in figure 9.6-(b), during the motion the system dissipates energy, consequently the time spent in the pinned-phase becomes larger and larger and eventually the inertia is not enough to overcome the static friction. The final pinned dynamics is therefore established at $t=t_{\text {PROJ }}^{*}$. As the lower inset in figure 9.6-(a) reveals, the contact line does not arrest at the zero equilibrium position but rather at a turning position (compatible with the prescribed hysteresis range), i.e. $e\left(t=t_{P R O J}^{*}\right) \neq 0, \theta\left(t=t_{P R O J}^{*}\right) \neq \theta_{s}$, thus meaning that some little potential energy will be still present in the system at the end of the dynamics, in analogy with dynamical systems subjected to dry friction.
One of the main limitations of the weakly nonlinear model is enlightened in figure 9.6-(b). Although the asymptotic analysis captures the initial nonlinear decaying trend of total energy density, computed as

$$
\begin{equation*}
E_{t o t}^{t_{i}}=\int_{\Omega} \mathbf{u}_{t_{i}}^{2} \mathrm{~d} \Omega+\int_{z=\eta_{0}}\left[\eta_{t_{i}}^{2}+\frac{1}{B o}\left(\frac{\partial \eta_{t_{i}}}{\partial x}\right)^{2}\right] \mathrm{d} x \tag{9.28}
\end{equation*}
$$

( $t_{i}$ designates the $i$-th time instant) it predicts the total arrest of the motion at $t=t_{W N L}^{*}$ and thus it fails in capturing the transition to the final pinned dynamics, where the fluid bulk keeps oscillating with smaller amplitudes and with a fixed contact line elevation. As a consequence, the total energy in this final state is not zero yet, but rather decays exponentially due to viscous bulk dissipation (the linear trend in figure 9.6-(b) for $t>t_{P R O}^{*}$, black dashed-dotted line), as correctly captured by the projection method.

As time progresses, note that the fraction of time spent in the pinned-phase (in green) increases while the time spent in the free-phase (in blue) decreases, resulting in a slow modulation of the instantaneous damping rate and instantaneous frequency see figure 9.7 , as also observed in experiments (Cocciaro et al., 1993). In particular, in the higher amplitude regime, the damping rate increases as the wave amplitude decreases until reaching a maximum value, after which it decreases to a nearly constant value. Interestingly, despite the lack of quantitative accuracy of the weakly nonlinear calculation, the predicted time of arrest $t_{W N L}^{*}$, seems to coincide with the maximum of the damping rate. Indeed, the time $t=t_{W N L}^{*}$, when the contact

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Figure 9.7 - (a) Nondimensional damping rate and (b) frequency modulation versus time. The damping rate is computed as the logarithmic decrement of the potential energy amplitude, shown in Fig. 9.6. The frequency is computed from the potential energy evaluating the period from peak to peak and the resulting value (triangle in (b)) is then roughly assigned to the mid-point of the corresponding time interval.
line is still oscillating, but with smaller amplitude, approximately predicts the beginning of the transition to the final pinned state, which will become fully established only at $t=t_{P R O J}^{*}$.
Once the pinned dynamic lasts the whole oscillating period, the damping rate is approximately constant and equal to the viscous damping coefficient of the first antisymmetric pinned mode (see figure 9.5-(g)). Concerning the frequency modulation in time, the weakly nonlinear theory gives an incorrect behaviour, with an abrupt and non-physical increase at $t=t_{W N L}^{*}$, corresponding to the finite time singularity. On the contrary, the projection scheme enables us to describe a smooth saturation from the characteristic value of the first antisymmetric free-mode, used to initialize the dynamics, to a final value, reached at $t=t_{\text {PROJ }}^{*}$ and corresponding to the natural oscillation frequency of the first pinned-mode.
Lastly, figure 9.8 provides a deeper insight on how the projection procedure works, showing the normalized mode amplitudes in the case reported in figure 9.6, with the series (9.18a)(9.18b) truncated at $N=30$ and starting, as initial condition, from a non zero amplitude of the first antisymmetric free-mode only, i.e. $A_{f_{n=1}}^{i . c .}$. It follows that all the other mode amplitudes, $A_{f_{n}}$, are initially zero, as visible in the inset in the bottom-left corner of figure 9.6 ( $f^{i . c .}$ \#1, where the superscript ${ }^{i . c .}$ stands for initial condition). At the first transition, mode $f_{n=1}$ is projected onto the pinned eigenmode basis ( $\mathrm{p} \# 2$ ), thereby exciting a certain number of pinned modes. In practice, at each projection, the system's total energy is transferred from an eigenvalue basis


Figure 9.8 - Main figure: first four mode amplitudes versus time and corresponding to the case of figure 9.6. The free-phase amplitudes are normalized in this plot with the initial condition (only $A_{f_{n=1}} \neq 0$ ), while the pinned-phase amplitude are rescaled with the value of $B_{p_{m=1}}$ computed at the first projection from the free-phase to the pinned-phase. The colors are chosen in agreement with figure 9.6-(a). Filled circles represent the amplitude values at the transition instant, while solid lines give their temporal evolutions according to their decay rates. The total $N=30$ amplitudes, computed at each projection time, $t=T_{f}$ (blue) and $T_{p}$ (green), are shown in the insets for different projection steps.
to the other and it is partitioned among the various modes. In the case of figure 9.8, given the prescribed initial state, the total energy is mainly exchanged between the two corresponding modes ( $f_{n=1}$ and $p_{m=1}$ ), with usually no more than 15 modes being excited, most of them having a negligible small amplitude and contributing only weakly to the overall dynamics. Given the prescribed initial condition, most of the energy is indeed contained in the first free-mode and in the corresponding pinned-mode.

A careful convergence and error analysis in relation to the truncation number of the series, $N$, is provided in the supplementary notes.

### 9.4 Extension to the fully nonlinear Jiang et al. model

The formalism of the projection scheme presented in $\$ 9.3 .1$ for Hocking's contact line model with hysteresis range, is based on the possibility to decompose the contact line law as the sequence of two (non-homogeneous) linear problems satisfied in each phase by the sum of a linear homogeneous solution and a particular static solution. For this reason, the extension to a fully nonlinear law, e.g. Jiang et al. cubic model (Jiang et al., 2004) may look challenging. Nevertheless, the continuous nonlinear law, e.g. cubic, can be split through a piecewise linear function (see figure 9.9) for which the projection approach is applicable.
In other words, the contact line law is split into a user-defined number of linear sub-phases

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having an equivalent Hocking's slope $(\alpha C a)_{i}$, in which the system can be still represented as a homogeneous (solution of the corresponding $i$-th generalized eigenvalue problem) plus a particular static solution,

$$
\begin{equation*}
\mathbf{q}^{i}=\theta_{i}^{ \pm} \mathbf{q}_{s}^{i}+\left(\sum_{n=1}^{N} C_{n}^{i} \hat{\mathbf{q}}_{n}^{i} e^{\lambda_{n}^{i}\left(t-T^{i}\right)}+\text { c.c. }\right) . \tag{9.29}
\end{equation*}
$$

As the contact line dynamics is advanced in time, a series of transitions occur when the contact line speed is $\dot{e}_{i}=v_{i}$ (see figure 9.9 ), and the $i$-th phase solution is progressively projected from one set of eigenmodes to next one ( $i+1$-th), following the prescribed contact line law. As the system dissipates energy during the motion, it will eventually end in the central nearly pinned-phase (zero equilibrium point, $\theta=\theta_{s}, \eta(x)=\eta_{0},\left.\partial_{t} \eta\right|_{x=1}=0$ ) where the motion decays exponentially owing to the viscous dissipation solely.
In contrast with Hocking's model with hysteresis, where only two phases are defined, here one has to solve numerically a certain number of eigenvalue problems, one for each sub-phase (see figure 9.9), thus yielding to a computationally more expensive calculation. However, the smooth cubic contact line law facilitates the projection algorithm, i.e. at each projection the energy is transferred only among few modes, allowing one to truncate the modal series to a lower number of natural modes. The competition of a higher number of modes (excited at each projection) in the case of Hocking's law with hysteresis, when compared to the smooth cubic law, is visible in figure 9.10-(a) and (c) (smoother time series) (Multimedia view) and figure 9.10-(b) and (d) (shakier time series) (Multimedia view).
Figure 9.10-(a) and (b) show the contact line elevation and speed computed via the projection method and weakly nonlinear model (see Viola and Gallaire (2018)) for Jing and Perlin's cubic law.

From a mathematical and physical perspective, the main difference introduced by Jiang et al. law with respect to the Hocking law with hysteresis, is the nature of the nonlinearity at zero contact line speed. The strong nonlinearity occurring at $\left.\partial_{t} \eta\right|_{x=1}=0$ introduces in the contact line motion a sort of solid-like friction effect at higher amplitudes (most active in the initial dynamics), which provokes the initial circular-piecewise-like behaviour in the phase portrait displayed in figure 9.10-(c). However, in this case, there exists only one possible final equilibrium state with $\left.e(t)\right|_{t \rightarrow+\infty}=0$ and $\left.\theta(t)\right|_{t \rightarrow+\infty}=\theta_{s}$, to which the system will tend asymptotically owing to the little viscous bulk dissipation. It follows that the contact line elevation and velocity are never exactly zero and therefore the phase portrait progressively turns into a classic logarithmic spiral, typical of linearly damped mechanical systems. Again, the limitations of the weakly nonlinear analysis, which predicts a final time of arrest and hence fails in describing the subsequent small oscillations, are overcome by the projection approach, as complementary semi-linear direct numerical simulations could confirm (see Appendix 9.6.2 and Integral Multimedia Movie 3).

On the other hand, as visible in figure 9.10-(f) (see also figure 9.6-(a)), when a pure hysteresis at zero speed is considered, there exists a final time of arrest for which $\dot{e}=0$ and $e=e_{e q}^{\text {final }} \neq e_{e q}^{\text {initial }}$ which results in a new final equilibrium. In the test cases of figures 7
and 11-(b) and (d), $e_{e q}^{\text {final }}$ is very small, leading to a final static contact angle $\theta^{\text {final }} \approx 91^{\circ}$ versus an initial static angle $\theta_{s}=90^{\circ}$. However, this may not be always the case as $e_{e q}^{\text {final }}$ mainly depends on the imposed initial condition and, particularly, on the width of the prescribed hysteresis range, $\Delta$. Although a thorough parametric analysis should be carried out in order to investigate the range of the possible $e_{e q}^{f i n a l}$, equation (9.25) says that once $\Delta$ is assigned, $\max \left(\left|e_{e q}^{f i n a l}\right|\right)$ is bounded by $F_{0} \Delta / 2$, with a maximum static angle $\theta=\theta_{s} \pm \Delta / 2$.


Figure 9.9 - Reduction of the Jiang et al. cubic model to a piecewise linear function via discretization of the cubic law in a set of linear steps with a given slope, $(\alpha C a)_{i}$ and intercept $\theta_{i}^{ \pm}$. The discretization introduces a series of transition conditions between two consecutive freephases at different contact line speeds, $\delta_{i}$. A non-uniform discretization better approximates the transition to the pinned-phase at a small velocity.

### 9.5 Conclusion

In this Chapter, we have presented a mathematical model based on successive linear eigenmode projections, which induce a loss of total energy and eventually contribute to the progressive nonlinear damping in the liquid motion. The projection scheme allowed us to describe and reproduce the nonlinear contact line dynamics and fluid bulk motion in confined basins and in the limit of small oscillation amplitudes, for which the expansion (9.13) holds. We have shown how, through the projection scheme, the actual dynamical change in the contact line boundary condition can be accounted for, thus overcoming the limitation of the asymptotic model. The computed instantaneous damping rate and frequency, which show a progressive transition from the initial stick-slip contact line motion, subjected to solid-like friction, to a final pinned state with fluid bulk oscillations damped by viscous dissipation solely, are in qualitative agreement with previous experimental observations (Cocciaro et al., 1993; Dollet et al., 2020).
In order to formalize the projection method presented here, a series of simplifications have been introduced throughout the Chapter. The static contact angle was assumed equal to $\pi / 2$,

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Figure 9.10 - (a) Contact line elevation versus time for pure water in a container of width $l=1.93 \mathrm{~cm}$ for which $B o=50, R e=7367$ and $C a=0.0068$. The contact line parameters for the Jiang et al. law are $\theta_{s}=\pi / 2, \beta=5.3$. We initialize the problem setting the complex amplitude of the first antisymmetric mode in order to have an initial contact line elevation and speed equal to 0 and 0.2 , respectively. Red and grey solid lines: weakly nonlinear model (WNL). Black solid line: projection scheme (PROJ). $t_{W N L}^{*}$ denotes the time of arrest computed solving amplitude equation (9.9) for the Jiang et al. model (see Viola and Gallaire (2018)). (c) Phase portrait: contact line velocity versus contact line elevation from PROJ. The results are obtained by discretizing the contact angle law in 61 sub-linear pieces ( 30 equispaced steps in $\left|\partial_{t} \eta\right|_{x=1} \in[0.01,0.2]$ and 31 in $\in[0,0.01)$ ), requiring the solution of 31 eigenvalue problem (the law is symmetric). The series are truncated to $N=10$ modes. The time evolutions were stopped at $t=50$ with a time step $\Delta t=0.005$. See also Appendix 9.6.2 and Integral Multimedia Movie 3 for a full free-surface dynamic representation and comparison with semi-linear direct numerical simulation (Multimedia view). (b) and (d) same as (a) and (c) with the contact line parameters for the Hocking law with hysteresis $\theta_{s}=\pi / 2, \alpha=123 \mathrm{rad}$ and $\Delta=18.5^{\circ}$. The series are truncated to $N=30$ modes. See also Integral Multimedia Movie 2 for full free surface dynamic representation. Multimedia view: https://doi.org/10.1063/5.0055898.2; https://doi.org/10.1063/5.0055898.3.
for which the free surface at rest is flat. Moreover, the contact line model was considered to be symmetric with respect to the zero contact line speed axis, although experimental evidence (see figure IV.2-(a)) shows that the advancing and receding dynamics usually differ from each other. Notwithstanding such idealizations, there are no actual restrictions in considering a non-flat static free surface, i.e. small oscillation on the top of a static meniscus $\eta_{0} \neq 0\left(\theta_{s} \neq \pi / 2\right)$ or in assuming non-symmetric advancing and receding phases (asymmetric contact line law).
Although the asymptotic model was shown to be only meaningful in the first phase of the overall dynamics, its greatest advantage, together with the reduced computational cost, is
the straightforward application to any nonlinear contact line model of experimental inspiration. We have thus shown how the projection method can be generalized to any smooth nonlinear function, e.g. a cubic law, preserving high accuracy in the final results, as shown by the comparison with semi-linear direct numerical simulations (see Appendix 9.6.2 and Integral Multimedia Movie 3). In reality, the static hysteresis range often exists, but at the same time, the angle does not vary linearly with the contact line speed (for $\dot{e} \neq 0$ ) as in Hocking's law, but rather nonlinearly and likely as in Jiang et al. model. Therefore, a combination of the two models, which together reduce to the nonlinear Dussan's law (Dussan, 1979), could be in principle more realistic.
Despite the two-dimensional idealization and semi-linear structure of the equations (owing to the small perturbation assumption), the contact line model is nonlinear, especially in the pure hysteresis case, and makes the system numerically stiff. As a consequence, a direct numerical simulation of the equations would require an implicit temporal integration scheme with a small time step in order to both preserve numerical stability and accuracy, thus yielding a high computational cost. Although the projection algorithm solves the same full set of equations, its computational cost is very advantageous with respect to a standard numerical time integration since only a few eigenvalue calculations are needed. Indeed, the temporal evolution is obtained by advancing in time linear quantities (known a priori from the eigen-calculation) according to their decay rate and oscillation frequency and whose corresponding amplitudes are easily computed at each transition by exploiting the bi-orthogonality relation of direct and adjoint modes (see also supplementary notes). Therefore, the projection approach can be also seen as an efficient physics-based reduced order model for the underlying small-amplitude nonlinear dynamics. A quantitative comparison with experimental observations was not possible due to the simplistic two-dimensional case considered, which drastically simplifies the formulation. In fact, the transition criterion for the consecutive projections is univocally defined by the motion of the contact line, which in two-dimension actually reduces to a contact point, following the prescribed law. In three-dimension, different points along the air-liquid-solid triple line have different velocities and the contact angle becomes a function of the contact line coordinate. Thus, the three-dimensionality breaks the uniqueness of the transition criterion, making the extension to three-dimensional vessels challenging. Nevertheless, we underly that the model could be straightforwardly applied to axisymmetric dynamics, where all the points along the interface perimeter behave in the same manner (two-dimensional-like model), as in the case of liquid oscillations in the U-tube configuration recently investigated in Dollet et al. (2020), for which quantitative comparison with experiments would be in principle possible.

### 9.6 Appendix

### 9.6.1 Convergence analysis of the eigen-calculation

The convergence analysis of the eigenvalue calculation in the case of Hocking's model with hysteresis, i.e. free-phase and pinned-phase, is given in Tab. 9.1 for three relevant eigenvalues,

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| $n_{x}=n_{z}$ | $\lambda_{f_{1}}$ | $\lambda_{f_{15}}$ | $\lambda_{f_{30}}$ | $\max \left(e_{\sigma}, e_{\omega}\right)$ |
| :---: | :---: | :---: | :---: | :---: |
| 50 | $-0.027+\mathrm{i} 1.29$ | $-1.200+\mathrm{i} 43.9$ | $-2.738+\mathrm{i} 124.9$ | - |
| 60 | $-0.027+\mathrm{i} 1.29$ | $-1.219+\mathrm{i} 44.1$ | $-3.138+\mathrm{i} 125.9$ | $14.6 \%$ |
| 70 | $-0.027+\mathrm{i} 1.29$ | $-1.224+\mathrm{i} 44.2$ | $-3.238+\mathrm{i} 126.6$ | $3.2 \%$ |
| 80 | $-0.027+\mathrm{i} 1.29$ | $-1.219+\mathrm{i} 44.2$ | $-3.299+\mathrm{i} 126.9$ | $1.9 \%$ |
|  | $\lambda_{p_{1}}$ | $\lambda_{p_{15}}$ | $\lambda_{p_{30}}$ |  |
| 50 | $-0.001+\mathrm{i} 1.47$ | $-0.474+\mathrm{i} 45.2$ | $-1.737+\mathrm{i} 125.8$ | - |
| 60 | $-0.001+\mathrm{i} 1.47$ | $-0.489+\mathrm{i} 45.3$ | $-1.844+\mathrm{i} 127.2$ | $6.2 \%$ |
| 70 | $-0.001+\mathrm{i} 1.47$ | $-0.497+\mathrm{i} 45.4$ | $-1.936+\mathrm{i} 127.8$ | $4.9 \%$ |
| 80 | $-0.001+\mathrm{i} 1.47$ | $-0.497+\mathrm{i} 45.4$ | $-1.992+\mathrm{i} 128.1$ | $2.9 \%$ |

Table 9.1 - Convergence analysis for the free and pinned eigenvalue calculations in the case of figure $9.10-b), d$ ) and $f$ ) with $B o=50, R e=7367, C a=0.0068$ and $\alpha=77 \mathrm{rad} . n_{x}$ and $n_{z}$ denote the number of grid points in the $x$ and $z$ direction, respectively. Three relevant eigenvalues, $\lambda=\sigma+\mathrm{i} \omega$ with $n, m=1,15$ and $30(N=30)$, are shown.
$n, m=1,15$ and 30 (note that the truncation number of the series is $N=30$ ) and four different mesh refinements. The maximum error is seen to be on the damping of mode $n, m=30$ and to decrease below $3 \%$ for $n_{x}=n_{z}=80$, which is considered a satisfactory trade-off between computation efficiency and accuracy. We underline that the time step used to march in time all our simulations was set to $\Delta t=0.005$. This value is ten times smaller than the shortest oscillation period, $T_{p_{m=30}} \approx 0.05$, hence ensuring that no modes are filtered out artificially.

### 9.6.2 Validation via comparison with semi-linear DNS



Figure 9.11 - Black solid line: free-surface shape corresponding to the nonlinear time evolution presented in figure 9.10 (Jiang et al. cubic contact line law) and obtained via projection algorithm at three different non-dimensional time instants, (a) $t=t_{0}=0$, (b) $t=t_{1}=1$ and (c) $t=t_{2}=2$. Red crosses: semi-linear direct numerical simulation, advanced in time with the same time step, i.e. $\Delta t=0.005$. See also Integral Multimedia Movie 3 associated with figure 9.10-(a) and (c).

The more sophisticated and discretized projection scheme employed in the case of the Jiang et al. cubic model is validated by comparison with semi-linear direct numerical simulation. Governing equations and boundary conditions (9.38)-(9.40e), linearized around the rest state $\mathbf{q}_{0}=\{\mathbf{0},-z, 0\}^{T}$, but subjected to the nonlinear contact line model, e.g. a cubic law, can be written in compact matrix form as,

$$
\begin{equation*}
\left(\partial_{t} \mathscr{B}-\mathscr{A}\right) \mathbf{q}_{1}=\mathscr{F}\left(\left.\frac{\partial \eta_{1}}{\partial t}\right|_{x= \pm 1}\right) \tag{9.30}
\end{equation*}
$$

where $\mathscr{B}$ and $\mathscr{A}$ are, respectively, the mass matrix and the stiffness matrix implemented numerically. The forcing vector $\mathscr{F}$ contains the nonlinear part of the contact line model and hence it has non-zero components at the contact line only.

$$
\mathscr{B}=\left[\begin{array}{ccc}
I_{\mathbf{u}} & 0 & 0  \tag{9.31}\\
0 & 0 & 0 \\
0 & 0 & I_{\eta}
\end{array}\right], \mathscr{A}=\left[\begin{array}{ccc}
R e^{-1} \Delta & -\nabla & 0 \\
\nabla^{T} & 0 & 0 \\
-I_{\left.w\right|_{z=0}} & 0 & 0
\end{array}\right]
$$

and $I_{\mathbf{u}}, I_{\eta}$ are the identity matrices associated to $\mathbf{u}$ and $\eta$.
We note that the very same matrices have been implemented numerically in order to solve the eigenvalue problem in the different phases of the projection algorithm, i.e. we solved $[(\sigma+\mathrm{i} \omega) \mathscr{B}-\mathscr{A}] \hat{\mathbf{q}}=\mathbf{0}$ with the proper boundary conditions.
Equation (9.30) is discretized in space by means of a pseudospectral Chebyshev collocation method and integrated in time with the implicit backward differentiation formula of order 2. The discretized state vector at the $n$-th time step is obtained by solving the nonlinear equation

$$
\begin{equation*}
\frac{3}{2} \mathscr{B} \mathbf{q}_{1}^{n}-2 \mathscr{B} \mathbf{q}_{1}^{n-1}+\frac{1}{2} \mathscr{B} \mathbf{q}_{1}^{n-2}-\Delta t \mathscr{A} \mathbf{q}_{1}^{n}=\Delta t \mathscr{F}\left(\left.\frac{\partial \eta_{1}}{\partial t}\right|_{x= \pm 1}\right) \tag{9.32}
\end{equation*}
$$

with $\left.\partial_{t} \eta_{1}\right|_{x= \pm 1}=\left.\left(\frac{3}{2} \eta_{1}^{n}-2 \eta_{1}^{n-1}+\frac{1}{2} \eta_{1}^{n-2}\right)\right|_{x= \pm 1}$.
The linearization around the base flow $\mathbf{q}_{0}$ implicitly assumes that nonlinear inertial and free surface curvature terms are negligible (only the contact line nonlinearity is kept for small oscillation amplitudes). As a consequence, the computational domain is kept fixed in time and it is defined by the flow configuration at rest. In figure 9.11, the interface shape computed via projection scheme and semi-linear direct numerical simulation is shown for three different and equispaced time instants of the nonlinear evolution discussed in figure 9.10-(a), (c) and (e), resulting in a quantitative very good agreement.

### 9.6.3 The crucial role of viscosity in the convergence of the projection scheme

To ease the mathematical treatment of the lateral wall boundary condition during the freephase, in this study a simple perfect slip condition was assumed. When the contact line is free to slide (free-phase), e.g. according to the Hocking linear law, the total damping coefficient is produced by the sum of four sources, namely (i) the fluid bulk, (ii) the solid bottom, (iii) the

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sidewall and (iv) the contact line (Case and Parkinson, 1957; Hocking, 1987; Miles, 1967):

$$
\begin{equation*}
\sigma_{\text {free }}=\sigma_{\text {bulk }}+\sigma_{\text {bottom }}+\sigma_{\text {side }}+\sigma_{c l} . \tag{9.33}
\end{equation*}
$$

In this work, $\sigma_{\text {side }}$ is overlooked owing to the imposition of a stress-free condition. Moreover, if the ratio $h / L$ is sufficiently large (as, for example, in the test cases proposed in our study), then the bottom term is negligible. In a vast range of typical lab-working conditions (except for highly viscous fluids or for very shallow fluid layers), the contact line dissipation represents the dominant term, i.e. $\sigma_{c l} \gg \sigma_{b u l k}$ (see Benjamin and Ursell (1954) and Hocking (1987) among other references), so that $\sigma_{f r e e} \approx \sigma_{c l}$. Hence, one may think that an inviscid model supplemented with a contact line law, as originally proposed by Hocking (Hocking, 1987), would be adequate to describe the free contact line dynamics and, at the same time, it would be much simpler than the present viscous formulation. The inviscid weakly nonlinear analysis presented in Viola et al. (2018), where only the leading order free contact line dynamics is described, was indeed formalized in this spirit.
Whereas this reasoning applies to the free-dynamics, it should be noted that during the pinned-phase, the contact line is at rest, so that no contact line dissipation takes place at all, i.e. $\sigma_{c l}=0$. During such a phase, the overall dissipation is given by (i) bulk, (ii) bottom (negligible in the deep water regime) and (iii) sidewall. Although we are neglecting this main contribution (iii) (see supplementary material for a thorough discussion about the stress-free vs. no-slip lateral wall condition accounting for (iii) during the pinned phase), the remaining bulk dissipation plays a crucial role in our projection scheme, that aims at describing the full dynamics, i.e. transition from stick-slip to pinned.

The bulk dissipation is indeed approximately proportional to the square of the wave natural frequency, $\omega^{2}$ (it was proven for a free contact line (Case and Parkinson, 1957; Miles, 1967), but it qualitatively applies for a pinned dynamics), meaning that the bulk damping can certainly be very small for the lowest frequency modes, yet, it tends to rapidly increase for higher frequency modes. Figure 9 (associated with the case discussed in figure 7) shows how the convergence of the projection algorithm is well achieved by retaining the first 30 free and pinned modes. This is possible exactly because the linear damping (bottom+bulk+contact line in the free phase and bottom+bulk in the pinned phase) damps out higher frequency modes rapidly enough, i.e. before they are projected again on the next phase, hence ensuring the series and thus the method convergence for a finite number of modes, $N$ (truncation number). On the contrary, if an inviscid model was to be considered, the free modes used to initialize the dynamics would project at the first transition on several pinned modes. None of the latter modes, due to the lack of dissipative sources, would die out during the pinned phase, hence all of them would project back to the next free phase modes. At each free-to-pinned projection, many unphysical pinned modes would be excited, preventing the convergence of the algorithm and describing an unrealistic "ultraviolent catastrophe". Furthermore, even if the loss of energy taking place at each projection and ascribed to the nonlinear static hysteresis range eventually led the contact line to pin, the final pinned oscillations would last indefinitely, as no dissipation is present without viscosity.

To conclude, keeping viscosity, albeit small, is not a mere choice, but rather a fundamental aspect that is required to ensure that the method works properly and that it is capable to reproduce a behaviour closer to the actual one observed in real experiments.

### 9.7 Supplementary Material

In §9.7.1 of the present supplementary notes we discuss more in detail the derivation of the adjoint problem and the adjoint modes, key point of the projection algorithm presented throughout the Chapter. In $\S 9.7 .2$, we carry out a thorough convergence analysis and error estimation, as the truncation number, $N$, used to describe the dynamics is varied. Lastly, in §9.7.3, further clarifications about the treatment of the lateral wall boundary condition, i.e. the employment of a stress-free model versus a no-slip condition at the wall for the pinned dynamics, are provided.

### 9.7.1 Global adjoint modes derivation and bi-orthogonality condition

In §IV A 2 of this Chapter, the following weighted inner product is introduced:

$$
\begin{equation*}
<\mathbf{w}, \mathbf{v}>_{E}=\int_{\Omega} \overline{\mathbf{u}}_{\mathbf{w}} \mathbf{u}_{\mathbf{v}} \mathrm{d} \Omega+\int_{z=0} \bar{\eta}_{\mathbf{w}} \eta_{\mathbf{v}} \mathrm{dx}+\frac{1}{B o} \int_{z=0} \frac{\partial \bar{\eta}_{\mathbf{w}}}{\partial x} \frac{\partial \eta_{\mathbf{v}}}{\partial x} \mathrm{dx} \tag{9.34}
\end{equation*}
$$

where $\mathbf{v}=\left\{\mathbf{u}_{\mathbf{v}}, p_{\mathbf{v}}, \eta_{\mathbf{v}}\right\}^{T}$ and $\mathbf{w}=\left\{\mathbf{u}_{\mathbf{w}}, p_{\mathbf{w}}, \eta_{\mathbf{w}}\right\}^{T}$ are two generic vectors, the bar designates the complex conjugate, $\Omega$ denotes the fluid bulk domain ( $d \Omega=d x d z$ ) and the subscript ${ }_{E}$ stands for energy. It follows that the energy norm of a generic vector $\mathbf{v}$ is

$$
\begin{equation*}
\langle\mathbf{v}, \mathbf{v}\rangle_{E}=\|\mathbf{v}\|_{E}^{2}=\underbrace{\int_{\Omega} \overline{\mathbf{u}}_{\mathbf{v}} \mathbf{u}_{\mathbf{v}} \mathrm{d} \Omega}_{\sim E_{k}^{v}}+\underbrace{\int_{z=0} \bar{\eta}_{\mathbf{v}} \eta_{\mathbf{v}} \mathrm{dx}}_{\sim E_{p g}^{\mathrm{v}}}+\underbrace{\frac{1}{B o} \int_{z=0} \frac{\partial \bar{\eta}_{\mathbf{v}}}{\partial x} \frac{\partial \eta_{\mathbf{v}}}{\partial x} \mathrm{dx}}_{\sim E_{p s}^{\mathbf{v}}} \tag{9.35}
\end{equation*}
$$

We can recognize that the three integrals in equation (9.35) represent a measure of the total energy density, given by the sum of kinetic, gravitational potential and surface potential energy densities, respectively, stored in $\mathbf{v}$.
Then the concept of adjoint modes, solutions of the adjoint problem, was invoked and it was stated that the direct modes, $\hat{\mathbf{q}}_{i}$, and the adjoint modes, $\hat{\mathbf{q}}_{j}^{\dagger}$, form a bi-orthogonal basis with respect to the weighted scalar product (9.34). Moreover, the adjoint modes can be normalized such that $<\hat{\mathbf{q}}_{j}^{\dagger}, \hat{\mathbf{q}}_{i}>_{E}=\delta_{i j}$, with $\delta_{i j}$ the Kronecker delta. Given a vector $\mathbf{v}$, which is represented as a linear combination of eigenvectors,

$$
\begin{equation*}
\mathbf{v}=\sum_{i=1}^{N} C_{i} \hat{\mathbf{q}}_{i} \tag{9.36}
\end{equation*}
$$

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the complex coefficients $C_{i}$ can be computed by projecting $\mathbf{v}$ onto the set of adjoint modes, which are bi-orthogonal to both the direct modes and their complex conjugates, that are both eigenvectors of the real sloshing operator, $\left(\partial_{t} \mathscr{B}-\mathscr{A}\right)=L$ (see also Appendix 9.6.2 of this Chapter):

$$
\begin{equation*}
<\hat{\mathbf{q}}_{j}^{\dagger}, \mathbf{v}>=<\hat{\mathbf{q}}_{j}^{\dagger}, \sum_{i=1}^{N} C_{i} \hat{\mathbf{q}}_{i}>=\sum_{i=1}^{N} C_{i}<\hat{\mathbf{q}}_{j}^{\dagger}, \hat{\mathbf{q}}_{i}>=C_{j} \in \mathbb{C} . \tag{9.37}
\end{equation*}
$$

and their value is univocally determined, i.e. it does not depend on the truncation number $N$. In this section of the supplementary notes we formally derive the adjoint problem used in this Chapter, we show how the adjoint modes are readily obtained from the direct modes (without any further computation) and we demonstrate that adjoint and direct modes form a bi-orthogonal basis with respect to a specific weighted scalar product, i.e. (9.34). To this end, let us briefly recall the linearized equations and boundary conditions governing the $\epsilon$-order problem, where the linearization is made around the rest state, $\mathbf{q}_{0}=\left\{\mathbf{u}_{0}, p_{0}, \eta_{0}\right\}^{T}=\{\mathbf{0},-z, 0\}^{T}$, with $\theta_{0}=\theta_{s}=\pi / 2$ :

$$
\begin{gather*}
\nabla \cdot \hat{\mathbf{u}}=0, \quad \lambda \hat{\mathbf{u}}+\nabla \hat{p}-\frac{1}{R e} \Delta \hat{\mathbf{u}}=\mathbf{0} \quad \text { on } \Omega,  \tag{9.38}\\
\lambda \hat{\eta}-\hat{w}=0 \quad \text { at } z=0, \tag{9.39}
\end{gather*}
$$

with $\lambda$ the eigenvalue, subjected to the following boundary conditions (b.c.) at the free surface, bottom, sidewalls and contact line,

$$
\begin{gather*}
\frac{\partial \hat{u}}{\partial z}+\frac{\partial \hat{w}}{\partial x}=0 \quad \text { at } z=0,  \tag{9.40a}\\
-\hat{p}+\hat{\eta}-\frac{1}{B o} \frac{\partial^{2} \hat{\eta}}{\partial x^{2}}+\frac{2}{R e} \frac{\partial \hat{w}}{\partial z}=0 \quad \text { at } z=0,  \tag{9.40b}\\
\hat{u}=\hat{w}=0 \quad \text { at } z=-\frac{h}{l},  \tag{9.40c}\\
\hat{u}=0, \quad \hat{w} \pm l_{s} \frac{\partial \hat{w}}{\partial x}=0 \quad \text { at } x= \pm 1,  \tag{9.40d}\\
\frac{\partial \hat{\eta}}{\partial x}=\mp\left(\sin \theta_{s}\right)^{-2} \hat{\alpha} \lambda \hat{\eta} \quad \text { at } x= \pm 1, z=0, \tag{9.40e}
\end{gather*}
$$

with $\mathbf{q}=\{\hat{\mathbf{u}}, \hat{p}, \hat{\eta}\}^{T} e^{\lambda t}+$ c.c. $=\hat{\mathbf{q}} e^{\lambda t}+$ c.c., $\mathbf{e}_{z}=\{0,0,1\}^{T}$ and where $\hat{\alpha} \in \mathbb{R}$ (constant) in (9.40e) is a generic slope for the linear Hocking's law. As described in Appendix 9.6.2 of this Chapter, governing equations (9.38)-(9.39) can be written in matrix compact form as

$$
\begin{equation*}
(\lambda \mathscr{B}-\mathscr{A}) \hat{\mathbf{q}}=L \hat{\mathbf{q}}=\mathbf{0} . \tag{9.41}
\end{equation*}
$$

which is forced to obey to the boundary conditions (9.40a)-(9.40e).

## Derivation of the adjoint problem

Let us introduce the following Hermitian, $H$, scalar product

$$
\begin{equation*}
<\mathbf{a}, \mathbf{b}>_{H}=\int_{\Omega}\left(\overline{\mathbf{u}}_{\mathbf{a}} \mathbf{u}_{\mathbf{b}}+\bar{p}_{\mathbf{a}} p_{\mathbf{b}}\right) \mathrm{d} \Omega+\int_{z=0} \bar{\eta}_{\mathbf{a}} \eta_{\mathbf{b}} \mathrm{dx} \tag{9.42}
\end{equation*}
$$

where $\mathbf{a}=\left\{\mathbf{u}_{\mathbf{a}}, p_{\mathbf{a}}, \eta_{\mathbf{a}}\right\}^{T}$ and $\mathbf{b}=\left\{\mathbf{u}_{\mathbf{b}}, p_{\mathbf{b}}, \eta_{\mathbf{b}}\right\}^{T}$ are two generic vectors. By definition, the adjoint operator, $L^{\dagger}$, of the direct operator, $L$, satisfies $<\mathbf{q}^{\dagger}, L \mathbf{q}>_{H}=<L^{\dagger} \mathbf{q}^{\dagger}, \mathbf{q}>_{H}$ (the hat symbol on $\hat{\mathbf{q}}$ is dropped out for convenience of notation). For any pair $\left(\mathbf{q}^{\dagger}, \mathbf{q}\right)$, the operator $L^{\dagger}$ is derived by integration by parts and the adjoint boundary conditions are chosen in order to cancel the boundary integrals coming from the integration by parts. Therefore, we first take the scalar product of $L \mathbf{q}$ with $\mathbf{q}^{\dagger}$, which is still unknown at this stage,

$$
\begin{equation*}
<\mathbf{q}^{\dagger}, L \mathbf{q}>_{H}=\int_{\Omega} \overline{\mathbf{u}}^{\dagger} \cdot\left(\lambda \mathbf{u}+\nabla p-R e^{-1} \Delta \mathbf{u}\right) \mathrm{d} \Omega+\int_{\Omega} \bar{p}^{\dagger}(\nabla \cdot \mathbf{u}) \mathrm{d} \Omega+\int_{z=0} \bar{\eta}^{\dagger}(\lambda \eta-w) \mathrm{dx}=0 \tag{9.43}
\end{equation*}
$$

and integrating by part

$$
\begin{align*}
& \int_{\Omega}\left(\lambda \overline{\mathbf{u}}^{\dagger}+\nabla \bar{p}^{\dagger}-R e^{-1} \Delta \overline{\mathbf{u}}^{\dagger}\right) \cdot \mathbf{u} \mathrm{d} \Omega+\int_{\Omega} p\left(\nabla \cdot \overline{\mathbf{u}}^{\dagger}\right) \mathrm{d} \Omega+ \\
+ & \int_{z=0} \bar{\eta}^{\dagger}(\lambda \eta-w) \mathrm{dx}-\int_{\partial \Omega} \overline{\mathbf{u}}^{\dagger} \cdot \tau_{n} \mathrm{dS}+\int_{\partial \Omega} \bar{\tau}_{n}^{\dagger} \cdot \mathbf{u} \mathrm{dS}=0 \tag{9.44}
\end{align*}
$$

where $\tau_{n}=-p \mathbf{n}+R e^{-1}\left(\nabla \mathbf{u}+\nabla^{T} \mathbf{u}\right) \cdot \mathbf{n}$ and $\tau_{n}^{\dagger}=-p^{\dagger} \mathbf{n}+R e^{-1}\left(\nabla \mathbf{u}^{\dagger}+\nabla^{T} \mathbf{u}^{\dagger}\right) \cdot \mathbf{n}$. The two volumeintegrals are null because they contain, in brackets, the bulk adjoint equations (momentum and continuity). The last two boundary integrals can be split into different contributions, i.e. solid bottom, lateral sidewalls and free surface. The contribution associated with the solid bottom is nullified if the adjoint problem satisfies (as the direct problem) the no-slip and no-penetration conditions, thus $\mathbf{u}=\mathbf{u}^{\dagger}=\mathbf{0}$ at $z=-h / l$. At the lateral solid wall at $x=1$, we have

$$
\begin{align*}
& -\int \bar{u}^{\dagger}\left(-p+2 R e^{-1} \partial_{x} u\right) \mathrm{dz}-\int \bar{w}^{\dagger} R e^{-1}\left(\partial_{z} u+\partial_{x} w\right) \mathrm{dz}+ \\
& \int\left(-\bar{p}^{\dagger}+2 R e^{-1} \partial_{x} \bar{u}^{\dagger}\right) u \mathrm{dz}+\int R e^{-1}\left(\partial_{z} \bar{u}^{\dagger}+\partial_{x} \bar{w}^{\dagger}\right) w \mathrm{dz} \tag{9.45}
\end{align*}
$$

The first and third integrals are zero if the adjoint problem satisfies the no-penetration condition along the lateral wall, $u=u^{\dagger}=0$ (it also follows that $\partial_{z} u=\partial_{z} u^{\dagger}=0$ in the second and fourth integrals), while the second and fourth boundary integrals cancel each other out if $w+l_{s} \partial_{x} w=w^{\dagger}+l_{s} \partial_{x} w^{\dagger}=0$. The same arguments are valid for the lateral wall at $x=-1$. The remaining contributions, together with the third integral in (9.44) are all evaluated at the free

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surface, $z=0$,

$$
\begin{equation*}
\int \bar{\eta}^{\dagger}(\lambda \eta-w) \mathrm{dx}-\int \bar{w}^{\dagger}\left(-p+2 R e^{-1} \partial_{z} w\right) \mathrm{dx}+\int w\left(-\bar{p}^{\dagger}+2 R e^{-1} \partial_{z} \bar{w}^{\dagger}\right) \mathrm{dx}=0 \tag{9.46}
\end{equation*}
$$

The terms between brackets in the second integrals can be replaced using the direct dynamic boundary condition,

$$
\begin{equation*}
\int \bar{\eta}^{\dagger}(\lambda \eta-w) \mathrm{dx}+\int \bar{w}^{\dagger}\left(\eta-B o^{-1} \partial_{x x} \eta\right) \mathrm{dx}+\int w\left(-\bar{p}^{\dagger}+2 R e^{-1} \partial_{z} \bar{w}^{\dagger}\right) \mathrm{dx}=0 \tag{9.47}
\end{equation*}
$$

In order to nullify the remaining boundary integrals we can assume that the adjoint problem satisfies

$$
\begin{equation*}
-p^{\dagger}+2 R e^{-1} \partial_{z} w^{\dagger}=+\left(\eta^{\dagger}-B o^{-1} \partial_{x x} \eta^{\dagger}\right) \tag{9.48}
\end{equation*}
$$

Substituting (9.48) in the third integral in (9.47), we obtain

$$
\begin{equation*}
\int\left[\left(\lambda \bar{\eta}^{\dagger}+\bar{w}^{\dagger}\right) \eta-B o^{-1}\left(\bar{w}^{\dagger} \partial_{x x} \eta+w \partial_{x x} \bar{\eta}^{\dagger}\right)\right] \mathrm{dx}=0 \tag{9.49}
\end{equation*}
$$

Equation (9.49) suggests that the adjoint kinematic equation is

$$
\begin{equation*}
\lambda \eta^{\dagger}=-w^{\dagger} \quad \text { at } z=0 \tag{9.50}
\end{equation*}
$$

Indeed, using such an adjoint kinematic relation, $\bar{w}^{\dagger}=-\lambda \bar{\eta}^{\dagger}$, together with the direct kinematic equation, $w=\lambda \eta$, and integrating by parts once the last two terms in brackets, we obtain

$$
\begin{equation*}
B o^{-1} \int\left(\partial_{x} \bar{\eta}^{\dagger} \partial_{x} \eta-\partial_{x} \eta \partial_{x} \bar{\eta}^{\dagger}\right) \mathrm{dx}+B o^{-1}\left[-\bar{\eta}^{\dagger} \partial_{x} \eta+\partial_{x} \bar{\eta}^{\dagger} \eta\right]_{x=-1}^{x=+1}=0 \tag{9.51}
\end{equation*}
$$

It is easy to recognize that the remaining boundary terms cancel out if the free-end edge contact line condition, $\partial_{x} \eta=\partial_{x} \eta^{\dagger}=0$, or the pinned-end edge condition, $\eta=\eta^{\dagger}=0$, are imposed at the contact line. If the linear Hocking's law is considered, then we have $\partial_{x} \eta=-\hat{\alpha} \lambda \eta$ and $\partial_{x} \bar{\eta}^{\dagger}=-\hat{\alpha} \lambda \bar{\eta}^{\dagger}$, which also nullify the boundary expression (9.51).
To summarize, the adjoint governing equations and boundary conditions are found to be:

$$
\begin{gather*}
\nabla \cdot \hat{\mathbf{u}}^{\dagger}=0, \quad \bar{\lambda} \hat{\mathbf{u}}^{\dagger}+\nabla \hat{p}^{\dagger}-\frac{1}{R e} \Delta \hat{\mathbf{u}}^{\dagger}=\mathbf{0} \quad \text { on } \Omega  \tag{9.52}\\
\bar{\lambda} \hat{\eta}^{\dagger}+\hat{w}^{\dagger}=0 \quad \text { at } z=0  \tag{9.53}\\
\frac{\partial \hat{u}^{\dagger}}{\partial z}+\frac{\partial \hat{w}^{\dagger}}{\partial x}=0 \quad \text { at } z=0  \tag{9.54a}\\
-\hat{p}^{\dagger}-\hat{\eta}^{\dagger}+\frac{1}{B o} \frac{\partial^{2} \hat{\eta}^{\dagger}}{\partial x^{2}}+\frac{2}{R e} \frac{\partial \hat{w}^{\dagger}}{\partial z}=0 \quad \text { at } z=0 \tag{9.54b}
\end{gather*}
$$

$$
\begin{gather*}
\hat{u}^{\dagger}=\hat{w}^{\dagger}=0 \quad \text { at } z=-\frac{h}{l}, \quad \hat{u}^{\dagger}=0, \quad \hat{w}^{\dagger}+l_{s} \frac{\partial \hat{w}^{\dagger}}{\partial x}=0 \quad \text { at } x= \pm 1  \tag{9.54c}\\
\frac{\partial \hat{\eta}^{\dagger}}{\partial x}=\mp\left(\sin \theta_{s}\right)^{-2} \hat{\alpha} \bar{\lambda} \hat{\eta}^{\dagger} \quad \text { at } z=0, x= \pm 1 \tag{9.54d}
\end{gather*}
$$

or, in matrix form, $\left(\bar{\lambda} \mathscr{B}^{\dagger}-\mathscr{A}^{\dagger}\right) \hat{\mathbf{q}}^{\dagger}=L^{\dagger} \hat{\mathbf{q}}^{\dagger}=\mathbf{0}$ (+b.c.), with $\mathscr{B}^{\dagger} \neq \mathscr{B}, \mathscr{A}^{\dagger} \neq \mathscr{A}$ and $\lambda^{\dagger}=\bar{\lambda}$.
Writing explicitly the direct mode as $\hat{\mathbf{q}}=(\hat{\mathbf{u}}, \hat{p}, \hat{\eta})^{T}=\hat{\mathbf{q}}_{r}+\mathrm{i} \hat{\mathbf{q}}_{i}$, it can be readily verified by substitution and comparison with the direct problem that vector $\hat{\mathbf{q}}^{\dagger}$,

$$
\hat{\mathbf{q}}=\left(\begin{array}{c}
\hat{\mathbf{u}}  \tag{9.55}\\
\hat{p} \\
\hat{\eta}
\end{array}\right)=\left(\begin{array}{c}
\hat{\mathbf{u}}_{R}+\mathrm{i} \hat{\mathbf{u}}_{I} \\
\hat{p}_{R}+\mathrm{i} \hat{p}_{I} \\
\hat{\eta}_{R}+\mathrm{i} \hat{\eta}_{I}
\end{array}\right), \quad \hat{\mathbf{q}}^{\dagger}=\left(\begin{array}{c}
\hat{\mathbf{u}}^{\dagger} \\
\hat{p}^{\dagger} \\
\hat{\eta}^{\dagger}
\end{array}\right)=\left(\begin{array}{c}
-\hat{\mathbf{u}}_{R}+\mathrm{i} \hat{\mathbf{u}}_{I} \\
-\hat{p}_{R}+\mathrm{i} \hat{p}_{I} \\
\hat{\eta}_{R}-\mathrm{i} \hat{\eta}_{I}
\end{array}\right)=\left(\begin{array}{c}
-\overline{\hat{\mathbf{u}}} \\
-\hat{\hat{p}} \\
\overline{\hat{\eta}}
\end{array}\right) \neq \hat{\mathbf{q}}
$$

is a solution of (9.52)-(9.54d) and therefore it represents the adjoint mode associated with the adjoint eigenvalue $\lambda^{\dagger}=\bar{\lambda}=\sigma-\mathrm{i} \omega$. In practice, the adjoint modes are directly obtained from the direct modes, without any further computation.

## Demonstration of the bi-orthogonality condition

Let us consider the direct problem $L \mathbf{q}_{i}=\mathbf{0}$ (again, the hat symbol is dropped out for convenience). Multiplying by $\mathbf{q}_{j}^{\dagger}$, integrating by parts and using the direct and adjoint boundary conditions (analogously to the previous section §9.7.1), we have

$$
\begin{align*}
&<\mathbf{q}_{j}^{\dagger}, L \mathbf{q}_{i}>_{H}=\int_{\Omega}\left(\lambda_{i} \overline{\mathbf{u}}_{j}^{\dagger}+\right.\left.\nabla \bar{p}_{j}^{\dagger}-R e^{-1} \Delta \overline{\mathbf{u}}_{j}^{\dagger}\right) \cdot \mathbf{u}_{i} \mathrm{~d} \Omega+\int_{\Omega}\left(\nabla \cdot \overline{\mathbf{u}}_{j}^{\dagger}\right) p_{i} \mathrm{~d} \Omega+ \\
&+\int_{z=0} \lambda_{i} \bar{\eta}_{j}^{\dagger} \eta_{i} \mathrm{dx}-B o^{-1} \int_{z=0} \underbrace{w_{i}}_{\lambda_{i} \eta_{i}} \partial_{x x} \bar{\eta}_{j}^{\dagger} \mathrm{dx}+ \\
&+\int_{z=0} \bar{w}_{j}^{\dagger} \eta_{i} \mathrm{dx}-B o^{-1} \int_{z=0} \underbrace{\bar{w}_{j}^{\dagger}}_{-\lambda_{j} \bar{\eta}_{j}^{\dagger}} \partial_{x x} \eta_{i} \mathrm{dx}= \\
&=<L^{\dagger} \mathbf{q}_{j}^{\dagger}, \mathbf{q}_{i}>_{H}=\int_{\Omega}\left(\lambda_{j} \overline{\mathbf{u}}_{j}^{\dagger}+\nabla \bar{p}_{j}^{\dagger}-R e^{-1} \Delta \overline{\mathbf{u}}_{j}^{\dagger}\right) \cdot \mathbf{u}_{i} \mathrm{~d} \Omega+\int_{\Omega}\left(\nabla \cdot \overline{\mathbf{u}}_{j}^{\dagger}\right) p_{i} \mathrm{~d} \Omega+ \\
&+\int_{z=0} \lambda_{j} \bar{\eta}_{j}^{\dagger} \eta_{i} \mathrm{dx}+\int_{z=0} \bar{w}_{j}^{\dagger} \eta_{i} \mathrm{dx} . \tag{9.56}
\end{align*}
$$

Integrating by parts once the two curvature-related integrals on the l.h.s. (with the imposition of the contact line boundary condition) and subtracting the r.h.s. (last line) from the l.h.s., we

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obtain the following condition,

$$
\begin{equation*}
\left(\lambda_{i}-\lambda_{j}\right)\left[\int_{\Omega} \overline{\mathbf{u}}_{j}^{\dagger} \mathbf{u}_{i} \mathrm{~d} \Omega+\left(\int_{z=0} \bar{\eta}_{j}^{\dagger} \eta_{i} \mathrm{dx}+\frac{1}{B o} \int_{z=0} \frac{\partial \bar{\eta}_{j}^{\dagger}}{\partial x} \frac{\partial \eta_{i}}{\partial x} \mathrm{dx}\right)\right]=\left(\lambda_{i}-\lambda_{j}\right)<\mathbf{q}_{j}^{\dagger}, \mathbf{q}_{i}>_{E}=0 \tag{9.57}
\end{equation*}
$$

which proves the bi-orthogonality property (if $i \neq j, \lambda_{i} \neq \lambda_{j}$ ) between direct and adjoint modes with respect to the weighted scalar product introduced in equation (9.34). The adjoint mode, derived directly from the direct mode as shown in equation (9.55), is then independently normalized such that $\left\langle\mathbf{q}_{j}^{\dagger}, \mathbf{q}_{i}>_{E}=\delta_{i j}\right.$.
We note that the initial Hermitian scalar product (9.42) used to derive the adjoint equations is different from that used by Viola and Gallaire (2018) (VG18) (and in Chapter 7 of this thesis), where the adjoint mode, solution of the corresponding adjoint equations, is found to be $\mathbf{q}_{V G 18}^{\dagger}=\overline{\mathbf{q}}$ and the associated bi-orthogonality condition reads

$$
\begin{equation*}
<\mathbf{q}_{j_{V G 18}}^{\dagger}, \mathbf{q}_{i}>_{V G 18}=\int_{\Omega} \overline{\mathbf{u}}_{j_{V G 18}}^{\dagger} \mathbf{u}_{i} \mathrm{~d} \Omega-\int_{z=0} \bar{\eta}_{j_{V G 18}}^{\dagger} \eta_{i} \mathrm{dx}-\frac{1}{B o} \int_{z=0} \frac{\partial \bar{\eta}_{j_{V G 18}}^{\dagger} \frac{\partial \eta_{i}}{\partial x} \mathrm{dx} . . . . .}{\partial x} \tag{9.58}
\end{equation*}
$$

The non-positiveness of $\langle\mathbf{w}, \mathbf{v}\rangle_{V G 18}$ makes it less suitable to define a norm. On the contrary, $\langle\mathbf{w}, \mathbf{v}\rangle_{E}$ is positive-definite and thus suitable to define a norm, i.e. the energy norm (9.35), which will be exploited in the next section to carry out a convergence analysis and error estimation, as the truncation number of the series, $N$, is varied.


Figure 9.12 - Relative error made at each projection versus time for a fixed $N=30$. Inset: relative error versus the number of modes $N$ for a fixed projection time, highlighted with a red circle in the main figure.

### 9.7.2 Convergence analysis: projection error vs. time and truncation number $N$

With reference to figure 6 of this Chapter, in this section we carry out a thorough convergence analysis and error estimation by changing the number of modes $N$ (truncation number) included in the series expansions. We recall that the results shown in figure 6 of the Chapter are computed considering pure water in a container of width $l=5 \mathrm{~cm}$ for which $B o=336$,
$R e=30717$ and $C a=0.011$. The static angle is set to $\theta_{s}=\pi / 2$ and the slope of the linear Hocking's law $\alpha$ is assumed to be 77 rad . The static contact angle hysteresis is fixed to $\Delta=20^{\circ}$. At each projection instant, $t_{i}$, the flow fields can be simultaneously expressed as a solution of the free-phase and of the pinned-phase. Hence, at every $t_{i}$ we can define two solution vectors, $\mathbf{v}_{f}^{i}$ and $\mathbf{v}_{p}^{i}$, associated with the transition between the two different phases. In other words, for a generic projection step, we have $\mathbf{v}_{\text {old }}^{i}$ and $\mathbf{v}_{\text {new }}^{i}$, corresponding to the solution before and after the projection step. Let us denote their difference as $\Delta \mathbf{v}^{i}=\mathbf{v}_{\text {new }}^{i}-\mathbf{v}_{\text {old }}^{i}$.

Using the energy norm defined by equation (9.35) and fixing the number of modes, i.e. $N=30$, we can compute the relative error introduced at each projection as

$$
\begin{equation*}
r_{i}\left(t_{i}\right)=\frac{\left\langle\left(\mathbf{v}_{\text {new }}^{i}-\mathbf{v}_{\text {old }}^{i}\right),\left(\mathbf{v}_{\text {new }}^{i}-\mathbf{v}_{\text {old }}^{i}\right)>_{E}\right.}{\left\langle\mathbf{v}_{\text {old }}^{i}, \mathbf{v}_{\text {old }}^{i}>_{E}\right.}=\frac{\left\langle\Delta \mathbf{v}^{i}, \Delta \mathbf{v}^{i}>_{E}\right.}{\left\langle\mathbf{v}_{\text {old }}^{i}, \mathbf{v}_{\text {old }}^{i}>_{E}\right.} \tag{9.59}
\end{equation*}
$$

In figure 9.12 the result of this procedure is depicted Figure 9.12 shows that the maximum relative error (in energy norm (9.35)) for a truncation number $N=30$ over the whole temporal evolution is less than $0.002 \%$, confirming the precision of the projection scheme. The inset in the top-left corner of Fig. 9.12 shows the error trend as a function of $N$ for a fixed projection instant ( $t_{i}=T_{f p}^{i}=72.73$ ), where the free-solution is projected onto the pinned or pinnedsolution.


Figure 9.13 - Relative error made at each projection versus time for a fixed $N=30$. Inset: relative error versus the number of modes $N$ for a fixed projection time, highlighted with a red circle in the main figure. The lateral walls are here modelled with a no-slip condition in the pinned-phases.

### 9.7.3 Treatment of the lateral wall boundary condition in the pinned-phase: freestress vs. no-slip

The choice of a free-stress $\left(l_{s} \gg 1, \partial_{x} w=0\right)$ boundary condition at the lateral solid walls, although physically unrealistic, is motivated by the need to simplify the wall and contact

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line treatment in the free-phase, which would be otherwise extremely complex. However, when the pinned-phase is considered one immediately notices imposing a no-slip boundary condition at the wall does not result in any stress-singularity, since the the contact line is pinned. Hence, the no-slip condition arises as a natural boundary condition along the entire lateral solid walls in the pinned phase. Letting unchanged the treatment of free-phase, we can first compute the eigenmodes for the pinned-solution by imposing the no-slip b.c. at the walls, then run the projection algorithm and eventually repeat the same convergence and error analysis presented in $\S 9.7 .2$. Figure 9.13 shows that in this case, the maximum error over the entire data sequence is still small, i.e. $<0.25 \%$. However, it is evident that the error (in energy norm (9.35)) is $10^{3}$ larger than the previous case with free-stress b.c. at the wall for both phases. This discrepancy is readily understood by inspecting the location of the major source of error, as depicted in figure 9.14. As the free-end (slip) phase is modelled with the


Figure 9.14 - (a) Absolute value of the difference of the vertical velocity field $\Delta w^{i}(x, z)=$ $w_{p}^{i}(x, z)-w_{f}^{i}(x, z)$, before and after projection. Only a portion of the right-half domain $(x \in[-1,1], z \in[-3,0])$ is shown. (b)-(d) Three different slices taken at $x_{*}$, as indicated by the dashed white lines in (a). The solution before, $w_{f}^{i}$, and after, $w_{p}^{i}$, projection are superimposed for comparison. The lateral walls are here modeled with a no-slip condition in the pinnedphases.
free-stress condition at $x=1$, at the projection instant, $t^{i}$, the flow solution $\mathbf{v}_{f}^{i}$ will have an uncontrolled non-zero tangential velocity at the wall, $w_{f}^{i}(x=1, z)$. If the no-slip condition is used in the pinned-representation, the solution vector after the projection step, $\mathbf{v}_{p}^{i}$, has a zero tangential velocity by construction, since the pinned-modes satisfy $\left.\hat{w}_{p_{m}}\right|_{x= \pm 1}=0$. It follows that the major source of error is localized along the lateral walls and it is due to the inability to
project between two basis satisfying different boundary conditions at the projection time. The projection algorithm is therefore mathematically ill-posed.
In spite of this evidence, the total error is kept always lower than $0.25 \%$ (at least in the present test case) by the projection method and hence it seems worth investigating how the no-slip condition, which does not neglect the Stokes boundary layer at the lateral walls, therefore being physically more relevant, modifies the damping coefficients of the pinned modes and consequently the overall dissipation during the nonlinear dynamics.
Figure 9.15 shows the contact line elevation and the total energy versus time computed with


Figure 9.15 - (a) Contact line elevation versus time. Comparison between the results obtained by modelling the lateral solid walls during the pinned-phase with a free-stress (black solid lines) and no-slip (light blue dash-dot lines). In the inset in the bottom-right corner, $t^{*}$ denotes the time of arrest computed in the two cases. (b) Total energy density versus time. The black solid lines correspond to the results shown in figure 6 of this Chapter.
the two different wall boundary conditions for the pinned-phase. We observe that the overall dissipation is mainly governed by the dissipation due to the sliding contact line and the static hysteresis (Coulomb-like friction) and the two predicted temporal evolution are indeed close to each other. However, as figure 9.15-(b) highlights, the main difference in the final trend, once the contact line is pinned. For $t>t^{*}$, the fluid bulk motion decays exponentially owing to the viscous dissipation only.

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| $\sigma_{p_{m}}^{F S}$ | $m=1$ | $m=2$ | $m=3$ | $m=4$ | $m=5$ | $m=6$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $-2.3 \times 10^{-4}$ | -0.0017 | -0.0044 | -0.0084 | -0.0136 | -0.0200 |
| $\sigma_{p_{m}}^{N S}$ | $m=1$ | $m=2$ | $m=3$ | $m=4$ | $m=5$ | $m=6$ |
|  | -0.0018 | -0.0031 | -0.0059 | -0.0100 | -0.0153 | -0.0219 |

Table 9.2 - Damping coefficient, $\sigma_{m}$, of the first six pinned-modes with free-stress (FS) and no-slip (NS) condition at the lateral walls. The values are computed for the same setting in figure 6 of this Chapter and recalled at the beginning of $\S 2$ of the present supplementary notes.

## 10 Stick-slip to stick transition induced by contact angle hysteresis: liquid oscillations in U-shaped tubes

In this Chapter 10, the nonlinear decay of oscillations of a liquid column in a $U$-shaped tube is investigated within the theoretical framework of the projection method formalized in Chapter 9. Starting from the full hydrodynamic system supplemented by a phenomenological contact line model, this physics-inspired method uses successive linear eigenmode projections to simulate the relaxation dynamics of liquid oscillations in the presence of sliding triple lines. Each projection is shown to eventually induces a rapid loss of total energy in the liquid motion, thus contributing to its nonlinear damping. A thorough quantitative comparison with experiments by Dollet et al. (2020) (Dollet et al., 2020) demonstrates that, in contradistinction with their simplistic one-degree-of-freedom model, the present approach not only describes well the transient stick-slip dynamics, but it also correctly captures the global stick-slip to stick transition, as well as the secondary bulk motion following the arrest of the contact line, which has been so far overlooked by existing theoretical analyses. This study offers therefore a further contribution to rationalizing the impact of contact angle hysteresis and its associated solidlike friction on the decay of liquid oscillations in the presence of sliding triple lines.

The Chapter is organized as follows. In $\$ 10.1$ we briefly summarize the experimental findings reported by Dollet et al. (2020) and comment on the advantages and limitations of the one-degree-of-freedom (ldof) system employed in their study to model the liquid oscillations. We present the full hydrodynamic system in $\$ 10.2$, while a numerical characterization in terms of oscillation frequencies and damping rates associated with the various dynamical phases is carried out in $\S 10.3$. The projection method is shortly recalled and described in $\S 10.4$. Results and comparison with experiments are given in $\S 10.5$, and final conclusions are outlined in §10.6.

### 10.1 Motivations and objectives

To clarify the role of the wetting properties on the damping of liquid oscillations, Dollet et al. (2020) studied the decay of oscillations of liquid columns in a U-shaped tube. They experimentally showed that in the presence of moving contact lines, oscillations are nonlinearly

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Figure 10.1 - (a) Rescaled interface height, $h=\bar{h} / h_{i n}$, for ethanol in the hydrophobic tube vs time $\bar{t}(s)$. Markers correspond to experiments, while the solid line is an exponential fit $h=\exp \left(-\sigma \omega_{0} \bar{t}\right)$, with the best fitting parameters, $\omega_{0}=11.1 \mathrm{rad} / \mathrm{s}$ and $\sigma=0.0415$. The inset displays a raw space-time diagram showing (as highlighted by yellow dashes) the draining top of the thin film deposited at the first descent. (b) Interface height $\bar{h}(t) \mathrm{mm}$ vs time $\bar{t}(s)$ for water and ethanol in the hydrophobic tube and for an initial elevation $h_{i n}=14.6 \mathrm{~mm}$. (c) Oscillation period, $T(s)$, vs liquid column length $l(m)$, as predicted by the pendulum analogy (black solid line), i.e. $T=2 \pi / \omega_{0}$ with $\omega_{0}^{2}=2 g / l$ and $g$ the gravity acceleration, and as measured experimentally (empty circles) for water in the hydrophobic tube and for an initial elevation $h_{\text {in }}=9.3 \mathrm{~mm}$ (as in panel (e)). (d) Rescaled interface height, $h$, vs time $\bar{t}(s)$, for water in the hydrophobic tube with a fixed liquid column length and at different initial elevation $h_{i n}$. The solid curves correspond to the predictions from equation (10.2) with $\sigma=0.06$ as a free fitting parameter common for all experiments and $\mu$ given by (10.1b). (e) Rescaled height $h$, with $h_{\text {in }}=9.3 \mathrm{~mm}$ vs the rescaled time $t=\omega_{0} \bar{t}$ for water in the hydrophobic tube at different liquid column lengths. The solid curve is given by equation (10.2) with $\sigma=0.06$ as determined from the best fit of the experimental data. (f) Phenomenological law used in the present work to model the apparent dynamic contact angle, $\theta$, vs the non-dimensional contact line speed, $C a \partial \eta / \partial t$, with $C a=v \rho \sqrt{g l / 2} / \gamma$ the Capillary number, $v$ the kinematic liquid viscosity, $\rho$ the liquid density and $\gamma$ the liquid-air surface tension.
damped, with a finite-time arrest and a dependence on initial conditions. Consistently with the theoretical analysis by Viola and Gallaire (2018), they also revealed that contact angle hysteresis can explain this behaviour and quantified the solid-like friction attributable to the contact angle hysteresis.
In the following, we will briefly summarize the experimental findings reported in Dollet et al. (2020), which have inspired the present study. For their experiments, Dollet et al. (2020) used two U-shaped glass tubes, one of which was rendered hydrophilic using plasma treatment,
and the other one hydrophobic by silanization with a silicon reagent. The two straight arms of the tubes, separated by a distance $R \approx 22.5 \mathrm{~mm}$ (the authors provided us with this value in a personal communication), have a constant inner radius $a=8.15 \pm 0.15 \mathrm{~mm}$ (see figure 10.2). They used two liquids, namely ultrapure water and absolute ethanol, whose wetting properties have been characterized by depositing droplets on glass slides treated similarly and simultaneously as the tubes, and slowly injecting or withdrawing liquid from these droplets. From the onset of contact line motion, they measured the values for the advancing, $\theta_{a}$, and receding, $\theta_{r}$, contact angles. Ethanol wetted perfectly the hydrophilic slide; for water, $\theta_{a}=(15 \pm 5)^{\circ}$ and no significant receding contact angle could be measured. On the hydrophobic slide, $\theta_{r}=(68 \pm 10)^{\circ}$ and $\theta_{a}=(93 \pm 2)^{\circ}$ for water, and for ethanol, $\theta_{r}=(28 \pm) 2^{\circ}$ and $\theta_{a}=(34 \pm 2)^{\circ}$.
In order to study the natural decay of the liquid oscillations, they injected a controlled volume in the tube, making a liquid column of length $l$ along the tube centerline. They then plugged one arm with a thin membrane under tension and injected through a flexible tube a controlled volume of air in the resulting trapped air pocket, creating an initial height imbalance $2 h_{\text {in }}$ between the two contact lines in the left and right straight arms of the tube. Lastly, by piercing the membrane with a needle, so as to ensure controlled initial conditions, they could record with a camera the subsequent oscillations of one of the two interfaces.
The relaxation of liquid oscillations in the hydrophilic tube, not reported here for the sake of brevity, was observed to be of exponential nature for both ethanol and water. More complex is instead the scenario when dealing with the hydrophobic tube. For this condition, the relevant results of their study are reported in figure 10.1. Panel (a) shows the relaxation dynamics for ethanol: the oscillations are exponentially damped, without dependence on the initial condition; a visual inspection of the raw space-time diagram highlights the draining top of the thin film deposited at the first descent; during most of the subsequent oscillations, the interface slides over this film.
Panel (b) shows the oscillation decay for both ethanol and water, and for the same liquid column length and initial elevation $h_{i n}$. For both liquids, the oscillation period, $T$, is well predicted by the theoretical dispersion relation, i.e. $T=2 \pi / \omega_{0}$, with $\omega_{0}^{2}=2 g / l$ (see also panel (c)), however, for water, the effect of wetting conditions is striking: despite the larger viscosity of ethanol, water oscillations are much more damped, with a finite-time contact line arrest, $t_{a r r}$, and a dependence of $t_{a r r}$ on the imposed initial condition, $h_{i n}$, as illustrated in panel (d).
To rationalize such nonlinear relaxation dynamics for the contact line, the authors employed the ldof model reminiscent of that of Viola et al. (2018). This simple model relies on two assumptions, namely (i) the tube curvature is neglected and (ii) the flow is hypothesized pluglike. It is difficult to rigorously justify (i), but (ii) appears reasonable as the Stokes boundary layer thickness in these experiments is of the order of $\sqrt{4 \pi v / T} \approx 0.4 \mathrm{~mm} \ll a(=8.15 \mathrm{~mm})$. The ldof model then reads:

$$
\begin{gather*}
\frac{d^{2} h}{d t^{2}}+2 \sigma \frac{d h}{d t}+h+\mu \operatorname{sign}\left(\frac{d h}{d t}\right)=0  \tag{10.1a}\\
h=\frac{\bar{h}}{h_{i n}}, \quad t=\omega_{0} \bar{t}, \quad \sigma=\frac{\bar{\sigma} \omega_{0}}{2 \pi \rho g a^{2}}, \quad \mu=\frac{\gamma\left(\cos \theta_{r}-\cos \theta_{a}\right)}{\rho g a h_{i n}}, \tag{10.1b}
\end{gather*}
$$

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with the initial conditions $h=1$ and $d h / d t=0$ at $t=0$ and with the bar symbol denoting dimensional quantities. Importantly in equation (10.1a), the linear damping coefficient $\sigma$ is considered as a free-fitting parameter. Although equation (10.1a) can be solved numerically or piecewise analytically, in the limit of small damping, i.e. $\sigma \ll 1$ and $\mu \ll 1$, an insightful solution can be obtained by applying the multiple scales method as outlined in Chapter 1 . The elevation $h(t)$ is expanded as $h_{0}+\epsilon h_{1} \ldots$, with $\epsilon \ll 1, h_{0}(t)=(1 / 2) A(T) e^{\mathrm{i} t}+$ c.c. and a slow time scale $\sim \epsilon t$ is introduced. Successively, the imposition of a solvability condition at order $\epsilon$ yields

$$
\begin{equation*}
h(t)=\left[-\frac{2 \mu}{\pi \sigma}+\left(1+\frac{2 \mu}{\pi \sigma}\right) e^{-\alpha t / 2}\right] \cos t \tag{10.2}
\end{equation*}
$$

if $t \leq t_{0}$, and $h=0$ if $t \geq t_{0}$, with $t_{0}=\frac{1}{\sigma} \log [1+(\pi \sigma / 2 \mu)]$ the time of arrest of the contact line oscillations. Equation 10.2 predicts an envelope shape that varies from the classical exponential damping as $\sigma \gg \mu$ (nearly linear dissipation) to a linear decay in time as $\mu \gg \sigma$ (solid-like friction). In spite of the strong oversimplifications, the ldof model predicts fairly well the experimental contact line dynamics once the damping $\sigma$ is fitted from experiments. In the experimental range of liquid column lengths explored, a unique value of $\sigma$, i.e. $\sigma=0.06$ (for water), allowed for a good overall comparison (see figure 10.1(e)).
To conclude this introductory section, one can state that the ldof nonlinear pendulum-like model is capable of reproducing the global features of the relaxation dynamics in the presence of contact angle hysteresis, hence providing a powerful tool to obtain a quick estimation, e.g., of the finite-time arrest.

Nevertheless, a few main drawbacks are worth to be commented on. Preceding the time of arrest, the contact line exhibits some transient stick-slip transitions (visible in figure 10.1(b) and (d)). As discussed in Chapter 9, each time that the contact line transiently reaches a zero speed (see figure 10.1(f)), the contact angle will have to adjust from $\theta_{a}$ to $\theta_{r}$ (or vice versa) while the contact line remains pinned; this dynamical variation obviously requires a certain time-interval to happen. Most importantly, after the time of arrest, the fluid bulk still exhibits oscillations, even if the contact line is pinned. As the latter is now fully fixed, these secondary oscillations are unaffected by nonlinear friction and, therefore, decay exponentially under the effect of pure linear viscous dissipation. Of course, such a global stick-slip to stick (pinned) dynamics cannot be captured by a simplistic ldof model, as it intrinsically calls for a modelization of the many degrees of freedom of the system. Lastly, the 1 dof model requires the fitting of the linear damping, $\sigma$, whose accurate computation can be very subtle. The linear damping englobes multiple dissipative effects: the dissipation occurring in the Stokes boundary laters at the tube walls, the one induced by three-dimensional effects in the curved part of the tube and, particularly, possible extra dissipation sources linked to the contact line motion, such as a dynamical contact angle variation at a non-zero contact line speed (see figure 10.1(f)) which is a ubiquitous feature of similar experiments (Cocciaro et al., 1993; Fiorini et al., 2022; Hocking, 1987; Jiang et al., 2004; Rio et al., 2005; Snoeijer and Andreotti, 2013; Xia and Steen, 2018).
With the aim of building a more refined model so as to overcome these limitations, in the following we will characterize the present $U$-tube dynamics by considering the full hydrody-
namic system of governing equations, to which we will apply the projection method developed in Chapter 9. The most interesting case of water oscillations in the hydrophobic tube will represent our experimental reference condition.

### 10.2 Full Hydrodynamic System


(b)


Figure 10.2 - Sketch of the U-tube configuration. (a) Full three-dimensional geometry (3D). (b) Two-dimensional (2D) view of the centerline plane. The tube radius is assumed constant and denoted by $a$. The length of the liquid column is $l$. $h$ indicates the height difference of the liquid column between the left and right straight channels. $g$ is the gravity acceleration. The advancing and receding dynamic contact angles are, respectively, $\theta_{a}$ and $\theta_{r}$, whereas the static contact angle is labelled as $\theta_{s}$ and it is in general $\neq 90^{\circ}$. (c) If the tube curvature is neglected, the 3D geometry can be reduced to an axisymmetric configuration, by considering only half of the liquid column, of length $l / 2$, and by imposing anti-symmetry conditions at the bottom boundary so as to restore the effect of the gravity term on the missing straight channel.

### 10.2.1 Governing equations

With regards to the experimental setup of Dollet et al. (2020) previously discussed, let us consider a U-shaped tube of radius $a$ and filled with a liquid column of length $l$, as illustrated in figure $10.2(\mathrm{a}, \mathrm{b})$. The section of the tube is assumed constant all over the tube length, a first geometrical approximation already dealt with by Dollet et al. (2020). The geometry of the problem remains intrinsically three-dimensional (3D). Nevertheless, by analogy with the approach employed by Iguchi et al. (1982) and Dollet et al. (2020), in the following, we neglect the tube curvature. This is certainly a strong a priori assumption, which appears worth to be discussed. Appendix is devoted to discussing, at least partially, its justification. Under this

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hypothesis, one may then imagine cutting the tube in half and unfolding it, so as to consider the $z$-axis as straight and only half of the liquid column, of length $l / 2$. At this stage, we have reduced the 3D geometry to an axisymmetric configuration, that can now be more easily described in cylindrical coordinates, Or $\phi z$. The origin of the cylindrical reference system is located at the intersection of the unperturbed free surface at $z=\eta$ with the centerline axis at $r=0$. The effect of the gravity term on the missing half of the domain can be correctly restored by considering proper anti-symmetry conditions on the bottom boundary at $z=-l / 2$ (figure 10.2(c)). The sudden sign switching of the effect of gravity in $z=-l / 2$ is consistent with neglecting the curvature in the U-turn region.
The viscous flow within the U-shape tube is thus governed by the incompressible NavierStokes equations

$$
\begin{equation*}
\nabla \cdot \mathbf{u}=0, \quad \frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u} \cdot \nabla) \mathbf{u}+\nabla p-\frac{1}{R e} \Delta \mathbf{u}=-1 \hat{\mathbf{e}}_{z} \tag{10.3}
\end{equation*}
$$

which are made nondimensional by using the container's characteristic length $l$ and the velocity $\sqrt{g l / 2}$ (figure 10.2). Consequently, the Reynolds number is defined as $R e=\frac{\sqrt{g(l / 2)^{3}}}{v}$ and the term $-1 \hat{\mathbf{e}}_{z}$ denotes the nondimensional gravity acceleration. In equation (10.3), $p(r, z, t)$ is the pressure field, whereas $\mathbf{u}(r, z, t)=\{u, w\}^{T}$ is the velocity field, with $u$ and $w$ the radial and axial velocity, respectively. Note that the dynamics is assumed axisymmetric and such assumption will be maintained throughout the manuscript. At the free surface, $z=\eta$, kinematic and dynamic boundary conditions hold,

$$
\begin{gather*}
\frac{D(\eta-z)}{D t}=\frac{\partial \eta}{\partial t}+u \frac{\partial \eta}{\partial r}-w=0  \tag{10.4a}\\
{\left[-p \mathbf{I}+\frac{1}{R e}\left(\nabla \mathbf{u}+\nabla^{T} \mathbf{u}\right)-\frac{1}{B o} \kappa(\eta) \mathbf{I}\right] \cdot \mathbf{n}=\mathbf{0}} \tag{10.4b}
\end{gather*}
$$

where $D / D t$ is the material derivative, $\mathbf{n}=\left(1+\eta_{r}^{2}\right)^{-1 / 2}\left\{-\eta_{r}, 1\right\}^{T}$ is unit vector normal to the interface, and $\kappa$ is the free surface curvature, $\kappa(\eta)=\left[\eta_{r r}+r^{-1} \eta_{r}\left(1+\eta_{r}^{2}\right)\right]\left(1+\eta_{r}^{2}\right)^{-3 / 2}$. The Bond number is defined as $B o=\frac{\rho g a^{2}}{\gamma}\left(\frac{l / 2}{a}\right)^{2}$, with $\gamma$ designating the air-liquid surface tension. As anticipated above, the restoring effect of the missing half of the tube, is reintroduced by imposing anti-symmetry conditions for $u$ and $w$ at the bottom boundary (see figure 10.2(c)). More precisely, we impose

$$
\begin{equation*}
u=\frac{\partial w}{\partial z}=0 \quad \text { at } z=-1 \tag{10.5}
\end{equation*}
$$

Moreover, owing to the axisymmetric assumption, the axis boundary condition imposes

$$
\begin{equation*}
u=\frac{\partial w}{\partial r}=0 \quad \text { at } r=0 \tag{10.6}
\end{equation*}
$$

### 10.2.2 Treatment of the sidewall: a macroscopic depth-dependent slip-length model

At the lateral wall, we adopt a slip-length model, thus assuming that the fluid speed relative to the solid wall is proportional to the viscous stress (Lauga et al., 2007; Navier, 1823) and that,
together with the no-penetration condition, provides the boundary conditions

$$
\begin{equation*}
u=0, \quad w+l_{s}(z) \frac{\partial w}{\partial x}=0 \quad \text { at } r=\frac{a}{l / 2} . \tag{10.7}
\end{equation*}
$$

Such a condition is indeed needed in order to regularize the stress singularity at the moving contact line (Davis, 1974; Huh and Scriven, 1971). It was hypothesized by Miles (1990) and Ting and Perlin (1995) that the phenomenological macroscopic slip length appearing in equation (10.7) should not be assumed constant along the wall, but rather spatially dependent on the position along the lateral wall and vanishing at a certain distance away from the contact line, where the flow obeys the no-slip condition. For this reason, we employ here a depthdependent slip length model as proposed by Bongarzone and Gallaire (2022), which has been shown to correctly estimate the linear dissipation occurring in the Stokes boundary layers at the lateral solid walls (see Appendix for further validations specific to present case). Briefly, we postulate that the slip length $l_{s}(z)$ is described by the exponential law

$$
\begin{equation*}
l_{s}(z)=l_{c l} \exp \left(-\frac{z}{\delta} \log \left(\frac{l_{\delta}}{l_{c l}}\right)\right), \quad z \in[-H, 0] \tag{10.8}
\end{equation*}
$$

In equation (10.8), $l_{c l}$ is the slip-length value at the contact line, $r=a /(l / 2)$ and $z=0$, whereas $l_{\delta}$ is its value at a distance $\delta$ below the contact line, $r=a /(l / 2)$ and $z=-\delta$, with $\delta$ representing the size of the slip region (Ting and Perlin, 1995). In principle, $l_{c l}, l_{\delta}$ and $\delta$ are all free parameters. However, keeping in mind that, macroscopically speaking, one aims at mimicking a stress-free condition in the vicinity of the contact line and a no-slip condition after a certain distance $\delta$, the natural choice is $l_{c l} \gg 1\left(\sim 10^{2} \div 10^{4}\right)$ and $l_{\delta} \ll 1\left(\sim 10^{-4} \div 10^{-6}\right)$. The range of values proposed in brackets is based on the sensitivity analysis reported in Bongarzone and Gallaire (2022), whereas the definition of the slip region penetration depth, $\delta$, as postulated by Miles (1990), is assumed of the order of the non-dimensional Stokes boundary layer thickness, i.e. $\delta \approx(l / 2)^{-1} \delta_{S t}=(l / 2)^{-1} \sqrt{2 v / \omega}$.

### 10.2.3 Phenomenological contact angle model and static meniscus

Lastly, to model the contact line motion, $z=\eta$ and $r=a /(l / 2)$, we include the phenomenological law of figure $10.1(\mathrm{f})$, which describes the nonlinear contact angle dynamic as a function of the contact line speed,

$$
\begin{equation*}
\frac{\partial \eta}{\partial r}= \pm \cot \theta, \quad \theta-\theta_{s}=\alpha C a \frac{\partial \eta}{\partial t}+\frac{\Delta}{2} \operatorname{sign}\left(\frac{\partial \eta}{\partial t}\right) \quad \text { (Hocking+hysteresis), } \tag{10.9}
\end{equation*}
$$

with $C a=v \rho \sqrt{g l / 2} / \gamma$ the Capillary number and with the value of $\alpha$ that will be discussed and specified in the next section. Note that this model has already been used in Chapter 9 and it results from a combination of the linear Hocking's law (Hocking, 1987), of slope $\alpha$, and a static contact angle hysteresis of range $\Delta$. In the rest of the paper, we will naively assume that the advancing and receding phases are completely symmetric and that the hysteresis range is centered around $\theta_{s}$, i.e. $\theta^{+}=\theta_{a}-\theta_{s}=\Delta / 2$ and $\theta^{-}=\theta_{r}-\theta_{s}=-\Delta / 2$, while being aware that the

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Water, $a=8.15 \mathrm{~mm}, \theta_{s}=80.5^{\circ}$


Figure 10.3 - Shape of the dimensional static meniscus, $\bar{\eta}_{0}$, computed numerically for $\theta_{s}=$ $\left(\theta_{a}+\theta_{r}\right) / 2=(93+68) / 2=80.5^{\circ}$ and for three different liquid column length, $l(\mathrm{~cm})$.
advancing and receding contact line dynamics are generally characterized by different value of $\alpha$, i.e. $\alpha_{A} \neq \alpha_{R}$ (Cocciaro et al., 1993; Cox, 1986; Dussan, 1979; Gennes, 1985; Rio et al., 2005; Voinov, 1976).
In the limit of small oscillation amplitudes and small static contact angle hysteresis, the fully nonlinear governing equations (10.3) together with their boundary conditions (10.4a)-(10.9) can be linearized around the rest state, characterized by zero velocity and pure hydrostatic pressure. With regards to the experiments by Dollet et al. (2020) for water in the hydrophobic tube, the measured advancing and receding contact angles are, respectively, $\theta_{a}=93^{\circ}$ and $\theta_{r}=68^{\circ}$. If we hypothesize the equilibrium angle $\theta_{s}$ to be the averaged value of $\theta_{a}$ and $\theta_{r}$, this amounts to $\theta_{s}=80.5^{\circ} \neq 90^{\circ}$, meaning that the static free surface is not flat. We, therefore, linearize the system of equations around an initially curved static meniscus, whose resulting axisymmetric shape, reported in figure 10.3, is computed as the solution of the following static equation:

$$
\begin{equation*}
\eta_{0}=\frac{1}{B o}\left[\frac{\eta_{0, r r}+r^{-1} \eta_{0, r}\left(1+\eta_{0, r}^{2}\right)}{\left(1+\eta_{0, r}^{2}\right)^{3 / 2}}\right] \text {, with }\left.\frac{\partial \eta_{0}}{\partial r}\right|_{r=0}=0,\left.\quad \frac{\partial \eta_{0}}{\partial r}\right|_{r=a /(l / 2)}=\cot \theta_{s} \text {, } \tag{10.10}
\end{equation*}
$$

Eq. (10.10) is nonlinear in $\eta_{0}$ and can be solved numerically using an iterative Newton method as described in Appendix A. 1 of Viola et al. (2018).

### 10.3 Natural properties of the system

Notwithstanding the linearization of the governing equations around the rest state, the system is still nonlinear owing to the hysteretic contact angle model (10.9). Nevertheless, it appears intuitive that the underlying contact line motion can be split into two distinct dynamical


Figure 10.4 - Eigenvalue spectrum associated with the two contact line boundary conditions, i.e. pinned (green markers) and free (blue markers), computed numerically by solving the generalized eigenvalue problem (10.16). For the case of a free contact line condition, the calculation here reported has been performed by imposing a value of $\alpha=0$. Both spectra are computed for a liquid column length $l=14.6 \mathrm{~cm}$.


Figure 10.5 - (a) Eigen-interface associated with the U-tube free mode computed in 10.4. The free surface dynamics in the free-phase consists of an upward-downward oscillation of a flat interface. (b) Eigen-interface associated with U-tube pinned mode computed in 10.4. The surface dynamics in the pinned-phase consists instead of an interface oscillating with a bell-like shape whose edges are anchored at the wall.
phases, namely a pinned-phase, described by the condition

$$
\begin{equation*}
\frac{\partial \eta}{\partial t}=0 \quad(\text { pinned }- \text { phase }) \tag{10.11}
\end{equation*}
$$

and a free-phase with

$$
\begin{equation*}
\frac{\partial \eta}{\partial r}+\alpha C a \frac{\partial \eta}{\partial t}=-\theta^{ \pm} \quad \text { (free-phase) } \tag{10.12}
\end{equation*}
$$

both evaluated at $r=a /(l / 2)$. The non-homogeneous term in the right-hand side of equation (10.12) will be dealt with within the formalism of the projection method. Let us ignore this term for the moment by rewriting

$$
\begin{equation*}
\frac{\partial \eta}{\partial r}+\alpha C a \frac{\partial \eta}{\partial t}=0 \tag{10.13}
\end{equation*}
$$

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Then, the system of governing equations closed by these two boundary conditions, taken independently, translate into two separated fully linear homogeneous problems, that can be both written in the form

$$
\begin{equation*}
\mathscr{B}_{f, p} \frac{\partial}{\partial t} \mathbf{q}_{f, p}=\mathscr{A}_{f, p} \mathbf{q}_{f, p} . \tag{10.14}
\end{equation*}
$$

with $\mathbf{q}_{f, p}=\left\{\mathbf{u}_{f, p}, p_{f, p}, \eta_{f, p}\right\}^{T}$ the state vector. The symbolic expressions of the mass matrix $\mathscr{B}_{f, p}$ and the stiffness matrix $\mathscr{A}_{f, p}$ are explicitly given in Chapter 9 , while the subscripts $f, p$ are here used to designate either the free $(f)$ or the pinned $(p)$ phase. By introducing the ansatz

$$
\begin{equation*}
\mathbf{q}_{f, p}=\hat{\mathbf{q}}_{f, p} e^{\lambda_{f, p} t}+c . c ., \tag{10.15}
\end{equation*}
$$

with $\lambda_{f, p}=-\sigma_{f, p}+\mathrm{i} \omega_{f, p}$, equation (10.14) reduces to the following generalized eigenvalue problem

$$
\begin{equation*}
\lambda_{f, p} \mathscr{B}_{f, p} \hat{\mathbf{q}}_{f, p}=\mathscr{A}_{f, p} . \tag{10.16}
\end{equation*}
$$

Matrices $\mathscr{A}_{f, p}$ and $\mathscr{B}_{f, p}$ are numerically discretized by means of a Chebyshev collocation method implemented in Matlab in the same fashion of Bongarzone and Gallaire (2022); Bongarzone et al. (2021c); Viola et al. (2018); Viola and Gallaire (2018); the resulting eigenvalue problem is also solved in Matlab via the built-in eigs function.
The eigenvalue spectrum associated with the solution of the two independent eigenvalue problems is reported in figure 10.4. This figure shows, for both wetting phases, a spectrum that contains two families of oscillating natural modes, namely a free/pinned U-tube mode and free/pinned capillary-gravity waves. However, these waves oscillate at a much larger frequency, at least ten times higher, than the fundamental U-tube mode, and are typically more damped than the U-tube mode. The latter mode, with its dynamical properties and structure, displayed in figure 10.5, is, therefore, the mode that is expected to govern the dynamics.
Hence, in the next two sub-sections we will carefully comment on the eigenvalue properties of such U-tube modes, tackled separately in the two dynamical phases. For simplicity, we will start from the pinned-phase, which appears easily describable from a numerical perspective. Successively, we will handle the free-phase, whose description hinges on the subtle modelling of the moving contact line and slip length conditions.

### 10.3.1 Pinned-phase

The dependence of the oscillation period and of the damping coefficient on the liquid column length for the U-tube pinned mode, as numerically computed, is shown in figure 10.6. Only one experimental value has been reported by Dollet et al. (2020) (in their Supplementary Material) and it seems in agreement with our trend, which is also reminiscent of that displayed in figure 10.1 (c), although no analytical dispersion relation exists for a pinned contact line.
More experimental values are available with regard to the damping coefficient. Although the experimental procedure followed by Dollet et al. (2020) in measuring these values does not allow for high accuracy, an overall fair agreement is found when compared with our numerical estimates.

In this regard, it is important to realize that a pinned contact line condition is mathematically fully compatible with a no-slip wall condition, i.e. no stress singularity needs to be resolved at the contact line, hence allowing one for a precise numerical estimation of the damping. If we ignore experimental errors and ensure numerical convergence, the main possible source of disagreement with these experiments is attributable to free surface contamination or threedimensional (3D) effects, overlooked by our ideal axisymmetric model, which neglects the tube curvature. To be sure that 3D effects are not important, in Appendix, we perform a full 3D eigenvalue calculation so as to refine the numerical values reported in figure 10.6. This calculation proves 3D corrections to be insignificant.


Figure 10.6 - Dimensional oscillation period, $T$, and damping coefficient, $\bar{\sigma}$, versus the water column length and associated with a pinned contact line dynamics of the fundamental U-tube mode. Green diamonds: values computed fully numerical eigenvalue calculation. White circles: values measured experimentally as reported in Dollet et al. (2020).

### 10.3.2 Free-phase

Ignoring dynamical contact angle variation: $\alpha=0$
In applying the ldof model, Dollet et al. (2020) used a non-dimensional linear damping coefficient $\sigma$ fitted from experiments and whose best-fit value amounts to 0.06 . This coefficient is difficult to estimate precisely, as it englobes several contributions, among which is the dissipation occurring in the laminar Stokes boundary layers at the lateral walls.
The numerical approach here employed, based on the slip length model previously discussed, provides a tool to compute the dissipation associated with the Stokes boundary layers (see Bongarzone and Gallaire (2022) for further details). By analogy with the pinned case, the dependence of the oscillation period and of the damping coefficient on the liquid column

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Figure 10.7 - (a) Dimensional oscillation period, $T$, and (b) damping coefficient, $\bar{\sigma}$, versus the water column length, $l(\mathrm{~cm})$ and associated with a free contact line dynamics of the fundamental U-tube mode. Blue diamonds: values computed fully numerical eigenvalue calculation by accounting for the variable slip length model discussed in equation (10.13) with $\alpha=0$. White circles in (a): values measured experimentally as reported in Dollet et al. (2020). The experimental range investigated in Dollet et al. (2020) is indicated by the grey arrow in (b). Within this range, the damping coefficient is nearly constant with the tube length.
length for the U-tube free mode is shown in figure 10.7. The numerics slightly overestimate the oscillation period, but overall it is in good agreement with the experiments. The fact that the experimental data are better described by the theoretical dispersion relation, which does not account for viscous dissipation, is however counter-intuitive. Pure viscous dissipation should indeed introduce a viscous correction to the natural frequency, which should result in a diminished value or, equivalently, in a higher oscillation period $T$. This may suggest that there is a second effect counteracting and compensating for such a viscous correction to the natural frequency. The Appendix shows that the curved part of the U-tube has an influence on the natural dynamics that can explain this argument.
When looking at the damping coefficient, we observe that within the experimental range of liquid column length, $l(\mathrm{~cm})$, considered, the damping $\sigma$ does not vary much with $l$, thus explaining why a single value of $\sigma$ fitted from experiments can allow a good match with those measurements. The present numerical calculation for the damping is also compared to the analytical estimate developed in Appendix, and that validates the numerical scheme.
Unfortunately, when taking the non-dimensional averaged value in the experimental range of length, this amounts to $\sigma \approx 0.027$, which is less than half the one needed for a good agreement with the data. We precise that the averaged value is computed as $\sigma=n_{i}^{-1} \sum_{i}^{n_{i}} \bar{\sigma}_{i} \sqrt{l_{i} / 2 g}$, with
$n_{i}$ the number of lengths $l$ used to sample the experimental range.
As discussed in Appendix, three-dimensional effects related to the tube curvature can produce an increase in the damping of a few percentages, not sufficient to explain such a mismatch. The extra dissipation missing in the modelization of the free phase must be therefore linked to the contact line dynamics.

## Accounting for dynamical contact angle variation: $\alpha \neq 0$



Figure 10.8 - Same as in figure 10.7 (here in $\sigma$-log scale), but with the light blue crosses indicating the values computed by also accounting for extra contact line dissipation produced by Hocking's law (Bongarzone et al., 2021c; Hocking, 1987) with $\alpha=200 \mathrm{rad}$ ). Within this range, the damping coefficient is nearly constant with the tube length, $l$, even for $\alpha=200 \mathrm{rad}$. The average value in this range is $\sigma \approx 0.06$, which matches the one used in figure 10.1 and obtained from the best-fit of the experiments.

In the experimental conditions considered, the extra contact line dissipation is well described by a linear damping coefficient. Hence, adopting a linear law for the dynamic contact angle variations with the contact line speed appears as the simplest and most natural choice. We, therefore, reintroduce the contact line parameter that characterizes the Hocking law, i.e. $\alpha \neq 0$. Recalling the contact line condition for the free-phase (10.13), one can see how a value of $\alpha=0$ would correspond to a contact line sliding over the solid substrate with a constant and zero slope (dashed lines in figure 10.1). On the other hand, the pinned condition (10.11) is nothing more than a limiting case of equation (10.12) with $\alpha \rightarrow+\infty$. We are supposing here to be in an intermediate situation where $\alpha$, sometimes also referred to as friction coefficient (Hamraoui et al., 2000) or mobility parameter $M$ (Xia and Steen, 2018), assumes a finite value

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different from zero.
Let us first blindly consider $\alpha$ as a free fitting parameter. A value of $\alpha=200 \mathrm{rad}$ leads to a non-dimensional averaged (in the experimental range of figure 10.8) damping coefficient of $\sigma=\bar{\sigma} \sqrt{l / 2 g} \approx 0.06$, which is exactly the value we need. If this procedure shows that a simple linear dynamic contact line model is sufficient to explain the missing dissipation, one can wonder whether the value of $\alpha$ used is admissible for the experimental condition discussed here.
Hamraoui et al. (2000) have studied the kinetics of capillary rise of pure water and pure ethanol as well as mixtures thereof that, under static conditions, wet glass capillary tubes in both dry and prewetting wall conditions. Specifically, they have postulated a dynamic contact angle term that is linearly dependent on the velocity of the capillary rise and whose correction, in this linear approximation, takes on the form of a three-phase line friction coefficient, $M$, equivalent to our parameter $\alpha$, up to a proper dimensionalization factor. The value of $M$ for ethanol, water and a water-ethanol mixture is reported in table 10.1.

| liquid | $\rho\left(\mathrm{kg} / \mathrm{m}^{3}\right)$ | $\gamma(\mathrm{N} / \mathrm{m})$ | $v\left(\mathrm{~m}^{2} / \mathrm{s}\right)$ | $M(\mathrm{~Pa} \mathrm{~s})$ | $\bar{\alpha}=\frac{M}{\gamma}(\mathrm{~s} / \mathrm{m})$ | $\alpha=\bar{\alpha} \frac{\gamma}{v \rho}(\mathrm{rad})$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| water | 1000 | 0.072 | $1.0 \times 10^{-6}$ | 0.2 | 6.25 | 200 |
| mixture | 983 | 0.050 | $1.0 \times 10^{-6}$ | 0.14 | 2.8 | 140 |
| ethanol | 786 | 0.022 | $1.4 \times 10^{-6}$ | 0.04 | 1.82 | 36 |

Table 10.1 - Value of the non-dimensional contact line parameter $\alpha$ for water, water-ethanol mixture and pure ethanol as measured by Hamraoui et al. (2000). The dimensional value of the friction coefficient $M$ (denoted by $\beta$ in their study) is here converted in the dimensional, $\bar{\alpha}$, and non-dimensional, $\alpha$, contact line parameter.

Particularly relevant to our study is the value measured by Hamraoui et al. (2000) for pure water amounts to $M=0.2$ Pa s, which translates into $\alpha=200$ rad, hence matching precisely the value necessary to match the data. As a side comment, the use of the coefficient $\alpha$ also produces an increase in the natural frequencies, thus bringing the numerics closer to the experimental values.
Through this careful comparison with experiments by Hamraoui et al. (2000) and Dollet et al. (2020), we have been capable of quantifying numerically the natural properties of the system in the two dynamical phases of interest, handled independently. All our estimates and hypotheses seem consistent with these measurements.
The idea is now to combine the two separated descriptions for the pinned-phase and freephase, so as to account for a dynamic change in the contact line boundary conditions and predict the nonlinear relaxation dynamics. This is done in the next section by employing the projection algorithm.


Figure 10.9 - (a) Axisymmetric meniscus modes associated with the free-phase and (b) with the pinned-phase. In (a), the angle, measured clockwise from the wall to the surface is 1 , whereas the contact line elevation is $F_{0}$. In (b), the angle is $1 / F_{0}$, whereas the contact line elevation is 1 . (c) Real part of the eigen-interface associated with the free and (d) pinned $U$-tube modes, with the corresponding eigenvalues, $\lambda_{f_{0}}=-\sigma_{f_{0}}+\mathrm{i} \omega_{f_{0}}$ and $\lambda_{p_{0}}=-\sigma_{p_{0}}+\mathrm{i} \omega_{p_{0}}$ reported on top. The free mode is normalized such that the contact line elevation is 1 , while the pinned mode is normalized such that the slope at the wall is 1 . For completeness, in (c), we have also reported the interface shape when $\alpha=0$ (thin blue line) as shown in figure 10.5(a). (e)-(i) Real part of the eigen-interface associated with the five least damped free and (j)-(n) pinned capillary-gravity waves. The same normalization as in (c) and (d) is employed.

### 10.4 Projection method

### 10.4.1 General formalism

A detailed step-by-step description of the projection algorithm has been already provided in Chapter 9. In this section, we recall the salient points of the method and we comment on the few differences intrinsic to specific dynamics of the problem here considered.

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When the contact line motion is schematized using Hocking's law amended with a static hysteresis range, we can identify two well-distinct phases of the dynamics, one in which the angle varies linearly with a slope $\alpha$ as a function of the contact line speed, Ca $\eta / \partial t$ (Hocking's linear law) and one in which the contact line is pinned at a certain elevation with zero velocity (static hysteresis) and the angle changes from $\theta_{s}+\theta^{+}$to $\theta_{s}+\theta^{-}\left(\Delta=\theta^{+}-\theta^{-}\right)$or vice versa. We denote these two phases as free, ${ }_{f}$, and pinned, $p$, phase, respectively.
The solution in these two phases is then expressed as the sum of the corresponding particular static solution (meniscus mode), $\mathbf{q}_{f_{s}}$ and $\mathbf{q}_{p_{s}}$ (the subscripts ${ }_{f, p_{s}}$ stand for free-static or pinnedstatic), and a truncated basis of linear eigenmodes, $\hat{\mathbf{q}}_{f_{n}}$ and $\hat{\mathbf{q}}_{p_{m}}$, weighted by their unknown amplitudes:

$$
\begin{gather*}
\mathbf{q}_{f}=\underbrace{\theta^{ \pm} \mathbf{q}_{f_{s}}}_{\text {free-end meniscus mode }}+\underbrace{\left(A_{0} \hat{\mathbf{q}} f_{0} e^{\lambda_{f_{0}}\left(t-T_{f}\right)}+c . c .\right)}_{\text {free-end U-tube mode }}+\underbrace{\left(\sum_{n=1}^{N_{f}} A_{f_{n}} \hat{\mathbf{q}}_{f_{n}} e^{\lambda_{f_{n}}\left(t-T_{f}\right)}+c . c .\right)}_{\text {free-end capillary-gravity waves }}  \tag{10.17a}\\
\mathbf{q}_{p}=\underbrace{e_{f p} \mathbf{q}_{p_{s}}}_{\text {pinned-end meniscus mode }}
\end{gather*}
$$

(10.17b)

All these ingredients are visually summarized in figure 10.9. As described in the previous section and in contradistinction with the two-dimensional system of Chapter 9, the present U-tube dynamics is characterized by two families of oscillating natural modes, namely a free/pinned U-tube mode and free/pinned capillary-gravity waves. However, these waves oscillate at a much larger frequency and are more damped than the U-tube modes. Accounting for them in the algorithm is useful if one is interested in capturing fast transients, but with the purpose of modelling the global dynamical features of the system, their inclusion in the analysis is not strictly necessary. Hereinafter we will ignore the capillary-gravity waves, and we will only retain the dominant free and pinned U-tube natural modes described in $\$ 10.3$ and here denoted by $\hat{\mathbf{q}}_{f_{0}}$ (free) and $\hat{\mathbf{q}}_{p_{0}}$ (pinned), with amplitudes $A_{0}$ and $B_{0}$, and eigenvalues $\lambda_{f_{0}}=-\sigma_{f_{0}}+\mathrm{i} \omega_{f_{0}}$ and $\lambda_{p_{0}}=-\sigma_{p_{0}}+\mathrm{i} \omega_{p_{0}}$, respectively.

Including a meniscus mode in the solution form (10.17a) associated with the free-phase, i.e. $\mathbf{q}_{f_{s}}$, is necessary in order to properly deal with the non-homogeneous term in the right-handside of the contact line condition (10.12). The particular solution resulting from this static forcing term, $-\theta^{ \pm}$, consists in a static meniscus modification $\eta_{f_{s}}$ (with $\mathbf{u}_{f_{s}}=\mathbf{0}$ ) that satisfies the linearized meniscus equation

$$
\begin{equation*}
\eta_{f_{s}}-\frac{1}{B o}\left[\frac{1}{\left(1+\eta_{0, r}^{2}\right)^{3 / 2}} \frac{\partial^{2} \eta_{f_{s}}}{\partial r^{2}}+\frac{\left(1+\eta_{0, r}^{2}\right)}{\left(1+\eta_{0, r}^{2}\right)^{5 / 2}} \frac{1}{r} \frac{\partial \eta_{f_{s}}}{\partial r}\right]=\text { constant, with }\left.\frac{\partial \eta_{f_{s}}}{\partial r}\right|_{r=a /(l / 2)}=-\theta^{ \pm} \tag{10.18}
\end{equation*}
$$

with the terms in brackets representing the first-order variation of the nonlinear curvature linearized around the static meniscus $\eta_{0}$ and applied to $\eta_{f_{s}}$. For the convenience of notation,
we encourage the reader to note that, in equation (10.18), we actually impose $\partial \eta_{f_{s}} / \partial r=-1$ instead of $-\theta^{ \pm}$, while keeping the term $\theta^{ \pm}$explicit in front of the particular solution in (10.17a). In Chapter 9, the constraint on the conservation of mass was used to set the value of the constant appearing in (10.18), by imposing $\int_{z=\eta_{0}} \eta_{f_{s}} \mathrm{~d} s=0$. Here, as we consider only half of the domain, this constraint must be relaxed, and the value of the constant let free. In practice, this is done by solving for $\left(\eta_{f_{s}}+\right.$ constant $)$, instead of for $\eta_{f_{s}}$ itself.

The pinned-condition (10.11) is homogeneous and it is explicitly accounted for in the corresponding eigenvalue problem. However, the condition $\partial \eta / \partial t$ also allows for a static particular solution with $\eta_{p_{s}}=$ constant at the contact line $r=a /(l / 2)$ (and with $\mathbf{u}_{p_{s}}=\mathbf{0}$ ). The meniscus mode for the pinned-phase is therefore computed as $\eta_{p_{s}}=\eta_{f_{s}} / F_{0}$, with $F_{0}$ the value of $\eta_{f_{s}}$ at the wall $r=a /(l / 2)$, so as to have a unitary value at $r=a /(l / 2)$ (see figure 10.9). This unitary value is weighted by the contact line elevation $e_{f p}$ in (10.17b), with $e_{f p}$ fixed during the pinned-phase and obtained as an output of the algorithm.

### 10.4.2 Workflow of the method



Figure 10.10 - Workflow of the projection algorithm (from (a) to (c)).

A visual workflow of the algorithm is illustrated in figure 10.10. Let us suppose to initialize the system in the upper free-phase (panel (a)) by assigning the amplitude of the free U-tube mode, $A_{0}$, at $t-T_{f}=0$. The system is let evolve in time according to (10.17a). When the contact line speed reaches the null value, we have the first transition, i.e. from free to pinned. At this time instant, $t=T_{p}$, we require the continuity of all variables of the system, i.e. $\mathbf{q}_{p}(0)=\mathbf{q}_{f}\left(T_{p}-T_{f}\right)$. This corresponds to imposing

$$
\begin{equation*}
\theta^{+} \mathbf{q}_{f_{s}}+\left(A_{0} \hat{\mathbf{q}}_{f_{0}} e^{\left(-\sigma_{f_{0}}+\mathrm{i} \omega_{f_{0}}\right)\left(T_{p}-T_{f}\right)}+c . c .\right)=e_{f p} \mathbf{q}_{p_{s}}+\left(B_{0} \hat{\mathbf{q}}_{p_{0}}+c . c .\right) \tag{10.19}
\end{equation*}
$$

which, using the fact that the contact line elevation at the end of the free-phase reads $\left(\hat{\eta}_{f_{0}}=1\right.$ at $r=a /(l / 2))$

$$
\begin{equation*}
e_{f p}=\theta^{+} F_{0}+\left(A_{0} e^{\left(-\sigma_{f_{0}}+\mathrm{i} \omega_{f_{0}}\right)\left(T_{p}-T_{f}\right)}+c . c .\right), \text { and } \eta_{p_{s}}=\eta_{f_{s}} / F_{0} \tag{10.20}
\end{equation*}
$$

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can be conveniently rewritten as

$$
\begin{equation*}
B_{0} \hat{\mathbf{q}}_{p_{0}}+c . c .=A_{0}\left(\hat{\mathbf{q}}_{f_{0}}-\mathbf{q}_{p_{s}}\right) e^{\left(-\sigma_{f_{0}}+\mathrm{i} \omega_{f_{0}}\right)\left(T_{p}-T_{f}\right)}+c . c .=\mathbf{f}_{f p} \tag{10.21}
\end{equation*}
$$

where the resulting term on the right-hand side is fully known.
The amplitude of the U-tube mode pertaining to the next pinned-phase, $B_{0}$, still unknown at this stage, is computed by projecting, with respect to a specific weighted inner product, the final-time free solution, $\mathbf{f}_{f p}$, on the initial-time pinned solution as

$$
\begin{equation*}
B_{0}=<\hat{\mathbf{q}}_{p_{0}}^{\dagger}, \mathbf{f}_{f p}>_{E} . \tag{10.22}
\end{equation*}
$$

with $\hat{\mathbf{q}}_{p_{0}}^{\dagger}$ the adjoint U-tube pinned-mode.
We are now in the pinned-phase (panel (b)). The initial contact angle is $\theta_{s}+\Delta / 2=\theta_{s}+\theta^{+}$, and the time-evolution of the system is described by (10.17b). The contact angle progressively changes with a fixed contact line elevation $e_{f p}$ and once it reaches the value $\theta_{s}-\Delta / 2=\theta_{s}+\theta^{-}$, the second transition occurs. We impose again the continuity of the flow variables, i.e. $\mathbf{q}_{f}(0)=$ $\mathbf{q}_{p}\left(T_{f}-T_{p}\right)$,

$$
\begin{equation*}
e_{f p} \mathbf{q}_{p_{s}}+\left(B_{0} \hat{\mathbf{q}}_{p_{0}} e^{\left(-\sigma_{p_{0}}+\mathrm{i} \omega_{p_{0}}\right)\left(T_{f}-T_{p}\right)}+c . c .\right)=\theta^{-} \mathbf{q}_{f_{s}}+\left(A_{0} \hat{\mathbf{q}}_{f_{0}}+c . c .\right) \tag{10.23}
\end{equation*}
$$

with

$$
\begin{equation*}
\theta^{-}=e_{f p} / F_{0}+\left(B_{0} e^{\left(-\sigma_{p_{0}}+\mathrm{i} \omega_{p_{0}}\right)\left(T_{f}-T_{p}\right)}+c . c .\right), \tag{10.24}
\end{equation*}
$$

so that equation (10.23) can be rearranged as

$$
\begin{equation*}
A_{0} \hat{\mathbf{q}}_{f_{0}}+c . c .=B_{0}\left(\hat{\mathbf{q}}_{p_{0}}-\mathbf{q}_{f_{s}}\right) e^{\left(-\sigma_{p_{0}}+\mathrm{i} \omega_{p_{0}}\right)\left(T_{f}-T_{p}\right)}+c . c .=\mathbf{f}_{p f .} \tag{10.25}
\end{equation*}
$$

We thus project the final-time pinned solution on the initial-time free solution, so as to determine the new amplitude $A_{0}$.

$$
\begin{equation*}
A_{0}=<\hat{\mathbf{q}}_{f_{0}}^{\dagger}, \mathbf{f}_{p f}>_{E} . \tag{10.26}
\end{equation*}
$$

with $\hat{\mathbf{q}}_{p_{0}}^{\dagger}$ the adjoint U-tube free-mode.
The system enters the lower free-phase (panel (c)) and the cycle is repeated over again. Each projection eventually induces a rapid loss of total energy in the liquid motion and contributes to its nonlinear damping. After a few cycles, the inertia of the oscillating liquid column will be no more sufficient to surpass the static solid-like friction and the system will get trapped in the pinned-phase. The secondary fluid bulk motion following the arrest of the contact line will decay exponentially under the effect of the linear viscous dissipation characteristic of the pinned dynamics.

### 10.4.3 E-norm inner product and definition of adjoint modes

We note that, owing to the axisymmetric configuration, the inner product employed in this context differs from that used in Chapter 9:

$$
\begin{equation*}
<\mathbf{w}, \mathbf{u}>_{E}=\int_{V} \overline{\mathbf{u}}_{\mathbf{w}} \mathbf{u}_{\mathbf{v}} r \mathrm{~d} r \mathrm{~d} z+\int_{z=\eta_{0}(r)}\left[\bar{\eta}_{\mathbf{w}} \eta_{\mathbf{v}}+\frac{1}{B o}\left(\frac{1}{\left(1+\eta_{0, r}^{2}\right)^{3 / 2}}\right) \frac{\partial \bar{\eta}_{\mathbf{w}}}{\partial r} \frac{\partial \eta_{\mathbf{v}}}{\partial r}\right] r \mathrm{~d} r \tag{10.27}
\end{equation*}
$$

where $\mathbf{v}=\left\{\mathbf{u}_{\mathbf{v}}, p_{\mathbf{v}}, \eta_{\mathbf{v}}\right\}^{T}$ and $\mathbf{w}=\left\{\mathbf{u}_{\mathbf{w}}, p_{\mathbf{w}}, \eta_{\mathbf{w}}\right\}^{T}$ are two generic vectors, the bar designates the complex conjugate and the subscript ${ }_{E}$ stands for energy. We recall that (10.27) represent the total energy norm, where the volume integral measures the kinetic energy, whereas the two boundary terms are, respectively, the gravitational and surface elastic potential energies. We also note that the surface integral associated with the surface energy (curvature term) is further weighted by $\left(1+\eta_{0, r}^{2}\right)^{-3 / 2}$, resulting from the linearization around an initially curved static meniscus, $\eta_{0}(r) \neq 0$.
As a final comment, in equations (10.22)-(10.26) we have invoked the concept of adjoint modes, solutions of the adjoint linearized homogeneous problem, whose formal derivation is given in the Chapter 9. In this regard, here we limit ourselves to reporting the final result, according to which

$$
\hat{\mathbf{q}}_{f, p}^{\dagger}=\left\{\begin{array}{l}
\hat{\mathbf{u}}^{\dagger}  \tag{10.28}\\
\hat{p}^{\dagger} \\
\hat{\eta}^{\dagger}
\end{array}\right\}_{f, p}=\left\{\begin{array}{l}
-\overline{\hat{\mathbf{u}}} \\
-\hat{\hat{p}} \\
\hat{\hat{\eta}}
\end{array}\right\}_{f, p} \neq \hat{\mathbf{q}}_{f, p}, \quad \lambda_{f, p}^{\dagger}=-\sigma_{f, p}-\mathrm{i} \omega_{f, p}=\bar{\lambda}_{f, p}
$$

The abovementioned supplementary notes also provide a demonstration that direct modes, $\hat{\mathbf{q}}_{f, p}$ and adjoint modes, $\hat{\mathbf{q}}_{f, p}$, form a bi-orthogonal basis with respect to the scalar product (10.27), with the adjoint modes that appear, therefore, as the most suitable choice for the projection step.

### 10.5 Comparison with experiments and results

### 10.5.1 Contact line dynamics and finite-time arrest

In this section, the most relevant results are discussed. First, we compare the contact line dynamics predicted by the projection method versus that predicted by the 1dof model and that measured experimentally by Dollet et al. (2020). This comparison is outlined in figure 10.11 for different initial contact line elevations, $h_{i n}$. The improvement brought by the present projection method is not striking from this comparison. Both the 1 dof model and the present model are in fairly good agreement with experiments. Nevertheless, we can spot, e.g. in panels ( $\mathrm{a}, \mathrm{b}, \mathrm{c}$ ), that our model seems to capture the stick-slip transitions preceding the contact line arrest. Those transitions are visible in the experiments and correspond to the dynamical phases where the contact line elevation remains approximately constant over a time interval,

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as indicated by the red arrows.


Figure 10.11 - Contact line elevation versus time for different initial conditions. Dashed line: ldof model. Red solid lines: predictions from the projection model. Markers: experiments by Dollet et al. (2020). We note that in performing the calculation, we have actually considered an effective tube length of 16.2 cm , where an excess length of $l^{\prime}=1.6 \mathrm{~cm}$ is introduced in order to into account the fact that the cross-section along the curved part of the tube is not constant due to the fabrication process. See Dollet et al. (2020) for further details.

The first interesting aspect highlighted by the projection model is related to the dependence of the finite-time arrest for the contact line, $t_{a r r}$, on the initial elevation, $h_{i n}$. The time arrest of the contact line is indicated in figure 10.11 by the vertical black dashed lines, while its dependence on $h_{i n}$ is characterized more in detail in figure 10.12, which shows how $t_{a r r}$ follows a step-like function.
From our knowledge, such a trend has not been reported in the literature yet, but it appears intuitively correct. Indeed, the arrest of the contact line occurs when, after a few oscillations cycles, the inertia of the system is no more sufficient to overcome this static friction. If for an initial condition $h_{i n}$, the time of arrest is $t_{\text {arr }}$, one can imagine that small variations of $h_{i n}$ will lead to the same $t_{\text {arr }}$. In order to prolongate in time the oscillatory contact line motion, the system needs to surpass this final energy barrier, which is only possible by starting from a sufficiently larger potential energy, and thus, from a larger $h_{i n}$.


Figure 10.12 - Finite time of arrest versus the imposed initial elevation. Black solid line: analytical prediction from the one-degree-of-freedom model proposed by Dollet et al. (2020). White triangles: experimental measurements by Dollet et al. (2020). Colored circles: projection method. The black dashed line only serves to guide the eyes.

### 10.5.2 Global damping properties and frequency modulation

As the projection method deals with the full hydrodynamic system, we have access to all the degrees of freedom of the system. Looking away from the contact line and rather focusing the attention, for example, on the centerline dynamics at $r=0$, then the useful insights brought by the present approach are evident. The centerline dynamics is of course affected by what happens at the contact line, but at the same time, it does not undergo a finite-time arrest. The associated time series, computed for different initial elevations, is reported in figure 10.13.
An inspection of this time-signal evolution reveals, consistently with previous experimental observations (Cocciaro et al., 1993), how the contact line arrest is followed by the secondary bulk motion characterized by an exponential relaxation with a constant damping coefficient (i.e. the final linear trend in the log-scale plot of figure 10.13), which is completely overlooked by the ldof model. By monitoring the nonlinear decay of such a signal, we can estimate the damping rate and the modulation of the oscillation frequency as a function of the timedependent oscillation amplitude
The result of this procedure is explained and illustrated in figure 10.14. Similarly to the weakly nonlinear analysis formalized by Viola and Gallaire (2018), the 1dof model predicts the initial increase in the damping rate, $D R(t)$, but it diverges around $t \approx t_{a r r}$. This finite-time singularity is not surprising as the contact line arrests at $t=t_{a r r}$, but it is only locally correct, and it does not represent a good description of the global damping rate. On the contrary, the damping rate resulting from the projection shows an increase as the wave amplitude decreases, until it reaches a maximum value, after which it decreases to a nearly constant value. Once the pinned dynamics is established, the damping rate is approximately constant and equal to the viscous damping coefficient of the pinned U-tube mode. Concerning the

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Figure 10.13 - Centerline free surface elevation, i.e. $r=0$ and $z=0$ (in log-scale), versus time for different initial elevations, $h_{i n}$. The grey solid lines show the actual signal produced by the projection method, while the coloured solid lines indicated the amplitude envelope only. The coloured dashed lines correspond to the analytical prediction given by the single-degree-of-freedom model employed by Dollet et al. (2020). An almost abrupt change in the trend of these signals is well visible. This is a clear sign of the final transition to a pinned contact line dynamics following the contact line arrest.
frequency modulation in time, we find a smooth evolution from the characteristic value of the initially dominant free U-tube mode to a final value, reached for $t \approx t_{\text {arr }}$ and corresponding to the natural oscillation frequency of the pinned U-tube mode. Although no experiments concerning the damping rate and frequency modulation in time were reported in Dollet et al. (2020), the initial and final values match well the experimental ones (as indicated in figure 10.14 by the values of $\omega_{\text {exp }}^{\text {free }}, \omega_{\text {exp }}^{\text {pinn }}$ and $\left.\sigma_{\text {exp }}^{\text {pinn }}\right)$, and the intermediate behaviour is fully consistent with that experimentally reported by Cocciaro et al. (1993).
We note that the centerline elevation, as the contact line elevation, is also a local measurement, but it is more representative of the overall dynamics. Similar trends for the damping and frequency are found by monitoring, e.g., the decay of the total energy (see Bongarzone et al. (2021c)), which represents instead a global measurement.

### 10.6 Conclusions

In this work, we have employed the projection method developed in Chapter 9 to study the natural relaxation dynamic of small amplitude liquid oscillations in a U-shaped tube, as experimentally investigated by Dollet et al. (2020).

First, we attempted to rationalize the linear dissipation properties of the system in both the free and pinned dynamical phases so as to explain the fitting parameter used in the 1 dof model of Dollet et al. (2020) (see equation (10.1a)). After having numerically estimated the contribution of the Stokes boundary layers and the effect of three-dimensionality, i.e. of the


Figure 10.14 - (a) Dimensional damping rate and (b) frequency modulation versus time at different initial conditions. The damping rate, $D R(t)$ is computed as the logarithmic decrement of the amplitude of the centerline free surface elevation, shown in figure 10.13. The frequency is computed from the same signal by evaluating the period from peak to peak, with the resulting value that is then roughly assigned to the midpoint of the corresponding time interval (coloured filled circles in (a) and (b)). The coloured solid lines represent the best fit of these time signals, whereas the coloured dashed lines correspond to the analytical prediction given by the single-degree-of-freedom model employed by Dollet et al. (2020).
tube curvature on the overall linear damping coefficients (see Appendix), a linear Hocking's law for the dynamic variation of the contact angle with the contact line speed has been accounted for in order to compensate for the missing dissipation, hence allowing for a good match with experiments. The combination of such a linear law with the static hysteresis range considered in Dollet et al. (2020) translates into the phenomenological nonlinear contact line model already used in Chapter 9.
The full hydrodynamic system, supplemented with this contact line model, has been then studied in the framework of the projection approach, so as to compare the resulting predictions with those from the simple ldof model employed in Dollet et al. (2020) and with their experimental measurements. When looking at the contact line dynamics only, the improvement brought by the present model is not striking. Both the ldof model and the present model are in fairly good agreement with experiments and well predict the contact line arrest. However, our model seems to correctly capture some of the stick-slip transitions occurring, in a more pronounced way, just before the finite-time arrest. If one is interested in having a quick estimation of the finite-time arrest for the contact line, we, therefore, recommend using the 1dof model.

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Nevertheless, although the peculiar contact line dynamics, with its stick-slip motion and finite-time arrest, is the main responsible for the initial nonlinear dissipation of the system, it is not fully representative of the global dynamics. Through the projection method, we have access to all the degrees of freedom of the system. This allowed us to explore, for example, the centerline dynamics, which is affected by what happens at the contact line but does not undergo a finite-time arrest. An inspection of this time-signal evolution reveals, consistently with previous experimental observations (Cocciaro et al., 1993), how the contact line arrest is followed by the secondary bulk motion characterized by an exponential relaxation. By monitoring the nonlinear decay of this such a signal obtained via the projection approach, we have been able to estimate the damping rate and the oscillation frequency (both amplitudedependent) of the system, hence correctly capturing the transition from an initial stick-slip motion to a final pinned dynamics, which has been so far overlooked by the theoretical analyses reported in the literature.
The projection method, here applied to the case of a piecewise linear contact line model, has already been generalized to any smooth non-linear contact line dynamics, e.g. a cubic law according to the Dussan model (see Bongarzone et al. (2021c)). Replacing the linear Hocking's law with a more sophisticated nonlinear law, e.g. cubic, and combining the latter with a range of static hysteresis is of interest and appears somewhat straightforward. Other future perspectives include the introduction in the model of small amplitude external forcing, i.e. axial time-harmonic excitations, and the more challenging extension to three-dimensional non-axisymmetric oscillatory dynamics, which is of great relevance for sloshing-related problems (Bongarzone et al., 2022a; Marcotte et al., 2023a,b) and in the description of oscillatory sessile drop dynamics (Amberg, 2022; Ludwicki et al., 2022; Noblin et al., 2004; Xia and Steen, 2018).

### 10.7 Appendix

### 10.7.1 Effect of the tube curvature on the damping

In this Appendix, we perform the full three-dimensional eigenvalue analysis for a pinned contact line. The latter condition is easier to resolve numerically, as no stress singularity emerges from the imposition of a no-slip wall. Although the flow dynamics for a moving contact line and the resulting damping properties may differ from the one considered here, the purpose of this appendix is simply to have a rough estimation of the effect of the curved part of the tube on the global linear damping coefficient. This computation serves us to partially justify the fundamental assumption of neglecting the tube curvature. With respect to the real experiment, we can only obtain a rough estimation, as the tube used by Dollet et al. (2020) shows a significantly smaller cross-section in its curved part than in its straight parts, where it is circular of uniform radius $a=8.15 \mathrm{~mm}$ within a few tens of microns. As it is difficult to measure this variation locally, the authors didn't report any information that could be used to mesh numerically the actual geometry. For these reasons, we will simply consider a constant cross-section of radius $a$.


Figure 10.15 - (a) Three-dimensional natural U-tube mode for a pinned contact line. The full domain has been resolved, but only a quarter of it is shown here for visualization purposes. (b) Axial velocity profile plotted at different sections along the tube, as indicated by the coloured arrows. The liquid column length in (a) and (b) has been set to $l=14.6 \mathrm{~cm}$. (c) Dimensional oscillation period, $T=2 \pi / \omega$, associated with the pinned contact line dynamics and as a function of the liquid column length, $l$. (d) Same as in (c), but for the dimensional damping coefficient. In (c) and (d), empty circles correspond to the present 3D calculation, black crosses are from the axisymmetric model discussed throughout the manuscript, while filled black diamonds are experimental measurements from Dollet et al. (2020). Only one measurement has been reported for the oscillation period.

Thus, the linearized governing equations with their boundary conditions have been implemented in the finite-element software COMSOL Multiphysics v5.6. To mesh the physical domain, we have adopted a hybrid quadrilateral-triangular mesh. Specifically, triangular
elements were used in the interior, while quadrilateral elements were adopted in the neighbourhood of the free surface, sidewalls and bottom, where, in addition, boundary layer refinements were used to better model the viscous Stokes boundary layers. The equations were manually written in their weak formulation using the Weak Form PDE tools available in the software. We used P2 for the velocity field and P1 elements for the pressure field, so as to avoid spurious pressure mode. The interface variable was discretized with P2 elements. Globally, the grid is made of approximately 300000 degrees of freedom, for which convergence was tested.
The results of this computation are reported in figure 10.15. Panel (a), gives a picture of the three-dimensional natural $U$-tube mode for a pinned contact line: the full domain has been resolved, but for visualization purposes, only a quarter of it is shown. The non-dimensional axial velocity profile is reported in panel (b) at different locations along the tube as indicated by the colored arrows. We can see how the effect of the curvature is locally important from the asymmetry in the velocity profile: the velocity is higher where the curvature is higher. This asymmetric profile gradually adapts to a symmetric plug-like flow in the straight arm of the tube, and eventually, it relaxes to a bell-like profile at the interface. This last profile seems peculiar, but it is consistent with the fact that the axial velocity at the surface equals the time derivative of the interface, which, for a pinned dynamics, has indeed a bell-like shape (see §10.3).
Although the curvature seems to affect the flow locally, figure 10.15(c) and (d) suggest that it does not significantly influence the eigenvalue properties of the system, i.e. the oscillation period (panel (c)) and the damping coefficient (panel (d)). Specifically, the oscillation period predicted by the axisymmetric model is only slightly larger than that predicted by the full 3 D calculation, and both trends, with respect to variations of the liquid column length, are consistent with the experimental measurements.
The damping coefficient is always larger than that computed via the axisymmetric model. This increase is attributable to three-dimensional effects, and it is a consequence of the slightly higher oscillation frequency. However, such an increase is bounded to less than $3 \%$ for the lengths $l$ considered. Hence, neglecting the curved part and employing a simplified axisymmetric model appears as a justifiable assumption for the geometrical and fluid properties examined in this work.

### 10.7.2 Theoretical estimate of the Stokes boundary layer contribution to the dissipation and comparison with the numerical slip-length model ${ }^{1}$

In the first part of Sec. 10.3.2, which deals with a description of the natural properties of the system in the free-phase, we have computed numerically the damping coefficient associated with the dissipation originating in the oscillating Stokes boundary layer at the lateral wall. This numerical estimate, based on the slip-length model (10.7)-(10.8), has provided a non-dimensional averaged damping value that amounts to $\sigma \approx 0.027$, which is less than half

[^3]the one needed for a good agreement with the data ( $\sigma \approx 0.6$ ). Such a disagreement has then motivated the introduction of an extra source of dissipation originating in the contact line region, which has eventually led to the desired value of $\sigma$.
The use of the phenomenological contact line model (10.13) and, specifically, of the chosen value of the contact line coefficient $\alpha \neq 0$, has already been justified throughout the manuscript. Nevertheless, it is still worth making sure that the original numerical estimate, obtained for $\alpha=0$, represents in the first place a good prediction of the lower bound for $\sigma$, so as to not overfit the value of $\alpha$ required to increase $\sigma$ up to the desired experimental value.

In this Appendix we therefore attempt in deriving an analytical estimation of the damping coefficient produced by the Stokes boundary dissipation. To this end, as in Sec. 10.3.2, we neglect the tube curvature and we assume a pure free-end edge contact line condition, i.e. $\alpha=0$. Additionally, for the sake of mathematical tractability, we ignore here the curvature of the static interface, i.e. $\eta_{0}(r)=0$, by taking $\theta_{s}=90^{\circ}$. Note that the experimentally measured value is $\theta_{s}=80.5^{\circ}$; this angle produces a static meniscus whose characteristic length is approximately $5-6 \%$ the tube radius, i.e. its influence is likely negligible (see Fig. 10.3). Note that the experimentally measured value is $\theta_{s}=80.5^{\circ}$, which is not far from $90^{\circ}$.
Under these hypotheses, the problem of free-phase $U$-tube oscillations is formally equivalent to the Stokes second problem for axial oscillations governed by

$$
\begin{equation*}
\frac{\partial w}{\partial t}=v\left(\frac{1}{r} \frac{\partial w}{\partial r}+\frac{\partial^{2} w}{\partial r^{2}}\right),\left.\quad w\right|_{r=a}=W \cos \omega_{0} t \tag{10.29}
\end{equation*}
$$

with the additional constraint the the axial velocity remains bounded for $r \rightarrow 0$. The solution of equation (10.29) gives the axisymmetric axial velocity profile inside the cylinder, i.e. for $0 \leq r \leq a$,

$$
\begin{equation*}
w(r, t)=W \operatorname{Real}\left[\frac{I_{0}\left(r \sqrt{\mathrm{i} \omega_{0} / v}\right)}{I_{0}\left(a \sqrt{\mathrm{i} \omega_{0} / v}\right)} e^{\mathrm{i} \omega_{0} t}\right] \tag{10.30}
\end{equation*}
$$

where $I_{0}$ is the modified Bessel function of the first kind.
We can then compute the total force exerted by the fluid on the lateral wall as

$$
\begin{equation*}
F=\left.\mu \frac{\partial w}{\partial r}\right|_{r=a}=(\pi a l) \mu W \text { Real }\left[\sqrt{\frac{\mathrm{i} \omega_{0}}{v}} \frac{I_{0}\left(r \sqrt{\mathrm{i} \omega_{0} / v}\right)}{I_{0}\left(a \sqrt{\mathrm{i} \omega_{0} / v}\right)}\right] \tag{10.31}
\end{equation*}
$$

where the term ( $\pi a l$ ) represents the total wall surface for half tube of radius $a$ and length $l / 2$. The associated power reads

$$
\begin{equation*}
P=\left.F \cdot w\right|_{r=a}=(\pi a l) \mu W^{2} \operatorname{Real}\left[\sqrt{\frac{\mathrm{i} \omega_{0}}{v}} \frac{I_{1}\left(a \sqrt{\mathrm{i} \omega_{0} / v}\right)}{I_{0}\left(a \sqrt{\mathrm{i} \omega_{0} / v}\right)}\right] \operatorname{Real}\left[e^{\mathrm{i} \omega_{0} t}\right] \tag{10.32}
\end{equation*}
$$

The power dissipated by viscous forces during the steady-state oscillatory motion can be expressed as

$$
\begin{equation*}
<\dot{E}>=-\frac{2 \pi}{\omega_{0}} \int_{0}^{\frac{2 \pi}{\omega_{0}}} P \mathrm{~d} t=-\frac{\omega_{0} \pi a l}{2} \mu W^{2} C \tag{10.33}
\end{equation*}
$$

Chapter 10. Stick-slip to stick transition induced by contact angle hysteresis: liquid oscillations in $U$-shaped tubes
with the auxiliary coefficient $C$ :

$$
\begin{equation*}
C=\int_{0}^{\frac{2 \pi}{\omega_{0}}} \operatorname{Real}\left[\sqrt{\frac{\mathrm{i} \omega_{0}}{v}} \frac{I_{1}\left(a \sqrt{\mathrm{i} \omega_{0} / v}\right)}{I_{0}\left(a \sqrt{\mathrm{i} \omega_{0} / v}\right)}\right] \operatorname{Real}\left[e^{\mathrm{i} \omega_{0} t}\right] \mathrm{d} t . \tag{10.34}
\end{equation*}
$$

In the potential flow limit, the U-tube linear dynamics is described by a plug flow with an interface rigidly oscillating in time at natural oscillation frequency $\omega_{0}^{2}=2 \mathrm{~g} / \mathrm{l}$ and without deforming in the radial direction. This simple dynamics can be described by introducing the generalized coordinate $q(t)$, such that the interface position $\eta$ and the axial velocity $w$ read, respectively, $\eta=q$ and $w=\dot{q}(t)$.
Let us now evaluate the total mechanical energy $E$, sum of the kinetic $(K)$ and potential $(P)$ energies, associated with the oscillatory motion:

$$
\begin{equation*}
E=K+P=\frac{\rho}{2} \int_{-l / 2}^{0} \int_{0}^{2 \pi} \int_{0}^{r} w^{2} r \mathrm{~d} r \mathrm{~d} \phi \mathrm{~d} z+\frac{\rho g}{2} \int_{0}^{2 \pi} \int_{0}^{a} \eta^{2} r \mathrm{~d} r \mathrm{~d} \phi=\frac{\rho g}{2} \pi a^{2}\left(\frac{\dot{q}^{2}}{\omega_{0}^{2}}+q^{2}\right) . \tag{10.35}
\end{equation*}
$$

Assuming the ansatz $q(t)=D_{q}(t) \cos \omega_{0} t$, one has that

$$
\begin{equation*}
E=\frac{\rho g \pi a^{2}}{2}\left[D_{q}^{2}+\dot{D}_{q}\left(\dot{D}_{q} \frac{\cos \omega_{0} t}{\omega_{0}^{2}}-D_{q} \frac{2 \sin \omega_{0} t \cos \omega_{0} t}{\omega_{0}^{2}}\right)\right] \approx \frac{\rho g \pi a^{2}}{2} D_{q}^{2} . \tag{10.36}
\end{equation*}
$$

with the last approximation on the right-hand side that holds for small damping, i.e. whenever $D_{q}(t)$ represents a slow time damping process over the characteristic fast time-scale typical of the oscillations at frequency, $1 / \omega_{0}$, so that $\dot{D}_{q} \ll D_{q} \sim \mathrm{O}(1)$. The time-derivative of the total energy then reads

$$
\begin{equation*}
\dot{E}=\rho g \pi a^{2} D_{q} \dot{D}_{q} . \tag{10.37}
\end{equation*}
$$

In contradistinction with the standard Stokes second problem, where the lateral wall is oscillating harmonically at a frequency $\omega_{0}$ with amplitude $W$, in the U-tube dynamics the sidewall is fixed and the liquid column is oscillating at frequency $\omega_{0}$ with amplitude $|w|=|\dot{q}|$. Recalling that $\langle\dot{E}\rangle=-\frac{\omega_{0} \pi a l}{2} \mu W^{2} C$, we can thus express $W^{2}$ as $|w|^{2}=|\dot{q}|^{2}=\omega_{0}^{2} D_{q}^{2}$. Lastly, by assuming that $\langle\dot{E}\rangle \approx \dot{E}$, one has that

$$
\begin{equation*}
\dot{E}=\rho g \pi a^{2} D_{q} \dot{D}_{q}=-\frac{\omega_{0}^{3} \pi a l}{2} \mu C D_{q}^{2}=\langle\dot{E}\rangle \quad \Longrightarrow \quad \dot{D}_{q}=-\frac{\omega_{0} v C}{\pi a} D_{q}, \tag{10.38}
\end{equation*}
$$

and

$$
\begin{equation*}
D_{q}=D_{q_{0}} \exp \left(\frac{\omega_{0} v C}{\pi a} t\right) \Rightarrow E=\underbrace{\frac{\rho g \pi a^{2}}{2} D_{q_{0}}^{2}}_{E_{0}} \exp \left(\frac{2 \omega_{0} v C}{\pi a} t\right), \tag{10.39}
\end{equation*}
$$

which eventually leads to the analytical estimation of the damping coefficient $\sigma$ as

$$
\begin{equation*}
\frac{E}{E_{0}}=\left(\frac{D_{q}}{D_{q_{0}}}\right)^{2}=\exp \left(\frac{2 \omega_{0} v C}{\pi a} t\right)=\exp \left(-2 \omega_{0} \sigma t\right) \quad \Rightarrow \quad \sigma=\frac{v C}{\pi a}, \tag{10.40}
\end{equation*}
$$

which must be compared with the numerical estimation reported in figure 10.7. This is done in figure 10.16. Both the theoretical and numerical models neglect the curvature of the tube and the extra contact line dissipation. We can see that the two predictions compare very well, hence confirming that the slip-length model (10.7)-(10.8) allows for a fair estimation of the Stokes boundary layer dissipation, as already suggested by the analysis of Bongarzone and Gallaire (2022). This calculation also further confirms that the laminar boundary layer dissipation alone is not sufficient to justify the experimentally fitted damping coefficient.
The effect of U-tube curvature on the damping has been discussed in Appendix. The increase in the damping attributable to the three-dimensionality of the flow in the U-turn region appears too small to close to the gap with experiments, hence reinforcing the hypothesis that the additional dissipation indeed comes from the contact line dynamics.


Figure 10.16 - (a) Non-dimensional, $\sigma$, and (b) dimensional, $\bar{\sigma}=\sigma \sqrt{2 g / l}$, damping coefficient versus the water column length, $l(\mathrm{~cm})$ and associated with a free contact line dynamics of the fundamental U-tube mode for $\alpha=0$ rad. Blue diamonds: values computed fully numerical eigenvalue calculation by accounting for the variable slip length model (10.13). The red solid lines correspond to the analytical estimate of the damping coefficient as estimated in this Appendix according to equation (10.40). The vertical black dashed lines in (a) and (b) indicate the length of the $U$-turn region, $\pi R \approx 7 \mathrm{~cm}$. Below this length, the liquid column is all contained in the U-turn region. In proximity and, particularly, below this limit value (as indicated by the grey-shaded regions), neglecting the curvature of the tube is no more a justifiable assumption.

## Conclusions and perspectives

Through this research, we have examined the self-sustained oscillations in cross-junction jets and the resonant dynamics of geometrically confined sloshing and Faraday waves. By drawing upon existing and homemade dedicated experiments, we have employed the tools of linear stability analysis and asymptotic techniques to establish comprehensive theoretical frameworks capable of explaining various flow features associated with these oscillatory dynamics. This has enabled us to create predictive models for the fundamental hydrodynamic processes involved, which is crucial to the effective design of related engineering devices.
Dedicated conclusions are outlined at the end of each Chapter and I encourage the reader to review them for more comprehensive remarks and debates. In the following, I will limit myself to giving only a very short summary, while focusing more on possible future perspectives that I personally consider of engineeristic relevance and scientific thrill.

## Cross-junction jets: 3D stability and complex jet networks

In Part I, we have described a feedback-free microfluidic oscillator based on two laminar impinging jets interacting within a cavity with no moving parts. Experiments and numerical simulations have been used to determine the region in the control parameter space where self-sustained oscillations manifest. Advances in the understanding of the physical mechanism behind these oscillations have been made by performing linear global stability and sensitivity analysis, which have identified a shear instability, located in the jet's interaction region, as the main candidate responsible for the emergence of the oscillatory regime observed in similar fluidic devices. Further interesting nonlinear features, involving symmetry-breaking and subcritical transitions, have also been described by means of the multiple-scales weakly nonlinear theory.
The idealized two-dimensional (2D) analysis of Chapter 3 could well reproduce some of the features observed experimentally, such as the scaling of the oscillation frequency. This has been shown to be proportional to the averaged flow velocity imposed at the symmetric inlets and inversely proportional to the distance between those inlets, irrespective of many other parameters, such as channel width, depth, length, and Reynolds number, among others. Nevertheless, a full three-dimensional (3D) global stability analysis appears necessary for a quantitative characterization of the stability properties of this category of low Reynolds
fluidic oscillators. This direction is already being pursued in the lab by one of my colleagues, which kindly provided me with some preliminary results, reported here in panels (a) and (b) of figure C.1.


Figure C. 1 - (a) Vertical velocity component associated with the non-oscillating global Mode A (as defined in Chapter 3) computed for $A R_{y}=s / w=9, A R_{z}=h / w=9$ and $R e=30$. (b) Oscillating global Mode B (as defined in Chapter 3) computed for the same control parameters. Both modes are unstable at $\left(A R_{y}, A R_{z}, R e\right)=(9,9,30)$. The colored regions correspond to negative and positive velocity contours of equal magnitude. Figures were provided by Timothée D. Salamon in personal communication. (c) Evolution of the dye concentration fields with time in a micro-oscillator structure with 3 inlet channels, for $R e=32$. The images are taken at regular time intervals during one oscillation (from left to right, top to bottom). The jets width is $100 \mu m$, the three jets are placed at $120^{\circ}$ angle on a circle of $800 \mu \mathrm{~m}$ in diameter, the output channels width is $2000 \mu \mathrm{~m}$, the thickness of the device is $525 \mu \mathrm{~m}$. These images are associated with a video winner of the 2019 American Physical Society's Division of Fluid Dynamics (DFD) Gallery of Fluid Motion Award, available online at https://doi.org/10.1103/APS.DFD.2019.GFM.V0036. (d,e) Base flow (left) and linear global Mode B (right), both represented as velocity field magnitude and computed for two-dimensional fluidic oscillators with 3 (d) and 4 (e) inlets for a Reynolds number $\operatorname{Re} \approx 30$ and a ratio $s / w=7$. For these parameters, Mode B is unstable.

Panels (a) and (b) show the vertical $y$-velocity component (imaginary part) associated with the 3D version of the non-oscillating global Mode A and the oscillating global Mode B, as defined in Chapter 3. Those modes are computed at Reynolds number $R e=30$ for a geometry with straight output channels of equal aspect ratios, $A R_{y}=s / w=9, A R_{z}=h / w=9$, for which both modes A and B are found to be unstable. These results reinforce the relevance of our previous 2D analysis and pave the way for thorough parametric stability analysis and investigation of nonlinear global mode interactions around codimension-2 points or, possibly, codimension-3 points and which might result in other symmetry-breaking conditions, hysteretic state transitions, vortex formations (Burshtein et al., 2019, 2021).

The development of such stability tools could be then used to explore the self-sustained oscillatory regime in more complex jet networks, as for example, the one reported in figure C.1(c). My preliminary stability calculations in 2D are in agreement with our experimental finding that oscillations appear for multiple interacting jets crossing at some specific angles (see figure C.1(d,e)). However, there is no guarantee that the same instability mechanism and frequency scaling will hold for these configurations, which are therefore worth to be investigated.
In the case of only two interacting jets, we have also shown how the range of Reynolds numbers for which oscillations are observed increases significantly when an expansion of the output channel is added. In this regard, one promising research direction could consist in optimizing the geometry of the microfluidic cavity and properly selecting the number of interacting jets, so as to achieve specific requirements, e.g. on the oscillation frequency, which could be of interest for hydrodynamic converters and switching devices in microfluidic circuitry.

## Streaming flow in sloshing and standing waves

In Part II, we have studied the harmonic and super-harmonic resonant sloshing dynamics in orbitally-shaken cylindrical reservoirs. An amplitude equation model, capable of predicting the finite wave amplitude saturation associated with different free surface patterns, has first been formalized for the case of a circular container's trajectory, which represents the usual forcing condition for shaken bioreactors. The analysis has then been applied to the case of longitudinal motions. This scenario is more interesting from the perspective of hydrodynamic instability, given the variety of wave regimes exhibited by the system, such as planar, irregular and swirling motions. Lastly, with a focus on harmonic resonances, we have bridged the gap between these two diametrically opposed shaking conditions by investigating the case of elliptic orbits. In particular, the counter-intuitive existence of stable swirling waves travelling in the opposite direction of the container motion has been demonstrated experimentally for the first time and successfully predicted by our weakly nonlinear model.
The qualitative analogies between the harmonic and super-harmonic system behaviours highlighted in Chapters 4 and 5 for rotary and longitudinal forcing, respectively, would suggest that such counter-propagating swirling waves could also be triggered by exciting the system elliptically in the vicinity of the super-harmonic resonance, thus calling for new experimental campaigns that would conclude this series of works on the surface hydrodynamics of these fundamental sloshing resonances in orbitally-shaken cylindrical containers.
The main limitations of the amplitude equation models developed in Part II are intrinsic to the fundamental assumption of an inviscid flow. This not only translates sometimes into an inaccurate prediction of the bound estimates between different wave regimes, as discussed in Chapters 5 and 6, but also precludes capturing the jump-down transition experimentally observed. In this regard, accounting for heuristic damping improves the predictions only partially, as typical analytical formulas (Case and Parkinson, 1957; Miles, 1967) are only valid
for small amplitude capillary-gravity waves, whereas the dissipation rates of forced wave motions are generally more complex and depend nonlinearly on the saturated wave amplitude (Bongarzone et al., 2022a; Raynovskyy and Timokha, 2020). A more rigorous viscous weakly nonlinear (WNL) analysis, in the same spirit as that developed in Chapter 7 of Part III, would indeed produce complex eigenfunctions eventually leading to complex-valued normal form coefficients so that the effective damping will be asymptotically proportional to the square of the wave amplitude through the cubic term in the amplitude equation.


Figure C. 2 - (a) Cartoon of the mean flow induced by swirling a glass of wine. Such flow can be split into a toroidal swirling flow and a poloidal recirculation. This is a modified version of a figure by W. Herreman available at https://perso.limsi.fr/wietze/sloshing.html. (b) Mean flow measured by Bouvard et al. (2017) by means of stroboscopic PIV left in the horizontal plane (toroidal) at an axial coordinate $z / R=-0.23$ below the free surface $(z=0)$, and right in the vertical plane (poloidal) at a phase $\pi / 2$. Parameters: container's radius $R=51.2 \mathrm{~mm}$, fluid viscosity $v=500 \mathrm{~mm}^{2} \mathrm{~s}^{-1}$, forcing amplitude $A / R=0.057$ and forcing frequency $\Omega / \omega_{1}=0.67$, with $A$ the radius of the container's trajectory and $\omega_{1}$ the lowest natural frequency.

Yet, what I personally consider the most exciting direction to pursue in this context is the modelling and prediction of the streaming flow in sloshing and Faraday waves.

## Eulerian and Lagrangian mean flow

The inviscid analysis of Part II ignores the presence of viscous boundary layers at the solid walls and free surface. As a direct consequence, the WNL time- and azimuthal-averaged mean flow reduces to a free-surface deformation only (further details in Chapter 4). This is in stark contrast with the evidence of the Eulerian mean flow, also known as viscous streaming flow, observed in experiments. Therefore, an inviscid model can characterize sufficiently well the free surface motion but it results in a poor approximation of the total velocity flow field, hence completely overlooking one of the essential points of interest for several applications: for example, in biology, the poloidal streaming flow experimentally described by Bouvard et al. (2017) (figure C.2) is of crucial importance in the design of shaken bioreactors, as it is this vertically recirculating flow that ensures good mixing, prevents sedimentation and enhances the gas transfer, so as to provide suitable oxygenation to the growing cell population.
For propagating waves, as for swirling waves in orbitally-shaken containers studied in Part II, the poloidal streaming flow is only one component of the total Lagrangian mean flow, whereas the other component is represented by the so-called Stokes drift (Bremer and Breivik, 2018), which, in first approximation, has a pure kinematic inviscid origin. On the contrary, for standing waves, e.g. for Faraday waves studied in Part III, the Eulerian streaming flow and Lagrangian mean flow coincide.

## Secondary drift instabilities

Instabilities of this streaming flow have been sometime shown to play an important role in the arising Faraday patterns and in their dynamics, namely in their drift (Martín et al., 2002; Vega et al., 2001) and mode interactions (Higuera and Knobloch, 2006; Higuera et al., 2002). For instance, the authors of a recent video winner of the 2022 American Physical Society's Division of Fluid Dynamics (DFD) Milton van Dyke Award (see figure C.3), have shown how, by tuning the driving amplitude at some specific driving frequency, the standing wave pattern in a thin annular container first enters a compression state and eventually starts to drift, as already observed by Douady et al. (1989). They have also shown how playing with the shape of the channel opens up many possibilities and applications, among which complex fluid networks (figure C.3(b)) and fluid pumps (figure C.3(c)).

Whether this symmetry breaking of the mean flow can be described by a coupled set of amplitude equations hinges again on the correct computation of the steady mean flow correction generated by each of the counterpropagating components of the leading-order standing wave.

## Influence of the contact line region on the streaming flow generation

The streaming flow is the result of a complex mechanism that couples the flow inside the bulk (far from the walls) and the boundary layers (near the walls) (Batchelor, 1967; Schlichting, 1932). More precisely, (i) the oscillatory flow in the bulk induces oscillating boundary layers;


Figure C. 3 - (a) Faraday waves corralled in an annulus of radial gap, $b$, comparable to the excited wavelength, $\lambda$, so as to align the wavelength along the annulus. By tuning the driving amplitude at some specific driving frequency, the standing wave pattern first enters a compression state and eventually starts to travel at a rate $\omega$. This rotation seems rooted in the underlying streaming flow due to oscillatory motions near the boundaries, which appears exacerbated by the meniscus. Playing with the shape of the channel opens up many possibilities and applications, among which complex fluid networks (b) and fluid pumps (c), where the rotation can be directed with ratcheted walls. All these figures have been extracted (and modified) by a video winner of the 2022 American Physical Society's Division of Fluid Dynamics (DFD) Milton van Dyke Award available online at https://doi.org/10.1103/APS.DFD.2022.GFM.V0040.
(ii) those then exert feedback on the bulk flow whose steady component originates in the streaming; (iii) lastly, the streaming is diffused into the bulk due to a viscous process (Nicolás and Vega, 2003). Hence, the standard approach, at least for weakly viscous fluid, uses the boundary layer theory and multiple scale analysis to calculate the nonlinear interaction in the viscous boundary layer that will force a mean flow in the bulk. The effect of the boundary layer can finally be written as boundary conditions for an effective streaming flow problem in the bulk, whose governing equations resemble the classical Navier-Stokes equations (Périnet
et al., 2017).
However, the main drawback of this method is that it ignores the dynamics of the meniscus region in the vicinity of the contact line, where the wave motion is also the most intense. Neglecting the features of the flow in this corner region may lead to a strong oversimplification. The key importance of this region has been highlighted by Bouvard et al. (2017) (see figure C.2(b)) in the case of swirling sloshing waves. Consistently, the authors of the video commented in figure C. 3 have found that symmetry-breaking drift of the standing wave pattern initially observed seems rooted in the underlying streaming flow due to oscillatory motions near the boundaries, which appears strongly exacerbated by the curved oscillating meniscus (see figure C.3(a)), as also suggested by Huang et al. (2020).

A correct description of the oscillating viscous wave flow near the contact line is, therefore, essential to achieve an accurate prediction of the streaming flow. The numerical tools developed in Part III and Part IV made several steps in this direction and could be used as fundamental building blocks of future research developments in this direction.
Specifically, a viscous WNL analysis has been formalized in Chapter 7 of Part III to study the weakly nonlinear coupling of sub-harmonic parametric waves and harmonic capillary waves produced by an axisymmetric oscillating meniscus, whose size and shape are initially set by adjusting the static contact angle by slightly under/overfilling the container while keeping the contact line fixed at the brim. Here the presence of the Stokes boundary layers is reintroduced without the need to resolve any stress singularity, as a pinned contact line allows one to rigorously impose a no-slip sidewall condition. In contradistinction with the inviscid analysis of Part II, such a WNL viscous analysis produced in fact a non-zero second-order Eulerian streaming flow, but at the time this work was carried out, I almost ignored the intriguing nature and practical relevance of such mean flow and I did not discuss nor investigate it more thoroughly. Yet, a pinned contact line dynamics does not seem of great relevance to the cases reported in figures C. 2 and C.3, where the streaming is enhanced by the oscillating meniscus in the corner region.
A moving contact line condition combined with a wall boundary layer description has been introduced for the first time in this thesis in Chapter 8, where we have investigated Faraday waves in Hele-Shaw cells by proposing a novel gap-averaged model accounting for inertial effects induced by the unsteady terms in the Navier-Stokes equations. Nevertheless, the HeleShaw approximation only gives an estimation of the global damping produced by boundary layers and contact line friction, but precludes one from describing the details of the flow in the neighbourhood of the contact line and it is not suitable the characterize the streaming flow.
Lastly, in Part IV, we have developed a physics-inspired mathematical model based on successive linear eigenmode projections to solve the relaxation dynamics of viscous capillary-gravity waves subjected to a nonlinear contact line model that accounts for nonlinear Coulomb solid-like friction. The framework of Chapter 9 considers a moving contact line showing static (hysteresis) and dynamic contact angle variations as a function of the contact line speed. Lastly, a curved static meniscus and the sidewall boundary layer are also introduced in Chapter 10 , where the contact line stress singularity has been resolved by adopting a macroscopic and phenomenological depth-dependent slip length model, which has been shown to give a good


Figure C. $4-$ (a) Linear viscous wave with $m=-1$ and plotted in the vertical plane for a zero phase. This eigenmode has a non-dimensional damping rate of 0.0414 and an oscillation frequency of 1.3316 . The color map represents the magnitude of the vertical velocity field, normalized by its maximum value, whereas arrows represent the vectorial field. Both (a) and (b) have been computed by accounting for a static contact angle of $45^{\circ}$. This produces an initially curved static meniscus $\eta$, better visible in figure C.5. (b) Viscous linear spectrum computed for the same fluid and geometrical parameters of Bouvard et al. (2017) and associated with an azimuthal wavenumber $m=0 . \sigma$ denotes the damping rate, while $\omega$ denotes the oscillation frequency, both made non-dimensional using the characteristic time-scale $\sqrt{R / g}$, with $R$ the container's radius and $g$ the gravity acceleration. The eigenvalues with $\omega \neq 0$ are viscous capillary-gravity waves, but those with $\omega=0$ are pure viscous modes (Martel and Knobloch, 1997), some of which are represented in figure C.5.
estimate of the associated dissipation (Bongarzone and Gallaire, 2022), in contrast with the slip length model employed by Viola and Gallaire (2018).

## The viscous modes conundrum

By means of these tools, we are now able to compute numerically the viscous linear spectrum by accounting for static meniscus, possibly contact angle (linearized) dynamics and viscous boundary layers. For example, let us consider the case of rotary sloshing studied by Bouvard et al. (2017). By formulating an asymptotic weakly nonlinear expansion as in Chapter 4, at leading order one would compute the time-oscillating swirling wave, rotating in the same direction of the container's direction of motion with an azimuthal structure defined by a wavenumber $m=1$ (see figure C.4(a)). At second-order, the oscillating wave, through quadratic nonlinearities in the governing equations, would produce a forcing term for $m=0$ and a zero oscillation frequency. The resolution of this second-order problem, through the inversion of the linearized operator, would then give the WNL time- and azimuthal-averaged Eulerian streaming flow.

Toroidal Viscous Modes


Poloidal Viscous Modes


Figure C. 5 - (a)-(c) Top-view of three toroidal (azimuthal) viscous modes visualized at the free surface, $z=\eta$. These modes have (a) zero, (b) one and (c) two radial nodes. (d)-(i) Poloidal viscous mode plotted in the vertical plane at a zero phase. The first three modes have (d) zero, (e) one and (f) two axial nodes with only one radial node. Modes (g)-(i) have the same numbers of axial nodes, but with two radial nodes. These modes have been computed for the very same configuration of figure C.4. The toroidal viscous modes essentially represent an Eulerian correction to the global toroidal Lagrangian mean flow, which is expected to be dominated by the Stokes drift. On the other hand, the poloidal viscous modes coincide with the global poloidal Lagrangian mean flow, up to a small viscous Stokes drift correction. The location of these modes on the viscous spectrum of figure C.4(b) is specified by the values of the associated damping coefficients, $\sigma$, reported in each panel.

Before going through this step it is instructive to examine the viscous linear spectrum for $m=0$, as a combination of those eigenmodes will dictate the structure of the mean flow response. Figure C.4(b) shows an example of such a spectrum, which displays two different classes of eigenvalues, namely, the standard viscous capillary-gravity waves, and the so-called viscous modes, as defined by Martel and Knobloch (1997). As the forcing has zero frequency, the only relevant family of modes for the calculation of the mean flow are the axisymmetric viscous modes. These modes can be further classified into two families, namely, toroidal modes (associated with a zero radial and axial flow) and poloidal modes (associated with zero azimuthal flow). Some of the least damped viscous modes are displayed in figure C.5.
A major issue in dealing with these modes lies in the fact that, for weakly viscous fluids, many of them decay more slowly than the gravity-capillary ones and must be therefore rigorously included in a weakly nonlinear analysis. A naive inversion of the second-order operator could indeed lead to a meaningless solution, as the operator, due to the small damping of some of the least damped viscous modes, may be nearly singular. In other words, a straightforward global resolvent response at $\omega=0$ could thus lead to a huge mean flow amplification, which would ruin the asymptotic expansion, with a consequent unphysical cubic nonlinear feedback on the wave amplitude saturation.
At this stage, one should wonder how many viscous modes shall be explicitly accounted for in a WNL analysis. Only the first few? Many? All of them?
According to the approach followed by Vega et al. (2001) and Higuera et al. (2002), the correct answer seems to be all of them, although using the numerics to pursue a similar approach and derive a system of coupled wave-mean flow amplitude equations, which also englobe all the details of the contact line dynamics, does not appear trivial at all.
What I believe to be a promising approach for this problem is a self-consistent model in the spirit of Mantič-Lugo and Gallaire (2016). The model consists of a decomposition of the full nonlinear Navier-Stokes equations in mean flow equation together with a linear perturbation equation around the mean flow, which are coupled, e.g., through the Reynolds stress. The full oscillating response and the resulting stress are approximated by the first harmonic calculated from the linear response to the external forcing around the aforementioned mean flow. This closed set of coupled equations can be solved in an iterative manner as partial nonlinearity is still preserved in the mean flow equation despite the assumed simplifications. In such a way, one could compute the mutual interaction of the wave with its own mean flow. The latter would contain the effect of all the viscous modes, hence hopefully providing a good description of the streaming flow.

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## Education

| Doctoral School of Mechanical Engineering | Jun 2019 - Current |
| :--- | ---: |
| École Polytechnique Fédérale de Lausanne (EPFL) | Lausanne, Switzerland |
| Estimated completion date: Sep 2023 |  |
| Provisional dissertation title: Self-sustained dynamics and forced resonant oscillations in flows: cross-junction |  |
| jets and sloshing liquids |  |
| Supervisor: Prof. François Gallaire |  |

## Master's Degree in Aerospace Engineering

Sep 2016 - Apr 2019
University of Pisa
Pisa, Italy
Thesis title: Sloshing waves and Faraday instability: contact line behaviour and static meniscus
Supervisor: Prof. Simone Camarri
Final Mark: 110/110 cum laude

- Research Internship at École Polytechnique Fédérale de Lausanne (EPFL) Seven months project on Sloshing wave dynamics and Faraday instability at Laboratory of Fluid Mechanics and Instabilities (LFMI).
Tutored by Prof. François Gallaire


## Bachelor's Degree in Aerospace Engineering

University of Pisa
Thesis title: Flow through a constant area duct with friction: Fanno flow
Supervisor: Prof. Maria Vittoria Salvetti
Final Mark: 108/110
Scientific High School Diploma
Sep 2008 - Jul 2013
I.I.S.S. Gandhi of Narni

Sep. 2018 - Mar. 2019
Lausanne, Switzerland

Sep 2013 - Oct 2016
Pisa, Italy

Final Mark: 86/100

## Licenses and certificates

## Deep Learning Specializations (Coursera)

Feb 2022
https:/ /www.coursera.org/account/accomplishments/specialization/certificate/WXQVWVW AF325 online

- Sequence Models
- Convolutional Neural Networks
- Improving Deep Neural Networks: Hyperparameter Tuning, Regularization and Optimization
- Structuring Machine Learning Projects
- Neural Networks and Deep Learning

Machine Learning (Coursera)
Jan 2022
https://www.coursera.org/account/accomplishments/certificate/8CDGUXB5BKTS online

[^4]
## Python for Data Science and Machine Learning (Learning \& Development)

École Polytechnique Fédérale de Lausanne (EPFL)
Python Fundamentals (Learning \& Development)
École Polytechnique Fédérale de Lausanne (EPFL)
Model Order Reduction Summer School (MORSS 2020)
Organized by École Polytechnique Fédérale de Lausanne (EPFL) and Eidgenössische Technische Hochschule (ETH)

International Summer School Complex Motion in Fluids
Technical University of Denmark (DTU)

21-23, Sep 2022
online
21-23, Feb 2022
online
$7-10$, Sep 2020
online

18-24, Aug 2019
Kysthusene Gilleleje, Denmark

## AWARDS

Gallery of Fluid Motion Award
Nov 2021
V0036: "Swinging Jets", DOI: https:/ / doi.org/10.1103/APS.DFD.2019.GFM.V0036
Seattle, WA, USA $72^{\text {th }}$ Annual Meeting of the APS Division of Fluid Dynamics (DFD)

## SCIENTIFIC PUBLICATIONS

## Peer-reviewed journal articles

- Marcotte, A., Gallaire, F. Bongarzone, A. (2023) Super-harmonically resonant swirling waves in longitudinally forced circular cylinders. Accepted in J. Fluid Mech. DOI: https:/ / doi.org/10.1017/jfm.2023.438
- Bongarzone, A., Viola, F., Camarri, S. Gallaire, F. 2022 b Sub-harmonic parametric instability in nearly-brimful circular cylinders: a weakly nonlinear analysis. J. Fluid Mech. 947, DOI: https:/ / doi.org/10.1017/jfm. 2022.600
- Bongarzone, A., Guido, M. . Gallaire F. 2022a An amplitude equation modeling the double-crest swirling in orbital shaken cylindrical containers. J. Fluid Mech. 943, DOI:
https:/ / doi.org/10.1017/jfm. 2022.440
- Bongarzone, A., Viola, F. Gallaire, F. 2021 Relaxation of capillary-gravity waves due to contact line nonlinearity: A projection method. Chaos 31 (12), 123124, DOI: https:/ / doi.org/10.1063/5.0055898
- Bongarzone, A., Bertsch, A., Renaud, P. Gallaire, F. 2021 Impinging planar jets: hysteretic behaviour and origin of the self-sustained oscillations. J. Fluid Mech. 913, DOI:
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- Bertsch, A., Bongarzone, A., Duchamp, M., Renaud, P. Gallaire, F. 2020a Feedback-free microfluidic oscillator with impinging jets. Phys. Rev. Fluids 5 (5), 054202, DOI: https:/ / doi.org/10.1103/PhysRevFluids.5.054202


## Submitted papers

- Bongarzone, A., Jouron, B., Viola, F., Gallaire, F. (2023c) A revised gap-averaged Floquet analysis for Faraday waves in Hele-Shaw cells. Under review in J. Fluid Mech. DOI:
https:/ / doi.org/10.48550/arXiv.2306.11501
- Marcotte, A., Gallaire, F. Bongarzone, A. (2023b) Swirling against the forcing: evidence of stable counter-directed sloshing waves in orbital-shaken reservoirs. Accepted in Phys. Rev. Fluids DOI: https:/ / doi.org/10.48550/arXiv.2302.14579
- Caruso Lombardi, F., Bongarzone, A., Zampogna, G. A., Gallaire, F., Camarri, S. Ledda P. G. (2023a) Three dimensional instability of the von Karman vortex street past a permeable circular cylinder: two-dimensional flow and DMD-based secondary stability analysis. Accepted in Phys. Rev. Fluids.


## Papers under preparation

- Bongarzone, A. Gallaire, F. (2023) Stick-slip to stick transition induced by contact angle hysteresis in U-shaped tubes: a projection method. In preparation for submission to Phys. Rev. Fluids.


## arXiv documents

- Bongarzone, A. Gallaire, F. (2022) Numerical estimate of the viscous damping of capillary-gravity waves: A macroscopic depth-dependent slip-length model. DOI:
https:/ /doi.org/10.48550/arXiv.2207.06907


## CONFERENCES CONTRIBUTED

A revised gap-averaged model of Faraday waves in Hele-Shaw cells
Jun 2023
$15^{\text {th }}$ SIG 33-ERCOFTAC Workshop
Symmetry-breaking swirling waves in longitudinally forced cylindrical containers
Alghero, Italy
Nov 2022
$75^{\text {th }}$ Annual Meeting of the APS Division of Fluid Dynamics (DFD) Indianapolis, IN, USA
Stick-slip to stick transition induced by contact angle hysteresis in U-shaped tubes: a projection method
$14^{\text {th }}$ European Fluid Mechanics Conference (EFMC14)
Amplitude equation model for prediction of super-harmonic double-crest wave dynamics in orbital shaken cylindrical containers

Nov 2021
Phoenix, AZ, USA
$74^{\text {th }}$ Annual Meeting of the APS Division of Fluid Dynamics (DFD)
The role of a capillary meniscus on the Faraday instability
Aug 2021
$25^{\text {th }}$ International Congress of Theoretical and Applied Mechanics (ICTAM) (speaker: F. Gallaire)
Impinging planar jets: hysteretic behaviour and origin of the self-sustained oscillations
Milano, Italy
$73^{\text {th }}$ Annual Meeting of the APS Division of Fluid Dynamics (DFD) (online)
Nov 2020

Nonlinear damping of sloshing motion caused by a piece-wise linear contact line model
Chicago, IL, USA
$73^{\text {th }}$ Annual Meeting of the APS Division of Fluid Dynamics (DFD) (online) (speaker: F. Gallaire) Chicago, IL, USA

Swinging jets (contribution V0036 to the Gallery of Fluid Motion contest)
$72^{\text {th }}$ Annual Meeting of the APS Division of Fluid Dynamics (DFD)

Nov 2019

Faraday instability: effect of the static meniscus (poster presentation)
$9^{\text {th }}$ International Summer School Complex Motion in Fluids

## INFORMAL TALKS AND SEMINARS

Super-harmonically resonant swirling waves in longitudinally forced cylinders
At Complex Fluids Group - Princeton University - hosted by Prof. H.A. Stone
At Brun Lab - Princeton University - hosted by Prof. P.-T. Brun
At Deike Lab - Princeton University - hosted by Prof. L. Deike

## Faraday waves

May 2022
At Gran Sasso Science Institute (GSSI)

Aug 2019
Kysthusene Gilleleje, Denmark

Nov 2022
Princeton, NJ, USA
Seattle, WA, USA

## TEACHING AND STUDENTS SUPERVISION

## Teaching Assistant

- Hydrodynamics Master course in Mechanical Engineering at EPFL 35 total hours

Spring 2022

- Numerical Flow Simulations Master course in Mechanical Engineering at EPFL

Fall 2020, 2021, 2022 130 total hours (softwares used: ANSYS - Workbench, Fluent, SpaceClaim)

- Numerical Methods in Biomechanics Master course in Mechanical Engineering at EPFL

Spring 2020, 2021 45 total hours (softwares used: COMSOL Multiphysics)

## Master Thesis Supervisor

- Tutored one visiting student from University of Pisa at EPFL

Sep 2021 - Mar 2022 Title of the project: Three- dimensional instability of the von Karman vortex street past a porous cylinder 85 total hours

- Tutored one student at EPFL

Spring 2021
Title of the project: Modeling hysteresis in orbital sloshing 120 total hours

## Semester Project Supervisor

- Tutored one Master student at EPFL

Spring 2023
Title of the project: Faraday waves in an annular Hele-Shaw cell 50 total hours

- Tutored one Master student at EPFL

Spring 2022
Title of the project: Capillary-gravity waves: effect of a circular corral 35 total hours

- Tutored one visiting Master student from École Polytechnique at EPFL

Spring 2021
Title of the project: Stability of fluidic oscillators 20 total hours

- Tutored one Master student at EPFL

Title of the project: Effect of a variable slip-length wall-condition on the damping of two-dimensional sloshing waves 30 total hours

## SKILLS

Languages: Italian (native), English (fluent), French (intermediate)
Programming: Matlab, Simulink, Mathematica, Python (NumPy, SciPy, Matplotlib, Pandas, TensorFlow, Jupyter)
Softwares: COMSOL, Nek5000, FreeFem++, ANSYS-Fluent, Paraview
Theoretical: finite elements, spectral and pseudospectral elements, finite differences, finite volumes, linear stability and asymptotic techniques (weakly-nonlinear multiple-scales analysis), reduced order models and decomposition techniques (POD, DMD)

Document Creation: Microsoft Office Suite (Excel, Word, PowerPoint), Adobe Creative Suite (Illustrator, Photoshop), LaTex, Overleaf

Lausanne, August 7, 2023
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[^0]:    ${ }^{1}$ Part of these notes was kindly provided by Yves-Marie Ducimetière in personal communication.

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