PROBABILISTIC FATIGUE LIFE MODEL OF COMPOSITE LAMINAS UNDER CYCLE LOADING

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Abstract: *A probabilistic model for estimating the fatigue life of composite laminas is developed in this paper. Based on the statistical characteristics of the fatigue life (mean value and standard deviation), a linear probabilistic model is proposed along with the statistical uncertainty, which allows for efficiently predicting the distribution of fatigue life at any stress levels for a constant stress ratio. By comparing predicted and experimental results for several composite materials, the proposed model is validated and its accuracy is further evaluated through the use of goodness-of-fit statistical tests. In addition, with a second-order polynomial model, the probabilistic constant life diagrams for different combinations of stress level and stress ratio are developed. Meanwhile, the probabilistic S-N curves for any stress level and stress ratio without experimental measurements are derived, and it is demonstrated that the probabilistic S-N curves agree well with the validation data, which could greatly save the cost of extra fatigue tests.*

Keywords: Fatigue; Composites; Probabilistic modeling; Statistical analysis.

1. Introduction

Fatigue failures occur very often in composite structures [1-3]. A large scatter in the fatigue properties has been observed during experimental investigations on fibrous composite materials [1], [4]. This kind of variability can be attributed to defects induced during manufacture and the inhomogeneity of composite materials. One of the most common approaches to calculate the fatigue strengths is using S-N curves [5, 6], which is a deterministic equation obtained by fitting the measured fatigue life data. Early in 1910, Basquin [7] suggested a power-law equation between fatigue strength and cycles to failure, which is still used by many researchers today. Generally, the fatigue performance of a transversely isotropic composite lamina can be revealed by the S-N curves established in the three principal directions of the materials, which is also an indispensable step in determining the fatigue life of the off-axis lamina [8-9]. Experimental tests on various stress levels are essential to derive the S-N curves, and to further consider the mean stress effect in fatigue life predictions, fatigue data for several R ($R=S_{min}/S_{max}$) values should be measured to construct the well-known constant life diagrams (CLD) [10-12]. However, to cope with the uncertainties introduced in the fatigue properties and derive a rational model of the fatigue life, statistical methods should be applied.

Statistical treatment of the fatigue strength and the developed stresses as random variables is suggested in probabilistic consideration of several fatigue failure criterions [8, 9]. Several stochastic variables, such as the elastic properties of the composite material, the geometrical characteristics and the applied loads, have been defined [13, 14] and introduced in a functional formulation that provides statistical descriptions of the developed stresses, which can be found in several works [15, 16]. However, despite a number of probabilistic fatigue life models have been proposed, very few works exist on the formulation of probabilistic CLD and even fewer for the probabilistic modeling of fatigue life with statistical characteristics of experimental data.

To deal with the problem at hand, a parameterization of the statistical characteristics of the fatigue life (mean value and standard deviation) with respect to the stress level *Sa*, and the ratio *R* based on experimental evidence is proposed in this work. Empirical models are developed including the statistical characteristics of fatigue life in closed-form equations, which is able to estimate the distribution of fatigue life for every combination of *Sa* and *R* and obtain probabilistic S-N curves as well as CLD diagrams.

2. Probabilistic fatigue life model

2.1 Linear probabilistic fatigue model

One of the most popular probabilistic models for the fatigue life data is the Lognormal distribution. Assuming that the fatigue life is described by the same type of parametric distribution independently of the applied stress amplitude *Sa* and the ratio *R*, for the Lognormal distribution, the fatigue life model can be written as

$$ln\, N = m + sZ, \tag{1}$$

where *N* is the allowable number of cycles at the stress amplitude *Sa* in a specific ratio *R*, *m* and *s* are the parameters of the Lognormal distribution, and *Z* is a random variable with the standard normal distribution. The mean value, α_1, and standard deviation, α_2, of the allowable number of cycles *N* can be expressed in terms of the parameters *m* and *s* [17]

$$\alpha_1 = e^{m+\frac{s^2}{2}}, \tag{2}$$

$$\alpha_2 = \sqrt{\left(e^{s^2} - 1\right)e^{2m+s^2}}. \tag{3}$$

Due to the limited number of experimental data, statistical uncertainty inserts in the model of Eq. (1). Thus, the variables α_1, α_2 can be thought of as random variables themselves. The interest is mainly focused to express α_1, α_2 (or their statistical characteristics, mean value $E(\alpha_i)$ and standard deviation $D(\alpha_i)$, if statistical uncertainty is considered) by means of simple algebraic equations with respect to the applied stress amplitude *Sa* and ratio *R* based on available experimental data.

For constant ratio *R*, a linear relation between the natural logarithm of α_i and the natural logarithm of the applied stress amplitude *Sa* is assumed as

$$ln\, \alpha_i = E(ln\, \alpha_i) + D(ln\, \alpha_i)Z_i, \quad i = 1,2, \tag{4}$$

where

$$E(ln\, \alpha_i) = c_i\, ln\, S_a + d_i, \quad i = 1,2, \tag{5}$$

where Z_i denotes random variables with the standard normal distribution, c_i and d_i are the regression coefficients. As $ln\alpha_i$ are direct statistical characteristics of testing data, they may be correlated variables. If they are correlated, accordingly the Z_i are also correlated ones with a correlation coefficient ρ.

Combining Eq. (4) and Eq. (5), we get

$$ln\,\alpha_i = c_i\,ln\,S_a + d_i + D(ln\,\alpha_i)Z_i, \quad i = 1{,}2, \tag{6}$$

where the term $D(ln\alpha_i)Z_i$ represents the random error accounting for the statistical uncertainty. Both α_1 and α_2 are random variables following the same distribution as N while the random component Z_i represents the statistical uncertainty inserted by the limited number of the experimental data. Ignoring the statistical uncertainty, the Eqs. (4)-(5) are greatly simplified and become

$$ln\,\alpha_i = c_i\,ln\,S_a + d_i, \quad i = 1{,}2. \tag{7}$$

When the statistical uncertainty is not considered, Eq. (7) can be used to determine the relation between $ln\alpha_i$ and $lnSa$. Once the linear relation is determined, the distribution of the fatigue life can be estimated by inserting Eq. (6) or Eq. (7) into Eqs. (1)-(3).

Through linear regression, we can easily obtain the slope and intercept of the proposed model in Eq. (7) (or Eq. (6) if considering the statistical uncertainty). Thus, for the case that the statistical uncertainty is omitted, the value of α_i for a specific stress level Sa can be directly computed using the proposed linear model, and then the parameters of the distribution can be further obtained. For Lognormal distribution, by solving Eqs. (2)-(3), the parameters m and s can be expressed as functions of α_i

$$m = \ln\left(\frac{\alpha_1^2}{\sqrt{\alpha_1^2+\alpha_2^2}}\right), \tag{8}$$

$$s = \sqrt{\ln\left(1+\frac{\alpha_2^2}{\alpha_1^2}\right)}. \tag{9}$$

Once the parameters of a distribution are determined, the cumulative distribution function (CDF) of the fatigue life N can be thereby obtained.

If the statistical uncertainty is considered, Eq. (6) is used for generating random numbers of α_i. As α_1 and α_2 are correlated variables, a covariance matrix (Cov) is required to generate random samples of α_i

$$Cov = \begin{bmatrix} D(ln\alpha_1)^2 & \rho D(ln\alpha_1)D(ln\alpha_2) \\ \rho D(ln\alpha_1)D(ln\alpha_2) & D(ln\alpha_2)^2 \end{bmatrix}. \tag{10}$$

The value of $D(ln\alpha_i)$ can be determined as the mean error of linear regression for each stress ratio, and the value of ρ can be estimated by bootstrap method. With these values, random samples for [α_1, α_2] with a given size of s_1 are generated. Then the parameters of Lognormal distribution and Weibull distribution corresponding to each pair of α_1 and α_2 are calculated by Eqs. (8)-(10). For each pair of obtained parameters ([m, s]), random numbers in the size of s_2 from the Lognormal distribution are generated with corresponding distribution parameters, which brings a group of random variables with the total size of s_1* s_2. By calculating the mean value and standard deviation of all generated random numbers of N, re-estimation of α_i is achieved on amplified samples where the statistical uncertainty is included. Then the parameters of distribution can be determined following Eqs. (8)-(9), on the basis of estimated values of α_i. In this way, the cumulative distribution function (CDF) of the fatigue life N, where the statistical uncertainty is considered, is obtained with calculated values of distribution parameters.

2.2 Probabilistic constant life model

To derive a general probabilistic constant life model, a second-order polynomial model is established to describe each curve in the probabilistic CLD graphs. For constant values of $ln\alpha_i$ (or $E(ln\alpha_i)$ if statistical uncertainty is considered), it can be written as

$$S_a = e_i R^2 + f_i R + g_i,\ \ i = 1{,}2, \tag{11}$$

The parameters in Eq. (11) are also assumed to be well described by second-order polynomial models in the following form:

$$\left.\begin{aligned} e_i &= m_i Y^2 + n_i Y + p_i \\ f_i &= s_i Y^2 + t_i Y + r_i \\ g_i &= u_i Y^2 + v_i Y + w_i \end{aligned}\right\},\ \ i = 1{,}2. \tag{12}$$

When omitting the statistical uncertainty,

$$Y = ln\alpha_i,\ \ i = 1{,}2, \tag{13}$$

and for the case that the statistical uncertainty is considered

$$Y = E(ln\alpha_i),\ \ i = 1{,}2. \tag{14}$$

Assuming that *N* obeys the lognormal distribution, the term *(ln(N)-m)/s* satisfies the standard normal distribution, which is

$$\frac{\ln(N)-m}{s} \sim N(0{,}1). \tag{15}$$

By means of the inverse relation of the standard normal cumulative distribution function, the *Pth* percentile point of fatigue data *r* can be calculated using the following formula:

$$r = \Phi^{-1}(P) = \frac{\ln(N)-m}{s}, \tag{16}$$

where $\Phi^{-1}(P)$ is the inverse of standard normal distribution for a certain probability *P*. The distribution parameters *m* and *s* can be expressed as functions of the stress level *Sa*, which are

$$m = F(S_a) = \ln\left(\frac{e^{2(c_1 lnS_a + d_1)}}{\sqrt{e^{2(c_1 ln\ \ {}_a + d_1)} + e^{2(c_2 lnS_a + d_2)}}}\right), \tag{17}$$

$$s = G(S_a) = \sqrt{\ln(1 + e^{2(c_2 ln\ \ {}_a + d_2 - c_1 ln\ \ {}_a - d_2)})}. \tag{18}$$

Substituting Eq.(17) and Eq. (18) into Eq. (16), there is

$$\ln(N) = F(S_a) + G(S_a) * r. \tag{19}$$

For a given value of *N* and failure probability *P*, the value of *r* can be obtained through Eq. (16), and the corresponding stress level *Sa* can be obtained by solving Eq. (19).

Furthermore, a probabilistic approach is introduced here to derive probabilistic *S-N* curves for a specific ratio *R* without experimental measurements including the statistical uncertainty. As the fatigue life *N* is assumed to satisfy Lognormal distribution, it should be pointed out that the distribution parameters *m* and *s* refer to the mean value and standard deviation of a normal distribution which *lnN* satisfies. With the properties of normal distribution, the probabilistic S-N curves can be derived once the distribution parameters *m* and *s* for each stress level are determined. For the case when omitting statistical uncertainty, the values of *m* and *s* can be directly derived once the coefficients of Eq. (12) are obtained through fitting. If the statistical uncertainty is considered, statistical methods proposed in the previous section should be conducted to generate random samples, and the values of *m* and *s* can be determined after re-estimation of α_i.

3. Validation of the proposed model

Experimental data on GFRP multidirectional laminate with a stacking sequence of [90/0/±45/0]S from the DOE/MSU database [18] are used for validation. First, the proposed model was examined on the tension-tension testing data of material DD16 for *R*=0.5, where data in three stress levels are used for fitting. Fig. 1(a) displays linear fitting results of this dataset, and the predicted distribution of fatigue life in the second stress level (marked red in Fig. 1(a)) was plotted in Fig. 1(b). Obviously, almost all the curves in Fig. 1(b) fit well with the experimental data, indicating the proposed model is capable of predicting the fatigue life distribution in the stress level without experimental measurement.

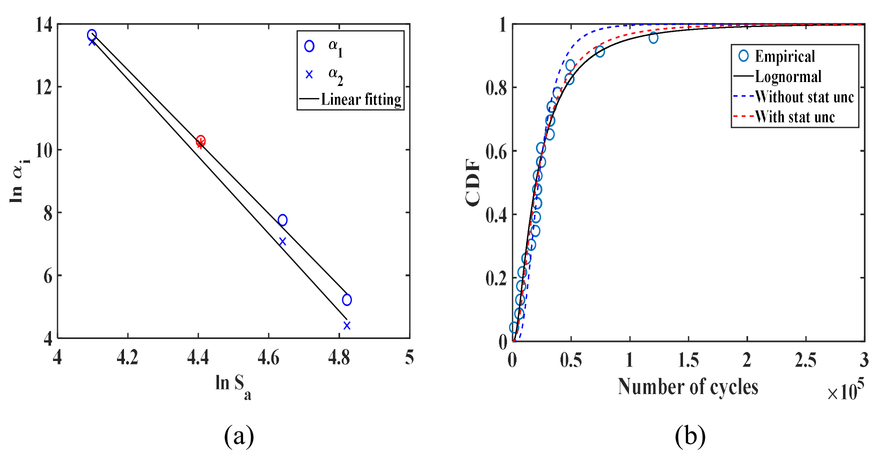
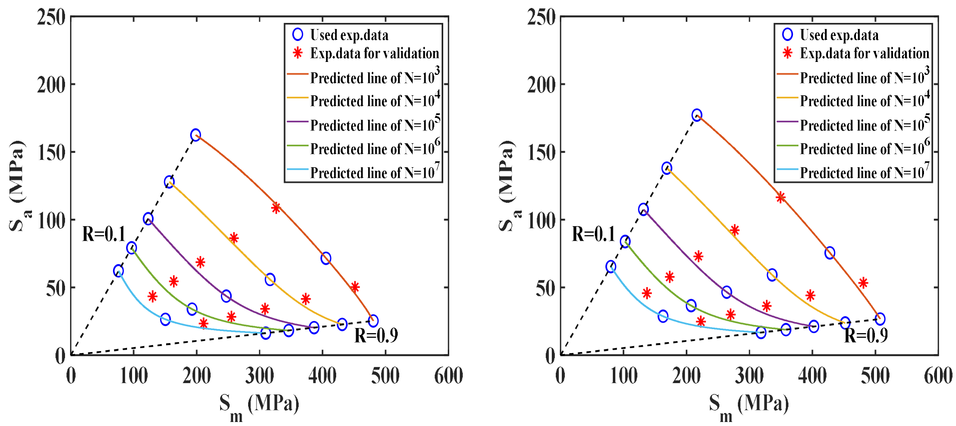


Figure 1. Fitting results on testing data of material DD16 for R=0.5: (a) linear regression results; (b) CDF assuming Lognormal distribution.

Fig. 2 presents the established probabilistic CLD model in and *Sm-Sa* plane. With the proposed second-order polynomial model, the fatigue behavior for any ratio *R* in the range of measured data (e.g., 0.1<*R*<0.9 for this dataset) can be described, and the mean stress effect on the fatigue behavior of the examined material can be reflected as well. Fig. 3 displays the probabilistic S-N curves for *R*=0.8, with the failure probability ranging from 2.28% to 97.72%. As most datapoints fall into the predicted band, the developed P-S-N curves are quite accurate. Besides, when considering statistical uncertainty, the derived S-N curves exhibits more variation and the observed wider range between the upper and lower bonds provides better predictions.



(a) (b)

Figure 2 Probabilistic CLD in in Sm- Sa plane of constant N for 0.1<R<0.9 under the failure probability: (a) P=50%; (b) P=90%.

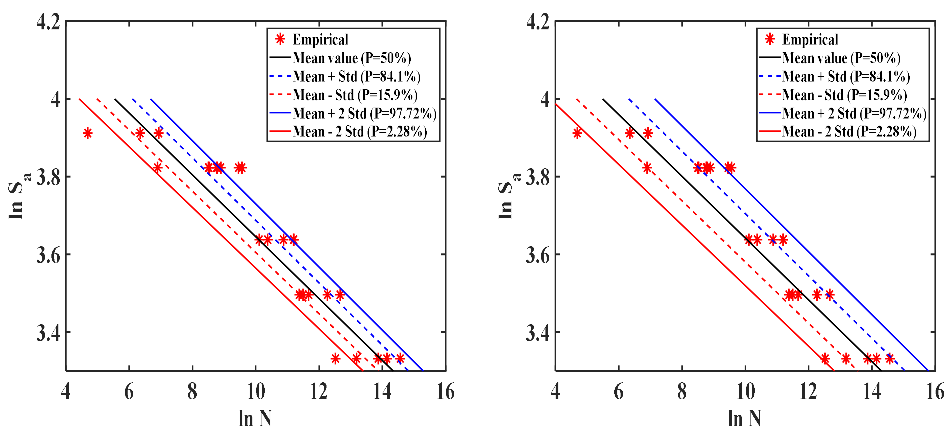


Fig. 3 Probabilistic S-N curves in a ln-ln plot for R=0.8: (a)omitting statistical uncertainty, (b) considering statistical uncertainty.

4. Conclusions

A probabilistic fatigue life model was proposed based on the statistical characteristics (mean value and the standard deviation) of the number of cycles to failure with respect to the applied stress level, *S* and stress ratio, *R*. The results obtained from this study can be summarized as follows:

- The relation between statistical characteristics of experimental data and the applied stress level was well described by the linear model. Statistical procedures of deriving the fatigue life distribution in certain stress level without measured data were suggested based on the proposed linear model, where the statistical uncertainty was included.
- By comparing predicted distributions with experimental data, it was concluded that both the quantity and the quality of used data have an impact on the performance of the proposed model.

- A probabilistic CLD model was developed without considering the statistical uncertainty, and proved quite successful for the estimation of the fatigue life under a certain failure probability, for different stress levels *Sa* and stress ratio *R* within the range of measured data.

- A probabilistic approach was introduced to develop probabilistic S-N curves for specific *R*-ratios considering the statistical uncertainty. The derived probabilistic S-N curves are well corroborated by the experimental data, while it was shown that the impact of data quantity and quality was reduced when the statistical uncertainty was included.

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