ALMOST CYCLIC REGULAR SEMISIMPLE ELEMENTS IN IRREDUCIBLE REPRESENTATIONS OF SIMPLE ALGEBRAIC GROUPS

D. M. TESTERMAN*

Ecole Polytechnique Fédérale de Lausanne
Institut de Mathématiques
Station 8
CH-1015 Lausanne, Switzerland
donna.testerman@epfl.ch

ALEXANDRE E. ZALESSKI

Department of Physics
Mathematics and Informatics
Academy of Sciences of Belarus
66 Prospekt Nezalejnasti
Minsk, 220000, Belarus
alexandre.zalesski@gmail.com

We dedicate this paper to the memory of James Humphreys. His books, research articles, and mathematical correspondence had a great influence on our own mathematics and will continue to train generations of young mathematicians. We are proud to be able to contribute to this issue.

Abstract. Let G be a simple linear algebraic group defined over an algebraically closed field of characteristic $p \geq 0$ and let ϕ be a nontrivial p-restricted irreducible representation of G. Let T be a maximal torus of G and $s \in T$. We say that s is *Ad-regular* if $\alpha(s) \neq \beta(s)$ for all distinct T-roots α and β of G. Our main result states that if all but one of the eigenvalues of $\phi(s)$ are of multiplicity 1 then, with a few specified exceptions, s is Ad-regular. This can be viewed as an extension of our earlier work, in which we show that, under the same hypotheses, either s is regular or G is a classical group and ϕ is "essentially" (a twist of) the natural representation of G.

1. Introduction

Fix an algebraically closed field F of characteristic $p \geq 0$ and let G be a simple linear algebraic group defined over F (hereafter referred to as an *algebraic group*). Recall that $g \in G$ is said to be *regular* if $\dim \mathsf{C}_G(g) = \mathrm{rank}(G)$; so in particular, for a semisimple element $s \in G$, s is regular if and only if $\mathsf{C}_G(s)^\circ$ is a maximal torus of G. Now let $\phi : G \to \mathrm{GL}(V)$ be a nontrivial rational representation (so V is a finite-dimensional FG-module). A starting point of our project is the following observation:

DOI: 10.1007/s00031-023-09793-5

*The first author acknowledges the support of the Swiss National Science Foundation, research grant number 200020_175571. In addition, this work was initiated while the second author visited the EPFL; he gratefully acknowledges this institute's support.

Received March 6, 2022. Accepted June 30, 2022.

Corresponding Author: D. M. Testerman, e-mail: donna.testerman@epfl.ch

($*$) Let $s \in G$ be semisimple. If $\phi(s)$ is a regular element in $\mathrm{GL}(V)$, then s is regular in G.

Note that the assumption on $\phi(s)$ in ($*$) is equivalent to saying that all eigenvalue multiplicities of $\phi(s)$ are equal to 1. For a simple algebraic group G, one might ask if there is a representation $\rho : G \to \mathrm{GL}(V)$ which characterizes regular semisimple elements $s \in G$ via the eigenvalue multiplicities of $\rho(s)$. With this in mind, in [12] we investigated semisimple elements s such that at most one eigenvalue multiplicity of $\rho(s)$ is greater than 1 (for ρ some nontrivial irreducible representation of G), with the aim of seeing if this property could be linked to some property similar to "regularity". We now define this property.

Let T be a maximal torus of G. As T meets every conjugacy class of semisimple elements of G, for our considerations we can assume that $s \in T$. Recall that the roots of G with respect to T are the nontrivial irreducible constituents of the restriction $\mathrm{Ad}|_T$, where Ad is the adjoint representation of G. Then s is regular if and only if $\alpha(s) \neq 1$ for every root α of the root system of G, see [10, Chap. III, §1, Cor. 1.7]. We are interested in a related, more restrictive condition on semisimple elements, given in the following.

Definition 1.1. Let $s \in T$. We say that s is *Ad-regular* if $\alpha(s) \neq \beta(s)$ whenever $\alpha \neq \beta$ are T-roots of G.

An Ad-regular element is regular as $\alpha(s) = 1$ implies $(-\alpha)(s) = 1$; however, there are many elements in T that are regular but not Ad-regular. Note that the property of being Ad-regular is independent of the choice of maximal torus containing the element, as it is equivalent to the element $\mathrm{Ad}(s)$ having exactly $\dim G - \mathrm{rank}(G) + 1$ distinct eigenvalues on the Lie algebra of G. Note as well that if H_{sc} is a simply connected simple algebraic group and $\pi : H_{sc} \to H$ is a central isogeny, then $s \in H_{sc}$ semisimple is (Ad-regular) regular if and only if $\pi(s)$ is (Ad-regular) regular. Since we are aiming to characterize these two properties of elements by the eigenvalue multiplicities in representations, in all of our results we will assume G to be a simply connected simple algebraic group. The analogous statements for other isogeny types follow immediately from these results.

Our first main result extends ($*$), by comparing s being Ad-regular in G with $\phi(s)$ being regular in $\mathrm{SL}(V)$ (where $\phi : G \to \mathrm{GL}(V)$ is some nontrivial p-restricted irreducible representation).

Theorem 1.2. *Let G be a simply connected simple algebraic group defined over F. Let $s \in G$ be a semisimple element and let ϕ be a nontrivial irreducible representation of G with p-restricted highest weight ω. Suppose that all of the eigenvalues of $\phi(s)$ are of multiplicity 1. Then s is Ad-regular unless one of the following holds:*

(1) $G = \mathsf{A}_n$, $n \geq 1$, *and* $\omega \in \{\varpi_1, \dots, \varpi_n\}$, *with* $p \neq 2$ *if* $n = 1$;
(2) $G = \mathsf{B}_n$, $n \geq 3$, *and* $\omega \in \{\varpi_1, \varpi_n\}$;
(3) $G = \mathsf{C}_n$, $n \geq 2$, *with*
  (a) $\omega = \varpi_1$, *or*
  (b) $\omega = \varpi_{n-1}$, *with* $p = 3$ *and* $n \geq 3$, *or*
  (c) $\omega = \varpi_n$, *with* $p \in \{2, 3\}$ *if* $n \geq 4$;
(4) $G = \mathsf{D}_n$, $n \geq 4$, *and* $\omega \in \{\varpi_1, \varpi_{n-1}, \varpi_n\}$;

(5) $G = \mathsf{E}_6$ *and* $\omega \in \{\varpi_1, \varpi_6\}$*;*
(6) $G = \mathsf{E}_7$ *and* $\omega = \varpi_7$*;*
(7) $G = \mathsf{F}_4$*,* $\omega = \varpi_4$ *and* $p = 3$*;*
(8) $G = \mathsf{G}_2$ *and either* $\omega = \varpi_1$*, or* $\omega = \varpi_2$ *and* $p = 3$*.*

(We refer the reader to the end of this section for an explanation of the notation used in the statement. For the reader's convenience the highest weights given in items (1)–(8) are collected in Table 4, at the end of this manuscript.)

The above result, as well as Theorems 1.4, 1.5, and 1.7 below, can be viewed as an identification result in the following sense: the existence of an element whose spectrum satisfies a specified criterion on a given representation either determines the representation or provides structural information about the element. We expect that Ad-regular elements can play a certain role in the general theory of algebraic groups and finite groups of Lie type. We mention the paper [9] of Seitz, in Lemma 2 of which he proves that $\alpha|_{T(q)} \neq \beta|_{T(q)}$ for the maximal tori $T(q)$ of finite quasisimple groups of Lie type $G(q)$, with $q > 5$. This allows him to obtain a certain upper bound for the homogeneous components of $T(q)$ on irreducible $G(q)$-modules. It was mentioned in [13, p. 179] that an analogous bound holds for the s-eigenspace dimensions, $s \in G$ semisimple, provided s is Ad-regular.

In fact, we are able to obtain nearly the same conclusion as in Theorem 1.2 under a weaker assumption, specifically, by allowing one of the eigenvalue multiplicities of $\phi(s)$ to be arbitrary (excluding the trivial case of s being central in G). We make a step in this direction in [12, Thm. 2], where we prove the following.

Theorem 1.3. *Let G be a simply connected simple algebraic group defined over F and let $s \in G$ be a noncentral nonregular semisimple element. Let V be a nontrivial irreducible FG-module with associated representation ϕ. Suppose that at most one eigenvalue of $\phi(s)$ is of multiplicity greater than* 1*. Then one of the following holds:*

(1) *G is of type A_n, B_n ($n \geq 3$ and $p \neq 2$), C_n ($n \geq 2$), or D_n ($n \geq 4$) and* $\dim V = n+1, 2n+1, 2n,$ *or* $2n$*, respectively;*
(2) *G is of type B_n, $n \geq 3$, $p = 2$ and* $\dim V = 2n$*;*
(3) *G is of type A_3 and* $\dim V = 6$*;*
(4) *G is of type C_2, $p \neq 2$ and* $\dim V = 5$*.*

In view of Theorem 1.3, in our consideration of semisimple elements having at most one eigenvalue multiplicity greater than 1 on some irreducible representation, we will henceforth focus our attention on regular elements. We have the following result.

Theorem 1.4. *Let G be a simply connected simple algebraic group defined over F, $s \in G$ a regular semisimple element, and let ϕ be a nontrivial irreducible representation with p-restricted highest weight ω. Suppose that at most one eigenvalue multiplicity of $\phi(s)$ is greater than* 1*. Then s is Ad-regular unless one of the following holds:*

(i) *G, p and ω are as in one of the cases* (1)–(8) *of the conclusion of Theorem* 1.2*;*
(ii) *$G = \mathsf{A}_n$ with either $\omega \in \{2\varpi_1, 2\varpi_n\}$, or $(n,p) = (2,3)$ and $\omega = \varpi_1 + \varpi_2$;*
(iii) *$G = \mathsf{B}_n$, $n \geq 3$, $p = 2$ and $\omega = \varpi_2$;*

(iv) *$G = \mathsf{C}_n$ with either $n \geq 3$, $(n, p) \neq (3, 3)$ and $\omega = \varpi_2$, or $n = 4$, $p \neq 2, 3$ and $\omega = \varpi_4$;*

(v) *$G = \mathsf{F}_4$ with either $p \neq 3$ and $\omega = \varpi_4$, or $p = 2$ and $\omega = \varpi_1$.*

The following result shows that the exceptions in Theorems 1.2 and 1.4 are genuine.

Theorem 1.5. *Let G be as above and let ϕ be a nontrivial irreducible representation of G with highest weight ω.*

(1) *Suppose that G, p, and ω are as in one of the conclusions* (1)–(8) *of Theorem* 1.2*. Then there exists a regular, not Ad-regular, semisimple element $s \in G$ such that all eigenvalue multiplicities of $\phi(s)$ are equal to* 1*.*

(2) *Suppose that G, p, and ω are as in one of the conclusions* (i)–(v) *of Theorem* 1.4*. Then there exists a regular, not Ad-regular, semisimple element $s \in G$ such that at most one eigenvalue multiplicity of $\phi(s)$ is greater than* 1*.*

There are many examples of irreducible representations ϕ and regular elements $s \in T$ such that $\phi(s)$ has exactly one eigenvalue of multiplicity greater than 1. However, these examples are within a rather narrow class of representations, as the following result shows.

Theorem 1.6 ([12, Thm. 1])**.** *Let G be a simply connected simple algebraic group defined over F and ϕ a nontrivial irreducible representation of G. Then the following statements are equivalent:*

(1) *There exists a noncentral semisimple element $s \in G$ such that at most one eigenvalue multiplicity of $\phi(s)$ is greater than* 1*.*

(2) *All nonzero weights of ϕ are of multiplicity* 1*.*

The irreducible representations ϕ whose nonzero weights are of multiplicity 1 have been determined in [11] and those with p-restricted highest weight are reproduced in Tables 2 and 3; see the end of the manuscript.

One can ask whether the assumption that ω be p-restricted can be dropped in Theorems 1.2 and 1.4. Note that if $\omega = p^a\lambda$ for some $a > 0$ and some p-restricted weight λ, then one can use Theorems 1.2 and 1.4 to obtain the eigenvalue multiplicities of semisimple elements on the associated irreducible G-module. For other non p-restricted weights, we have the following result.

Theorem 1.7. *Let G, s be as in Theorem* 1.2 *and assume* $\mathrm{char}(F) = p > 0$*. Let ϕ be an irreducible representation of G with highest weight $\sum p^i\lambda_i$, where all λ_i are p-restricted weights and at least two of them are nonzero. Suppose that at most one of the eigenvalue multiplicities of $\phi(s)$ is greater than* 1*. Then either s is Ad-regular or for each i with $\lambda_i \neq 0$, we have $(G, \lambda_i) = (G, \omega)$ for (G, ω) as in items* (1)–(8) *of Theorem* 1.2*. Moreover,*

(i) *if $(G, p) = (\mathsf{C}_n, 2)$, then for all i, $(\lambda_i, \lambda_{i+1}) \neq (\varpi_n, \varpi_1)$;*

(ii) *if $(G, p) = (\mathsf{G}_2, 2)$, then for all i, $(\lambda_i, \lambda_{i+1}) \neq (\varpi_1, \varpi_1)$;*

(iii) *if $(G, p) = (\mathsf{G}_2, 3)$, then for all i, $(\lambda_i, \lambda_{i+1}) \neq (\varpi_2, \varpi_1)$.*

Acknowledgements. We would like to acknowledge an extremely careful reading of the submitted manuscript by the referee. Taking into account his or her detailed comments and suggestions enabled us to significantly improve the exposition, clarify the statements of the results, and eliminate certain errors. We would also like to thank Gunter Malle for reading a previous version of the paper and pointing out some inaccuracies in the text.

Notation. We fix throughout an algebraically closed field F of characteristic $p \geq 0$. By abuse of notation, when we write $p > p_0$ for some prime p_0, we allow the case $\mathrm{char}(F) = 0$.

Throughout the paper, G is a simple simply connected linear algebraic group defined over F. (As mentioned before, we will suppress the adjective "linear".) All G-modules considered are rational finite-dimensional FG-modules.

We fix a maximal torus T in G, which in turn defines the roots of G as well as the weights of G-modules and representations. Recall that the T-weights of a G-module V are the irreducible constituents of the restriction of V to T. When T is fixed (which we assume throughout the paper), we omit the reference to T and write "weights" in place of "T-weights". The set of weights of a G-module V is denoted by $\Omega(V)$. Recall that the roots of G are the nonzero weights of the adjoint representation, equivalently of the G-module $\mathrm{Lie}(G)$. We denote the set of roots by Φ or $\Phi(G)$. The weights lying in the $\mathbb{Z}$-span of Φ are called *radical.* We fix a base of the root system and the simple roots of $\Phi(G)$ are denoted by $\alpha_1, \dots, \alpha_n$ and ordered as in [1]; we denote the associated set of positive roots by Φ^+. For a G-module V, $t \in T$ and $\mu \in \Omega(V)$, $\mu(t)$ is an eigenvalue of t on V, $(-\mu)(t) = t^{-1}$, and $(\mu + \nu)(t) = \mu(t)\nu(t)$ for $\nu \in \Omega(V)$. As every semisimple element of G is conjugate to an element of T, in most results which follow, the assumption that s is semisimple is replaced by the assumption that $s \in T$. We write T_{reg} to denote the set of regular elements in T.

The set $\Omega = \mathrm{Hom}(T, F^\times)$ (the rational homomorphisms of T to $F^\times$) is called the *weight lattice*, which is a free $\mathbb{Z}$-module of finite rank called *the rank of* G. The Weyl group of G is denoted by W (or $W(G)$), where $W(G) = N_G(T)/T$ so the conjugation action of $N_G(T)$ on T yields an action of $W(G)$ on T and on the T-weights. The W-orbit of $\mu \in \Omega$ is denoted by $W\mu$. Minuscule weights are defined and tabulated in [2, Chap. VIII, §7.3].

We let $\{\varpi_1, \dots, \varpi_n\}$ be the set of fundamental dominant weights with respect to the choice of simple roots. These form a $\mathbb{Z}$-basis of Ω, so every $\nu \in \Omega$ can be expressed in the form $\nu = \sum a_i \varpi_i$ ($a_i \in \mathbb{Z}$); the set of ν with $a_i \geq 0$ for all i is denoted by Ω^+. If $p > 0$, we set Ω_p^+ to be the set of weights $\nu = \sum a_i \varpi_i$ with $0 \leq a_i < p$ for all i. If $\mathrm{char}(F) = 0$ we let $\Omega_p^+ = \Omega^+$. The weights of Ω^+, respectively Ω_p^+, are called *dominant*, respectively *p-restricted* weights (so that if $\mathrm{char}(F) = 0$, all dominant weights are by definition p-restricted weights).

There is a standard partial ordering of elements of Ω; namely, for $\mu, \mu' \in \Omega$ we write $\mu \prec \mu'$ or $\mu' \succ \mu$ if and only if $\mu' - \mu$ lies in the $\mathbb{Z}_{\geq 0}$-span of Φ^+, and $\mu \neq \mu'$. We write $\mu \preceq \mu'$ or $\mu' \succeq \mu$ if and only if either $\mu = \mu'$ or $\mu \prec \mu'$. For $\mu, \nu \in \Omega^+$, with $\mu \preceq \nu$, we say μ *is subdominant to* ν. Every irreducible G-module V has a weight ω such that $\mu \prec \omega$ for every $\mu \in \Omega(V)$ with $\mu \neq \omega$. The weight ω is called the *highest weight of* V. There is a bijection between Ω^+ and the set of isomorphism classes of irreducible G-modules, so for $\omega \in \Omega^+$ we denote by V_ω the irreducible G-module with highest weight ω. An irreducible G-module is called *p-restricted* if its highest weight is p-restricted. The zero weight space of V is sometimes denoted by V^T. In general, for $S \subseteq G$ we set $V^S = \{v \in V : sv = v \text{ for all } s \in S\}$. The maximal root of $\Phi(G)$ is denoted by ϖ_a; this is the highest weight of the adjoint module $\mathrm{Lie}(G)$ and of a composition factor V_a of $\mathrm{Lie}(G)$.

Because of natural isomorphisms between certain indecomposable root systems, we will assume throughout that the rank of G is as in Table 1 below. For brevity

TABLE 1. rank(G).

G	rank(G)
A_n	$n \geq 1$
B_n	$n \geq 3$
C_n	$n \geq 2$
D_n	$n \geq 4$

we write $G = \mathsf{A}_n$ to say that G is a simple simply connected algebraic group of type A_n, and similarly for the other types. For classical groups G, the module with highest weight ϖ_1 is called *the natural module* with one exception, namely the natural module for $G = \mathsf{B}_n$ when $p = 2$ is a reducible $(2n+1)$-dimensional module with two composition factors V_{ϖ_1} and V_0.

In our proofs, we regularly use so-called "Bourbaki weights", which are elements of a $\mathbb{Z}$-lattice containing Ω with basis $\varepsilon_1, \varepsilon_2, \ldots$; the explicit expressions of the fundamental weights and the simple roots of G in terms of the ε_i are given in [1, Planches I–XIII].

If $h : G \to G$ is a surjective algebraic group homomorphism and $\phi : G \to \mathrm{GL}(V)$ is a representation of G then the h-twist ϕ^h of ϕ is defined as the mapping $g \to \phi(h(g))$ for $g \in G$, and the associated G-module is denoted as V^ϕ. Consider now the case when $\mathrm{char}(F) = p > 0$ and let $Fr : G \to G$ be the Frobenius morphism arising from the mapping $x \mapsto x^p$ $(x \in F)$. For a G-module V and integers $k \geq 0$, the modules V^{Fr^k} are called *Frobenius twists of* V; if V is irreducible with highest weight ω then the highest weight of V^{Fr^k} is $p^k\omega$. Every $\omega = \sum a_i\varpi_i \in \Omega^+$ has a unique p-adic expansion $\omega = \lambda_0 + p\lambda_1 + \cdots + p^k\lambda_k$ for some k, where $\lambda_0, \ldots, \lambda_k \in \Omega_p^+$. The Steinberg tensor product theorem then implies that $V_\omega \cong V_{\lambda_0} \otimes V_{\lambda_1}^{\mathrm{Fr}} \otimes \cdots \otimes V_{\lambda_k}^{\mathrm{Fr}^k}$.

We now introduce two main notions used throughout the text. A matrix $M \in M_{n\times n}(F)$ is called *cyclic* if, for some $v \in F^n$, the space of column vectors over F is spanned by the vectors $v, Mv, M^2v, \ldots$; this is equivalent to the minimal polynomial of M being equal to the characteristic polynomial. In particular, a semisimple element $g \in \mathrm{GL}_n(F)$ is cyclic if and only if all eigenvalue multiplicities equal 1. We will say M is *almost cyclic* if, for some $\lambda \in F$, M is conjugate to a matrix $\mathrm{diag}(\lambda{\cdot}\mathrm{Id}, M_1)$, where M_1 is a cyclic matrix. Hence, a diagonalizable matrix M is almost cyclic if and only if at most one eigenvalue multiplicity is greater than 1. This leads us to introduce the following terminology to be used throughout.

Definition 1.8. Let $g \in G$ and let V be a G-module with associated representation $\phi : G \to \mathrm{GL}(V)$. We say that g is *cyclic on* V, respectively *almost cyclic on* V if $\phi(g)$ is cyclic, respectively almost cyclic.

The elements satisfying Definition 1.8 which we consider in this paper will always be semisimple.

2. Preliminaries

We start by recalling a characterization of regular semisimple elements, as mentioned in the Introduction. We will use this frequently without direct reference.

Lemma 2.1. [10, Chap. III, §1, Cor. 1.7] *Let H be a semisimple algebraic group, $T \leq H$ a maximal torus and $s \in T$. Then the following conditions are equivalent:*

(1) *s is regular in H;*
(2) *if $gs = sg$ for $g \in H$, then g is semisimple;*
(3) *for all roots $\alpha \in \Phi(H)$, $\alpha(s) \neq 1$.*

The following result concerns assertion (*) in the Introduction; the first statement of Lemma 2.2 rephrases (*) in terms of Definition 1.8.

Lemma 2.2. *Let V, M be nontrivial G-modules and $s \in T$.*

(1) *If s is cyclic on V, then s is regular.*
(2) *Assume $s \notin Z(G)$. If s is almost cyclic on $V \otimes M$, then s is regular.*

Proof. (1) Let ρ be the representation afforded by V. If s is cyclic on V then $\mathsf{C}_{\mathrm{GL}(V)}(\rho(s))$ consists of semisimple elements. So the claim follows from Lemma 2.1.

(2) By [12, Lem. 1], s is cyclic on V, so the result follows from (1). □

Definition 2.3. Let V be a G-module, $\mu, \nu \in \Omega(V)$, and $s \in T$. We say that s *separates the weights* μ, ν of V if $\mu(s) \neq \nu(s)$. If this holds for every pair of weights $\mu, \nu \in \Omega(V)$, we say that s *separates the weights of* V.

If s separates the weights of V then the eigenvalue multiplicities of s acting on V are simply the weight multiplicities of V.

Lemma 2.4. *Let V, V_1, V_2 be nontrivial G-modules. Let $s \in T \setminus Z(G)$ and assume that s is almost cyclic on V.*

(1) *Suppose that $V = V_1 \otimes V_2$. Then s is cyclic on V_1 and on V_2, all weights of V_1 and V_2 are of multiplicity 1, and s is regular.*
(2) *Suppose that $\Omega(V_1) + \Omega(V_2) = \Omega(V)$. Then s separates the weights of V_i, for $i = 1, 2$.*

Proof. This follows from [12, Lems. 1 and 3]. □

We will rely on the following important result [7, Thm. 1] describing precisely, in many cases, the set $\Omega(V)$. We require an additional notation, defined in loc. cit.; we write $e(G)$ for the maximum of the squares of the ratios of the lengths of the roots in $\Phi(G)$. So $e(G) = 1$ for G of type $\mathsf{A}_n, \mathsf{D}_n$, or E_n, $e(G) = 2$ for G of type $\mathsf{B}_n, \mathsf{C}_n$, or F_4, and $e(G) = 3$ for G of type G_2.

Theorem 2.5 ([7, Thm. 1]). *Assume that $p = 0$ or $p > e(G)$. Let λ be a p-restricted weight. Then $\Omega(V_\lambda) = \{w(\mu) \mid \mu \in \Omega^+, \mu \preceq \lambda, w \in W\}$.*

As an immediate Corollary of Theorem 2.5, we have the following.

Lemma 2.6. *Assume that $p = 0$ or $p > e(G)$. Let $\lambda, \mu \in \Omega^+$ with λ p-restricted, and V_λ, respectively, V_μ the associated irreducible G-modules. Then the following hold.*

1. *If $\mu \prec \lambda$ then $\Omega(V_\mu) \subseteq \Omega(V_\lambda)$.*
2. *If $\lambda + \mu$ is p-restricted then $\Omega(V_{\lambda+\mu}) = \Omega(V_\lambda \otimes V_\mu) = \Omega(V_\lambda) + \Omega(V_\mu)$.*
3. *If λ is radical then some root is a weight of V_λ; otherwise $\Omega(V_\lambda)$ contains some minuscule weight.*

We begin our considerations of Ad-regular elements and their action in certain representations by considering the action on V_a, the irreducible G-module with highest weight the highest root.

Lemma 2.7. *Assume that $p = 2$ if $G = \mathsf{A}_1$, $p \neq 3$ if $G = \mathsf{A}_2$ or G_2, and $p \neq 2$ if $G = \mathsf{B}_n$, C_n or F_4. Let $s \in T \setminus Z(G)$. Then s is Ad-regular if and only if s is almost cyclic on V_a.*

Proof. If $G = \mathsf{A}_1$ and $p = 2$, then all nonidentity semisimple elements of G act cyclically on V_a (which is just a twist of the natural representation of G) and all nonidentity semisimple elements are Ad-regular. So here the statement is clear.

Now turn to the other cases and suppose that s is Ad-regular, so that $\alpha(s) \neq \beta(s)$ for roots $\alpha, \beta \in \Phi(G)$, with $\alpha \neq \beta$. Then, with one exception, Theorem 2.5 implies that $\Omega(V_a) = \Phi(G) \cup \{0\}$ and so s is almost cyclic on V_a. The exception occurs when $G = \mathsf{G}_2$ and $p = 2$. But here as well, we have $\Omega(V_a) = \Phi(G) \cup \{0\}$; see, for example, [5].

Now suppose that s is almost cyclic on V_a (and $\mathrm{rank}(G) > 1$). Then, in all cases considered here, the zero weight has multiplicity at least 2 on V_a. Moreover, Theorem 1.2 implies that s is regular and so $\alpha(s) \neq 1$ for all $\alpha \in \Phi(G)$, and we conclude that $\alpha(s) \neq \beta(s)$ for all roots $\alpha, \beta \in \Phi(G)$, with $\alpha \neq \beta$; that is, s is Ad-regular. $\square$

Remark 2.8. The exceptions in Lemma 2.7 are genuine. Indeed, if $p \neq 2$ and $G = \mathrm{SL}_2(F)$, then elements of T which are not Ad-regular are of the form $s = \mathrm{diag}(c, c^{-1})$ with $c^4 = 1$, since $\alpha(s) = c^2$. We choose c such that $c^2 = -1$ and then s is noncentral, regular, and almost cyclic on V_a. If $G = \mathsf{C}_n, n \geq 2$, with $p = 2$, then $\varpi_a = 2\varpi_1$. Then V_a is the Frobenius twist of V_{ϖ_1}, so s is almost cyclic on V_a if and only if s is almost cyclic on V_{ϖ_1}. However, it is easy to construct nonregular elements $s \in T$ that are almost cyclic on V_{ϖ_1}. For $G = \mathsf{G}_2$, $p = 3$, see Proposition 4.9(4), and for $G = \mathsf{F}_4$, $p = 2$, see Remark 4.4(1). For $G = \mathsf{A}_2$ with $p = 3$ take $s = \mathrm{diag}(a, -a, -a^{-2})$ for $a \in F^\times$, so $\alpha_1(s) = (-\alpha_1)(s)$. Then the Jordan normal form of s on V_a is $\mathrm{diag}(1, -1, -1, a^3, -a^3, a^{-3}, -a^{-3})$. This matrix is almost cyclic if $a^4 \neq 1$. Finally, for $G = \mathsf{B}_n, n \geq 3$ and $p = 2$, see Lemma 3.15 and Remark 3.16.

Lemma 2.9 and Remark 2.10 will not be required in what follows, but give some additional information about the relationship between a semisimple element being regular and its action on V_a.

Lemma 2.9. *Let $s \in T$. Let ω be the highest short root and $V = V_\omega$.*

1. *If s is regular, then $V^s = V^T$ and $V_a^s = V_a^T$.*
2. *Suppose that $V_a^s = V_a^T$. If $p > e(G)$ or if $G = \mathsf{G}_2$ and $p = 2$, then s is regular in G.*

Proof. By Lemma 2.1, s is regular if and only if $\alpha(s) \neq 1$ for all $\alpha \in \Phi(G)$. For (1), we use the fact that all nonzero weights of V_a and V are roots. For (2), using Theorem 2.5, one observes that for $p > e(G)$, $\Omega(V_a) = \Phi(G) \cup \{0\}$. As mentioned in the proof of Lemma 2.7, this also is true for $G = \mathsf{G}_2$, when $p = 2$. So $V_a^T = V_a^s$ implies $\alpha(s) \neq 1$ for $\alpha \in \Phi(G)$, and hence s is regular. $\square$

Remark 2.10. If $p = e(G)$, there exists a nonregular element $s \in T$, such that $V_a^s = V_a^T$. See Proposition 4.9(5), for $G = \mathsf{G}_2$ with $p = 3$ and Remark 4.4(2) for $G = \mathsf{F}_4$ with $p = 2$. Now let $G = \mathsf{C}_n$ with $p = 2$. Then $V_a = V_{2\varpi_1}$ and hence $\Omega(V_a) = \{\pm 2\varepsilon_1, \ldots, \pm 2\varepsilon_n\}$. So $V_a^T = \{0\}$, and $V_a^s = V_a^T$ if and only if $V_a^s = \{0\}$, equivalently, if and only if 1 is not an eigenvalue of s on V_{ϖ_1}. As $n \geq 2$, there are nonregular elements $s \in T$ such that $V_{\varpi_1}^s = \{0\}$. For $G = \mathsf{B}_n$, when $p = 2$, the nonzero weights of V_a are $\{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}$. Choose s such that $\varepsilon_1(s) = 1$, so s is not regular, but $(\varepsilon_i \pm \varepsilon_j)(s) \neq 1$ for all $i \neq j$.

3. Classical groups

In this section, we prove Theorems 1.2, 1.4, and 1.5 for classical groups. In view of Theorem 1.6, we must examine the modules V_ω, for $\omega \in \Omega_p^+$, all of whose nonzero weights are of multiplicity 1. These are listed in Tables 2 and 3. Recall that we have fixed a maximal torus T of G, and we assume that our semisimple element belongs to T.

3.1. Groups of type A_n

Throughout this subsection, G is of type A_n.

Lemma 3.1. *Let $\omega \in \Omega^+$ and $n \geq 2$.*

1. *If $\omega \succ 2\varpi_1$ then $\omega \succeq \varpi_1 + \varpi_2 + \varpi_n$.*
2. *If $\omega \succ \varpi_1 + \varpi_n$, then when $n \geq 3$ we have $\omega \succeq \varpi_2 + \varpi_{n-1}$, and when $n = 2$ we have $\omega \succeq \nu$, for some $\nu \in \{3\varpi_1, 3\varpi_2\}$.*

Proof. Let $\omega \succ 2\varpi_1$. By [15, Lem. 3.2], we can assume that $\omega = 2\varpi_1 - \varpi_{j-1} + \varpi_j + \varpi_k - \varpi_{k+1}$ for some $j, k \in \{1, \ldots, n\}$, $j \leq k$. (We set $\varpi_0 = 0 = \varpi_{n+1}$.) It follows that $k = n$ and $j \in \{1, 2\}$, and in both cases (1) easily follows.

Now, let $\omega \succ \varpi_1 + \varpi_n$. As above, we can assume that $\omega = \varpi_1 + \varpi_n - \varpi_{j-1} + \varpi_j + \varpi_k - \varpi_{k+1}$. Then $j \in \{1, 2\}$ and $k \in \{n-1, n\}$. If $n \geq 3$, then $\omega \in \{2\varpi_1 + 2\varpi_n, \varpi_2 + \varpi_{n-1}, \varpi_2 + 2\varpi_n, 2\varpi_1 + \varpi_{n-1}\}$. As $2\varpi_1 + 2\varpi_n \succ \varpi_2 + 2\varpi_n \succ \varpi_2 + \varpi_{n-1}$ and $2\varpi_1 + \varpi_{n-1} \succ \varpi_2 + \varpi_{n-1}$, the result follows. If $n = 2$ then $\omega \in \{2\varpi_1 + 2\varpi_2, 3\varpi_2, 3\varpi_1\}$ and $2\varpi_1 + 2\varpi_2 \succ 3\varpi_i$, $i = 1, 2$, and the result of (2) follows. $\square$

Proposition 3.2. *Let $G = \mathsf{A}_n$, with $n \geq 1$, $s \in T_{\mathrm{reg}}$ and let $\omega \in \Omega_p^+$.*

1. *Assume that $\omega \notin \{0, \varpi_1, \ldots, \varpi_n, 2\varpi_1, 2\varpi_n\}$ and that s is almost cyclic on V_ω. Then s is Ad-regular, unless $(n, p) = (2, 3)$ and $\omega = \varpi_1 + \varpi_2$.*
2. *If $\omega \in \{2\varpi_1, 2\varpi_n\}$ and s is cyclic on V_ω, then s is Ad-regular.*
3. *If $(n, p) = (2, 3)$ and $\omega = \varpi_1 + \varpi_2$, with s cyclic on V_ω, then s is Ad-regular.*

Proof. Set $V = V_\omega$. Recall that the T-roots of G are $\varepsilon_i - \varepsilon_j$ for $i, j \in \{1, \dots, n+1\}$, $i \neq j$. Then s regular in G implies that $\alpha(s) \neq 1$ for all $\alpha \in \Phi(G)$.

For (1), suppose the contrary, that is, s is almost cyclic on V_ω and s is not Ad-regular, so that $\alpha(s) = \beta(s)$ for some roots $\alpha \neq \beta$. As W acts transitively on the roots, we can assume that $\alpha = \alpha_1 = \varepsilon_1 - \varepsilon_2$.

If $n = 1$ then $\beta = -\alpha_1$ and $\alpha(s) = \beta(s)$ implies $(2\alpha_1)(s) = 1$, whence $\alpha_1(s) = -1$. Let $\omega = a\varpi_1$, where by hypothesis $3 \leq a \leq p-1$. Then the weights of V are $\omega - i\alpha_1$ for $0 \leq i \leq a$, and $(\omega - i\alpha_1)(s) = (-1)^i\omega(s)$, contradicting that s is almost cyclic on V.

Now let $n \geq 2$. Recall that W acts on $\varepsilon_1, \dots, \varepsilon_{n+1}$ as the symmetric group S_{n+1} does. Since $\beta = \varepsilon_i - \varepsilon_j$ for some $i \neq j$, we can assume that $\beta \in \{-(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_1 - \varepsilon_3), \pm(\varepsilon_2 - \varepsilon_3), \varepsilon_3 - \varepsilon_4\}$ (where $\varepsilon_3 - \varepsilon_4$ occurs only if $n \geq 3$). As s is regular, $\alpha - \beta$ is not a root, so $\beta \notin \{\varepsilon_1 - \varepsilon_3, -\varepsilon_2 + \varepsilon_3\}$, and hence $\beta \in \{-(\varepsilon_1 - \varepsilon_2), -(\varepsilon_1 - \varepsilon_3), \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4\} = \{-\alpha_1, -\alpha_1 - \alpha_2, \alpha_2, \alpha_3\}$. In addition, as s is regular, if $\beta = -\alpha$ then $p \neq 2$ and $\alpha_1(s) = -1$.

We refer to [12, Def. 3] for the notion of "weights of level i". By the hypotheses on ω and [12, Lem. 12], ω is a weight of level at least 2. Then [12, Lem. 11] implies that there exists a weight μ of level 2 with $\mu \in \Omega(V)$; that is, $\mu \in \Omega(V) \cap \{\varpi_1 + \varpi_i, \varpi_i + \varpi_n : i = 1, \dots, n\}$. Moreover, again by [12, Lem. 11], all weights subdominant to μ also lie in $\Omega(V)$. We consider in turn each possibility for the weight μ.

Suppose first that $\mu = \varpi_1 + \varpi_i$ for $1 < i < n$. Then $\mu - \alpha_1 - \alpha_2 - \cdots - \alpha_i = \varpi_{i+1} \in \Omega(V)$ and $\varpi_{i+1} - \alpha$, $\varpi_{i+1} - \beta \in W\mu$. In addition, both $\varpi_{i+1} - \alpha_{i+1} - \alpha$ and $\varpi_{i+1} - \alpha_{i+1} - \beta$ lie in $\Omega(V)$, as long as $\beta \in \{-\alpha_1, \alpha_2\}$ when $i = 2$. Since s is almost cyclic on V, we deduce that either $i = 2$ and $\beta \in \{-\alpha_1 - \alpha_2, \alpha_3\}$ or $(\varpi_{i+1} - \alpha)(s) = (\varpi_{i+1} - \alpha - \alpha_{i+1})(s)$, so $\alpha_{i+1}(s) = 1$, contradicting the fact that s is regular. So it remains to consider the case where $i = 2$, $\mu = \varpi_1 + \varpi_2$, $n \geq 3$ and $\beta \in \{-\alpha_1 - \alpha_2, \alpha_3\}$. We give the details for the case $\beta = -\alpha_1 - \alpha_2$. Here $\eta = \mu - \alpha_1 - 2\alpha_2 - \alpha_3 \in \Omega(V)$ and $\eta - \alpha, \eta - \beta \in \Omega(V)$. Hence we deduce as above that $\varpi_{i+1}(s) = \eta(s)$ and so $(\alpha_2 + \alpha_3)(s) = 1$, contradicting that s is regular. The case $i = 2$ and $\beta = \alpha_3$ is entirely similar.

Suppose now that $\mu = \varpi_1 + \varpi_n$. Then all roots are weights of V_ω, and since $\alpha(s) = \beta(s)$, we have $(-\alpha)(s) = (-\beta)(s)$, whence $\alpha(s) = \beta(s) = -1$ and $(\alpha + \beta)(s) = 1$. Then $\alpha + \beta$ is not a root as $s \in T_{\mathrm{reg}}$. Therefore, $\beta \in \{-\alpha_1, \alpha_3\}$. If $\beta = \alpha_3$, so $n \geq 3$, then $\alpha_1 + \alpha_2 + \alpha_3$ is a root and $(\alpha_1 + \alpha_2 + \alpha_3)(s) = \alpha_2(s)$. As before, we deduce that $\alpha_2(s) = \alpha_1(s) = -1$, and then $(\alpha_1 + \alpha_2)(s) = 1$, whence a contradiction. So we are left with the case where $\beta = -\alpha$ and $\alpha(s) = (-\alpha)(s) = -1$. By Lemma 2.7 (and the hypothesis of (1)), we may assume that $\omega \neq \varpi_1 + \varpi_n$. By Lemma 3.1, $\omega \succeq \nu$, where $\nu \in \{3\varpi_1, 3\varpi_2\}$ if $n = 2$, and $\nu = \varpi_2 + \varpi_{n-1}$ if $n \geq 3$. As usual, Theorem 2.5 implies that all weights subdominant to ν lie in $\Omega(V)$.

Now if $n \geq 3$, so $\nu = \varpi_2 + \varpi_{n-1}$, then set $\eta = \nu - \alpha_1 - 2\alpha_2 - \alpha_3 - \cdots - \alpha_{n-1}$. Note that $\eta \pm \alpha_1$ and $\eta - \alpha_n \pm \alpha_1$ lie in $\Omega(V)$. But then we deduce that $\alpha_n(s) = 1$, giving the usual contradiction. If $n = 2$ and $\nu = 3\varpi_1$, we have $\nu, \nu - \alpha_1, \nu - 2\alpha_1, \nu - 3\alpha_1 \in \Omega(V)$, with $\nu(s) = (\nu - 2\alpha_1)(s)$ and $(\nu - \alpha_1)(s) = (\nu - 3\alpha_1)(s)$, implying as usual that $\nu(s) = (\nu - \alpha_1)(s)$, which is impossible as $\alpha_1(s) = -1$. The case $\nu = 3\varpi_2$ is analogous.

Note that the cases considered above cover also the weights $\mu = \varpi_i + \varpi_n$, for $2 \leq i < n$, by considering the dual V^*; so (again invoking duality) it remains to consider the case $\mu = 2\varpi_1$. Then Lemma 3.1 shows that $\omega \succeq \varpi_1+\varpi_2+\varpi_n$. If $n \geq 4$, we then have that $\omega \succ \varpi_3+\varpi_n$, so that $\Omega(V_{\varpi_3+\varpi_n}) \subset \Omega(V_\omega)$ and the result follows from previously considered cases. If instead $n = 3$, so that $\omega \succeq \varpi_1 + \varpi_2 + \varpi_3$, we consider the different possibilities for β, as before; recall that $\alpha = \alpha_1$ and $\beta \in \{-\alpha_1, -\alpha_1 - \alpha_2, \alpha_2, \alpha_3\}$. Setting $\nu = \varpi_1 + \varpi_2 + \varpi_3$, and $\eta = \nu - \alpha_1 - \alpha_2 - \alpha_3$ and $\eta' = \nu - \alpha_1 - 2\alpha_2 - \alpha_3$, we have that $\eta - \alpha, \eta - \beta, \eta' - \alpha, \eta' - \beta \in \Omega(V)$ and deduce that $\eta(s) = \eta'(s)$ and so $\alpha_2(s) = 1$ giving the usual contradiction. Finally, if $n = 2$, write $\nu = \varpi_1 + 2\varpi_2$ so that $\omega \succeq \nu$. We argue as in the previous cases. As before, since s is regular and $\alpha(s) = \beta(s)$, $\alpha - \beta$ is not a root. So taking $\alpha = \alpha_1$, we have that $\beta \in \{-\alpha_1, \alpha_2, -\alpha_1 - \alpha_2\}$. Set $\eta = \nu - \alpha_1 - \alpha_2$ and $\eta' = \nu - 2\alpha_1 - 2\alpha_2$. Then one checks that $\eta - \alpha, \eta - \beta, \eta' - \alpha, \eta' - \beta \in \Omega(V)$. So we deduce that $\eta(s) = \eta'(s)$ and so $(\alpha_1 + \alpha_2)(s) = 1$, a contradiction.

Turning to the proof of (2), we assume now that s is regular and cyclic on $V = V_\omega$, with $\omega \in \{2\varpi_1, 2\varpi_n\}$. By duality, we may assume that $\omega = 2\varpi_1$; note that $p \neq 2$. Suppose the contrary, that s is not Ad-regular, and let α, β be as in the proof of (1). Here, we only need to find distinct weights $\eta, \eta' \in \Omega(V)$ such that $\eta(s) = \eta'(s)$. It is straightforward to find such a pair for each choice of β (recall that $\alpha = \alpha_1$). For example, if $\beta = -\alpha_1$, then $(2\alpha_1)(s) = 1$ and so we may take $\eta = \omega$ and $\eta' = \omega - 2\alpha_1$. Or if $\beta = \alpha_2$ (so $n \geq 2$), we take $\eta = \omega - 2\alpha_1$ and $\eta' = \omega - \alpha_1 - \alpha_2$. The remaining cases are left to the reader.

To conclude, we turn to (3). Here, as $\Omega(V) = \{0\} \cup \Phi(G)$, and s is cyclic on V, we see that $\alpha(s) \neq \beta(s)$ for all $\alpha, \beta \in \Phi(G)$, $\alpha \neq \beta$; thus s is Ad-regular as claimed. $\square$

Corollary 3.3. *Theorem* 1.2 *and Theorem* 1.4 *hold for* $G = \mathsf{A}_n$.

Proof. For Theorem 1.2, we assume that $s \in T$ is cyclic on V_ω, for some $\omega \in \Omega_p^+$, $\omega \neq 0$. In addition, we assume that either $p = 2$ when $n = 1$ or $\omega \notin \{\varpi_1, \dots, \varpi_n\}$. First assume $n = 1$ and consider the case $\omega = \varpi_1$, when by assumption $p = 2$. As s is cyclic on V_ω, we have $\alpha_1(s) \neq 1$ and so s is Ad-regular. Now if $\omega \neq \varpi_1$, again as s is cyclic on V_ω, Theorem 1.3 implies that s is regular and Proposition 3.2 gives the desired conclusion. So we now assume $n \geq 2$; Theorem 1.3 implies that s is regular and then we use Proposition 3.2 to see that s is Ad-regular.

For Theorem 1.4, we assume that $s \in T_{\mathrm{reg}}$ is almost cyclic on V_ω, for some $\omega \in \Omega_p^+$, $\omega \neq 0$. We suppose further that either $p = 2$ if $n = 1$ or $\omega \notin \{2\varpi_1, 2\varpi_n, \varpi_1, \dots$ $\dots, \varpi_n\}$. In addition, we assume that if $n = 2$ and $\omega = \varpi_1 + \varpi_2$, then $p \neq 3$. Now if $n = 1$, with $p = 2$ and so $\omega = \varpi_1$, then s is almost cyclic on V_ω if and only if $s \neq 1$, if and only if s is cyclic on V_ω, and if and only if s is Ad-regular. For $n \geq 2$, Proposition 3.2 gives the result. $\square$

We now investigate the weights in Theorem 1.2(1) and Theorem 1.4(ii).

Lemma 3.4. *Let* $G = \mathsf{A}_n$ *and* $\omega \in \Omega_p^+$, $\omega \neq 0$.

(1) *Assume that* $p \neq 2$ *if* $n = 1$. *If* $\omega \in \{\varpi_1, \dots, \varpi_n\}$, *then there exists* $s \in T$ *with* s *not Ad-regular and* s *cyclic on* V_ω.

(2) *If $\omega \in \{2\varpi_1, 2\varpi_n\}$, then there exists $s \in T_{\mathrm{reg}}$, with s not Ad-regular and s almost cyclic on V_ω.*

Proof. (2) By Theorem 1.3, it suffices to find $s \in T$, which is almost cyclic on V_ω and yet not Ad-regular. Here we have $p \neq 2$ and the result follows from Remark 2.8 when $n = 1$. So we now take $n \geq 2$. By duality, it suffices to consider $\omega = 2\varpi_1$. Choose $T \leq \mathrm{SL}_{n+1}(F)$ to be the group of diagonal matrices $t = \mathrm{diag}(t_1, \dots, t_{n+1})$ of determinant 1, and set $\varepsilon_i(t) = t_i$, for $i = 1, \dots, n+1$.

Suppose first that $n \geq 3$. Let $s = \mathrm{diag}(a^3, a^2, a, b_1, \dots, b_{n-2}) \in \mathrm{SL}_{n+1}(F)$ for some $a, b_1, \dots, b_{n-2} \in F^\times$. Then $\alpha_1(s) = \alpha_2(s) = a$ so s is not Ad-regular. The weights of V_ω are $2\varepsilon_i$ and $\varepsilon_i + \varepsilon_j$ for $i, j \in \{1, \dots, n\}$, $i \neq j$. So the eigenvalues of s on V_ω are a^6, a^4, a^2, b_i^2 for $i = 1, \dots, n-2$ and $a^5, a^4, a^3, a^k b_i, b_i b_j$ for $1 \leq i < j \leq n-2$, $k = 1, 2, 3$. It is straightforward to see that there exist $a, b_1, \dots, b_{n-2} \in F^\times$ such that a^4 is the only eigenvalue of s on V_ω of multiplicity greater than 1, whence the result in this case.

If $n = 2$, then let $s = \mathrm{diag}(a, -a, -a^{-2})$. Then $\alpha_1(s) = (-\alpha_1)(s) = -1$ and the eigenvalues of s on V_ω are $a^2, a^2, a^{-4}, -a^2, -a^{-1}, a^{-1}$. So s is almost cyclic on V_ω for a suitable choice of $a \in F^\times$, and yet s is not Ad-regular.

(1) A similar construction can be used when $\omega \in \{\varpi_1, \dots, \varpi_n\}$. The assumption on the characteristic in case $n = 1$ is necessary as all nonidentity semisimple elements in $\mathrm{SL}_2(F)$ are Ad-regular when $\mathrm{char}(F) = 2$. □

We note that Remark 2.8 and Lemma 3.4 establish the statements of Theorem 1.5 for the groups of type A_n.

3.2. Groups of type B_n, C_n, and D_n

In this section, we prove the main results for classical groups not of type A_n. Recall that we assume that $n \geq 3$ when $G = \mathsf{B}_n$, $n \geq 2$ when $G = \mathsf{C}_n$, and $n \geq 4$ when $G = \mathsf{D}_n$.

The weights of V_{ϖ_1} are $\pm\varepsilon_1, \dots, \pm\varepsilon_n$ and, additionally, 0 when G is of type B_n and $p \neq 2$. So every element $s \in T$ can be written as $s = \mathrm{diag}(a_1, \dots, a_n, 1, a_n^{-1}, \dots \dots, a_1^{-1})$, with respect to some basis of V_{ϖ_1}, where 1 must be dropped unless $G = \mathsf{B}_n$ with $p \neq 2$.

Lemma 3.5. *Let $G = \mathsf{B}_n$, $n \geq 3$, or D_n, $n \geq 4$, or $G = \mathsf{C}_n$, $n \geq 3$ and $p = 2$. Let $\omega \in \Omega_p^+$, $\omega \neq 0$, and $s \in T_{\mathrm{reg}}$. Suppose that s is almost cyclic on V_ω. Then s is Ad-regular unless one of the following holds:*

(i) *$G = \mathsf{B}_n$ and $\omega \in \{\varpi_1, \varpi_n\}$;*
(ii) *$G = \mathsf{B}_n$, $p = 2$ and $\omega = \varpi_2$;*
(iii) *$G = \mathsf{C}_n$ and $\omega \in \{\varpi_1, \varpi_2, \varpi_n\}$; or*
(iv) *$G = \mathsf{D}_n$ and $\omega \in \{\varpi_1, \varpi_{n-1}, \varpi_n\}$.*

Proof. By Theorem 1.6, all nonzero weights of V_ω are of multiplicity 1. The nontrivial p-restricted irreducible G-modules with this property are listed in Tables 2 and 3. We find that $\omega \in \{\varpi_1, 2\varpi_1, \varpi_2, \varpi_n\}$ for $G = \mathsf{B}_n$, $\omega \in \{\varpi_1, 2\varpi_1, \varpi_2, \varpi_{n-1}, \varpi_n\}$ for $G = \mathsf{D}_n$, and $\omega \in \{\varpi_1, \varpi_2, \varpi_n\}$ for $G = \mathsf{C}_n$. Lemma 2.7 then gives the result unless $G = \mathsf{B}_n$ or D_n and $\omega = 2\varpi_1$.

So suppose that $p \neq 2$, $\omega = 2\varpi_1$ and s is almost cyclic on V_ω. Note that $\varpi_2 = 2\varpi_1 - \alpha_1$, so by Lemma 2.6(1), $\Omega(V_{\varpi_2}) \subset \Omega(V_{2\varpi_1})$. Moreover, the 0 weight of $V_{2\varpi_1}$ has multiplicity strictly greater than 1 (see Table 3). Finally, as $V_a = V_{\varpi_2}$ and $p \neq 2$, we have $\Phi(G) \subset \Omega(V_{2\varpi_1})$. Now suppose that s is not Ad-regular; then Lemma 2.7 implies that s is not almost cyclic on V_{ϖ_2}, and since here as well the 0 weight occurs with multiplicity at least 2, we have that there exist $\alpha, \beta \in \Phi(G)$, $\alpha \neq \beta$ with $\alpha(s) = \beta(s) \neq 1$. But this then implies that s is not almost cyclic on $V_{2\varpi_1}$, giving the result. □

Corollary 3.6. *Theorem* 1.2 *and Theorem* 1.4 *hold for* $G = \mathsf{B}_n$, $n \geq 3$, $G = \mathsf{D}_n$, $n \geq 4$, *and* $G = \mathsf{C}_n$ *with* $n \geq 3$ *and* $p = 2$.

Proof. For Theorem 1.2, we take $s \in T$ and $\omega \in \Omega_p^+$, $\omega \neq 0$, such that s is cyclic on V_ω. Then all weights of V_ω occur with multiplicity 1 and so ω is as in Table 2. As all of these weights appear in the conclusion of Theorem 1.2, the result holds.

For Theorem 1.4, we take ω as before and suppose that $s \in T_{\mathrm{reg}}$, with s almost cyclic on V_ω. Then Lemma 3.5 gives the result. □

We now turn to the consideration of the group $G = \mathsf{C}_n$, when $p \neq 2$. Our first result gives some information about the associated weight lattice Ω.

Lemma 3.7. *Let* Φ *be of type* C_n *and let* $\omega \in \Omega^+$. *Suppose that* $\omega \notin \{0, 2\varpi_1\}$ *and* ω *is not a fundamental dominant weight. Then one of the following holds:*

(i) $\omega \succeq \varpi_1 + \varpi_2$;
(ii) $n \geq 3$ *and* $\omega \succeq \varpi_1 + \varpi_3$; *or*
(iii) $n = 2$ *and* $\omega \succeq 2\varpi_2$.

Proof. If ω is not radical (so $\omega \succ \varpi_1$ by [12, Lem. 12]), then by [14, Lem. 2.2] we see that (i) holds. So suppose now that ω is radical and set $\omega = \sum c_i\varpi_i$, with $\sum c_i \geq 2$. Consider first the case $n = 2$. If $c_2 \geq 2$, then $\omega - 2\varpi_2$ is a radical dominant weight, so $\omega \succeq 2\varpi_2$. If $c_2 = 1$, then c_1 is even and $c_1 \geq 2$ by hypothesis, so again $\omega \succ \omega - \alpha_1 = (c_1 - 2)\varpi_1 + 2\varpi_2 \succeq 2\varpi_2$. So finally, suppose that $c_2 = 0$, so that $c_1 \geq 4$ is even. Then $\omega \succeq \omega - 2\alpha_1 = (c_1 - 4)\varpi_1 + 2\varpi_2 \succeq 2\varpi_2$.

Consider now the case $n \geq 3$. If $c_1c_3 \neq 0$ then $\omega - \varpi_1 - \varpi_3$ is a dominant radical weight, so $\omega - \varpi_1 - \varpi_3 \succeq 0$, whence the result. Next suppose that $c_ic_j \neq 0$ for some $i < j$, with i, j both odd and $(i, j) \neq (1, 3)$. As $\varpi_i \succ \varpi_{i-2}$ for $i \geq 2$, it follows that $\omega \succ \omega' = \omega + \varpi_1 + \varpi_3 - \varpi_i - \varpi_j$, and ω' is a dominant radical weight having nonzero coefficient of ϖ_1 and ϖ_3. By the previous argument, $\omega' \succeq \varpi_1 + \varpi_3$, whence the result in this case. The same argument works if $c_i \geq 2$ for some odd $i \geq 3$, as $\varpi_{i-2} = \varpi_i - \beta$ for some $\beta \in \Phi^+$, and so $c_i\varpi_i = (c_i - 1)\varpi_i + \varpi_i = (c_i - 1)\varpi_i + \varpi_{i-2} + \beta$, and we are in the previous case. We are left with the case where $c_i = 0$ for all odd i and either $c_i \geq 2$ for some even i or $c_ic_j \neq 0$ for some $i < j$ with i, j both even. In each case, it is a straightforward check to see that $\omega \succeq 2\varpi_2 \succ \varpi_1 + \varpi_3$ as required. □

Proposition 3.8. *Let* $G = \mathsf{C}_n$, $n \geq 3$, $p \neq 2$, *and* $s \in T$. *Let* $\omega \in \Omega_p^+$, $\omega \neq 0$, *and suppose that* s *is almost cyclic on* V_ω. *Then* s *is Ad-regular unless one of the following holds:*

(1) $\omega \in \{\varpi_1, \varpi_2\}$;
(2) $\omega = \varpi_{n-1}$, *with* $p = 3$ *and* $n \geq 4$; *or*
(3) $\omega = \varpi_n$, *with* $p = 3$ *if* $n \geq 5$.

Proof. We assume that $\omega \neq \varpi_1$. Then by Theorem 1.3, s is regular so $\alpha(s) \neq 1$ for every root α. By Theorem 1.6, ω occurs in Table 2 or 3. The case $\omega = 2\varpi_1 = \varpi_a$ is settled in Lemma 2.7, so we assume as well that $\omega \neq 2\varpi_1$. Inspecting the entries of Tables 2 and 3, we see that either the statement of the result holds or we have $p > 3$ and $\omega \in \{\varpi_{n-1} + \frac{p-3}{2}\varpi_n,\ \frac{p-1}{2}\varpi_n\}$. So consider these remaining configurations. By Theorem 3.7, either $\omega \succeq \varpi_1 + \varpi_2$ or $\omega \succeq \varpi_1 + \varpi_3$. By Lemma 2.5, we have $\Omega(V_{\varpi_1+\varpi_2}) \subseteq \Omega(V)$, respectively $\Omega(V_{\varpi_1+\varpi_3}) \subseteq \Omega(V)$.

Suppose that s is not Ad-regular, so that we have $\alpha(s) = \beta(s)$ for some $\alpha, \beta \in \Phi(G)$, $\alpha \neq \beta$. The roots of G are $\pm 2\varepsilon_i, \pm\varepsilon_j \pm \varepsilon_k$ for $i, j, k \in \{1, \dots, n\}$. As s is regular, we have $\varepsilon_i(s) \neq \pm 1$ for all i and $(\varepsilon_j \pm \varepsilon_k)(s) \neq 1$, for all $j \neq k$. Using the Weyl group, we can assume that s satisfies one of the following conditions:

(i) $(2\varepsilon_1)(s) = (2\varepsilon_2)(s)$;
(ii) $(2\varepsilon_1)(s) = (\varepsilon_2 - \varepsilon_1)(s)$;
(iii) $(2\varepsilon_1)(s) = (\varepsilon_2 - \varepsilon_3)(s)$;
(iv) $n \geq 4$ and $(\varepsilon_1 - \varepsilon_2)(s) = (\varepsilon_3 - \varepsilon_4)(s)$; or
(v) $(2\varepsilon_1)(s) = (-2\varepsilon_1)(s)$.

We now consider the two cases $\omega \succeq \varpi_1 + \varpi_2$ or $\omega \succeq \varpi_1 + \varpi_3$ separately.

Suppose first that $\omega \succeq \varpi_1 + \varpi_2 \succ \varpi_1$. Then V_ω has weights $\pm\varepsilon_i$ $(i = 1, \dots, n)$ and $\pm 2\varepsilon_i \pm \varepsilon_j$, for $1 \leq i \neq j \leq n$. If (i) holds, $(2\varepsilon_1)(s) = (2\varepsilon_2)(s)$, so $(2\varepsilon_1 \pm \varepsilon_3)(s) = (2\varepsilon_2 \pm \varepsilon_3)(s)$. Since s is almost cyclic on V_ω, we have $(2\varepsilon_1 + \varepsilon_3)(s) = (2\varepsilon_1 - \varepsilon_3)(s)$, whence $\varepsilon_3(s) = \pm 1$, contradicting s regular. If (ii) holds, $(2\varepsilon_1)(s) = (\varepsilon_2 - \varepsilon_1)(s)$, so that $(2\varepsilon_1 - \varepsilon_2)(s) = (-\varepsilon_1)(s)$. As $\pm(2\varepsilon_1 - \varepsilon_2)$, $\pm\varepsilon_1$ are weights of V_ω, it follows that $(-\varepsilon_1)(s) = -1 = \varepsilon_1(s)$, again a contradiction. If (iii) holds, $(2\varepsilon_1)(s) = (\varepsilon_2 - \varepsilon_3)(s)$, so that $(2\varepsilon_1 - \varepsilon_2)(s) = (-\varepsilon_3)(s)$ and we get a contradiction as in (ii). Note that $\varpi_1 + \varpi_2 \succ \varpi_3$, so V_ω contains the weights of V_{ϖ_3}. These include the weights $\pm(\varepsilon_1 - \varepsilon_3 + \varepsilon_4)$, if $n \geq 4$. If (iv) holds, then $(-\varepsilon_1 + \varepsilon_3 - \varepsilon_4)(s) = (-\varepsilon_2)(s)$, and $(\varepsilon_1 - \varepsilon_3 + \varepsilon_4)(s) = \varepsilon_2(s)$; as usual, s almost cyclic on V_ω implies that $\varepsilon_2(s) = (-\varepsilon_2)(s)$, again a contradiction. Finally, if (v) holds, so that $(2\varepsilon_1)(s) = (-2\varepsilon_1)(s)$, then $(2\varepsilon_1 + \varepsilon_2)(s) = (-2\varepsilon_1 + \varepsilon_2)(s)$ and $(-2\varepsilon_1 + \varepsilon_3)(s) = (2\varepsilon_1 + \varepsilon_3)(s)$, which then implies that $\varepsilon_2(s) = \varepsilon_3(s)$ so $(\varepsilon_2 - \varepsilon_3)(s) = 1$, contradicting that s is regular.

Suppose now that $\omega \succeq \varpi_1 + \varpi_3 \succ 2\varpi_1 \succ \varpi_2$. Then $\Omega(V_\lambda) \subseteq \Omega(V_\omega)$ for $\lambda \in \{\varpi_1 + \varpi_3, 2\varpi_1, \varpi_2\}$; in particular, $\pm 2\varepsilon_i \pm \varepsilon_j \pm \varepsilon_k$ are weights of V_ω for i, j, k distinct, as are $\pm 2\varepsilon_i$ for $1 \leq i \leq n$. If (i) holds, then $(2\varepsilon_1)(s) = (2\varepsilon_2)(s)$ and $(-2\varepsilon_1)(s) = (-2\varepsilon_2)(s)$. As usual, s almost cyclic on V_ω implies that $(2\varepsilon_1)(s) = (-2\varepsilon_2)(s)$, so that we have $(2\varepsilon_1 + \varepsilon_2 + \varepsilon_3)(s) = (\varepsilon_3 - \varepsilon_2)(s)$. Similarly, we find that $(-2\varepsilon_1 + \varepsilon_2 - \varepsilon_3)(s) = (-\varepsilon_2 - \varepsilon_3)(s)$ and we deduce that $\varepsilon_3(s) = (-\varepsilon_3)(s)$ contradicting that s is regular. If (ii) holds, $(2\varepsilon_1)(s) = (\varepsilon_2 - \varepsilon_1)(s)$, then $(2\varepsilon_1 + \varepsilon_2 + \varepsilon_3)(s) = (-\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3)(s)$ and $(2\varepsilon_1 + \varepsilon_2 - \varepsilon_3)(s) = (-\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3)(s)$. As $2\varepsilon_1 + \varepsilon_2 \pm \varepsilon_3$ and $-\varepsilon_1 + 2\varepsilon_2 \pm \varepsilon_3$ are weights of $V_{\varpi_1+\varpi_3}$, it follows that $(2\varepsilon_1 + \varepsilon_2 + \varepsilon_3)(s) = (2\varepsilon_1 + \varepsilon_2 - \varepsilon_3)(s)$, whence $(2\varepsilon_3)(s) = 1$, a contradiction. If (iii) holds, $(2\varepsilon_1)(s) = (\varepsilon_2 - \varepsilon_3)(s)$, then $(2\varepsilon_1 - \varepsilon_2 + \varepsilon_3)(s) = 1$ and $(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3)(s) = (-2\varepsilon_3)(s)$. The first equality means that the multiplicity of the eigenvalue 1 is greater than 1 and then the second equality implies that $(-2\varepsilon_3)(s) = 1$, a contradiction. If (iv) holds, so that $n \geq 4$ and $(\varepsilon_1 - \varepsilon_2)(s) = (\varepsilon_3 - \varepsilon_4)(s)$, then $(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3)(s) = (\varepsilon_1 - \varepsilon_4)(s)$ and $(-\varepsilon_1 + \varepsilon_3 - 2\varepsilon_4)(s) = (-\varepsilon_2 - \varepsilon_4)(s)$. As s is almost cyclic on V_ω, we have $(\varepsilon_1 - \varepsilon_4)(s) = (-\varepsilon_2 - \varepsilon_4)(s)$, whence $\varepsilon_1(s) = (-\varepsilon_2)(s)$, giving the usual contradiction. Finally, if (v) holds, then $(2\varepsilon_1)(s) = (-2\varepsilon_1)(s)$, and both weights $\pm 2\varepsilon_1$ occur in $V_{2\varpi_1}$. We have as well $(2\varepsilon_1 + \varepsilon_2 + \varepsilon_3)(s) = (-2\varepsilon_1 + \varepsilon_2 + \varepsilon_3)(s)$. So we deduce that $(2\varepsilon_1 + \varepsilon_2 + \varepsilon_3)(s) = (2\varepsilon_1)(s)$ giving $(\varepsilon_2 + \varepsilon_3)(s) = 1$ and the final contradiction. □

Corollary 3.9. *Theorem* 1.2 *and Theorem* 1.4 *hold for $G = \mathsf{C}_n$, $n \geq 3$ and $p \neq 2$.*

Proof. We first let $s \in T$ and $\omega \in \Omega_p^+$, $\omega \neq 0$, such that s is cyclic on V_ω. Then all weights of V_ω occur with multiplicity 1 and so ω appears in Table 2. In addition, Proposition 3.8 applies. So either s is Ad-regular or one of the following holds:

(i) $\omega = \varpi_1$;
(ii) $\omega = \varpi_2$ and $(n, p) = (3, 3)$;
(iii) $n = 3$ and $\omega = \varpi_3$; or
(iv) $n \geq 4$, $p = 3$ and $\omega \in \{\varpi_{n-1}, \varpi_n\}$.

These are precisely the weights listed in the statement of Theorem 1.2.

Now for Theorem 1.4, we let ω be as above, but this time we take $s \in T_{\mathrm{reg}}$ and suppose that s is almost cyclic on V_ω. Then Proposition 3.8 applies and we find that either s is Ad-regular or ω is as listed in (i)–(iv) above, or $\omega = \varpi_2$, for $(n, p) \neq (3, 3)$, or $n = 4$ and $\omega = \varpi_4$ with $p \neq 3$. Hence we have the result of Theorem 1.4. □

We now consider the group C_2.

Lemma 3.10. *Let $G = \mathsf{C}_2$, and let $s \in T_{\mathrm{reg}}$. Then,*

(1) *s is cyclic on V_{ϖ_1} and almost cyclic on V_{ϖ_2}. If s is not cyclic on V_{ϖ_2} then $p \neq 2$ and -1 is an eigenvalue of s on V_{ϖ_2} of multiplicity* 2.
(2) *There exists $s' \in T_{\mathrm{reg}}$, with s' not Ad-regular, and s' cyclic on V_{ϖ_1} and on V_{ϖ_2}.*
(3) *Let $\omega \in \Omega_p^+$, $p \neq 2$, $\omega \notin \{0, \varpi_1, \varpi_2\}$. If s is almost cyclic on V_ω, then s is Ad-regular.*

Proof. The proof of (1) is straightforward and left to the reader. For (2), define s' by $\varepsilon_1(s') = a$, $\varepsilon_2(s') = a^3$, where $a \in F^\times$ and $a^{24} \neq 1$. Then s' is cyclic on V_{ϖ_1} (and hence regular) and is not Ad-regular as $(2\varepsilon_1)(s') = (\varepsilon_2 - \varepsilon_1)(s')$. The eigenvalues of s' on V_{ϖ_2} are $1, a^{\pm 2}, a^{\pm 4}$ (where 1 is to be dropped if $p = 2$), so s' is cyclic on V_{ϖ_2}.

(3) As usual, Theorem 1.6 implies that ω appears in Table 2 or Table 3; since the weight $\omega = 2\varpi_1$ is handled in Lemma 2.7, we have to examine the cases with $p \neq 2$ and $\omega \in \{2\varpi_2, \varpi_1 + \frac{p-3}{2}\varpi_2, \frac{p-1}{2}\varpi_2\}$.

Suppose that s is not Ad-regular. As in the proof of Proposition 3.8, we can assume that one of the following holds:

(i) $(2\varepsilon_1)(s) = (2\varepsilon_2)(s)$;
(ii) $(2\varepsilon_1)(s) = (\varepsilon_2 - \varepsilon_1)(s)$; or
(iii) $(2\varepsilon_1)(s) = (-2\varepsilon_1)(s)$.

By Lemma 3.7, either $\omega \succeq \varpi_1 + \varpi_2 \succ \varpi_1$ or $\omega \succeq 2\varpi_2$.

Suppose first that $\omega \succeq \varpi_1 + \varpi_2$. Note that $\pm 2\varepsilon_1 \pm \varepsilon_2, \pm\varepsilon_1 \pm 2\varepsilon_2, \pm\varepsilon_1, \pm\varepsilon_2 \in \Omega(V_{\varpi_1+\varpi_2})$. If (i) holds, then $(2\varepsilon_1 - \varepsilon_2)(s) = \varepsilon_2(s)$ and $(-2\varepsilon_1 + \varepsilon_2)(s) = -\varepsilon_2(s)$. As s is almost cyclic on V_ω, it follows that $\varepsilon_2(s) = -\varepsilon_2(s)$, which is a contradiction as $2\varepsilon_2$ is a root. If (ii) holds, then $(2\varepsilon_1)(s) = (\varepsilon_2 - \varepsilon_1)(s)$ implies that $(2\varepsilon_1 - \varepsilon_2)(s) = (-\varepsilon_1)(s)$ and $(-2\varepsilon_1 + \varepsilon_2)(s) = \varepsilon_1(s)$. As s is almost cyclic on V_ω and $\pm(2\varepsilon_1 - \varepsilon_2)$, $\pm\varepsilon_1 \in \Omega(V_{\varpi_1+\varpi_2})$, it follows that $\varepsilon_1(s) = (-\varepsilon_1)(s)$, whence $(2\varepsilon_1)(s) = 1$, which contradicts s being regular. Finally, if (iii) holds, so that $(2\varepsilon_1)(s) = (-2\varepsilon_1)(s)$, then $(2\varepsilon_1 \pm \varepsilon_2)(s) = (-2\varepsilon_1 \pm \varepsilon_2)(s)$ which implies as before that $\varepsilon_2(s) = (-\varepsilon_2)(s)$, giving the usual contradiction.

Suppose that $\omega \succeq 2\varpi_2$. As $2\varpi_2 \succ 2\varpi_1 \succ \varpi_2$, we observe that $\pm 2\varepsilon_1 \pm 2\varepsilon_2$, $\pm 2\varepsilon_1$, $\pm 2\varepsilon_2$, $\pm\varepsilon_1 \pm \varepsilon_2$ are weights of V_ω. If (i) holds, then $\pm(2\varepsilon_1 - 2\varepsilon_2)(s) = 1$ and hence the multiplicity of the eigenvalue 1 of s on V_ω is at least 2. In addition, $(2\varepsilon_1)(s) = (2\varepsilon_2)(s)$, so $(2\varepsilon_2)(s) = 1$ as s is almost cyclic on V_ω. This is a contradiction. If (ii) holds, that is $(2\varepsilon_1)(s) = (\varepsilon_2 - \varepsilon_1)(s)$, then $(-2\varepsilon_1)(s) = (-\varepsilon_2 + \varepsilon_1)(s)$. So $(\varepsilon_2 - \varepsilon_1)(s) = -1$ and the multiplicity of -1 as an eigenvalue of s on V_ω is at least 2. As $\pm(2\varepsilon_1 - 2\varepsilon_2) \in \Omega(V_{2\varpi_2})$ and $(2\varepsilon_1 - 2\varepsilon_2)(s) = 1$, we conclude that 1 is an eigenvalue of s on V_ω of multiplicity at least 2. This is a contradiction, as s is almost cyclic on V_ω. Finally, if (iii) holds, then we have $(2\varepsilon_1)(s) = (-2\varepsilon_1)(s)$, and so $(2\varepsilon_1 + 2\varepsilon_2)(s) = (-2\varepsilon_1 + 2\varepsilon_2)(s)$, from which we deduce that $(2\varepsilon_2)(s) = 1$, a contradiction. □

Corollary 3.11. *Theorem* 1.2 *and Theorem* 1.4 *hold for* $G = \mathsf{C}_2$.

Proof. Recall that for $s \in T$ and $\omega \in \Omega_p^+$, $\omega \neq 0$, if s is almost cyclic on V_ω, then ω appears in Table 2 or Table 3. In particular, for $p = 2$, the only weights which need to be considered are ϖ_1 and ϖ_2 and both of these are in the statements of the theorems. Henceforth we assume that $p \neq 2$. If $s \in T$ is cyclic on V_ω for some $\omega \in \Omega_p^+$, $\omega \neq 0$, then again consulting Table 2, we find that $\omega = \varpi_1$ or $\omega = \varpi_2$, as in the conclusion of Theorem 1.2. So finally we are left to consider the case where $s \in T_{\mathrm{reg}}$, $\omega \in \Omega_p^+$, and s is almost cyclic on V_ω. Then here Lemma 3.10(3) shows that either s is Ad-regular or $\omega \in \{0, \varpi_1, \varpi_2\}$ as claimed in Theorem 1.4. □

To complete our consideration of the classical type groups, we must establish the existence of elements as claimed in Theorem 1.5 for the groups of type B_n, C_n, and D_n.

Lemma 3.12. *Let* $G = \mathsf{B}_n$ *with* $n \geq 3$, *or* $G = \mathsf{C}_n$, *with* $p = 2$ *and* $n \geq 3$. *Then there exists* $s \in T_{\mathrm{reg}}$ *such that s is not Ad-regular, but cyclic on each of the modules* V_{ϖ_1} *and* V_{ϖ_n}. *Moreover, if* $G = \mathsf{B}_n$, *then we may choose s so that s is cyclic on* $V_{\varpi_1} \oplus V_{\varpi_n} \oplus \delta_{p,2} V_0$.

Proof. For $0 \leq i \leq n-3$, let $b_i \in F^\times$ be a primitive p_i-th root of unity where $p_0, \ldots, p_{n-3}$ are distinct primes greater than $\max\{p, 11\}$. Set

$$X = \mathrm{diag}(b_0, (b_0)^3, (b_0)^5, b_1, \ldots, b_{n-3}).$$

Now set $t = \mathrm{diag}(X, 1, X^{-1})$, where the 1 is to be dropped if $G = \mathsf{C}_n$. For a suitable choice of basis for the natural module for G, we have $t \in \mathrm{SO}_{2n+1}(F)$, respectively $t \in \mathrm{Sp}_{2n}(F)$, for $G = \mathsf{B}_n$, respectively $G = \mathsf{C}_n$.

We first note that t is cyclic on V_{ϖ_1}. In addition, $(\varepsilon_2-\varepsilon_1)(t) = b_0^2 = (\varepsilon_3-\varepsilon_2)(t)$, so t is not Ad-regular.

Consider first the case $G = \mathsf{B}_n$ and let $s \in G$ be a semisimple element acting on V_{ϖ_1} as t does. In particular, s is regular (since cyclic on V_{ϖ_1}), but s is not Ad-regular. Now the weights of $V_{\varpi_1} \oplus V_{\varpi_n} \oplus \delta_{p,2}V_0$ are $0, \pm\varepsilon_i$ $(1 \le i \le n)$, $\frac{1}{2}(\pm\varepsilon_1 \pm \cdots \pm \varepsilon_n)$. Hence, the remaining claim will follow if we show that the elements $\pm\varepsilon_i(s), 1, (\frac{1}{2}(\pm\varepsilon_1 \pm \cdots \pm \varepsilon_n))(s)$ are distinct. As $|t|$ is odd, we may assume that $|s|$ is odd as well. Hence, it suffices to show that the eigenvalues $2\varepsilon_i(s), 1, (\pm\varepsilon_1 \pm \cdots \pm \varepsilon_n)(s)$ are distinct. The elements $(\pm\varepsilon_1 \pm \cdots \pm \varepsilon_n)(s)$ have the form $(b_0)^{\pm m}x$, for $m \in \{1, 3, 7, 9\}$ and for some $x \in \{b_1^{\pm 1} \cdots b_{n-3}^{\pm 1}\}$. As $p_i > 11$, all these elements are distinct, and differ from $1, (b_0)^{\pm 2}, (b_0)^{\pm 6}, b_0^{10}$, and $b_1^{\pm 2}, \ldots, b_{n-3}^{\pm 2}$, whence the result.

The argument for $G = \mathsf{C}_n$ is entirely similar as the weights of V_{ϖ_n} are $\pm\varepsilon_1 \pm \cdots \pm \varepsilon_n$. $\square$

Lemma 3.13. *Let $G = \mathsf{D}_n$, $n \ge 4$. Then there exists $s \in T_{\mathrm{reg}}$ such that s is not almost cyclic on V_{ϖ_2} (equivalently, s is not Ad-regular), but cyclic on $V_{\varpi_1} \oplus V_{\varpi_{n-1}} \oplus V_{\varpi_n} \oplus V_0$.*

Proof. As $\varpi_a = \varpi_2$, by Lemma 2.7, $s' \in T$ is almost cyclic on V_{ϖ_2} if and only if s' is Ad-regular. We now exhibit the existence of the s as claimed in the statement.

The group $G = \mathsf{D}_n$ is a maximal rank subgroup of B_n generated by root subgroups corresponding to long roots. The element s introduced in Lemma 3.12 is therefore contained in a subgroup $G = \mathsf{D}_n$, and s is regular as an element of D_n, since it is regular in B_n. Moreover, we have seen in the proof of Lemma 3.12 that $(\varepsilon_2-\varepsilon_1)(s) = (\varepsilon_3-\varepsilon_2)(s)$; as $\varepsilon_2-\varepsilon_1$ and $\varepsilon_3-\varepsilon_2$ are long roots of B_n, they are roots of the subgroup D_n. So s is not Ad-regular in D_n. Furthermore, for the purposes of this proof, let $\eta_1, \ldots, \eta_n$ denote the fundamental dominant weights of B_n. Then, the irreducible module for B_n with highest weight η_n restricts to D_n as $V_{\varpi_{n-1}} \oplus V_{\varpi_n}$. In addition, $V_{\eta_1}|_G = V_{\varpi_1} \oplus (1-\delta_{p,2})V_0$. Now Lemma 3.12 shows that s is cyclic on $V_{\eta_1} \oplus V_{\eta_n} \oplus \delta_{p,2}V_0$, and so s is cyclic on $V_{\varpi_1} \oplus V_{\varpi_{n-1}} \oplus V_{\varpi_n} \oplus V_0$, as claimed. $\square$

Remark 3.14. Note that the above two results provide a proof of Theorem 1.5(1) for $G = \mathsf{B}_n$, $n \ge 3$, $G = \mathsf{D}_n$ with $n \ge 4$, and $G = \mathsf{C}_n$ for $n \ge 3$ and $p = 2$.

Lemma 3.15. *Let $G = \mathsf{C}_n$, $n \ge 2$. Then there exists $s \in T_{\mathrm{reg}}$, not Ad-regular, with s almost cyclic on V_{ϖ_2}.*

Proof. Define s by $\varepsilon_1(s) = a$, $\varepsilon_2(s) = a^3$, $\varepsilon_{i+2}(s) = b_i$, for $a, b_1, \ldots, b_{n-2} \in F^\times$. Then s is not Ad-regular as $(2\varepsilon_1)(s) = a^2 = (\varepsilon_2-\varepsilon_1)(s)$, and $2\varepsilon_1, \varepsilon_2-\varepsilon_1$ are roots of G. Moreover, one may choose the a, b_i such that s is regular. The eigenvalues of s on V_{ϖ_2} are $1, a^{\pm 2}, a^{\pm 4}, a^{\pm 1}b_i^{\pm 1}, a^{\pm 3}b_i^{\pm 1}, b_i^{\pm 1}b_j^{\pm 1}, i \ne j$. We may choose $a, b_1, \ldots, b_{n-2} \in F^\times$ such that the eigenvalues different from 1 are distinct. Indeed, the nonzero weights of V_{ϖ_2} are roots and by choosing s regular, we know that the multiplicity of the eigenvalue 1 is precisely the multiplicity of the zero weight in V_{ϖ_2}, while all other weights have multiplicity 1. Hence s is almost cyclic on V_{ϖ_2}. $\square$

Remark 3.16. A similar argument works for $G = \mathsf{B}_n$, with $p = 2$; for this case, set $\varepsilon_1(s) = a$ and $\varepsilon_2(s) = a^2$ and the remaining values as for C_n.

Lemma 3.17. *Let $G = \mathsf{C}_n$, $n \geq 2$. Then there exists $s \in T_{\text{reg}}$, s not Ad-regular, which separates the weights of V_{ϖ_k}, for $k = 1, \ldots, n$.*

Proof. Let $a, b_i \in F^\times$, for $1 \leq i \leq n$, and define $s \in T$ by $\varepsilon_i(s) = b_i$ for $1 \leq i \leq n$ and in addition $b_1 = a$ and $b_2 = a^3$. We first note that s is not Ad-regular as $2\varepsilon_1(s) = (\varepsilon_2 - \varepsilon_1)(s)$. We now show that by choosing appropriate values for the b_i, the element s separates the weights of V_{ϖ_k}, as claimed. This will also show that s is regular as s is then cyclic on V_{ϖ_1}.

The nonzero weights of V_{ϖ_k} are $\pm\varepsilon_{i_1} \pm \cdots \pm \varepsilon_{i_j}$ for $j \leq k$ with $k - j$ even and $1 \leq i_1 < \cdots < i_j \leq n$. This follows from the fact that V_{ϖ_j} occurs as a composition factor of $\wedge^j(V_{\varpi_1})$, for all $1 \leq j \leq n$, and the weights subdominant to ϖ_k, for $k \geq 2$, are of the form ϖ_{k-2i} for $0 \leq i \leq \lfloor k/2 \rfloor$. A weight $\pm\varepsilon_{i_1} \pm \cdots \pm \varepsilon_{i_j}$ takes the value $b_{i_1}^{\pm 1} \cdots b_{i_j}^{\pm 1} = xy$, on s, where $x = a^{\pm m}$ for some $0 \leq m \leq 4$ and y is a product of certain $b_i^{\pm 1}$, with $i > 2$. (In particular, if $n = 2$ then the weights of V_{ϖ_k} take values among $a^{\pm 1}, a^{\pm 3}, a^{\pm 2}, a^{\pm 4}, 1$ on s.) For a suitable choice of a and b_i, these values are distinct. For example, we can choose b_i to be a primitive p_i-th root of unity for $3 \leq i \leq n$ for primes $p_3 > \cdots > p_n > \max\{2, p\}$, and a to be a primitive p_0-th root of unity for a prime $p_0 > \max\{p, p_3, 7\}$. Therefore, s separates the weights of V_{ϖ_k} for every fixed k. □

Proposition 3.18. *Let $G = \mathsf{C}_n, n \geq 2$, $p \neq 2$.*

(1) *Let $\omega \in \Omega_p^+$ be as follows:*
   (i) *$\omega = \varpi_1$;*
   (ii) *$\omega = \varpi_{n-1}$ with $n \geq 3$ and $p = 3$; or*
   (iii) *$\omega = \varpi_n$ with $n \geq 3$ and $p = 3$ if $n \geq 4$.*

   Then there exists $s \in T_{\text{reg}}$, with s not Ad-regular but cyclic on V_ω.

(2) *Let $\omega = \varpi_4$, with $n = 4$ and $p \neq 3$. Then there exists $s \in T_{\text{reg}}$, with s not Ad-regular but almost cyclic on V_ω.*

Proof. This follows from Lemma 3.17 and Tables 2 and 3. Indeed, if s separates the weights of V then s is cyclic on V whenever all weights of V are of multiplicity 1, and almost cyclic if exactly one weight is of multiplicity greater than 1. □

Lemma 3.19. *Theorem* 1.5 *is true if G is of classical type.*

Proof. For G of type A_n, this was established in Section 3.1. For Theorem 1.5(1), we refer to Remark 3.14 and note that it remains to consider $G = \mathsf{C}_2$ and $G = \mathsf{C}_n$ with $n \geq 3$ and $p \neq 2$. For $G = \mathsf{C}_2$, we must establish the existence of $s \in T$, cyclic on V_{ϖ_1} and on V_{ϖ_2}, with s not Ad-regular. This is covered by Lemma 3.10(2).

For $G = \mathsf{C}_n$, $n \geq 3$ and $p \neq 2$, we must establish the existence of $s \in T$, s not Ad-regular, with s cyclic on

(a) V_{ϖ_1},
(b) V_{ϖ_3} when $n = 3$, and
(c) $V_{\varpi_{n-1}}$ or V_{ϖ_n}, when $p = 3$.

(If necessary, we may have different s for each of the modules.) This is precisely the statement of Proposition 3.18(1).

Now we turn to the list of modules to be treated for Theorem 1.5(2). For $n \geq 3$, $\omega = \varpi_2$, and $G = \mathsf{C}_n$, or $G = \mathsf{B}_n$ with $p = 2$, the result follows from Lemma 3.15

and Remark 3.16. For $G = \mathsf{C}_4$ with $\omega = \varpi_4$, $p \neq 2, 3$, the result follows from Proposition 3.18(2). $\square$

4. Groups of exceptional types and Theorem 1.7

In this section, we will establish all of the main results in case G is of exceptional type and then deduce Theorem 1.7.

Lemma 4.1. *Theorem* 1.4 *holds for G of exceptional type.*

Proof. Let $\omega \in \Omega_p^+$ and $s \in T$ be almost cyclic on V_ω. Then Theorem 1.6 and Tables 2 and 3 imply that either $\omega = \varpi_a$, or ω is the highest short root (in case $G = \mathsf{G}_2$ or F_4), or ω is a minuscule weight. The minuscule weights appear in the conclusion of Theorem 1.4, as well as the highest short roots. Moreover, Lemma 2.7 shows that if s is almost cyclic on V_a, then either s is Ad-regular or $(G, p) = (\mathsf{G}_2, 3)$ or $(\mathsf{F}_4, 2)$. The latter configurations are also in the list of exceptions in the statement of Theorem 1.4, hence the result. $\square$

Lemma 4.2. *Theorem* 1.2 *holds for G of exceptional type.*

Proof. Let $\omega \in \Omega_p^+$ and $s \in T$ be cyclic on V_ω. Then all weights of V_ω have multiplicity 1 and now comparing the statement of Theorem 1.2 with Table 2 gives the result. $\square$

We are left with proving Theorem 1.5; we treat each of the groups in turn.

Proposition 4.3. *Let G be of type F_4. There exists $s \in T_{\mathrm{reg}}$ such that s is almost cyclic on V_{ϖ_4} but not Ad-regular. If $p = 3$ then s can be chosen to be cyclic on V_{ϖ_4}.*

Proof. Let $V = V_{\varpi_4}$. Recall that G contains a subsystem subgroup H of type B_4, generated by the long root subgroups of G, so $T \leq H$. For the purposes of this proof, let η_i, $1 \leq i \leq 4$ denote the fundamental dominant weights of $H = \mathsf{B}_4$ (with respect to an appropriate choice of Borel subgroup of H). Then the nonzero weights of $V|_H$ are exactly the nonzero weights of the FH-module $V_{\eta_1} \oplus V_{\eta_4}$. By comparison of the dimensions we conclude that the nontrivial composition factors of $V|_H$ are V_{η_1} and V_{η_4}, and there are $(1-\delta_{p,3})(1+\delta_{p,2})$ trivial composition factors.

Now let $s \in H$ be as in Lemma 3.12; then, $\alpha(s) = \beta(s)$, for α and β distinct roots of H, so s is not Ad-regular in F_4. Moreover, s is cyclic on $V_{\eta_1} \oplus V_{\eta_4}$ and so almost cyclic on V. Then Theorem 1.3 implies that s is regular. If $p = 3$, then $V|_H = V_{\eta_1} \oplus V_{\eta_4}$ and Lemma 3.12 shows that s is cyclic on V. $\square$

Remark 4.4. Let $G = \mathsf{F}_4$, $p = 2$.

(1) There exists an element $s \in T_{\mathrm{reg}}$, s not Ad-regular, which is almost cyclic on $V_a = V_{\varpi_1}$ (and not almost cyclic on V_{ϖ_4}). Recall that identifying T with $(F^\times)^4$ and ε_i with the i-th coordinate function for $1 \leq i \leq 4$, the roots of F_4 are $\pm\varepsilon_i$, $\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$ (short) and $\pm\varepsilon_i \pm \varepsilon_j$, $1 \leq i < j \leq 4$ (long). For $1 \leq i \leq 3$, let $a_i \in F^\times$ be a primitive p_i-th root of unity, where $p_1 = 5, p_2 = 7, p_3 = 11$. Define $s \in T$ by $\varepsilon_1(s) = a_1$, $\varepsilon_2(s) = a_2$, $\varepsilon_3(s) = a_1a_2$, $\varepsilon_4(s) = a_3$. The nonzero weights of V_a are $\pm\varepsilon_i \pm \varepsilon_j$ $(1 \leq i < j \leq 4)$ so s is almost cyclic on V_a. In addition, the weights $\frac{1}{2}(\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4)$ and $\frac{1}{2}(-\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4)$ differ by $\varepsilon_1+\varepsilon_2-\varepsilon_3$ and so take the same value on s. But $(\pm\frac{1}{2}(\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4))(s) = (\pm\frac{1}{2}(-\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4))(s) = (\sqrt{a_3})^{\pm 1}$,

and so s is not almost cyclic on V_{ϖ_4}. Note that s is regular as $\alpha(s) \neq 1$ for every root α, but it is not Ad-regular. This justifies one of the claims of Remark 2.8, and together with Proposition 4.3, completes the proof of Theorem 1.5 for the group $G = \mathsf{F}_4$.

(2) There exists a nonregular element $s \in T$ such that $V_a^s = V_a^T$. Indeed, let $\varepsilon_i(s) = t_i \in F^\times$, where $t_1 = 1$. Then s is not regular. The weights of $V_a = V_{\varpi_1}$ are $0, \pm\varepsilon_i \pm \varepsilon_j$, so the eigenvalues of s on the weight spaces for nonzero weights of V_a are $(\pm\varepsilon_i \pm \varepsilon_j)(s)$, that is, $t_i^{\pm 1}, t_i^{\pm 1} t_j^{\pm 1}$, where $i, j = 2, 3, 4, i \neq j$. For an appropriate choice of $t_i \in F^\times$, these eigenvalues differ from 1, and the claim follows. This proves a claim in Remark 2.10 for the group $G = \mathsf{F}_4$.

Proposition 4.5. *Let $G = \mathsf{E}_6$. For $i = 1, 6$, there exists $s_i \in T$, not Ad-regular, which is cyclic on V_{ϖ_i}.*

Proof. It suffices to consider $V = V_{\varpi_1}$. Let $H \leq G$ be a subsystem subgroup of type D_5. For the purposes of this proof, let η_i, for $1 \leq i \leq 5$, denote the fundamental dominant weights of H (with respect to an appropriate choice of maximal torus and Borel subgroup of H). Then, we have that $V|_H = V_{\eta_1} \oplus V_{\eta_4} \oplus V_0$. (Here we have made a choice of the labeling of the roots of H.)

Now let $s \in H$ be as in Lemma 3.13. Then s is cyclic on $V_{\eta_1} \oplus V_{\eta_4} \oplus V_{\eta_5} \oplus V_0$, so in particular on V. As well, s is not Ad-regular in H and since roots of H are roots of G, s is not Ad-regular. $\square$

Proposition 4.6. *Let G be of type E_6. For $i = 1, 6$, there exists $s_i \in T_{\mathrm{reg}}$ such that s_i is not Ad-regular but almost cyclic on V_{ϖ_i}.*

Proof. It suffices to establish the result for V_{ϖ_1}. Let $H \leq G$ be of type F_4. Let $s \in H$ be as constructed in the proof of Proposition 4.3. For the purposes of this proof, let η_i, $1 \leq i \leq 4$ denote the fundamental dominant weights of H (with respect to a fixed choice of Borel subgroup and maximal torus). Then $V_{\varpi_1}|_H = V_{\eta_4} \oplus (1+\delta_{p,3}) V_0$. Since s is almost cyclic on V_{η_4}, and cyclic if $p = 3$, and the zero weight has multiplicity 2 in V_{η_4} if $p \neq 3$, we see that s is almost cyclic on V_{ϖ_1}. We claim that s is regular but not Ad-regular. Recall that $V_a = V_{\varpi_2}$. We note that

$$V_a|_H = \begin{cases} V_{\eta_1} \oplus V_{\eta_4}, & \text{if } p \neq 2, 3; \\ V_{\eta_1} \oplus V_{\eta_4} \oplus V_0, & \text{if } p = 3; \\ V_{\eta_1} \oplus V_{\eta_4} \oplus V_{\eta_4}, & \text{if } p = 2. \end{cases}$$

To see that s is regular, we need to show that the multiplicity of the eigenvalue 1 for s on V_a is equal to $\dim T = 6$. Note that s is regular in H and so the eigenvalue 1 on V_{η_1} has multiplicity $4 - 2\delta_{p,2}$. Also, the nonzero weights of V_{η_4} are roots of H and the zero weight has multiplicity $2 - \delta_{p,3}$ in V_{η_4}; so again the eigenvalue 1 has multiplicity $2 - \delta_{p,3}$ in V_{η_4}. In each of the above decompositions, we find that the eigenvalue 1 for s on V_a has multiplicity 6.

Finally, to see that s is not Ad-regular, we appeal to Lemma 2.7. It suffices to see that s is not almost cyclic on V_a. Recall that $\eta_4 \prec \eta_1$. Using Theorem 2.5 and the above decompositions of $V_a|_H$, we find that all nonzero weights of V_{η_4} occur with multiplicity 2 in V_a, whence the result. $\square$

We will use the following result when treating the group $G = \mathsf{E}_7$.

Proposition 4.7. *Let $G = \mathsf{A}_n$, $n \geq 4$. Then there exists $s \in T_{\mathrm{reg}}$, s not Ad-regular, such that s is not almost cyclic on $V_{\varpi_1+\varpi_n}$ and cyclic on $V = V_{\varpi_2} \oplus V_{\varpi_{n-1}}$.*

Proof. We start by considering a certain subvariety of F^m. Let $m \geq 4$. Set $X = \{(x_1, \dots, x_m) \in F^m \mid x_1 \cdots x_m = 1\}$. Then X is isomorphic to the principal open subset $D(f) \subset F^{m-1}$ defined by the function $f \in F[T_1, \dots, T_{m-1}]$, $f = T_1 \cdots T_{m-1}$. Indeed, the associated F-algebra of X is isomorphic to the algebra $F[T_1, \dots, T_{m-1}, (T_1 \cdots T_{m-1})^{-1}]$ which is (isomorphic to) the localization of the algebra $F[T_1, \dots, T_{m-1}]$ with respect to the multiplicative set $S = \{(T_1 \cdots T_{m-1})^\ell \mid \ell \geq 0\}$. In particular, X is irreducible of dimension $m-1$. Now consider the closed subset Y of X, $Y = \{(x_1, \dots, x_m) \in X \mid x_1^2 = x_2x_3\}$. Then the associated affine F-algebra is $F[X]/(T_1^2 - T_2T_3)$. Since $F[X]$ is a localization of $F[T_1, \dots, T_{m-1}]$ and $T_1^2 - T_2T_3$ is an irreducible polynomial in $F[T_1, \dots T_{m-1}]$, Y is also an irreducible subvariety of F^m, of dimension $m-2$.

We now apply the above reasoning to a maximal torus of $G = \mathrm{SL}_{n+1}(F)$, namely the usual torus of diagonal matrices of determinant 1. Set $s = \mathrm{diag}(d_1, \dots, d_{n+1}) \in T$, where $d_i \in F^\times$ and $\prod_{i=1}^{n+1} d_i = 1$. If we assume further that $d_1^2 = d_2d_3$ then $(\varepsilon_1 - \varepsilon_2)(s) = (\varepsilon_3 - \varepsilon_1)(s)$ and so s is not almost cyclic on $V_{\varpi_1+\varpi_n}$. Let $Y = \{s \in T \mid d_1^2 = d_2d_3\}$ so that by the discussion of the first paragraph, Y is an irreducible closed subvariety of T of dimension $n-1$. The module V has 1-dimensional T-weight spaces. For weights $\lambda, \mu \in \Omega(V)$, with $\lambda \neq \mu$, let $K_{\lambda\mu} = \ker(\lambda - \mu)$. The set $K := \bigcup_{\lambda,\mu} K_{\lambda\mu}$ is a proper closed subset of T. Since Y is irreducible, if $Y \subseteq K$, then there exist $\lambda, \mu \in \Omega(V)$, with $\lambda \neq \mu$, such that $Y \subseteq \ker(\lambda - \mu)^\circ$ and dimension considerations show that these two sets must be equal. But the weights of V are of the form $\pm(\varepsilon_i + \varepsilon_j)$, $1 \leq i < j \leq n+1$. So the possible weight differences $\lambda - \mu$ (up to a sign) are $\varepsilon_i + \varepsilon_j \pm (\varepsilon_k + \varepsilon_\ell)$, $\varepsilon_j - \varepsilon_\ell$, $2\varepsilon_i + \varepsilon_j + \varepsilon_\ell$ and $2(\varepsilon_i + \varepsilon_j)$, for i, j, k, ℓ distinct. So there are no distinct weights λ, μ with $Y = \ker(\lambda - \mu)^\circ$. Hence Y does not lie in K and the elements in $Y \setminus K$ are almost cyclic, indeed cyclic, on V and not almost cyclic on $V_{\varpi_1+\varpi_n}$. So these elements are regular by Lemma 2.2 and not Ad-regular by Lemma 2.7. □

We now use the preceding result to establish the following.

Proposition 4.8. *Let $G = \mathsf{E}_7$. There exists an element $s \in T_{\mathrm{reg}}$, with s not Ad-regular, and such that s is cyclic on V_{ϖ_7}.*

Proof. Set $V = V_{\varpi_7}$. Let $H \leq G$ be a maximal rank subgroup of type A_7 with $T \leq H$, and for the purposes of this proof, let η_i, $1 \leq i \leq 7$, denote the fundamental dominant weights of H (with respect to an appropriate choice of maximal torus T_H and Borel subgroup). Then $V|_H = V_{\eta_2} \oplus V_{\eta_6}$. Let $s \in T_H$ be as in Proposition 4.7, so that s is cyclic on V and not Ad-regular in H. Then s is not Ad-regular as the roots of H are roots of G. Finally, Lemma 2.2 shows that s is regular. □

For the proof of Theorem 1.5, it remains to treat the group $G = \mathsf{G}_2$. We investigate in detail the two modules corresponding to the fundamental dominant weights.

Proposition 4.9. *Let $G = \mathsf{G}_2$, $V_i = V_{\varpi_i}$ for $i = 1, 2$.*

1. *Let $s \in T_{\mathrm{reg}}$ and suppose that s is not cyclic on V_1. Then $p \neq 2$ and the eigenvalues (with multiplicities) of s on V_1 are $1, -1, -1, -b, b^{-1}, b, -b^{-1}$, where $b \in F^\times$, $b^2 \neq 1$. In particular, s is almost cyclic on V_1 if and only if $b^2 \neq -1$.*
2. *Let $s \in T_{\mathrm{reg}}$ and assume that $p = 3$. Then s is almost cyclic on V_2 and cyclic on V_1. More precisely, if s is not cyclic on V_2 then, for some $c \in F^\times, c^4 \neq 1$, the eigenvalues (with multiplicities) of s on V_2 are $1, -1, -1, c^{\pm 3}, -c^{\pm 3}$, and those on V_1 are $1, c^{\pm 1}, -c^{\pm 1}, -c^{\pm 2}$.*
3. *There exists $s \in T_{\mathrm{reg}}$, not Ad-regular, with s cyclic on V_1.*
4. *Let $p = 3$. Then there exists $s \in T_{\mathrm{reg}}$, not Ad-regular, with s cyclic on V_2.*
5. *Let $p = 3$. Then there exists a nonregular element $s \in T$, with $V_a^s = V_a^T$.*

Proof. Recall that we are using the Bourbaki labeling of roots and so α_1 is a short root in $\Phi(G)$. Fix $s \in T$ and set $\alpha_1(s) = c$, $\alpha_2(s) = b$, for some $c, b \in F^\times$.

(1) Here we suppose s to be regular and not cyclic on V_1. The weights of V_1 are the short roots, and the weight 0 if $p \neq 2$, all with multiplicity 1. Then $(\alpha_1 + \alpha_2)(s) = cb$, $(2\alpha_1 + \alpha_2)(s) = c^2 b$ so the eigenvalues of s on V_1 are 1 (if $p \neq 2$), $c^{\pm 1}, (cb)^{\pm 1}, (c^2 b)^{\pm 1}$. Since s is not cyclic on V_1, at least two of the eigenvalues are equal. As s is regular, $\alpha(s) \neq 1$ for every root α, in particular, $1 \notin \{c, cb, c^2 b\}$. As $W(G)$ is transitive on the short roots, we can assume that $c \in \{c^{-1}, (cb)^{\pm 1}, (c^2 b)^{\pm 1}\}$. Now if $c \in \{(cb)^{\pm 1}, (c^2 b)^{\pm 1}\}$, then $1 \in \{b, c^2 b, cb, c^3 b\}$. But $\alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2$ are roots, so this contradicts s being regular. So we have $c = c^{-1}$. Then $c^2 = 1$ and $c = \alpha_1(s) \neq 1$, so $p \neq 2$ and $c = -1$. Then the eigenvalues of s on V_1 are $1, -1, -1, -b, -b^{-1}, b, b^{-1}$, where $b \neq \pm 1$ as s is regular, establishing the first claim. Finally, s is almost cyclic on V_1 if and only if $b^2 \neq -1$.

(2) Here we suppose s to be regular and $p = 3$. The weights of V_2 are the long roots and the weight 0, so the eigenvalues of s on V_2 are $1, b^{\pm 1}, (c^3 b)^{\pm 1}, (c^3 b^2)^{\pm 1}$. Note that s regular implies that $b \neq 1$. As above, if s is not cyclic on V_2, we can assume that $\alpha_2(s) \in \{-\alpha_2(s), \pm(3\alpha_1 + \alpha_2)(s), \pm(3\alpha_1 + 2\alpha_2)(s)\}$. As s is regular, $\alpha_2(s) \neq (-3\alpha_1 - \alpha_2)(s)$ and $\alpha_2(s) \neq (3\alpha_1 + 2\alpha_2)(s)$. If $\alpha_2(s) = (3\alpha_1 + \alpha_2)(s)$ then $(3\alpha_1)(s) = 1$, so $c^3 = 1$, and if $\alpha_2(s) = (-3\alpha_1 - 2\alpha_2)(s)$, then $(cb)^3 = 1$. But since $p = 3$, this implies that $c = 1$, respectively $cb = 1$, contradicting that s is regular. So these two cases are ruled out, and we are left with the case where $\alpha_2(s) = -\alpha_2(s)$. So $b^2 = 1$, and hence $b = -1$, and $c^2 \neq 1$ since s is regular.

We now have that the eigenvalues of s on V_2 are $1, -1, -1, c^3, -c^3, c^{-3}, -c^{-3}$. Suppose for a contradiction that s is not almost cyclic on V_2; then either $c^3 = -1$ or $c^6 = -1$. In the first case, $c = -1$, and hence $cb = 1$ contradicting s regular. Thus $c^6 = -1$, equivalently, $c^2 = -1$. But then, $(2\alpha_1 + \alpha_2)(s) = c^2 b = 1$, again contradicting that s is regular. Hence s is almost cyclic on V_2 as claimed. Note that $c^2 \neq -1$, else $(2\alpha_1 + \alpha_2)(s) = 1$. Now the eigenvalues of s on V_1 are $1, c, c^{-1}, -c, -c^{-1}, -c^2, -c^{-2}$, and hence s is cyclic on V_1.

(3) Let $c = b$, so that the eigenvalues of s on V_1 are $c^{\pm 3}, c^{\pm 2}, c^{\pm 1}, 1$, where 1 is to be dropped if $p = 2$; they are distinct if $c^k \neq 1$ for $k \leq 6$. So choosing c to be an appropriate root of unity, we have that s is cyclic on V_1 (hence regular) and is not Ad-regular as $\alpha_1(s) = \alpha_2(s) = c$.

(4) Take again $c = b$, so that the eigenvalues of s on V_2 are $c^{\pm 1}, c^{\pm 4}, c^{\pm 5}, 1$. Then choosing c appropriately, we see that s is cyclic on V_2, regular and not Ad-regular, as in (3).

The final statement (5) is straightforward, as we may take $c = 1$ so that s is not regular, while the nonzero weights of V_a are $\pm(3\alpha_1 + 2\alpha_2)$, $\pm(3\alpha_1 + \alpha_2)$ and $\pm\alpha_2$; so that choosing b such that $b^2 \neq 1$, we have $V_a^s = V_a^T$. $\square$

We have now completed the proof of Theorem 1.5 for the exceptional groups. Finally, we conclude the article with the following.

Proof of Theorem 1.7. Let ϕ_i be the irreducible representation of G with highest weight λ_i. By Lemma 2.4, $\phi_i(s)$ is cyclic for every i. Therefore, by Theorem 1.2, s is Ad-regular unless, possibly, for every i, (G, λ_i) is as in cases (1) – (8) there. Also, Theorem 1.6 implies that all nonzero weights of V_ω have multiplicity one, and we then apply [11, Thm. 2(2)] and [16, Prop. 2] to justify the conditions given in (i)–(iii) of the statement. $\square$

5. Tables

Tables 2 and 3 are taken from [11, Tables 1,2]. We recall here our convention for reading the tables in case $\mathrm{char}(F) = 0$: for a natural number a the expressions $p > a$, $p \geq a$ or $p \neq a$ are to be interpreted as the absence of any restriction, that is, a is allowed to be any natural number. Note further that when a weight λ has coefficients expressed in terms of p, we are assuming that $\mathrm{char}(F) = p > 0$.

TABLE 2. Non-trivial irreducible p-restricted G-modules V_ω with all weights of multiplicity 1.

G	ω
A_1	$a\varpi_1, 1 \leq a < p$
$\mathsf{A}_n, n > 1$	$a\varpi_1, b\varpi_n,\ 1 \leq a, b < p$ $\varpi_i,\ 1 < i < n$ $c\varpi_i + (p-1-c)\varpi_{i+1},\ 1 \leq i < n,\ 0 \leq c < p$
$\mathsf{B}_n, n > 2$	$\varpi_1,\ \varpi_n$
$\mathsf{C}_n, n > 1, p = 2$	$\varpi_1,\ \varpi_n$
$\mathsf{C}_2, p > 2$	$\varpi_1,\ \varpi_2,\ \varpi_1 + \frac{p-3}{2}\varpi_2,\ \frac{p-1}{2}\varpi_2$
$\mathsf{C}_3, p > 2$	ϖ_3
$\mathsf{C}_n, n > 2, p > 2$	$\varpi_1,\ \varpi_{n-1} + \frac{p-3}{2}\varpi_n,\ \frac{p-1}{2}\varpi_n$
$\mathsf{D}_n, n > 3$	$\varpi_1,\ \varpi_{n-1},\ \varpi_n$
E_6	$\varpi_1,\ \varpi_6$
E_7	ϖ_7
$\mathsf{F}_4, p = 3$	ϖ_4
$\mathsf{G}_2, p \neq 3$	ϖ_1
$\mathsf{G}_2, p = 3$	$\varpi_1,\ \varpi_2$

TABLE 3. Irreducible p-restricted G-modules V_ω with nonzero weights of multiplicity 1 and whose zero weight has multiplicity greater than 1.

G	conditions	ω	weight 0 multiplicity
A_n	$n > 1, (n,p) \neq (2,3)$	$\varpi_1 + \varpi_n$	$\begin{cases} n-1 & \text{if } p \mid (n+1), \\ n & \text{if } p \nmid (n+1) \end{cases}$
A_3	$p > 3$	$2\varpi_2$	2
B_n	$n > 2$	ϖ_2	$\begin{cases} n-\gcd(2,n) & \text{if } p = 2, \\ n & \text{if } p \neq 2 \end{cases}$
	$n > 2$	$2\varpi_1$	$\begin{cases} n & \text{if } p \mid (2n+1), \\ n+1 & \text{if } p \nmid (2n+1) \end{cases}$
C_n	$n > 2, (n,p) \neq (3,3)$	ϖ_2	$\begin{cases} n-2 & \text{if } p \mid n, \\ n-1 & \text{if } p \nmid n \end{cases}$
	$n > 1$	$2\varpi_1$	n
C_2	$p \neq 5$	$2\varpi_2$	2
C_4	$p \neq 2, 3$	ϖ_4	2
D_n	$n > 3$	ϖ_2	$\begin{cases} n-\gcd(2,n) & \text{if } p = 2, \\ n & \text{if } p \neq 2 \end{cases}$
	$n > 3$	$2\varpi_1$	$\begin{cases} n-2 & \text{if } p \mid n, \\ n-1 & \text{if } p \nmid n \end{cases}$
E_6		ϖ_2	$\begin{cases} 5 & \text{if } p = 3, \\ 6 & \text{if } p \neq 3 \end{cases}$
E_7		ϖ_1	$\begin{cases} 6 & \text{if } p = 2, \\ 7 & \text{if } p \neq 2 \end{cases}$
E_8		ϖ_8	8
F_4		ϖ_1	$\begin{cases} 2 & \text{if } p = 2, \\ 4 & \text{if } p \neq 2 \end{cases}$
	$p \neq 3$	ϖ_4	2
G_2	$p \neq 3$	ϖ_2	2

TABLE 4. Highest weights of the nontrivial p-restricted modules in cases (1)–(8) of Theorem 1.2.

A_1 $p \neq 2$	A_n $n \geq 2$	B_n $n \geq 3$	C_2	C_3 $p \neq 2, 3$	C_n $n \geq 3$, $p = 2$	C_n $n \geq 3$, $p = 3$
ϖ_1	$\varpi_1, \dots, \varpi_n$	ϖ_1, ϖ_n	ϖ_1, ϖ_2	ϖ_1, ω_3	ϖ_1, ϖ_n	$\varpi_1, \varpi_{n-1}, \varpi_n$
D_n $n \geq 4$	E_6	E_7	F_4 $p = 3$	G_2 $p \neq 3$	G_2 $p = 3$	
$\varpi_1, \varpi_{n-1}, \varpi_n$	ϖ_1, ϖ_6	ϖ_7	ϖ_4	ϖ_1	ϖ_1, ϖ_2	

References

[1] N. Bourbaki, *Groupes et Algebres de Lie, Chapitres* 4*,* 5 *et* 6, Masson, Paris, 1981.

[2] N. Bourbaki, *Groupes et Algebres de Lie, Chapitres* 7 *et* 8, Hermann, Paris, 1975.

[3] J. Humphreys, *Modular Representations of Finite Groups of Lie Type*, Cambridge Univ. Press, Cambridge, 2006.

[4] F. Lübeck, *Small degree representations of finite Chevalley groups in defining characteristic*, LMS J. Comput. Math. **4** (2001), 135–169.

[5] F. Lübeck, *Tables of weight multiplicities*, `http://www.math.rwth-aachen.de/~Frank.Luebeck/chev/WMSmall/index.html`.

[6] G. Malle, D. Testerman, *Linear Algebraic Groups and Finite Groups of Lie Type*, Cambridge Univ. Press, Cambridge, 2011.

[7] А. А. Премет, *Веса инфинитезимально неприводимых представлений групп Шевалле над полем простой характеристики*, Матем. сб. **133(175)** (1987), ном. 2(6), 167–183. Engl. transl.: A. A. Premet, *Weights of infinitesimally irreducible representations of Chevalley groups over a field of prime characteristic*, Math. USSR Sb. **61** (1988), no. 1, 167–183.

[8] G. M. Seitz, *The Maximal Subgroups of Classical Algebraic Groups*, Memoirs Amer. Math. Soc. **365**, Amer. Math. Soc., Providence, 1987.

[9] G. M. Seitz, *Bounds for dimensions of weight spaces of maximal tori*, in: *Linear Algebraic Groups and their Representations* (Los Angeles, CA, 1992), Contemp. Math., Vol. 153, Amer. Math. Soc., Providence, RI, 1993, pp. 157–161.

[10] T. Springer, R. Steinberg, *Conjugacy classes*, in: *Seminar on Algebraic Groups and Related Finite Groups*, Lect. Notes in Math., Vol. 131, Springer Verlag, Berlin, 1970, pp. 167–266.

[11] D. Testerman, A. Zalesski, *Subgroups of simple algebraic groups containing maximal tori, and representations with multiplicity* 1 *nonzero weights*, Transform. Groups **20** (2015), 831–861.

[12] D. Testerman, A. E. Zalesski, *Spectra of nonregular elements in irreducible representations of simple algebraic groups*, North-Western European J. Math. **7** (2021), 185–212.

[13] A. Zalesski, *On eigenvalues of group elements in representations of simple algebraic groups and finite Chevalley groups*, Acta Appl. Math. **108** (2009), 175–195.

[14] A. Zalesski, *Unisingular representations of finite symplectic groups*, Comm. Algebra **50** (2022), 1697–1719.

[15] A. Zalesski, *Singer cycles in* 2*-modular representations of* $\mathrm{GL}_{n+1}(2)$, Archiv der Math. **110** (2018), 433–446.

[16] А. Е. Залесский, И. Супруненко, *Представления размерностей* $(p^n \mp 1)/2$ *симплектической группы степени* $2n$ *над полем характеристики* p, Вести АН БССР, Сер. физ.-мат. навук (1987), ном. 6, 9–15. Engl. transl.: A. Zalesski, I. Suprunenko, *Representations of dimensions* $(p^n \mp 1)/2$ *of the symplectic group of degree* $2n$ *over a field of characteristic* p, `arXiv:2108.10650v2` (2021).

Funding Information Open Access funding provided by EPFL Lausanne.