ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Nonsmooth and nonenergetic solutions of PDE's through convex integration

Master thesis
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Summary

In this Master thesis we explore the convex integration method by S. Müller and V. Šverák and its applications to partial differential equations. In particular, we use it to build very irregular solutions to elliptic systems. We also apply this method to build very weak solutions to elliptic scalar-valued partial differential equations. For this, we also use staircase laminates invented by D. Faraco.

Résumé

Dans ce projet de Master, nous explorons la méthode d'intégration convexe de S. Müller et V. Šverák et l'application de celle-ci aux équations différentielles partielles. En particulier, nous construisons des solutions très irrégulières à des systèmes elliptiques. Nous appliquons également cette méthode afin de construire des solutions très faibles à des équations différentielles scalaires elliptiques. Pour ce faire, nous utilisons les *staircase laminates* inventés par D.Faraco.

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Chapter 1

Introduction

Two decades ago, S. Müller and V. Šverák published a groundbreaking paper with the title "Convex integration for Lipschitz mappings and counterexamples to regularity", where they introduced a general method for constructing solutions to partial differential equations (PDE), known as convex integration. Since then, the field of convex integration has continued growing and many open problems have been solved using these methods.

In the first part of the thesis (Chapters 2 - 5) we will explore the convex integration method by Müller and Šverák. We will investigate their work in which they show existence of very irregular solutions to elliptic systems as well as the subsequent work by L. Székelyhidi in 2004. In a second part (Chapter 6), we will go through the method known as L^p-convex integration, invented by D. Faraco. We use this method to prove existence of very weak solutions to elliptic scalar valued boundary value problems. Below, we give a chapter-by-chapter account of the content of the present thesis.

Chapter 2: We begin by introducing some notations and conventions. We also present the notion of rank-one convexity and laminates.

Chapter 3: In this chapter, we introduce the basic building blocks of the convex integration scheme by Müller and Šverák [21]. This scheme is used to solve differential inclusions.

Chapter 4: The aim of this chapter is to prove existence of very irregular solutions to elliptic systems. For this, we use the convex integration scheme from Chapter 3 and the notion of T_N-configurations. We are particularly interested in the Euler-Lagrange equation generated by a quasiconvex functional. We also address Evans' partial regularity theorem and the necessity of some of the assumptions in this result. The main reference for this chapter is [21].

Chapter 5: Here we investigate the case of polyconvex functionals. In the same way as we proved existence of very irregular solutions in Chapter 4, we prove existence of very irregular solutions to the Euler-Lagrange equation generated by a polyconvex functional. We follow the work from Székelyhidi [24].

Chapter 6: In this chapter, we address another convex integration scheme known as L^p convex integration. It is based on the notion of staircase laminates invented by Faraco in [11]. We use this method to prove existence of very weak solutions to elliptic scalar-valued PDE's.

Chapter 7: We conclude the thesis by recalling the central observations.

Chapter 2

Preliminaries

The goal of this chapter is to introduce some basic notions which will be used throughout the thesis.

2.1 Notation and conventions

We first specify some notation:

- Ω : throughout the entirety of this document, unless stated otherwise, $\Omega \subset \mathbb{R}^n$ denotes an arbitrary convex domain (e.g. a ball),
- $\mathcal{P}(S)$: denotes the set of all probability measures on a set S,
- $\mathcal{L}^n$: denotes the n-dimensional Lebesgue measure,
- Given two measurable spaces $(X, \mathcal{G})$, $(Y, \mathcal{H})$, a measure μ on X and a measurable map $T \colon X \to Y$, the pushforward measure $T_{\#}\mu$ (which is a measure on Y) is defined as
$$T_{\#}\mu(A) = \mu(T^{-1}(A)) \quad \forall A \in \mathcal{H},$$
- Du : denotes the gradient of the map u,
- ∂_y : denotes the derivative with respect to the variable y,
- $\operatorname{diam}(\Omega)$: denotes the diameter of the set Ω, i.e.
$$\operatorname{diam}(\Omega) := \sup_{x,y \in \Omega} |x - y|,$$
- e_j : denotes the j-th vector of the canonical basis of $\mathbb{R}^n$ or $\mathbb{R}^m$, i.e. the vector taking the value 1 in its j-th entry and 0 in all other entries,

- Proj_A : denotes the orthogonal projection onto a linear subspace A,
- $M^{m\times n}$: denotes the set of $m \times n$ matrices,
- $S^{n\times n}$: denotes the set of $n \times n$ symmetric matrices,
- $C^0(\overline{\Omega})$: denotes the set of continuous functions, endowed with the norm

$$\|u\|_{C^0(\overline{\Omega})} := \sup_{x\in\overline{\Omega}} |u(x)|,$$

- $C^\alpha(\overline{\Omega})$: denotes Hölder spaces ($0 < \alpha < 1$), endowed with the norm

$$\|u\|_{C^\alpha(\overline{\Omega})} := \|u\|_{C^0(\overline{\Omega})} + [u]_{C^\alpha(\overline{\Omega})}$$

  where

$$[u]_{C^\alpha(\overline{\Omega})} := \sup_{x,y\in\overline{\Omega}, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha},$$

- $L^p(\Omega)$: denotes L^p spaces ($1 \le p \le \infty$), endowed with the norm

$$\|u\|_{L^p(\Omega)} := \begin{cases} \left(\int_\Omega |u(x)|^p \, dx\right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty; \\ \operatorname{ess\,sup} |u| & \text{if } p = \infty; \end{cases}$$

  where

$$\operatorname{ess\,sup} v := \inf\{\alpha \in \mathbb{R} : v(x) < \alpha \text{ for a.e. } x \in \Omega\},$$

- $W^{k,p}(\Omega)$: denotes Sobolev spaces ($k \ge 1$ integer, $1 \le p \le \infty$) , endowed with the norm

$$\|u\|_{W^{k,p}(\Omega)} := \sum_{|\alpha|\le k} \|\partial_\alpha u\|_{L^p(\Omega)}.$$

  and

- $W^{k,p}_{loc}(\Omega)$: denotes the space of measurable functions u such that $u \in W^{k,p}(\omega)$ for all $\omega \Subset \Omega$.

All the functional spaces have their vector-valued counterpart and will be denoted by an extra $;\mathbb{R}^m$. For example, the Sobolev space of $\mathbb{R}^m$-valued maps will be denoted by $W^{k,p}(\Omega, \mathbb{R}^m)$. The notation of the norms remains the same as in the scalar-valued case. For more details about functional spaces such as the set of continuous functions, Hölder spaces, L^p spaces and Sobolev spaces, we refer to [4, 10].

To finish this section, we also clarify some conventions:

- For a product
$$\prod_{a\in\mathcal{A}} a,$$
when $\mathcal{A}$ is empty we will mean
$$\prod_{a\in\mathcal{A}} a = 1.$$

- For a vector-valued map $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ we denote its components as $u^1, \ldots, u^m$, meaning that $u = (u^1, \ldots, u^m)$. Moreover, unless stated otherwise, we take as a convention that a vector-valued map is a function from a subset of $\mathbb{R}^n$ to $\mathbb{R}^m$.

2.2 Rank-one convexity

Here we present the notion of rank-one convexity. We say that two distinct matrices A and B are rank-one connected if $\operatorname{rank}(A - B) \leq 1$. When this is the case, the segment

$$[A, B] := \{tA + (1-t)B : t \in [0,1]\}$$

is called a rank-one segment.

Definition 2.2.1. *Let $f\colon M^{m\times n} \to \mathbb{R}$. We say that f is rank-one convex if for all $A, B \in M^{m\times n}$, where $\operatorname{rank}(B) = 1$, we have that*

$$t \mapsto f(A + tB) \quad \textit{is convex.}$$

This definition allows us to define rank-one convex hulls of sets in $M^{m\times n}$.

Definition 2.2.2. *Let K be a compact subset of $M^{m\times n}$. The rank-one convex hull $K^{rc} \subset M^{m\times n}$ of K is defined as*

$$K^{rc} = \{X \in M^{m\times n} : \forall f\colon M^{m\times n} \to \mathbb{R} \textit{ rank-one convex } f \leq 0 \textit{ on } K \textit{ implies } f(X) \leq 0\}.$$

For an open set $O \subset M^{m\times n}$, we define the rank-one convex hull

$$O^{rc} = \bigcup_{K \textit{is a compact subset of } O} K^{rc}.$$

A consequence of this definition is that the rank-one convex hull of an open set is open. Before going further, we point out the following fact: the rank-one convex hull of a set K is not the smallest set containing all rank-one connections of K. In fact, even though a set K contains no rank-one connections, its rank-one convex hull may be larger than K. An example of this is provided in Section 4.4.

2.3 Laminates

In this section we introduce the concept of laminates, which is a special class of probability measures.

Definition 2.3.1. *The barycenter of a probability measure $\nu \in \mathcal{P}(M^{m\times n})$ is*

$$\bar{\nu} = \int_{M^{m\times n}} X \, d\nu(X) \in M^{m\times n}.$$

Definition 2.3.2. *A measure $\nu \in \mathcal{P}(M^{m\times n})$ is a laminate if*

$$f(\bar{\nu}) \leq \int_{M^{m\times n}} f \, d\nu \quad \forall f \colon M^{m\times n} \to \mathbb{R} \text{ rank-one convex.}$$

Moreover, we denote the set of all laminates supported in a compact set K by $\mathcal{P}^{rc}(K)$.

Definition 2.3.3. *Let $O \subset M^{m\times n}$ be an open subset. Assume $\nu = \sum_{j=1}^{r} \lambda_j \delta_{A_j}$, $A_j \in O$, $A_j \neq A_k$ if $j \neq k$. The measure $\nu' \in \mathcal{P}(M^{m\times n})$ can be obtained from ν by an elementary splitting in O if, for some $j \in \{1, \dots, r\}$ and some $\lambda \in [0,1]$, there exists a rank-one segment $[B_1, B_2] \subset O$ containing A_j with $(1-s)B_1 + sB_2 = A_j$ such that*

$$\nu' = \nu + \lambda\lambda_j[(1-s)\delta_{B_1} + s\delta_{B_2} - \delta_{A_j}].$$

Remark 2.3.4. *In the definition above, if ν is a laminate, then the measure ν' obtained as an elementary splitting of ν is also a laminate.*

Definition 2.3.5. *We say that ν is a laminate of finite order in O if there exists $\nu_1, \dots, \nu_m$ such that*

- $\nu_1 = \delta_A$ *for some* $A \in O$;
- $\nu_m = \nu$ *and*
- ν_{j+1} *can be obtained from* ν_j *by an elementary splitting in* O, $\forall j = 1, \dots, m-1$.

When $O = M^{m\times n}$, laminates of finite order in O are simply called laminates of finite order. We denote laminates of finite order in O by $\mathcal{L}(O)$.

The following result is a consequence of [17, Corollary 4.11] and provides a characterisation of rank-one convex hulls.

Proposition 2.3.6. *For any compact set K it holds that $K^{rc} = \{\overline{\nu} : \nu \in \mathcal{P}^{rc}(K)\}$.*

Finally, we state the following result which allows us to approximate laminates by laminates of finite order.

Theorem 2.3.7. *Let K be a compact subset of $M^{m\times n}$ and let $\nu \in \mathcal{P}^{rc}(K)$. Let $O \subset M^{m\times n}$ be an open set such that $K^{rc} \subset O$. Then there is a sequence $\{\nu_j\}_{j=1}^{\infty} \subset \mathcal{L}(O)$ of laminates of finite order in O such that $\bar{\nu}_j = \bar{\nu}$ for all j and $\nu_j \overset{*}{\rightharpoonup} \nu$.*

Proof. We refer to [21, Theorem 2.1] for a proof. ∎

Chapter 3

Building blocks of convex integration

In this chapter we introduce the main building blocks of the convex integration techniques developed by Müller and Šverák in [21]. The goal of the techniques presented in this chapter is to solve a differential inclusion of the form

$$Du \in E$$

where $E \subset M^{m\times n}$ is some set. To be precise, we are interested in finding Sobolev functions $u\colon \Omega \to \mathbb{R}^m$ such that

$$Du(x) \in E \quad \text{for a.e. } x \in \Omega.$$

We begin by stating and proving some basic and fundamental results in Section 3.1. Then, we investigate the case where E is open in Section 3.2. Finally, we consider the more complicated case when the set E is compact in Section 3.3. The main reference for this chapter is the paper by Müller and Šverák [21].

3.1 Basic building blocks

Definition 3.1.1. *A Lipschitz mapping $u\colon \Omega \to \mathbb{R}^m$ is called piecewise affine if there exists a countable system of mutually disjoint open sets $\Omega_j \subset \Omega$, such that $\mathcal{L}^n(\partial\Omega_j) = 0$ for all j, which cover Ω up to a set of measure 0, and the restriction of u to each of the sets Ω_j is affine.*

The following lemma is crucial for the whole chapter. All results in the remainder of this chapter follow from this lemma. Essentially, it says that affine maps whose gradient is contained in a rank-one segment $[A, B]$ (meaning A and B are rank-one connected) can be approximated by a piecewise affine map u with gradients close to A or B. In addition, the lemma gives an idea of how the values of Du are distributed, meaning it gives a good idea of what the measure $(Du)_{\#}\mathcal{L}^n\,|_{\Omega}$ looks like.

Lemma 3.1.2. *Let $A, B \in M^{m\times n}$ with $\operatorname{rank}(A-B) = 1$, $b \in \mathbb{R}^m$, $0 < \lambda < 1$ and $C = (1-\lambda)A + \lambda B$. Then, for any $0 < \delta < |A-B|/2$, $0 < \alpha < 1$ and $\varepsilon > 0$, there is a piecewise affine Lipschitz mapping $u\colon \Omega \to \mathbb{R}^m$ such that*

- $\|u - (Cx+b)\|_{C^\alpha(\overline{\Omega})} < \varepsilon$,
- $u(x) = Cx + b$ *on* $\partial\Omega$ *and*
- $\operatorname{dist}(Du(x), \{A, B\}) < \delta$ *a.e. in* Ω.

In addition,

$$\mathcal{L}^n\left(\{x \in \Omega, |Du(x) - A| < \delta\}\right) = (1-\lambda)\mathcal{L}^n(\Omega) \;\; \textit{and}$$
$$\mathcal{L}^n\left(\{x \in \Omega, |Du(x) - B| < \delta\}\right) = \lambda\mathcal{L}^n(\Omega).$$

The proof of this lemma starts with strong additional assumptions. Under these assumptions, we first see that the proof can be carried out in the case where Ω is a special domain. We use a Vitali covering argument to generalise the result to arbitrary Ω. Then we relax the assumptions further to get the result.

Proof. Step 1: (We begin by proving the result under very strong additional assumptions.) Assume $A = -\lambda a \otimes e_j$ and $B = (1-\lambda)a \otimes e_j$ for some fixed $j = 1, \ldots, n$, $a \in \mathbb{R}^m$ and $C = 0$. Define the functions $h\colon \mathbb{R} \to \mathbb{R}$ and $w\colon \mathbb{R}^n \to \mathbb{R}^m$, by

$$h(s) = \frac{|s| + (2\lambda - 1)s}{2}$$

and

$$w(x) = a \max\left(0, 1 - \sum_{k=1, k\neq j}^{n} |x_k| - h(x_j)\right).$$

From the definition of w, it is clear that

$$|w(x) - w(y)| \leq |a| \sum_{k=1}^{n} |x_k - y_k|.$$

Take δ' small (to be chosen later) and set

$$v(x) = \delta' w\left(x_1, \ldots, x_{j-1}, \frac{x_j}{\delta'}, x_{j+1}, \ldots, x_n\right).$$

Define the set $\Lambda = \{x \in \Omega : |v(x)| > 0\}$. We observe that, for all $x, y \in \Lambda$,

$$\begin{aligned} |v(x) - v(y)| &\le |a|\delta' \left(\sum_{k=1, k\neq j}^{n} \underbrace{|x_k - y_k|}_{\le 2} \right) + |a| \underbrace{|x_j - y_j|}_{\le \frac{\delta'}{\lambda(1-\lambda)}} \\ &\le 2|a|\delta' \sum_{k=1, k\neq j}^{n} |x_k - y_k|^{\alpha} + |a| \left(\frac{\delta'}{\lambda(1-\lambda)} \right)^{1-\alpha} |x_j - y_j|^{\alpha}. \end{aligned}$$

It follows from this inequality that by taking δ' small enough, we get $[v]_{C^{\alpha}(\overline{\Lambda})} \le \varepsilon/4$ and $\|v\|_{L^{\infty}(\Omega)} \le \varepsilon/4$. Now, we compute the partial derivatives of v. For $i \neq j$, we have

$$\partial_{x_i} v(x) = \begin{cases} -\delta' a \,\mathrm{sign}\,(x_i) & \text{for a.e. } x \in \Lambda; \\ 0 & \text{for a.e. } x \notin \Lambda. \end{cases}$$

For $i = j$, we have

$$\partial_{x_j} v(x) = \begin{cases} -a \dfrac{\mathrm{sign}\,(x_j/\delta') + (2\lambda - 1)}{2} & \text{for a.e. } x \in \Lambda; \\ 0 & \text{for a.e. } x \notin \Lambda. \end{cases}$$

In other words,

$$\partial_{x_j} v(x) = \begin{cases} -\lambda a & \text{for a.e. } x \in \Lambda \text{ such that } \mathrm{sign}\,(x_j) \ge 0; \\ (1-\lambda) a & \text{for a.e. } x \in \Lambda \text{ such that } \mathrm{sign}\,(x_j) < 0; \\ 0 & \text{for a.e. } x \notin \Lambda. \end{cases}$$

The computations of $\partial_{x_i} v$ (for $i \neq j$) and $\partial_{x_j} v$ above imply that

$$\mathrm{dist}\,(Dv(x), \{A, B\}) \le (n-1)|a|\delta' \quad \text{a.e. in } \Lambda.$$

Now we may take δ' small enough to obtain $(n-1)|a|\delta' < \delta$. Notice that the two sets

$$\Lambda_A := \{x \in \Lambda, |Dv(x) - A| < \delta\} \quad \text{and} \quad \Lambda_B := \{x \in \Lambda, |Dv(x) - B| < \delta\}$$

are disjoint. To get an idea of what the sets Λ, Λ_A and Λ_B look like, Figure 3.1 provides an example. We would now like to compute the size of these two sets. In order to do that,

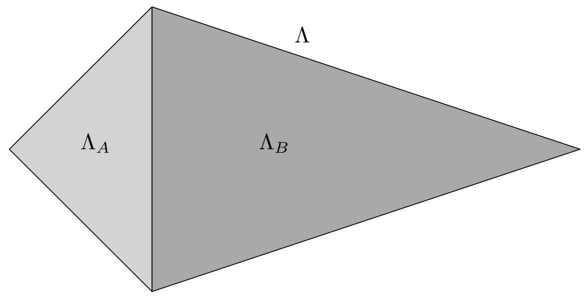

Figure 3.1: Illustrative example of the sets Λ, Λ_A and Λ_B in the two-dimensional case with λ approximately equal to 3/4. This drawing in based on [22, Figure 9.4].

we notice that $x = (x_1, \dots, x_n) \in \Lambda$ if and only if

$$|x_j| + (2\lambda - 1)x_j < \underbrace{2\delta' \left(1 - \sum_{k=1, k\neq j}^{n} |x_k|\right)}_{=:M(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)} \tag{3.1}$$

which is equivalent to

$$x_j \in \left(-\frac{M}{2(1-\lambda)}, \frac{M}{2\lambda}\right), \tag{3.2}$$

where M is the function defined in (3.1). Notice that the positive part of the interval in (3.2) is equal to $(1-\lambda)$ times the length of the entire interval, while the negative part of this interval is equal to λ times the length of the entire interval. Then, for a.e. point $x \in \Lambda$, only the value of $\partial_{x_j} v(x)$ determines whether $x \in \Lambda_A$ or $x \in \Lambda_B$. Recalling the formula for $\partial_{x_j} v(x)$ above we see that for a.e. $x \in \Lambda$,

$$\begin{aligned} x \in \Lambda_A &\Leftrightarrow x_j \geq 0, \\ x \in \Lambda_B &\Leftrightarrow x_j < 0. \end{aligned}$$

Thus, by Fubini's theorem

$$\begin{aligned} \mathcal{L}^n(\Lambda_A) &= \int_\Lambda 1_A \, dx = \int_{\{\sum_{k=1,k\neq j}^n |x_k|<1\}} \left(\int_0^{\frac{M}{2\lambda}} 1 \, dx_j\right) dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_n \\ &= \int_{\{\sum_{k=1,k\neq j}^n |x_k|<1\}} \left((1-\lambda) \int_{\frac{M}{2(\lambda-1)}}^{\frac{M}{2\lambda}} 1 \, dx_j\right) dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_n \\ &= (1-\lambda) \int_\Lambda 1 \, dx = (1-\lambda) \mathcal{L}^n(\Lambda) \end{aligned}$$

and it follows that

$$\mathcal{L}^n(\Lambda_B) = \mathcal{L}^n(\Lambda) - \mathcal{L}^n(\Lambda_A) = \lambda\mathcal{L}^n(\Lambda).$$

In other words, we have proved

$$\begin{aligned}&\mathcal{L}^n(\{x \in \Lambda, |Dv(x) - A| < \delta\}) = (1-\lambda)\mathcal{L}^n(\Lambda) \text{ and}\\ &\mathcal{L}^n(\{x \in \Lambda, |Dv(x) - B| < \delta\}) = \lambda\mathcal{L}^n(\Lambda).\end{aligned}$$

as wished. This proves the result in the special case where $\Omega = \Lambda$ and $b = 0$ because $v = 0$ on $\partial\Lambda$ and $\|v\|_{C^\alpha(\overline{\Lambda})} \le \varepsilon/2$. To finish this step, we know from the Vitali covering theorem (see [22, Theorem A.15]) that there exists $\{a_i\}_{i=1}^\infty$ and $\{r_i\}_{i=1}^\infty \subset (0,1)$ such that with

$$\Lambda_i = a_i + r_i\Lambda \quad \forall i \ge 1,$$

the sets Λ_i are mutually disjoint and

$$\mathcal{L}^n\left(\Omega \setminus \bigcup_{i=1}^\infty \Lambda_i\right) = 0.$$

Then define $u \colon \Omega \to \mathbb{R}^m$ by

$$u(x) = r_i v\left(\frac{x - a_i}{r_i}\right) + b \quad \text{for } x \in \Lambda_i.$$

Since the sets $\{\Lambda_i\}_{i=1}^\infty$ are mutually disjoint and $v = 0$ on $\partial\Lambda$, u is continuous. From the fact that $[v]_{C^\alpha(\overline{\Lambda})} \le \varepsilon/4$, we deduce that $[u - b]_{C^\alpha(\overline{\Lambda_i})} \le \varepsilon/4$ for all i. Now all the desired properties of u follow directly from the properties of v, except the fact that $\|u - b\|_{C^\alpha(\overline{\Omega})}$ is small enough. To prove it, let $x, y \in \Omega$. Then there are $i_x, i_y \ge 1$ such that $x \in \Lambda_{i_x}$, $y \in \Lambda_{i_y}$. If $i_x = i_y$, then

$$|u(x) - u(y)| \le \frac{\varepsilon}{4}|x - y|^\alpha. \tag{3.3}$$

Now assume $i_x \ne i_y$. Consider the straight line from x to y and denote its intersection with $\partial\Lambda_{i_x}$ as z_x and its intersection with $\partial\Lambda_{i_y}$ as z_y. Then, since $u(z_x) = u(z_y) = b$,

$$|u(x)-u(y)| \le |u(x)-u(z_x)|+|u(y)-u(z_y)| \le \frac{\varepsilon}{4}\left(|x - z_x|^\alpha + |y - z_y|^\alpha\right) \le \frac{\varepsilon}{2}|x-y|^\alpha. \tag{3.4}$$

Thus by (3.3) and (3.4), $[u - b]_{C^\alpha(\overline{\Omega})} \le \varepsilon/2$ and since $\|u - b\|_{L^\infty(\Omega)} \le \varepsilon/4$, we have

$$\|u - b\|_{C^\alpha(\overline{\Omega})} < \varepsilon$$

as wished. This finishes Step 1.

Step 2: (Now we slightly relax the assumptions.) Assume $A = -\lambda a \otimes z$ $B = (1-\lambda)a \otimes z$ for some $a \in \mathbb{R}^m$, $z \in \mathbb{R}^n$ and $C = 0$. Actually, then the result follows immediately from the previous step by bilinearity of the tensor product.

Step 3: (We relax the assumptions further to treat the general case.) We no longer assume that A and B have specific forms as in Step 1 and 2. Since $\operatorname{rank}(A - B) = 1$, there are $w \in \mathbb{R}^m, v \in \mathbb{R}^n$ such that $B - A = w \otimes v$. Define $A' = A - C$ and $B' = B - C$. We immediately see that $A' = -\lambda w \otimes v$ and $B' = (1-\lambda)w \otimes v$. Applying the previous step gives us a piecewise affine mapping u and the mapping $x \mapsto Cx + u(x)$ finishes the proof in this case. □

Before going further, let us point out the following:

Remark 3.1.3. *In the paper by Müller and Šverák [21], Lemma 3.1.2 is stated in a different way, using the notion of C^0-approximations. This is also the case of several of the following results which originate from the same paper. Moreover, in the paper by Müller and Šverák, this lemma and the corresponding following results never mention the Hölder bound as done in e.g. [25] and in this document.*

Lemma 3.1.4. *Let $\nu \in \mathcal{P}(M^{m\times n})$ be a laminate of finite order and let $A = \bar{\nu}$ be the barycenter of ν. Write ν as*

$$\nu = \sum_{j=1}^{r} \lambda_j \delta_{A_j}$$

with $\lambda_j > 0$ for all j and $A_i \neq A_j$ whenever $i \neq j$. Let

$$\delta_1 = \min\left\{ \frac{|A_i - A_j|}{2} : 1 \le i < j \le r \right\}.$$

Then for each $b \in \mathbb{R}^m$, each $0 < \delta < \delta_1$, $0 < \alpha < 1$ and each $\varepsilon > 0$, there is a piecewise affine mapping $u\colon \Omega \to \mathbb{R}^m$ such that

- $\|u - (Ax + b)\|_{C^\alpha(\overline{\Omega})} < \varepsilon$,
- *$u = Ax + b$ on $\partial\Omega$ and*
- $\operatorname{dist}(Du(x), \{A_1, \dots, A_r\}) < \delta$ *a.e. in* Ω.

In addition,

$$\mathcal{L}^n\left(\{x \in \Omega : \operatorname{dist}(Du(x), A_j) < \delta\}\right) = \lambda_j \mathcal{L}^n(\Omega), \quad \forall j = 1, \dots, r.$$

Proof. This follows by induction from Lemma 3.1.2. □

3.2 Open inclusions

In this section we are interested in differential inclusions into open sets. The two next results answer the following vague question:

If U is an open set and $v\colon \Omega \to \mathbb{R}^m$ satisfies $Dv(x) \in U^{rc}$ for a.e. $x \in \Omega$, does there exist $u\colon \Omega \to \mathbb{R}^m$ arbitrarily close to v with respect to the C^α norm such that $Du \in U$ on a large portion of the Ω ?

The answer is yes as the following results show. Lemma 3.2.1 proves it in the case where u is affine and Corollary 3.2.2 generalizes it to the case where u is a piecewise affine Lipschitz mapping.

Lemma 3.2.1. *Let $K \subset M^{m\times n}$ be a compact set and let $U \subset M^{m\times n}$ be an open set containing K. Let $\nu \in \mathcal{P}^{rc}(K)$ and denote $A = \bar{\nu}$. Let $b \in \mathbb{R}^m$. Then, for any given $\varepsilon, \delta > 0$ and $0 < \alpha < 1$, there is a piecewise affine mapping u such that*

- $\|u - (Ax + b)\|_{C^\alpha(\overline{\Omega})} < \varepsilon$,
- $u = Ax + b$ *on* $\partial\Omega$,
- $Du(x) \in U^{rc}$ *a.e. in* Ω *and*
- $\mathcal{L}^n\left(\{x \in \Omega : Du(x) \in U\}\right) > (1-\delta)\mathcal{L}^n\left(\Omega\right)$.

Proof. Since U^{rc} is open and $K^{rc} \subset U^{rc}$, Theorem 2.3.7 gives us that there exists a laminate μ of finite order supported in U^{rc} such that $\bar{\mu} = \bar{\nu}$ and

$$\mu(U) > 1 - \delta. \tag{3.5}$$

We may write it as $\mu = \sum_{j=1}^r \lambda_j \delta_{A_j}$ where the matrices A_j are assumed to be distinct. Define

$$\delta_1 = \min\left\{\frac{|A_i - A_j|}{2} : 1 \le i < j \le r\right\} > 0.$$

Choose $0 < \delta < \delta_1$ so that for each A_k belonging to U we have $\operatorname{dist}(A_k, \partial U) > \delta$ and for all A_k, we have $\operatorname{dist}(A_k, \partial(U^{rc})) > \delta$. From (3.5) we know that

$$\sum_{k\in\{1,\dots,r\} \text{ such that } A_k \in U} \lambda_k > 1 - \delta. \tag{3.6}$$

From Lemma 3.1.4 we know that the mapping $x \mapsto Ax + b$ admits an approximation given by a piecewise affine mapping u satisfying

- $\|u - (Ax + b)\|_{C^\alpha(\overline{\Omega})} < \varepsilon$,
- $u = Ax + b$ on $\partial\Omega$ and

- $\operatorname{dist}(Du(x), \{A_1, \ldots, A_r\}) < \delta$ a.e. in Ω.

In addition, we know that

$$\mathcal{L}^n\left(\{x \in \Omega : \operatorname{dist}(Du(x), A_j) < \delta\}\right) = \lambda_j \mathcal{L}^n(\Omega), \quad \forall j = 1, \ldots, r.$$

Hence, $Du(x) \in U^{rc}$ for a.e. $x \in \Omega$ and by (3.6) we have

$$\mathcal{L}^n\left(\{x \in \Omega : Du(x) \in U\}\right) > (1-\delta)\mathcal{L}^n(\Omega)$$

as desired. This finishes the proof. $\square$

Corollary 3.2.2. *Let $O \subset M^{m\times n}$ be an open bounded set. Let $u\colon \Omega \to \mathbb{R}^m$ be a piecewise affine Lipschitz mapping such that $Du(x) \in O^{rc}$ for a.e. $x \in \Omega$. Then for any $\varepsilon, \delta > 0$, there is a piecewise affine Lipschitz mapping $v\colon \Omega \to \mathbb{R}^m$ such that*

- $\|u - v\|_{C^\alpha(\overline{\Omega})} < \varepsilon$,
- *$v = u$ on $\partial\Omega$,*
- *$Dv(x) \in O^{rc}$ a.e. x in Ω and*
- $\mathcal{L}^n\left(\{x \in \Omega : Dv(x) \in O\}\right) > (1-\delta)\mathcal{L}^n(\Omega)$.

Proof. Since u is a piecewise affine Lipschitz mapping, there is a countable collection of mutually disjoint open sets $\{\Omega_j\}_{j\in J}$ such that

$$\mathcal{L}^n\left(\Omega \setminus \bigcup_{j\in J} \Omega_j\right) = 0 \tag{3.7}$$

and u is affine on each Ω_j. Thus, for all $j \in J$, there is A_j and b_j such that

$$u\,|_{\Omega_j} = A_j x + b_j.$$

Due to the fact that $Du(x) \in O^{rc}$ for a.e. $x \in \Omega$, we have $A_j \in O^{rc}$ for all $j \in J$. By definition of rank-one convex hulls, for each j, there is a compact set $K_j \Subset O$ such that $A_j \in K_j^{rc}$. By Proposition 2.3.6, there exists $\nu_j \in \mathcal{P}^{rc}(K_j)$ such that $\overline{\nu}_j = A_j$. Applying the previous lemma, we get a piecewise affine Lipschitz mapping $v_j\colon \Omega_j \to \mathbb{R}^m$ satisfying

- $\|v_j - (A_j x + b_j)\|_{C^\alpha(\overline{\Omega}_j)} < \varepsilon/2$,
- $Dv_j(x) \in O^{rc}$ a.e. in Ω_j,
- $u = A_j x + b_j$ on $\partial\Omega_j$ and
- $\mathcal{L}^n\left(\{x \in \Omega_j : Dv_j(x) \in O\}\right) > (1-\delta)\mathcal{L}^n(\Omega_j)$.

Define $v\colon \Omega \to \mathbb{R}^m$ as $v(x) = v_j(x)$ whenever $x \in \Omega_j$. By (3.7), v is well-defined up to a set of measure 0 and since v is piecewise affine on each Ω_j, v is a piecewise affine Lipschitz mapping. Finally, it follows from the properties of each v_j described above that the piecewise affine Lipschitz mapping v satisfies the properties in the statement of the corollary. With one exception, all the properties follow directly from the properties of each v_j. Indeed, only the fact that the Hölder norm $\|u - v\|_{C^\alpha(\overline{\Omega})}$ is small is not a direct consequence of the properties of each v_j. In order to prove that $\|u - v\|_{C^\alpha(\overline{\Omega})} < \varepsilon$, we proceed in the exact same way as in the proof of Lemma 3.1.2. This finishes the proof. $\square$

Given the question stated in the beginning of the section that we have answered with the two previous results it is natural to ask the following more demanding question:

If U is an open set and $v\colon \Omega \to \mathbb{R}^m$ satisfies $Dv(x) \in U^{rc}$ for a.e. $x \in \Omega$, does there exist $u\colon \Omega \to \mathbb{R}^m$ arbitrarily close to v with respect to the C^α norm such that $Du(x) \in U$ for a.e. $x \in \Omega$?

Again, the answer is yes and this is the content of the following theorem.

Theorem 3.2.3. *Let $O \subset M^{m\times n}$ be an open bounded set. Let $u_0\colon \Omega \to \mathbb{R}^m$ be a piecewise affine Lipschitz mapping such that $Du_0(x) \in O^{rc}$ for a.e. $x \in \Omega$. Then, for each $\varepsilon > 0$ and $0 < \alpha < 1$, there exists a piecewise affine mapping $u\colon \Omega \to \mathbb{R}^m$ such that*

- $\|u_0 - u\|_{C^\alpha(\overline{\Omega})} < \varepsilon$,
- $u = u_0$ *on* $\partial\Omega$ *and*
- $Du(x) \in O$ *a.e. in* Ω.

The idea of the proof is to apply the previous results inductively. We build a sequence of mappings $\{u_k\}_{k=1}^\infty$ such that

$$\mathcal{L}^d\left(\{x \in \Omega : Du_k(x) \in O^{rc} \setminus O\}\right) \to 0 \tag{3.8}$$

as $k \to \infty$. For each k, we build u_{k+1} from u_k by applying the previous results to u_k restricted to the set of points $x \in \Omega$ such that $Du_k(x) \in O^{rc} \setminus O$. This yields (3.8). Therefore, we expect that the limit function of this sequence should satisfy the desired properties.

Proof. Let $\delta > 0$ be close to 0. To prove the theorem, we construct a sequence of piecewise affine Lipschitz mappings $\{u_k\}_{k=1}^\infty$, from Ω to $\mathbb{R}^m$ such that for all $k \geq 1$:

- $\|u_k - u_{k-1}\|_{C^\alpha(\overline{\Omega})} < 2^{-k}\varepsilon$,
- $u_k = u_0$ on $\partial\Omega$,
- $Du_k(x) \in O^{rc}$ for a.e. $x \in \Omega$ and

- $\mathcal{L}^n(\{x \in \Omega : Du_k(x) \in O\}) > (1-\delta^k)\mathcal{L}^n(\Omega)$.

The existence of u_1 follows from Corollary 3.2.2. To complete the inductive step, assume that there is a function u_k satisfying the conditions above and build a function u_{k+1} with the desired properties. Since u_k is piecewise affine, there is a collection of open sets $\{\Omega_{k,j}\}_{j\in J_k}$ such that

$$\mathcal{L}^n\left(\Omega \setminus \bigcup_{j\in J_k} \Omega_{k,j}\right) = 0$$

and u_k is affine on each $\Omega_{k,j}$. Then let $\widetilde{J}_k \subset J_k$ be the collection of indices j such that

$$Du_k|_{\Omega_{k,j}} \in O^{rc} \setminus O.$$

For future arguments, define

$$\Omega_k := \bigcup_{j\in J_k\setminus \widetilde{J}_k} \Omega_{k,j}.$$

By the assumptions on u_k, we obtain that

$$\mathcal{L}^n\left(\bigcup_{j\in \widetilde{J}_k} \Omega_{k,j}\right) < \delta^k \mathcal{L}^n(\Omega).$$

By applying Lemma 3.2.1 to each $u_k|_{\Omega_{k,j}}$ where $j \in \widetilde{J}_k$, we can build a piecewise affine Lipschitz mapping $u_{k,j} \colon \Omega_{k,j} \to \mathbb{R}^m$ such that

- $\|u_{k,j} - u_k\|_{C^\alpha(\overline{\Omega_{k,j}})} < 2^{-(k+2)}\varepsilon$,
- $u_{k,j} = u_k$ on $\partial\Omega_{k,j}$,
- $Du_{k,j}(x) \in O^{rc}$ for a.e. $x \in \Omega_{k,j}$ and
- $\mathcal{L}^n(\{x \in \Omega_{k,j} : Du_k(x) \in O\}) > (1-\delta)\mathcal{L}^n(\Omega_{k,j})$.

By defining the mapping $u_{k+1} \colon \Omega \to \mathbb{R}^m$ by

$$u_{k+1}(x) = \begin{cases} u_{k,j}(x) & \text{if } x \in \Omega_{k,j} \text{ for some } j \in \widetilde{J}_k; \\ u_k(x) & \text{otherwise;} \end{cases}$$

we get the desired function. In particular, to prove that

$$\|u_{k+1} - u_k\|_{C^\alpha(\overline{\Omega})} < 2^{-(k+1)}\varepsilon,$$

we proceed in the exact same way as in the proof of Lemma 3.1.2. The other properties follow directly from the properties of the functions $u_{k,j}$ and u_k. We now have a sequence $\{u_k\}_{k=1}^{\infty}$ with the desired properties. It follows from the above properties that this sequence converges in $C^\alpha(\overline{\Omega};\mathbb{R}^m)$ to some $u \in C^\alpha(\overline{\Omega};\mathbb{R}^m)$ and that $\|u-u_0\|_{C^\alpha(\overline{\Omega})} \leq \varepsilon$. Since $u_k = u_0$ on $\partial\Omega$ for all k, we get that $u = u_0$ on $\partial\Omega$. Now let us prove that u is piecewise affine. By the construction above $u = u_k$ on Ω_k. Since each u_k is piecewise affine and the sets Ω_k cover Ω up to a set of Lebesgue measure 0, we deduce that u is piecewise affine. Finally, let us show that $Du(x) \in O$ for a.e. $x \in \Omega$. From the construction above, $Du_k(x) \in O$ for a.e. $x \in \Omega_k$. Since, as we have already said, the sets Ω_k cover Ω up to a set of Lebesgue measure 0, we deduce that $Du(x) \in O$ for a.e. $x \in \Omega$. Thus u satisifies all the desired properties and this finishes the proof. □

Finally, the following lemma follows from Theorem 3.2.3, Lemma 3.1.4 and Lemma 3.2.1.

Lemma 3.2.4. *Let $\nu \in \mathcal{P}(M^{m\times n})$ be a purely atomic laminate i.e.*

$$\nu = \sum_{j=1}^{r} \lambda_j \delta_{A_j}$$

with $\lambda_j > 0$ for all j and $A_i \neq A_j$ whenever $i \neq j$. Let $A = \bar{\nu}$ the barycenter. Let

$$\delta_1 = \min\left\{\frac{|A_i - A_j|}{2} : 1 \leq i < j \leq r\right\}.$$

Then, for each $b \in \mathbb{R}^m$, each $0 < \delta < \delta_1$, each $\varepsilon > 0$, and $0 < \alpha < 1$ there exists a piecewise affine mappings u satisfying

- $\|u - (Ax+b)\|_{C^\alpha(\overline{\Omega})} < \varepsilon$,
- $u = Ax + b$ *on* $\partial\Omega$,
- $\operatorname{dist}(Du(x), \{A_1, \ldots, A_r\}) < \delta$ *a.e. in* Ω, *and*

$$\mathcal{L}^n\left(\{x \in \Omega : \operatorname{dist}(Du(x), A_j) < \delta\}\right) = \lambda_j \mathcal{L}^n(\Omega), \quad \forall j = 1, \ldots, r.$$

3.3 Compact inclusions

In this section, we investigate the case where the set is compact. Let us first heuristically explain why this case is more complicated than when the set is open. Recall that Lemmas 3.1.2 and 3.1.4 are the starting point of all the results in the previous section. Notice that in these two results we obtain a piecewise affine Lipschitz mapping whose gradients is contained in small sets around the support of the laminate. Moreover, the size of this set can be made as small as one wishes. Since for points in open sets we may find small neighbourhoods contained in the open set, it is no surprise that the previous results gave us Theorem 3.2.3. Indeed, by taking the small sets around the support of the laminate to be sufficiently small, the gradient must belong to the open set. For the same reason, we can also convince ourselves that the case of a compact set (possibly with empty interior) is more involved. To overcome this difficulty, we introduce the notion of in-approximation coming from M. Gromov [14] which allows to approximate compact sets by open sets in such a way that the results from the previous section can be applied inductively to obtain results for compact sets.

Definition 3.3.1. *Assume K compact. We say that a sequence of equibounded open sets $\{U_i\}_{i=1}^{\infty}$ is an in-approximation of K if $U_i \subset U_{i+1}^{rc}$ for all $i \geq 1$ and*

$$\sup_{X \in U_i} \operatorname{dist}(X, K) \to 0 \quad \text{as } i \to \infty. \tag{3.9}$$

Remark 3.3.2. *Condition (3.9) in the definition above can be replaced by the following statement: for any sequence $\{X_i\}_{i=1}^{\infty}$ such that $X_i \in U_i$ for all $i \geq 1$, every accumulation point of $\{X_i\}_{i=1}^{\infty}$ is in K.*

To conclude this section and the current chapter, we will prove the following theorem:

Theorem 3.3.3. *Assume that a compact set $K \subset M^{m\times n}$ admits an in-approximation by open sets $\{U_i\}_{i=1}^{\infty}$ in the sense above. Then for any $\varepsilon > 0$ and any $v \in C^1(\overline{\Omega}; \mathbb{R}^m)$ satisfying $Dv(x) \in U_1$ in Ω there exists a piecewise affine mapping u such that*

- $\|u - v\|_{C^\alpha(\overline{\Omega})} < \varepsilon$,
- $u = v$ *on* $\partial\Omega$ *and*
- $Du(x) \in K$ *a.e. in* Ω.

Before proving this theorem, we need the following approximation result:

Proposition 3.3.4. *Let $u \in C^1(\overline{\Omega}; \mathbb{R}^m)$. Then, for any $0 < \alpha < 1$ and $\varepsilon > 0$ there is a piecewise affine Lipschitz mapping $v \colon \Omega \to \mathbb{R}^m$ such that*

- $\|u - v\|_{C^\alpha(\overline{\Omega})} < \varepsilon$,
- $u = v$ *on* $\partial\Omega$ *and*
- $\|Du - Dv\|_{L^\infty(\Omega)} < \varepsilon$.

Remark 3.3.5. *In the previous proposition, if we make the additional assumption that $U \subset M^{m\times n}$ is an open set such that $Du(x) \in U$ for all $x \in \overline{\Omega}$, then we deduce also that $Du(x) \in U$ a.e. in Ω.*

Proof. Step 1: (We begin by proving a preliminary result) We prove that: for any $\varepsilon, \delta > 0$, there exists $v \colon \Omega \to \mathbb{R}^m$ such that

- $\|u - v\|_{C^\alpha(\overline{\Omega})} < \varepsilon$,

- $v = u$ on $\partial\Omega$,
- there exists an open subset $\widetilde{\Omega} \subset \Omega$ such that $\mathcal{L}^n\left(\partial\widetilde{\Omega}\right) = 0$ satisfying

$$\mathcal{L}^n(\widetilde{\Omega}) > (1-\delta)\mathcal{L}^n(\Omega)$$

  such that $v\,|_{\widetilde{\Omega}}$ is piecewise affine and
- $\|Dv - Du\|_{L^\infty(\Omega)} < \varepsilon$.

We see directly that by using a Vitali covering argument, we can reduce ourselves to the case where $\Omega = (0,1)^n$. Let ε_0 be small (to be selected later). Then split Ω into smaller n-dimensional open mutually disjoint cubes $\{Q_i\}_{i=1}^N$ such that each Q_i is centered in a point p_i and

$$\mathcal{L}^n\left(\Omega \setminus \bigcup_{i=1}^N Q_i\right) = 0.$$

Moreover, we select these cubes sufficiently small so that for all $i = 1, \dots, N$ and all $x, y \in Q_i$:

- $|u(x) - u(y)| < \varepsilon_0$,
- $|u(x) - u(y) - \langle Du(y), x - y\rangle| < \varepsilon_0|x-y|$ and
- $|Du(x) - Du(y)| < \varepsilon_0$.

Before going further, we introduce the following notation: for an n-dimensional open cube

$$Q = \prod_{i=1}^n (a_i, b_i),$$

we denote by Q^γ the following set:

$$Q^\gamma = \prod_{i=1}^n (a_i + \gamma, b_i - \gamma).$$

Then for each $i = 1, \dots, N$, define the affine map $P_i\colon Q_i \to \mathbb{R}^m$ by

$$P_i(x) = u(p_i) + \langle Du(p_i), x - p_i\rangle.$$

Take

$$\gamma_i = \frac{\operatorname{diam}(Q_i)}{K}$$

for some large K (to be chosen later). For each i, let $\varphi_i \in C_0^\infty(Q_i)$ be such that

- $\varphi_i = 1$ on $Q_i^{\gamma_i}$,

- $\varphi_i = 0$ on $Q_i \setminus Q_i^{\frac{\gamma_i}{2}}$, and
- $|D\varphi_i| < \dfrac{C}{\gamma_i}$ for some C independent of γ_i and i.

Define
$$v_i = \varphi_i P_i + (1-\varphi_i)u$$
for all $1 \le i \le N$. Then,
$$|v_i(x) - u(x)| = |\varphi_i(x)P_i(x) - \varphi_i(x)u(x)| < |P_i(x) - u(x)| < \varepsilon_0 |x - p_i|.$$
Moreover,
$$\begin{aligned} Dv_i - Du &= P_i D\varphi_i + \varphi_i DP_i - uD\varphi_i - \varphi_i Du \\ &= D\varphi_i(P_i - u) + \varphi_i(DP_i - Du). \end{aligned}$$
Therefore,
$$\begin{aligned} |Dv_i - Du| &\le C\frac{K}{\operatorname{diam}(Q_i)}\varepsilon_0 \operatorname{diam}(Q_i) + \varepsilon_0 \\ &= CK\varepsilon_0 + \varepsilon_0 = (CK+1)\varepsilon_0. \end{aligned}$$
Take K sufficiently large to guarantee $\mathcal{L}^n(Q_i^{\gamma_i}) > (1-\delta)\mathcal{L}^n(Q_i)$ for all i. Then select ε_0 sufficiently small to obtain $(CK+1)\varepsilon_0 < \varepsilon$. Then by defining $v\colon \Omega \to \mathbb{R}^m$ as
$$v(x) = v_i(x) \quad \text{for all } x \in Q_i$$
the claim is proved with
$$\widetilde{\Omega} = \bigcup_{i=1}^{N} Q_i^{\gamma_i}.$$

Step 2: (We now prove the proposition) It suffices to apply the previous claim inductively to obtain a sequence of functions $\{v_k\}_{k=1}^{\infty}$ satisfying

- $\|v_k - v_{k-1}\|_{C^\alpha(\overline{\Omega})} < 2^{-k}\varepsilon$, for all $k \ge 2$, $\|v_1 - u\|_{C^\alpha(\overline{\Omega})} < 2^{-1}\varepsilon$,
- $v_k = u$ on $\partial\Omega$,
- for all $k \ge 1$, there exists an open subsets $\Omega_k \subset \Omega$ such that $\mathcal{L}^n(\partial\Omega_k) = 0$ satisfying
$$\mathcal{L}^n(\Omega_k) > (1-\delta^k)\mathcal{L}^n(\Omega)$$
such that $v_k|_{\Omega_k}$ is piecewise affine and
- $\|Dv_k - Dv_{k-1}\|_{L^\infty(\Omega)} < 2^{-k}\varepsilon$, for all $k \ge 2$, $\|Dv_1 - Du\|_{L^\infty(\Omega)} < 2^{-1}\varepsilon$.

The case $k = 1$ follows from step 1. Now assume we have v_k satisfying the properties above and prove the existence of v_{k+1}. Apply the claim to the function $v_k\,|_{\Omega\setminus\Omega_k}$ in order to get a function $w \in C^1(\overline{\Omega} \setminus \Omega_k, \mathbb{R}^m)$ such that

- $\|w - v_k\|_{C^\alpha(\overline{\Omega}\setminus\Omega_k)} < 2^{-(k+2)}\varepsilon$,
- $w = v_k$ on $\partial(\Omega \setminus \Omega_k)$,
- there exists an open subset $\widetilde{\Omega} \subset \Omega \setminus \Omega_k$ such that $\mathcal{L}^n\left(\partial\widetilde{\Omega}\right) = 0$ satisfying

  $$\mathcal{L}^n(\widetilde{\Omega}) > (1-\delta)\mathcal{L}^n(\Omega \setminus \Omega_k)$$

  such that $w\,|_{\widetilde{\Omega}}$ is piecewise affine and
- $\|Dw - Dv_k\|_{L^\infty(\Omega\setminus\Omega_k)} < 2^{-(k+2)}\varepsilon$.

To finish, it suffices to define $v_{k+1}\colon \Omega \to \mathbb{R}^m$ as

$$v_{k+1}(x) = \begin{cases} v_k(x) & \text{if} \quad x \in \Omega_k; \\ w(x) & \text{if} \quad x \in \Omega \setminus \Omega_k. \end{cases}$$

Then

$$\|v_{k+1} - v_k\|_{C^\alpha(\overline{\Omega})} < 2^{-(k+1)}\varepsilon$$

by the same argument as in the proof of Lemma 3.1.2. We see that the sequence $\{v_k\}_{k=1}^\infty$ converges in $C^\alpha(\overline{\Omega}; \mathbb{R}^m)$ to some $v \in C^\alpha(\overline{\Omega}; \mathbb{R}^m)$. Finally, we have the other desired properties because

$$\begin{aligned} \|u - v\|_{C^\alpha(\overline{\Omega})} &\leq \lim_{k\to\infty} \|u - v_k\|_{C^\alpha(\overline{\Omega})} \leq \varepsilon, \\ \|Du - Dv\|_{L^\infty(\Omega)} &\leq \lim_{k\to\infty} \|Du - Dv_k\|_{L^\infty(\Omega)} \leq \varepsilon \end{aligned}$$

and $v = u$ on $\partial\Omega$ because $v_k = u$ on $\partial\Omega$ for all k. $\square$

We can now prove Theorem 3.3.3. There are two main ideas which make up this proof. On one hand, we use the fact that we have an in-approximation combined with Theorem 3.2.3 to build a sequence of functions $\{u_i\}_{i=1}^\infty$ such that $Du_i \in U_i$ a.e. for all i. On the other hand, to guarantee that we obtain the differential inclusion in the end, we will build the sequence in such a way that $Du_i \to Du$ in L^1. To obtain such convergence, we use convolutions. For each i, we select ε_i small enough to satisfy $\|Du_i - Du_i * \rho_{\varepsilon_i}\|_{L^1(\Omega)} < 2^{-i}$. Finally, by using the fact that $D(u_i - u) * \rho_{\varepsilon_i} = (u_i - u) * D\rho_{\varepsilon_i}$, we see that it suffices to ensure that $\|u_i - u\|_{L^\infty(\Omega)}$ converges to 0 in a suitable way.

Proof of Theorem 3.3.3. Step 1 (Construction): Let $\rho\colon \mathbb{R}^n \to \mathbb{R}$ be smooth, nonnegative, supported in $B_1(0)$ such that $\int_{\mathbb{R}^n} \rho \, dx = 1$. Define $\rho_\varepsilon\colon \mathbb{R}^n \to \mathbb{R}$ as

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right).$$

In this proof, we will use convolutions between mappings u from Ω to $\mathbb{R}^m$ with $u = v$ on $\partial\Omega$ and mollifiers ρ_ε. With the standard definition of convolutions, this is not well defined and therefore we introduce the following convention. Let $w\colon \mathbb{R}^n \to \mathbb{R}^m$ be a compactly supported Lipschitz mapping such that $w = v$ on $\overline{\Omega}$. Then when writing $u * \rho_\varepsilon$, we use the standard definition of convolution on $\mathbb{R}^n$ where u is extended by w outside $\overline{\Omega}$. We also want to point out that in some cases in this proof we use convolutions between mappings from Ω to $\mathbb{R}^m$ which vanish on $\partial\Omega$ and mollifiers ρ_ε. In these cases, we extend the mappings by 0 outside $\overline{\Omega}$ and use the classical definition of convolution. Now, let us start with the proof. Choose ε_0 (to be determined later) and by Proposition 3.3.4 and Remark 3.3.5, we find a piecewise affine Lipschitz $u_1\colon \Omega \to \mathbb{R}^m$ with

- $\|u_1 - v\|_{C^\alpha(\overline{\Omega})} < \varepsilon_0$ in Ω,
- $u_1 = v$ on $\partial\Omega$ and
- $Du_1(x) \in U_1$ a.e. in Ω.

Then choose $0 < \varepsilon_1 < \min(2^{-1}, \varepsilon_0)$ so that $\|Du_1 * \rho_{\varepsilon_1} - Du_1\|_{L^1(\Omega)} \leq 2^{-1}$. Since $U_1 \subset U_2^{rc}$, Theorem 3.2.3 guarantees that there exists a piecewise affine Lipschitz mapping $u_2\colon \Omega \to \mathbb{R}^m$ such that

- $\|u_2 - u_1\|_{C^\alpha(\overline{\Omega})} < \dfrac{\varepsilon_1}{2}$ in Ω,
- $u_2 = v$ on $\partial\Omega$ and
- $Du_2(x) \in U_2$ a.e. in Ω.

Then select $0 < \varepsilon_2 < \min(2^{-2}, \varepsilon_1)$ such that

$$\|Du_2 * \rho_{\varepsilon_2} - Du_2\|_{L^1(\Omega)} \leq 2^{-2}.$$

By applying Theorem 3.2.3, in the same manner as we just did above, we can find a sequence of piecewise affine Lipschitz mappings $\{u_i\}_{i=1}^\infty$ from Ω to $\mathbb{R}^m$ and a decreasing sequence $\{\varepsilon_i\}_{i=1}^\infty$ such that

- $0 < \varepsilon_i < 2^{-i}$,
- $Du_i(x) \in U_i$ a.e. in Ω,
- $u_i = v$ on $\partial\Omega$,
- $\|Du_i * \rho_{\varepsilon_i} - Du_i\|_{L^1(\Omega)} \leq 2^{-i}$ and

- $\|u_{i+1} - u_i\|_{C^\alpha(\overline{\Omega})} \leq 2^{-i}\varepsilon_i$.

We see that the sequence $\{u_i\}_{i=1}^\infty$ converges in $C^\alpha(\overline{\Omega}; \mathbb{R}^m)$ to some $u \in C^\alpha(\overline{\Omega}; \mathbb{R}^m)$. It is clear that the mappings u_i are uniformly Lipschitz (because the sets U_i are equibounded, see Definition 3.1.1). Thus, $u\colon \Omega \to \mathbb{R}^m$ is Lipschitz. Moreover, by selecting the elements of the sequence $\{\varepsilon_i\}_{i=0}^\infty$ small enough, we get $\|u - v\|_{C^\alpha(\overline{\Omega})} < \varepsilon$. In addition, since $u_i = v$ on $\partial\Omega$ for all i, we get $u = v$ on $\partial\Omega$. This achieves the first step.

Step 2 (Proof that $Du \in K$): Now we prove that $Du(x) \in K$ for a.e. $x \in \Omega$. We begin by proving that $Du_i \to Du$ in $L^1(\Omega)$. First, we see that we have

$$\begin{aligned}\|Du_i - Du\|_{L^1(\Omega)} \leq \underbrace{\|Du_i - Du_i * \rho_{\varepsilon_i}\|_{L^1(\Omega)}}_{\leq 2^{-i} \to 0 \text{ as } i \to \infty} &+ \|Du_i * \rho_{\varepsilon_i} - Du * \rho_{\varepsilon_i}\|_{L^1(\Omega)} \\ &+ \underbrace{\|Du * \rho_{\varepsilon_i} - Du\|_{L^1(\Omega)}}_{\to 0 \text{ as } i \to \infty},\end{aligned}$$

so it suffices to prove that the second term converges to 0 as $i \to \infty$. Inside Ω, we have

$$D(u_i - u) * \rho_{\varepsilon_i} = (u_i - u) * D\rho_{\varepsilon_i},$$

and thus the remaining term can be estimated by

$$\begin{aligned}&\|(u_i - u) * D\rho_{\varepsilon_i}\|_{L^1(\Omega)} \\ &\leq \| \, D\rho_{\varepsilon_i}\|_{L^1(B_1)} \|u_i - u\|_{L^1(\Omega)} \\ &\leq \frac{C_1}{\varepsilon_i} \|u_i - u\|_{L^\infty(\Omega)}\end{aligned}$$

for some C_1 depending only on Ω and the choice of ρ. In addition,

$$\|u_i - u\|_{L^\infty(\Omega)} \leq \sum_{j \geq i} \|u_{j+1} - u_j\|_{L^\infty(\Omega)} \leq \sum_{j=i}^\infty 2^{-j}\varepsilon_j < 2\frac{\varepsilon_i}{2^i}.$$

Therefore,

$$\|Du_i * \rho_{\varepsilon_i} - Du * \rho_{\varepsilon_i}\|_{L^1(\Omega)} \leq \frac{2C_1}{2^i} = 2^{1-i}C_1$$

which converges to 0 as $i \to \infty$. This proves that $Du_i \to Du$ in $L^1(\Omega)$. Then notice that for a.e. $x \in \Omega$,

$$\operatorname{dist}(Du(x), K) \leq |Du(x) - Du_i(x)| + \operatorname{dist}(Du_i(x), K) \quad \text{for all } i \geq 1.$$

Now realise that since $\{U_i\}_{i=1}^\infty$ is an in-approximation we have

$$\sup_{X \in U_i} \operatorname{dist}(X, K) \to 0 \quad \text{ as } i \to \infty.$$

Thus $\operatorname{dist}(Du_i(x), K) \to 0$ uniformly in x as $i \to \infty$. Combined with the fact that

$$Du_i \to Du \text{ in } L^1(\Omega),$$

we get

$$\int_\Omega \operatorname{dist}(Du(x), K)\, dx = 0 \quad \text{which implies} \quad Du(x) \in K \text{ for a.e. } x \in \Omega,$$

as wished. This finishes this step and the proof. $\square$

To conclude, let us make a comment about the proof above. When reading the proof, it is easy to think of $\{u_k\}_{k=1}^\infty$ as a simple approximating sequence which approximates u. It is useful to point out that while it is true that $\{u_k\}_{k=1}^\infty$ approximates u, the construction of the sequence $\{u_k\}_{k=1}^\infty$ is much more subtle than a standard approximating sequence. Indeed, for a general approximating sequence $u_k \to u$ in $C^\alpha(\overline{\Omega}; \mathbb{R}^m)$ we do not get $Du_k \to Du$ in $L^1(\Omega; M^{m\times n})$. Why do we obtain this in the proof above and how it is related to the fact that $\{u_k\}_{k=1}^\infty$ is a more subtle construction than a standard approximating sequence? Briefly put, this comes from the fact that for any given u_{k_0} in the sequence and any small η, we are able to construct the rest of the sequence (i.e. the sequence $\{u_k\}_{k>k_0}$) in such a way that the limit function u satisfies $\|u_{k_0} - u\|_{L^\infty(\Omega)} < \eta$. More precisely, in the proof above, we want to make $\|Du_{k_0} * \rho_{\varepsilon_{k_0}} - Du * \rho_{\varepsilon_{k_0}}\|_{L^\infty(\Omega)} \to 0$ as $k_0 \to \infty$. As we can see in the proof above, in order to guarantee this, we need to show that

$$\frac{C_1}{\varepsilon_{k_0}} \|u_{k_0} - u\|_{L^\infty(\Omega)} \to 0.$$

This is where the fact of being able to make $\|u_{k_0} - u\|_{L^\infty(\Omega)}$ as small as one wants becomes useful. Indeed, we choose the rest of the sequence (i.e. the sequence $\{u_k\}_{k>k_0}$) in such a way that $\|u_{k_0} - u\|_{L^\infty(\Omega)} < 2^{1-k_0}\varepsilon_{k_0}$. The fact that $2^{1-k_0} \to 0$ as $k_0 \to \infty$ then allows to conclude.

Chapter 4

Application to quasiconvex functionals

In this chapter we are going to apply the techniques from the previous chapter to quasiconvex functionals. In Section 4.1 we introduce the notion of quasiconvexity. In particular, we state Evans' partial regularity theorem [9, Theorem 1] which applies to the minimization problem

$$\min_u \mathcal{I}(u) := \int_\Omega F(Du)\,dx \tag{4.1}$$

where u belongs to a Sobolev space and F is uniformly quasiconvex. In particular, Evans' theorem in particular states that if $F \in C^\infty(M^{m\times n})$ then the minimizer of (4.1) is smooth except possibly in a subset of null Lebesgue measure. In addition, it is a fact that all stationary points of the functional $\mathcal{I}$ solve the Euler-Lagrange equation

$$\operatorname{div} DF(Du) = 0. \tag{4.2}$$

The objective of the remainder of the chapter is to show that there are very irregular solutions to the equation (4.2) (which are also stationary points to $\mathcal{I}$). This shows that the assumption of being a global minimizer in Evans' theorem is essential. In particular, the assumption that u is a stationary point of $\mathcal{I}$ is not sufficient. Indeed, we prove the following theorem from [21]:

Theorem 4.0.1. *There exists a smooth, strongly quasiconvex function $F_0 \colon M^{2\times 2} \to \mathbb{R}$, with D^2F_0 uniformly bounded in $M^{2\times 2}$, four matrices $A_1, A_2, A_3, A_4 \in M^{2\times 2}$ and $\delta, \eta > 0$ such that the following is true: Let $F \colon M^{2\times 2} \to \mathbb{R}$ be a C^2 function satisfying*

$$DF(A_j) = DF_0(A_j) \quad \textit{and} \quad |D^2F(A_j) - D^2F_0(A_j)| < \delta \textit{ for all } j = 1,2,3,4. \tag{4.3}$$

Then for each piecewise C^1 function $v\colon \Omega \to \mathbb{R}^2$ satisfying $|Dv| < \eta$ a.e. in Ω and any $\varepsilon > 0$, there exists a piecewise affine Lipschitz mapping $u\colon \Omega \to \mathbb{R}^m$ satisfying the following properties:

- $\|u - v\|_{C^\alpha(\overline{\Omega})} < \varepsilon$,
- *$u = v$ on $\partial\Omega$ and*
- *u is not C^1 on any open subset of Ω and is a weak solution of the equation*

$$\operatorname{div} DF(Du) = 0 \quad \text{in } \Omega. \tag{4.4}$$

The main idea behind the proof is to see the Euler-Lagrange equation (4.2) as a differential inclusion. Indeed, in Section 4.2 we show that the equation can be seen as a differential inclusion. In Section 4.3, we present some preliminary tools. Then, we introduce the notion of T_N-configuration in Section 4.4. This is a set of N points in $M^{m\times n}$ which has particular geometric properties. In Section 4.5, we build the function F_0 of Theorem 4.0.1. Finally, we prove Theorem 4.0.1 in Section 4.6. The main reference for this chapter is [21] and unless explicitly stated otherwise, all results and proofs come from this paper.

4.1 Quasiconvexity

In this section we introduce the notion of quasiconvexity and some of its properties. For more information about quasiconvexity, we refer to [22, Chapter 5].

A C^2-function $F\colon M^{m\times n} \to \mathbb{R}$ is said to be quasiconvex if

$$\int_\Omega [F(A + D\psi) - F(A)]\, dx \geq 0 \quad \forall A \in M^{m\times n},\ \forall \psi \in C_0^\infty(\Omega, \mathbb{R}^m).$$

Moreover, F is uniformly quasiconvex if there is $\gamma > 0$ such that

$$\int_\Omega [F(A + D\psi) - F(A)]\, dx \geq \gamma \int_\Omega |D\psi|^2\, dx \quad \forall A \in M^{m\times n},\ \forall \psi \in C_0^\infty(\Omega, \mathbb{R}^m).$$

It is a well-known fact that under the growth assumption

$$|F(A)| \leq C(1 + |A|^2) \quad \forall A \in M^{m\times n},$$

the functional

$$I(u) = \int_\Omega F(Du)\, dx$$

is weakly lower semicontinuous on $W^{1,2}(\Omega, \mathbb{R}^m)$ if and only if F is quasiconvex. We refer the reader to [22, Theorem 5.16 and Proposition 5.18] for a proof of this.

We can also define quasiconvexity for functions defined on $S^{k\times k}$ (for some $k \geq 1$). Let $F\colon S^{k\times k} \to \mathbb{R}$. Then we say that $F\colon S^{k\times k} \to \mathbb{R}$ is quasiconvex on $S^{k\times k}$ if

$$\int_\Omega \left[F(A + D^2\psi) - F(A)\right]\, dx \geq 0 \quad \forall A \in S^{k\times k}, \forall \psi \in C_0^\infty(\Omega).$$

In addition, we say that F is strongly quasiconvex on $S^{k\times k}$ if

$$\int_\Omega \left[F(A + D^2\psi) - F(A)\right]\, dx \geq \gamma \int_\Omega |D^2\psi|^2\, dx \quad \forall A \in S^{k\times k}, \forall \psi \in C_0^\infty(\Omega).$$

We also state the following characterisation of quasiconvexity [21, Lemma 2.1]:

Lemma 4.1.1. *Let $\mathbb{T}^n$ be a flat n-dimensional torus. A function $F\colon M^{m\times n} \to \mathbb{R}$ is quasiconvex if and only if*

$$\int_{\mathbb{T}^n} [F(A + D\varphi) - F(A)] \geq 0 \quad \forall A \in M^{m\times n}, \forall \varphi \in C^\infty(\mathbb{T}^n, \mathbb{R}^m).$$

Finally, we recall Evans partial regularity theorem (see [9, Theorem 1] and [22, Theorem 5.22.]).

Theorem 4.1.2. *Suppose $F \in C^2(M^{m\times n})$, uniformly quasiconvex, satisfying*

$$|D^2F(A)| \leq C.$$

Let $u \in W^{1,\infty}(\Omega, \mathbb{R}^m)$ be a minimizer of $\mathcal{I}$. Then there exists a relatively closed subset $\Sigma_u \subset \Omega$ of null Lebesgue measure such that

$$Du \in C^\alpha(\Omega \setminus \Sigma_u, M^{m\times n})$$

for each $0 < \alpha < 1$. In addition, if $F \in C^\infty(M^{m\times n})$ then $u \in C^\infty(\Omega \setminus \Sigma_u, \mathbb{R}^m)$.

4.2 The Euler-Lagrange equation as a differential inclusion

In this section we reformulate equation (4.2) as a differential inclusion to be able to apply the convex integration techniques from the previous chapter. In fact, we will show that this equation is equivalent to the differential inclusion into the set

$$K_F := \left\{ \begin{pmatrix} X \\ DF(X)J \end{pmatrix} : X \in M^{2\times 2} \right\} \subset M^{4\times 2}. \tag{4.5}$$

Let $F \in C^1(M^{2\times 2})$. We consider the equation (4.2) and notice that it implies

$$\operatorname{curl}(DF(Du)J) = 0, \quad \text{where} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.6}$$

Since Ω is assumed to be convex (e.g. a ball), and (4.6) holds, we get that there exists $\widetilde{u}\colon \Omega \to \mathbb{R}^2$ such that

$$D\widetilde{u} = DF(Du)J.$$

Define the function $w\colon \Omega \to \mathbb{R}^4$ by

$$w = \begin{pmatrix} u \\ \widetilde{u} \end{pmatrix}.$$

Then clearly

$$Dw = \begin{pmatrix} Du \\ D\widetilde{u} \end{pmatrix} = \begin{pmatrix} Du \\ DF(Du)J \end{pmatrix} \in K_F$$

as wished. We now prove the reverse implication. Assume we have a Lipschitz function $w\colon \Omega \to \mathbb{R}^4$ such that

$$Dw(x) \in K_F \quad \text{for a.e } x \in \Omega.$$

Then let

$$u = \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \quad \text{and} \quad \widetilde{u} = \begin{pmatrix} w^3 \\ w^4 \end{pmatrix}.$$

We obtain that $D\widetilde{u} = DF(Du)J$ and conclude that

$$\operatorname{div} DF(Du) = \operatorname{curl}(DF(Du)J) = \operatorname{curl}(D\widetilde{u}) = 0.$$

Thus, we have proved that equation (4.2) is equivalent to the differential inclusion (4.6).

4.3 Preliminary tools

In this section we introduce some tools which will be of great value in the following sections. We start with a lemma which allows us to extend quasiconvex functions defined on $S^{2\times 2}$ to quasiconvex functions defined on $M^{2\times 2}$. Let us introduce the following notation: for any $X \in M^{k\times k}$ (for some $k \geq 1$), we define its symmetric part as

$$X_{sym} = \frac{X + X^T}{2}.$$

We also define the antisymmetric part as

$$X_{asym} = \frac{X - X^T}{2}.$$

Lemma 4.3.1. *Let $f\colon S^{2\times 2} \to \mathbb{R}$ be a smooth function such that $|D^2 f| \leq c$ in $S^{2\times 2}$. Assume that f is strongly quasiconvex on $S^{2\times 2}$ in the sense that for some $\gamma > 0$ we have*

$$\int_{\mathbb{R}^2} \big(f(A + D^2\phi) - f(A)\big)\, dx \geq \gamma \int_{\mathbb{R}^2} |D^2\phi|^2\, dx$$

for $A \in S^{2\times 2}$ and all smooth, compactly supported $\phi\colon \mathbb{R}^2 \to \mathbb{R}$. Then for sufficiently large $\kappa > 0$, the function $\widetilde{f}\colon M^{2\times 2} \to \mathbb{R}$ defined by $\widetilde{f}(X) = f(X_{sym}) + \kappa |X_{asym}|^2$ is strongly quasiconvex on $M^{2\times 2}$.

To prove this lemma, we need the following proposition. Let us introduce the following notation:

$$D^\perp \varphi = JD\varphi, \quad \text{where} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Proposition 4.3.2 (Helmholtz Decomposition)**.** *Let $\mathbb{T}^2$ be the two-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$. Let $\varphi\colon \mathbb{T}^2 \to \mathbb{R}^2$ be a smooth vector field. Then there are two smooth scalar functions $\phi, \eta \in C^\infty(\mathbb{T}^2)$, and a vector $a \in \mathbb{R}^2$ such that*

$$\varphi = D\phi + D^\perp \eta + a. \tag{4.7}$$

In addition, the following equality is satisfied

$$\int_{\mathbb{T}^2} |D\varphi|^2\, dx = \int_{\mathbb{T}^2} |D^2\phi|^2\, dx + \int_{\mathbb{T}^2} |D^2\eta|^2\, dx. \tag{4.8}$$

Proof. We write the functions φ, ϕ and η as Fourier series on $\mathbb{T}^2$:

$$\varphi(x) = \sum_{\nu\in\mathbb{Z}^2} \begin{pmatrix} \varphi_\nu^1 \\ \varphi_\nu^2 \end{pmatrix} \mathrm{e}^{2\pi i\nu\cdot x},$$

$$\phi(x) = \sum_{\nu\in\mathbb{Z}^2} \phi_\nu \mathrm{e}^{2\pi i\nu\cdot x}$$

and

$$\eta(x) = \sum_{\nu\in\mathbb{Z}^2} \eta_\nu \mathrm{e}^{2\pi i\nu\cdot x}$$

where $\varphi_\nu \in \mathbb{C}^2$ and $\phi_\nu, \eta_\nu \in \mathbb{C}$ for all $\nu \in \mathbb{Z}^2$. Then

$$D\phi(x) = 2\pi i \sum_{\nu\in\mathbb{Z}^2} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \phi_\nu \mathrm{e}^{2\pi i\nu\cdot x}$$

and

$$D^\perp\eta(x) = 2\pi i \sum_{\nu\in\mathbb{Z}^2} \begin{pmatrix} -\nu_2 \\ \nu_1 \end{pmatrix} \eta_\nu \mathrm{e}^{2\pi i\nu\cdot x}.$$

Thus, in order for (4.7) to be satisfied, we need to choose $a = \varphi_0$ and the following system to be satisfied for all $\nu \in \mathbb{Z}^2 \setminus \{0\}$:

$$\begin{cases} \varphi_\nu^1 & = 2\pi i(\nu_1\phi_\nu - \nu_2\eta_\nu); \\ \varphi_\nu^2 & = 2\pi i(\nu_2\phi_\nu + \nu_1\eta_\nu). \end{cases} \tag{4.9}$$

This means that

$$\begin{cases} \phi_\nu &= \dfrac{1}{2\pi i(\nu_1^2+\nu_2^2)}(\varphi_\nu^1\nu_1 + \varphi_\nu^2\nu_2); \\ \\ \eta_\nu &= \dfrac{1}{2\pi i(\nu_1^2+\nu_2^2)}(\varphi_\nu^2\nu_1 - \varphi_\nu^1\nu_2); \end{cases} \tag{4.10}$$

for all $\nu \in \mathbb{Z}^2 \setminus \{0\}$. By choosing the coefficients ϕ_ν and η_ν so that the system above is solved, (4.7) holds. Let us now prove that $\phi, \eta \in C^\infty(\mathbb{T}^2)$. Since φ is smooth, we obtain that (for instance by [12, Corollary 2.11])

$$\sum_{\nu\in\mathbb{Z}^2} |\varphi_\nu^1| < \infty \quad \text{and} \quad \sum_{\nu\in\mathbb{Z}^2} |\varphi_\nu^2| < \infty.$$

By the equations in (4.10), this means that

$$\sum_{\nu\in\mathbb{Z}^2} |\phi_\nu| < \infty \quad \text{and} \quad \sum_{\nu\in\mathbb{Z}^2} |\eta_\nu| < \infty.$$

Thus, (by e.g. [12, Corollary 2.10]) we obtain that $\phi, \eta \in C(\mathbb{T}^2)$. This argument can be applied to derivatives of φ, ϕ and η. Thus, in the end we obtain $\phi, \eta \in C^\infty(\mathbb{T}^2)$.

Finally, it remains to prove (4.8). First, we notice that

$$D\varphi(x) = 2\pi i \sum_{\nu\in\mathbb{Z}^2} \begin{pmatrix} \varphi_\nu^1\nu_1 & \varphi_\nu^1\nu_2 \\ \varphi_\nu^2\nu_1 & \varphi_\nu^2\nu_2 \end{pmatrix} \mathrm{e}^{2\pi i\nu\cdot x},$$

$$D^2\phi(x) = 2\pi i \sum_{\nu\in\mathbb{Z}^2} \begin{pmatrix} \nu_1^2 & \nu_1\nu_2 \\ \nu_1\nu_2 & \nu_2^2 \end{pmatrix} \phi_\nu \mathrm{e}^{2\pi i\nu\cdot x}$$

and

$$D^2\eta(x) = 2\pi i \sum_{\nu\in\mathbb{Z}^2} \begin{pmatrix} \nu_1^2 & \nu_1\nu_2 \\ \nu_1\nu_2 & \nu_2^2 \end{pmatrix} \eta_\nu \mathrm{e}^{2\pi i\nu\cdot x}.$$

Then, by the Parseval identity (in the first and last equality),

$$\begin{aligned} \int_{\mathbb{T}^2} |D\varphi|^2\,dx &= (2\pi)^2 \sum_{\nu\in\mathbb{Z}^2} \left(|\varphi_\nu^1\nu_1|^2 + |\varphi_\nu^1\nu_2|^2 + |\varphi_\nu^2\nu_1|^2 + |\varphi_\nu^2\nu_2|^2\right) \\ &\overset{(4.9)}{=} (2\pi)^2 \sum_{\nu\in\mathbb{Z}^2} (\nu_1^2+\nu_2^2)|\nu_1\phi_\nu - \nu_2\eta_\nu|^2 + (2\pi)^2 \sum_{\nu\in\mathbb{Z}^2} (\nu_1^2+\nu_2^2)|\nu_2\phi_\nu + \nu_1\eta_\nu|^2 \\ &= (2\pi)^2 \sum_{\nu\in\mathbb{Z}^2} \left(\nu_1^4 + 2\nu_1^2\nu_2^2 + \nu_2^4\right)|\phi_\nu|^2 \\ &\quad + (2\pi)^2 \sum_{\nu\in\mathbb{Z}^2} \left(\nu_1^4 + 2\nu_1^2\nu_2^2 + \nu_2^4\right)|\eta_\nu|^2. \\ &= \int_{\mathbb{T}^2} |D^2\phi|^2\,dx + \int_{\mathbb{T}^2} |D^2\eta|^2\,dx. \end{aligned}$$

This proves (4.8) and finishes the proof. $\square$

Proof of Lemma 4.3.1. We will prove that for κ large enough we get

$$\int_{\mathbb{T}^2} (\widetilde{f}(A+D\varphi) - \widetilde{f}(A))\,dx \geq \frac{\gamma}{2}\int_{\mathbb{T}^2} |D\varphi|^2\,dx, \quad \forall A \in M^{2\times 2}, \forall \varphi \in C^\infty(\mathbb{T}^2, \mathbb{R}^2). \tag{4.11}$$

Take $\varphi \in C^\infty(\mathbb{T}^2, \mathbb{R}^2)$ arbitrary. By Proposition 4.3.2, there are two functions $\phi, \eta \in \mathbb{C}^\infty(\mathbb{T}^2)$ and a vector $a \in \mathbb{R}^2$ such that

$$\varphi = D\phi + D^\perp \eta + a.$$

Then $D\varphi = D^2\phi + DD^\perp\eta$. Define $Y\colon \mathbb{T}^2 \to S^{2\times 2}$ as $Y = (DD^\perp\eta)_{sym}$. We find that

$$\int_{\mathbb{T}^2} |Y|^2\,dx = \frac{1}{2}\int_{\mathbb{T}^2} |D^2\eta|^2\,dx = \int_{\mathbb{T}^2} |(DD^\perp\eta)_{asym}|^2\,dx. \tag{4.12}$$

In order to establish (4.11), we write the left-hand side as the sum of three integrals:

$$\begin{aligned}\int_{\mathbb{T}^2} (\widetilde{f}(A+D\varphi) - \widetilde{f}(A)) &= \underbrace{\int_{\mathbb{T}^2} \left[f(A_{sym} + D^2\phi + Y) - f(A_{sym} + D^2\phi)\right]\,dx}_{=I} \\ &\quad + \underbrace{\int_{\mathbb{T}^2} \left[f(A_{sym} + D^2\phi) - f(A_{sym})\right]\,dx}_{=II} \\ &\quad + \underbrace{\int_{\mathbb{T}^2} \kappa|A_{asym} + (DD^\perp\eta)_{asym}|^2 - \kappa|A_{asym}|^2\,dx}_{=III}.\end{aligned}$$

Now we look at each of these integrals individually. Thanks to Proposition 4.3.2 we see immediately that

$$II \geq \gamma \int_{\mathbb{T}^2} |D^2\phi|^2\,dx$$

because f is quasiconvex on $S^{2\times 2}$. Then we see that

$$III = \kappa \int_{\mathbb{T}^2} |Y|^2\,dx.$$

Finally, we take care of I for which we obtain the following:

$$\begin{aligned}
I &= \int_{\mathbb{T}^2} \left[f(A_{sym} + D^2\phi + Y) - f(A_{sym} + D^2\phi) - Df(A_{sym} + D^2\phi)Y \right] dx \\
&\quad + \int_{\mathbb{T}^2} \left[Df(A_{sym} + D^2\phi) - Df(A_{sym}) \right] Y \, dx \\
&\geq - \int_{\mathbb{T}^2} \frac{c}{2} |Y|^2 \, dx - \int_{\mathbb{T}^2} c |D^2\phi| |Y| \, dx \\
&\geq - \int_{\mathbb{T}^2} \frac{c}{2} |Y|^2 + \frac{c^2}{2\gamma} |Y|^2 + \frac{\gamma}{2} |D^2\phi|^2 \, dx.
\end{aligned}$$

In the end we obtain (by taking κ large enough)

$$\begin{aligned}
I + II + III &\geq \frac{\gamma}{2} \int_{\mathbb{T}^2} |D^2\phi|^2 \, dx + \left(\kappa - \frac{c}{2} - \frac{c^2}{2\gamma} \right) \int_{\mathbb{T}^2} |Y|^2 \, dx \\
&\overset{(4.8)}{=} \frac{\gamma}{2} \int_{\mathbb{T}^2} |D\varphi|^2 \, dx - \frac{\gamma}{2} \int_{\mathbb{T}^2} |D^2\eta|^2 \, dx + \left(\kappa - \frac{c}{2} - \frac{c^2}{2\gamma} \right) \int_{\mathbb{T}^2} |Y|^2 \, dx \\
&\overset{(4.12)}{=} \frac{\gamma}{2} \int_{\mathbb{T}^2} |D\varphi|^2 \, dx + \left(\kappa - \frac{c}{2} - \frac{c^2}{2\gamma} - \gamma \right) \int_{\mathbb{T}^2} |Y|^2 \, dx \geq \frac{\gamma}{2} \int_{\mathbb{T}^2} |D\varphi|^2 \, dx.
\end{aligned}$$

This proves the result. □

4.4 T_N-configurations

Now let us introduce the notion of T_N-configurations.

Definition 4.4.1. *We say that $X_1, \ldots, X_N \in M^{m\times n}$ form a T_N-configuration if*

$$\operatorname{rank}(X_i - X_j) > 1 \quad \textit{for all} \quad i, j = 1, \ldots, N,$$

and if there exist rank-one matrices $C_1, \ldots, C_N$ satisfying

$$\sum_k C_k = 0, \tag{4.13}$$

$\kappa_1, \ldots, \kappa_N > 0$ and a matrix $P \in M^{m\times n}$ such that the following equalities hold:

$$\begin{aligned}
X_1 &= P + \kappa_1 C_1, \\
X_2 &= P + C_1 + \kappa_2 C_2, \\
&\vdots \\
X_n &= P + \sum_{i=1}^{n-1} C_i + \kappa_n C_n, \quad \forall\, 1 \leq n \leq N.
\end{aligned} \tag{4.14}$$

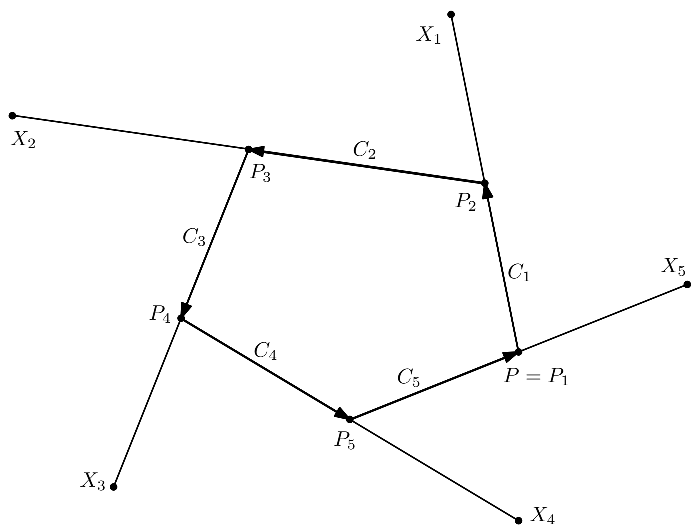

Figure 4.1: Example of a T_N-configuration (in this case $N = 5$). Notice that all segments joining two points represent a rank-one connection.

Subsequent to this definition, we notice that the following four matrices $\{X_i^0\}_{i=1}^4$ form a T_4-configuration:

$$X_1^0 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \\ 0 & -1 \\ 3 & 0 \end{pmatrix}, \quad X_2^0 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 3 \\ 1 & 0 \end{pmatrix}, \quad X_3^0 = \begin{pmatrix} -3 & 0 \\ 0 & 1 \\ 0 & 1 \\ -3 & 0 \end{pmatrix} \text{ and } \quad X_4^0 = \begin{pmatrix} -1 & 0 \\ 0 & -3 \\ 0 & -3 \\ -1 & 0 \end{pmatrix}. \tag{4.15}$$

This particular T_N-configuration will be particularly frequent in the remainder of this chapter. For future reference, we also define $\{A_i\}_{i=1}^4$ as

$$A_1 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \quad A_4 = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}. \tag{4.16}$$

See Figure 4.1 for an illustration of the definition of T_N-configurations. Moreover, we define P_n for $n = 1, \dots, N$ as

$$P_n = P + \sum_{i=1}^{n-1} C_i. \tag{4.17}$$

In the next lemma we prove that T_N-configurations have very special geometric structure. Indeed, from the definition, we see that T_N-configurations contain no rank-one connections. Despite this, T_N-configurations have a nontrivial rank-one convex hull, as the following lemma shows.

Lemma 4.4.2. *If $X_1, \dots, X_N$ form a T_N-configuration, the rank-one convex hull of the set $\{X_1, \dots, X_N\}$ contains the points $\{P_i\}_{i=1}^N$.*

Proof. For any rank-one convex function $f\colon M^{m\times n} \to \mathbb{R}$ such that $f \le 0$ on $\{X_1, \dots, X_N\}$, we have

$$f(P_{i+1}) \le \frac{1}{\kappa_i} f(X_i) + \left(1 - \frac{1}{\kappa_i}\right) f(P_i) \le \left(1 - \frac{1}{\kappa_i}\right) f(P_i)$$

where the indices are considered to be modulo N. By induction we get that $f(P_i) \le 0$ for all $i = 1, \dots, N$. □

Remark 4.4.3. *In the previous lemma, when X equals one of the points P_i, we can compute the laminate μ supported in $\{X_1, \dots, X_N\}$ such that $\overline{\mu} = X$. Notice that such a laminate exists by Proposition 2.3.6. Since we will be particularly interested in T_4-configurations, we consider the specific case $N = 4$. Moreover, without loss of generality, take $i = 1$ and notice that for each j modulo 4, we have (with $\beta_j = 1 - 1/\kappa_j$)*

$$P_{j+1} = (1 - \beta_j) X_j + \beta_j P_j.$$

Continuing this argument further yields

$$P_1 = \sum_{i=1}^{4} \frac{(1-\beta_i)\beta_1\beta_2\beta_3\beta_4}{\beta_1 \dots \beta_i (1 - \beta_1\beta_2\beta_3\beta_4)} X_i.$$

Thus, the desired laminate μ is

$$\mu = \sum_{i=1}^{4} \underbrace{\frac{(1-\beta_i)\beta_1\beta_2\beta_3\beta_4}{\beta_1 \dots \beta_i (1 - \beta_1\beta_2\beta_3\beta_4)}}_{=:\mu_i} \delta_{X_i}.$$

By direct computations, when $X_i = X_i^0$ for all i, defined in (4.15), we have

$$\mu_l > \frac{16}{15} \cdot \frac{1}{8}, \ \text{for all } l = 2, 3, 4. \tag{4.18}$$

Proposition 4.4.4. *Recall the matrices X_1^0, X_2^0, X_3^0, X_4^0 defined in (4.15). In a neighbourhood of $(X_1^0, X_2^0, X_3^0, X_4^0) \in (\mathbb{R}^{4\times 2})^4$, there exists a 24-dimensional smooth manifold whose elements are T_4-configurations. We denote this manifold $\mathcal{M}$. In addition, there are well-defined smooth maps $\pi_k\colon \mathcal{M} \to M^{4\times 2}$ (k = 1, …, 4) such that for all*

$$(X_1, X_2, X_3, X_4) \in \mathcal{M},$$

$P_k = \pi_k(X_1, X_2, X_3, X_4)$ corresponds to the P_k defined in (4.17) for the T_4-configuration X_1, X_2, X_3, X_4.

The strategy of the proof of this proposition is to parametrise a set of T_4-configurations close to $X_1^0, X_2^0, X_3^0, X_4^0$. To this end, we let $C_1^0, C_2^0, C_3^0, C_4^0$ be the rank-one matrices of the T_4-configuration $X_1^0, X_2^0, X_3^0, X_4^0$. Then, we parametrise the set of rank-one matrices C_1, C_2, C_3, C_4 near to $C_1^0, C_2^0, C_3^0, C_4^0$ which also satisfy (4.13). We use this parametrisation combined with the equations (4.14) to parametrise a set of T_4-configurations close to $X_1^0, X_2^0, X_3^0, X_4^0$. By making sure that the parametrisation is locally a homeomorphism, we show existence of the smooth maps $\pi_k \colon \mathcal{M} \to M^{4\times 2}$. We also deduce that $\mathcal{M}$ is 24-dimensional because the dimension of the space of all parameters in the parametrisation is 24-dimensional.

Proof. Begin by observing that the rank-one matrices C_1^0, C_2^0, C_3^0, C_4^0 of the T_4-configuration $(X_1^0, X_2^0, X_3^0, X_4^0)$ can be written as

$$C_1^0 = f_1 \otimes e_1, \quad C_2^0 = f_2 \otimes e_2, \quad C_3^0 = -f_1 \otimes e_1, \quad \text{and} \quad C_4^0 = -f_2 \otimes e_2,$$

where $f_1 = (2,0,0,2)$, $f_2 = (0,2,2,0)$. Then notice that any 4-tuple of rank-one matrices in a neighbourhood of $(C_1^0, C_2^0, C_3^0, C_4^0)$ can be parametrized by

$$C_1 = (f_1 + a_1) \otimes (e_1 + \beta_1 e_2), \tag{4.19a}$$
$$C_2 = (f_2 + a_2) \otimes (e_2 - \beta_2 e_1), \tag{4.19b}$$
$$C_3 = (-f_1 + a_3) \otimes (e_1 + \beta_3 e_2) \text{ and} \tag{4.19c}$$
$$C_4 = (-f_2 + a_4) \otimes (e_2 - \beta_4 e_1), \tag{4.19d}$$

where $a_1, \ldots, a_4 \in \mathbb{R}^4$, $\beta_1, \ldots, \beta_4 \in \mathbb{R}$. Since the 4-tuple $(C_1, \ldots, C_4)$ must also satisfy the closing condition

$$\sum_{i=1}^{4} C_i = 0, \tag{4.20}$$

the parameters $a_1, \ldots, a_4, \beta_1, \ldots \beta_4$ must also satisfy the following two equations:

$$\left(\sum_{i=1}^{4} C_i\right) e_1 = a_1 + a_3 + (\beta_4 - \beta_2) f_2 - \beta_2 a_2 - \beta_4 a_4 = 0 \text{ and}$$
$$\left(\sum_{i=1}^{4} C_i\right) e_2 = a_2 + a_4 + (\beta_1 - \beta_3) f_1 + \beta_1 a_1 + \beta_3 a_3 = 0.$$

Hence a_3 and a_4 can be written as

$$a_3 = -a_1 - (\beta_4 - \beta_2) f_2 + \beta_2 a_2 + \beta_4 a_4 \text{ and}$$
$$a_4 = -a_2 - (\beta_1 - \beta_3) f_1 - \beta_1 a_1 - \beta_3 a_3.$$

Therefore, the set of 4-tuples of rank-one matrices satisfying (4.20) can be parametrized by $a_1, a_2 \in \mathbb{R}^4$, $\beta_1, \dots, \beta_4 \in \mathbb{R}$ as in (4.19) but where a_3 and a_4 have been replaced by the expressions above. In other words,

$$\begin{aligned} C_1 &= (f_1 + a_1) \otimes (e_1 + \beta_1 e_2), \\ C_2 &= (f_2 + a_2) \otimes (e_2 - \beta_2 e_1), \\ C_3 &= (-f_1 - a_1 - (\beta_4 - \beta_2) f_2 + \beta_2 a_2 + \beta_4 a_4) \otimes (e_1 + \beta_3 e_2) \text{ and} \\ C_4 &= (-f_2 - a_2 - (\beta_1 - \beta_3) f_1 - \beta_1 a_1 - \beta_3 a_3) \otimes (e_2 - \beta_4 e_1). \end{aligned} \tag{4.21}$$

Now we would like to parametrize the set of T_4-configurations by plugging (4.21) into (4.14). To be precise, we define the map

$$\Phi \colon M^{4\times 2} \times (\mathbb{R}^4)^2 \times \mathbb{R}^4 \times \mathbb{R}^4 \to \{(X_1, \dots, X_4) \in (M^{4\times 2})^4 : \{X_k\}_{k=1}^4 \text{ forms a } T_4\text{-configuration}\}$$

as

$$\Phi(P, a_1, a_2, \beta_1, \beta_2, \beta_3, \beta_4, \kappa_1, \kappa_2, \kappa_3, \kappa_4) = \begin{pmatrix} P + \kappa_1 C_1 \\ P + C_1 + \kappa_2 C_2 \\ P + C_1 + C_2 + \kappa_3 C_3 \\ P + C_1 + C_2 + C_3 + \kappa_4 C_4 \end{pmatrix}.$$

Then we compute the differential of Φ at $(0,0,0,0,0,0,0,2,2,2,2)$:

$$D\Phi(0,0,0,0,0,0,0,2,2,2,2)[P, a_1, a_2, \beta_1, \beta_2, \beta_3, \beta_4, \kappa_1, \kappa_2, \kappa_3, \kappa_4].$$

Using Maple, we show that the differential has full rank (i.e. 24). In particular, this means that in a neighbourhood of

$$(0,0,0,0,0,0,0,2,2,2,2)$$

Φ is an immersion. Since every immersion is locally an embedding, we conclude that by taking $\mathcal{U} \subset M^{4\times 2} \times (\mathbb{R}^4)^2 \times \mathbb{R}^4 \times \mathbb{R}^4$ small enough, $\mathcal{M} := \Phi(\mathcal{U})$ is the desired manifold. Finally, we prove the existence of the smooth maps π_k. Indeed, the homeomorphism $\Phi^{-1} \colon \mathcal{M} \to \mathcal{U}$ allows us to recover the matrices P, C_1, C_2, C_3 and C_4 and thus also the matrices P_k for all k. More precisely, we define $\pi_k = \widetilde{\pi}_k \circ \Phi^{-1}$, in which

$$\widetilde{\pi}_k \colon M^{4\times 2} \times (\mathbb{R}^4)^2 \times \mathbb{R}^4 \times \mathbb{R}^4 \to M^{4\times 2}$$

is defined by

$$\widetilde{\pi}_k(P, a_1, a_2, \beta_1, \beta_2, \beta_3, \beta_4, \kappa_1, \kappa_2, \kappa_3, \kappa_4) = P + \sum_{i=1}^{k-1} C_i,$$

where each C_i is the expression defined in (4.21). $\square$

We conclude this section by defining some more maps. We define the projections

$$\phi_k \colon \mathcal{M} \to M^{4\times 2} \quad \text{as} \quad \phi_k(X_1, \ldots, X_4) = X_k$$

for all $k = 1, \ldots, 4$. We also define the maps $\pi'_k \colon \mathcal{M} \to (T_{X_1^0}K)^\perp$ such that $\pi'_k(X_1, \ldots, X_4)$ is the orthogonal projection of $\pi_k(X_1, \ldots, X_4)$ onto $(T_{X_k^0}K)^\perp$. In other words,

$$\pi'_k = \mathrm{Proj}_{(T_{X_k^0}K)^\perp} \circ \pi_k.$$

4.5 Construction of a special quasiconvex function

The objective of this section is to build the function F_0 of Theorem 4.0.1. To do so, the crucial point is to have T_4-configurations whose elements are all contained in the set K_{F_0} (see (4.5) to recall the definition of K_{F_0}). Therefore, we define the 16-dimensional manifold

$$\mathcal{K}_F = K_F \times K_F \times K_F \times K_F.$$

embedded in $(\mathbb{R}^{4\times 2})^4$. To be more precise, we will build an F_0 such that the T_4-configuration given by (4.15) is contained in $\mathcal{K}_{F_0}$. Unfortunately, this is not enough. We also need a non-degeneracy condition to be satisfied. Before stating it, we introduce the notion of transversality (the following definition and theorem come from [19, Chapter 6]):

Definition 4.5.1. *Let $\mathcal{A}$ be a smooth manifold and consider two embedded smooth submanifolds, $\mathcal{S}$, $\mathcal{S}' \subset \mathcal{A}$. Then we say that $\mathcal{S}$ and $\mathcal{S}'$ intersect transversely if at each $p \in \mathcal{S} \cap \mathcal{S}'$, the tangent spaces $T_p\mathcal{S}$ and $T_p\mathcal{S}'$ together span $T_p\mathcal{A}$.*

Let us now make a few remarks.

Remark 4.5.2. *If for some $p \in \mathcal{S} \cap \mathcal{S}'$ the tangent $T_p\mathcal{S}$ and $T_p\mathcal{S}'$ together span $T_p\mathcal{A}$, we will say that $\mathcal{S}$ and $\mathcal{S}'$ intersect transversely at p.*

Remark 4.5.3. *In this section, the manifold denoted $\mathcal{A}$ in the previous definition will always be $(M^{4\times 2})^4$.*

Remark 4.5.4. *Notice that in Definition 4.5.1, the condition that the tangent spaces $T_p\mathcal{S}$ and $T_p\mathcal{S}'$ together span $T_p\mathcal{A}$ is an open condition. In other words, if for some $p \in \mathcal{S} \cap \mathcal{S}'$, $T_p\mathcal{S}$ and $T_p\mathcal{S}'$ together span $T_p\mathcal{A}$ then there is an open neighbourhood $\mathcal{U} \subset \mathcal{S} \cap \mathcal{S}'$ of p such that for all $p' \in \mathcal{U}$, $T_{p'}\mathcal{S}$ and $T_{p'}\mathcal{S}'$ together span $T_{p'}\mathcal{A}$. In particular, this means that if $\mathcal{S}$ and $\mathcal{S}'$ intersect transversely at some point p then $\mathcal{S}$ and $\mathcal{S}'$ intersect transversely locally around p.*

A nice feature of manifolds that intersect transversely is that their intersection is also a manifold. Indeed, the following result holds (see [19, Theorem 6.30] for a proof).

Theorem 4.5.5. *Suppose $\mathcal{S}, \mathcal{S}' \subset \mathcal{A}$ are embedded submanifolds. If $\mathcal{S}$ intersects $\mathcal{S}'$ transversely, then $\mathcal{S} \cap \mathcal{S}'$ is an embedded submanifold of $\mathcal{A}$ with*

$$\dim(\mathcal{S} \cap \mathcal{S}') = \dim(\mathcal{A}) - \dim(\mathcal{S}) - \dim(\mathcal{S}').$$

We can now state the non-degeneracy condition:

Definition 4.5.6. *We say that condition (C) is satisfied if*

(i) $\mathcal{M}$ and $\mathcal{K}_F$ intersect transversely at $(X_1^0, X_2^0, X_3^0, X_4^0)$ and

(ii) after replacing $\mathcal{M}$ with a small neighbourhood of $(X_1^0, X_2^0, X_3^0, X_4^0)$ in $\mathcal{M}$, the map (ϕ_k, π_k') is, for each k, a nondegenerate diffeomorphism from $\mathcal{M} \cap \mathcal{K}_F$ to a neighbourhood of $(X_k^0, \pi_k'(X_1^0, X_2^0, X_3^0, X_4^0))$ in $K \times (T_{X_k^0} K)^\perp$.

Remark 4.5.7. *By Theorem 4.5.5 and Remark 4.5.4 , the condition (i) in the definition above implies that $\mathcal{M} \cap \mathcal{K}_F$ is locally a manifold around $(X_1^0, X_2^0, X_3^0, X_4^0)$.*

Before stating the main result of this section, we introduce the map $\theta \colon M^{2\times 2} \to M^{2\times 2}$ defined by

$$\theta(X) = TXJ^T \quad \text{where} \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now, the goal of the remainder of this section is to prove the two following lemmas:

Lemma 4.5.8. *There exists a smooth, strongly quasiconvex function $F_1 \colon M^{2\times 2} \to \mathbb{R}$ with uniformly bounded $D^2 F_1$ which satisfies $F_1(\theta(X)) = F_1(X)$ and*

$$DF_1(A_1) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Remark 4.5.9. *It should be noted that F_1 has the property $DF_1(\theta(X)) = \theta(DF_1(X))$. As a consequence, Lemma 4.5.8 implies that the set K_{F_1} contains the T_4-configuration X_1^0, X_2^0, X_3^0 and X_4^0. However, as mentioned in the beginning of this section, we also need the non-degeneracy condition (C) to be satisfied. We will see that whether (C) is satisfied or not depends only on the values of the second derivatives $D^2F(A_j)$ of the function F. The following lemma tells us for which values $D^2F(A_j)$, (C) is satisfied.*

Lemma 4.5.10. *Assume that $DF(A_k) = DF_1(A_k)$ for $k = 1, \ldots, 4$. Then condition (C) above is satisfied for generic values of $D^2F(A_k)$, $k = 1, \ldots, 4$, i.e. there is a residual set such that for all 4-tuples $(D^2F(A_1), D^2F(A_2), D^2F(A_3), D^2F(A_4))$ belonging to this residual set, (C) is satisfied.*

This lemma in particular tells us that if the function F_1 given by Lemma 4.5.8 does not satisfy (C), then we may take a smooth, compactly supported, function $V\colon M^{2\times 2} \to \mathbb{R}$ such that $F_0 := F_1 + V$ is smooth, strongly quasiconvex and satisfies (C). Indeed choose a smooth V such that $\|V\|_{C(M^{2\times 2})}$ is small, $DV(A_j) = 0$ for all j and $D^2V(A_j)$ such that F_0 satisfies (C) in virtue of Lemma 4.5.10. Since $DV(A_j) = 0$ for all j, K_{F_0} still contains the T_4-configuration given by X_1^0, X_2^0, X_3^0 and X_4^0. The fact that F_0 is strongly quasiconvex follows from the fact that $\|V\|_{C(M^{2\times 2})}$ is small and F_1 is strongly quasiconvex.

Proof of Lemma 4.5.8. We begin by defining $f_0\colon S^{2\times 2} \to \mathbb{R}$ by

$$f_0(X) = \begin{cases} \det(X) & \text{if } X \text{ is positive definite;} \\ 0 & \text{otherwise.} \end{cases}$$

We claim the following which we will prove later:

Claim: f_0 is quasiconvex.

Now take ω to be a non-negative smooth function that satisfies the following properties:

(i) ω is supported in $B_{1/8}$,

(ii) $\displaystyle\int_{S^{2\times 2}} \omega \, dX = 1$,

(iii) $\displaystyle\int_{S^{2\times 2}} X\omega \, dX = 0$ and

(iv) $\displaystyle\int_{S^{2\times 2}} \det X\omega(X) \, dX = 0$.

To show that such a function exists, it suffices to check that there exist two positive functions $\omega_1, \omega_2 \in C_0^\infty(B_{1/8})$ such that,

$$\int_{S^{2\times 2}} X\omega_i(X) \, dX = 0 \quad \text{and} \quad (-1)^i \int_{S^{2\times 2}} \det X\omega_i(X) \, dX < 0 \quad \text{for } i = 1, 2.$$

Then, by considering convex combinations of the form $\lambda\omega_1 + (1-\lambda)\omega_2$, we find a function ω such that

$$\int_{S^{2\times 2}} X\omega(X) \, dX = 0 \quad \text{and} \quad \int_{S^{2\times 2}} \det X\omega(X) \, dX = 0.$$

Finally, it then suffices to normalise the function ω to get property (ii) and hence the desired function. Let us therefore verify that ω_1 and ω_2 exist. To show ω_1 exists, it suffices to take some arbitrary φ smooth, non-negative, radially symmetric, function centred at the origin. Then consider the function of the form $\omega_1(X) = \varphi(X - Y_1) + \varphi(X - Y_2)$ where

$$Y_1 = \begin{pmatrix} \frac{1}{32} & 0 \\ 0 & \frac{1}{32} \end{pmatrix} \quad \text{and} \quad Y_2 = -Y_1.$$

Under the assumption that φ is supported in a sufficiently small ball around the origin, it is clear that

$$\int_{S^{2\times 2}} \det(X)\omega_1(X)\,dX > 0.$$

It only remains to check that (iii) holds, which however follows from the fact that w_1 is even. The construction of ω_2 is similar. This proves existence of ω with the desired properties. Then define a new function $f_1\colon S^{2\times 2} \to \mathbb{R}$ as

$$f_1(X) = \max(f_0(X), |X|^2 - 100).$$

The fact that f_1 is quasiconvex on $S^{2\times 2}$ follows from the fact that the maximum of quasiconvex functions is quasiconvex. Then define $f_2\colon S^{2\times 2} \to \mathbb{R}$ by $f_2 = f_1 * \omega$. Note that $f_2(X) = f_0(X)$ when $|X| < 9$ and $B_{1/8}(X)$ is contained in the set of symmetric positive definite matrices. The fact that f_2 is quasiconvex on $S^{2\times 2}$ follows from the fact that a convolution between a positive mollifier and a quasiconvex function is quasiconvex. Take $\gamma > 0$ (to be chosen later) and define $f_3\colon S^{2\times 2} \to \mathbb{R}$ as $f_3(X) = f_2(X) + \gamma|X|^2$, so that f_3 is strongly quasiconvex on $S^{2\times 2}$. Then let $\widetilde{f}_3$ be the strongly quasiconvex extension to $M^{2\times 2}$ given by Lemma 4.3.1. Finally, define $f_4\colon M^{2\times 2} \to \mathbb{R}$ as

$$f_4(X) = \sum_{k=0}^{3} \widetilde{f}_3(\theta^{-k}(X) - H) \quad \text{where } H = \begin{pmatrix} \frac{5}{4} & 0 \\ 0 & -\frac{5}{4} \end{pmatrix}.$$

Now, by definition of f_4, we have $f_4(\theta(X)) = f_4(X)$. From this it follows that

$$Df_4(\theta(X)) = \theta(Df_4(X)). \tag{4.22}$$

Finally, $Df_4(A_1)$ can now be computed explicitely and after long but simple computations, we find that

$$D\widetilde{f}_3(A_1) = \begin{pmatrix} \frac{1}{4} + 14\gamma & 0 \\ 0 & \frac{7}{4} + 2\gamma \end{pmatrix}.$$

Then we consider functions $F_{\alpha,\beta}\colon M^{2\times 2} \to \mathbb{R}$ of the form

$$F_{\alpha,\beta}(X) = \frac{1}{2}\alpha|X|^2 + \beta f_4(X) \quad (\text{where } \alpha, \beta > 0).$$

Now we can find $\alpha, \beta, \gamma > 0$ such that

$$DF_{\alpha,\beta}(A_1) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Notice that, due to (4.22), we have

$$DF_{\alpha,\beta}(\theta(X)) = \theta(DF_{\alpha,\beta}(X))$$

so $F_{\alpha,\beta}$ is the required function. This finishes the core of the proof and only the proof of the claim remains:

Proof of the Claim: Let $B_1(0)$ denote the ball of radius 1 centered in 0. We wish to prove that

$$\int_{B_1(0)} [f_0(A + D^2\phi) - f_0(A)]\,dx \geq 0, \quad \forall A \in S^{2\times 2}, \forall \varphi \in C_0^\infty(\Omega). \tag{4.23}$$

We begin by the specific case where $I = A$. Define $u_0 \colon B_1(0) \to \mathbb{R}$ as

$$u_0(x) = \frac{|x|^2}{2}$$

and $u \colon B_1(0) \to \mathbb{R}$ as

$$u(x) = u_0(x) + \phi(x).$$

Then we define $\varphi \colon B_1(0) \to \mathbb{R}$ as $\varphi = Du$ and the set

$$E := \{x \in B_1(0) : D\varphi \text{ is positive definite}\} \subset B_1(0).$$

Equation (4.23) is then equivalent to

$$\int_E \det(D\varphi)\,dx \geq \mathcal{L}^n\left(B_1(0)\right). \tag{4.24}$$

It suffices to verify that $B_1(0) \subset \varphi(E)$ because

$$\int_E \det(D\varphi)\,dx = \int_E |\det(D\varphi)|\,dx = \int_{\varphi(E)} 1\,dx = \mathcal{L}^n\left(\varphi(E)\right).$$

Thus, let $b \in B_1(0)$ be arbitrary and $a \in \overline{B_1(0)}$ be a point where the function $x \mapsto u(x) - b\cdot x$ attains the minimum in $\overline{B_1(0)}$. The minimum is negative and on $\partial B_1(0)$ the map is strictly positive. Thus $a \in B_1(0)$. Since the minimum is attained in a, $\varphi(a) = b$ and $a \in E$. Thus $B_1(0) \subset \varphi(E)$ as desired. This proves (4.24). Thus (4.23) holds for $A = I$. This means that, for any open domain $\Omega' \subset S^{2\times 2}$, we have (by the same argument as in the proof of [22, Lemma 5.2])

$$\int_{\Omega'} [f_0(I + D^2\phi) - f_0(I)]\,dx \geq 0, \quad \forall \varphi \in C_0^\infty(\Omega). \tag{4.25}$$

Now we prove the general case. If A is not positive definite, then (4.23) holds because $f_0 \geq 0$ and $f_0(A) = 0$. If A is positive definite, then (4.23) follows from the following computation:

$$\begin{aligned}
&\int_{B_1(0)} \left[f_0(A + D^2\phi(x)) - f_0(A)\right] \, dx \\
&= \det(A^{1/2}) \int_{B_1(0)} \left[f_0(I + A^{-1}D^2\phi(x)) - f_0(I)\right] \det(A^{1/2}) \, dx \\
&= \det(A^{1/2}) \int_{A^{1/2}(B_1(0))} \Big[f_0(I + A^{-1}D^2\phi(A^{-1/2}x)) - f_0(I)\Big] \, dx \\
&= \det(A^{1/2}) \int_{A^{1/2}(B_1(0))} \left[f_0(I + D^2\eta(x)) - f_0(I)\right] \, dx \overset{(4.25)}{\geq} 0,
\end{aligned}$$

where $\eta\colon A^{1/2}(B_1(0)) \to \mathbb{R}$ is defined by

$$\eta(x) = \phi(A^{-1/2}x)$$

and $A^{1/2}$ denotes the symmetric positive definite matrix such that $A^{1/2}A^{1/2} = A$ (and $A^{-1/2}$ is its inverse). $\square$

We will now prove Lemma 4.5.10. The strategy of the proof is to find a nontrivial polynomial Q in the values of $D^2F(A_i)$ such that whenever $Q \neq 0$, it implies that (C) is satisfied.

Proof of Lemma 4.5.10. First we must compute the tangent spaces $T_{(X_1^0,X_2^0,X_3^0,X_4^0)}\mathcal{M}$ and $T_{(X_1^0,X_2^0,X_3^0,X_4^0)}\mathcal{K}$. We find that the tangent space $T_{(X_1^0,X_2^0,X_3^0,X_4^0)}\mathcal{M}$ can be identified with 4-tuples $(Z_1, \dots, Z_4)$, where each Z_i is a 4 by 2 matrix of the form

$$Z_1 = \begin{pmatrix} p_{11} + 2a_{11} + \kappa_1' & p_{12} + 2\beta_1' \\ p_{21} + 2a_{21} & p_{22} \\ p_{31} + 2a_{31} & p_{32} \\ p_{41} + 2a_{41} + \kappa_1' & p_{42} + 2\beta_1' \end{pmatrix},$$

$$Z_2 = \begin{pmatrix} p_{11} + a_{11} & p_{12} + a_{12} + \beta_1' \\ p_{21} + a_{21} - 2\beta_2' & p_{22} + a_{22} + \kappa_2' \\ p_{31} + a_{31} - 2\beta_2' & p_{32} + a_{32} + \kappa_2' \\ p_{41} + a_{41} & p_{42} + a_{42} + \beta_1' \end{pmatrix},$$

$$Z_3 = \begin{pmatrix} p_{11} - a_{11} - \kappa_3' & p_{12} + a_{12} - 2\beta_3' + \beta_1' \\ p_{21} - a_{21} + \beta_2' - 2\beta_4' & p_{22} + a_{22} \\ p_{31} - a_{31} + \beta_2' - 2\beta_4' & p_{32} + a_{32} \\ p_{41} - a_{41} - \kappa_3' & p_{42} + a_{42} - 2\beta_3' + \beta_1' \end{pmatrix} \text{ and}$$

$$Z_4 = \begin{pmatrix} p_{11} & p_{12} - a_{12} + \beta_3' - \beta_1' \\ p_{21} + \beta_4' & p_{22} - a_{22} - \kappa_4' \\ p_{31} + \beta_4' & p_{32} - a_{32} - \kappa_4' \\ p_{41} & p_{42} - a_{42} + \beta_3' - \beta_1' \end{pmatrix}.$$

The tangent space $T_{(X_1^0,X_2^0,X_3^0,X_4^0)}\mathcal{K}$ consists of 4-tuples of the following form

$$\left(\begin{pmatrix} M_1 \\ D^2F(A_1)M_1J \end{pmatrix}, \begin{pmatrix} M_2 \\ D^2F(A_2)M_2J \end{pmatrix}, \begin{pmatrix} M_3 \\ D^2F(A_3)M_3J \end{pmatrix}, \begin{pmatrix} M_4 \\ D^2F(A_4)M_4J \end{pmatrix}\right)$$

where $M_1, M_2, M_3, M_4 \in M^{2\times 2}$. Now we claim the following which we shall prove later:

Claim: If the following system of equations

$$Z_j = \begin{pmatrix} M_j \\ D^2F(A_j)M_jJ \end{pmatrix}, \quad j = 1, \dots, 4, \tag{4.26}$$

$$\begin{pmatrix} p_{31} & p_{32} \\ p_{41} & p_{42} \end{pmatrix} = D^2F(A_1)\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} J \quad \textit{and} \tag{4.27}$$

$$M_1 = 0, \tag{4.28}$$

has no nontrivial solutions, then $\mathcal{M}$ and $\mathcal{K}_F$ intersect transversely at $(X_1^0, X_2^0, X_3^0, X_4^0)$ and the map (ϕ_1, π_1') is locally a nondegenerate diffeomorphism from $\mathcal{M} \cap \mathcal{K}_F$ to a neighbourhood of $(X_k^0, \pi_k'(X_1^0, X_2^0, X_3^0, X_4^0))$ in $K \times (T_{X_k^0}K)^\perp$.

This system is equivalent to a linear system with 40 equations and 40 unknowns. The determinant of the matrix of this linear system is a polynomial in the entries of $D^2F(A_j)$, $j = 1, 2, 3, 4$. We call this polynomial Q_1. We can prove that Q_1 is not identically zero by noticing that with

$$D^2F(A_1) = I, \quad D^2F(A_2) = I, \quad D^2F(A_3) = 0, \quad \text{and} \quad D^2F(A_4) = I$$

the system admits no nontrivial solution. In a similar way, for each $k = 2, 3, 4$ there is a polynomial Q_k in the entries of $D^2F(A_j)$ such that $Q_k \neq 0$ implies that $\mathcal{M}$ and $\mathcal{K}_F$ intersect transversely at $(X_1^0, X_2^0, X_3^0, X_4^0)$ and the map $(X_1, \dots, X_4) \mapsto (X_k, \pi_k'(X_1, X_2, X_3, X_4))$ is a nondegenerate diffeomorphism between a neighbourhood of $(X_1^0, \dots, X_4^0) \in \mathcal{M} \cap \mathcal{K}_F$ and a neighbourhood of $(X_k^0, \pi_k'(X_1^0, X_2^0, X_3^0, X_4^0)) \in K \times (T_{X_k^0}K)^\perp$. Finally, define $Q = Q_1Q_2Q_3Q_4$. We notice that $Q \neq 0$ implies (C). Since Q is a polynomial in the entries of $D^2F(A_j)$ that is not identically zero, we have finished the proof once the claim is proved.

Proof of the Claim: First we prove that if the system does not have any nontrivial solutions then $\mathcal{M}$ and $\mathcal{K}_F$ intersect transversely. We would like to show that $T_{(X_1^0,X_2^0,X_3^0,X_4^0)}\mathcal{M}$ and $T_{(X_1^0,X_2^0,X_3^0,X_4^0)}\mathcal{K}_F$ span $(M^{4\times 2})^4$. Define

$$W := \{(Z_1, Z_2, Z_3, Z_4) \in T_{(X_1^0,X_2^0,X_3^0,X_4^0)}\mathcal{M} : (4.27) \text{ holds and } Z_1 = 0\}.$$

It is enough to prove that $W + T_{(X_1^0,X_2^0,X_3^0,X_4^0)}\mathcal{K}_F$ spans $(M^{4\times 2})^4$. Since

$$\dim(W) + \dim(T_{(X_1^0,X_2^0,X_3^0,X_4^0)}\mathcal{K}_F) = 32$$

and both W and $T_{(X_1^0,X_2^0,X_3^0,X_4^0)}\mathcal{K}_F$ are linear spaces, it suffices to show that all vectors belonging to W are linearly independent from all vectors in $T_{(X_1^0,X_2^0,X_3^0,X_4^0)}\mathcal{K}_F$. In other words, we would like to find that (4.26) has no nontrivial solutions. To this end, we assume that (4.26) holds and we prove that the solution is the trivial one. Notice that (4.27) already holds by definition of W and since we assume (4.26), $Z_1 = 0$ implies (4.28). Thus all equations (4.26), (4.27) and (4.28) hold. By the claim, we get that the solution is the trivial one. Thus, $\mathcal{M}$ and $\mathcal{K}_F$ intersect transversely at $(X_1^0, X_2^0, X_3^0, X_4^0)$ and by Theorem 4.5.5 and Remark 4.5.4, $\mathcal{M} \cap \mathcal{K}_F$ is locally an 8-dimensional manifold in a neighbourhood of $(X_1^0, X_2^0, X_3^0, X_4^0)$. Now we prove that the system having no nontrivial solutions also implies that the map

$$(X_1, \dots, X_4) \mapsto (X_1, \pi_1'(X_1, X_2, X_3, X_4)) \tag{4.29}$$

is a nondegenerate diffeomorphism between a neighbourhood of $(X_1^0, \dots, X_4^0) \in \mathcal{M} \cap \mathcal{K}_F$ and a neighbourhood of $(X_1^0, (P_1^0)') \in K \times (T_{X_1^0}K)^\perp$. Since the neighbourhood around $(X_1^0, X_2^0, X_3^0, X_4^0)$ may be taken as small as we wish it suffices to show that the differential of the map in (4.29) is bijective. Since both $\mathcal{M} \cap \mathcal{K}_F$ and $K \times (T_{X_1^0}K)^\perp$ are 8-dimensional manifolds, it suffices to verify that the differential is injective. First notice that the differential is given by

$$D\pi_1' = \mathrm{Proj}_{(T_{X_1^0}K)^\perp} \circ D\pi_1. \tag{4.30}$$

To prove that the differential is injective, we assume that

$$Z_1 = 0, \quad D\pi_1'(Z_1, Z_2, Z_3, Z_4) = 0$$

for some $(Z_1, Z_2, Z_3, Z_4) \in T_{(X_1^0,X_2^0,X_3^0,X_4^0)}\mathcal{M} \cap \mathcal{K}_F$ and show that this implies

$$Z_1 = Z_2 = Z_3 = Z_4 = 0.$$

From $D\pi_1'(Z_1, Z_2, Z_3, Z_4) = 0$, we deduce that

$$D\pi_1(Z_1, Z_2, Z_3, Z_4) \in T_{X_1^0}K. \tag{4.31}$$

Since the differential of the map π_1 is

$$D\pi_1(Z_1, Z_2, Z_3, Z_4) = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \\ p_{31} & p_{32} \\ p_{41} & p_{42} \end{pmatrix},$$

(4.31) gives us that (4.27) holds. Then we choose M_1, M_2, M_3, M_4 such that (4.26) holds (this is possible since $Z_j \in T_{X_j^0}K$ for all $j = 1, 2, 3, 4$). Finally (4.28) holds since $Z_1 = 0$. Since all equations (4.26), (4.27) and (4.28) hold, we conclude that

$$Z_1 = Z_2 = Z_3 = Z_4 = 0$$

as desired. □

We recall that this last lemma implies that there exists a strongly quasiconvex function F_0 which satisfies (C) such that K_{F_0} contains the T_4-configuration given by X_1^0, X_2^0, X_3^0 and X_4^0.

4.6 Proof of the main theorem

Finally, in this last section of the chapter, the goal is to prove Theorem 4.0.1 stated at the beginning of this chapter. First, however, we build a suitable in-approximation.

Lemma 4.6.1. *Using the notation above, assume that condition (C) is satisfied. Let $r > 0$. Then there exists an in-approximation $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of the set*

$$K_r = \bigcup_{j=1}^{4} \{X \in M^{4\times 2}, |X - X_j^0| \leq r\} \cap K_F$$

such that $\mathcal{U}_1$ contains a small neighbourhood of the rank-one convex hull of the points $P_1^0, \ldots, P_4^0$.

Before proving this we need the following result from linear algebra (see [26, Chapter 0] for a proof and other related results):

Lemma 4.6.2. *Let A, B, C, D be four $k \times k$ matrices and suppose that D is invertible. Then the determinant of the block matrix*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is

$$\det(M) = \det(A - BD^{-1}C)\det(D). \tag{4.32}$$

Proof of Lemma 4.6.1 . We consider maps $\Phi_k^\lambda \colon \mathcal{M}_F \cap \mathcal{K} \to M^{4\times 2}$ for $k = 1, \ldots, 4$ and $\lambda \in (0, 1)$ defined by

$$\Phi_k^\lambda = (1 - \lambda)\pi_k + \lambda\phi_k.$$

We begin with the following claim which we prove later:

Claim: There exists $0 < \Lambda < 1$ *and an open set* $\mathcal{O} \subset \mathcal{M} \cap \mathcal{K}_F$ *such that* $\Phi_k^\lambda |_{\mathcal{O}}$ *is an open map for all* $k = 1, \ldots, 4$ *and* $\Lambda < \lambda < 1$.

Now take a sequence of open sets $\{\mathcal{O}_j\}_{j=1}^\infty$ such that $\mathcal{O}_j \subset \mathcal{O}$ for all $j \geq 1$ and $\overline{\mathcal{O}}_j \subset \mathcal{O}_{j+1}$ for all $j \geq 1$. Then take an increasing sequence of numbers $\{\lambda_j\}_{j=1}^\infty$ such that $\lambda_1 = 0$ and $\lambda_j > \Lambda$ for all $j \geq 2$. Then, for each $k = 1, \ldots, 4$ and $j \geq 0$, define

$$\mathcal{U}_{k,j} = \{\Phi_k^{\lambda_j}(X_1, X_2, X_3, X_4) : (X_1, X_2, X_3, X_4) \in \mathcal{O}_j\}$$

and

$$\mathcal{U}_j = \bigcup_{k=1}^4 \mathcal{U}_{k,j}.$$

By the claim, $\mathcal{U}_j$ is open for all $j \geq 1$ and since the map π_1 is an open map, $\mathcal{U}_1$ is open too. The fact that $\mathcal{U}_j \subset \mathcal{U}_{j+1}^{co}$ for all $j \geq 1$ is the content of the following lemma. The fact that

$$\sup_{X \in \mathcal{U}_j} \operatorname{dist}(X, K_r) \to 0$$

follows from the definition of the maps Φ_k^λ. Thus $\{\mathcal{U}_j\}_{j=1}^\infty$ is an in-approximation of the set K_r. Finally, since $\mathcal{U}_1$ contains $P_1^0, P_2^0, P_3^0, P_4^0$ and $\mathcal{U}_1 \subset \mathcal{U}_2^{rc}$ we have that $\mathcal{U}_2^{rc}$ contains a neighbourhood of the rank-one convex hull of $\{P_1^0, P_2^0, P_3^0, P_4^0\}$. Replacing $\mathcal{U}_1$ by this neighbourhood gives us the desired in-approximation.

Proof of the Claim: Without loss of generality, we take $k = 1$. Take $\mathcal{O}$ to be a small neighbourhood of $(X_1^0, X_2^0, X_3^0, X_4^0)$ in $\mathcal{M} \cap \mathcal{K}_F$. Notice that for all $(X_1, X_2, X_3, X_4) \in \mathcal{O}$, X_1 and P_1 can be written as

$$X_1 = X_1^0 + X(X_1) + \xi(X_1) \tag{4.33}$$

where $X(X_1) \in T_{A_1}K$ and $\xi(X(X_1)) \in (T_{A_1}K)^\perp$, and

$$P_1(X_1, X_2, X_3, X_4) = P_1^0 + \eta(X_1, X_2, X_3, X_4) + Y(X_1, X_2, X_3, X_4)$$

where $Y(X_1, X_2, X_3, X_4) \in (T_{A_1}K)^\perp$ and $\eta(X_1, X_2, X_3, X_4) \in T_{A_1}K$. It is clear that X is a diffeomorphism when taking $\mathcal{O}$ small enough. Moreover, from condition (C) we see that the map $(X_1, X_2, X_3, X_4) \mapsto (X_1, Y(X_1, X_2, X_3, X_4))$ is a diffeomorphism (provided $\mathcal{O}$ is small enough) Since X is a diffeomorphism the previous observation implies that $(X_1, X_2, X_3, X_4) \mapsto (X(X_1), Y(X_1, X_2, X_3, X_4))$ is also a diffeomorphism. Thus (X, Y) can be taken as coordinates in $\mathcal{O}$ (provided it is small enough). In these coordinates, the map Φ_1^λ takes the form

$$\begin{aligned}\Phi_1^\lambda(X,Y) &= (1-\lambda)(P_1^0 + Y + \eta(X,Y)) + \lambda(X_1^0 + X + \xi(X)) \\ &= (1-\lambda)P_1^0 + \lambda X_1^0 \\ &\quad + \underbrace{\lambda X + (1-\lambda)\eta(X,Y)}_{\in T_{A_1}K} \\ &\quad + \underbrace{(1-\lambda)Y + \lambda\xi(X)}_{\in (T_{A_1}K)^\perp}\end{aligned}$$

and its differential in matrix form with respect to (X,Y) is

$$\begin{pmatrix} \lambda I + (1-\lambda)\partial_X\eta & (1-\lambda)\partial_Y\eta \\ \lambda\partial_X\xi & (1-\lambda)I \end{pmatrix}.$$

To show that the map $\Phi_1^\lambda|_{\mathcal{O}}$ is an open map, we show that the determinant is nonzero in $\mathcal{O}$ (where $\mathcal{O}$ does not depend on λ). Observe that

$$\begin{aligned}&\det\begin{pmatrix} \lambda I + (1-\lambda)\partial_X\eta & (1-\lambda)\partial_Y\eta \\ \lambda\partial_X\xi & (1-\lambda)I \end{pmatrix} \\ &\overset{(4.32)}{=} \det(\lambda I + (1-\lambda)\partial_X\eta - \lambda\partial_Y\eta\partial_X\xi)\underbrace{\det((1-\lambda)I)}_{\neq 0}.\end{aligned}$$

Therefore, it suffices to show that

$$\det(\lambda(I - \partial_Y\eta\partial_X\xi) + (1-\lambda)\partial_X\eta) \neq 0 \quad \text{ in } \mathcal{O}.$$

Define[1]

$$M = 2\max(|\partial_Y\eta(0,0)|_\infty, |\partial_X\eta(0,0)|_\infty, 1).$$

Then there is $\delta_1 > 0$ such that

$$|\partial_Y\eta(X,Y)|_\infty < M \quad \text{and} \quad |\partial_X\eta(X,Y)|_\infty < M \quad \text{for all} \quad (X,Y) \in B_{\delta_1}(0) \times B_{\delta_1}(0).$$

Before going further, we prove that $\partial_X\xi(0) = 0$ in coordinates. From (4.33) we obtain

$$\underbrace{\frac{X_1 - X_1^0}{|X_1 - X_1^0|}}_{\to v \in T_{A_1}K} = \underbrace{\frac{1}{|X_1 - X_1^0|}X(X_1)}_{\in T_{A_1}K} + \underbrace{\frac{1}{|X_1 - X_1^0|}\xi(X_1)}_{\in (T_{A_1}K)^\perp}$$

and therefore

$$\lim_{X_1 \to X_1^0} \frac{\xi(X_1)}{|X_1 - X_1^0|} = 0.$$

[1] For all matrices $A \in M^{n\times m}$, we define $|A|_\infty := \max_{1\leq i\leq n, 1\leq j\leq m} |A_{ij}|$

This is equivalent to $\partial_X \xi(0) = 0$ in coordinates. Therefore, for any $\varepsilon > 0$ (to be chosen later) there is $\delta_2 > 0$ such that $|\partial_X \xi(X)| < \varepsilon$ for all $X \in B_{\delta_2}(0)$. Define $\delta = \min(\delta_1, \delta_2)$. Taking some Λ (to be chosen later), we get for all $\lambda > \Lambda$ and all $(X, Y) \in B_\delta(0) \times B_\delta(0)$ that

$$|\partial_Y \eta(X,Y)\partial_X \xi(X,Y)|_\infty < \varepsilon M \quad \text{and} \quad |(1-\lambda)\partial_X \eta(X,Y)|_\infty < (1-\Lambda)M.$$

Finally, taking $\varepsilon = (8M)^{-1}$ and $\Lambda = 1 - (8M)^{-1}$, we get that for all $\lambda > \Lambda$ and all $(X,Y) \in B_\delta(0) \times B_\delta(0)$

$$|\partial_Y \eta(X,Y)\partial_X \xi(X,Y)|_\infty < 1/8 \quad \text{and} \quad |(1-\lambda)\partial_X \eta(X,Y)|_\infty < 1/8.$$

Thus,

$$\det(\lambda(I - \partial_Y \eta \partial_X \xi) + (1-\lambda)\partial_X \eta) \neq 0$$

for all $(X,Y) \in B_\delta(0) \times B_\delta(0)$ and $\lambda > \Lambda$. This proves that the map $\Phi_1^\lambda |_{\mathcal{O}}$ is an open map as wished. □

Lemma 4.6.3. *Using the notation introduced in the previous proof, the following is true. For each integer $j \geq 0$, the set $\mathcal{U}_j$ is contained in $\mathcal{U}_{j+1}^{rc}$ and each $A \in \mathcal{U}_{j,k}$ is the center of mass of a laminate $\mu = \sum_{l=1}^4 \mu_l \delta_{Y_j}$ with $Y_l \in \mathcal{U}_{l,j+1}$. In addition, when λ_j is sufficiently close to 1 and $\mathcal{O}$ is sufficiently small (but independent of j), the following is true:*

$$\mu_k \geq 1 - (\lambda_{j+1} - \lambda_j), \tag{4.34}$$
$$|Y_k - A| \leq 2|X_k^0 - P_k^0|(\lambda_{j+1} - \lambda_j) \quad \textit{and} \tag{4.35}$$
$$\mu_l \geq (\lambda_{j+1} - \lambda_j)/8 \quad \textit{for all } l \neq k. \tag{4.36}$$

Proof. Without loss of generality, assume that $A \in \mathcal{U}_{j,1}$. Then there exists

$$(X_1, X_2, X_3, X_4) \in \mathcal{O}_j$$

such that

$$A = (1-\lambda_j)\pi_1(X_1, X_2, X_3, X_4) + \lambda_j X_1.$$

Since $\{X_i\}_{i=1}^4$ forms a T_4-configuration there exists $P \in M^{4\times 2}$, $\{\kappa_i\}_{i=1}^4$ and rank-one matrices $\{C_i\}_{i=1}^4 \subset M^{4\times 2}$ such that

$$X_i = P + \sum_{l=1}^{i-1} C_l + \kappa_i C_i \quad \text{for all } i = 1, \ldots, 4.$$

Then, for each $i = 1, \ldots, 4$ set

$$Y_i = (1-\lambda_{j+1})\pi_i(X_1, X_2, X_3, X_4) + \lambda_{j+1} X_l.$$

This means

$$Y_i = P + \sum_{l=1}^{i-1} C_l + \lambda_{j+1}\kappa_i C_i \quad \text{for all } i = 1, \dots, 4,$$

which shows that $\{Y_i\}_{i=1}^4$ forms a T_4-configuration. Now define

$$P_1 = \pi_1(X_1, X_2, X_3, X_4)$$

and see that Y_1 and P_1 are rank-one connected. Therefore, the following probability measure is a laminate of finite order:

$$\widetilde{\mu} = \frac{\lambda_j}{\lambda_{j+1}}\delta_{Y_1} + \left(1 - \frac{\lambda_j}{\lambda_{j+1}}\right)\delta_{P_1}. \tag{4.37}$$

We compute its barycenter:

$$\begin{aligned}\overline{\widetilde{\mu}} &= \frac{\lambda_j}{\lambda_{j+1}}Y_1 + \left(1 - \frac{\lambda_j}{\lambda_{j+1}}\right)P_1 \\ &= \frac{\lambda_j}{\lambda_{j+1}}((1-\lambda_{j+1})P_1 + \lambda_{j+1}X_1) + \left(1 - \frac{\lambda_j}{\lambda_{j+1}}\right)P_1 \\ &= (1-\lambda_j)P_1 + \lambda_j X_1 \\ &= A.\end{aligned}$$

Before going further we notice that (4.35) holds due to the following computation:

$$\begin{aligned}|Y_1 - A| &= |(1-\lambda_{j+1})P_1 + \lambda_{j+1}X_1 - (1-\lambda_j)P_1 + \lambda_j X_1| \\ &= |X_1 - P_1|(\lambda_{j+1} - \lambda_j) \le 2|X_1^0 - P_1^0|(\lambda_{j+1} - \lambda_j)\end{aligned}$$

where the last inequality hold due to the fact that $\mathcal{O}$ is assumed to be sufficiently small. Since $\{Y_i\}_{i=1}^4$ forms a T_4-configuration, Proposition 2.3.6 guarantees that there is a laminate α supported in $\{Y_l\}_{l=1}^4$ such that $\overline{\alpha} = P_1$. By Remark 4.4.3 and (4.18), α takes the form

$$\alpha = \sum_{l=1}^{4} \alpha_l \delta_{Y_l}$$

where

$$\alpha_l \ge \frac{16}{15} \cdot \frac{1}{8} \text{ for all } l = 2, 3, 4, \tag{4.38}$$

under the assumption that λ_j is sufficiently large and $\mathcal{O}$ is sufficiently small. To define the desired laminate μ, we need to replace δ_{P_1} in (4.37) by α. Set

$$\mu = \frac{\lambda_j}{\lambda_{j+1}}\delta_{Y_1} + \left(1 - \frac{\lambda_j}{\lambda_{j+1}}\right)\alpha$$

The fact that μ is a laminate comes from the fact that $\widetilde{\mu}$ is a laminate, α is a laminate with barycenter P_1 and $\operatorname{rank}(P_1 - Y_1) = 1$. From (4.38) and the assumption that λ_j is close to 1, we conclude that (4.36) holds. Moreover, (4.34) holds from a direct computation. □

We can now prove Theorem 4.0.1. Let us first briefly present the idea behind the proof. We will build a sequence $\{u_i\}_{i=1}^{\infty}$ such that $Du_i \in \mathcal{U}_i$ a.e. for all i. For each i, we build u_{i+1} from u_i by adding more oscillations at each step. We expect the limit of this sequence to be the desired function. Take $V \subset \Omega$ such that $u_i|_V$ is affine. Let $A = Du_i|_V \in \mathcal{U}_i$. By Lemma 4.6.3, there exists a laminate μ supported in $\mathcal{U}_{i+1}$ with $\overline{\mu} = A$ and $\mu(\mathcal{U}_{l,i+1}) > 0$ for all l (provided i is large enough). By applying Lemma 3.2.4, we get a function $w\colon V \to \mathbb{R}^4$ such that instead of being affine, the gradient takes values in all of the sets $\mathcal{U}_{l,i+1}$. This adds oscillations and by applying the same argument to each affine part of u_i, we get a function u_{i+1} with more oscillations. By imposing the sets V to become smaller and smaller as $i \to \infty$, we expect the limit function to not be C^1 on any open set.

Proof of Theorem 4.0.1. Let F be an arbitrary function satisfying the assumptions in the statement of the theorem. Since $DF(A_j) = DF_0(A_j)$ for all j, we know that $\mathcal{M}$ and $\mathcal{K}_F$ intersect in the same point as $\mathcal{M}$ and $\mathcal{K}_{F_0}$ (i.e. in $(X_1^0, X_2^0, X_3^0, X_4^0)$). Then we see from the proof of Lemma 4.5.10 that condition (C) is preserved under small perturbations of the value of the second derivatives of F_0. Thus, there is a $\delta > 0$ such that for all $F \in C^2(M^{2\times 2})$ satisfying (4.3) the condition (C) still holds. Let $r > 0$, then Lemma 4.6.1 gives us an in-approximation $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of the set

$$K_r = \bigcup_{j=1}^{4} \{X \in M^{4\times 2}, |X - X_j^0| \leq r\} \cap K_F.$$

The lemma also gives us that the set $\mathcal{U}_1$ contains the origin and therefore there exists $\varepsilon > 0$ such that $B_\varepsilon(0) \subset \mathcal{U}_1$. We could already now invoke Theorem 3.3.3 to get a solution to the differential inclusion

$$Dw(x) \in K_r \quad \text{for a.e. } x \in \Omega,$$

and hence a solution to the PDE under consideration. However, this does not automatically gives us a solution which is nowhere C^1. To get such a solution, we revisit the proof of Theorem 3.3.3 and make a more explicit construction. Let $\{\lambda_j\}_{j=1}^{\infty}$ be a sequence as in the proof of Lemma 4.6.1. We may assume that it satisfies

$$\prod_{j=3}^{\infty} (1 - (\lambda_j - \lambda_{j-1})) \geq \frac{1}{2}.$$

Let $r > 0$ be small. Now, let $\phi\colon M^{4\times 2} \to \mathbb{R}$ be a continuous function which is identically equal to 1 in $B_{2r}(0)$ and vanishes outside $B_{3r}(0)$. Then, for each $l = 1, 2, 3, 4$, define $\phi_l(X) = \phi(X - X_l^0)$. To prove the theorem we would like to find a piecewise affine $u\colon \Omega \to \mathbb{R}^2$ such that

- $\|u - v\|_{C^\alpha(\overline{\Omega})} < \varepsilon$,
- $u = v$ on $\partial\Omega$,
- u is not C^1 on any open set and
- u is a weak solution to the equation $\operatorname{div} DF(Du) = 0$.

Let v be a function as in the statement of the theorem and define $\widetilde{w}$ by

$$\widetilde{w} = \begin{pmatrix} v \\ 0 \end{pmatrix}.$$

We now define a sequence of functions $w_j \colon \Omega \to \mathbb{R}^4$ and a sequence of families $\mathcal{F}_j$ of open subsets of Ω satisfying the following properties:

(i) The sets in $\mathcal{F}_j$ are open, mutually disjoint, contained in Ω together with their closures, and cover Ω up to a set of measure 0.

(ii) Each set in $\mathcal{F}_{j+l}$ is contained in a set of $\mathcal{F}_j$ for all $j, l \geq 1$.

(iii) $\sup\{\operatorname{diam}(V) : V \in \mathcal{F}_j\} \to 0$ as $j \to \infty$.

(iv) Dw_j is constant on each $V \in \mathcal{F}_j$ for all $j \geq 1$.

(v) $Dw_j \in \mathcal{U}_j$ a.e. in Ω,

(vi)
- $\|w_1 - \widetilde{w}\|_{C^\alpha(\overline{\Omega})} < \varepsilon/2$,
- $\|w_{j+1} - w_j\|_{C^\alpha(\overline{\Omega})} < 2^{-(j+2)}\varepsilon$, $\forall j \geq 1$ and
- $w_j = \widetilde{w}$ on $\partial\Omega$ $\forall j \geq 1$.

In addition, the following properties hold for j large enough:

(vii) we have

$$\int_\Omega |Dw_{j+1} - Dw_j| \, dx \leq L(\lambda_{j+1} - \lambda_j)\mathcal{L}^2(\Omega).$$

and

(viii) for each $V \in \mathcal{F}_j$ and each $l = 1, 2, 3, 4$, we have

$$\int_V \phi_l(Dw_{j+1}) \, dx \geq \frac{1}{8}(\lambda_{j+1} - \lambda_j)\mathcal{L}^2(V), \tag{4.39}$$

$$\int_V \phi_l(Dw_{j+1}) \, dx \geq (1 - (\lambda_{j+1} - \lambda_j)) \int_V \phi_l(Dw_j) \, dx. \tag{4.40}$$

In order to build $\{w_j\}_{j=1}^{\infty}$ and $\{\mathcal{F}_j\}_{j=1}^{\infty}$, we proceed by induction. The existence of w_1 and $\mathcal{F}_1$ follow from Theorem 3.2.3. Now, to take care of the inductive step, assume that we are already given w_j, $\mathcal{F}_j$ and wish to build w_{j+1} and $\mathcal{F}_{j+1}$. Let $V \in \mathcal{F}_j$ be arbitrary and assume $Dw_j = A \in \mathcal{U}_j$ in V. Let $k \in \{1,2,3,4\}$ be such that $A \in \mathcal{U}_{k,j}$. By Lemma 4.6.3 there exists a laminate

$$\mu = \sum_{l=1}^{4} \mu_l \delta_{Y_l} \quad \text{with } Y_l \in \mathcal{U}_{l,j+1} \tag{4.41}$$

such that $\overline{\mu} = A$. Moreover, when λ_j is close enough to 1 (i.e. j is large enough), we have

$$\mu_k \geq 1 - (\lambda_{j+1} - \lambda_j), \tag{4.42}$$

$$|Y_k - A| \leq 2|X_k^0 - P_k^0|(\lambda_{j+1} - \lambda_j) \quad \text{and} \tag{4.43}$$

$$\mu_l \geq (\lambda_{j+1} - \lambda_j)/8 \quad \text{for all } l \neq k. \tag{4.44}$$

Then by Lemma 3.2.4 we can define a new function $w_{j+1}^V \colon V \to \mathbb{R}^4$ such that

- $\|w_{j+1}^V - w_j\|_{C^\alpha(\overline{V})} < 2^{-(j+3)}\varepsilon$,
- $Dw_{j+1}^V(x) \in \mathcal{U}_{j+1}$ for a.e. $x \in V$,
- $w_{j+1}^V = w_j$ on ∂V and
- $\mathcal{L}^2\left(\{x \in V : Dw_{j+1}^V \in \mathcal{U}_{l,j+1}\}\right) = \mu_l \mathcal{L}^2(V)$ for each $l = 1,2,3,4$.

We choose a family $\mathcal{F}_{j+1}^V$ of mutually disjoint open sets of radius less that $\frac{1}{j+1}$ which cover V up to a set of measure 0 and such that Dw_{j+1}^V is constant on each of these sets. We can now define

$$\mathcal{F}_{j+1} = \bigcup_{V \in \mathcal{F}_j} \mathcal{F}_{j+1}^V$$

and $w_{j+1} \colon \Omega \to \mathbb{R}^4$ by $w_{j+1}(x) = w_{j+1}^V(x)$ if $x \in V$ for all $V \in \mathcal{F}_j$. Finally, we check that all the properties (i) - (viii) hold. We realise that the properties (i) - (vi) hold directly from the above construction. To prove (vii), we observe that for any $j \geq 1$ and $V \in \mathcal{F}_j$ (where we take k to be such that $Dw_j|_V \in \mathcal{U}_{k,j}$ as in the construction above)

$$\begin{aligned}
\int_V |Dw_{j+1} - Dw_j|\,dx &= \int_{\{x \in V : Dw_{j+1}(x) \in \mathcal{U}_{k,j+1}\}} |Dw_{j+1} - Dw_j|\,dx \\
&\quad + \int_{\{x \in V : Dw_{j+1}(x) \notin \mathcal{U}_{k,j+1}\}} |Dw_{j+1} - Dw_j|\,dx \\
&\overset{(4.42),(4.43)}{\leq} 4|X_1^0 - P_1^0|(\lambda_{j+1} - \lambda_j)\mathcal{L}^2(V) \\
&\quad + (\lambda_{j+1} - \lambda_j) \max_{1 \leq i,j \leq 4} |X_i^0 - X_j^0| \mathcal{L}^2(V)
\end{aligned}$$

so by taking $L = 4|X_1^0 - P_1^0| + \max_{1\le i,j\le 4}|X_i^0 - X_j^0|$, we get (vii). Finally, we prove (viii). Let $j \geq 1$ and $V \in \mathcal{F}_j$ be arbitrary. When j is large enough, inequalities (4.42) and (4.44) imply

$$\int_V \phi_l(Dw_{j+1})\,dx \geq \frac{1}{8}(\lambda_{j+1} - \lambda_j)\mathcal{L}^2\,(V) \quad \forall l = 1,2,3,4.$$

This proves the first inequality of (viii). Now, let us prove the second one. Take k such that $Dw_j|_V \in \mathcal{U}_{k,j}$. Then (4.42) implies

$$\int_V \phi_k(Dw_{j+1})\,dx \geq (1 - (\lambda_{j+1} - \lambda_j))\int_V \phi_k(Dw_j)\,dx.$$

This inequality is also true if we replace k with some $l \neq k$, which follows from the fact that

$$\int_V \phi_l(Dw_j)\,dx = 0$$

and $\phi_l \geq 0$. Thus (viii) holds. Now that we have established the (i) - (viii), we conclude the proof of the theorem. Let $w_\infty = \lim_{j\to\infty} w_j$. We notice that due to (vii) and the fact that $\lambda_j \to 1$, $Dw_j \to Dw_\infty$ in $L^1(\Omega; M^{4\times 2})$. From the fact that $\{\mathcal{U}_j\}_{j=1}^\infty$ is an in-approximation, (v) and the fact that $Dw_j \to Dw_\infty$ in $L^1(\Omega; M^{4\times 2})$, we deduce that $Dw_\infty(x) \in K_F$ for a.e. $x \in \Omega$ and hence w_∞ is also Lipschitz. Finally, for j sufficiently large and for all $V \in \mathcal{F}_j$,

$$\begin{aligned}
\int_V \phi_l(Dw_\infty)\,dx &= \lim_{m\to\infty}\int_V \phi_l(Dw_m)\,dx \\
&\geq \lim_{m\to\infty}(1 - (\lambda_m - \lambda_{m-1}))\dots(1 - (\lambda_{j+2} - \lambda_{j+1}))\int_V \phi_l(Dw_{j+1})\,dx \\
&\geq \frac{1}{2}\int_V \phi_l(Dw_{j+1})\,dx \\
&\geq \frac{1}{16}(\lambda_{j+1} - \lambda_j)\mathcal{L}^2\,(V) > 0 \quad \text{for all } l = 1,\dots,4.
\end{aligned}$$

Therefore the essential oscillation of Dw_∞ over any open set is at least

$$\max_{1\le i,j\le 4}\frac{|X_i^0 - X_j^0|}{2}.$$

Thus w_∞ is not C^1 in any open subset of Ω. Finally, viewing w_∞ as

$$w_\infty = \begin{pmatrix} u \\ \widetilde{u} \end{pmatrix}$$

where $u, \widetilde{u} \colon \Omega \to \mathbb{R}^2$, we see that u solves the equation (4.4) and is not C^1 in any open set. This proves the theorem. □

Chapter 5

Application to polyconvex functionals

In this chapter we will apply the convex integration techniques from Chapter 3 to polyconvex functionals. In particular, we will prove the following theorem from Székelyhidi [24, Theorem 1].

Theorem 5.0.1. *Let Ω be the unit ball in $\mathbb{R}^2$. There exists a smooth, strongly polyconvex function $F\colon M^{2\times 2} \to \mathbb{R}$ with bounded second derivatives, such that the corresponding elliptic system*

$$\operatorname{div} DF(Du) = 0 \tag{5.1}$$

admits weak solutions $u\colon \Omega \to \mathbb{R}^2$, which are Lipschitz, but not C^1 in any open subset of Ω. Moreover, F can be chosen so that these weak solutions are weak local minimizers of the functional

$$\mathcal{I}(u) = \int_\Omega F(Du)\,dx,$$

which means that $\mathcal{I}(u) \leq \mathcal{I}(u+\varphi)$ for all $\varphi \in W^{1,\infty}(\Omega)$ such that $\|\varphi\|_{W^{1,\infty}(\Omega)} < \varepsilon$ for some small ε.

The notion of polyconvex functions is recalled in Section 5.1. Then, we prove Theorem 5.0.1 in Section 5.2. The main reference for this entire chapter is the paper by Székelyhidi [24]. We would also like to state that we reuse the same notation as in the previous chapter. In particular, $\mathcal{M}$ is some manifold consisting of T_N-configurations, $\mathcal{K}_F = (K_F)^N$ and the maps ϕ_k, π_k remain the same.

5.1 Polyconvexity

In this section, we introduce the concept of polyconvexity.

Definition 5.1.1. *A function $F\colon M^{2\times 2} \to \mathbb{R}$ is called polyconvex if there exists a convex function $G\colon M^{2\times 2} \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ such that*

$$F(X) = G(X, \det X) \quad \forall X \in M^{2\times 2}.$$

Remark 5.1.2. *Due to the fact that $M^{2\times 2} \times \mathbb{R} \cong \mathbb{R}^5$, the function G in the definition above will be interpreted as a function from $\mathbb{R}^5$ to $\mathbb{R}$.*

It is a well-known fact that polyconvexity implies quasiconvexity under the assumption that F never takes the value $+\infty$. As a consequence, the results stated for quasiconvex functionals in the previous chapter remain true for polyconvex functionals.

Finally, we say that a function $F\colon M^{2\times 2} \to \mathbb{R}$ is strongly polyconvex if there exists a polyconvex $F_0\colon M^{2\times 2} \to \mathbb{R}$ and $\varepsilon > 0$ such that

$$F(X) = F_0(X) + \frac{\varepsilon}{2}|X|^2.$$

For more about polyconvexity, we refer to [22, Chapter 6].

5.2 Proof of the main theorem

The goal of this section is to prove Theorem 5.0.1. The proof is very similar to the one of Theorem 4.0.1 from the previous chapter. However, in the opinion of the author, the proof presented in this section, due to Székelyhidi, offers another perspective.

As in the previous chapter, we need a nondegeneracy condition.

Definition 5.2.1 (*Condition (C)*)**.** *Suppose $F \in C^2(M^{2\times 2})$ is such that K_F contains a T_N-configuration $\{Z_i\}_{i=1}^N$. In addition, suppose that $\mathcal{M}$ is a manifold whose elements are T_N-configurations and $(Z_1, \dots, Z_N) \in \mathcal{M}$. If $\mathcal{M}$ and $\mathcal{K}_F$ intersect transversely at $(Z_1, \dots, Z_N)$ and for each $k = 1, \dots, N$, the maps*

$$\pi_k\colon \mathcal{M} \to M^{4\times 2}$$

are local submersions on $\mathcal{M} \cap \mathcal{K}_F$, then F is said to satisfy condition (C) at $(Z_1, \dots, Z_N)$.

For the moment, we assume that the nondegeneracy condition (C) is satisfied and prove that under the simple condition that K_F contains a T_N-configuration, we can build a very irregular solution the equation (5.1).

Proposition 5.2.2. *Suppose $F \in C^2(M^{2\times 2})$ is such that K_F contains a T_N-configuration $\{Z_1^0, \dots, Z_N^0\}$ and suppose F satisfies condition (C) at $(Z_1^0, \dots, Z_N^0)$. Then there exists another T_N-configuration $\{\widetilde{Z}_1^0, \dots, \widetilde{Z}_N^0\}$ arbitrarily close to $\{Z_1^0, \dots, Z_N^0\}$ such that the following holds. Let*

$$\widetilde{P}_1 \in \{\widetilde{Z}_1^0, \dots, \widetilde{Z}_N^0\}^{rc}$$

as defined in (4.17). Then for any $\varepsilon > 0$ there exists a Lipschitz map $w \colon \Omega \to \mathbb{R}^4$ with the following properties:

- *$Dw(x) \in K_F \cap \left(\bigcup_{k=1}^N B_\varepsilon(\widetilde{Z}_k^0)\right)$ a.e. in Ω,*
- *the function $u = (w^1, w^2)$ is a weak solution of (5.1),*
- *$\|w - \widetilde{P}_1 x\|_{C^\alpha(\overline{\Omega})} < \varepsilon$, $w(x) = \widetilde{P}_1 x$ on $\partial\Omega$ and*
- *Dw has essential oscillation of order 1 in any subdomain of Ω, so that w is nowhere C^1.*

Remark 5.2.3. *Notice that the above proposition has no assumption on F other than it satisfying (C) and being of C^2-regularity. This means that once we have proved this theorem, it only remains to prove that there exists a polyconvex function F satisfying (C) such that the set K_F contains a T_N-configuration.*

The proof of this proposition is similar to that of Theorem 4.0.1.

Proof. The proof will be split into 3 steps.

Step 1 (Define the preliminary notions): Due to condition (C) and using similar arguments as in the proof of Theorem 4.0.1 we can prove that there is a T_N-configuration $\{\widetilde{Z}_1^0, \dots, \widetilde{Z}_N^0\}$ arbitrarily close to $\{Z_1^0, \dots, Z_N^0\}$ for which there is a neighbourhood

$$\overline{\mathcal{O}} \subset \mathcal{M} \cap \mathcal{K}_F \cap (B_\varepsilon(\widetilde{Z}_1^0) \times \ldots \times B_\varepsilon(\widetilde{Z}_N^0)) \tag{5.2}$$

and $0 < \Lambda < 1$ such that the maps

$$\Phi_{\lambda_i}^k := \lambda \phi_k + (1 - \lambda)\pi_k$$

are submersions when restricted to $\mathcal{O}$, for all $k = 1, \dots, N$ and $\Lambda < \lambda < 1$. Then, let $\{\lambda_j\}_{j=1}^\infty$ be an increasing sequence such that $\lambda_1 = 0$, $\lambda_j > \Lambda$ for all $j \geq 2$ and $\lambda_j \to 1$. Then define the maps Φ_i^k for $i \geq 1$ and $k = 1, \dots, N$ as

$$\Phi_i^k := \Phi_{\lambda_i}^k.$$

Finally, let $\{\mathcal{O}_i\}_{i=1}^\infty$ be a sequence of open sets such that

$$\overline{\mathcal{O}}_i \subset \mathcal{O}_{i+1} \subset \mathcal{O} \subset \mathcal{M} \cap \mathcal{K}_F \cap (B_\varepsilon(\widetilde{Z}_1^0) \times \ldots \times B_\varepsilon(\widetilde{Z}_N^0))$$

for all $i \geq 1$. For all $i \geq 1$, $k = 1, \ldots, N$, define $\mathcal{U}_{i,k} = \Phi_i^k(\mathcal{O}_i)$ and $\mathcal{U}_i = \cup_{k=1}^N \mathcal{U}_{i,k}$. Since the maps Φ_i^k are submersions, they are open maps, hence all the sets $\mathcal{U}_{i,k}$ and $\mathcal{U}_i$ are open. We wish to show that $\{\mathcal{U}_i\}_{i=1}^\infty$ is an in-approximation of the set

$$K_F \cap \left(\bigcup_{k=1}^N B_\varepsilon(\widetilde{Z}_k^0) \right).$$

Due to the openness of the sets $\mathcal{U}_i$, (5.2) and the fact that $\lambda_i \to 1$, it only remains to show that for any point A in $\mathcal{U}_i$, there is a laminate supported in $\mathcal{U}_{i+1}$ with barycenter A. This will be done in Step 2. Notice that by definition of λ_1, $\widetilde{P}_1 \in \mathcal{U}_1^{rc}$.

Step 2 : We now describe a process of choosing laminates and, with the use of Lemma 3.2.4, we modify functions. Assume that $A \in \mathcal{U}_i$. Without loss of generality, we assume $A \in \mathcal{U}_{i,1}$. By definition of the sets $\mathcal{U}_i$ and $\mathcal{U}_{i,k}$ in the previous step, there exists a T_N-configuration $(Z_1, \ldots, Z_N) \in \mathcal{O}_i$ such that A is contained in $[P_1, Z_1]$ with

$$A = \lambda_i Z_1 + (1 - \lambda_i) P_1.$$

Consider N points $\{\widetilde{Z}_k\}_{k=1}^N$ defined as

$$\widetilde{Z}_k = \lambda_{i+1} Z_k + (1 - \lambda_{i+1}) P_k$$

for all $k = 1, \ldots, N$. Then, by definition, $\widetilde{Z}_k \in \mathcal{U}_{i+1,k}$ for all $k = 1, \ldots, N$, since $(Z_1, \ldots, Z_N) \in \mathcal{O}_i \subset \mathcal{O}_{i+1}$. We see that $\{\widetilde{Z}_1, \ldots, \widetilde{Z}_N\}$ is a T_N-configuration such that $\pi_k(\widetilde{Z}_1, \ldots, \widetilde{Z}_N) = P_k$ for all $k = 1, \ldots, N$. Thus, there is a laminate

$$\nu = \sum_{k=1}^N \nu_k \delta_{\widetilde{Z}_k}$$

with barycenter P_1. The fact that $\nu_k > 0$ for all $k = 1, \ldots, N$ follows from computations similar to the ones in Remark 4.4.3. Then we define

$$\mu = \frac{\lambda_i}{\lambda_{i+1}} \delta_{\widetilde{Z}_1} + \left(1 - \frac{\lambda_i}{\lambda_{i+1}}\right) \nu.$$

We see that μ is a laminate supported in $\mathcal{U}_{i+1}$ with barycenter A. This proves that $\{\mathcal{U}_i\}_{i=1}^\infty$ is an in-approximation of

$$K_F \cap \left(\bigcup_{k=1}^N B_\varepsilon(\widetilde{Z}_k^0) \right).$$

Note that

$$\mu(\mathcal{U}_{i+1,1}) > \frac{\lambda_i}{\lambda_{i+1}}.$$

For any subdomain $\Omega' \subset \Omega$, Lemma 3.2.4 proves that there exists a piecewise affine Lipschitz function $w \colon \Omega' \to \mathbb{R}^4$ such that

- $w(x) = Ax$ on $\partial\Omega'$ and $Dw(x) \in \mathcal{U}_{i+1}$ in Ω',
- $\|w - Ax\|_{C^\alpha(\overline{\Omega'})} < 2^{-(i+1)}\varepsilon$,
- $\mathcal{L}^2\left(\{x \in \Omega' : Dw(x) \in \mathcal{U}_{i+1,1}\}\right) > \dfrac{\lambda_i}{\lambda_{i+1}}\mathcal{L}^2\left(\Omega'\right)$ and
- $\displaystyle\int_{\Omega'} |Dw - A|\, dx \leq C(\lambda_{i+1} - \lambda_i)\mathcal{L}^2\left(\Omega'\right)$ for some C independent of i and Ω'.

The first three properties follow from Lemma 3.2.4 and the fact that the sets $\mathcal{U}_{i,k}$ are open. To prove the last property, we notice that (for some constants C, C_1, C_2):

$$\begin{aligned}\int_{\Omega'} |Dw - A|\, dx &= \int_{\Omega' \cap \{Dw \in \mathcal{U}_{i+1,1}\}} |Dw - A|\, dx + \int_{\Omega' \cap \{Dw \notin \mathcal{U}_{i+1,1}\}} |Dw - A|\, dx \\ &\leq C_1\mathcal{L}^2\left(\Omega'\right)(\lambda_{i+1} - \lambda_i) + C_2\mathcal{L}^2\left(\Omega'\right)\left(1 - \frac{\lambda_i}{\lambda_{i+1}}\right) \\ &\leq C\mathcal{L}^2\left(\Omega'\right)(\lambda_{i+1} - \lambda_i).\end{aligned} \tag{5.3}$$

Step 3 (Description of the inductive scheme): In this step, we define a sequence of functions using Step 2. Define $w_0 \colon \Omega \to \mathbb{R}^2$ as $w_0(x) = \widetilde{P}_1 x$. In order to define an inductive scheme, assume that we have a piecewise affine Lipschitz function $w_i \colon \Omega \to \mathbb{R}^2$ such that $Dw_i(x) \in \mathcal{U}_i$ for a.e. $x \in \Omega$. Hence, there exist open sets $\{\Omega^i_j\}_{j=1}^\infty$ such that

$$\mathcal{L}^2\left(\Omega \setminus \bigcup_{j=1}^\infty \Omega^i_j\right) = 0$$

and for all $j \geq 1$, $w_i|_{\Omega^i_j}$ is an affine function. Without loss of generality, we may assume that $\operatorname{diam}(\Omega^i_j) \leq 1/i$ for all $j \geq 1$. Moreover, as in the proof of Theorem 4.0.1, the sets $\{\Omega^i_j\}_{j=1}^\infty$ can be chosen in such a way that for each j there is k such that $\Omega^i_j \subset \Omega^{i-1}_k$. Now we apply Step 2 to each $w_i|_{\Omega^i_j}$ to get a new function $w_{i+1} \colon \Omega \to \mathbb{R}^4$ in the same way as in the proof of Theorem 4.0.1. By construction, the sequence $\{w_i\}_{i=1}^\infty$ converges to some $w \in C^\alpha(\overline{\Omega}; \mathbb{R}^4)$. Now we prove that the sequence $\{w_i\}_{i=1}^\infty$ converges in $W^{1,1}(\Omega; \mathbb{R}^4)$. In order to show it, we prove that it is a Cauchy sequence. For each i let $\{\Omega^i_j\}_{j=1}^\infty$ be the collection of open sets given above. By Step 2, and more precisely by (5.3), we have

$$\int_{\Omega^i_j} |Dw_{i+1} - Dw_i|\, dx \leq C\mathcal{L}^2\left(\Omega^i_j\right)(\lambda_{i+1} - \lambda_i).$$

Since the sets Ω^i_j cover Ω up to a set of measure 0, we have

$$\int_{\Omega} |Dw_{i+1} - Dw_i|\, dx \leq C\mathcal{L}^2\left(\Omega\right)(\lambda_{i+1} - \lambda_i).$$

Since $\lambda_i \to 1$, this shows that $w \in W^{1,1}(\Omega;\mathbb{R}^4)$ and $Dw_i \to Dw$ in $L^1(\Omega; M^{4\times 2})$. By the construction described above, w also satisfies

- $w(x) = P_0 x$ on $\partial\Omega$ and $Dw(x)$,
- $\|w - P_0 x\|_{C^\alpha(\overline{\Omega})} < \varepsilon$ and
- $Dw(x) \in K_F \cap \left(\bigcup_{k=1}^N B_\varepsilon(\widetilde{Z}_k^0)\right)$ for a.e. $x \in \Omega$.

Finally, we show that Dw has essential oscillation of order 1 in any open set. Take an arbitrary open subset $\Omega' \subset \Omega$. By taking i_0 sufficiently large, there exists a j such that $\Omega_j^{i_0} \subset \Omega'$. By construction of the function w_{i_0+1}, there exists an $\varepsilon' > 0$ such that for all $k = 1,\dots,N$

$$\mathcal{L}^2\left(\{x \in \Omega_j^{i_0} : Dw_{i_0+1}(x) \in \mathcal{U}_{i_0+1,k}\}\right) > \varepsilon' \mathcal{L}^2\left(\Omega_j^{i_0}\right). \tag{5.4}$$

This follows from the choice of laminates in Step 2. Now for all $i > i_0$ and any l such that $\Omega_l^i \subset \Omega_j^{i_0}$, we have

$$\mathcal{L}^2\left(\{x \in \Omega_l^i : Dw_{i+1}(x) \in \mathcal{U}_{i+1,k}\}\right) > \frac{\lambda_i}{\lambda_{i+1}} \mathcal{L}^2\left(\{x \in \Omega_l^i : Dw_i(x) \in \mathcal{U}_{i,k}\}\right). \tag{5.5}$$

This inequality holds because w_i is affine on Ω_l^i. If $Dw_i|_{\Omega_l^i} \in \mathcal{U}_{i,k}$ then (5.5) holds by the construction described in Step 2. If $Dw_i|_{\Omega_l^i} \notin \mathcal{U}_{i,k}$, then the right-hand-side of (5.5) is zero and the left-hand-side is strictly positive by the construction described above. Thus, (5.5) holds. Since there is a collection $\{\Omega_l^i\}_{l\in J}$ of open sets $\Omega_l^i \subset \Omega_j^{i_0}$ which cover $\Omega_j^{i_0}$ up to set of measure 0, we get that

$$\mathcal{L}^2\left(\{x \in \Omega_j^{i_0} : Dw_{i+1}(x) \in \mathcal{U}_{i+1,k}\}\right) > \frac{\lambda_i}{\lambda_{i+1}} \mathcal{L}^2\left(\{x \in \Omega_j^{i_0} : Dw_i(x) \in \mathcal{U}_{i,k}\}\right).$$

Applying this inequality inductively and then (5.4), we obtain that

$$\begin{aligned}\mathcal{L}^2\left(\{x \in \Omega_j^{i_0} : Dw_i(x) \in \mathcal{U}_{i,k}\}\right) &> \frac{\lambda_{i-1}}{\lambda_i} \cdots \frac{\lambda_{i_0+1}}{\lambda_{i_0+2}} \mathcal{L}^2\left(\{x \in \Omega_j^{i_0} : Dw_{i_0+1}(x) \in \mathcal{U}_{i+1,k}\}\right) \\ &> \frac{\lambda_{i-1}}{\lambda_i} \cdots \frac{\lambda_{i_0+1}}{\lambda_{i_0+2}} \varepsilon' \mathcal{L}^2\left(\Omega_j^{i_0}\right) = \frac{\lambda_{i_0+1}}{\lambda_i} \varepsilon' \mathcal{L}^2\left(\Omega_j^{i_0}\right).\end{aligned}$$

Letting $i \to \infty$ yields

$$\mathcal{L}^2\left(\{x \in \Omega' : Dw(x) \in B_\varepsilon(\widetilde{Z}_k^0)\}\right) \geq \lambda_{i_0+1} \varepsilon' \mathcal{L}^2\left(\Omega_j^{i_0}\right),$$

for all $k = 1,\dots,N$. Since this holds for any open subsets $\Omega' \subset \Omega$, we conclude that Dw has essential oscillation of order 1 in any subdomain of Ω. This finishes the proof. $\square$

As already mentioned in Remark 5.2.3, we need to prove that there exists a polyconvex function F such that the set K_F contains a T_N-configuration. We will show that in fact there exists a polyconvex F such that K_F contains a T_5-configuration. Note that there is no polyconvex function F such that K_F contains a T_4-configuration (see [18, Proposition 3.11]).

Lemma 5.2.4. *There exists a smooth, strongly polyconvex function $F\colon M^{2\times 2} \to \mathbb{R}$ and a T_5-configuration $\{Z_i\}_{i=1}^5 \subset M^{4\times 2}$ such that $\{Z_i\}_{i=1}^5 \subset K_F$. Moreover F can be chosen so that $D^2F(X_i)$ is positive definite for each i, where*

$$Z_i = \begin{pmatrix} X_i \\ Y_i \end{pmatrix}.$$

The main idea of the proof of this lemma is to prove that the existence of F follows from a system of inequalities being solvable.

Proof. Let us introduce the following notation: $\widetilde{X} = (X, \det X) \in \mathbb{R}^5$ for $X \in M^{2\times 2}$. By definition, K_F contains a T_5-configuration $\{Z_i\}_{i=1}^5$ where

$$Z_i = \begin{pmatrix} X_i \\ Y_i \end{pmatrix}, \quad X_i \in M^{2\times 2}, Y_i \in M^{2\times 2}$$

if and only if

$$Y_i = DF(X_i)J \quad \forall i = 1, \ldots, 5. \tag{5.6}$$

In addition, $F\colon M^{2\times 2} \to \mathbb{R}$ is strongly polyconvex if there is a convex function $G\colon \mathbb{R}^5 \to \mathbb{R}$ and some $\varepsilon > 0$ such that

$$F(X) = \frac{\varepsilon}{2}|X|^2 + G(X, \det X). \tag{5.7}$$

Hence, (5.6) is satisfied if and only if

$$\partial_X G(X_i, \det X_i) + \partial_d G(X_i, \det X_i) \operatorname{cof} X_i = -Y_i J - \varepsilon X_i, \tag{5.8}$$

where ∂_d is the derivative with respect to the determinant term in G. We claim the following:

Claim 1: If we have real numbers $\{c_i\}_{i=1}^5 \subset \mathbb{R}$ and $\{B_i\}_{i=1}^5$, $\{\widetilde{X}_i\}_{i=1}^5 \subset \mathbb{R}^5$ such that

$$c_j > c_i + \langle B_i, \widetilde{X}_j - \widetilde{X}_i \rangle \quad \text{for all } i \neq j, \tag{5.9}$$

then there exists a smooth convex function $G\colon \mathbb{R}^5 \to \mathbb{R}$ such that

$$G(\widetilde{X}_i) = c_i \quad \text{and} \quad DG(\widetilde{X}_i) = B_i \tag{5.10}$$

for all $i = 1, \dots, 5$.

Now we use this claim to prove the existence of a convex function G and a T_5-configuration satisfying (5.8). By combining (5.8), (5.9) and (5.10), we obtain

$$\begin{aligned} c_j &\overset{(5.9)}{>} c_i + \langle B_i, \widetilde{X}_j - \widetilde{X}_i \rangle \\ &\overset{(5.10)}{=} c_i + \langle DG(\widetilde{X}_i), \widetilde{X}_j - \widetilde{X}_i \rangle \\ &= c_i + \langle \partial_X G(\widetilde{X}_i), X_j - X_i \rangle + \partial_d G(\widetilde{X}_i)(\det X_j - \det X_i) \\ &\overset{(5.8)}{=} c_i - \langle Y_i J + \varepsilon X_i + \partial_d G(\widetilde{X}_i) \operatorname{cof} X_i, X_j - X_i \rangle + \partial_d G(\widetilde{X}_i)(\det X_j - \det X_i), \end{aligned} \tag{5.11}$$

for all $i \neq j$. Writing $d_i = \partial_d G(\widetilde{X}_i)$, we get[1]

$$c_i - c_j + d_i \det(X_j - X_i) + \langle X_i - X_j, Y_i J \rangle < \varepsilon \langle X_i, X_i - X_j \rangle \quad \forall i \neq j. \tag{5.12}$$

Thus, if

$$c_i - c_j + d_i \det(X_j - X_i) + \langle X_i - X_j, Y_i J \rangle < 0 \tag{5.13}$$

for all $i \neq j$, then we can select $\varepsilon > 0$ small enough to get (5.12). This shows that if we can find $\{c_i\}_{i=1}^5, \{d_i\}_{i=1}^5 \subset \mathbb{R}$ and a T_5-configuration $\{Z_i\}_{i=1}^5$ such that (5.13) holds, then there exists a smooth, strongly polyconvex function $F\colon M^{2\times 2} \to \mathbb{R}$ such that $\{Z_i\}_{i=1}^5$ is contained in K_F. Indeed, if (5.13) holds, then there exists $\varepsilon > 0$ such that (5.12) holds. By (5.11) and Claim 1, there exists a convex function G such that (5.8) holds. The quantities $\{c_i\}_{i=1}^5, \{d_i\}_{i=1}^5 \subset \mathbb{R}$ and $\{Z_i\}_{i=1}^5$ can be found with the help of a computer. We refer to [24, 20] for an example of a solution. Now, it still remains to show that the function F can be chosen in such a way that for all $i = 1, \dots, 5$, $D^2F(X_i)$ is positive definite. This follows from the next claim:

Claim 2: In Claim 1, the function G can be chosen in such a way that for all $i = 1, \dots, 5$, $D^2F(X_i)$ is positive definite, where F is defined by (5.7).

Now it only remains to prove Claim 1 and Claim 2.

Proof of Claim 1: Assume $\{c_i\}_{i=1}^5 \subset \mathbb{R}$ and $\{B_i\}_{i=1}^5$, $\{\widetilde{X}_i\}_{i=1}^5 \subset \mathbb{R}^5$ such that

$$c_j > c_i + \langle B_i, \widetilde{X}_j - \widetilde{X}_i \rangle \quad \text{for all } i \neq j.$$

Define $G_0\colon \mathbb{R}^5 \to \mathbb{R}$ by

$$G_0(X) = \max_{i=1,\dots,5} \left(c_i + \langle B_i, \widetilde{X} - \widetilde{X}_i \rangle \right). \tag{5.14}$$

[1] Here we use the fact that the following formula holds for all $A, B \in M^{2\times 2}$:

$$\det(A - B) = \det A - \det B - \langle \operatorname{cof} B, A - B \rangle.$$

Due to (5.9), G_0 is affine and thus differentiable in a neighbourhood of $\widetilde{X}_i$. In particular, $DG_0(\widetilde{X}_i) = B_i$. Then select a mollifier $\phi \in C_0^\infty(\mathbb{R}^5)$ supported in a small ball around 0 such that

$$\int_{\mathbb{R}^5} \phi(\widetilde{Y})\, d\widetilde{Y} = 1 \quad \text{and} \quad \int_{\mathbb{R}^5} \widetilde{Y}\phi(\widetilde{Y})\, d\widetilde{Y} = 0. \tag{5.15}$$

Then the mollified function $(\phi * G_0)$ is smooth and, in a neighbourhood of $\widetilde{X}_i$, we have

$$\begin{aligned}
(\phi * G_0)(\widetilde{X}) &= \int_{\mathbb{R}^5} \phi(\widetilde{Y}) G_0(\widetilde{X} - \widetilde{Y}) d\widetilde{Y} \\
&= \int_{\mathbb{R}^5} \phi(\widetilde{Y})(c_i + \langle B_i, (\widetilde{X} - \widetilde{Y}) - \widetilde{X}_i\rangle) d\widetilde{Y} \\
&= \int_{\mathbb{R}^5} \phi(\widetilde{Y})(c_i + \langle B_i, \widetilde{X} - \widetilde{X}_i\rangle) d\widetilde{Y} - \underbrace{\int_{\mathbb{R}^5} \phi(\widetilde{Y})\langle B_i, \widetilde{Y}\rangle d\widetilde{Y}}_{=0} \\
&= c_i + \langle B_i, \widetilde{X} - \widetilde{X}_i\rangle = G_0(\widetilde{X}).
\end{aligned}$$

Thus $G = \phi * G_0$ is the desired function and this proves Claim 1.

Proof of Claim 2: The proof of this second claim is similar to the proof of the first claim. However, we define the function G_0 in a different way. First, we define $\Psi \colon M^{2\times 2} \to \mathbb{R}$ by

$$\Psi(X) = \begin{cases} \gamma |X|^2 & \text{if } |X| < \delta; \\ \gamma\delta |X| & \text{otherwise;} \end{cases}$$

for some $\gamma, \delta > 0$ to be chosen later. Define $G_0 \colon \mathbb{R}^5 \to \mathbb{R}$ as

$$G_0(\widetilde{X}) = \max_{i=1,\dots,5} \Big(c_i + \langle B_i, \widetilde{X} - \widetilde{X}_i\rangle + \Psi(X - X_i) - \phi * \Psi(0) \Big). \tag{5.16}$$

Since G_0 is the maximum of convex functions, G_0 is convex. Since (5.9) holds, for any $\gamma > 0$, there exists $\delta > 0$ such that

$$c_j > c_i + \langle B_i, \widetilde{X}_j - \widetilde{X}_i\rangle + \Psi(X - X_i) \quad \forall i \neq j. \tag{5.17}$$

Then, define $G = \phi * G_0$. In the same way as in the proof of the previous claim, now due to (5.17), in a neighbourhood of $\widetilde{X}_i$

$$G(\widetilde{X}) = c_i + \langle B_i, \widetilde{X} - \widetilde{X}_i\rangle + \phi * \Psi(X - X_i) - \phi * \Psi(0).$$

Thus,

$$G(\widetilde{X}_i) = c_i + \phi * \Psi(0) - \phi * \Psi(0) = c_i.$$

In a neighbourhood of $\widetilde{X}_i$,

$$DG(\widetilde{X}) = B_i + \phi * D\Psi(X - X_i) = B_i,$$

since $D\Psi(X) = 2\gamma X$ in a neighbourhood of 0 combined with (5.15). By definition of F (see (5.7)), in a neighbourhood of $\widetilde{X}_i$, we have

$$\begin{aligned}
F(X) &= \frac{\varepsilon}{2}|X|^2 + G(X, \det X) \\
&= \frac{\varepsilon}{2}|X|^2 + \langle B_i, \widetilde{X} - \widetilde{X}_i\rangle + \phi * \Psi(X - X_i) - \phi * \Psi(0) \\
&= \frac{\varepsilon}{2}|X|^2 + \langle \partial_X G(\widetilde{X}_i), X - X_i\rangle + \partial_d G(\widetilde{X}_i)(\det X - \det X_i) + \phi * \Psi(X - X_i) \\
&\quad - \phi * \Psi(0) \\
&= \frac{\varepsilon}{2}|X|^2 + \langle Y_i J + \varepsilon X_i, \widetilde{X} - \widetilde{X}_i\rangle + d_i \det(X - X_i) + \phi * \Psi(X - X_i) - \phi * \Psi(0).
\end{aligned}$$

Finally, we only have to prove that $D^2F(X_i)$ is positive definite for all i. After standard computations, we find that

$$D^2F(X_i)[Z, Z] = \varepsilon|Z|^2 + 2d_i \det Z + 2\gamma|Z|^2 = 2d_i \det Z + (2\gamma + \varepsilon)|Z|^2.$$

By choosing $\gamma > \max_{i=1,\dots,5} |d_i|$, $D^2F(X_i)$ is positive definite for all i. This finishes the proof of Claim 2 and therefore also finishes the proof of the lemma. □

In addition to the results that we have already proved, we need to be able to build the function F such that (C) is satisfied. As in the previous chapter, in the case of polyconvex functions, it suffices to show that for any function F_0 such that K_{F_0} contains a T_N-configuration, perturbing it infinitesimally is enough to get a function which satisfies (C). This is the content of the following theorem and provides us with the remaining piece of the proof of Theorem 5.0.1.

Theorem 5.2.5. *Suppose $F_0 \in C^2(M^{2\times 2})$ is such that K_{F_0} contains a T_N-configuration. Then for any $\delta > 0$, there exists $F \in C^2(M^{2\times 2})$ with $\|D^2F - D^2F_0\|_{C^0(M^{2\times 2})} < \delta$ such that K_F contains the same T_N-configuration and moreover F satisfies the non-degeneracy condition (C).*

We do not prove this result here (we refer to [24] for a proof), but we give the ideas of the proof. Let $Z_1, \dots, Z_N$ be the T_N-configuration in the statement of the theorem and

$$Z_i = \begin{pmatrix} X_i \\ Y_i \end{pmatrix}, \quad X_i \in M^{2\times 2}, Y_i \in M^{2\times 2}.$$

As for Lemma 4.5.10, the main point is that we can perturb F_0 with some V satisfying $DV(X_i) = 0$ for all i and $D^2V(X_i)$ chosen suitably. Then K_F, where $F = F_0 + V$, still contains the T_N-configuration $Z_1, \dots, Z_N$. The goal is to prove that for generic values of second derivatives $D^2F(X_i)$ we have

$$\begin{aligned}
T_{(Z_1,\dots,Z_N)}\mathcal{K}_F + T_{(Z_1,\dots,Z_N)}\mathcal{M} = (M^{4\times 2})^N \quad &\text{and} \\
\dim(\mathrm{Im}(D\pi_k|_{T_{(Z_1,\dots,Z_N)}\mathcal{K}_F})) = 8 \quad &\text{for all } k = 1, \dots, 8.
\end{aligned}$$

By dimension-counting arguments, this turn out to be equivalent to showing

$$T_{(Z_1,\ldots,Z_N)}\mathcal{K}_F + \ker D\pi_k = (M^{4\times 2})^N$$

for all k. We can restrict our attention to the case $k = 1$. Recall that $T_{(Z_1,\ldots,Z_N)}\mathcal{K}_F = V_1 \times \ldots \times V_N$ where

$$V_i = \left\{ \begin{pmatrix} M_i \\ D^2F(X_i)M_i \end{pmatrix} : M_i \in M^{2\times 2} \right\} \qquad \text{for all } i = 1, \ldots, N.$$

Finally, we show that $\ker D\pi_1$ contains a $4N$-dimensional subspace L such that for generic values of $D^2F(X_i)$

$$L \cap (V_1 \times \ldots \times V_N) = \emptyset.$$

This implies $L + (V_1 \times \ldots \times V_N) = (M^{4\times 2})^N$ which allows us to conclude.

We are now in position to prove the main theorem stated at the beginning of this chapter. The proof is short because all the groundwork has been carried out in the other results in this section.

Proof of Theorem 5.0.1. First, by Lemma 5.2.4, there is a polyconvex smooth function such that K_F contains a T_5-configuration $\{Z_i\}_{i=1}^5$. Moreover, we may suppose that $D^2F(X_i)$ is positive definite for each $i = 1, \ldots, 5$. In virtue of Theorem 5.2.5, we may suppose that (C) is fulfilled. Finally, we conclude by applying Proposition 5.2.2. Notice that by taking $\varepsilon > 0$ small enough, the solution obtained from Proposition 5.2.2 is a weak local minimizer of the functional $\mathcal{I}$. $\square$

Chapter 6

Nonenergetic solutions to PDE's

In the previous chapters, convex integration techniques have been used to build very irregular Lipschitz functions which solve some PDE. As we could see in the previous chapters, this is achieved by introducing more and more oscillations at each step. However, convex integration techniques can also be used to build mappings which solve some PDE such that the gradient belongs to some L^p space but not to L^q for some $q > p$. This technique is called L^p-convex integration and was invented by Faraco in [11] using so-called staircase laminates. In this chapter we will use the technique invented by Faraco and subsequently further developed and used by Astala-Conti-Faraco-Maggi-Székelyhidi [2, 6, 7, 11]. In this chapter we are interested in equations of the form

$$\operatorname{div} Df(Du) = 0 \tag{6.1}$$

where $f \in C^\infty(\mathbb{R}^2)$ is uniformly convex and has uniformly bounded Hessian. In other words, there are $0 < \lambda < \Lambda < \infty$ such that

$$\lambda I \leq D^2 f(x) \leq \Lambda I \quad \forall x \in \mathbb{R}^2. \tag{6.2}$$

Under these assumptions, it is a well known fact from De Giorgi's theorem [8] that for $u \in W^{1,2}(\Omega)$ being a solution to (6.1) is equivalent to it being a minimizer of the following energy functional

$$\mathcal{E}(u) = \int_\Omega f(Du)\, dx$$

in $W^{1,2}(\Omega)$. However, equation (6.1) when intended weakly makes sense for $u \in W^{1,1}(\Omega)$ under the assumption that $f \in C^\infty(\mathbb{R}^2)$ and (6.2) holds, whereas the functional $\mathcal{E}$ requires u to belong to $W^{1,2}(\Omega)$. In this chapter, we prove that there are very weak solutions to (6.1) belonging to $u \in W^{1,1}(\Omega)$ but not to $W^{1,2}(\Omega)$. Since the energy functional $\mathcal{E}$ requires u to belong to $W^{1,2}(\Omega)$, this justifies the notion of nonenergetic solutions:

Definition 6.0.1. *Let $f \in C^\infty(\mathbb{R}^2)$ be a uniformly convex function with uniformly bounded Hessian. For $1 \le p < 2$, we call $u \in W^{1,p}(\Omega)$ a nonenergetic solution to (6.1) if it solves (6.1) in a distributional sense but does not belong to the Sobolev space $W^{1,2}(\Omega)$. To be precise, $u \in W^{1,p}(\Omega) \setminus W^{1,2}(\Omega)$ is a nonenergetic solution if*

$$\int_\Omega Df(Du) \cdot D\varphi \, dx = 0 \quad \forall \varphi \in C_0^\infty(\Omega).$$

The main theorem that we shall prove in this chapter is

Theorem 6.0.2. *There is a uniformly convex function $f \in C^\infty(\mathbb{R}^2)$ with uniformly bounded Hessian, i.e. there exists $0 < \lambda < \Lambda$ so that*

$$\lambda I \le D^2 f \le \Lambda I,$$

for which the equation (6.1) admits a nonenergetic solution.

Now, we present a selection of existing results about very weak solutions to PDE's. In 1964, Serrin [23] proved:

Theorem 6.0.3. *For all $p > 1$, there exists $A \in C(\mathbb{R}^n, M^{n\times n})$ elliptic such that the equation*

$$\operatorname{div}(A(x) \cdot Du(x)) = 0, \quad x \in \mathbb{R}^n \tag{6.3}$$

admits solutions $u \colon \mathbb{R}^n \to \mathbb{R}$ belonging to $W^{1,1}(\mathbb{R}^n)$ but not to $W^{1,p}(\mathbb{R}^n)$.

One corollary of Serrin's result is the optimality of De Giorgi's strategy in [8]. In 2008, Brezis [3, 1] proved:

Theorem 6.0.4. *Assume $A \in C^0(\overline{\Omega}, M^{n\times n})$ and $u \in W^{1,p}(\Omega)$ for some $p > 1$. If u is a weak solution of*

$$\operatorname{div}(A(x) \cdot Du(x)) = 0, \quad x \in \Omega$$

then $u \in W^{1,q}_{loc}(\Omega)$ for every $q < \infty$. Moreover,

$$\|u\|_{W^{1,q}(\omega)} \le C \|u\|_{W^{1,p}(\Omega)}$$

for every $\omega \Subset \Omega$, where C depends only on $N, \lambda, p, q, \omega, \Omega$ and A.

Related results can be found in [5, 13, 15, 16]. In 2008, Astala, Faraco and Székelyhidi [2] proved:

Theorem 6.0.5. *For every $\alpha \in (0, 1)$ there exists a measurable mapping*

$$A \colon \Omega \to \left\{ \begin{pmatrix} \frac{1}{\sqrt{K}} & 0 \\ 0 & \sqrt{K} \end{pmatrix}, \begin{pmatrix} \sqrt{K} & 0 \\ 0 & \frac{1}{\sqrt{K}} \end{pmatrix} \right\}$$

and a function $u \in C^{1,\alpha}(\overline{\Omega})$ *such that* $u \in W^{2,q}(\Omega)$ *for all* $q < \frac{2K}{K+1}$,

$$\langle A(x), D^2u \rangle = 0 \tag{6.4}$$

in the sense of distributions, but for every disc $B = B(x_0, r) \subset \Omega$

$$\int_B |D^2u|^{\frac{2K}{K+1}}\, dx = \infty.$$

Despite the fact that the proof of Theorem 6.0.5 by Astala, Faraco and Székelyhidi [2] is close to our proof of Theorem 6.0.2 and the fact that in the classical case the two equations (6.1) and (6.4) coincide when $A = D^2f(Du)$, we could find no trivial way to pass from one to the other. Therefore, from the point of view of the PDE's under consideration, our result is different and new. However, we will see that the differential inclusion used to prove Theorem 6.0.2 is extremely similar to the one used by Astala, Faraco and Székelyhidi.

We point out that contrary to the results by Serrin and Astala-Faraco-Székelyhidi where the PDE under consideration is linear, the PDE in Theorem 6.0.2 is nonlinear.

In Section 6.1 we state some preliminary results. Then we define the geometric setup in Section 6.2. In Section 6.3 we go through the strategy of the proof of Theorem 6.0.2. Finally, we prove Theorem 6.0.2 in Section 6.4.

6.1 Preliminary results

As in Theorems 4.0.1 and 5.0.1, a central part of the proof of Theorem 6.0.2, is to reformulate equation (6.1) as a differential inclusion. By the arguments of Section 4.2 we find that the PDE (6.1) is equivalent to the differential inclusion

$$Du(x) \in \left\{ \begin{pmatrix} a & b \\ \partial_b f(a,b) & -\partial_a f(a,b) \end{pmatrix} : a, b \in \mathbb{R} \right\} \quad \text{for a.e. } x \in \Omega.$$

We take f to be in the form

$$f(a,b) = \varphi(a) + \frac{1}{2}b^2 \tag{6.5}$$

for some uniformly convex $\varphi \in C^\infty(\mathbb{R})$. Then the differential inclusion becomes

$$Du(x) \in \left\{ \begin{pmatrix} a & b \\ b & -\varphi'(a) \end{pmatrix} : a, b \in \mathbb{R} \right\} =: K \quad \text{for a.e. } x \in \Omega.$$

Notice that the set K is a subset of $S^{2\times 2}$. More precisely, K is a 2-dimensional manifold embedded in $S^{2\times 2}$ (which is 3-dimensional). This makes the set K possible to visualize. We also see that the variable b makes no difference in K in the sense that

$$\begin{pmatrix} a & b_0 \\ b_0 & -\varphi'(a) \end{pmatrix} \in K \text{ for some } b_0 \quad \text{if and only if} \quad \begin{pmatrix} a & 0 \\ 0 & -\varphi'(a) \end{pmatrix} \in K.$$

Therefore, we can visualise K as the 1-dimensional manifold

$$\{(a, -\varphi'(a)) \in \mathbb{R}^2 : a \in \mathbb{R}\}$$

embedded in $\mathbb{R}^2$. We will take φ to be of the form

$$\varphi(x) = \begin{cases} \dfrac{1}{2}C_1 x^2 & \text{if } x \in (\frac{1}{5}, \infty); \\ \dfrac{1}{2}C_2 x^2 & \text{if } x \in (-\infty, 0); \\ \text{smooth extension} & \text{if } x \in [0, \frac{1}{5}]; \end{cases} \tag{6.6}$$

for constants C_1, C_2 such that $0 < C_1 < 1$ and $C_2 > 1$. Since K is contained in $S^{2\times 2}$, we state some convex integration results for symmetric matrices. The following two results come from [2]. Due to their similarity to the results in Chapter 3, we choose not to present the proofs here. The following lemma is very similar to Lemma 3.1.2.

Lemma 6.1.1. *Let $\alpha \in (0,1)$, $\varepsilon, \delta > 0$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $A, B \in S^{n\times n}$ with $\operatorname{rank}(A - B) = 1$ and suppose $C = \lambda A + (1-\lambda)B$ for some $\lambda \in (0,1)$. Then there exists a piecewise affine Lipschitz mapping $u\colon \Omega \to \mathbb{R}^n$ such that*

- $\|u - Cx\|_{C^\alpha(\overline{\Omega})} < \varepsilon$,
- $u(x) = Cx$ *if* $x \in \partial\Omega$,
- *the following equalities hold:*

$$\mathcal{L}^n\left(\{x \in \Omega : |Du(x) - A| < \delta\}\right) = \lambda \mathcal{L}^n(\Omega),$$
$$\mathcal{L}^n\left(\{x \in \Omega : |Du(x) - B| < \delta\}\right) = (1-\lambda)\mathcal{L}^n(\Omega),$$

  and

- *there is an $f \in W^{2,\infty}(\Omega)$ piecewise quadratic such that $u = Df$ and $f = \dfrac{\langle Cx, x\rangle}{2}$ on $\partial\Omega$.*

What distinguishes Lemma 6.1.1 from Lemma 3.1.2 is the existence of a piecewise quadratic mapping $f \in W^{2,\infty}(\Omega)$ in Lemma 6.1.1. The idea behind the proof of this lemma is to construct a sequence of functions $\{f_n\}_{n=1}^{\infty} \subset W^{2,\infty}(\Omega)$ such that the gradients are always close to the segment $[A, B]$ and

$$\mathcal{L}^n\left(\{x \in \Omega : \operatorname{dist}\left(Df_k(x), \{A, B\}\right) > \delta\}\right) \to 0.$$

as $k \to \infty$. Then we take f to be the limit of the sequence $\{f_n\}_{n=1}^{\infty}$ in $W^{2,\infty}(\Omega)$. The only subtle point now is that we can not yet guarantee that

$$\mathcal{L}^n\left(\{x \in \Omega : |Du(x) - A| < \delta\}\right) = \lambda \mathcal{L}^n\left(\Omega\right),$$
$$\mathcal{L}^n\left(\{x \in \Omega : |Du(x) - B| < \delta\}\right) = (1-\lambda) \mathcal{L}^n\left(\Omega\right).$$

Indeed, at this point, we only have these equalities up to some small error. To solve this issue, we use the same argument as above but by either replacing A by some nearby $\hat{A}$ or replacing B by some nearby $\hat{B}$. This allows us to prove Lemma 6.1.1. The next proposition is similar to Lemma 3.1.4 (the existence of a piecewise quadratic f distinguishes them) and follows from Lemma 6.1.1 by induction.

Proposition 6.1.2. *Let $\mu = \sum_{i=1}^{N} \alpha_i \delta_{A_i} \in \mathcal{L}(S^{n\times n})$ be a laminate of finite order such that $\{A_i\}_{i=1}^{N} \subset S^{n\times n}$ and with barycenter $\overline{\mu} = A$. Then, for any $\alpha \in (0,1)$,*

$$0 < \delta < \min_{1 \le i < j \le N} \frac{|A_i - A_j|}{2}$$

and every bounded open set $\Omega \subset \mathbb{R}^n$, there exists a piecewise affine Lipschitz mapping $u \colon \Omega \to \mathbb{R}^n$ such that

- $\|u - Cx\|_{C^{\alpha}(\overline{\Omega})} < \varepsilon$,
- $u(x) = Cx$ *if* $x \in \partial\Omega$,
- *the following equalities hold:*

  $$\mathcal{L}^n\left(\{x \in \Omega : |Du(x) - A_i| < \delta\}\right) = \alpha_i \mathcal{L}^n\left(\Omega\right) \quad \textit{for all } i = 1, \dots, N,$$

  and
- *there is an* $f \in W^{2,\infty}(\Omega)$ *piecewise quadratic such that* $u = Df$ *and* $f = \dfrac{\langle Cx, x\rangle}{2}$ *on* $\partial\Omega$.

Remark 6.1.3. *The last point in each of the two previous results implies that $Du(x) \in S^{n\times n}$ for a.e. $x \in \Omega$.*

6.2 Geometric setup

In this section, we define some sets which will be used in the proof of Theorem 6.0.2. We refer to Figure 6.1 for an illustration of all the sets that we define below. Recall the

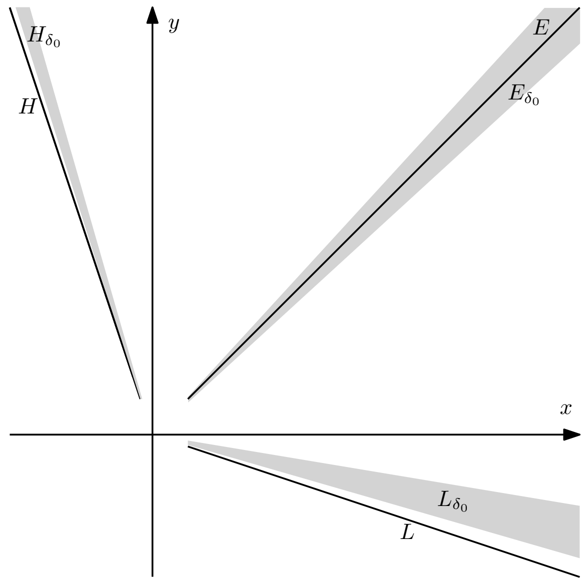

Figure 6.1: Illustration of the sets L, H, E, L_{δ_0}, H_{δ_0}, E_{δ_0} with respect to the variables x and y.

constants C_1 and C_2 from (6.6). We define

$$L := \left\{ \begin{pmatrix} x & b \\ b & y \end{pmatrix} : x \geq \frac{1}{5}, y = -C_1 x, -1 \leq b \leq 1 \right\},$$
$$H := \left\{ \begin{pmatrix} x & b \\ b & y \end{pmatrix} : x < 0, \frac{1}{5} \leq y = -C_2 x, -1 \leq b \leq 1 \right\} \quad \text{and}$$
$$E := \left\{ \begin{pmatrix} x & b \\ b & y \end{pmatrix} : x \geq \frac{1}{5}, y = x, -1 \leq b \leq 1 \right\}.$$

For future reference, for all $R > 0$, we define

$$\widetilde{B}_R := \left\{ \begin{pmatrix} x & b \\ b & y \end{pmatrix} : x^2 + y^2 < R^2, -1 < b < 1 \right\}.$$

Define $K' := L \cup H$. In the proof of Theorem 6.0.2, we will build solutions to

$$Du(x) \in K' \text{ for a.e. } x \in \Omega. \tag{6.7}$$

By definition of the sets L and H, we have $K' \subset K$ so that (6.7) implies that

$$Du(x) \in K \text{ for a.e. } x \in \Omega.$$

For all $\delta_0 > 0$, we define the sets

$$E_{\delta_0} := \left\{ \begin{pmatrix} x & b \\ b & y \end{pmatrix} : x \geq \frac{1}{5}, y = (1+\delta)x \text{ for some } \delta \in (-\delta_0, \delta_0), -1 < b < 1 \right\},$$

$$L_{\delta_0} := \left\{ \begin{pmatrix} x & b \\ b & y \end{pmatrix} : x \geq \frac{1}{5}, y = (\delta - C_1)x \text{ for some } \delta \in \left(\frac{1}{2}\delta_0, 2\delta_0\right), -1 < b < 1 \right\} \quad \text{and}$$

$$H_{\delta_0} := \left\{ \begin{pmatrix} x & b \\ b & y \end{pmatrix} : x < 0, \frac{1}{5} \leq y = -(C_2+\delta)x \text{ for some } \delta \in \left(\frac{1}{2}\delta_0, 2\delta_0\right), -1 < b < 1 \right\}.$$

6.3 Strategy of the proof

In this section we briefly give an intuition behind the proof of Theorem 6.0.2. To do this, we first present an incorrect attempt at a proof which gives the main ideas. Then we discuss how we can adapt this attempt at a proof to turn it into a correct working proof. We take A to be the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In order to try to build a solution to (6.7), we use Proposition 6.1.2. First, however, notice that the rank-one directions of the set of diagonal matrices are

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We split the matrix A into two other rank-one connected matrices. Indeed, notice that

$$A = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 1 & 0 \\ 0 & -C_1 \end{pmatrix} \quad \text{with } \lambda = \frac{1+C_1}{2+C_1}.$$

Notice that the second matrix in the sum above belongs to the set K'. Then we split the first matrix into two other rank-one connected matrices as follows:

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \eta \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + (1-\eta) \begin{pmatrix} -\frac{2}{C_2} & 0 \\ 0 & 2 \end{pmatrix} \quad \text{where } \eta = \frac{C_2+2}{2C_2+2}.$$

Here again the second matrix belongs to the set K'. We could go on like this by splitting the first matrix in the same way as we split A. From now on, let us for simplicity denote matrices of the form

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

by (x, y). Using this notation, we have proved the following: there exists a laminate of finite order μ such that $\overline{\mu} = A$ which takes the form

$$\mu = \frac{C_2+2}{2C_2+2}\frac{1+C_1}{2+C_1}\delta_{(2,2)} + \frac{1}{2+C_1}\delta_{(1,-C_1)} + \frac{C_2}{2C_2+2}\frac{1+C_1}{2+C_1}\delta_{(-\frac{2}{C_2},2)}.$$

Actually, the following result can be proved using the same approach as above:

Proposition 6.3.1. *Let $a > 0$. Then there is a laminate of finite order μ taking the form*

$$\mu_a = \frac{C_2+2}{2C_2+2}\frac{1+C_1}{2+C_1}\delta_{(2a,2a)} + \frac{1}{2+C_1}\delta_{(a,-C_1 a)} + \frac{C_2}{2C_2+2}\frac{1+C_1}{2+C_1}\delta_{(-\frac{2a}{C_2},2a)}$$

such that $\overline{\mu} = (a,a)$. In addition, we have

$$\int_{S^{2\times 2}} |x_{11}|^p \, d\mu_a(x) = \frac{2^p(C_2 + C_2^{1-p} + 2)(1+C_1) + 2C_2 + 2}{(2+C_1)(2C_2+2)}|a|^p \text{ and}$$
$$\int_{S^{2\times 2}} |x_{22}|^p \, d\mu_a(x) = \frac{2^p(1+C_1) + C_1^p}{2+C_1}|a|^p.$$

Using this result, we can build a sequence of laminates $\{\mu^{(k)}\}_{k=1}^\infty$ with interesting properties:

Proposition 6.3.2. *There exists a sequence of laminates of finite order $\{\mu^{(k)}\}_{k=0}^\infty$ such that for all $k \geq 0$, $\mu^{(k+1)}$ is obtained from $\mu^{(k)}$ by two elementary splittings and $\mu^{(0)} = \delta_{(1,1)}$. Moreover, the sequence can be chosen so that the following holds: for all $p \geq 1$ we have*

$$\int_{S^{2\times 2}} |x_{11}|^p \, d\mu^{(k)}(x), \int_{S^{2\times 2}} |x_{22}|^p \, d\mu^{(k)}(x) \sim \sum_{j=1}^{k-1} \left(2^p \frac{C_2+2}{2C_2+2}\frac{1+C_1}{2+C_1}\right)^j$$

asymptotically[1] *as $k \to \infty$.*

Proof. First we define $\mu^{(0)} = \delta_{(1,1)}$ and $\mu^{(1)} = \mu_1$ defined in the previous result. Then assume we are given a laminate of finite order $\mu^{(k)}$ which can be written in the form

$$\mu^{(k)} = \nu + \left(\frac{C_2+2}{2C_2+2}\frac{1+C_1}{2+C_1}\right)^k \delta_{(2^k,2^k)}$$

for some measure ν with $\nu(\{(2^k,2^k)\}) = 0$. Then, we define $\mu^{(k+1)}$ as

$$\mu^{(k+1)} = \nu + \left(\frac{C_2+2}{2C_2+2}\frac{1+C_1}{2+C_1}\right)^k \mu_{2^k}.$$

Thus,

$$\int_{S^{2\times 2}} |x_{11}|^p \, d\mu^{(k+1)}(x) = \int_{S^{2\times 2}} |x_{11}|^p \, d\mu^{(k)}(x) - \left(\frac{C_2+2}{2C_2+2}\frac{1+C_1}{2+C_1}\right)^k 2^{kp}$$

[1]When writing that $a_k \sim b_k$ asymptotically as $k \to \infty$ for two sequences $\{a_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$, we mean that
$$\frac{a_k}{b_k} \to 1 \quad \text{as } k \to \infty.$$

$$
\begin{aligned}
&+\left(\frac{C_2+2}{2C_2+2}\frac{1+C_1}{2+C_1}\right)^k \int_{S^{2\times 2}} |x_{11}|^p d\mu_{2^k}(x) \\
&= \int_{S^{2\times 2}} |x_{11}|^p \, d\mu^{(k)}(x) \\
&+2^{kp}\left(\frac{C_2+2}{2C_2+2}\frac{1+C_1}{2+C_1}\right)^k \left(\frac{(2^p(C_2+C_2^{1-p}+2)-(2C_2+2))(1+C_1)}{(2+C_1)(2C_2+2)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{S^{2\times 2}} |x_{22}|^p \, d\mu^{(k+1)}(x) &= \int_{S^{2\times 2}} |x_{22}|^p \, d\mu^{(k)}(x) - \left(\frac{C_2+2}{2C_2+2}\frac{1+C_1}{2+C_1}\right)^k 2^{kp} \\
&+\left(\frac{C_2+2}{2C_2+2}\frac{1+C_1}{2+C_1}\right)^k \int_{S^{2\times 2}} |x_{22}|^p d\mu_{2^k}(x) \\
&= \int_{S^{2\times 2}} |x_{22}|^p \, d\mu^{(k)}(x) \\
&+2^{kp}\left(\frac{C_2+2}{2C_2+2}\frac{1+C_1}{2+C_1}\right)^k \left(\frac{2^p(1+C_1)+C_1^p}{2+C_1}-1\right).
\end{aligned}
$$

□

Remark 6.3.3. *The sequence of laminates built as in the proof above (or similarly) are known as staircase laminates.*

A nice feature of the sequence of laminates constructed in the previous proposition is that as $k \to \infty$, more and more mass of the measure belongs to the set K'. Now let $\{u_k\}_{k=1}^{\infty}$ be the sequence of piecewise affine Lipschitz functions generated by the sequence of laminates $\{\mu^{(k)}\}_{k=1}^{\infty}$. In an ideal world, we would have

$$
\frac{(Du_k)_{\#}\mathcal{L}^2|_{\Omega}}{\mathcal{L}^2(\Omega)} = \mu^{(k)} \tag{6.8}
$$

for all $k \geq 1$, Then, under suitable assumptions on the constants C_1 and C_2, the limit function is such that

$$
Du(x) \in K' \quad \text{for a.e. } x \in \Omega.
$$

Figure 6.2 illustrates this situation. However, Proposition 6.1.2 does not really give us (6.8). Indeed, given a laminate of finite order μ, then the function given by Proposition 6.1.2 is not such that the gradient only takes values that belong to the support of μ. We can only say that the function can be taken so that the gradient belongs to arbitrary small balls around the support of μ. This is illustrated by Figure 6.3. Thus, the problem of the approach suggested up to now in this chapter is that there is no way to guarantee that

$$
Du(x) \in K' \quad \text{for a.e. } x \in \Omega.
$$

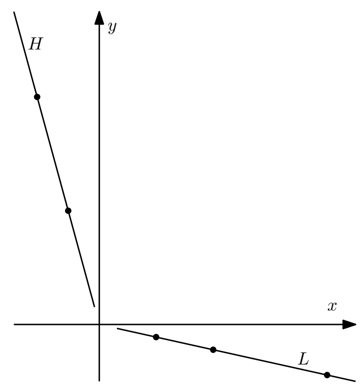

Figure 6.2: In an ideal world, the gradient takes values only belonging to the support of the measure, i.e. the dots in the drawing above.

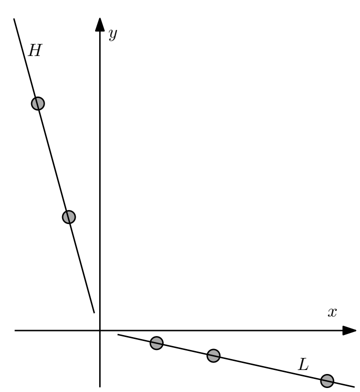

Figure 6.3: We only know that the gradient takes values arbitrarily close (marked in gray) to the support of the measure.

As already said, in general we can only say that for some arbitrarily small ε,

$$\operatorname{dist}\big(Du(x), K'\big) < \varepsilon \quad \text{for a.e. } x \in \Omega.$$

In the remainder of the section, we propose a way to correct the suggested approach in order to obtain a working proof. This remains on an heuristic level and the rigorous proof is carried out in the next section. As in the case of compact inclusions in Section 3.3, we approximate the set K' by open sets and use an inductive scheme. To do it, we take a decreasing sequence $\{\delta_k\}_{k=1}^{\infty}$ which converges to 0. In addition, we assume that all elements of this sequence are small. The first step consists in taking the laminate μ_1 given by Proposition 6.3.1 but where C_1 has been replaced by $C_1 - \delta_1$ and C_2 by $C_2 + \delta_1$. Then apply Proposition 6.1.2 together with this laminate μ_1 to get a function $u_1 \colon \Omega \to \mathbb{R}^2$ such that

$$Du_1(x) \in E_{\delta_1} \cup L_{\delta_1} \cup H_{\delta_1} \quad \text{for a.e. } x \in \Omega.$$

By Proposition 6.1.2, we can assume u_1 to be such that the measure

$$\frac{(Du_1)_{\#}\mathcal{L}^2|_{\Omega}}{\mathcal{L}^2(\Omega)}$$

looks similar to the laminate μ_1 mentioned earlier. Since this laminate is equal to the laminate $\mu^{(1)}$ given by Proposition 6.3.2, we obtain that

$$\frac{(Du_1)_{\#}\mathcal{L}^2|_{\Omega}}{\mathcal{L}^2(\Omega)}$$

looks similar to $\mu^{(1)}$. We also point out that the sets L_{δ_1} and H_{δ_1} are close to our desired set K'. The next step consists of two parts:

Part 1 (main iteration): We consider the open set $\Omega' \subset \Omega$ on which $Du_1 \in E_{\delta_1}$ a.e. Then for each affine part of the mapping $u_1|_{\Omega'}$ we do the following: let $\widetilde{\Omega} \subset \Omega'$ be an open set such that $u_1|_{\widetilde{\Omega}}$ is affine. Assume that

$$Du_1|_{\widetilde{\Omega}} = \begin{pmatrix} x & b \\ b & y \end{pmatrix}.$$

This means that $y = (1+\delta)x$ for some $\delta \in (-\delta_1, \delta_1)$. Then let μ be the laminate of the form

$$\mu = \lambda_1 \delta_{Z_1} + \lambda_2 \delta_{Z_2} + \lambda_3 \delta_{Z_3}$$

given by the same reasoning which allowed us to prove Proposition 6.3.1 but where

$$Z_1 = \begin{pmatrix} 2x & b \\ b & 2x \end{pmatrix}, \quad Z_2 = \begin{pmatrix} x & b \\ b & -(C_1 - \delta_2)x \end{pmatrix}, \quad Z_3 = \begin{pmatrix} -\frac{2x}{C_2+\delta_2} & b \\ b & 2x \end{pmatrix},$$

such that $\overline{\mu} = Du_1|_{\widetilde{\Omega}}$. By assumption $Du_1|_{\widetilde{\Omega}}$ is close to $(2,2)$ and therefore μ looks similar to μ_2 given by Proposition 6.3.1. In addition, we see that μ is supported in $E_{\delta_2} \cup L_{\delta_2} \cup H_{\delta_2}$. Then, by applying Proposition 6.1.2 we get a mapping $v \colon \widetilde{\Omega} \to \mathbb{R}^2$ such that

$$Dv(x) \in E_{\delta_2} \cup L_{\delta_2} \cup H_{\delta_2}$$

for a.e. $x \in \widetilde{\Omega}$. Moreover, the measure

$$\frac{(Dv)_{\#}\mathcal{L}^2|_{\widetilde{\Omega}}}{\mathcal{L}^2(\widetilde{\Omega})}$$

looks similar to μ_2. Doing this for all affine parts of $u_1|_{\Omega'}$ we get a piecewise affine function $v \colon \Omega' \to \mathbb{R}^2$ such that the measure

$$\frac{(Dv)_{\#}\mathcal{L}^2|_{\Omega'}}{\mathcal{L}^2(\Omega')}$$

looks similar to μ_2.

Part 2 (correction procedure): We consider the open set Ω'' for which $Du_1 \in L_{\delta_1} \cup H_{\delta_1}$ a.e. As in Part 1, we consider subsets $\widetilde{\Omega} \subset \Omega''$ such that $u_1|_{\widetilde{\Omega}}$ is affine. Without loss of generality, assume that

$$Du_1|_{\widetilde{\Omega}} = \begin{pmatrix} x & b \\ b & y \end{pmatrix} \in L_{\delta_1}.$$

This means that $y = (\delta - C_1)x$ for some $\delta \in (\delta_1/2, 2\delta_1)$. We consider a laminate μ of the form

$$\mu = \lambda \delta_{Z_1} + (1-\lambda)\delta_{Z_2}$$

where

$$Z_1 = \begin{pmatrix} x & b \\ b & x \end{pmatrix}, \quad Z_2 = \begin{pmatrix} x & b \\ b & (\delta_2 - C_1)x \end{pmatrix}.$$

Notice that under our assumption that the elements δ_k are small, λ is small. Due to Proposition 6.1.2, we get an approximation $w \colon \widetilde{\Omega} \to \mathbb{R}^2$ of $u_1|_{\widetilde{\Omega}}$ such that

$$Dw \in E_{\delta_2} \cup L_{\delta_2}$$

and the measure

$$\frac{(Dw)_{\#}\mathcal{L}^2|_{\widetilde{\Omega}}}{\mathcal{L}^2(\widetilde{\Omega})}$$

looks similar to μ. Since λ is small and the matrix

$$\begin{pmatrix} x & b \\ b & (\delta_2 - C_1)x \end{pmatrix}$$

is close to the matrix

$$\begin{pmatrix} x & b \\ b & y \end{pmatrix}$$

the measure

$$\frac{(Dw)_{\#}\mathcal{L}^2|_{\widetilde{\Omega}}}{\mathcal{L}^2(\widetilde{\Omega})}$$

turns out to look similar to the measure

$$\frac{(Du_1|_{\widetilde{\Omega}})_{\#}\mathcal{L}^2|_{\widetilde{\Omega}}}{\mathcal{L}^2(\widetilde{\Omega})}$$

except for very little mass. The case when $Du_1|_{\widetilde{\Omega}} \in H_{\delta_1}$ is analogous. In the end, this gives us a function $w\colon \Omega'' \to \mathbb{R}^2$ such that the measure

$$\frac{(Dw)_{\#}\mathcal{L}^2|_{\Omega''}}{\mathcal{L}^2(\Omega'')}$$

looks similar to the measure

$$\frac{(Du_1)_{\#}\mathcal{L}^2|_{\Omega''}}{\mathcal{L}^2(\Omega'')}$$

except for very little mass. On a heuristic level, we assume that this mass is negligible.

Then we define the function u_2 such that $u_2 = v$ on Ω' and $u_2 = w$ on Ω''. Finally, from the fact that the measure

$$\frac{(Dv)_{\#}\mathcal{L}^2|_{\Omega'}}{\mathcal{L}^2(\Omega')}$$

is close to μ_2 and that the measure

$$\frac{(Dw)_{\#}\mathcal{L}^2|_{\Omega''}}{\mathcal{L}^2(\Omega'')}$$

looks similar to

$$\frac{(Du_1)_{\#}\mathcal{L}^2|_{\Omega''}}{\mathcal{L}^2(\Omega'')},$$

we see, by recalling the proof of Proposition 6.3.2, that the measure

$$\frac{(Du_2)_{\#}\mathcal{L}^2|_{\Omega}}{\mathcal{L}^2(\Omega)}$$

must look similar to the laminate $\mu^{(2)}$. Then we do the same thing for u_2 and so on. This gives us a sequence of mappings $\{u_k\}_{k=1}^{\infty}$. We expect (at least on an heuristic level for the time being) that the measures

$$\frac{(Du_k)_{\#}\mathcal{L}^2|_{\Omega}}{\mathcal{L}^2(\Omega)}$$

will look similar to the measures $\mu^{(k)}$ described in Proposition 6.3.2. Therefore, we expect to be able to choose C_1 and C_2 such that the limit function u^1 belongs to $W^{1,1}(\Omega)$ but not to $W^{1,2}(\Omega)$. In addition, since $\mathcal{L}^2\left(\{x \in \Omega : Du_k(x) \in E_{\delta_k}\}\right) \to 0$ as $k \to \infty$ and $L_{\delta_k} \cup H_{\delta_k}$ approach K' in some sense, we expect that

$$Du(x) \in K' \quad \text{for a.e. } x \in \Omega.$$

6.4 Proof of the main theorem

In this section we prove Theorem 6.0.2. However, before going to the proof, we first rigorously define the main iteration (Part 1 in the previous section) of the proof as well as the correction procedures (Part 2 in the previous section). These concepts were introduced in the previous section on an heuristic level.

Main iteration

We define a few quantities. For all $1 \le p \le 2$ we define

$$\begin{aligned}
\alpha_p &:= 2^p \frac{C_2 + 2}{2C_2 + 2} \frac{1 + C_1}{2 + C_1}, \\
\beta_p &:= \frac{1}{2 + C_1} + 2^p \frac{C_2^{1-p}}{2C_2 + 2} \frac{1 + C_1}{2 + C_1} \text{ and} \\
\gamma_p &:= \frac{C_1^p}{2 + C_1} + 2^p \frac{C_2}{2C_2 + 2} \frac{1 + C_1}{2 + C_1}.
\end{aligned}$$

For future reference, we define

$$\begin{aligned}
c_\alpha &:= \frac{1}{2} \inf_{1 \le p \le 2} \alpha_p = \frac{C_2 + 2}{2C_2 + 2} \frac{1 + C_1}{2 + C_1} > 0, \\
C_\alpha &:= 2 \sup_{1 \le p \le 2} \alpha_p = 8 \frac{C_2 + 2}{2C_2 + 2} \frac{1 + C_1}{2 + C_1} < \infty, \\
c_\beta &:= \frac{1}{2} \inf_{1 \le p \le 2} \beta_p > 0, \\
C_\beta &:= 2 \sup_{1 \le p \le 2} \beta_p < \infty, \\
c_\gamma &:= \frac{1}{2} \inf_{1 \le p \le 2} \gamma_p > 0 \text{ and} \\
C_\gamma &:= 2 \sup_{1 \le p \le 2} \gamma_p < \infty.
\end{aligned}$$

The use of these constants will for example be that we will be able to say that for a $\widetilde{\alpha}_p$ close to α_p for any $p \in [1,2]$, we have

$$c_\alpha \leq \widetilde{\alpha}_p \leq C_\alpha.$$

The main iteration of the proof is contained in the following lemma:

Lemma 6.4.1. *Let $0 < \delta_1 < \delta_0/10$ and $u\colon \Omega \to \mathbb{R}^2$ a piecewise affine Lipschitz map such that*

$$Du(x) \in E_{\delta_0} \cap \widetilde{B}_R \quad \text{for a.e. } x \in \Omega$$

for some $R > 0$. Then there exist $\overline{\delta} > 0$ and $B > 0$ such that if $\delta_0 < \overline{\delta}$, then for any $\varepsilon > 0$ there exists a piecewise affine Lipschitz map $v\colon \Omega \to \mathbb{R}^2$ satisfying:

- $\|v - u\|_{C^\alpha(\overline{\Omega})} < \varepsilon$,
- *$v = u$ on $\partial\Omega$ and,*
- *$Dv(x) \in (E_{\delta_1} \cup L_{\delta_1} \cup H_{\delta_1}) \cap \widetilde{B}_{2R}$ for a.e. $x \in \Omega$.*

In addition, there are two open sets Ω_1 and Ω_2 covering Ω up to a set of measure 0 such that

$$\begin{aligned} Dv(x) &\in E_{\delta_1} \quad \text{for a.e. } x \in \Omega_1, \\ Dv(x) &\in L_{\delta_1} \cup H_{\delta_1} \quad \text{for a.e. } x \in \Omega_2, \end{aligned}$$

for which we have

$$\int_{\Omega_1} |\partial_{x_1} v^1|^p \, dx = \widetilde{\alpha}_p^{(1)} \int_\Omega |\partial_{x_1} u^1|^p \, dx, \tag{6.9}$$

$$\int_{\Omega_2} |\partial_{x_1} v^1|^p \, dx = \widetilde{\beta}_p \int_\Omega |\partial_{x_1} u^1|^p \, dx, \tag{6.10}$$

$$\int_{\Omega_1} |\partial_{x_2} v^2|^p \, dx = \widetilde{\alpha}_p^{(2)} \int_\Omega |\partial_{x_2} u^2|^p \, dx, \tag{6.11}$$

$$\int_{\Omega_2} |\partial_{x_2} v^2|^p \, dx = \widetilde{\gamma}_p \int_\Omega |\partial_{x_2} u^2|^p \, dx, \tag{6.12}$$

for some $\widetilde{\alpha}_p^{(1)}$, $\widetilde{\alpha}_p^{(2)}$, $\widetilde{\beta}_p$, $\widetilde{\gamma}_p$ which satisfy

$$|\widetilde{\alpha}_p^{(1)} - \alpha_p|, |\widetilde{\alpha}_p^{(2)} - \alpha_p|, |\widetilde{\beta}_p - \beta_p|, |\widetilde{\gamma}_p - \gamma_p| \leq B\delta_0.$$

Remark 6.4.2. *When applying this lemma, in addition to having a piecewise affine mapping u, we will also have a specific collection of mutually disjoint open sets $\mathcal{F}$ covering Ω up to a set of measure 0, such that for all $V \in \mathcal{F}$, $u|_V$ is affine. In addition to the above lemma giving us a mapping v, it also yields a new collection of mutually disjoint open sets $\mathcal{F}'$ such that for all $V' \in \mathcal{F}'$, $v|_{V'}$ is affine and there is some $V \in \mathcal{F}$ such that $V' \subset V$.*

Remark 6.4.3. *By definition of the sets L_{δ_1}, H_{δ_1}, E_{δ_1} and the fact that*

$$Dv(x) \in L_{\delta_1} \cup H_{\delta_1} \cup E_{\delta_1}$$

it follows that $\|\partial_{x_1} v^2\|_{L^\infty(\Omega)}, \|\partial_{x_2} v^1\|_{L^\infty(\Omega)} < 1$ in the previous lemma.

Remark 6.4.4. *In the remainder of this section, whenever writing $\overline{\delta}$ or B, we intend the quatities given by this lemma.*

Proof. Assume $u \colon \Omega \to \mathbb{R}^2$ is affine and let

$$Du = Z = \begin{pmatrix} x & b \\ b & y \end{pmatrix} \in E_{\delta_0}$$

where $y = (1+\delta)x$ for some $\delta \in (-\delta_0, \delta_0)$. Using the same argument as for Proposition 6.3.1, we can prove that there exists a laminate of finite order μ of the form

$$\mu = \lambda_1 \delta_{Z_1} + \lambda_2 \delta_{Z_2} + \lambda_3 \delta_{Z_3}$$

where

$$Z_1 = \begin{pmatrix} 2x & b \\ b & 2x \end{pmatrix}, \quad Z_2 = \begin{pmatrix} x & b \\ b & -(C_1 - \delta_1)x \end{pmatrix}, \quad Z_3 = \begin{pmatrix} -\frac{2x}{C_2+\delta_1} & b \\ b & 2x \end{pmatrix},$$

and

$$\begin{aligned}
\lambda_1 &= \frac{C_2 + \delta_1 + 2}{2C_2 + 2\delta_1 + 2} \frac{1 + \delta + C_1 - \delta_1}{2 + C_1 - \delta_1}, \\
\lambda_2 &= \frac{1 - \delta}{2 + C_1 - \delta_1}, \\
\lambda_3 &= \frac{C_2 + \delta_1}{2C_2 + 2\delta_1 + 2} \frac{1 + \delta + C_1 - \delta_1}{2 + C_1 - \delta_1}.
\end{aligned}$$

By choosing $\overline{\delta}$ small enough and imposing $\delta_0 < \overline{\delta}$ and $\delta_1 < \delta_0/10$, we have

$$\begin{aligned}
&\left| \lambda_1 - \frac{C_2 + 2}{2C_2 + 2} \frac{1 + C_1}{2 + C_1} \right| \leq B_1 \delta_0, \\
&\left| \lambda_2 - \frac{1}{2 + C_1} \right| \leq B_2 \delta_0 \text{ and} \\
&\left| \lambda_3 - \frac{C_2}{2C_2 + 2} \frac{1 + C_1}{2 + C_1} \right| \leq B_3 \delta_0
\end{aligned}$$

for some B_1, B_2, B_3 which depend only on C_1 and C_2.

By Proposition 6.1.2, there is a piecewise affine Lipschitz map $v \colon \Omega \to \mathbb{R}^2$ such that

- $\|v-u\|_{C^\alpha(\overline{\Omega})} < \varepsilon$,
- $v = u$ on $\partial\Omega$ and
- $Dv(x) \in (L_{\delta_1} \cup H_{\delta_1} \cup E_{\delta_1}) \cap \widetilde{B}_{2R}$ for a.e $x \in \Omega$.

The existence of the open sets Ω_1 and Ω_2 in the statement comes from the fact that v is piecewise affine. Now it remains to prove that the equalities (6.9), (6.10), (6.11) and (6.12) hold. By Proposition 6.1.2, v can be chosen such that

$$\left|\int_{\Omega_1} |\partial_{x_1} v^1|^p \, dx - \lambda_1 |2x|^p \mathcal{L}^2(\Omega)\right| < \eta$$
$$\left|\int_{\Omega_2} |\partial_{x_1} v^1|^p \, dx - \left(\lambda_2 |x|^p + \lambda_3 \left|\frac{2x}{C_2+\delta_1}\right|^p\right) \mathcal{L}^2(\Omega)\right| < \eta$$
$$\left|\int_{\Omega_1} |\partial_{x_2} v^2|^p \, dx - \lambda_1 |2x|^p \mathcal{L}^2(\Omega)\right| < \eta$$
$$\left|\int_{\Omega_2} |\partial_{x_2} v^2|^p \, dx - (\lambda_2 |(C_1-\delta_1)x|^p + \lambda_3 |2x|^p) \mathcal{L}^2(\Omega)\right| < \eta$$

for some arbitrarily small $\eta > 0$. Then, we notice that

$$\lambda_1 |2x|^p \mathcal{L}^2(\Omega) = 2^p \lambda_1 |x|^p \mathcal{L}^2(\Omega) = 2^p \lambda_1 \int_\Omega |\partial_{x_1} u^1|^p \, dx,$$
$$\left(\lambda_2 |x|^p + \lambda_3 \left|\frac{2x}{C_2+\delta_1}\right|^p\right) \mathcal{L}^2(\Omega) = \left(\lambda_2 + \frac{2^p\lambda_3}{(C_2+\delta_1)^p}\right) |x|^p \mathcal{L}^2(\Omega)$$
$$= \left(\lambda_2 + \frac{2^p\lambda_3}{(C_2+\delta_1)^p}\right) \int_\Omega |\partial_{x_1} u^1|^p \, dx,$$
$$\lambda_1 |2x|^p \mathcal{L}^2(\Omega) = 2^p \lambda_1 |x|^p \mathcal{L}^2(\Omega) = \frac{2^p\lambda_1}{(1+\delta)^p} |y|^p \mathcal{L}^2(\Omega) = \frac{2^p\lambda_1}{(1+\delta)^p} \int_\Omega |\partial_{x_2} u^2|^p \, dx \text{ and}$$
$$(\lambda_2 |(C_1-\delta_1)x|^p + \lambda_3 |2x|^p) \mathcal{L}^2(\Omega) = \left(\left(\frac{C_1-\delta_1}{1+\delta}\right)^p \lambda_2 + \frac{2^p\lambda_3}{(1+\delta)^p}\right) |y|^p \mathcal{L}^2(\Omega)$$
$$= \left(\left(\frac{C_1-\delta_1}{1+\delta}\right)^p \lambda_2 + \frac{2^p\lambda_3}{(1+\delta)^p}\right) \int_\Omega |\partial_{x_2} u^2|^p \, dx.$$

We see that for $\overline{\delta}$ small enough we have

$$|2^p\lambda_1 - \alpha_p| \leq 2^p B_1 \delta_0 \leq 4 B_1 \delta_0,$$
$$\left|\left(\lambda_2 + \frac{2^p\lambda_3}{(C_2+\delta_1)^p}\right) - \beta_p\right| \leq B_2\delta_0 + 4^p B_3 \delta_0 \leq (16 B_3 + B_2)\,\delta_0,$$
$$\left|\frac{2^p\lambda_1}{(1+\delta)^p} - \alpha_p\right| \leq 4^p B_1 \delta_0 \leq 16 B_1 \delta_0 \text{ and}$$

$$\left| \left(\left(\frac{C_1 - \delta_1}{1+\delta} \right)^p \lambda_2 + \frac{2^p \lambda_3}{(1+\delta)^p} \right) - \gamma_p \right| \leq 2^p B_2 \delta_0 + 4^p B_3 \delta_0 \leq (4B_2 + 16B_3)\delta_0.$$

Then, take B to be

$$B := 2 \max \{4B_1, 16B_3 + B_2, 16B_1, 4B_2 + 16B_3\}.$$

Since B_1, B_2, B_3 depend only on C_1 and C_2, B depends only on C_1 and C_2. Finally, by taking η small enough, we have that

$$\begin{aligned}
\int_{\Omega_1} |\partial_{x_1} v^1|^p \, dx &= \widetilde{\alpha}_p^{(1)} \int_\Omega |\partial_{x_1} u^1|^p \, dx, \\
\int_{\Omega_2} |\partial_{x_1} v^1|^p \, dx &= \widetilde{\beta}_p \int_\Omega |\partial_{x_1} u^1|^p \, dx, \\
\int_{\Omega_1} |\partial_{x_2} v^2|^p \, dx &= \widetilde{\alpha}_p^{(2)} \int_\Omega |\partial_{x_2} u^2|^p \, dx \text{ and} \\
\int_{\Omega_2} |\partial_{x_2} v^2|^p \, dx &= \widetilde{\gamma}_p \int_\Omega |\partial_{x_2} u^2|^p \, dx,
\end{aligned}$$

for some $\widetilde{\alpha}_p^{(1)}$, $\widetilde{\alpha}_p^{(2)}$, $\widetilde{\beta}_p$, $\widetilde{\gamma}_p$ which satisfy

$$|\widetilde{\alpha}_p^{(1)} - \alpha_p|, |\widetilde{\alpha}_p^{(2)} - \alpha_p|, |\widetilde{\beta}_p - \beta_p|, |\widetilde{\gamma}_p - \gamma_p| \leq B\delta_0.$$

This proves the result in the case where u is affine. In order to prove this result in the general case where we only know u to be piecewise affine, we apply the argument above to each affine part of u. This finishes the proof. □

Correction procedures

The correction procedures are described by the following two lemmas. The first one corrects vertical elementary splittings and the second one corrects horizontal elementary splittings. Figure 6.4 illustrates Lemma 6.4.5, i.e. a vertical correction procedure.

Lemma 6.4.5. *Let $0 < \delta_1 < \delta_0/10$ and $u \colon \Omega \to \mathbb{R}^2$ a piecewise affine Lipschitz mapping such that*

$$Du(x) \in L_{\delta_0} \cap \widetilde{B}_R \quad \text{for a.e. } x \in \Omega$$

for some $R > 0$. For any $\xi > 1$, there exists $\overline{\delta}_\xi > 0$ such that if $\delta_0 < \overline{\delta}_\xi$ then for any $\varepsilon > 0$ there exists a piecewise affine Lipschitz map $v \colon \Omega \to \mathbb{R}^2$ satisfying:

- $\|v - u\|_{C^\alpha(\overline{\Omega})} < \varepsilon$,
- $v = u$ *on* $\partial\Omega$,

- $Dv(x) \in (E_{\delta_1} \cup L_{\delta_1}) \cap \widetilde{B}_{2R}$ *for a.e.* $x \in \Omega$ *and*
- $|\partial_{x_1} v^1(x)| > (1 - 2\delta_0)|\partial_{x_1} u^1(x)|$ *for a.e.* $x \in \Omega$.

In addition, there are two open sets Ω_1 *and* Ω_2 *covering* Ω *up to a set of measure 0 such that*

$$\begin{aligned} Dv(x) &\in E_{\delta_1} \text{ for a.e. } x \in \Omega_1, \\ Dv(x) &\in L_{\delta_1} \text{ for a.e. } x \in \Omega_2, \end{aligned}$$

for which we obtain

$$\mathcal{L}^2(\Omega_2) > .(1 - 2\delta_0)\mathcal{L}^2(\Omega) \tag{6.13}$$

and the following inequalities are fulfilled:

$$\int_{\Omega_1} |\partial_{x_1} v^1|^p \, dx \leq 2\delta_0 \int_{\Omega} |\partial_{x_1} u^1|^p \, dx, \tag{6.14}$$

$$(1 - 2\delta_0) \int_{\Omega} |\partial_{x_1} u^1|^p \, dx \leq \int_{\Omega_2} |\partial_{x_1} v^1|^p \, dx \leq \int_{\Omega} |\partial_{x_1} u^1|^p \, dx, \tag{6.15}$$

$$\int_{\Omega_1} |\partial_{x_2} v^2|^p \, dx \leq \frac{4\delta_0}{C_1^p} \int_{\Omega} |\partial_{x_2} u^2|^p \, dx, \tag{6.16}$$

$$(1 - 2\delta_0) \int_{\Omega} |\partial_{x_2} u^2|^p \, dx \leq \int_{\Omega_2} |\partial_{x_2} v^2|^p \, dx \leq \xi \int_{\Omega} |\partial_{x_2} u^2|^p \, dx. \tag{6.17}$$

Proof. Assume that $u\colon \Omega \to \mathbb{R}^2$ is affine and let

$$Du = Z = \begin{pmatrix} x & b \\ b & y \end{pmatrix} \in L_{\delta_0}$$

where $y = (\delta - C_1)$ for some $\delta \in (\delta_0/2, 2\delta_0)$. We notice that there is a laminate of finite order μ satisfying $\overline{\mu} = Z$ of the form

$$\mu = \lambda \delta_{Z_1} + (1 - \lambda)\delta_{Z_2}$$

where

$$Z_1 = \begin{pmatrix} x & b \\ b & x \end{pmatrix}, \quad Z_2 = \begin{pmatrix} x & b \\ b & (\delta_1 - C_1)x \end{pmatrix}$$

and

$$\lambda = \frac{\delta - \delta_1}{1 - \delta_1 + C_1}.$$

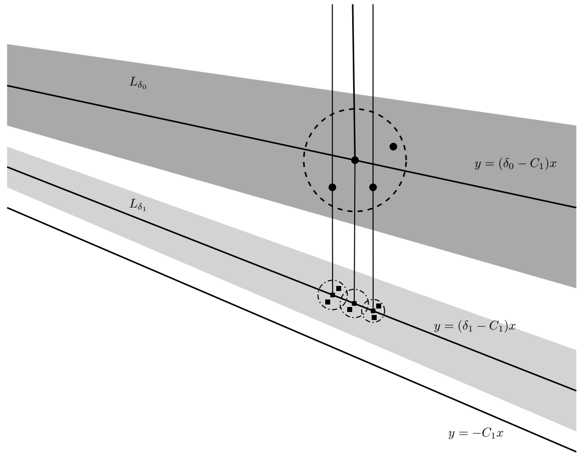

Figure 6.4: Illustration of the vertical correction procedures: the dots represent the distribution of the gradients of some function u. Then the vertical correction procedure given by Lemma 6.4.5 is applied. Visually, this consists in splitting each dot along vertical lines between the sets L_{δ_1} and E_{δ_1}. This gives us a new function v. It is clear that for a very large portion of the $x \in \Omega$ such that $Du(x) \in L_{\delta_0}$, we then have $Dv(x) \in L_{\delta_1}$. The distribution of the gradients of v is represented by squares in this figure.

Using the fact that $\delta \in (\delta_0/2, 2\delta_0)$, we deduce that (since we already may choose $\overline{\delta}_\xi$ so small that it implies $1 - \delta_1 + C_1 > 1$):

$$0 \leq \lambda \leq \frac{2\delta_0 - \delta_1}{1 - \delta_1 + C_1} \leq 2\delta_0.$$

By applying Proposition 6.1.2 we obtain a piecewise affine Lipschitz mapping $v \colon \Omega \to \mathbb{R}^2$ satisfying:

- $\|v - u\|_{C^\alpha(\overline{\Omega})} < \varepsilon$,
- $v = u$ on $\partial\Omega$,
- $Dv(x) \in (E_{\delta_1} \cup L_{\delta_1}) \cap \widetilde{B}_{2R}$ for a.e. $x \in \Omega$ and
- $|\partial_{x_1} v^1(x)| > (1 - 2\delta_0)|\partial_{x_1} u^1(x)|$ for a.e. $x \in \Omega$.

The existence of the open sets Ω_1 and Ω_2 in the statement follows from the fact that v is piecewise affine. The fact that $\mathcal{L}^2(\Omega_2) > (1 - 2\delta_0)\mathcal{L}^2(\Omega)$ comes from Proposition 6.1.2 combined with the fact that $\lambda \leq 2\delta_0$. Now it remains to prove that the inequalities (6.14), (6.15), (6.16) and (6.17) are fulfilled. From Proposition 6.1.2, it follows that v can be chosen such that

$$\begin{aligned}
&\left|\int_{\Omega_1} |\partial_{x_1} v^1|^p \, dx - \lambda |x|^p \mathcal{L}^2(\Omega)\right| < \eta, \\
&\left|\int_{\Omega_2} |\partial_{x_1} v^1|^p \, dx - (1-\lambda) |x|^p \mathcal{L}^2(\Omega)\right| < \eta, \\
&\left|\int_{\Omega_1} |\partial_{x_2} v^2|^p \, dx - \lambda |x|^p \mathcal{L}^2(\Omega)\right| < \eta, \\
&\left|\int_{\Omega_2} |\partial_{x_2} v^2|^p \, dx - (1-\lambda) |(\delta_1 - C_1) x|^p \mathcal{L}^2(\Omega)\right| < \eta,
\end{aligned}$$

for some arbitrarily small $\eta > 0$. Then we notice that

$$\begin{aligned}
\lambda |x|^p \mathcal{L}^2(\Omega) &= \lambda \int_\Omega |\partial_{x_1} u^1|^p \, dx \quad \textit{and} \\
(1-\lambda) |x|^p \mathcal{L}^2(\Omega) &= (1-\lambda) \int_\Omega |\partial_{x_1} u^1|^p \, dx.
\end{aligned}$$

This proves that (6.14) and (6.15) are fulfilled. For the remaining inequalities to prove, we notice that

$$\begin{aligned}
&\lambda |x|^p \mathcal{L}^2(\Omega) = \lambda \left|\frac{y}{\delta - C_1}\right|^p \mathcal{L}^2(\Omega) = \lambda \frac{|y|^p}{(C_1 - \delta)^p} \mathcal{L}^2(\Omega) = \frac{\lambda}{(C_1-\delta)^p} \int_\Omega |\partial_{x_2} u^2|^p \, dx \quad \text{and} \\
&(1-\lambda) |(\delta_1 - C_1) x|^p \mathcal{L}^2(\Omega) = (1-\lambda)(C_1 - \delta_1)^p \left|\frac{y}{\delta - C_1}\right|^p \mathcal{L}^2(\Omega) \\
&\qquad = (1-\lambda) \underbrace{\left(\frac{C_1 - \delta_1}{C_1 - \delta}\right)^p}_{\geq 1} \int_\Omega |\partial_{x_2} u^2|^p \, dx.
\end{aligned}$$

From these computations, it follows that by choosing $\overline{\delta}_\xi$ small enough, we obtain

$$\begin{aligned}
&0 \leq \frac{\lambda}{(C_1 - \delta)^p} \leq \frac{4\delta_0}{C_1^p} \quad \text{and} \\
&(1 - 2\delta_0) \leq (1-\lambda) \left(\frac{C_1 - \delta_1}{C_1 - \delta}\right)^p \leq \xi.
\end{aligned}$$

This proves (6.16) and (6.17). This finishes the proof when u is assumed to be affine. The case when u is piecewise affine follows from the affine case by applying the above argument for each affine part of u. This finishes the proof. □

We also have a similar result for the horizontal correction procedure. Since the strategy of the proof is similar to the previous one, we do not present a proof for this lemma.

Lemma 6.4.6. *Let $0 < \delta_1 < \delta_0/10$ and $u\colon \Omega \to \mathbb{R}^2$ a piecewise affine Lipschitz mapping such that*

$$Du(x) \in H_{\delta_0} \cap \widetilde{B}_R \quad \text{for a.e. } x \in \Omega$$

for some $R > 0$. For any $\xi > 1$, there exists $\overline{\delta}_\xi > 0$ such that if $\delta_0 < \overline{\delta}_\xi$ then for any $\varepsilon > 0$ there exists a piecewise affine Lipschitz map $v\colon \Omega \to \mathbb{R}^2$ satisfying:

- $\|v - u\|_{C^\alpha(\overline{\Omega})} < \varepsilon$,
- *$v = u$ on $\partial\Omega$ and*
- *$Dv(x) \in (E_{\delta_1} \cup H_{\delta_1}) \cap \widetilde{B}_{2R}$ for a.e. $x \in \Omega$.*

In addition, there are two open sets Ω_1 and Ω_2 covering Ω up to a set of measure 0 such that

$$\begin{aligned} Dv(x) &\in E_{\delta_1} \text{ for a.e. } x \in \Omega_1, \\ Dv(x) &\in H_{\delta_1} \text{ for a.e. } x \in \Omega_2, \end{aligned}$$

for which we obtain

$$\begin{aligned} &|\partial_{x_1} v^1(x)| > |\partial_{x_1} u^1(x)| \text{ for a.e. } x \in \Omega_2, \\ &\mathcal{L}^2(\Omega_2) > (1 - 2\delta_0)\mathcal{L}^2(\Omega) \end{aligned}$$

and the following inequalities are fulfilled:

$$\begin{aligned} \int_{\Omega_1} |\partial_{x_1} v^1|^p \, dx &\le 4\delta_0 C_2^p \int_\Omega |\partial_{x_1} u^1|^p \, dx, \\ (1 - 2\delta_0) \int_\Omega |\partial_{x_1} u^1|^p \, dx \le \int_{\Omega_2} |\partial_{x_1} v^1|^p \, dx &\le \xi \int_\Omega |\partial_{x_1} u^1|^p \, dx, \\ \int_{\Omega_1} |\partial_{x_2} v^2|^p \, dx &\le 2\delta_0 \int_\Omega |\partial_{x_2} u^2|^p \, dx, \\ (1 - 2\delta_0) \int_\Omega |\partial_{x_2} u^2|^p \, dx \le \int_{\Omega_2} |\partial_{x_2} v^2|^p \, dx &\le \int_\Omega |\partial_{x_2} u^2|^p \, dx. \end{aligned}$$

To simplify the proof of the main theorem we present the following corollary which allows to perform both correction procedures simultaneously. Combining Lemma 6.4.5 and Lemma 6.4.6, we obtain:

Corollary 6.4.7. *Let $\delta_1 < \delta_0/10$ and $u\colon \Omega \to \mathbb{R}^2$ a piecewise affine Lipschitz mapping such that*

$$Du(x) \in (L_{\delta_0} \cup H_{\delta_0}) \cap \widetilde{B}_R \quad \text{for a.e. } x \in \Omega$$

for some $R > 0$. For any $\xi > 1$, there exists $\overline{\delta}_\xi > 0$ such that if $\delta_0 < \overline{\delta}_\xi$ then for any $\varepsilon > 0$, there exists a piecewise affine Lipschitz map $v\colon \Omega \to \mathbb{R}^2$ satisfying:

- $\|v-u\|_{C^\alpha(\overline{\Omega})} < \varepsilon$,
- $v = u$ *on* $\partial\Omega$ *and*
- $Dv(x) \in (E_{\delta_1} \cup H_{\delta_1} \cup L_{\delta_1}) \cap \widetilde{B}_{2R}$ *for a.e.* $x \in \Omega$.

In addition, there are two open sets Ω_1 *and* Ω_2 *covering* Ω *up to a set of measure 0 such that*

$$\begin{aligned} &Dv(x) \in E_{\delta_1} \text{ for a.e. } x \in \Omega_1,\\ &Dv(x) \in L_{\delta_1} \cup H_{\delta_1} \text{ for a.e. } x \in \Omega_2 \end{aligned}$$

for which we obtain

$$|\partial_{x_1} v^1(x)| > (1-2\delta_0)|\partial_{x_1} u^1(x)| \text{ for a.e. } x \in \Omega_2, \tag{6.18}$$

$$\mathcal{L}^2(\Omega_2) > .(1-2\delta_0)\mathcal{L}^2(\Omega) \tag{6.19}$$

and the following inequalities are fulfilled:

$$\begin{aligned} \int_{\Omega_1} |\partial_{x_1} v^1|^p\,dx &\le 4\delta_0 C_2^p \int_\Omega |\partial_{x_1} u^1|^p\,dx,\\ (1-2\delta_0)\int_\Omega |\partial_{x_1} u^1|^p\,dx \le \int_{\Omega_2} |\partial_{x_1} v^1|^p\,dx &\le \xi \int_\Omega |\partial_{x_1} u^1|^p\,dx,\\ \int_{\Omega_1} |\partial_{x_2} v^2|^p\,dx &\le \frac{4\delta_0}{C_1^p}\int_\Omega |\partial_{x_2} u^2|^p\,dx,\\ (1-2\delta_0)\int_\Omega |\partial_{x_2} u^2|^p\,dx \le \int_{\Omega_2} |\partial_{x_2} v^2|^p\,dx &\le \xi \int_\Omega |\partial_{x_2} u^2|^p\,dx. \end{aligned}$$

Remark 6.4.8. *As for Lemma 6.4.1, when applying this corollary, in addition to having a piecewise affine u, we will also have a specific collection of mutually disjoint open sets* $\mathcal{F}$ *covering* Ω *up to a set of measure 0 so that for all* $V \in \mathcal{F}$, $u|_V$ *is affine. We will also assume that when applying this corollary, it is applied to each* $u|_V$, $V \in \mathcal{F}$ *individually. This in particular means that in addition to (6.19) being satisfied we also have*

$$\mathcal{L}^2(V \cap \Omega_2) > .(1-2\delta_0)\mathcal{L}^2(V) \text{ for all } V \in \mathcal{F}. \tag{6.20}$$

Moreover, this corollary then yields a new collection of mutually disjoint open sets $\mathcal{F}'$ *such that for all* $V' \in \mathcal{F}'$, $v|_{V'}$ *is affine and there is some* $V \in \mathcal{F}$ *such that* $V' \subset V$.

Remark 6.4.9. *In the corollary above, by definition of the sets* E_{δ_1}, L_{δ_1} *and* H_{δ_1} *and that fact that*

$$Dv(x) \in L_{\delta_1} \cup H_{\delta_1} \cup E_{\delta_1} \text{ for a.e. } x \in \Omega,$$

it follows that $\|\partial_{x_1} v^2\|_{L^\infty(\Omega)}, \|\partial_{x_2} v^1\|_{L^\infty(\Omega)} < 1$.

Remark 6.4.10. *From now on, whenever we write $\overline{\delta}_\xi$ for some $\xi > 1$, we intend the $\overline{\delta}_\xi$ given by Corollary 6.4.7.*

We can now prove the main theorem.

Proof of Theorem 6.0.2. Step 1 (Setup): To lighten the notation, we assume $\mathcal{L}^2(\Omega) = 1$. As in the proof of Theorem 3.3.3, let $\rho\colon \mathbb{R}^n \to \mathbb{R}$ be smooth, nonnegative, supported in $B_1(0)$ such that $\int_{\mathbb{R}^n} \rho \, dx = 1$. Define $\rho_\varepsilon\colon \mathbb{R}^n \to \mathbb{R}$ as

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right).$$

We will use convolutions between mappings u from Ω to $\mathbb{R}^2$ with $u = Cx$ on $\partial\Omega$ for some $C \in S^{2\times 2}$ and mollifiers ρ_ε. With the standard definition of convolutions, this is not well defined and therefore we introduce the following convention. When writing $u * \rho_\varepsilon$, we use the standard definition of convolution on $\mathbb{R}^2$ where u is extended by Cx outside $\overline{\Omega}$. We now define some needed ingredients of the proof. Recall that we defined the quantity α_p as

$$\alpha_p = 2^p \frac{C_2 + 2}{2C_2 + 2} \frac{1 + C_1}{2 + C_1}$$

for all $p \in [1, 2]$. We fix the constants $0 < C_1 < 1$ and $1 < C_2 < \infty$ such that $\alpha_{1+\gamma} < 1$ for some $0 < \gamma < 1$ and $\alpha_2 > 1$. The existence of such constants C_1 and C_2 follows from the fact that for any $p \in [1, 2]$, α_p depends continuously on C_1 and C_2, the fact that $\alpha_1 \to 1$ if $C_1, C_2 \to 1$ and $\alpha_1 \to 1/2$ if $C_1 \to 0$, $C_2 \to \infty$. Therefore, we can find C_1 and C_2 such that $\alpha_1 < 1$ and $\alpha_2 = 2\alpha_1 > 1$. Finally the fact that $\alpha_{1+\gamma} = 2^\gamma \alpha_1$ allows us to find a $0 < \gamma < 1$ such that $\alpha_{1+\gamma} < 1$. Then, take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let

$$0 < \delta < \min\left(\frac{1 - \alpha_{1+\gamma}}{2B}, \frac{\alpha_2 - 1}{2B}\right) \tag{6.21}$$

and so small that the following implications hold for all $p \in [1, 2]$:

$$|\widetilde{\alpha}_p - \alpha_p| < B\delta \Rightarrow c_\alpha < \widetilde{\alpha}_p < C_\alpha, \tag{6.22}$$

$$|\widetilde{\beta}_p - \beta_p| < B\delta \Rightarrow c_\beta < \widetilde{\beta}_p < C_\beta, \tag{6.23}$$

$$|\widetilde{\gamma}_p - \gamma_p| < B\delta \Rightarrow c_\gamma < \widetilde{\gamma}_p < C_\gamma. \tag{6.24}$$

Let $\{\xi_j\}_{j=1}^\infty$ be a sequence such that $\xi_j > 1$ for all $j \geq 1$ and

$$\prod_{j=1}^\infty \xi_j \leq 2. \tag{6.25}$$

Finally, choose a sequence $\{\delta_j\}_{j=0}^{\infty}$ in such a way that the following properties are satisfied:

$$\delta_j < 2^{-(2j+2)} \quad \forall j \geq 0, \tag{6.26}$$

$$\delta_j < \overline{\delta}_{\xi_{j+1}} \quad \forall j \geq 0, \tag{6.27}$$

$$\delta_j < \min(\delta, \overline{\delta}) \quad \forall j \geq 0, \tag{6.28}$$

$$\prod_{j=1}^{\infty}(1 - 2\delta_j) > \frac{1}{2}, \tag{6.29}$$

$$\delta_j \leq \frac{(\xi_{j+1} - 1)c_\alpha^{j+1}}{8C_2^2 C_\beta \sum_{l=1}^{j} C_\alpha^{l-1}} \quad \forall j \geq 1, \tag{6.30}$$

$$\delta_j \leq \frac{C_1^2(\xi_{j+1} - 1)c_\alpha^{j+1}}{8C_\gamma \sum_{l=1}^{j} C_\alpha^{l-1}} \quad \forall j \geq 1. \tag{6.31}$$

Step 2 (First iteration of the scheme): In this step, we build the function u_1 of our sequence $\{u_k\}_{k=1}^{\infty}$. We will describe the properties that we would like this sequence to have in the next step of the proof. Let $0 < \varepsilon_0 < 2^{-1}$. By Lemma 6.4.1, there exists a piecewise affine Lipschitz map $u_1 \colon \Omega \to \mathbb{R}^2$ such that

$$\|u_1 - Ax\|_{C^\alpha(\overline{\Omega})} < \varepsilon_0 \quad \text{and} \quad u_1 = Ax \text{ on } \partial\Omega.$$

In addition, there are two open sets $\Omega_{1,1}$, $\Omega_{1,2}$ covering Ω up to a measure 0 set i.e.

$$\mathcal{L}^2\left(\Omega \setminus (\Omega_{1,1} \cup \Omega_{1,2})\right) = 0$$

such that

$$Du_1 \in E_{\delta_1} \cap \widetilde{B}_4 \text{ for a.e. } x \in \Omega_{1,1} \quad \text{and}$$
$$Du_1 \in (L_{\delta_1} \cup H_{\delta_1}) \cap \widetilde{B}_4 \text{ for a.e. } x \in \Omega_{1,2}.$$

Moreover, by Lemma 6.4.1, for all $p \in [1, 2]$, the following equalities are fulfilled

$$\int_{\Omega_{1,1}} |\partial_{x_1} u_1^1|^p = \alpha_{p,1}^{(1)},$$
$$\int_{\Omega_{1,2}} |\partial_{x_1} u_1^1|^p = \beta_{p,1},$$
$$\int_{\Omega_{1,1}} |\partial_{x_2} u_1^2|^p = \alpha_{p,1}^{(2)},$$
$$\int_{\Omega_{1,2}} |\partial_{x_2} u_1^2|^p = \gamma_{p,1},$$

for some $\alpha_{p,1}^{(1)}$, $\beta_{p,1}$, $\alpha_{p,1}^{(2)}$, $\gamma_{p,1}$ which satisfy

$$|\widetilde{\alpha}_p^{(1)} - \alpha_p|, |\widetilde{\alpha}_p^{(2)} - \alpha_p|, |\widetilde{\beta}_p - \beta_p|, |\widetilde{\gamma}_p - \gamma_p| \leq B\delta_0 \leq B\delta.$$

Lemma 6.4.1 also gives us a collection $\mathcal{F}_1$ of mutually disjoint open sets covering Ω such that $u_1|_V$ is affine for all $V \in \mathcal{F}_1$ (see Remark 6.4.2).

Then select $0 < \varepsilon_1 < \min(\varepsilon_0, 2^{-2})$ such that $\|Du_1 * \rho_{\varepsilon_1} - Du_1\|_{L^1(\Omega)} < 2^{-1}$.

Step 3 (Description of the inductive scheme): In this step we describe the properties of the sequence of functions $\{u_k\}_{k=1}^{\infty}$ that we want to build. In Step 4, we will then show that there is an inductive step such that for all k we can build u_{k+1} from u_k. We will build a sequence of piecewise affine Lipschitz mappings $\{u_k\}_{k=1}^{\infty}$, two sequences of open sets $\{\Omega_{k,1}\}_{k=1}^{\infty}$ and $\{\Omega_{k,2}\}_{k=1}^{\infty}$ such that the following holds for all $k \geq 1$:

$$\begin{aligned}
&\mathcal{L}^2\left(\Omega \setminus (\Omega_{k,1} \cup \Omega_{k,2})\right) = 0,\\
&Du_k(x) \in E_{\delta_k} \cap \widetilde{B}_{2^{k+1}} \text{ for a.e } x \in \Omega_{k,1},\\
&Du_k(x) \in (L_{\delta_k} \cup H_{\delta_k}) \cap \widetilde{B}_{2^{k+1}} \text{ for a.e } x \in \Omega_{k,2}.
\end{aligned}$$

In addition, for all $p \in [1, 2]$, there are sequences $\{\alpha_{p,j}^{(1)}\}_{j=1}^{\infty}$, $\{\beta_{p,j}\}_{j=1}^{\infty}$, $\{\alpha_{p,j}^{(2)}\}_{j=1}^{\infty}$, $\{\gamma_{p,j}\}_{j=1}^{\infty}$ such that the following inequalities are satisfied for all $k \geq 1$:

$$\prod_{j=1}^{k} \alpha_{p,j}^{(1)} \leq \int_{\Omega_{k,1}} |\partial_{x_1} u_k^1|^p \, dx \leq \prod_{j=1}^{k} \xi_j \alpha_{p,j}^{(1)}, \tag{6.32}$$

$$\prod_{j=1}^{k-1}(1 - 2\delta_j) \left(\sum_{j=1}^{k} \beta_{p,j} \prod_{l=1}^{j-1} \alpha_{p,l}^{(1)} \right) \leq \int_{\Omega_{k,2}} |\partial_{x_1} u_k^1|^p \, dx \leq \prod_{j=1}^{k} \xi_j \left(\sum_{j=1}^{k} \beta_{p,j} \prod_{l=1}^{j-1} \alpha_{p,l}^{(1)} \right), \tag{6.33}$$

$$\prod_{j=1}^{k} \alpha_{p,j}^{(2)} \leq \int_{\Omega_{k,1}} |\partial_{x_2} u_k^2|^p \, dx \leq \prod_{j=1}^{k} \xi_j \alpha_{p,j}^{(2)}, \tag{6.34}$$

$$\prod_{j=1}^{k-1}(1 - 2\delta_j) \left(\sum_{j=1}^{k} \gamma_{p,j} \prod_{l=1}^{j-1} \alpha_{p,l}^{(2)} \right) \leq \int_{\Omega_{k,2}} |\partial_{x_2} u_k^2|^p \, dx \leq \prod_{j=1}^{k} \xi_j \left(\sum_{j=1}^{k} \gamma_{p,j} \prod_{l=1}^{j-1} \alpha_{p,l}^{(2)} \right), \tag{6.35}$$

and

$$|\widetilde{\alpha}_{p,j}^{(1)} - \alpha_p|, |\widetilde{\alpha}_{p,j}^{(2)} - \alpha_p|, |\widetilde{\beta}_{p,j} - \beta_p|, |\widetilde{\gamma}_{p,j} - \gamma_p| \leq B\delta \quad \forall j \geq 1.$$

We also have

$$\|\partial_{x_1} u_k^2\|_{L^\infty(\Omega)}, \|\partial_{x_2} u_k^1\|_{L^\infty(\Omega)} < 1, \forall k \geq 1. \tag{6.36}$$

Moreover, for each $k \geq 1$ there exists a collection of mutually disjoint open sets $\mathcal{F}_k$ that cover Ω up to a set of measure 0 and so that for each $V \in \mathcal{F}_k$, $u_k|_V$ is affine. We build the sequences $\{u_k\}_{k=1}^\infty$ and $\{\mathcal{F}_k\}_{k=1}^\infty$ in such a way that for $j > l$ every set in $\mathcal{F}_j$ is contained in a set of $\mathcal{F}_l$. In addition, for any $V \in \mathcal{F}_k$ such that $V \subset \Omega_{k,2}$, we will have that

$$\begin{aligned} &\mathcal{L}^2\left(V \cap \Omega_{k+1,2}\right) > (1-2\delta_k)\mathcal{L}^2\left(V\right) \quad \text{and} \\ &\quad |\partial_{x_1} u^1_{k+1}(x)| > (1-2\delta_k)|\partial_{x_1} u^1_k(x)| \quad \text{for a.e. } x \in V \cap \Omega_{k+1,2}. \end{aligned} \tag{6.37}$$

Finally, we also build the sequence $\{u_k\}_{k=1}^\infty$ in such a way that for another decreasing sequence $\{\varepsilon_k\}_{k=1}^\infty$ (which satisfies $0 < \varepsilon_k < \min(2^{-(k+1)}, \varepsilon_{k-1})$) we have

$$\|u_k - u_{k-1}\|_{C^\alpha(\overline{\Omega})} < 2^{-(k-1)}\varepsilon_{k-1} \quad \text{and} \quad \|Du_k * \rho_{\varepsilon_k} - Du_k\|_{L^1(\Omega)} < 2^{-k}$$

for all $k \geq 2$.

Now we have described all the properties that we would like our sequence $\{u_k\}_{k=1}^\infty$ to fulfill. In the next step, we will show that such a sequence indeed can be constructed by showing the existence of an inductive step which allows to build u_{k+1} from u_k. The reader who would first like to see why the sequence described above allows us to prove the theorem can skip Step 4 and pass directly to the conclusion in Step 5.

Step 4 (Inductive step): We assume that we are given $u_k\colon \Omega \to \mathbb{R}^2$ piecewise affine and $\Omega_{k,1}$, $\Omega_{k,2}$ such that they cover Ω up to a set of measure 0 such that

$$\begin{aligned} &Du_k(x) \in E_{\delta_k} \cap \widetilde{B}_{2^{k+1}} \text{ for a.e } x \in \Omega_{k,1}, \\ &Du_k(x) \in (L_{\delta_k} \cup H_{\delta_k}) \cap \widetilde{B}_{2^{k+1}} \text{ for a.e } x \in \Omega_{k,2}. \end{aligned}$$

In addition, we also assume the existence of $\{\alpha^{(1)}_{p,j}\}_{j=1}^k$, $\{\beta_{p,j}\}_{j=1}^k$, $\{\alpha^{(2)}_{p,j}\}_{j=1}^k$, $\{\gamma_{p,j}\}_{j=1}^k$ such that

$$\prod_{j=1}^k \alpha^{(1)}_{p,j} \leq \int_{\Omega_{k,1}} |\partial_{x_1} u^1_k|^p\,dx \leq \prod_{j=1}^k \xi_j \alpha^{(1)}_{p,j}, \tag{6.38}$$

$$\prod_{j=1}^{k-1}(1-2\delta_j)\left(\sum_{j=1}^k \beta_{p,j} \prod_{l=1}^{j-1} \alpha^{(1)}_{p,l}\right) \leq \int_{\Omega_{k,2}} |\partial_{x_1} u^1_k|^p\,dx \leq \prod_{j=1}^k \xi_j \left(\sum_{j=1}^k \beta_{p,j} \prod_{l=1}^{j-1} \alpha^{(1)}_{p,l}\right), \tag{6.39}$$

$$\prod_{j=1}^k \alpha^{(2)}_{p,j} \leq \int_{\Omega_{k,1}} |\partial_{x_2} u^2_k|^p\,dx \leq \prod_{j=1}^k \xi_j \alpha^{(2)}_{p,j}, \tag{6.40}$$

$$\prod_{j=1}^{k-1}(1-2\delta_j)\left(\sum_{j=1}^k \gamma_{p,j} \prod_{l=1}^{j-1} \alpha^{(2)}_{p,l}\right) \leq \int_{\Omega_{k,2}} |\partial_{x_2} u^2_k|^p\,dx \leq \prod_{j=1}^k \xi_j \left(\sum_{j=1}^k \gamma_{p,j} \prod_{l=1}^{j-1} \alpha^{(2)}_{p,l}\right), \tag{6.41}$$

and

$$|\alpha^{(1)}_{p,j} - \alpha_p|, |\alpha^{(2)}_{p,j} - \alpha_p|, |\beta_{p,j} - \beta_p|, |\gamma_{p,j} - \gamma_p| \leq B\delta \quad \forall 1 \leq j \leq k.$$

Moreover, we assume that $\mathcal{F}_k$ is a collection of mutually disjoint open sets that cover Ω up to a set of measure 0 and for all $V \in \mathcal{F}_k$, $u_k|_V$ is affine. Finally, we also assume that we have a decreasing sequence $\{\varepsilon_j\}_{j=1}^k$ such that

$$\|u_j - u_{j-1}\|_{C^\alpha(\overline{\Omega})} < 2^{-(j-1)}\varepsilon_{j-1} \quad \text{and} \quad \|Du_j * \rho_{\varepsilon_j} - Du_j\|_{L^1(\Omega)} < 2^{-j}$$

for all $2 \leq j \leq k$. Now we can begin constructing the new function u_{k+1}. It will be done in two parts as described in Section 6.3. Let $\mathcal{F}_k^{(1)}$ be the collection of all $V \in \mathcal{F}_k$ such that $u_k|_V \in \Omega_{k,1}$ and $\mathcal{F}_k^{(2)}$ the collection of all $V \in \mathcal{F}_k$ such that $u_k|_V \in \Omega_{k,2}$. Firstly, by applying Lemma 6.4.1 to $u_k|_{\Omega_{k,1}}$, we obtain a piecewise affine Lipschitz map $v\colon \Omega_{k,1} \to \mathbb{R}^2$ with

$$\|v - u_k\|_{C^\alpha(\overline{\Omega_{k,1}})} < 2^{-(k+1)}\varepsilon_k, \tag{6.42}$$

$$v = u \quad \text{on } \partial\Omega_{k,1}, \tag{6.43}$$

and two open sets Ω_1' and Ω_2' such that

$$\begin{aligned}
&\mathcal{L}^2\left(\Omega_{k,1} \setminus (\Omega_1' \cup \Omega_2')\right) = 0,\\
&Dv(x) \in E_{\delta_{k+1}} \cap \widetilde{B}_{2^{k+2}} \text{ for a.e } x \in \Omega_1',\\
&Dv(x) \in (L_{\delta_{k+1}} \cup H_{\delta_{k+1}}) \cap \widetilde{B}_{2^{k+2}} \text{ for a.e } x \in \Omega_2'.
\end{aligned}$$

Moreover, by Lemma 6.4.1, the following equalities hold:

$$\int_{\Omega_1'} |\partial_{x_1} v^1|^p \, dx = \alpha_{p,k+1}^{(1)} \int_{\Omega_{k,1}} |\partial_{x_1} u_k^1|^p \, dx, \tag{6.44}$$

$$\int_{\Omega_2'} |\partial_{x_1} v^1|^p \, dx = \beta_{p,k+1} \int_{\Omega_{k,1}} |\partial_{x_1} u_k^1|^p \, dx, \tag{6.45}$$

$$\int_{\Omega_1'} |\partial_{x_2} v^2|^p \, dx = \alpha_{p,k+1}^{(2)} \int_{\Omega_{k,1}} |\partial_{x_2} u_k^2|^p \, dx, \tag{6.46}$$

$$\int_{\Omega_2'} |\partial_{x_2} v^2|^p \, dx = \gamma_{p,k+1} \int_{\Omega_{k,1}} |\partial_{x_2} u_k^2|^p \, dx, \tag{6.47}$$

for some $\alpha_{p,k+1}^{(1)}$, $\alpha_{p,k+1}^{(2)}$, $\beta_{p,k+1}$, $\gamma_{p,k+1}$ which satisfy

$$|\alpha_{p,k+1}^{(1)} - \alpha_p|, |\alpha_{p,k+1}^{(2)} - \alpha_p|, |\beta_{p,k+1} - \beta_p|, |\gamma_{p,k+1} - \gamma_p| \leq B\delta_k \leq B\delta.$$

Finally, as we mentioned in Remark 6.4.2 we get a collection $\mathcal{G}_1$ of mutually disjoint open sets covering $\Omega_{k,1}$ such that for all $V \in \mathcal{G}_1$, $v|_V$ is affine and every set in $\mathcal{G}_1$ is contained in a set belonging to $\mathcal{F}_k^{(1)}$. Secondly, by applying Corollary 6.4.7 to $u_k|_{\Omega_{k,2}}$ and, we obtain a piecewise affine Lipschitz map $w\colon \Omega_{k,2} \to \mathbb{R}^2$ such that

$$\|w - u_k\|_{C^\alpha(\overline{\Omega_{k,2}})} < 2^{-(k+1)}\varepsilon_k, \tag{6.48}$$

$$w = u_k \quad \text{on } \partial\Omega_{k,2} \tag{6.49}$$

and two open sets Ω_1'' and Ω_2'' such that

$$\begin{aligned}
&\mathcal{L}^2\left(\Omega_{k,1} \setminus (\Omega_1'' \cup \Omega_2'')\right) = 0,\\
&Dw(x) \in E_{\delta_{k+1}} \cap \widetilde{B}_{2^{k+2}} \text{ for a.e } x \in \Omega_1'',\\
&Dw(x) \in (L_{\delta_{k+1}} \cup H_{\delta_{k+1}}) \cap \widetilde{B}_{2^{k+2}} \text{ for a.e } x \in \Omega_2''.
\end{aligned}$$

By Corollary 6.4.7 and (6.20), for all $V \in \mathcal{F}_k$ such that $V \subset \Omega_{k,2}$ (i.e. for all $V \in \mathcal{F}_k^{(2)}$),

$$\begin{aligned}
&\mathcal{L}^2\left(V \cap \Omega_2''\right) > (1-2\delta_k)\mathcal{L}^2(V) \quad \text{and}\\
&|\partial_{x_1} u_{k+1}^1(x)| > (1-2\delta_k)|\partial_{x_1} u_k^1(x)| \quad \text{for a.e. } x \in V \cap \Omega_2''.
\end{aligned} \tag{6.50}$$

Moreover, by Corollary 6.4.7 and the fact that $\delta_k < \overline{\delta}_{\xi_{k+1}}$, the following inequalities hold for all $p \in [1,2]$:

$$\int_{\Omega_1''} |\partial_{x_1} w^1|^p \, dx \le 4\delta_k C_2^p \int_{\Omega_{k,2}} |\partial_{x_1} u_k^1|^p \, dx, \tag{6.51}$$

$$(1-2\delta_k)\int_{\Omega_{k,2}} |\partial_{x_1} u_k^1|^p \, dx \le \int_{\Omega_2''} |\partial_{x_1} w^1|^p \, dx \le \xi_{k+1} \int_{\Omega_{k,2}} |\partial_{x_1} u_k^1|^p \, dx, \tag{6.52}$$

$$\int_{\Omega_1''} |\partial_{x_2} w^2|^p \, dx \le \frac{4\delta_k}{C_1^p} \int_{\Omega_{k,2}} |\partial_{x_2} u_k^2|^p \, dx, \tag{6.53}$$

$$(1-2\delta_k)\int_{\Omega_{k,2}} |\partial_{x_2} u_k^2|^p \, dx \le \int_{\Omega_2''} |\partial_{x_2} w^2|^p \, dx \le \xi_{k+1} \int_{\Omega_{k,2}} |\partial_{x_2} u_k^2|^p \, dx. \tag{6.54}$$

Finally, as mentioned in Remark 6.4.8 we get a collection $\mathcal{G}_2$ of mutually disjoint open sets covering $\Omega_{k,2}$ such that for all $V \in \mathcal{G}_2$, $w|_V$ is affine and every set in $\mathcal{G}_2$ is contained in a set belonging to $\mathcal{F}_k^{(2)}$.

Define $\Omega_{k+1,1} = \Omega_1' \cup \Omega_1''$, $\Omega_{k+1,2} = \Omega_2' \cup \Omega_2''$ and $u_{k+1}\colon \Omega \to \mathbb{R}^2$ as

$$u_{k+1}(x) = \begin{cases} v(x) & \text{if } x \in \Omega_{k,1};\\ w(x) & \text{if } x \in \Omega_{k,2}. \end{cases}$$

This in particular means that

$$Du_{k+1}(x) \in E_{\delta_{k+1}} \cap \widetilde{B}_{2^{k+2}}$$

for a.e. $x \in \Omega_{k+1,1}$ and

$$Du_{k+1}(x) \in (L_{\delta_{k+1}} \cup H_{\delta_{k+1}}) \cap \widetilde{B}_{2^{k+2}}$$

for a.e. $x \in \Omega_{k+1,2}$. Define

$$\mathcal{F}_{k+1} := \mathcal{G}_1 \cup \mathcal{G}_2.$$

From (6.50), for all $V \in \mathcal{F}_k$ such that $V \subset \Omega_{k,2}$, we have

$$\begin{aligned} &\mathcal{L}^2\left(V \cap \Omega_{k+1,2}\right) > (1-2\delta_k)\mathcal{L}^2\left(V\right) \quad \text{and} \\ &\quad |\partial_{x_1} u^1_{k+1}(x)| > (1-2\delta_k)|\partial_{x_1} u^1_k(x)| \quad \text{ for a.e. } x \in V \cap \Omega_{k+1,2}. \end{aligned} \tag{6.55}$$

This proves (6.37) for u_{k+1} By (6.42) and (6.48)

$$\|u_{k+1} - u_k\|_{C^\alpha(\overline{\Omega})} < 2^{-k}\varepsilon_k. \tag{6.56}$$

Then, select $0 < \varepsilon_{k+1} < \min(2^{-(k+2)}, \varepsilon_k)$ such that

$$\|Du_{k+1} * \rho_{\varepsilon_{k+1}} - Du_{k+1}\|_{L^1(\Omega)} < 2^{-(k+1)}. \tag{6.57}$$

By Lemma 6.4.1 and Corollary 6.4.7 (see Remarks 6.4.3 and 6.4.9), we have that

$$\|\partial_{x_1} u^2_{k+1}\|_{L^\infty(\Omega)}, \|\partial_{x_2} u^1_{k+1}\|_{L^\infty(\Omega)} < 1. \tag{6.58}$$

Now, in order to prove that u_{k+1} satisfies the inequalities (6.32), (6.33), (6.34) and (6.35), we estimate the following four quantities:

(i) $\displaystyle\int_{\Omega_{k+1,1}} |\partial_{x_1} u^1_{k+1}|^p \, dx$,

(ii) $\displaystyle\int_{\Omega_{k+1,2}} |\partial_{x_1} u^1_{k+1}|^p \, dx$,

(iii) $\displaystyle\int_{\Omega_{k+1,1}} |\partial_{x_2} u^2_{k+1}|^p \, dx$ and

(iv) $\displaystyle\int_{\Omega_{k+1,2}} |\partial_{x_2} u^2_{k+1}|^p \, dx$.

All these estimates are based on the fact that

$$\Omega_{k+1,1} = \Omega_1' \cup \Omega_1'' \quad \text{and} \quad \Omega_{k+1,2} = \Omega_2' \cup \Omega_2''$$

are disjoint unions combined with the inequalities obtained from Lemma 6.4.1 and Corollary 6.4.7, i.e. (6.44), (6.45), (6.46), (6.47), (6.51), (6.52), (6.53) and (6.54). In these estimates, we will see that the effect of the correction due to Corollary 6.4.7 is, in some sense, negligible. This is due to (6.30) and (6.31) which give us that δ_k is small. Recall that in the inequalities given by Corollary 6.4.7, i.e. (6.51), (6.52), (6.53) and (6.54), both δ_k and ξ_{k+1} appear. Since we already know that ξ_{k+1} is close to 1, by taking δ_k small enough, we expect the influence of the correction procedure to be negligible.

For (i), we have:

$$\int_{\Omega_{k+1,1}} |\partial_{x_1} u^1_{k+1}|^p \, dx = \int_{\Omega'_1} |\partial_{x_1} v^1|^p \, dx + \int_{\Omega''_1} |\partial_{x_1} w^1|^p \, dx.$$

Therefore,

$$\begin{aligned}
\int_{\Omega_{k+1,1}} |\partial_{x_1} u^1_{k+1}|^p \, dx &\overset{(6.44),(6.51)}{\leq} \alpha^{(1)}_{p,k+1} \int_{\Omega_{k,1}} |\partial_{x_1} u^1_k|^p \, dx + 4\delta_k C_2^p \int_{\Omega_{k,2}} |\partial_{x_1} u^1_k|^p \, dx \\
&\overset{(6.38),(6.39)}{\leq} \alpha^{(1)}_{p,k+1} \prod_{j=1}^{k} \xi_j \alpha^{(1)}_{p,j} + 4\delta_k C_2^p \prod_{j=1}^{k} \xi_j \left(\sum_{j=1}^{k} \beta_{p,j} \prod_{l=1}^{j-1} \alpha^{(1)}_{p,l} \right) \\
&\overset{(6.30)}{\leq} \alpha^{(1)}_{p,k+1} \prod_{j=1}^{k} \xi_j \alpha^{(1)}_{p,j} + (\xi_{k+1} - 1) c_\alpha^{k+1} \\
&\overset{(6.22)}{\leq} \alpha^{(1)}_{p,k+1} \prod_{j=1}^{k} \xi_j \alpha^{(1)}_{p,j} + (\xi_{k+1} - 1) \alpha^{(1)}_{p,k+1} \prod_{j=1}^{k} \xi_j \alpha^{(1)}_{p,j} \\
&\leq \prod_{j=1}^{k+1} \xi_j \alpha^{(1)}_{p,j}.
\end{aligned}$$

and

$$\int_{\Omega_{k+1,1}} |\partial_{x_1} u^1_{k+1}|^p \, dx \geq \alpha^{(1)}_{p,k+1} \int_{\Omega_{k,1}} |\partial_{x_1} u^1_k|^p \, dx \overset{(6.38)}{\geq} \prod_{j=1}^{k+1} \alpha^{(1)}_{p,j}.$$

Thus,

$$\prod_{j=1}^{k+1} \alpha^{(1)}_{p,j} \leq \int_{\Omega_{k+1,1}} |\partial_{x_1} u^1_{k+1}|^p \, dx \leq \prod_{j=1}^{k+1} \xi_j \alpha^{(1)}_{p,j}. \tag{6.59}$$

For (ii), we have:

$$\int_{\Omega_{k+1,2}} |\partial_{x_1} u^1_{k+1}|^p \, dx = \int_{\Omega'_2} |\partial_{x_1} v^1|^p \, dx + \int_{\Omega''_2} |\partial_{x_1} w^1|^p \, dx.$$

Therefore

$$\int_{\Omega_{k+1,2}} |\partial_{x_1} u^1_{k+1}|^p \, dx \overset{(6.45),(6.52)}{\leq} \beta_{p,k+1} \int_{\Omega_{k,1}} |\partial_{x_1} u^1_k|^p \, dx + \xi_{k+1} \int_{\Omega_{k,2}} |\partial_{x_1} u^1_k|^p \, dx$$

$$
\begin{aligned}
&\overset{(6.38),(6.39)}{\leq} \beta_{p,k+1}\prod_{j=1}^{k}\xi_j\alpha_{p,j}^{(1)} + \xi_{k+1}\prod_{j=1}^{k}\xi_j\left(\sum_{j=1}^{k}\beta_{p,j}\prod_{l=1}^{j-1}\alpha_{p,l}^{(1)}\right)\\
&= \prod_{j=1}^{k}\xi_j\left(\beta_{p,k+1}\prod_{j=1}^{k}\alpha_{p,j}^{(1)}\right) + \prod_{j=1}^{k+1}\xi_j\left(\sum_{j=1}^{k}\beta_{p,j}\prod_{l=1}^{j-1}\alpha_{p,l}^{(1)}\right)\\
&\leq \prod_{j=1}^{k+1}\xi_j\left(\sum_{j=1}^{k+1}\beta_{p,j}\prod_{l=1}^{j-1}\alpha_{p,l}^{(1)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\Omega_{k+1,2}}|\partial_{x_1}u_{k+1}^1|^p\,dx &\overset{(6.45),(6.52)}{\geq} \beta_{p,k+1}\int_{\Omega_{k,1}}|\partial_{x_1}u_k^1|^p\,dx + (1-2\delta_k)\int_{\Omega_{k,2}}|\partial_{x_1}u_k^1|^p\,dx\\
&\overset{(6.38),(6.39)}{\geq} \beta_{p,k+1}\prod_{j=1}^{k}\alpha_{p,j}^{(1)} + (1-2\delta_k)\prod_{j=1}^{k-1}(1-2\delta_j)\left(\sum_{j=1}^{k}\beta_{p,j}\prod_{l=1}^{j-1}\alpha_{p,l}^{(1)}\right)\\
&\geq \beta_{p,k+1}\prod_{j=1}^{k}\alpha_{p,j}^{(1)} + \prod_{j=1}^{k}(1-2\delta_j)\left(\sum_{j=1}^{k}\beta_{p,j}\prod_{l=1}^{j-1}\alpha_{p,l}^{(1)}\right)\\
&\geq \prod_{j=1}^{k}(1-2\delta_j)\left(\sum_{j=1}^{k+1}\beta_{p,j}\prod_{l=1}^{j-1}\alpha_{p,l}^{(1)}\right).
\end{aligned}
$$

Thus,

$$
\prod_{j=1}^{k}(1-2\delta_j)\left(\sum_{j=1}^{k+1}\beta_{p,j}\prod_{l=1}^{j-1}\alpha_{p,l}^{(1)}\right) \leq \int_{\Omega_{k+1,2}}|\partial_{x_1}u_{k+1}^1|^p\,dx \leq \prod_{j=1}^{k+1}\xi_j\left(\sum_{j=1}^{k+1}\beta_{p,j}\prod_{l=1}^{j-1}\alpha_{p,l}^{(1)}\right). \tag{6.60}
$$

For (iii), we have:

$$
\int_{\Omega_{k+1,1}}|\partial_{x_2}u_{k+1}^2|^p\,dx = \int_{\Omega_1'}|\partial_{x_2}v^2|^p\,dx + \int_{\Omega_1''}|\partial_{x_2}w^2|^p\,dx
$$

Therefore

$$
\begin{aligned}
\int_{\Omega_{k+1,1}}|\partial_{x_2}u_{k+1}^2|^p\,dx &\overset{(6.46),(6.53)}{\leq} \alpha_{p,k+1}^{(2)}\int_{\Omega_{k,1}}|\partial_{x_2}u_k^2|^p\,dx + \frac{4\delta_k}{C_1^p}\int_{\Omega_{k,2}}|\partial_{x_2}u_k^2|^p\,dx\\
&\overset{(6.41),(6.40)}{\leq} \alpha_{p,k+1}^{(2)}\prod_{j=1}^{k}\xi_j\alpha_{p,j}^{(2)} + \frac{4\delta_k}{C_1^p}\prod_{j=1}^{k}\xi_j\left(\sum_{j=1}^{k}\gamma_{p,j}\prod_{l=1}^{j-1}\alpha_{p,l}^{(2)}\right)
\end{aligned}
$$

$$\overset{(6.31)}{\leq} \alpha^{(2)}_{p,k+1} \prod_{j=1}^{k} \xi_j \alpha^{(2)}_{p,j} + (\xi_{k+1} - 1) c_\alpha^{k+1}$$

$$\overset{(6.22)}{\leq} \alpha^{(2)}_{p,k+1} \prod_{j=1}^{k} \xi_j \alpha^{(2)}_{p,j} + (\xi_{k+1} - 1) \alpha^{(2)}_{p,k+1} \prod_{j=1}^{k} \xi_j \alpha^{(2)}_{p,j}$$

$$\leq \prod_{j=1}^{k+1} \xi_j \alpha^{(2)}_{p,j}$$

and

$$\int_{\Omega_{k+1,1}} |\partial_{x_2} u^2_{k+1}|^p \, dx \geq \alpha^{(2)}_{p,k+1} \int_{\Omega_{k,1}} |\partial_{x_2} u^2_k|^p \, dx \overset{(6.40)}{\geq} \prod_{j=1}^{k+1} \alpha^{(2)}_{p,j}.$$

Thus,

$$\prod_{j=1}^{k+1} \alpha^{(2)}_{p,j} \leq \int_{\Omega_{k+1,1}} |\partial_{x_2} u^2_{k+1}|^p \, dx \leq \prod_{j=1}^{k+1} \xi_j \alpha^{(2)}_{p,j}. \tag{6.61}$$

For (iv), we have:

$$\int_{\Omega_{k+1,2}} |\partial_{x_2} u^2_{k+1}|^p \, dx = \int_{\Omega_2'} |\partial_{x_2} v^2|^p \, dx + \int_{\Omega_2''} |\partial_{x_2} w^2|^p \, dx$$

Therefore

$$\int_{\Omega_{k+1,2}} |\partial_{x_2} u^2_{k+1}|^p \, dx \overset{(6.47),(6.54)}{\leq} \gamma_{p,k+1} \int_{\Omega_{k,1}} |\partial_{x_2} u^2_k|^p \, dx + \xi_{k+1} \int_{\Omega_{k,2}} |\partial_{x_2} u^2_k|^p \, dx$$

$$\overset{(6.40),(6.41)}{\leq} \gamma_{p,k+1} \prod_{j=1}^{k} \xi_j \alpha^{(2)}_{p,j} + \xi_{k+1} \prod_{j=1}^{k} \xi_j \left(\sum_{j=1}^{k} \gamma_{p,j} \prod_{l=1}^{j-1} \alpha^{(2)}_{p,l} \right)$$

$$\leq \prod_{j=1}^{k} \xi_j \left(\gamma_{p,k+1} \prod_{j=1}^{k} \alpha^{(2)}_{p,j} \right) + \prod_{j=1}^{k+1} \xi_j \left(\sum_{j=1}^{k} \gamma_{p,j} \prod_{l=1}^{j-1} \alpha^{(2)}_{p,l} \right)$$

$$\leq \prod_{j=1}^{k+1} \xi_j \left(\sum_{j=1}^{k+1} \gamma_{p,j} \prod_{l=1}^{j-1} \alpha^{(2)}_{p,l} \right)$$

and

$$\int_{\Omega_{k+1,2}} |\partial_{x_2} u^2_{k+1}|^p \, dx \overset{(6.47),(6.54)}{\geq} \gamma_{p,k+1} \int_{\Omega_{k,1}} |\partial_{x_2} u^2_k|^p \, dx + (1 - 2\delta_k) \int_{\Omega_{k,2}} |\partial_{x_2} u^2_k|^p \, dx$$

$$
\begin{aligned}
&\overset{(6.40),(6.41)}{\geq} \gamma_{p,k+1} \prod_{j=1}^{k} \alpha_{p,j}^{(2)} + (1-2\delta_k) \prod_{j=1}^{k-1} (1-2\delta_j) \left(\sum_{j=1}^{k} \gamma_{p,j} \prod_{l=1}^{j-1} \alpha_{p,l}^{(2)} \right) \\
&\geq \gamma_{p,k+1} \prod_{j=1}^{k} \alpha_{p,j}^{(2)} + \prod_{j=1}^{k} (1-2\delta_j) \left(\sum_{j=1}^{k} \gamma_{p,j} \prod_{l=1}^{j-1} \alpha_{p,l}^{(2)} \right) \\
&\geq \prod_{j=1}^{k} (1-2\delta_j) \left(\sum_{j=1}^{k+1} \gamma_{p,j} \prod_{l=1}^{j-1} \alpha_{p,l}^{(2)} \right).
\end{aligned}
$$

Thus,

$$
\prod_{j=1}^{k} (1-2\delta_j) \left(\sum_{j=1}^{k+1} \gamma_{p,j} \prod_{l=1}^{j-1} \alpha_{p,l}^{(2)} \right) \leq \int_{\Omega_{k+1,2}} |\partial_{x_2} u_{k+1}^2|^p \, dx \leq \prod_{j=1}^{k+1} \xi_j \left(\sum_{j=1}^{k+1} \gamma_{p,j} \prod_{l=1}^{j-1} \alpha_{p,l}^{(2)} \right). \tag{6.62}
$$

Finally, by (6.55), (6.56), (6.57), (6.58), (6.59), (6.60), (6.61) and (6.62), u_{k+1} satisfies the desired properties described in Step 3. As a consequence, we also deduce that we can build a sequence $\{u_k\}_{k=1}^{\infty}$ of mappings which satisfies the properties stated in Step 3.

Step 5: (Conclusion) In this step we conclude the proof. From the previous steps, we now have a sequence of mappings $\{u_k\}_{k=1}^{\infty}$, two sequences of open sets $\{\Omega_{k,1}\}_{k=1}^{\infty}$ and $\{\Omega_{k,2}\}_{k=1}^{\infty}$ such that the following holds:

$$
\begin{aligned}
&\mathcal{L}^2 \left(\Omega \setminus (\Omega_{k,1} \cup \Omega_{k,2}) \right) = 0, \\
&Du_k(x) \in E_{\delta_k} \cap \widetilde{B}_{2^{k+1}} \text{ for a.e } x \in \Omega_{k,1} \text{ and} \\
&Du_k(x) \in (L_{\delta_k} \cup H_{\delta_k}) \cap \widetilde{B}_{2^{k+1}} \text{ for a.e } x \in \Omega_{k,2}.
\end{aligned}
$$

for all $k \geq 1$. By (6.32),(6.33), (6.34) and (6.35) combined with (6.25) and (6.29), the following inequalities are satisfied for all $k \geq 1$ and all $p \in [1,2]$:

$$
\prod_{j=1}^{k} \alpha_{p,j}^{(1)} \leq \int_{\Omega_{k,1}} |\partial_{x_1} u_k^1|^p \, dx \leq 2 \prod_{j=1}^{k} \alpha_{p,j}^{(1)}, \tag{6.63}
$$

$$
\frac{1}{2} \sum_{j=1}^{k} \beta_{p,j} \prod_{l=1}^{j-1} \alpha_{p,l}^{(1)} \leq \int_{\Omega_{k,2}} |\partial_{x_1} u_k^1|^p \, dx \leq 2 \sum_{j=1}^{k} \beta_{p,j} \prod_{l=1}^{j-1} \alpha_{p,l}^{(1)}, \tag{6.64}
$$

$$
\prod_{j=1}^{k} \alpha_{p,j}^{(2)} \leq \int_{\Omega_{k,1}} |\partial_{x_2} u_k^2|^p \, dx \leq 2 \prod_{j=1}^{k} \alpha_{p,j}^{(2)}, \tag{6.65}
$$

$$\frac{1}{2}\sum_{j=1}^{k}\gamma_{p,j}\prod_{l=1}^{j-1}\alpha_{p,l}^{(2)} \le \int_{\Omega_{k,2}} |\partial_{x_2} u_k^2|^p\,dx \le 2\sum_{j=1}^{k}\gamma_{p,j}\prod_{l=1}^{j-1}\alpha_{p,l}^{(2)}. \tag{6.66}$$

Since $\sup_{j\ge 1}\alpha_{1+\gamma,j}^{(1)} < 1$ (which follows from the fact that $\alpha_{1+\gamma} < 1$ and the choice of δ), the inequalities above imply

$$\sup_{k\ge 1}\left(\|\partial_{x_1} u_k^1\|_{L^{1+\gamma}(\Omega)} + \|\partial_{x_2} u_k^2\|_{L^{1+\gamma}(\Omega)}\right) < \infty.$$

Combined with the the fact that $\|\partial_{x_1} u_k^2\|_{L^\infty(\Omega)}, \|\partial_{x_2} u_k^1\|_{L^\infty(\Omega)} < 1$ for all $k \ge 1$ (see (6.36)), we deduce that there is a constant Q such that

$$\sup_{k\ge 1}\|Du_k\|_{L^{1+\gamma}(\Omega)} \le Q. \tag{6.67}$$

In addition, it follows from the construction that there is an $u \in C^\alpha(\overline{\Omega};\mathbb{R}^2)$ such that $u_k \to u$ in $C^\alpha(\overline{\Omega};\mathbb{R}^2)$. By (6.67), $\{u_k\}_{k=1}^\infty$ converges weakly in $W^{1,1+\gamma}(\Omega;\mathbb{R}^2)$ to some function $v \in W^{1,1+\gamma}(\Omega;\mathbb{R}^2)$, up to a subsequence. Thus, $u = v$, which proves that $u \in W^{1,1+\gamma}(\Omega;\mathbb{R}^2)$.

Now, we claim the following, which we will prove later.

Claim 1: $Du(x) \in K'$ for a.e. $x \in \Omega$.

By the claim we deduce that u^1 solves the equation (6.1) since $K' \subset K$. Due to the fact that $u \in W^{1,1+\gamma}(\Omega;\mathbb{R}^2)$, we have $u^1 \in W^{1,1}(\Omega;\mathbb{R}^2)$ as desired.

Now, we claim the following which we will prove later.

Claim 2: $u^1 \notin W^{1,2}(\Omega)$.

This claim allows us to conclude that u^1 is a nonenergetic solution of (6.1) as wished. It only remains to prove Claim 1 and Claim 2.

Proof of Claim 1: We begin by showing that $Du_k \to Du$ in $L^1(\Omega)$ as $k \to \infty$. To prove it, we use the same strategy as in the proof of Theorem 3.3.3. We have

$$\begin{aligned}\|Du_k - Du\|_{L^1(\Omega)} \le \underbrace{\|Du_k - Du_k * \rho_{\varepsilon_k}\|_{L^1(\Omega)}}_{\le 2^{-i}\to 0 \text{ as } i\to\infty} + \|Du_k * \rho_{\varepsilon_k} - Du * \rho_{\varepsilon_k}\|_{L^1(\Omega)} \\ + \underbrace{\|Du * \rho_{\varepsilon_k} - Du\|_{L^1(\Omega)}}_{\to 0 \text{ as } k\to\infty},\end{aligned}$$

so it suffices to prove that the second term converges to 0 as $k \to \infty$. Inside Ω, we have

$$D(u_k - u) * \rho_{\varepsilon_k} = (u_k - u) * D\rho_{\varepsilon_k}$$

and thus the remaining term can be estimated by

$$\begin{aligned}
&\|(u_k - u) * D\rho_{\varepsilon_k}\|_{L^1(\Omega)} \\
&\leq \| D\rho_{\varepsilon_k}\|_{L^1(B_1)}\|u_k - u\|_{L^1(\Omega)} \\
&\leq \frac{Q_1}{\varepsilon_k}\|u_k - u\|_{L^\infty(\Omega)}
\end{aligned}$$

for some Q_1. In addition,

$$\|u_k - u\|_{L^\infty(\Omega)} \leq \sum_{j\geq k}\|u_{j+1} - u_j\|_{L^\infty(\Omega)} \leq \sum_{j=k}^{\infty} 2^{-j}\varepsilon_j < 2\frac{\varepsilon_k}{2^k}.$$

Therefore,

$$\|Du_k * \rho_{\varepsilon_k} - Du * \rho_{\varepsilon_k}\|_{L^1(\Omega)} \leq \frac{2Q_1}{2^k} \leq 2^{1-k}Q_1$$

which converges to 0 as $k \to \infty$. This proves that $Du_i \to Du$ in $L^1(\Omega; M^{2\times 2})$. Now, we will use this fact to show that $Du(x) \in K' \cup E$ for a.e. $x \in \Omega$. For each $k \geq 1$, define the sets

$$\mathcal{U}_k := (E_{\delta_k} \cup L_{\delta_k} \cup H_{\delta_k}) \cap \widetilde{B}_{2^{k+1}}.$$

It follows from the construction described earlier in the proof that

$$Du_k(x) \in \mathcal{U}_k \quad \text{for a.e. } x \in \Omega$$

for all $k \geq 1$. In addition, by direct computations, we notice that

$$\sup_{X\in\mathcal{U}_k} \operatorname{dist}\left(X, K' \cup E\right) \leq 2^{k+2}\delta_k \overset{(6.26)}{\leq} \frac{1}{2^k} \to 0$$

as $k \to \infty$. As in the proof of Theorem 3.3.3, this allows us to conclude that

$$Du(x) \in K' \cup E \quad \text{for a.e. } x \in \Omega.$$

Now, it only remains to show that $Du(x) \in K'$ for a.e. $x \in \Omega$. In order to prove this, it suffices to show that $\mathcal{L}^2\left(\{x \in \Omega : Du(x) \in E\}\right) = 0$. Notice that for $\varepsilon > 0$ sufficiently small, we have for any $k \geq 1$:

$$\begin{aligned}
&\{x \in \Omega : Du(x) \in E\} \\
&\subset \{x \in \Omega : |Du_k(x) - Du(x)| > \varepsilon, Du(x) \in E\} \\
&\qquad \cup \{x \in \Omega : |Du_k(x) - Du(x)| \leq \varepsilon, Du(x) \in E\} \\
&\subset \{x \in \Omega : |Du_k(x) - Du(x)| > \varepsilon\} \cup \{x \in \Omega : |Du_k(x) - Du(x)| \leq \varepsilon, \partial_{x_1}u_k^1, \partial_{x_2}u_k^2 \geq 0\}
\end{aligned}$$

$$\subset \{x \in \Omega : |Du_k(x) - Du(x)| > \varepsilon\} \cup \{x \in \Omega : \partial_{x_1} u_k^1, \partial_{x_2} u_k^2 \geq 0\}$$
$$\subset \{x \in \Omega : |Du_k(x) - Du(x)| > \varepsilon\} \cup \Omega_{k,1}.$$

Thus

$$\begin{aligned}&\mathcal{L}^2\left(\{x \in \Omega : Du(x) \in E\}\right)\\ &\leq \mathcal{L}^2\left(\{x \in \Omega : |Du_k(x) - Du(x)| > \varepsilon\}\right) + \mathcal{L}^2\left(\Omega_{k,1}\right).\end{aligned}$$

The first term converges to 0 as $k \to \infty$ because $Du_k \to Du$ in measure. The second term converges to 0 as $k \to \infty$ by combining (6.63) with the fact that $\partial_{x_1} u_k^1 > 1/5$ a.e. on $\Omega_{k,1}$ for all $k \geq 1$. Thus $\mathcal{L}^2\left(\{x \in \Omega : Du(x) \in E\}\right) = 0$ which proves that

$$Du(x) \in K' \text{ for a.e. } x \in \Omega.$$

This proves the Claim 1.

Proof of Claim 2: Let $k \geq 1$ and $V \in \mathcal{F}_k$ such that $V \subset \Omega_{k,2}$, i.e. $V \in \mathcal{F}_k^{(2)}$. By (6.37),

$$\mathcal{L}^2\left(V \cap \Omega_{k+1,2}\right) > (1 - 2\delta_k)\mathcal{L}^2\left(V\right).$$

Then since the open sets of $\mathcal{F}_{k+1}$ cover Ω and every set of $\mathcal{F}_{k+1}$ is contained in a set of $\mathcal{F}_k$, we deduce that there exists a countable subset of $\mathcal{F}_{k+1}$ consisting of open sets which cover $V \cap \Omega_{k,2}$ up to a set of measure 0. Then again, by applying (6.37) to each one of these sets, we deduce that

$$\mathcal{L}^2\left(V \cap \Omega_{k+1,2} \cap \Omega_{k+2,2}\right) > (1 - 2\delta_k)(1 - 2\delta_{k+1})\mathcal{L}^2\left(V\right).$$

By induction, we have

$$\mathcal{L}^2\left(V \cap \bigcap_{l>k} \Omega_{l,2}\right) > \prod_{i=k}^{\infty}(1 - 2\delta_i)\mathcal{L}^2\left(V\right).$$

Due to (6.29), we have

$$\mathcal{L}^2\left(V \cap \bigcap_{l>k} \Omega_{l,2}\right) > \frac{1}{2}\mathcal{L}^2\left(V\right).$$

Define

$$W_V^k := V \cap \bigcap_{l>k} \Omega_{l,2}.$$

By (6.37)

$$|\partial_{x_1} u_{j+1}^1(x)| \geq (1 - 2\delta_j)|\partial_{x_1} u_j^1(x)| \quad \text{for a.e. } x \in W_V^k \text{ and all } j \geq k.$$

As a consequence, by (6.29),

$$|\partial_{x_1} u_l^1(x)| \geq \left(\prod_{j=k}^{l-1}(1-2\delta_j)\right)|\partial_{x_1} u_k^1(x)| \geq \frac{1}{2}|\partial_{x_1} u_k^1(x)| \tag{6.68}$$

for a.e. $x \in W_V^k$, for all $l > k$. Since $Du_j \to Du$ in $L^1(\Omega; M^{2\times 2})$, there exists a subsequence such that the gradients converge a.e. on Ω. Therefore, due to (6.68),

$$|\partial_{x_1} u^1(x)| \geq \frac{1}{2}|\partial_{x_1} u_k^1(x)| \quad \text{for a.e. } x \in W_V^k.$$

Thus,

$$\int_{W_V^k} |\partial_{x_1} u^1|^2 \, dx \geq \frac{1}{4}\int_{W_V^k} |\partial_{x_1} u_k^1|^2 \, dx.$$

Finally, since $W_V^k \subset V$, $\partial_{x_1} u_k^1$ is constant on V and $\mathcal{L}^2\left(W_V^k\right) > \mathcal{L}^2\left(V\right)/2$, we obtain

$$\int_V |\partial_{x_1} u^1|^2 \, dx \geq \int_{W_V^k} |\partial_{x_1} u^1|^2 \, dx \geq \frac{1}{4}\int_{W_V^k} |\partial_{x_1} u_k^1|^2 \, dx \geq \frac{1}{8}\int_V |\partial_{x_1} u_k^1|^2 \, dx.$$

Since this last inequality holds for all $V \in \mathcal{F}_k$ such that $V \subset \Omega_{k,2}$ and because $\Omega_{k,2}$ can be covered by mutually disjoint open sets in $\mathcal{F}_k$ up to a set of measure 0, we deduce that

$$\int_\Omega |\partial_{x_1} u^1|^2 \, dx \geq \int_{\Omega_{k,2}} |\partial_{x_1} u^1|^2 \, dx \geq \frac{1}{8}\int_{\Omega_{k,2}} |\partial_{x_1} u_k^1|^2 \, dx \overset{(6.64)}{\geq} \frac{1}{16}\sum_{j=1}^{k} \beta_{2,j} \prod_{l=1}^{j-1} \alpha_{2,l}^{(1)}.$$

As $\inf_{j\geq 1} \alpha_{2,j}^{(1)} > 1$ and $\beta_{2,j} > c_\beta > 0$, we obtain

$$\int_{\Omega_{k,2}} |\partial_{x_1} u_k^1|^2 \, dx \to \infty \quad \text{as } k \to \infty.$$

Thus,

$$\int_\Omega |\partial_{x_1} u^1|^2 \, dx = \infty$$

and hence $u^1 \notin W^{1,2}(\Omega)$. This proves Claim 2 and finishes the proof of the theorem.

□

Finally, we point out two corollaries of the scheme introduced in the proof above. First, we see that by choosing C_1 and C_2 in an suitable fashion, we can prove the following:

Corollary 6.4.11. *For any $p \in [1, 2)$, there is an $f \in C^\infty(\mathbb{R}^2)$ uniformly convex and with uniformly bounded Hessian, i.e. there exists $0 < \lambda < \Lambda$ such that*

$$\lambda I \leq D^2 f(x) \leq \Lambda I \quad \forall x \in \mathbb{R}^2,$$

for which the equation

$$\operatorname{div} Df(Du) = 0$$

admits a nonenergetic solution belonging to the Sobolev space $W^{1,p}(\Omega)$.

The scheme can be adapted to prove:

Corollary 6.4.12. *There is an $f \in C^2(\mathbb{R}^2)$ uniformly convex and with uniformly bounded Hessian, i.e. there exists $0 < \lambda < \Lambda$ such that*

$$\lambda I \leq D^2 f(x) \leq \Lambda I \quad \forall x \in \mathbb{R}^2,$$

for which the equation

$$\operatorname{div} Df(Du) = 0$$

admits infinitely many nonenergetic solutions.

Chapter 7

Conclusion

We summarise the main points of the thesis in this chapter. In Chapter 3, we introduced the convex integration method by Müller and Šverák. This methods allows us to solve differential inclusions

$$Du \in K.$$

Subsequent to this, in Chapter 4, we applied the convex integration method to elliptic systems, in particular to the Euler-Lagrange equation generated by a quasiconvex functional, i.e

$$\operatorname{div} DF(Du) = 0$$

where F is quasiconvex. In particular, we proved the existence of very irregular solutions in Theorem 4.0.1. Important aspects of the proof of this result were to write the Euler-Lagrange equation in the form of a differential inclusion and the notion of T_N-configurations. Then in Chapter 5, we proved the same result for the Euler-Lagrange equation generated by a polyconvex functional. Finally, in Chapter 6, we introduced L^p-convex integration invented by Faraco. A central concept of this method are staircase laminates. We applied this method to the following scalar-valued elliptic PDE:

$$\operatorname{div} Df(Du) = 0$$

where f has uniformly bounded Hessian and is uniformly convex. In Theorem 6.0.2, we proved the existence of very weak solutions.

To summarise this thesis in one sentence: we have explored the convex integration method and how we can use it to build counterexamples to regularity and integrability questions.

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