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Semiclassical methods in conformal field theories scrutinized by the epsilon-expansion

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When a wise man does not understand, he says: “I do not understand.”
The fool and the uncultured are ashamed of their ignorance.
They remain silent when a question could bring them wisdom.
— Frank Herbert, *The Godmakers*

À mes parents.

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Sincerely, Gil

Foreword

The material of this thesis is strongly based on the three following papers:

[1] G. Badel, G. Cuomo, A. Monin, and R. Rattazzi, "The Epsilon Expansion Meets Semiclassics," *JHEP* **11** (2019) 110, `arXiv:1909.01269 [hep-th]`,

[2] G. Badel, G. Cuomo, A. Monin, and R. Rattazzi, "Feynman diagrams and the large charge expansion in $3-\varepsilon$ dimensions," *Phys. Lett.* **B802** (2020) 135202, `arXiv:1911.08505 [hep-th]`,

[3] G. Badel, A. Monin, and R. Rattazzi, "Identifying Large Charge Operators," `arXiv:2207.08919 [hep-th]`.

The chapters are organised as follows:

- Chapter 1 is a review of existing literature on the large charge expansion in Conformal Field Theories,
- Chapter 2 is mostly based on [1], with part of the development of section 2.4 being also taken from [3]. Its final section 2.5 is based on [2],
- Chapter 3 is based on the second half of [3],
- Chapter 4 is based on the first half of [3].
- Finally, the conclusion is a summary of results and a discussion of some possible extensions of this work, including some research other authors based on [1, 2].

Abstract

Conformal Field Theories (CFTs) are crucial for our understanding of Quantum Field Theory (QFT). Because of their powerful symmetry properties, they play the role of signposts in the space of QFTs. Any method that gives us information about their structure, and lets us compute their observables, is therefore of great interest. In this thesis we explore the large quantum number sector of CFTs, by describing a semiclassical expansion approach. The idea is to describe the theory in terms of fluctuations around a classical background, which corresponds to a superfluid state of finite charge density. We detail the implementation of the method in the case of $U(1)$-invariant lagrangian CFTs defined in the epsilon-expansion. After introducing the method for generic correlators, we illustrate it by performing the computation of several observables.

First, we compute the scaling dimension of the lowest operator having a given large charge n under the $U(1)$ symmetry. We demonstrate how the semiclassical result in this case bridges the gap between the naive diagrammatic computation (which fails at too large n) and the general large-charge expansion of CFTs (which is only valid for n large enough).

Second, we apply the method to the computation of 3- and 4-point functions involving the same operator. This lets us derive some of the OPE (Operator Product Expansion) coefficients.

Finally, we consider the rest of the spectrum of charge-n operators, and propose a way to classify them by studying their free-theory equivalent. In the free theory, we construct the complete set of primary operators with number of derivatives bounded by the charge. We also find a mapping between the excited states of the superfluid and the vacuum states of standard quantization, which is valid when the spin of said states is bounded by the square root of the charge.

Keywords: large charge, epsilon-expansion, conformal field theories, semiclassical expansion, effective field theory, superfluid, Feynman diagrams, primary operators, vortices.

Résumé

Les théories de champs conformes (Conformal Field Theories, CFTs) sont cruciales pour notre compréhension de la théorie quantique des champs (Quantum Field Theory, QFT). En raison de leurs puissantes propriétés de symétrie, elles jouent le rôle de repères d'orientation dans l'espace des QFTs. Toute méthode qui nous donne des informations sur leur structure, et nous permet de calculer leurs observables, est donc d'un grand intérêt. Dans cette thèse, nous explorons le secteur à grands nombres quantiques des CFTs, en décrivant une approche par expansion semiclassique. L'idée est de décrire la théorie en termes de fluctuations autour d'un fond classique, qui correspond à un état superfluide de densité de charge finie. Nous détaillons l'implémentation de la méthode dans le cas de CFTs lagrangiennes $U(1)$-invariantes définies dans l'expansion en epsilon. Après avoir présenté la méthode pour des corrélateurs génériques, nous l'illustrons en effectuant le calcul de plusieurs observables.

Tout d'abord, nous calculons la dimension d'échelle de l'opérateur le plus bas ayant une grande charge n donnée pour la symétrie $U(1)$. Nous démontrons comment le résultat semiclassique dans ce cas comble le fossé entre le calcul diagrammatique naïf (qui échoue pour une trop grande charge n) et l'expansion à grande charge générale des CFTs (qui n'est valable que pour une charge n suffisamment grande).

Deuxièmement, nous appliquons la méthode au calcul des fonctions à 3 et 4 points impliquant le même opérateur. Ceci nous permet de dériver certains des coefficients de l'OPE (Operator Product Expansion, expansion de produit d'opérateurs).

Enfin, nous considérons le reste du spectre des opérateurs de charge n, et proposons un moyen de les classifier en étudiant leur équivalent en théorie libre. Dans la théorie libre, nous construisons la totalité de l'ensemble des opérateurs primaires dont le nombre de dérivées est borné par la charge. Nous trouvons également une correspondance entre les états excités du superfluide et les états du vide de la quantification standard, qui est valide lorsque le spin de ces états est borné par la racine carrée de la charge.

Contents

Introduction

The mathematical framework of Quantum Field Theory (QFT) is ubiquitous in contemporary theoretical physics. It was developed throughout the 20th century, the first instance of a QFT being Quantum Electrodynamics (QED), devised to describe processes involving light and electrons. In the modern point of view, it is understood as the logical consequence of the union of the fundamental principles of special relativity (Lorentz invariance) and of quantum mechanics. One of the features of this language is that it naturally depicts the observed phenomenon of particle creation and annihilation, while quantum mechanics alone can only account for a fixed number of particles. Therefore, it is obviously the appropriate framework to understand particle physics, in particular experiments consisting of colliding particles at relativistic speeds. However, over the decades this methodology has proven to be capable of building very accurate models in various areas of theoretical physics.

One prominent example of a QFT is the Standard Model of particle physics (SM), the current most fundamental theory of Nature. Let us cite one of its many great successes by mentioning the hypothesizing of the Higgs field and its associated scalar boson [4, 5, 6, 7, 8, 9, 10, 11, 12], whose existence was confirmed little more than ten years ago [13, 14]. This model has been tested with an enormous amount of experimental data, showing a very high accuracy [15]. On the other hand, the contradictions between this model and the experiment are what drives the theoretical research of possible physics Beyond the Standard Model (BSM). Many of those models are phrased as QFTs as well. This includes, to name a few, the composite Higgs [16], supersymmetry (SUSY) [17], extra dimensions [18, 19, 20, 21], and many proposals for the identity of Dark Matter [22, 23].

Another domain that makes extensive use of quantum field theory is cosmology, for the description of phenomena in the early universe such as inflation [24] and baryogenesis [25, 26]. One further example of a discipline that heavily relies on QFT as a tool (often in its non-relativistic version, NRQFT) is condensed matter physics [27]. The remarkable fact about these examples is that they cover a wide range of energy scales (or equivalently, length scales, since the two notions are inversely proportional because of the uncertainty principle).

Renormalization and scale invariance

However, QFT did not immediately impose itself as a universally accepted paradigm. One of the obstacles it had to face was that its perturbative computations had an apparently meaningless infinite result. The cause is that quantum field theory describes systems of infinitely many interacting quantum degrees of freedom.

The solution to this problem, called *renormalization* actually turned out as an important physical feature of QFT [28, 29, 30]. First, one has to artificially modify the theory by introducing a parameter which regularizes the divergences (for example a modification of the dimension of spacetime as $d = 4 - \varepsilon$). Then, the predictions of the modified theory can be compared to a fixed value given at energy scale M. Adjusting the parameters of the theory (couplings) for a few of these predictions to be correct, one indeed obtains a theory capable of predicting other finite observables accurately as the regularization is removed ($\varepsilon \to 0$).

The choice of a renormalization scale M, which was made arbitrarily, should not affect the physical predictions of the theory. Therefore, choosing a different scale M' can only imply different values for the renormalized couplings of the theory in such a way that the final predictions for all observables are unaffected. We say that the variants of the theory renormalized at different scales are related by a *renormalization group* (RG) transformation.

The invariance of the physical observables under the RG transformations is a powerful set of constraints that lead to the *Callan-Symanzik equations* [31, 32]. These inform us about the real dependence of the observables on the positions or momenta they take as input. In other words, the Callan-Symanzik equations describe the modifications of the dynamics of the system if we “zoom” in or out to a different scale.

One key component of the Callan-Symanzik equations are the β-functions, which describe the rate of change of the couplings under a change of renormalization scale M. Their solution can be expressed by defining a notion of *running coupling*, which depends on the scale of momenta at play. For example, this explains the observed phenomenon that in QED, the electron charge appears to be higher in experiments at high-energy such as collisions, than it is in low-energy, large-distance experiments.

More than its relevance to physically correct predictions, the RG has a huge influence on our current understanding of quantum field theory as a whole. The running of the couplings at varying scales can be seen as a trajectory inside the space of all quantum field theories. The action of the renormalization group can then be expressed as a flow within this space, naturally called *Renormalization Group flow.* Thus, the RG flow provides a way to explore this space of theories, and study some general properties they have. In particular, some theories share a similar low-energy limit, when one considers observables computed at momenta far below any characteristic scale of the theory (such as masses of particles, for example). This can be seen by noticing that their RG flow in the IR all converge towards the same theory. We say these theories belong to the same *universality class.*

The convergence point is a fixed point of the renormalization group. This means a theory for which the couplings values are such that the beta functions vanish. Physically, it means that this theory is invariant under scaling transformations. For this reason, scaling-invariant theories are of great interest since they are the fixed points of different RG flows. One application of that is the study of continuous phase transitions in statistical mechanics and of quantum critical points of condensed matter systems. Indeed, at their critical conditions, these systems have an infinite correlation length, meaning they have no characteristic length scale – they become scale invariant. Thus they can be studied by scale-invariant QFTs, and the corresponding RG flows.

Conformal Field Theories

It is generically the case that scaling-invariant QFTs actually possess a larger spacetime symmetry group, called the conformal group [33, 34, 35, 36][1]. It is formed by complementing the Poincaré group (made of translations, rotations and boosts) with scaling and with the special conformal transformations. In general, conformal transformations can be defined as the spacetime transformations that locally modify length scales without affecting the angles. In $d = 2$ dimensions the conformal group is larger, in fact infinite-dimensional, since any holomorphic transformation is conformal. We call QFTs equipped with this enhanced symmetry Conformal Field Theories (CFTs).

Besides the insight they offer into the RG and into continuous phase transitions, CFTs have many other important applications. They are one of the basic tools of String Theory, where Polyakov's world-sheet quantum field theory reduces to a CFT once gauge-fixed [47]. They are one of the two sides of the AdS/CFT correspondence, which is a duality between string theories in Anti-de Sitter $(d+1)$-dimensional spacetime (AdS) and Conformal Field Theories defined on the d-dimensional boundary of AdS [48, 49, 50]. This equivalence is one of the most powerful tools at our disposal to study string theory, which is currently our best candidate for a theory of quantum gravity. The great advantage is that it provides a non-perturbative way to define string theory in an asymptotically AdS spacetime, by looking at it from the CFT side of the correspondence, which is more under control. As an even more recent application, let us cite the development of the formalism of Celestial Amplitudes, a holographic equivalence between gravitational scattering amplitudes in asymptotically flat $(d+2)$-dimensional spacetime and a CFT

[1] This has been proven to be always true in $d = 2$ dimensions under some technical assumptions [37, 38] and proven perturbatively in $d = 4$ [39, 40], with good progress towards a non-perturbative proof [41, 42, 43]. There is currently no known proof in other dimensions. Notice that there are known examples of physically meaningful scale-invariant but non-conformal theories, for which some of the assumptions are broken [44, 45]. We point the interested reader to the review [46] for precise statements.

living on the d-dimensional celestial sphere [51, 52, 53, 54].

Thankfully, CFTs are among the best-understood of all quantum field theories. Indeed, the consequence of their extended spacetime symmetries is some very stringent constraints on the form of correlators. The functional form of 2- and 3-point functions is completely fixed, up to some normalization constants. 4- and higher-point functions can be reduced to 3-point functions using the Operator Product Expansions (OPE). In other words, all n-point functions can be deduced from the knowledge of the spectrum of operator scaling dimensions and OPE coefficients. We call *CFT data* this set of numbers which completely characterises a conformal field theory. In $d = 2$ dimensions, the infinite-dimensional algebra leads to some models being completely solvable. An other useful consequence of scale invariance is the existence of radial quantization, and the fact that it provides a correspondence between quantum states and local operators.

Before discussing in more detail the techniques used to study Conformal Field Theories specifically, let us review the computational techniques and limitations that exist in general QFTs, and more precisely the distinction between weakly coupled and strongly coupled observables.

Weakly coupled versus strongly coupled observables

Despite its many successes, performing Quantum Field Theory computations is by no means a trivial task, and in some cases, there is currently no known analytical method to derive the answer.

One major simplifying property is when the theory at hand is *weakly coupled*. To best explain what that means, we refer to the Path Integral (PI) formulation of QFT [55, 56, 57], which is a generalization of the PI formalism introduced by Feynman in quantum mechanics [58, 59]. Consider a single scalar field $\phi(t, \vec{x})$ in a d-dimensional spacetime ($t \in \mathbb{R}, \vec{x} \in \mathbb{R}^{d-1}$), then its dynamics is described by an action functional $S[\phi]$. In a classical field theory, the physically valid field evolutions are those which are stationary points (also called *saddles*) of the action functional. A unique, deterministic evolution is then selected once we fix boundary conditions for the field at initial and final times, $\phi(t_i, \vec{x})$ and $\phi(t_f, \vec{x})$. On the other hand in the quantum case, a path integral describes the *probability amplitude* of evolution from an inital field configuration $\phi_i(\vec{x}) = \phi(t_i, \vec{x})$ to a final one $\phi_f(\vec{x}) = \phi(t_f, \vec{x})$ as

$$\left\langle \phi_f \left| e^{-iH(t_f - t_i)} \right| \phi_i \right\rangle = \int_{\phi_i}^{\phi_f} \mathcal{D}\phi \; e^{iS[\phi]} \,. \tag{1}$$

This can be interpreted as a weighted average over all arbitrary field evolutions with boundary conditions ϕ_i and ϕ_f. More general observables are obtained by inserting local operators into this path integral.

We say a path integral is *weakly coupled* if it can be approximated by a systematic expansion around some leading classical field evolution $\phi_{cl.}(t, \vec{x})$. In that case the main contribution to the PI $\mathcal{I}_{cl.}$ is given by simply evaluating the action on $\phi_{cl.}$. Then, quantum corrections $\mathcal{I}_{qu.}$ can be computed by quantizing the fluctuations around $\phi_{cl.}$. We call this a *semiclassical expansion.* Note that finite solutions to the equations of motion differ largely from the trivial vacuum solution. Thus they can be considered as being made of a large number of quanta. In cases where such an expansion is not possible, or useless since all terms have the same order of magnitude, the PI is called *strongly coupled*. Notice that the property of being weakly or strongly coupled is defined for a particular path integral within a QFT that can involve different sectors and regimes. Indeed, a given QFT can have some PI that are weakly coupled and some other that are strongly coupled, depending on the operators inserted into those path integrals and the boundary conditions. For example, Quantum Chromodynamics (QCD), the QFT that describes strong interaction, is strongly coupled at energy scale $E \sim$ GeV. However, to describe long distance physics, it is possible to integrate out the degrees of freedom with large quantum fluctuations, and be left with a weakly coupled Effective Field Theory (EFT), which corresponds to the low-energy excitations of the pions.

Interestingly, the path integral point of view brings an explanation to the emergence of a classical behaviour in a quantum mechanical system. Indeed, in the weakly coupled case, we can distinguish observables for which the quantum part is a small correction to the classical part $\mathcal{I}_{qu.} \ll \mathcal{I}_{cl.}$, and call them *classical observables.* On the other hand, observables for which these two parts are comparable, or the quantum component dominates, are called *quantum observables.* In the strongly coupled case, since there is no semiclassical expansion, all observable are of the quantum mechanical type.

Let us consider an example. In the simplest case of a free scalar theory, where the action is quadratic, all path integrals can be computed exactly since they are gaussian. The result always corresponds to a semiclassical expansion; all PIs are weakly coupled. When the boundary states are the vacuum (or made of a few excitations of it), the semiclassical saddle around which the expansion is performed is the trivial vacuum solution $\phi_{cl.} \equiv 0$. As a consequence, the classical part of those path integrals is zero, thus these observables are quantum mechanical. However, it is also possible to choose boundary states (analogous to the coherent states of the harmonic oscillator) for which the expansion is around a non-trivial saddle $\phi_{cl.} \neq 0$, and has a leading classical result.

Perturbation theory

Many interacting QFTs fall into the strongly coupled regime for all their observables. When that is the case, the only way to obtain the physical predictions of the theory is to resort to numerical techniques, like simulation on a lattice. Hamiltonian truncation is an other numerical, non-perturbative framework which currently receives a lot of attention.

The only situation where a weak coupling expansion is generally available, is for systems that are well approximated by a free theory. There, observables which involve a small number of excitations of the vacuum are given at leading order by the free theory quantum prediction, corrected by some subleading terms stemming from the interactions, and that are suppressed by a small parameter λ. Consider for example the free scalar perturbed by a $\lambda\phi^4$ interaction term in 4 dimensions

$$S[\phi] = \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right) . \tag{2}$$

Then, for example the 2-to-2 scattering (S-matrix element) can be computed (in either path integral formulation or canonical quantization) from

$$\begin{aligned}\lim_{(t_f-t_i)\to\infty} & \left\langle \vec{p}_C, \vec{p}_D \left| e^{-iH(t_f-t_i)} \right| \vec{p}_A, \vec{p}_B \right\rangle \\ =& 4E_AE_B(2\pi)^6\Big(\delta^{(3)}(\vec{p}_A-\vec{p}_C)\delta^{(3)}(\vec{p}_B-\vec{p}_D)+\delta^{(3)}(\vec{p}_A-\vec{p}_D)\delta^{(3)}(\vec{p}_B-\vec{p}_C)\Big) \\ & - i\lambda(2\pi)^4\delta^{(3)}(\vec{p}_C+\vec{p}_D-\vec{p}_B-\vec{p}_B) \;+\; O(\lambda^2)\,,\end{aligned} \tag{3}$$

where the state $|\vec{p}_A,\vec{p}_B\rangle$,$|\vec{p}_C,\vec{p}_D\rangle$ are asymptotic 2-particle states. The first line of the result is the leading, quantum contribution (since we are expanding around the vacuum $\phi_{cl.} \equiv 0$) while the second line is the leading quantum correction (tree level). The $O(\lambda^2)$ stands for higher-order corrections (loops) in the small-λ expansion.

Again, let us stress that this expansion around the vacuum solution is not a valid way to compute all path integrals of the theory. For example, consider as initial state a single, virtual particle with energy nm in its rest frame, that decays into a large number n of real bosons at rest in that frame. This 1-to-n process can be represented by the diagram of figure 1.

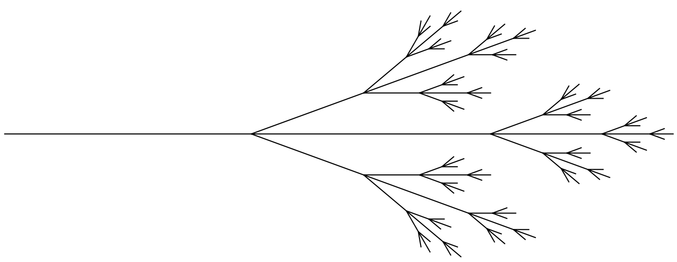


Figure 1: Tree-level diagram for the decay of one very off-shell particle into n real particles.

The amplitude for this decay at tree-level is given as [60, 61, 62, 63]

$$\mathcal{A}^{\text{tree}}(1^* \to n) = n!\left(\frac{\lambda}{2(4!)m^2}\right)^{\frac{n-1}{2}} \tag{4}$$

where the factor $n!$ comes from the combinatorics of this diagram with many legs. Loop corrections can also be computed, and assuming $n \gg 1$, the leading term at loop level k is proportional to $\lambda^k n^{2k}$. Neglecting the subleading terms at each loop level, one gets

$$\mathcal{A}^{\text{leading loops}}(1^* \to n) = \mathcal{A}^{\text{tree}}(1^* \to n) \left(1 + B_d \lambda n^2 + \frac{1}{2} B_d^2 \lambda^2 n^4 + \cdots + \frac{1}{k!} (B_d \lambda n^2)^k + \dots \right) \tag{5}$$

where B_d is a constant that depends on the dimension. Taking this result at face value, it seems to indicate that this observable is strongly coupled if $\lambda n^2 \gtrsim 1$, since quantum corrections then become important. However, the precise form of the coefficients means the leading term of all loop levels can be resummed, and the amplitude can be rewritten as

$$\begin{aligned} \mathcal{A}^{\text{leading loops}}(1^* \to n) &= n! \left(\frac{\lambda}{2(4!)m^2}\right)^{\frac{n-1}{2}} e^{B_d \lambda n^2} \\ &\approx \frac{1}{\lambda} \sqrt{2\pi} \left(\frac{1}{2(4!)m^2}\right)^{\frac{n-1}{2}} e^{\frac{1}{\lambda}(\lambda n \ln(\lambda n) - \lambda n + B_d \lambda^2 n^2) + \frac{1}{2} \ln(\lambda n)} . \end{aligned} \tag{6}$$

This new form is symptomatic of a semiclassical expansion, that can be performed in the double scaling limit

$$\lambda \to 0, \quad n \to \infty, \quad \lambda n = \text{ fixed.} \tag{7}$$

The system remains weakly coupled in that regime, even if $\lambda n \gg 1$, albeit in a non-trivial way. The difficult problem of finding a suitable classical solution $\phi_{cl.}$ in order to perform the semiclassical expansion has been studied in [64] (see [65] for a more recent review; notice that some technical difficulties remain to be clarified [66, 67]). Pushing this expansion to further orders should restore the terms we neglected above at each loop level.

Semiclassical methods represent one of the main tools to investigate non-perturbative phenomena in QFT. Besides multiparticle production, we can also cite vacuum decay [68, 69], instantons [70], and topological defects [71, 72, 73] as a partial list of the different declinations of this methodology.

Symmetries and spontaneous symmetry breaking

It is generally the case that finding a suitable saddle over which to build the semiclassical expansion is hard. However, like many things in QFT, it can be considerably simplified by exploiting the symmetries of the theory. The states and operators of the theory are classified according to representations of the symmetry group, and the way they transform is specified by their quantum numbers (also called *charges*). While the vacuum is generally invariant under those transformations, non-trivial solutions to the classical equations of motion are transformed into different solutions. Indeed as we noted, they are associated with states composed by many quanta, which in general have large quantum numbers. Thus the search for semiclassical saddles is guided by the symmetries of the theory, and the quantum numbers of the observable of interest. The symmetries for which the observable has vanishing quantum numbers will also leave invariant the associated saddle.

On the other hand, the fact that the saddle has some non-vanishing quantum number implies that the semiclassical expansion around it spontaneously breaks part of the symmetry. An important general consequence of spontaneous symmetry breaking is Goldstone's theorem, which states that for each broken continuous symmetry in the semiclassical expansion (each symmetry which is no longer manifest), there will be one ungaped bosonic degree of freedom. Moreover, when the non-vanishing quantum numbers are large, the energy gaps of the other degrees of freedom become large as well. This indicates that, for energies which are not far above the saddle energy compared to those gaps, an effective description may be available for the Goldstone modes, in which the gaped degrees of freedom are integrated out.

In fact, one can expect that most observables with large quantum numbers, being associated with states made of many quanta, have a weakly coupled semiclassical expansion for their path integral, around a non-trivial saddle with the same quantum numbers. Even in cases where we do not succeed in finding a good saddle in the complete theory, for example when we don't have a lagrangian description for the theory, the symmetry breaking pattern can guide us towards a useful effective description for the Goldstone modes.

Recently, this general expectation has been realized in the context of Conformal Field Theories [74, 75, 76], in an approach called the *Large Charge Expansion*. This is a simplifying limit for correlators involving two operators with large quantum numbers, which can be placed at the origin and at infinity modulo conformal transformations. Conformal symmetry is advantageous in this context because of the state-operator correspondence deriving from radial quantization, which is not present in a generic QFT. Thanks to this correspondence, the operators with large quantum numbers can be translated to states used as the endpoints of the path integrals, more precisely the lowest-energy states sharing the same quantum numbers as the operators. In this limit, the boundary states can be traded for the simplest possible states with the correct quantum numbers, for which the other symmetries are unbroken. This means a simple saddle can be found which respects the unbroken symmetries. In particular, in the usual conformal mapping of the theory to the cylinder, one can assume the saddle to have a homogeneous charge density. This simple classical solution can be interpreted as a superfluid state, making a nice connection with condensed matter physics, whose intuition will prove useful in this analysis.

To motivate the interest of this method, let us now come back to the subject of Conformal Field Theories, and present the existing computational techniques in that domain.

Computation techniques in CFT

Like any quantum field theory, some CFTs can be studied in perturbation theory. In particular, weakly coupled CFTs that are the IR fixed point of a renormalization group flow. In that case as we mentioned earlier, thanks to the universality property, any QFT flowing to that fixed point can be equivalently used.

On the other hand, many interesting CFTs are strongly coupled. As for generic QFT, a lattice simulation or hamiltonian truncation can be a good way to make progress in that case. Except there are CFTs which don't even have a lagrangian description, being defined solely by their CFT data, and therefore are out of reach of these methods.

However, thanks to the properties of CFTs, some powerful methods are available to investigate them. The AdS/CFT duality can be used, since subject to some additional constraints the dual to a strongly coupled CFT is a weakly coupled theory in AdS. However, not all CFTs possess such a holographic dual, hence this method is not very general. We also have at our disposal a very powerful non-perturbative method called the Conformal Bootstrap [77, 78, 34, 36, 79]. The idea is to use the general properties of CFTs (unitarity, associativity of the operator product expansion, and existence of a traceless stress-energy tensor) to impose constraints on the spectrum of the theory and the values of OPE coefficients. This turns out to be sufficient to numerically compute the CFT data of some theories with a huge precision.

Any new means of investigation of CFTs is in our interest, especially if non-perturbative. For example, the early applications of the conformal bootstrap focused on operators of the lowest scaling dimensions in conformal theories. Hence, they are well complemented by methods focusing on large scaling dimensions. The large charge expansion framework falls into that category, since operators with large quantum numbers also have large scaling dimensions.

Structure of the thesis

This thesis reports a study that was performed with the goal of investigating the large-charge sector of Conformal Field Theories in some particular cases where a perturbative expansion is available. More precisely, our research focused on the scalar $U(1)$ model in $4-\varepsilon$ or $3-\varepsilon$ dimensions at the Wilson-Fisher fixed point. The advantage of this theory is that it is a UV-complete lagrangian description of a CFT. In the $\varepsilon \ll 1$ limit, observables can be calculated using perturbation theory.

The first quantity of which we present the calculation is the scaling dimension of the lowest-dimensional operator with charge n under the $U(1)$ symmetry, which we denote ϕ^n. Very much like in the case of multiparticle amplitudes described above, a naive computation, performed in the usual way of quantizing the theory around the trivial vacuum, does not lead to a meaningful perturbative expansion when operators with large $U(1)$ charge n are involved. This is a consequence of the combinatorics of Feynman diagrams at play. Indeed, one notices in this case that the leading large-n parts of the coefficients of the perturbative powers series in ε scale like $\varepsilon^k n^{k+1}$, rendering the expansion not reliable if the product εn is not small.

However, it is possible to organize the computation as a semiclassical expansion, by considering the double scaling limit

$$\varepsilon \to 0, \quad n \to \infty, \quad \varepsilon n = \text{ fixed.} \tag{8}$$

This means that the leading contribution to large-n observables is given by evaluating them classically, using a non-trivial solution to the classical equations of motion (saddle) as input. One can interpret this classical solution as representing the collective effect of many quanta being excited to reach that high quantum number. We will see that in this case, the classical solution corresponds to a superfluid state.

Subleading contributions can then be systematically computed by quantizing the theory around the semiclassical background, describing the fields as a quantum fluctuation added to the semiclassical solution. This results in a perturbative expansion where the quantum corrections can be represented and computed as Feynman diagrams for the fluctuations. The result takes the form of a power series in ε, where εn appears as a finite parameter.

Thanks to this technique, observables involving operators with large quantum numbers can be computed systematically, and here in a more direct way than what can be done for generic Conformal Field Theories, where the large charge expansion relies on the effective field theory of Goldstone bosons. Thus, this study offers more insight into the properties of this expansion, and demonstrates explicitly how it takes into account the gaped degrees of freedom when the computation is performed in the UV-complete picture. Our results can be expanded in the small εn limit where they match the standard perturbative computation, as well as in the large εn limit, where they match with the large charge EFT results. Thus they furnish the correct interpolation between those two regimes of the theory.

This thesis is outlined as follows. In the first chapter, we will introduce the subject of the large charge sector of CFTs, reviewing the recent literature and focusing on the EFT approach. The other chapters present original research projects. Chapter 2 describes the use of the semiclassical method when applied to the ε-expansion of $U(1)$ Wilson-Fisher fixed point to compute the anomalous dimension of the ϕ^n operator. This was performed both in the ϕ^4 theory in $4-\varepsilon$ dimensions [1], and in the ϕ^6 theory in $3-\varepsilon$ dimensions [2]. We comment on some applications of our results, including boosting perturbative Feynman diagram computations up to five loops, and comparison with Montecarlo simulations at $\varepsilon = 1$. Chapter 3 focuses on the $d = 4 - \varepsilon$ theory with quartic interaction, and presents the computation of three- and four-point functions of scalar operators. We explain how OPE coefficients can be derived from the conformal block decomposition of the latter [3]. In chapter 4, we present an investigation of the spectrum of fluctuations around the semiclassical vacuum, which is also presented in [3]. In free theory and quantization around the trivial vacuum, we classified the spectrum in terms of primary and descendent operators, providing a full basis of all primaries which have a total number of derivatives lower than the charge. We also described a mapping between trivial vacuum excitations and fluctuations around the semiclassical saddle, and discussed the limit of validity of this mapping at large spins. Finally, we conclude the dissertation by summarizing the results and proposing future work directions. Appendices provide some technical precisions and detailed derivations.

1 Conformal Field Theories at large charge

In this chapter, we review the basics of the semiclassical large charge expansion in CFTs. This idea was proposed in [74, 76, 75], and the discussion in this chapter is also based on [80, 81]. This study was motivated by the previous observation of simplifying limits in the large quantum sector of different CFTs [82, 83, 84, 85]. Furthermore, some results of the conformal bootstrap at large spin indicated that might be a general property of the structures of CFTs [86, 87].

As is the case in the rest of this thesis, we discuss the case of operators charged under an abelian $U(1)$ internal symmetry group. It can be generalized to larger internal groups and to operators with large spin. We also focus on the problem of computing the scaling dimension Δ_n of the lowest scalar operator $\mathcal{O}_n$ with given charge n under the $U(1)$ symmetry. The method can also be used to compute more observables of the CFT, for example the OPE coefficients of $\mathcal{O}_n$. On the other hand, we consider a general dimension d, since the next chapters will consider theories in different number of dimensions.

This approach is based on two key ideas:

1. The operator-state correspondence, which is a feature of radial quantization of conformal field theories, and
2. The description of the sector of states with large charge n by the effective field theory of a generalized superfluid.

1.1 Leading behaviour: dimensional analysis

The leading contribution to the scaling dimension Δ_n can be estimated directly by a simple dimensional argument, which is the following. In a CFT defined on the d-dimensional euclidean plane $\mathbb{R}^d$, one can apply radial quantization and use Weyl invariance to map the theory to the cylinder $\mathbb{R} \times S^{d-1}$. Parametrizing $\mathbb{R}^d$ by polar coordinates (r, θ), where

θ collectively denotes the coordinates on S^{d-1}, and $\mathbb{R} \times S^{d-1}$ by (τ, θ), the mapping is simply given by

$$r = R\, e^{\tau/R} \tag{1.1}$$

with R the sphere radius. The volume of the sphere is then

$$R^{d-1}\Omega_{d-1} = R^{d-1}\frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{1.2}$$

where Ω_{d-1} is the volume of the unit sphere.

In radial quantization, the state-operator correspondence associates to any local operator acting on the vacuum, a state in the Hilbert space, such that the energy of the state is proportional to the scaling dimension of the operator. We denote $|n\rangle$ the state created by operator $\mathcal{O}_n$. It has the same quantum numbers as $\mathcal{O}_n$, and is thus the lowest-energy charge-n state, with energy

$$E_n = \frac{\Delta_n}{R}. \tag{1.3}$$

At large charge n, we can assume that the state $|n\rangle$ has an important semi-classical part, corresponding to a condensed matter state with homogeneous charge density ρ_n up to the characteristic energy scale

$$\rho_n = \frac{n}{R^{d-1}\Omega_{d-1}} \quad \Longrightarrow \quad (\rho_n)^{\frac{1}{d-1}} \sim \frac{n^{\frac{1}{d-1}}}{R}.$$

This scale is parametrically larger than the scale associated to the sphere $\frac{1}{R}$. Thus the same scale should[1] be associated to the energy density $\mathcal{E}_n$, which yields

$$\begin{aligned}
\rho_n &= \frac{n}{R^{d-1}\Omega_{d-1}} &&\Longrightarrow && (\rho_n)^{\frac{1}{d-1}} \sim \frac{n^{\frac{1}{d-1}}}{R} \\
\mathcal{E}_n &= \frac{\Delta_n}{R^{d}\Omega_{d-1}} &&\Longrightarrow && (\mathcal{E}_n)^{\frac{1}{d}} \sim \frac{(\Delta_n)^{\frac{1}{d}}}{R} \\
(\rho_n)^{\frac{1}{d-1}} &\sim (\mathcal{E}_n)^{\frac{1}{d}} &&\Longrightarrow && \Delta_n \sim Q^{\frac{d}{d-1}}.
\end{aligned} \tag{1.4}$$

This power law is the leading contribution to the dimension at large charge n.

1.2 Corrections: superfluid EFT for Goldstone bosons

To estimate the behaviour of corrections to (1.4), it is necessary to analyse the semiclassical expansion of path integrals in the theory. Note that some of the steps will be illuminated in section 2.4 when we describe the analog computation in the ε-expansion in which all

[1] There are exceptions to this rule, in which case the leading scaling turns out to be different from (1.4). This happens in supersymmetric theories that have a manifold of exactly flat directions [74].

the steps are fully explicit.

All observables of the CFT can be related to correlators. Therefore, our goal is to understand $(N+2)$-point functions of the form

$$\langle \mathcal{O}_{-n}(\tau_f, \theta_f)\ \mathcal{O}_N(\tau_N, \theta_N) \dots \mathcal{O}_1(\tau_1, \theta_1)\ \mathcal{O}_n(\tau_i, \theta_i) \rangle, \tag{1.5}$$

where $\mathcal{O}_{-n}$ is the hermitian conjugate operator to $\mathcal{O}_n$, with opposite charge, and the other $\mathcal{O}_i$ are any operators with finite quantum numbers. Using the conformal symmetry, we can move the point $x_i \to 0$, equivalently $\tau_i \to -\infty$ and then the point $x_f \to \infty$, so $\tau_f \to +\infty$, and the state-operator correspondence will then translate the operators $\mathcal{O}_n, \mathcal{O}_{-n}$ into states

$$\lim_{\tau_i \to -\infty} \mathcal{O}_n(\tau_i, \theta_i)|0\rangle = e^{\Delta_n \tau_i / R}|n\rangle \equiv |n, \tau_i\rangle \tag{1.6}$$

$$\lim_{\tau_f \to \infty} \langle 0|\mathcal{O}_{-n}(\tau_f, \theta_f) = \langle n|e^{-\Delta_n \tau_f / R} \equiv \langle n, \tau_f| \tag{1.7}$$

In fact, since $|n\rangle$ is the state with lowest energy with charge n, we can make use of any (non-orthogonal) charge-n state $|\psi_n\rangle$, since they have the same asymptotic behaviour

$$\lim_{\tau_i \to -\infty} e^{H\tau_i}|\psi_n\rangle = e^{\Delta_n \tau_i}\langle n|\psi_n\rangle\ |n\rangle \tag{1.8}$$

$$\lim_{\tau_f \to \infty} \langle \psi_n|e^{-H\tau_f} = e^{-\Delta_n \tau_f}\langle \psi_n|n\rangle\ \langle n|. \tag{1.9}$$

This means the computation of (1.5) is equivalent to computing

$$\langle \psi_n|\mathcal{O}_N(\tau_N, \theta_N) \dots \mathcal{O}_1(\tau_1, \theta_1)|\psi_n\rangle. \tag{1.10}$$

In particular, the desired scaling dimension is given by

$$\langle \psi_n|e^{-H(\tau_f - \tau_i)}|\psi_n\rangle = \mathcal{N}\, e^{-\Delta_n(\tau_f - \tau_i)/R}, \tag{1.11}$$

where $\mathcal{N}$ is a normalization constant that does not depend on the τ's.

This is where the assumption of the superfluid effective description comes in. For large n, we expect that the correlator (1.5) can be approximated by a semiclassical path integral, whose lagrangian we can construct by an Effective Field Theory argument, following constraints dictated by the pattern of symmetry breaking. In this case, in line with the picture of section 1.1 we expect a homogeneous state on the sphere, such that the rotation group $SO(d)$ is preserved. Clearly the two insertion points at $\tau_{f,i} \to \pm\infty$ break translations and special conformal transformations, and the charge density breaks the internal $U(1)$ symmetry, but a combination of time translation and charge Q is preserved

$$\bar{H} \equiv H + \mu R\, Q, \tag{1.12}$$

which means the symmetry breaking pattern is

$$SO(d+1,1) \times U(1) \to SO(d) \times \bar{H}. \tag{1.13}$$

This is the simplest[2] and most natural symmetry breaking pattern, and it is known to correspond to the EFT of a superfluid [88, 89, 90]. The constant μ is the chemical potential.

This symmetry breaking pattern, with one broken internal symmetry generator, indicates the presence of exactly one Goldstone boson, all other degrees of freedom having a finite gap[3]. If we consider excitations with low enough energy, we can therefore assume that these degrees of freedom are integrated out, and only the scalar Goldstone, which we denote $\chi(\tau,\theta)$ is relevant. The remaining symmetry (1.12) suggests an expansion of the form $\chi(\tau,\theta) = -i\mu\tau + \pi(\tau,\theta)$, where the action of the $U(1)$ symmetry is a constant shift in the field χ. Once the effective lagrangian for the field χ is built, Noether's theorem can be used to derive the relation between μ and n.

The shift symmetry means that the EFT lagrangian must be constructed as a function of $\partial_\mu\chi$. The effective action for the Goldstone mode can be systematically constructed, taking into account all constraints from symmetries, using the Callan-Coleman-Wess-Zumino (CCWZ) construction [92, 93, 94, 75], yielding

$$\begin{aligned} S[\chi] = &- c_1 \int \mathrm{d}\tau\, \mathrm{d}^{d-1}\theta\ \sqrt{g}(\partial\chi)^d \\ &+ c_2 \int \mathrm{d}\tau\, \mathrm{d}^{d-1}\theta\ \sqrt{g}(\partial\chi)^d \left[\frac{\mathcal{R}}{(\partial\chi)^2} + (d-1)(d-2)\frac{[\nabla_\mu(\partial\chi)]^2}{(\partial\chi)^4}\right] \\ &- c_3 \int \mathrm{d}\tau\, \mathrm{d}^{d-1}\theta\ \sqrt{g}(\partial\chi)^d \left[\mathcal{R}_{\mu\nu}\frac{\partial^\mu\chi\partial^\nu\chi}{(\partial\chi)^4} + (d-1)(d-2)\frac{[\partial^\mu\chi\nabla_\mu(\partial\chi)]^2}{(\partial\chi)^6}\right. \\ &\qquad\qquad \left. + (d-2)\nabla_\mu\left[\frac{\partial^\mu\chi\partial^\nu\chi}{(\partial\chi)^2}\right]\frac{\nabla_\nu(\partial\chi)}{(\partial\chi)^3}\right] + O\left((\partial\chi)^d\frac{\nabla^4}{(\partial\chi)^4}\right), \end{aligned} \tag{1.14}$$

where $(\partial\chi) = (-g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi)^{1/2}$, and ∇_μ is the standard covariant derivative, $\mathcal{R}_{\mu\nu}$ is the Ricci tensor and $\mathcal{R}$ the Ricci scalar deriving from the cylinder metric $g_{\mu\nu}$. Note also that we discarded total derivatives as well as terms vanishing under the leading order

[2]In theories with large internal symmetry groups, there exist symmetry breaking patterns with homogeneous states which are not a superfluid, for instance it could be a Fermi liquid. In those cases, the construction of the EFT does not follow the same rules as with superfluids, and it is less understood how to proceed [75].

[3]Assuming there is no other symmetry, such as supersymmetry, to prevent it [91].

equations of motion (given by the first line only) [95]. The c_i's are Wilson coefficients, which are not determined by the EFT but depend on the specific underlying theory.

We are now fully armed to compute the path integral (1.11). The choice of the boundary state $|\psi_n\rangle$ decides what boundary conditions are set for the path integral with action (1.14). As we said, we make the choice which corresponds to a homogeneous field value, obtaining the path integral

$$\langle\psi_n|e^{-H(\tau_f-\tau_i)}|\psi_n\rangle = \int \mathcal{D}\chi \exp\left(-S[\chi] - i\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \int \mathrm{d}^{d-1}\theta\sqrt{g}\frac{n}{R^{d-1}\Omega_{d-1}}\dot{\chi}\right) \quad (1.15)$$

where the second term comes from the boundary conditions. This path integral can now be semiclassically expanded. The first step is searching for a saddle of the equations of motion, and as expected it is given by $\chi = -i\mu\tau + \pi_0$. Variating the action with respect to the boundary values, one gets the condition that fixes the relation between μ and the charge, as

$$\frac{n}{R^{d-1}\Omega_{d-1}} = i\frac{\partial\mathcal{L}}{\partial\dot{\chi}} = c_1 d\mu^{d-1} - c_2(d-2)\mu^{d-3}\mathcal{R} + O(\mu^{d-5}), \quad (1.16)$$

which at large n can be solved perturbatively to find

$$R\mu = \left(\frac{n}{c_1 d\Omega_{d-1}}\right)^{\frac{1}{d-1}}\left[1 + \frac{c_2(d-2)^2}{c_1 d}\left(\frac{n}{c_1 d\Omega_{d-1}}\right)^{-\frac{2}{d-1}} + O\left(\left(\frac{n}{c_1 d\Omega_{d-1}}\right)^{-\frac{4}{d-1}}\right)\right]. \quad (1.17)$$

The leading order contribution to the dimension Δ_n is now given by evaluating the action in (1.15) on the solution $\chi = -i\mu\tau + \pi_0$, yielding

$$\Delta_n^{(0)} = \alpha_1\, n^{\frac{d}{d-1}} + \alpha_2\, n^{\frac{d-2}{d-1}} + O\left(n^{\frac{d-4}{d-1}}\right), \quad (1.18)$$

where the α's are constants depending on the Wilson coefficients as

$$\alpha_1 = \frac{c_1(d-1)\Omega_{d-1}}{(c_1 d\,\Omega_{d-1})^{\frac{d}{d-1}}}, \qquad \alpha_2 = \frac{c_2(d-1)(d-2)\Omega_{d-1}}{(c_1 d\,\Omega_{d-1})^{\frac{d-2}{d-1}}}. \quad (1.19)$$

We retrieve the leading order result (1.4).

The NLO correction in the semiclassical expansion is obtained by considering fluctuations $\pi(\tau,\theta)$ around the semiclassical saddle $\chi = -i\mu\tau + \pi(\tau,\theta)$. Plugging this into the action (1.14), we can expand in a series in inverse powers of μ, since μ is large at large n (1.17), yielding at quadratic order

$$S[\pi] = \frac{d(d-1)}{2}c_1\mu^{d-2}\int \mathrm{d}\tau\, \mathrm{d}^{d-1}\theta\, \sqrt{g}\left(\dot{\pi}^2 + \frac{1}{d-1}g^{ij}\partial_i\pi\partial_j\pi + O\left(\frac{\nabla^4}{\mu^2}\right)\right). \quad (1.20)$$

This action describes a phonon with speed of sound $c_s = \frac{1}{\sqrt{d-1}}$. This quadratic action has for spectrum

$$\omega_\ell = \frac{J_\ell}{\sqrt{d-1}} + O\left(\frac{J_\ell^3 R^2}{n^{\frac{2}{d-1}}}\right), \tag{1.21}$$

where

$$J_\ell^2 = \frac{\ell(\ell+d-2)}{R^2} \tag{1.22}$$

is the ℓth eigenvalue of the laplacian on the sphere $\Delta_{S^{d-1}}$. This eigenvalue has multiplicity

$$n_{\ell,d} = \frac{(2\ell+d-2)\Gamma(\ell+d-2)}{\Gamma(\ell+1)\Gamma(d-1)}. \tag{1.23}$$

The leading quantum correction to the scaling dimension Δ_n is given by the fluctuation determinant arising from the gaussian path integral of action (1.20)

$$\begin{aligned}\Delta_n^{(1)} &= \frac{R}{2(\tau_f-\tau_i)} \log\det\left[-\partial_\tau^2 - \frac{1}{d-1}\Delta_{S^{d-1}} + O\left(\frac{\nabla^4}{\mu^2}\right)\right] \\ &= \frac{R}{2}\sum_\ell n_{\ell,d}\omega_\ell \\ &= \beta_0 + \beta_1 n^{-\frac{2}{d-1}} + \dots\end{aligned} \tag{1.24}$$

where the β's are functions of the dimension and Wilson coefficient, similarly to (1.19). Note however that the dispersion relation at leading order (1.21) depends only on the dimension, being in fact a consequence of conformal invariance, thus β_0 is also a function of the dimension only. We will not detail the computation here, the interested reader can consult ref. [81]. Note however that it is very close in philosophy to the computation we are going to do in full detail in the next chapter (section 2.4.5).

The contributions of the result sum to

$$\begin{aligned}\Delta_n &= \Delta_n^{(0)} + \Delta_n^{(1)} + \dots \\ &= n^{\frac{d}{d-1}}\left[\alpha_1 + \alpha_2\, n^{-\frac{2}{d-1}} + \dots\right] + n^0\left[\beta_0 + \beta_1 n^{-\frac{2}{d-2}} + \dots\right] + \dots\end{aligned} \tag{1.25}$$

following this pattern, the two-loop term $\Delta_n^{(2)}$ would scale like $n^{-\frac{d}{d-1}}$ (up to logarithms of n).

Let us give the explicit result in the most common dimensions. For $d=3$

$$\Delta_n\Big|_{d=3} = \alpha_1 n^{\frac{3}{2}} + \alpha_2 n^{\frac{1}{2}} - 0.0937255 + \alpha_3 n^{-\frac{1}{2}} + \beta_1 n^{-1} + \dots \tag{1.26}$$

We can see that there is a single contribution to the n^0 term, coming from $\beta_0 n^0 = 0.0937255$. This is always the case when d is odd. As said above, this coefficient does not depend on the Wilson coefficients, but only on the dimension d, thus it is a truly universal prediction of the methodology. This property will be checked by the explicit computation in the ε-expansion. Note also that $\alpha_1, \alpha_2, \alpha_3, \beta_1$ depend only on 3 Wilson coefficients c_1, c_2, c_3, thus there is a nontrivial universal relationship between these four α, β coefficients.

On the other hand for $d = 4$, we have that the β_0 term contribution mixes with that of α_3 (same for β_1 with α_4). In fact, it is possible to show that $\beta_{0,1}$ have simple poles in the limit $d \to 4$. This divergent contribution, coming from a quantum loop, can be taken care of by renormalizing the Wilson coefficients. After this process, it yields a logarithm of the UV cutoff scale μ. The final result takes the form

$$\Delta_n\Big|_{d=4} = \alpha_1 n^{\frac{4}{3}} + \alpha_2 n^{\frac{2}{3}} - \frac{1}{48\sqrt{3}} \log n + \alpha_3 + O(n^{-\frac{2}{3}}, n^{-\frac{2}{3}} \log n). \tag{1.27}$$

There is again a universal component, the $\log n$ term.

In the following chapters of the thesis, we are going to work in the ε-expansion, specifically in dimension either $d = 4 - \varepsilon$ or $d = 3 - \varepsilon$. In both cases the large charge limit will match with the general result (1.25).

A very similar approach can be used to compute correlators with insertions (1.5). Again, we will not detail this further, but the method is equivalent to what we will do in chapter 3 in ε-expansion.

2 The ε-expansion meets semiclassics: dimension of ϕ^n

2.1 Introduction

In this thesis, we would like to present a different approach to explore the large charge sector of conformal field theories. While the previous chapter investigated the information provided about this class of observables by the EFT formalism, we now demonstrate it in the context of the ε-expansion [1, 2] (also see [96, 97] for related works). The construction will parallel that of chapter 1, with the difference that one does not need to construct an effective field theory. In this case, one way to retrieve the Wilson coefficients of the corresponding EFT is by matching the results of this chapter with the general prediction (1.25). More specifically, throughout the next three chapters we shall consider two variants of $U(1)$-invariant CFTs defined by a lagrangian in ε-expansion, which are weakly coupled at small $\varepsilon \ll 1$.

1. the complex massless scalar with a quartic interaction in $4-\varepsilon$ euclidean dimensions given by lagrangian

$$\mathcal{L} = \partial\bar{\phi}\partial\phi + \frac{\lambda_0}{4}\left(\bar{\phi}\phi\right)^2. \tag{2.1}$$

where λ_0 is the bare coupling. We will first consider general coupling, but we shall later derive more specific results by focusing on the Wilson-Fisher fixed point. Renormalized field and coupling are defined according to

$$\phi = Z_\phi[\phi], \quad \lambda_0 = M^\varepsilon \lambda Z_\lambda, \tag{2.2}$$

where M is the sliding scale. Throughout the thesis we will adopt the minimal subtraction scheme, where Z_ϕ and Z_λ are expressed as an ascending series of pure poles. In particular we have

$$\log Z_\lambda = \sum_k \frac{z_k(\lambda)}{\varepsilon^k} = \frac{c_{11}\lambda + c_{12}\lambda^2 + \dots}{\varepsilon} + \frac{c_{22}\lambda^2 + \dots}{\varepsilon^2} + \dots, \tag{2.3}$$

where [98]

$$z_1(\lambda) = 5\frac{\lambda}{(4\pi)^2} - \frac{15}{2}\frac{\lambda^2}{(4\pi)^4} + O\left(\frac{\lambda^3}{(4\pi)^6}\right). \tag{2.4}$$

Notice moreover that $Z_\phi = 1$ up to two-loop corrections

$$Z_\phi = 1 - \frac{\lambda^2}{(16\pi^2)^2 8\varepsilon} + O(\lambda^3). \tag{2.5}$$

Using (2.2) one can easily show that the β-function equals

$$\frac{\partial\lambda}{\partial \log M} \equiv \beta(\lambda) = -\varepsilon\lambda + \beta_4(\lambda), \tag{2.6}$$

with

$$\beta_4(\lambda) = \lambda^2\frac{\partial z_1}{\partial\lambda} = 5\frac{\lambda^2}{(4\pi)^2} - 15\frac{\lambda^3}{(4\pi)^4} + O\left(\frac{\lambda^4}{(4\pi)^6}\right). \tag{2.7}$$

At the Wilson-Fisher fixed point, defined by $\lambda = \lambda_*$ such that $\beta(\lambda_*) = 0$, the theory is invariant under conformal transformations. The fixed point coupling λ_* is non-trivially determined by the space-time dimensionality

$$\frac{\lambda_*}{(4\pi)^2} = \frac{\varepsilon}{5} + \frac{3}{25}\varepsilon^2 + O(\varepsilon^3). \tag{2.8}$$

2. the complex massless scalar with a sextic interaction in $3-\varepsilon$ euclidean dimensions given by lagrangian

$$\mathcal{L} = \partial\bar{\phi}\partial\phi + \frac{\lambda_0^2}{36}\left(\bar{\phi}\phi\right)^3. \tag{2.9}$$

The renormalization of the fields and coupling is also introduced with (2.2), but of course the renormalization constants Z_ϕ, Z_λ are different. In this theory the fields renormalization $Z_\phi = 1$ up to 4 loops. The β-function is given by [99]

$$\frac{\partial\lambda}{\partial \log M} \equiv \beta(\lambda) = \lambda\left[-\varepsilon + \frac{7\lambda^2}{48\pi^2} + O\left(\frac{\lambda^4}{(4\pi)^4}\right)\right]. \tag{2.10}$$

For $\varepsilon \ll 1$, this implies the existence of an IR-stable fixed point at

$$\frac{\lambda_*^2}{(4\pi)^2} = \frac{3}{7}\varepsilon + O\left(\varepsilon^2\right). \tag{2.11}$$

The goal of this chapter is to introduce the semiclassical saddle expansion for computing correlators involving operators with large charge n in these weakly-coupled CFTs. Although we will formulate the method in general terms in section 2.4, the concrete computations of this chapter focus on a simple example, plausibly the simplest one in the reach of this method, namely the scaling dimension Δ_{ϕ^n} of the operator ϕ^n. This quantity is derived from the asymptotic behaviour of the 2-point function $\left\langle\bar{\phi}^n(x_f)\phi^n(x_i)\right\rangle$.

Example of computations of higher-point functions are discussed in the next chapter.

The main conceptual result of this dissertation is that the CFT data (scaling dimensions and OPE coefficients), for operators charged with a large quantum number n under the $U(1)$ internal symmetry, can be computed through a systematic expansion around a non-trivial trajectory, when considered in the double scaling limit

$$\begin{aligned} \lambda \sim \varepsilon \to 0 \\ n \to \infty \end{aligned} \qquad \lambda n = \text{ fixed.} \tag{2.12}$$

For instance, in this chapter we shall study the scaling dimension of the lowest-dimensional operator of charge n, ϕ^n. This will yield a result of the form

$$\Delta_{\phi^n} = \frac{1}{\lambda_*}\Delta_{-1}(\lambda_* n) + \Delta_0(\lambda_* n) + \lambda_* \Delta_1(\lambda_* n) + \ldots \tag{2.13}$$

with λ_* the fixed point coupling (either (2.8) or (2.11) depending in which theory we work), and with Δ_{k-1} representing the k-th loop contribution in the semiclassical expansion. This result will be made concrete through the explicit computation of the leading and subleading terms, Δ_{-1} and Δ_0, in the two theories listed above.

The equation (2.13) shows that in the double scaling limit, ε (or equivalently λ_*) remains the loop expansion parameter, while the effects of large n are controlled by the classical parameter $\lambda_* n$. Our system, when weakly coupled around the vacuum, thus remains weakly coupled also at large n. However our result applies equally well to large and to small $\lambda_* n$, where one can also compute using Feynman diagrams. On the one hand this illustrates that the poor behaviour of standard perturbation theory as $\lambda_* n$ is increased is simply tied to a poor choice of the path integral trajectory around which to expand. On the other hand it allows to compare our semiclassical computation to the results obtained using Feynman diagrams.

The simplicity of this approach, we believe, illuminates previous literature in related but different contexts. As concerns multilegged scattering amplitude discussed in the introduction, the structure of our computation is precisely the same, and precisely identical in the emergence of a double scaling limit, $\lambda \to 0$ with λn fixed. This indicates a sort of universality in the structure of multilegged observables, with λn acting like a sort of 't Hooft coupling. On the CFT side, our result directly connects to preceding work on the general properties of large charge operators [74, 75, 100]. In that context, it shows more concretely how the superfluid configuration of the leading trajectory emerges and it offers a concrete "UV" complete realization of the effective field theory describing the superfluid. In particular the parameter λn controls the occurrence of the pure superfluid regime: at small λn the leading trajectory corresponds to a superfluid interacting with a light radial excitation, while at large λn the latter decouples. In our amusingly simple scenario, the parameter λn thus seems to play a role similar to the 't Hooft coupling in

AdS/CFT, where it controls the gap between stringy and supergravity modes.

A comment is in order regarding the claim that we compute the dimension of operator ϕ^n. Indeed, in an interacting theory, it is not trivial to make sense of a field expression such as ϕ^n, and even in perturbation theory the interpretation may depend on the chosen renormalization procedure. However, the scaling dimensions are well-defined physical quantities in a Conformal Field Theory, which is precisely the case that we want to consider. For small λn, when diagrammatic perturbation theory holds, ϕ^n is the operator of lowest dimension with $U(1)$ charge n (indeed, any other operator with charge n, e.g. $\phi^{n-2}(x)(\partial\phi(x))^2$, clearly possesses a larger scaling dimension in the free limit, and for small enough λn the ordering is not affected). Therefore it does not mix in MS renormalization, and the perturbatively renormalized operator with a well-defined scaling dimension is indeed $[\phi^n]$. Throughout this thesis, we conventionally extend this idea to generic λn, *defining* the operator ϕ^n to be the lowest dimension charge n operator. Level crossing may in principle occur at finite λn, but that would unavoidably be associated with a non-analyticity in the dependence on λn of the minimal dimension at fixed charge. The result we shall obtain with our semiclassical method is however analytic at positive λn and matches the dimension of ϕ^n at small λn. That however does not imply that the field expression for the lowest dimension charge n operator remains ϕ^n for all values of λn, but it tells us that ϕ^n is indeed the "equivalent" of that operator in the small λn limit[1]. It should however become clear from our discussion that the precise form of the lowest dimension operator is a separate issue and does not affect the semiclassical computation of its scaling dimension[2].

This chapter is organised as follows. We begin by focusing on the complex scalar with quartic interaction described by the lagrangian (2.1) in $4-\varepsilon$ euclidean dimensions. In a first approach we do not tune the coupling to the Wilson-Fisher fixed point, to see what can be said of the semiclassical expansion for non-conformal theory. We start in section 2.2 by reviewing the computation of Δ_{ϕ^n} in the usual regime of finite n and $\lambda \ll 1$, where perturbation theory is valid. We detail the diagrammatic computation at two-loop, and we present a diagrammatic analysis of the higher-loop contributions, which shows perturbation theory is limited to small λn, and gives hints that a semiclassical expansion exists beyond. In section 2.3 we derive the existence of the semiclassical expansion from general arguments, and argue the expansion takes the form (2.13). We perform said expansion but only for the leading term Δ_{-1} and in the perturbative regime ($\lambda n \ll 1$). This approach is not tractable outside of this regime, thus does not provide any improvement over the direct diagrammatic computation.

Starting from section 2.4, we specialize to the Wilson-Fisher fixed point (2.8), and use

[1]We will use the same idea of free-theory representent to classify charge-n operators in chapter 4.

[2]It matters for the computation of the normalization of the correlator, and thus for the computation of higher point functions. Notice we do not compute this effect in chapter 3, assuming we work with canonically normalized operators. It remains an open question for future work.

the additional properties of the conformal theory to simplify the computation. The key features of CFTs that are exploited here are radial quantization and the associated state-operator correspondence. The conformal mapping to the cylinder also helps simplifying the discussion, and reveals that the semiclassical saddle can be seen as a superfluid state. We perform the explicit calculation of the first two leading terms in (2.13) for arbitrary values of $\lambda_* n$. Some direct applications of the results are analyzed in section 2.4.6. We not only find perfect agreement with the diagrammatical result in the limit $\lambda n \ll 1$, but are also able to combine our result with finite order calculations and predict expansion coefficients that are beyond the order reached by each method when taken individually. The results also match with the large charge EFT results in the limit $\lambda n \gg 1$. Our systematic expansion in $d = 4 - \varepsilon$ also invites a comparison with the results of Monte Carlo simulations in $d = 3$. While we are aware that taking $\varepsilon = 1$ is a significant stunt, we nonetheless find the comparison encouraging already with the first two orders we computed. This warrants computation of the next order, Δ_1.

Finally, in section 2.5, we generalize these results to the complex scalar with sextic interaction in $3 - \varepsilon$ dimensions. Further generalisations to non-abelian symmetry groups and different theories will be discussed in the conclusion of the thesis.

2.2 Perturbation theory around the vacuum

2.2.1 Renormalization of operators

In this chapter (except section 2.5) we will consider the massless $U(1)$ symmetric theory with quartic interaction in $d = 4 - \varepsilon$ dimensional euclidean space with bare lagrangian given in (2.1). We will first consider general coupling, but we shall later derive more specific results by focusing on the Wilson-Fisher fixed point (2.8). The common procedure to compute scaling dimensions of operators is to work in renormalized perturbation theory and to perform the renormalization of the operators of interest [56, 101]. In all computations of this thesis we work in the minimal subtraction (MS) scheme defined by (2.2 – 2.8). Similarly to ϕ, all renormalized operators are denoted with square brackets

$$\mathcal{O}(x) = Z_{\mathcal{O}}\, [\mathcal{O}](x), \tag{2.14}$$

where the renormalization constant $Z_{\mathcal{O}}$ is also an ascending series of pure poles in ε. For $\varepsilon \ll 1$ the theory is weakly coupled. As we will show in the next subsection, this does not prevent perturbation theory around the vacuum to break down for specific observables.

2.2.2 Diagrammatic computation of the anomalous dimension

We will study the scaling dimension of the simplest operator with $U(1)$ charge[3] n ($-n$), denoted by $[\phi^n]$ ($[\bar{\phi}^n]$) and related to the bare field by

$$(\phi(x))^n = Z_{\phi^n}[\phi^n](x)\,. \tag{2.15}$$

where Z_{ϕ^n} is the renormalization factor in MS scheme. The anomalous dimension is then given by

$$\gamma_{\phi^n} = \frac{\partial \log Z_{\phi^n}}{\partial\lambda}\,\left[-\varepsilon\lambda + \beta_4(\lambda)\right]\,, \tag{2.16}$$

and is finite as the poles cancel to the relevant order in λ. For arbitrary λ, γ_{ϕ^n} is scheme dependent, and thus unphysical, beyond leading order. That can easily be seen by changing the scheme according to $[\phi^n] \to f(\lambda)[\phi^n]$ and $Z_{\phi^n} \to Z_{\phi^n}/f(\lambda)$, with $f(\lambda)$ a power series with finite coefficients. In the new scheme the anomalous dimension is modified according to $\gamma_{\phi^n} \to \gamma_{\phi^n} - \beta(\partial_\lambda \ln f)$. On the other hand $\beta(\lambda_*) = 0$, so that γ_{ϕ^n} is scheme independent and physical at the fixed point. Indeed, a straightforward solution of the Callan-Symanzik equation for $\langle[\bar{\phi}^n][\phi^n]\rangle$ shows that the operator's physical dimension at the fixed point is

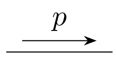
$$\Delta_{\phi^n} = n(d/2-1) + \gamma_{\phi^n}(\lambda_*)\,. \tag{2.17}$$

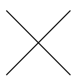
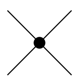
We now perform this computation explicitly at 2-loops. We shall need this in order to compare to the results of the more powerful method we shall develop in the next sections. For simplicity, we work in momentum space and we consider an insertion of the operator ϕ^n within n equal incoming momenta p. We want to compute, according to the definitions (2.2),(2.15):

$$\langle \phi^n \bar{\phi}(p)\bar{\phi}(p)\ldots\bar{\phi}(p)\rangle = Z_{\phi^n} Z_\phi^n \langle [\phi^n]\,[\bar{\phi}](p)\,[\bar{\phi}](p)\ldots[\bar{\phi}](p)\rangle \tag{2.18}$$

and find the right renormalization constant Z_{ϕ^n} such that $\langle [\phi^n]\,[\bar{\phi}](p)\,[\bar{\phi}](p)\ldots[\bar{\phi}](p)\rangle$ is finite in the MS scheme.

The Feynman rules of renormalized perturbation theory are:



$= \dfrac{1}{p^2}$  $= -\lambda$  $= -\delta_\lambda$ (2.19)

where $\delta_\lambda = \frac{5\lambda^2}{16\pi^2\varepsilon}$ is the coupling counterterm at one-loop in MS [98]. The ϕ^n operator will be represented by a crossed vertex and normalized to

$$\otimes = 1. \tag{2.20}$$

[3] In our conventions, $\phi, \bar{\phi}$ have charge, respectively, 1 and -1.

All diagrams to two-loop are displayed in figure 2.1. To lighten the notation, we don't draw the incoming lines if they are directly connected to the ϕ^n operator, only those connected to other vertices are shown.

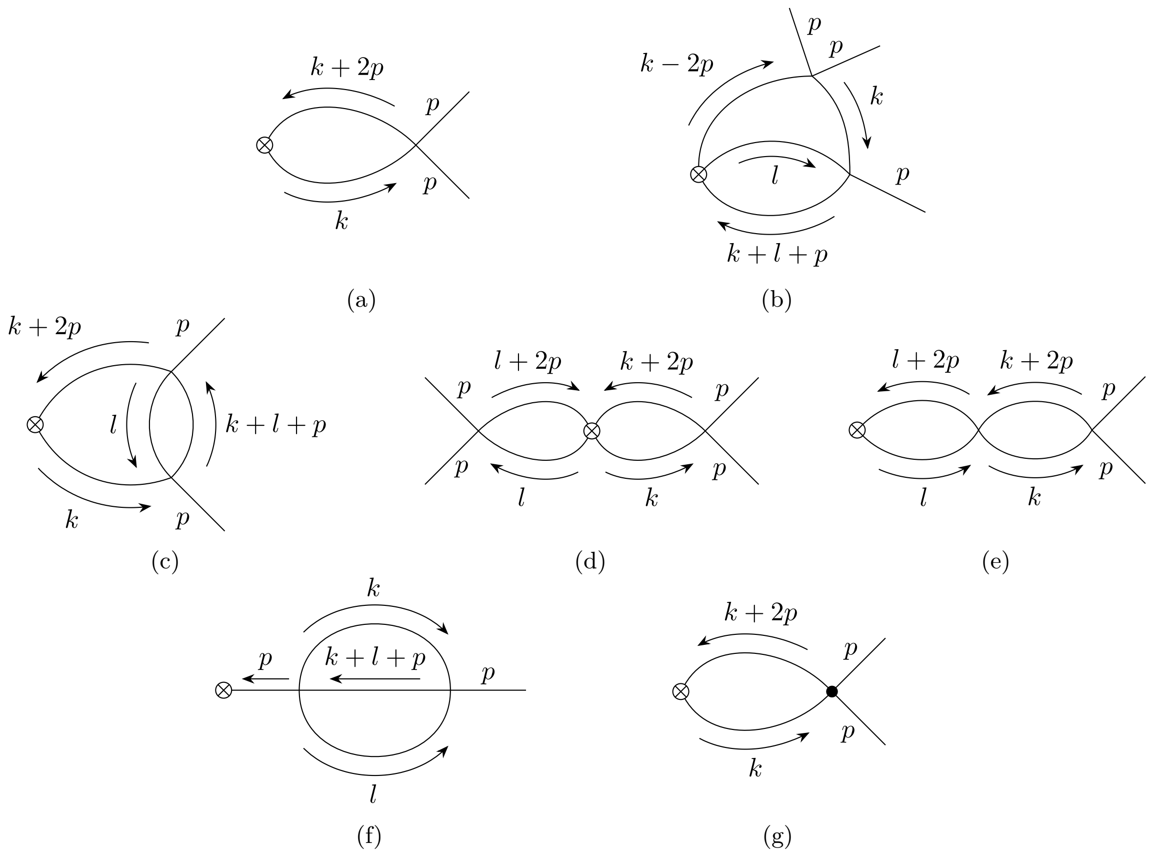


Figure 2.1: Feynman diagrams that contribute at two-loop.

The one-loop diagram is:

$$
\begin{aligned}
\text{(a)} &= \frac{n(n-1)}{2}\frac{1}{2}(-\lambda)\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2}\frac{1}{(k+2p)^2} \\
&= -\frac{\lambda}{16\pi^2}\frac{n(n-1)}{4}\left(\frac{2}{\varepsilon}+2-\gamma+\log\left(\frac{\pi M^2}{p^2}\right)\right)+O(\varepsilon)
\end{aligned}
\tag{2.21}
$$

where in the first line, the first factor $\frac{n(n-1)}{2}$ indicates the number of ways the external momenta can be connected to form this diagram: one has to chose 2 momenta among n. The next factor $\frac{1}{2}$ is the usual symmetry factor, then comes the vertex, and finally the loop integral. In the result, M is the scale introduced in (2.2).

Six diagrams have to be computed at two-loop level. We need only the divergent piece of these diagrams. The procedure to compute the first two diagrams is described in [56].

The last diagram includes the one-loop counterterm δ_λ.

$$
\begin{aligned}
(\mathrm{b}) &= \frac{n(n-1)(n-2)}{2}\frac{1}{2}(-\lambda)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\int\frac{\mathrm{d}^dl}{(2\pi)^d}\frac{1}{k^2}\frac{1}{(k-2p)^2}\frac{1}{l^2}\frac{1}{(k+l+p)^2} \\
&= \frac{\lambda^2}{(16\pi^2)^2}\frac{n(n-1)(n-2)}{4}\left(\frac{2}{\varepsilon^2}+\frac{5-2\gamma+2\log\left(\frac{\pi M^2}{p^2}\right)}{\varepsilon}\right)+O(\varepsilon^0) \qquad (2.22)
\end{aligned}
$$

$$
\begin{aligned}
(\mathrm{c}) &= \frac{n(n-1)}{2}(-\lambda)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\int\frac{\mathrm{d}^dl}{(2\pi)^d}\frac{1}{k^2}\frac{1}{(k+2p)^2}\frac{1}{l^2}\frac{1}{(k+l+p)^2} \\
&= \frac{\lambda^2}{(16\pi^2)^2}\frac{n(n-1)}{2}\left(\frac{2}{\varepsilon^2}+\frac{5-2\gamma+2\log\left(\frac{\pi M^2}{p^2}\right)}{\varepsilon}\right)+O(\varepsilon^0) \qquad (2.23)
\end{aligned}
$$

$$
\begin{aligned}
(\mathrm{d}) &= \frac{n(n-1)(n-2)(n-3)}{8}\frac{1}{4}(-\lambda)^2\left(\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{k^2}\frac{1}{(k+2p)^2}\right)^2 \\
&= \frac{\lambda^2}{(16\pi^2)^2}\frac{n(n-1)(n-2)(n-3)}{8}\left(\frac{1}{\varepsilon^2}+\frac{2-\gamma+\log\left(\frac{\pi M^2}{p^2}\right)}{\varepsilon}\right)+O(\varepsilon^0) \qquad (2.24)
\end{aligned}
$$

$$
\begin{aligned}
(\mathrm{e}) &= \frac{n(n-1)}{2}\frac{1}{4}(-\lambda)^2\left(\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{k^2}\frac{1}{(k+2p)^2}\right)^2 \\
&= \frac{\lambda^2}{(16\pi^2)^2}\frac{n(n-1)}{2}\left(\frac{1}{\varepsilon^2}+\frac{2-\gamma+\log\left(\frac{\pi M^2}{p^2}\right)}{\varepsilon}\right)+O(\varepsilon^0) \qquad (2.25)
\end{aligned}
$$

$$
\begin{aligned}
(\mathrm{f}) &= n\frac{1}{2}(-\lambda)^2\frac{1}{p^2}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\int\frac{\mathrm{d}^dl}{(2\pi)^d}\frac{1}{k^2}\frac{1}{l^2}\frac{1}{(k+l+p)^2} \\
&= -\frac{\lambda^2}{(16\pi^2)^2}\frac{n}{4\varepsilon}+O(\varepsilon^0) \qquad (2.26)
\end{aligned}
$$

$$
\begin{aligned}
(\mathrm{g}) &= \frac{n(n-1)}{2}\frac{1}{2}(-\delta_\lambda)\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{k^2}\frac{1}{(k+2p)^2} \\
&= -\frac{\lambda^2}{(16\pi^2)^2}\frac{5n(n-1)}{4}\left(\frac{2}{\varepsilon^2}+\frac{2-\gamma+\log\left(\frac{\pi M^2}{p^2}\right)}{\varepsilon}\right)+O(\varepsilon^0) \qquad (2.27)
\end{aligned}
$$

Summing all contributions we get:

$$
\begin{aligned}
&(2.20)+(2.21)+(2.22)+(2.23)+(2.24)+(2.25)+(2.26)+(2.27)\\
&=\left(1-\frac{\lambda n(n-1)}{(16\pi^2)2\varepsilon}+\frac{\lambda^2}{(16\pi^2)^2}\left(\frac{n^4-2n^3-9n^2+10n}{8\varepsilon^2}+\frac{n^3-n^2-n}{4\varepsilon}\right)\right)\\
&\quad\times\left(1-\frac{\lambda n(n-1)\left(2-\gamma+\log\left(\frac{\pi M^2}{p^2}\right)\right)}{4(16\pi^2)}\right)+O(\varepsilon,\lambda^2\varepsilon^0)
\end{aligned}
\tag{2.28}
$$

where the result, following (2.18), has been factored as $Z_{\phi^n}Z_\phi^n$, which contains only poles according to MS prescription, times the finite value of $\langle[\phi^n]\,[\bar\phi](p)\,[\bar\phi](p)\ldots[\bar\phi](p)\rangle$. This lets us compute the renormalization factor Z_{ϕ^n} using (2.5):

$$
Z_{\phi^n}=1-\frac{\lambda n(n-1)}{(16\pi^2)2\varepsilon}+\frac{\lambda^2}{(16\pi^2)^2}\left(\frac{n^4-2n^3-9n^2+10n}{8\varepsilon^2}+\frac{2n^3-2n^2-n}{8\varepsilon}\right). \tag{2.29}
$$

The anomalous dimension γ_{ϕ^n} is computed using (2.16) and yields

$$
\gamma_{\phi^n}=n\left[\frac{\lambda}{16\pi^2}\,\frac{(n-1)}{2}-\left(\frac{\lambda}{16\pi^2}\right)^2\frac{2n^2-2n-1}{4}\right]. \tag{2.30}
$$

When the theory is considered at the fixed point (2.8) this implies

$$
\Delta_{\phi^n}=n\left[\left(\frac{d}{2}-1\right)+\frac{\varepsilon}{10}\,(n-1)-\frac{\varepsilon^2}{100}(2n^2-8n+5)\right]. \tag{2.31}
$$

2.2.3 Anomalous dimension of large charge operators

The result of the previous subsection is valid as long as $n\geq 4$ because of the four connections in diagram (2.1d). Notice this diagram has a leading contribution at $n\gg 1$ scaling like $\lambda^2 n^4$. On the other hand, in (2.30) the leading contribution at two loops scales like $\lambda^2 n^3$, indicating there is some kind of cancellation happening when one takes the logarithm in (2.16).

We now focus on $n\gg 1$, the regime of large charge or many legs, and proceed to a diagrammatic analysis of the growth of the multiplicity factors with n, see figure 2.2. Considering any loop order $k\ll n$, one finds that the leading contributions to Z_{ϕ^n} scale like $\lambda^k n^{2k}$, and come from the daisy diagrams in the leftmost column of figure 2.2. Similarly, all contributions scaling from $\lambda^k n^{2k}$ down to $\lambda^k n^{k+2}$ come from "disconnected" diagrams that are combinations of lower-loop connected diagrams, see for example the diagram in the center of the figure. This is because the largest contribution from a fully-connected diagram comes from the type of diagrams in the top row for which the number of legs picked from the ϕ^n equals $k+1$, and scale as $\lambda^k n^{k+1}$. In fact, a more

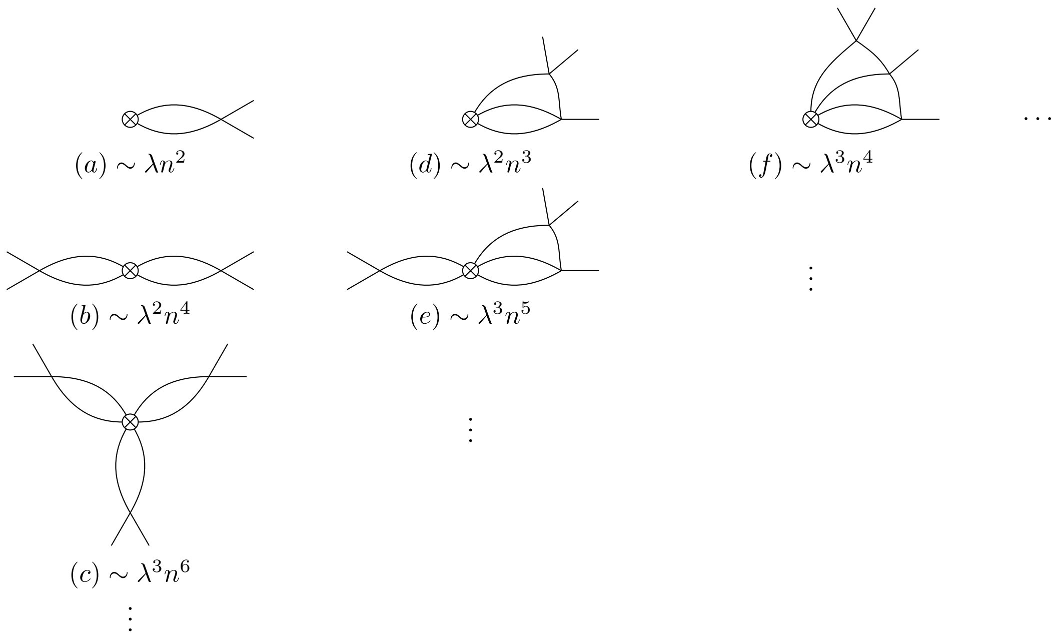


Figure 2.2: Some characteristic Feynman diagrams that appear with the ϕ^n operator.

detailed analysis shows that at any given loop order k, the terms with powers of n between $k+2$ and $2k$ simply exponentiate from lower loop terms[4]. As a consequence, in the expansion of $\ln Z_{\phi^n}$, and thus of γ_{ϕ^n}, those highest powers cancel, and the leading remaining contribution at order k scales like the connected diagram, $\lambda^k n^{k+1}$. That is

$$\gamma_{\phi^n} = n \sum_{k=1} \lambda^k P_k(n) \,, \tag{2.32}$$

with P_k a polynomial of degree k. Note that the leading $\lambda^k n^{k+1}$ term is not only derived from the fully-connected diagram, but also receives contributions from disconnected ones. Thus while this exponentiation explains how higher terms cancel, it does not allow to skip the computation of disconnected diagrams for remaining terms.

In truth we have explicitly checked (2.32) only up to four loops, but in the next section we shall give a general argument bypassing the diagrammatic analysis. The above result shows that, no matter how weakly coupled the theory is, for sufficiently large λn, perturbation theory breaks down. The series in eq. (2.32) can also be organized in terms of leading and subleading n-powers, in close analogy with leading and subleading logs in the RG resummation

$$\gamma_{\phi^n} = n \sum_{\kappa=0} \lambda^\kappa F_\kappa(\lambda n) \,. \tag{2.33}$$

Very much like for the RG, this alternative rewriting of the series suggests an alternative

[4]As an illustration, it is simple to check that the sum over daisy diagrams exponentiate the λn^2 contribution from the single petal diagram (a).

loop expansion, performed after resumming (or straight out computing) all powers of λn. Again, the physics underlying this alternative interpretation will be made manifest in the next sections. Notice in passing, and consistently with the results in the next section, that the leading-n contribution $F_0(\lambda n)$ is unaffected by changes in the subtraction scheme, like for instance $\lambda \to \lambda + a\lambda^2$ or $Z_{\phi^n} \to Z_{\phi^n}(1 + bn^2\lambda)$, the latter corresponding to a simple reshuffling of the finite terms in the daisy diagram (a).

2.3 Semiclassical approach in non-conformal theory

2.3.1 Semiclassical expansion of the path integral

We now consider a first semiclassical method with which it is straightforward to come up using the intuition of the path integral. In this section we still consider a general weak coupling λ, not set at the Wilson-Fisher fixed point.

The scaling dimension of $[\phi^n]$ can also be directly computed by considering the two-point function

$$\langle \bar{\phi}^n(x_f)\phi^n(x_i)\rangle \equiv \frac{\int \mathcal{D}\phi\mathcal{D}\bar{\phi}\,\bar{\phi}^n(x_f)\phi^n(x_i)\exp\left[-\int \mathcal{L}\right]}{\int \mathcal{D}\phi\mathcal{D}\bar{\phi}\,\exp\left[-\int \mathcal{L}\right]} \equiv Z_{\phi^n}^2\langle[\bar{\phi}^n](x_f)[\phi^n](x_i)\rangle\,. \tag{2.34}$$

The above integral can be cast in a form which exhibits its semiclassical nature in the small λ regime independently of the size of n. First it is convenient to rescale the field $\phi \to \phi/\sqrt{\lambda_0}$ to exhibit λ_0 as the loop counting parameter

$$\int \mathcal{L} \to \frac{1}{\lambda_0}\int\left[\partial\bar{\phi}\partial\phi + \frac{1}{4}\left(\bar{\phi}\phi\right)^2\right] \equiv \frac{S}{\lambda_0}\,. \tag{2.35}$$

Secondly $\bar{\phi}^n(x_f)\phi^n(x_i)$ can be brought up in the exponent, obtaining

$$Z_{\phi^n}^2\lambda_0^n\langle[\bar{\phi}^n](x_f)[\phi^n](x_i)\rangle = \frac{\int \mathcal{D}\phi\mathcal{D}\bar{\phi}\,e^{-\frac{1}{\lambda_0}\left[\int \partial\bar{\phi}\partial\phi+\frac{1}{4}\left(\bar{\phi}\phi\right)^2-\lambda_0 n\left(\ln\bar{\phi}(x_f)+\ln\phi(x_i)\right)\right]}}{\int \mathcal{D}\phi\mathcal{D}\bar{\phi}\,e^{-\frac{1}{\lambda_0}\left[\int \partial\bar{\phi}\partial\phi+\frac{1}{4}\left(\bar{\phi}\phi\right)^2\right]}}\,. \tag{2.36}$$

The dependence on λ_0 and n, shows that we can perform the path integral using a saddle point expansion in the limit of small λ_0, while keeping $\lambda_0 n$ fixed. This limit thus encompasses the case where $\lambda_0 n$ is (arbitrarily) large[5]. Independently of the detailed form of the field configuration furnishing the steepest descent, the right hand side of eq. (2.36) will then take the form

$$\lambda_0^{-1/2}\,e^{\frac{1}{\lambda_0}\Gamma_{-1}(\lambda_0 n, x_{fi})+\Gamma_0(\lambda_0 n, x_{fi})+\lambda_0\Gamma_1(\lambda_0 n, x_{fi})+\ldots}\,,\qquad x_{fi} = x_f - x_i. \tag{2.37}$$

[5]Of course we are making here a formal statement by using the bare coupling, which is a power series in the renormalized coupling. In terms of renormalized quantities the limit is thus $\lambda(M)$ small with $\lambda(M)n$ fixed.

The factor $\lambda_0^{-1/2}$ is understood as follows. The path integral in the denominator is computed through a saddle-point expansion around the trivial point $\phi = \bar{\phi} = 0$, while the action of the path integral in the numerator is stationary on a continuous family of nontrivial configurations with $\phi, \bar{\phi} \neq 0$ and parametrized by the zero mode associated to the corresponding spontaneous breaking of the $U(1)$ symmetry. As the integral over the zero mode is clearly independent of the value of the action, this results in a mismatch of the powers of $\lambda_0^{1/2}$ in between the numerator and the denominator, leading to (2.37) [6].

Now, notice that by using Stirling's formula the expression $\lambda_0^{n+1/2} n!$ can be written in the same form as the exponential factor in eq. (2.37). It is then convenient to redefine the Γ_k's so as to factor out a $\lambda_0^{n+1/2} n!$ in the the exponential factor in eq. (2.37) and rewrite that equation as

$$Z_{\phi^n}^2 \lambda_0^n \langle [\bar{\phi}^n](x_f)[\phi^n](x_i)\rangle = \lambda_0^n n!\, e^{\frac{1}{\lambda_0}\Gamma_{-1}(\lambda_0 n, x_{fi}) + \Gamma_0(\lambda_0 n, x_{fi}) + \lambda_0 \Gamma_1(\lambda_0 n, x_{fi}) + \ldots} \tag{2.38}$$

Comparing to eq. (2.34), we deduce that the exponential factor in eq. (2.38) coincides at weak coupling and finite n with the loop expansion we discussed in the previous section. In particular, given

$$D(x) = \frac{1}{\Omega_{d-1}(d-2)(x^2)^{d/2-1}} = \langle \bar{\phi}(x)\phi(0)\rangle_{free}\,, \qquad \Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{2.39}$$

one has

$$\lim_{\lambda_0 \to 0} e^{\frac{1}{\lambda_0}\Gamma_{-1}(\lambda_0 n, x_{fi}) + \Gamma_0(\lambda_0 n, x_{fi}) + \lambda_0 \Gamma_1(\lambda_0 n, x_{fi}) + \ldots} = D(x_{fi})^n\,. \tag{2.40}$$

Moreover one has that the $\lambda_0^\kappa \Gamma_\kappa$'s must possess a power series expansion in λ_0 with fixed n. Renormalization is simply performed by separating out the UV divergent part in each term in the exponent

$$\lambda_0^\kappa \Gamma_\kappa(\lambda_0 n, x_{fi}) = \lambda^\kappa \Gamma_\kappa^{div}(\lambda n, \lambda) + \lambda^\kappa \Gamma_\kappa^{ren}(\lambda n, \lambda, x_{fi}, M) \tag{2.41}$$

where of course $\lambda \equiv \lambda(M)$ and where the resulting $\lambda^\kappa \bar{\Gamma}_\kappa$ behave like power series at

[6] The situation is fully analogous to the following example involving two dimensional integrals:

$$I(\lambda, n) = \frac{\int_C dz d\bar{z} (z\bar{z})^n \exp\left\{-\frac{1}{\lambda}\left[z\bar{z} + \frac{1}{4}(z\bar{z})^2\right]\right\}}{\int_C dz d\bar{z} \exp\left\{-\frac{1}{\lambda}\left[z\bar{z} + \frac{1}{4}(z\bar{z})^2\right]\right\}} = \frac{\int_C dz d\bar{z} \exp\left\{-\frac{1}{\lambda}\left[z\bar{z} + \frac{1}{4}(z\bar{z})^2 - \lambda n \log(z\bar{z})\right]\right\}}{\int_C dz d\bar{z} \exp\left\{-\frac{1}{\lambda}\left[z\bar{z} + \frac{1}{4}(z\bar{z})^2\right]\right\}}.$$

The integral in the denominator is performed in an expansion around $z = \bar{z} = 0$ and is thus proportional to λ due to the gaussian integration on the two directions of the plane. The exponent in the numerator is instead stationary on the whole circle defined by $z\bar{z} = \sqrt{1 + 2\lambda n} - 1$; in this case, while the integral over the radial direction produces a factor of $\sqrt{\lambda}$, angular integration gives an overall factor of 2π. The full result, for arbitrary λn, is thus proportional to $\lambda^{-1/2}$:

$$I(\lambda, n) = \sqrt{\frac{2\pi}{\lambda}} \frac{e^{-\frac{\lambda n + \sqrt{1+2\lambda n} - 1}{2\lambda}} \left(\sqrt{1 + 2\lambda n} - 1\right)^{n + \frac{1}{2}}}{(1 + 2\lambda n)^{1/4}} \left[1 + O(\lambda)\right].$$

$\lambda = 0$. From eqs. (2.36,2.38) we can then write

$$Z^2_{\phi^n} = e^{\sum_{\kappa=-1} \lambda^\kappa \Gamma^{div}_\kappa(\lambda n, \lambda)} \equiv e^{\sum_{\kappa=-1} \lambda^\kappa \bar{\Gamma}^{div}_\kappa(\lambda n)} \tag{2.42}$$

and

$$\langle [\bar{\phi}^n](x_f)[\phi^n](x_i)\rangle = n!\, e^{\sum_{\kappa=-1} \lambda^\kappa \Gamma^{ren}_\kappa(\lambda n, \lambda, x_{fi}, M)} \equiv n!\, e^{\sum_{\kappa=-1} \lambda^\kappa \bar{\Gamma}^{ren}_\kappa(\lambda n, x_{fi}, M)}\,. \tag{2.43}$$

where, in the rightmost expressions, we rearranged the expansion in λ using the (asymptotic) power series expansion of the $\lambda^\kappa \Gamma_\kappa$. Eq. (2.42) provides a formal proof of eqs. (2.32,2.33). In the above expression the $\bar{\Gamma}_\kappa$ represents the $(\kappa+1)$-loop correction to the saddle point approximation. In particular $\bar{\Gamma}^{div}_{-1}(\lambda n)$ and $\bar{\Gamma}^{ren}_{-1}$, represent the leading semiclassical contribution, the exponent at the saddle point[7]. However, they fully determine the leading-n contribution $F_0(\lambda n)$ in eq. (2.33), thus resumming at once the largest powers of n up to arbitarily high-loop orders in the stardard diagrammatic approach! The remarkable result highlighted by our formal derivation and by eq. (2.33), is that the result is organized as a 't Hooft expansion in which λn is the fixed 't Hooft coupling while $\lambda \ll 1$ and $n \gg 1$.

The rest of the chapter is devoted to explicitly deriving these expressions, at leading (LO) and next-to-leading (NLO) order in the λ expansion with λn fixed. In the next subsection we will perform a warm up computation by working at small but fixed λn. In the later sections we shall develop the case of arbitrary λn by focussing on the Wilson-Fisher fixed point, where conformal invariance permits to tackle some technical difficulties in the computation.

2.3.2 Semiclassics at small fixed λn

At small λn ordinary perturbation theory works. In this case the path integral eq. (2.34) can be computed by expanding around the trivial background $\phi = \bar{\phi} = 0$. In that case the insertions of ϕ^n and $\bar{\phi}^n$, are not included in the exponent (as the exponent of eq. (2.36) is singular at $\phi = \bar{\phi} = 0$) and are purely determined by the quantum fluctuation $\delta\phi$ around the trivial solution, i.e. $\phi \equiv 0 + \delta\phi$. The loop expansion is purely generated by the small quartic term $\lambda\phi^4$. For instance, working at order λ one finds

$$\langle \bar{\phi}^n(x_f)\phi^n(x_i)\rangle = \frac{n!\left[1 - \frac{\lambda n(n-1)}{2(4\pi)^2}\left(\frac{2}{\varepsilon} + \log x^2_{fi} + 1 + \gamma + \log\pi\right) + O\left(\frac{\lambda^2}{(4\pi)^4}\right)\right]}{\left[\Omega_{d-1}(d-2)\right]^n \left(x^2_{fi}\right)^{n\left(\frac{d}{2}-1\right)}}. \tag{2.44}$$

compatibly with the one-loop contribution to γ_{ϕ^n} derived in section 2.

[7] As we shall illustrate in a moment and, as it must be according to our derivation, the divergent part appears from purely classical properties of the saddle point solution.

As λn grows, the fluctuations of $\bar{\phi}^n(x_f)\phi^n(x_i)$ become significant, and for sufficiently large λn they cannot be captured by perturbation theory. However eq. (2.36) invites us to perform the computation around the stationary points of

$$S_{eff} \equiv \int d^d x \left[\partial\bar{\phi}\partial\phi + \frac{1}{4}\left(\bar{\phi}\phi\right)^2\right] - n\lambda_0\left(\log\bar{\phi}(x_f) + \log\phi(x_i)\right). \tag{2.45}$$

The equations of motion defining the stationary configuration include the operator insertions as a source

$$\begin{aligned} \partial^2\phi(x) - \frac{1}{2}\phi^2(x)\bar{\phi}(x) &= -\frac{\lambda_0 n}{\bar{\phi}(x_f)}\delta^{(d)}(x - x_f), \\ \partial^2\bar{\phi}(x) - \frac{1}{2}\phi(x)\bar{\phi}^2(x) &= -\frac{\lambda_0 n}{\phi(x_i)}\delta^{(d)}(x - x_i). \end{aligned} \tag{2.46}$$

Before discussing the details of the general computation, it is instructive to discuss the solution of (2.46) for small λn. Namely, we compute the function $\Gamma_{-1}(\lambda n)$ in (2.38) to order $O\left(\lambda^2 n^2/(4\pi)^4\right)$ and we check that the result agrees with (2.44). As we work at first order in the coupling, in what follows we will take $\lambda_0 = \lambda$. Now, for small λn the equations (2.46) can be solved perturbatively; to this aim, it is convenient to expand the fields as

$$\phi = (\lambda n)^{1/2}\left[\phi^{(0)} + \phi^{(1)} + \ldots\right], \qquad \bar{\phi} = (\lambda n)^{1/2}\left[\bar{\phi}^{(0)} + \bar{\phi}^{(1)} + \ldots\right], \tag{2.47}$$

where $\phi^{(k)}, \bar{\phi}^{(k)} = O\left(\lambda^k n^k\right)$. At the zeroth order, the equations of motion read

$$\begin{aligned} \partial^2\phi^{(0)}(x) &= -\frac{1}{\bar{\phi}^{(0)}(x_f)}\delta^{(d)}(x - x_f), \\ \partial^2\bar{\phi}^{(0)}(x) &= -\frac{1}{\phi^{(0)}(x_i)}\delta^{(d)}(x - x_i), \end{aligned} \tag{2.48}$$

whose solution is uniquely defined up to one free parameter and has the form

$$\begin{aligned} \phi^{(0)}(x) &= \frac{c_0}{\Omega_{d-1}(d-2)}\frac{1}{|x - x_f|^{d-2}}, \\ \bar{\phi}^{(0)}(x) &= \frac{\bar{c}_0}{\Omega_{d-1}(d-2)}\frac{1}{|x - x_i|^{d-2}}; \end{aligned} \tag{2.49}$$

with the parameters c_0 and $\bar{c}_0$ related by

$$c_0\bar{c}_0 = \Omega_{d-1}(d-2)|x_f - x_i|^{d-2}. \tag{2.50}$$

Notice that on the saddle-point, i.e. on the solution of (2.48), the fields ϕ and $\bar{\phi}$ are analytically continued away from the original integration contour, since they are not related by complex conjugation. As a consequence, the fields appearing in the source terms in the right hand side of (2.48) have a finite value and no regularization procedure is needed to find the solution (2.49). Finally, the arbitrariness in the solution is related to the symmetry $(\phi, \bar{\phi}) \to (\alpha\phi, \alpha^{-1}\bar{\phi})$ of the action (2.45) analytically continued to arbitrary values of the fields. The one free parameter in the solution precisely corresponds to the presence of the one zero mode we mentioned before.

The next to leading contribution is determined by

$$\begin{aligned}
\partial^2 \phi^{(1)}(x) &= \frac{\lambda n}{2}\left[\phi^{(0)}(x)\right]^2 \bar{\phi}^{(0)}(x) + \frac{\bar{\phi}^{(1)}(x_f)}{\left[\bar{\phi}^{(0)}(x_f)\right]^2}\delta^{(d)}(x-x_f), \\
\partial^2 \bar{\phi}^{(1)}(x) &= \frac{\lambda n}{2}\left[\bar{\phi}^{(0)}(x)\right]^2 \phi^{(0)}(x) + \frac{\phi^{(1)}(x_i)}{\left[\phi^{(0)}(x_i)\right]^2}\delta^{(d)}(x-x_i).
\end{aligned} \tag{2.51}$$

The solution reads

$$\begin{aligned}
\phi^{(1)}(x) &= -\frac{\lambda n}{2}\int d^d y D(x-y)\left[\phi^{(0)}(y)\right]^2 \bar{\phi}^{(0)}(y) - D(x-x_f)\frac{\bar{\phi}^{(1)}(x_f)}{\left[\bar{\phi}^{(0)}(x_f)\right]^2}, \\
\bar{\phi}^{(1)}(x) &= -\frac{\lambda n}{2}\int d^d y D(x-y)\left[\bar{\phi}^{(0)}(y)\right]^2 \phi^{(0)}(y) - D(x-x_i)\frac{\phi^{(1)}(x_i)}{\left[\phi^{(0)}(x_i)\right]^2},
\end{aligned} \tag{2.52}$$

where $\phi^{(1)}(x_i)$ and $\bar{\phi}^{(1)}(x_f)$ satisfy

$$\frac{\phi^{(1)}(x_i)}{c_0} + \frac{\bar{\phi}^{(1)}(x_f)}{\bar{c}_0} = -\frac{\lambda n}{2}\int d^d y D^2(x_i-y)D^2(x_f-y). \tag{2.53}$$

There is a one parameter arbitrariness in the solution due to the aforementioned symmetry. The integrals are formally divergent in $d=4$ and thus are performed via standard dimensional regularization techniques. Plugging the solution in the action (2.45), we find

$$\begin{aligned}
S_{eff} &= \lambda n - \lambda n \log\left[\frac{\lambda n}{\Omega_{d-1}(d-2)}\frac{1}{(x_{fi}^2)^{d/2-1}}\right] \\
&+ \lambda^2 n^2 \left(\frac{1}{16\pi^2 \varepsilon} + \frac{1+\gamma+\log\pi}{32\pi^2}\right) + \frac{\lambda^2 n^2}{32\pi^2}\log x_{fi}^2 \,.
\end{aligned} \tag{2.54}$$

$e^{-S_{eff}/\lambda}$ must represent the leading term

$$\lambda^n n! e^{\frac{\Gamma_{-1}}{\lambda}} \tag{2.55}$$

in eq. (2.38) with Γ_{-1} expanded up to $O(\lambda^2 n^2)$. It is easy to see it does. In particular, $\log n! \approx n\log n - n$ ensures that Γ_{-1} has a well defined power series in λn as expected.

The correlator, according to eqs. (2.34,2.36), then reads

$$\langle\bar{\phi}^n(x_f)\phi^n(x_i)\rangle = \frac{n^n e^{-n}\exp\left[-\lambda n^2\left(\frac{1}{16\pi^2\varepsilon}+\frac{1+\gamma+\log\pi}{32\pi^2}\right)\right]}{\left[\Omega_{d-1}(d-2)\right]^n \left(x_{fi}^2\right)^{n\left(\frac{d}{2}-1\right)+\frac{\lambda n^2}{32\pi^2}}}. \tag{2.56}$$

This expression[8] reproduces the result of the standard perturbative computation (2.44) up to subleading terms at large n. Remarkably, the $O(\lambda n^2)$ correction to the scaling dimension results in (2.44) from a genuine one-loop computation, while it results in (2.56) from the *classical* solution of the saddle point equations (2.46). According to our discussion, the subleading $O(\lambda n)$ contribution to γ_{ϕ^n} in eq. (2.44), would instead arise from the first quantum correction around the saddle, i.e. from Γ_0 in eq. (2.38). Our alternative semiclassical computation shows that the $O(\lambda n^2)$ contribution to γ_{ϕ^n} is a genuinely classical contribution, while the $O(\lambda n)$ is intrinsically quantum. The emergence of classical physics in the presence of large quantum numbers, n in this case, is a crucial fact of physics. Our case here is closely analogous to the relation between the classical approximation to the squared angular momentum, ℓ^2, and the exact quantum result, $\ell(\ell+1)$ (see ref. [75] for an illustration).

2.3.3 Semiclassics at finite λn

Finding the solution of (2.46) is in general a technically challenging task, but symmetries can help tackle the difficulties. In the case at hand the relevant ones are $U(1)$ symmetry, rotational invariance and dilations. The latter is lacking in our theory away from the Wilson-Fisher fixed point, and we can now explain how this is a problem. Starting with $U(1)$, the conservation of the associated Noether current

$$j_\mu = \bar{\phi}\partial_\mu\phi - \phi\partial_\mu\bar{\phi}. \tag{2.57}$$

provides powerful insight. The field insertions in (2.45) act as a source for the current (2.57). Indeed, from the equations of motion (2.46) we get

$$\partial_\mu j^\mu = n\delta^{(d)}(x-x_i) - n\delta^{(d)}(x-x_f). \tag{2.58}$$

We can then use Gauss law to determine the flux of the current through a sphere centered at x_i with radius r:

$$\oint_{x_i} d\Omega_{d-1}\, r^{d-1} j^\mu(x) n_\mu(x) = n\,\theta\left(|x_f-x_i|-r\right), \tag{2.59}$$

[8] This expression was recently derived also in [96], where the authors considered the correlator in the $\lambda\to 0$ limit with λn^2 fixed, clearly corresponding to small λn. This is just a particular limit of the general formula (2.43), as our approach makes clear.

where $n_\mu(x)$ is the unit vector orthogonal to the sphere at point x. Sufficiently close to the point x_i, i.e. for $|x - x_i| \ll |x_f - x_i|$, we expect the solution of (2.46) to be approximately spherically symmetric. In this regime, we then conclude from eq. (2.59) that the current is given by

$$j_\mu(x) = \frac{n}{\Omega_{d-1}} \frac{(x - x_i)_\mu}{|x - x_i|^d} \left[1 + O\left(\frac{|x - x_i|}{|x_f - x_i|}\right)\right] . \tag{2.60}$$

This equation provides a simple constraint involving both ϕ and $\bar{\phi}$. Unfortunately it is not enough to fix their coordinate dependence. In fact, even in the regime $|x-x_i| \ll |x_f - x_i|$, where spherical symmetry is expected, the radial dependence of the solution is non-trivial, as one can convince oneself by making eq. (2.52) explicit. The origin of such a complicated dependence is the lack of dilation invariance of generic $\lambda\phi^4$ in d-dimension. Notice, instead that in the free case, where dilations are a symmetry, the solution displays a simple scaling behaviour [102]. Working in strictly $d = 4$, where ϕ^4 is scale invariant is also not an option, because of the need for regulation[9]. We thus conclude that the only way forward in order to more easily derive the solution is to work directly at the Wilson-Fisher fixed point, where we can profit from the bonus of scale invariance. That also matches well, and not unrelatedly, the fact that only at the fixed point is the anomalous dimension a fully physical quantity.

2.4 Finite λn in the conformal theory

We now consider the theory at its Wilson-Fisher fixed point, where the spacetime symmetry is enhanced to the conformal group. We exploit the consequences of conformal symmetry, especially the possibility to use radial quantization and the state-operator correspondence, to simplify the problem. Relatedly, conformal invariance allows to map our theory from the plane to the cylinder

$$\mathbb{R}^d \to \mathbb{R} \times S^{d-1} , \tag{2.61}$$

in such a way that the dilations on the plane are mapped to time translations on the cylinder. Correspondingly, the spectrum of operator dimensions on the plane, the eigenvalues of the dilation charge D, are mapped to the energy spectrum on the cylinder, the eigenvalues of H_{cyl}. Our goal of computing the dimension of $[\phi^n]$ is thus mapped into the computation of the energy of the corresponding state on the cylinder. The advantage offered by this viewpoint is that time translations on the cylinder, unlike dilations on the plane, are a symmetry also away from the fixed point. When mapping our semiclassical computation to the cylinder, we will thus have an additional symmetry controlling the classical solution, even away from criticality. In other words, while in the approach of

[9] If we contented ourselves with the leading semiclassical approximation we could work in $d = 4$ and regulate ϕ^n by point splitting.

the previous section, a simple scaling ansatz for the radial dependence of the solution is inconsistent, given the lack of scale invariance in the regulated theory, on the cylinder it is possible to consistently look for a solution that is stationary in time. That enormously simplifies our task. Of course, we must stress that this very non trivial simplification only works at the fixed point.

2.4.1 Weyl map to the cylinder

The idea of mapping the theory to the cylinder was introduced in section 1.1. One can also denote the position on the sphere by a unit vector $\vec{n} \in \mathbb{R}^d$, such that the mapping is summed up as

$$x^\mu = R\, e^{\tau/R}\, n^\mu(\theta), \quad \tau = R \ln\left(\frac{|x|}{R}\right), \quad |n| = 1. \tag{2.62}$$

The cylinder metric is then related to the flat one by a Weyl rescaling

$$ds^2_{cyl} = d\tau^2 + R^2 d\Omega^2_{d-1} = \frac{R^2}{r^2} ds^2_{flat}\,. \tag{2.63}$$

The fields are also rescaled by a Weyl factor. A primary operator $\mathcal{O}$ on the plane with scaling dimension $\Delta_\mathcal{O}$ has a counterpart denoted $\hat{\mathcal{O}}$ on the cylinder given by

$$\hat{\mathcal{O}}(\tau, \Omega_{d-1}) = r^{\Delta_\mathcal{O}} \mathcal{O}(x). \tag{2.64}$$

The bare fields appearing in the action are rescaled with their engineering dimension

$$\hat{\phi}(\tau, \Omega_{d-1}) = r^{\frac{d-2}{2}} \phi(x) \tag{2.65}$$

such that the bare action of the theory on the cylinder reads[10]

$$S_{cyl} = \int d^d x \sqrt{g} \left[g^{\mu\nu} \partial_\mu \bar{\phi} \partial_\nu \phi + m^2 \bar{\phi}\phi + \frac{\lambda_0}{4} \left(\bar{\phi}\phi\right)^2 \right], \tag{2.66}$$

where the mass term $m^2 = \left(\frac{d-2}{2R}\right)^2$ arises from the $\mathcal{R}(g)\bar{\phi}\phi$ coupling to the Ricci scalar which is enforced by conformal invariance[11] [103]. Weyl invariance[12] at the fixed point ensures that the flat space theory (2.1) is equivalent[13] to the one on the cylinder described by (2.66).

[10]From this point forward we will be working with canonically normalized fields, which is to say

$$\langle 0 | [\mathcal{O}](x) [\mathcal{O}](y) | 0 \rangle = (x - y)^{-2\Delta_\mathcal{O}}.$$

[11]Hence, at the fixed point, m^2 is not renormalized by loop effects.

[12]The Weyl anomaly does not affect correlation functions of local operators [34].

[13]Note that we dropped a factor of r in front of the interaction term in (2.66) compared to what is dictated by Weyl invariance. This however does not affect the conclusion that both theories are equivalent at their fixed point, which is the IR-fixed point of both RG flows.

A particularly simple application of this mapping is the two-point function of a scalar primary operator $\mathcal{O}$ of scaling dimension $\Delta_{\mathcal{O}}$ and its conjugate

$$\langle \hat{\mathcal{O}}^\dagger(x_f)\hat{\mathcal{O}}(x_i)\rangle = |x_f|^{\Delta_{\mathcal{O}}}|x_i|^{\Delta_{\mathcal{O}}}\langle \mathcal{O}^\dagger(x_f)\mathcal{O}(x_i)\rangle \equiv \frac{|x_f|^{\Delta_{\mathcal{O}}}|x_i|^{\Delta_{\mathcal{O}}}}{|x_f-x_i|^{2\Delta_{\mathcal{O}}}}. \tag{2.67}$$

Now, the limit $x_i \to 0$ on the plane translates to $\tau_i \to -\infty$ on the cylinder and the above equation becomes

$$\langle \mathcal{O}^\dagger(x_f)\mathcal{O}(x_i)\rangle_{\text{cyl}} \overset{\tau_i\to-\infty}{=} e^{-E_{\mathcal{O}}(\tau_f-\tau_i)}, \qquad E_{\mathcal{O}} = \Delta_{\mathcal{O}}/R\,. \tag{2.68}$$

More precisely one can check that the rate of approach to the above limiting result is controlled by $e^{\tau_i/R}$. So that the above equation holds with exponential precision for $|\tau_i/R| \gg 1$. By eq. (2.68) the action of $\mathcal{O}(x_i)$ at $\tau_i \to -\infty$ simply creates a state on the cylinder with energy $\Delta_{\mathcal{O}}/R$ and carrying all the global quantum numbers of $\mathcal{O}$. This is an instance of the operator state correspondence, which greatly illuminates many aspects of conformal field theory when viewed on the cylinder. The same result can be obtained by sending the point x_f to infinity ($\tau_f \to +\infty$).

In the following, we shall consider $\mathcal{O} = \phi^n$, and $\mathcal{O}^\dagger = \bar{\phi}^n$. By the same argument as just above, the two-point function $\langle \bar{\phi}^n(x_f)\phi^n(x_i)\rangle$, with $\tau_{f,i} = \pm T/2$, for $T \to \infty$ directly yields the scaling dimension Δ_{ϕ^n}

$$\langle \bar{\phi}^n(x_f)\phi^n(x_i)\rangle \overset{T\to\infty}{=} \mathcal{N}e^{-E_{\phi^n}T}, \qquad E_{\phi^n} = \Delta_{\phi^n}/R, \tag{2.69}$$

where the (divergent) coefficient $\mathcal{N}$ is independent of T.

To compute the two point function we can then proceed with the methodology discussed in section 2.3.1. The result will have the structure of eq. (2.38). Upon separating out the divergent and finite part of the $\lambda_0^\kappa\Gamma_\kappa$'s, we will have a T independent divergent piece determining the normalization factor $\mathcal{N}$, while the T dependent part will be finite when written in terms of $\lambda(M)$ and linear in T for $T \gg R$. The linearity in T will follow provided the solution is stationary in time, which it will be, thanks to time translation invariance of the action regardless of the theory being at the fixed point. Similarly to eq. (2.43) we shall thus have

$$\begin{aligned}\Delta_{\phi^n} = RE_{\phi^n} &= \frac{1}{\lambda_0}e_{-1}(\lambda_0 n, d) + e_0(\lambda_0 n, d) + \lambda_0 e_1(\lambda_0 n, d) + \ldots \\ &= \frac{1}{\lambda}\bar{e}_{-1}(\lambda n, RM, d) + \bar{e}_0(\lambda n, RM, d) + \lambda\bar{e}_1(\lambda n, RM, d) + \ldots\end{aligned} \tag{2.70}$$

where $\lambda \equiv \lambda(M)$ and $\bar{e}_k$ is defined from the e_k's analogolously to $\bar{\Gamma}_k$ in eq. (2.42). By choosing $\lambda = \lambda_*$ the dependence on RM will have to drop by scale invariance giving a result of the form

$$\Delta_{\phi^n} = \frac{1}{\lambda_*}\Delta_{-1}(\lambda_* n) + \Delta_0(\lambda_* n) + \lambda_* \Delta_1(\lambda_* n) + \dots . \tag{2.71}$$

2.4.2 Semiclassical Saddle

We now proceed to the explicit implementation of the semiclassical method. We do this in a slightly more general case, considering $(N+2)$-point correlators of the form[14]

$$\langle \bar{\phi}^n(x_f)\mathcal{O}_N(x_N)\dots\mathcal{O}_1(x_1)\phi^n(x_i)\rangle, \tag{2.72}$$

where the $\mathcal{O}_i$ are operators with no large quantum numbers. The trick is then to use conformal invariance to send the insertion of ϕ^n ($\bar{\phi}^n$) to the origin $x_i \to 0$ (to infinity $x_f \to \infty$), which on the cylinder translates to infinite past and future times $\tau_{i,f} \to \pm\infty$. In terms of the variables on the cylinder the correlator (2.72) has the form

$$\langle \hat{\bar{\phi}}^n(\tau_f)\hat{\mathcal{O}}_N(\tau_N)\dots\hat{\mathcal{O}}_1(\tau_1)\hat{\phi}^n(\tau_i)\rangle e^{-\Delta_{\phi^n}\tau_f}e^{-\Delta_N\tau_N}\dots e^{-\Delta_1\tau_1}e^{-\Delta_{\phi^n}\tau_i}\,, \tag{2.73}$$

where for simplicity we did not indicate the dependence of the operators on the angular coordinates. Δ_i denotes the scaling dimension of operator $\mathcal{O}_i$. Using the operator-state correspondence (see section 4.2.2 for more details) we define the normalized[15] charge-n state associated to the ϕ^n operator

$$|n\rangle = \frac{(4\pi)^{n/2}}{\sqrt{n!}}\phi^n(0)|0\rangle = \frac{(4\pi)^{n/2}}{\sqrt{n!}}\lim_{\tau\to-\infty} e^{-\Delta_{\phi^n}\tau_i}\hat{\phi}^n(\tau_i)|0\rangle\,, \tag{2.74}$$

and its conjugate

$$\langle n| = \frac{(4\pi)^{n/2}}{\sqrt{n!}}\lim_{\tau_f\to\infty}\langle 0|e^{\Delta_{\phi^n}\tau_f}\hat{\bar{\phi}}^n(\tau_f)\,. \tag{2.75}$$

Eq. (2.72) is thus related to cylinder correlators according to

$$\lim_{x_f\to\infty}\frac{(4\pi)^n}{n!}x_f^{2\Delta_{\phi^n}}\langle\bar{\phi}^n(x_f)\mathcal{O}_N(x_N)\dots\mathcal{O}_1(x_1)\phi^n(0)\rangle = \langle n|\hat{\mathcal{O}}_N(\tau_N)\dots\hat{\mathcal{O}}_1(\tau_1)|n\rangle\prod_{j=1}^N e^{-\Delta_j\tau_j}. \tag{2.76}$$

The state $|n\rangle$, corresponding to operator ϕ^n, is the state of charge n with the lowest

[14]More precisely, we compute the correlator of renormalized operators. We drop the square brackets in the remainder of this chapter for readability.

[15]We define the renormalized $[\phi^n]$ to be normalized in such a way that the state $|n\rangle$ has unit norm.

energy. Then, for any charge-n state $|\psi_n\rangle$ with non-zero overlap with $|n\rangle$, we have

$$\lim_{\tau_i \to -\infty} e^{H\tau_i}|\psi_n\rangle = \lim_{\tau_i \to -\infty} e^{\Delta_{\phi^n}\tau_i}|n\rangle\langle n|\psi_n\rangle \tag{2.77}$$

$$\lim_{\tau_f \to \infty} \langle\psi_n|e^{-H\tau_f} = \lim_{\tau_f \to \infty} e^{-\Delta_{\phi^n}\tau_f}\langle\psi_n|n\rangle\langle n|, \tag{2.78}$$

where H is the Hamiltonian on the cylinder.

Therefore we can also write

$$\langle n|\hat{\mathcal{O}}_N(\tau_N)\dots\hat{\mathcal{O}}_1(\tau_1)|n\rangle = \lim_{\substack{\tau_f \to \infty \\ \tau_i \to -\infty}} \frac{\langle\psi_n|e^{-H\tau_f}\hat{\mathcal{O}}_N(\tau_N)\dots\hat{\mathcal{O}}_1(\tau_1)e^{H\tau_i}|\psi_n\rangle}{\langle\psi_n|e^{-H(\tau_f-\tau_i)}|\psi_n\rangle}. \tag{2.79}$$

The right hand side can be represented by a path integral. For that purpose, it is useful to introduce polar coordinates for the fields

$$\hat{\phi} = \frac{\rho}{\sqrt{2}}e^{i\chi}, \quad \hat{\bar{\phi}} = \frac{\rho}{\sqrt{2}}e^{-i\chi}\,, \tag{2.80}$$

and single out their zero modes on the sphere

$$\chi = \chi_0 + \chi_\perp, \quad \int \chi(\vec{n})d\Omega_{d-1} = \chi_0\Omega_{d-1}, \quad \int \chi_\perp(\vec{n})d\Omega_{d-1} = 0\,, \tag{2.81}$$

$$\rho = \rho_0 + \rho_\perp, \quad \int \rho(\vec{n})d\Omega_{d-1} = \rho_0\Omega_{d-1}, \quad \int \rho_\perp(\vec{n})d\Omega_{d-1} = 0\,. \tag{2.82}$$

with $\mathrm{d}\Omega_{d-1} = \mathrm{d}^{d-1}\theta\,\sqrt{g}$ the volume element of S^{d-1}. A convenient choice for the state $|\psi_n\rangle$ is then

$$\langle\rho,\chi|\psi_n\rangle = \delta(\rho_0 - f)\delta(\rho_\perp)\delta(\chi_\perp)e^{in\chi_0}, \tag{2.83}$$

with f a constant whose value will be suitably decided below. This represents a state of homogeneous charge density. As a result eq. (2.79) can be recast as

$$\langle n|\hat{\mathcal{O}}_N\dots\hat{\mathcal{O}}_1|n\rangle \underset{\substack{\tau_f \to \infty \\ \tau_i \to -\infty}}{=} \mathcal{Z}^{-1}\int d\chi_i d\chi_f e^{-\frac{in(\chi_f-\chi_i)}{\Omega_{d-1}}} \int_{\substack{\rho(\tau_i)=f \\ \chi(\tau_i)=\chi_i}}^{\substack{\rho(\tau_f)=f \\ \chi(\tau_f)=\chi_f}} \mathcal{D}\rho\mathcal{D}\chi\,\hat{\mathcal{O}}_N\dots\hat{\mathcal{O}}_1\,e^{-S[\rho,\chi]}, \tag{2.84}$$

with

$$\mathcal{Z} = \int d\chi_i d\chi_f e^{-\frac{in(\chi_f-\chi_i)}{\Omega_{d-1}}} \int_{\substack{\rho(\tau_i)=f \\ \chi(\tau_i)=\chi_i}}^{\substack{\rho(\tau_f)=f \\ \chi(\tau_f)=\chi_f}} \mathcal{D}\rho\mathcal{D}\chi\,e^{-S[\rho,\chi]}\,. \tag{2.85}$$

and where the action is given by

$$S[\rho,\chi] = \int d\tau d\Omega_{d-1}\left[\frac{1}{2}(\partial\rho)^2 + \frac{1}{2}\rho^2(\partial\chi)^2 + \frac{1}{2}m^2\rho^2 + V_{int}(\rho)\right] \tag{2.86}$$

with $m = \frac{d}{2} - 1$ and

$$V_{int}(\rho) = \frac{\lambda}{16}\rho^4\,. \tag{2.87}$$

The saddle point is fixed by two conditions, corresponding to the variation of the action with respect to ϕ in the bulk and on the boundary. The latter, in view of eq. (2.83), reduces to variation with respect to the zero modes of χ, χ_i and χ_f. From the bulk we have

$$\begin{aligned} \partial_\mu\left(\sqrt{g}g^{\mu\nu}\rho^2\partial_\nu\chi\right) &= 0, &(2.88)\\ -\partial^2\rho+\rho\left[(\partial\chi)^2+m^2\right]+\partial_\rho V_{int}(\rho) &= 0 &(2.89)\end{aligned}$$

with $g_{\mu\nu}$ the metric on the cylinder. The first equation, corresponding to variation with respect to χ, coincides with $U(1)$ current conservation. The variation at the boundaries gives instead

$$(\rho^2\dot{\chi})(\tau_i)=(\rho^2\dot{\chi})(\tau_f)=-\frac{in}{\Omega_{d-1}}, \tag{2.90}$$

which fixes the charge to be n and spatially homogeneous at the boundaries. Equations (2.88), (2.89), (2.90) along with the constraint (2.83) have the simple solution

$$\rho_S(\tau)=f,\quad \chi_S(\tau)=-i\mu(\tau-\tau_i)+\chi_i, \tag{2.91}$$

with μ and f satisfying

$$\begin{aligned} \mu^2-m^2 &= \frac{1}{f}\frac{\partial V_{int}(f)}{\partial f}=\frac{\lambda}{4}f^2\,, &(2.92)\\ f^2\mu &= \frac{n}{\Omega_{d-1}}. &(2.93)\end{aligned}$$

A few comments are in order. The last two equations determine the "suitable" value of f, we alluded to below its definition in (2.83). It is only for this specific choice of f in (2.83) that the saddle point equations have a solution with a simple linear time dependence. Other choices would give solutions with a more complicated behaviour near the boundaries, but for $(\tau_f-\tau_i)\to\infty$ the result for (2.84) would be the same.

Given the constraint $f^2\geq 0$, imposed by the boundary condition $\rho_i=\rho_f=f\in\mathbb{R}$, these equations admit a unique solution for f^2 and μ. On this profile χ is analytically continued to the complex plane (see the comments below (2.49)). Notice that the condition $f^2\geq 0$ implies that the solution for μ is discontinuous at $\lambda_0 n=0$. This can be seen easily substituting (2.93) into (2.92)

$$\mu(\mu^2-m^2)=\frac{\lambda_0 n}{4R^{d-1}\Omega_{d-1}}\quad\text{with}\quad n/\mu\geq 0, \tag{2.94}$$

where the last inequality follows from the reality condition on f. Its solution is

$$\mu=\mu_4(\lambda n,d)=\frac{(d-2)}{2}\frac{\left(3^{1/3}+\left[\frac{9\lambda n\Gamma(d/2)}{2\pi^{d/2}(d-2)^3}-\sqrt{\left(\frac{9\lambda n\Gamma(d/2)}{2\pi^{d/2}(d-2)^3}\right)^2-3}\right]^{2/3}\right)}{3^{2/3}\left[\frac{9\lambda n\Gamma(d/2)}{2\pi^{d/2}(d-2)^3}-\sqrt{\left(\frac{9\lambda n\Gamma(d/2)}{2\pi^{d/2}(d-2)^3}\right)^2-3}\right]^{1/3}} \tag{2.95}$$

where we have defined a function denoted μ_4 (4 as in quartic) to distinguish from the different formula μ_6 which is obtained in the case of the sextic interaction (see section 2.5). It is then obvious that the, otherwise analytical, solution of (2.94) satisfies $\mu(\lambda_0 n)=-\mu(-\lambda_0 n)$, implying the existence of a discontinuity for $\lambda_0 n\to 0$, where $\mu\simeq \mathrm{sgn}(n)\left[m+O\left(\lambda_0 n\right)\right]$. As a consequence of the latter, also the scaling dimension Δ_{ϕ^n} will be non-analytic at $\lambda n=0$. This reflects the physical fact that the scaling dimension of ϕ^n and the operator with opposite charge, $\bar{\phi}^n$, are the same; as the expansion (2.32) contains odd powers of n, the physical scaling dimension cannot be continous at $n=0$. In the following, we implicitly consider only $n>0$.

Notice also that the solution (2.91) is invariant under the combination $H-\mu Q$ of time translations and $U(1)$ charge rotations, which means it describes a superfluid phase [88, 89], with homogeneous charge density $j_0=\mu f^2$ and chemical potential given by μ. Finally notice that, while $\chi_f-\chi_i=-i\mu(\tau_f-\tau_i)$ is fixed by (2.91), the zero mode χ_i is not: integrating over it guarantees that correlators respect charge conservation.

Expanding around the saddle we can systematically compute any observable as a power series in λ with coefficients that are themselves functions of λn. For instance, given

$$\lim_{\substack{\tau_f\to\infty\\ \tau_i\to-\infty}}\langle\psi_n|e^{-H(\tau_f-\tau_i)}|\psi_n\rangle=e^{-\Delta_{\phi^n}(\tau_f-\tau_i)}|\langle n|\psi_n\rangle|^2, \tag{2.96}$$

and its path integral representation (2.85), the evaluation of the action on the saddle point immediately gives the scaling dimension of ϕ^n at leading order shown in (2.70). Same goes for the $\hat{\mathcal{O}}_i$ insertions, which are local functions of ρ and χ. At the leading order the correlator (2.72) is then simply given by the product of the $\hat{\mathcal{O}}_i$ computed on the saddle. The computation of higher order terms require to consider fluctuations around the semiclassical background.

2.4.3 Fluctuations

Expanding the fields in (2.84) around the saddle

$$\rho=\rho_S+r,\quad \chi=\chi_S+\frac{\pi}{f}. \tag{2.97}$$

we can now write

$$\langle n|\hat{\mathcal{O}}_N \dots \hat{\mathcal{O}}_1|n\rangle = \frac{\int d\chi_i \int \mathcal{D}r\mathcal{D}\pi\, \hat{\mathcal{O}}_N \dots \hat{\mathcal{O}}_1 e^{-\hat{S}[r,\pi]}}{2\pi \int \mathcal{D}r\mathcal{D}\pi\, e^{-\hat{S}[r,\pi]}}, \tag{2.98}$$

where the action for the fluctuations is given by

$$\hat{S}[r,\pi] = \int d\tau d\Omega_{d-1} \left(\mathcal{L}_2 + \mathcal{L}_{int}\right), \tag{2.99}$$

with

$$\mathcal{L}_2 = \frac{1}{2}(\partial r)^2 + \frac{1}{2}(\partial\pi)^2 - 2i\mu r\dot{\pi} + \frac{1}{2}\left[V''_{int}(f) - (\mu^2 - m^2)\right] r^2, \tag{2.100}$$

and

$$\mathcal{L}_{int} = \frac{1}{f}\left[r(\partial\pi)^2 - i\mu r^2\dot{\pi}\right] + \frac{r^2(\partial\pi)^2}{2f^2} + \left[V_{int}(f+r) - \left(V_{int}(f) + V'_{int}(f)r + \frac{1}{2}V''_{int}(f)r^2\right)\right]. \tag{2.101}$$

The canonically conjugated momenta[16] forming pairs (r, P) and (π, Π) are

$$P = i\dot{r}, \quad \Pi = i\dot{\pi}\left(1 + \frac{r}{f}\right)^2 + 2\mu r\left(1 + \frac{r}{2f}\right). \tag{2.102}$$

These variables can be expanded in harmonic modes as

$$\begin{pmatrix} r(\tau,\vec{n}) \\ \pi(\tau,\vec{n}) \end{pmatrix} = \sum_{\ell=0}^{\infty}\sum_{\vec{m}} \begin{pmatrix} r_{\ell\vec{m}}(\tau) \\ \pi_{\ell\vec{m}}(\tau) \end{pmatrix} Y_{\ell\vec{m}}(\vec{n}), \quad \begin{pmatrix} P(\tau,\vec{n}) \\ \Pi(\tau,\vec{n}) \end{pmatrix} = \sum_{\ell=0}^{\infty}\sum_{\vec{m}} \begin{pmatrix} P_{\ell\vec{m}}(\tau) \\ \Pi_{\ell\vec{m}}(\tau) \end{pmatrix} Y^*_{\ell\vec{m}}(\vec{n}), \tag{2.103}$$

where $Y_{\ell\vec{m}}(\vec{n})$ are the spherical harmonics in $d-1$ dimensions[17] satisfying

$$\Delta_{S^{d-1}} Y_{\ell\vec{m}}(\vec{n}) = -J_\ell^2 Y_{\ell\vec{m}}(\vec{n}), \tag{2.105}$$

where $\Delta_{S^{d-1}}$ is the Laplacian on the sphere S^{d-1} and where J_ℓ^2 is the eigenvalue of the $SO(d)$ casimir

$$J_\ell^2 = \ell(\ell + d - 2), \tag{2.106}$$

The $Y_{\ell\vec{m}}(\vec{n})$ also satisfy the normalization and completeness conditions

$$\int Y_{\ell\vec{m}}(\vec{n}) Y^*_{\ell'\vec{m}'}(\vec{n}) d\Omega_{d-1} = \delta_{\ell\ell'}\delta_{\vec{m}\vec{m}'}, \tag{2.107}$$

[16] The presence of "i" in front of time derivatives is because we work in Euclidean time.

[17] $\vec{m}$ is a multi-index taking

$$N_{\ell,d} = (2\ell + d - 2)\frac{(\ell + d - 3)!}{(d-2)!\ell!} \tag{2.104}$$

different values.

and

$$\sum_{\ell=0}^{\infty}\sum_{\vec{m}} Y_{\ell\vec{m}}(\vec{n})Y^*_{\ell\vec{m}}(\vec{n}') = \delta^{(S^{d-1})}(\vec{n}-\vec{n}'). \tag{2.108}$$

Notice in particular that $Y_{0\vec{0}} = 1/\sqrt{\Omega_{d-1}}$. The harmonic modes are canonical variables satisfying equal-time commutation relations

$$\begin{aligned}
&[r(\tau,\vec{n}),P(\tau,\vec{n}')] = i\delta(\vec{n}-\vec{n}') \Leftrightarrow [r_{\ell\vec{m}}(\tau),P_{\ell'\vec{m}'}(\tau)] = i\delta_{\ell\ell'}\delta_{\vec{m}\vec{m}'},\\
&[\pi(\tau,\vec{n}),\Pi(\tau,\vec{n}')] = i\delta(\vec{n}-\vec{n}') \Leftrightarrow [\pi_{\ell\vec{m}}(\tau),\Pi_{\ell'\vec{m}'}(\tau)] = i\delta_{\ell\ell'}\delta_{\vec{m}\vec{m}'},
\end{aligned} \tag{2.109}$$

with the other commutators vanishing.

Linearized fluctuations

In section 4.3.1 we will need the modes evolving according to the full lagrangian. To set the basis of the semiclassical expansion we must however consider the modes of the quadratic Lagrangian (2.100)

$$\mathcal{L}_2 = \frac{1}{2}(\partial r)^2 + \frac{1}{2}(\partial\pi)^2 - 2i\mu r\dot{\pi} + \frac{1}{2}M^2 r^2, \tag{2.110}$$

with

$$M^2 = V''_{int}(f) - f^{-1}V'_{int}(f) = V''_{int}(f) - (\mu^2-m^2)\,. \tag{2.111}$$

At this order the canonical momenta are

$$\tilde{P} = i\dot{r}, \quad \tilde{\Pi} = i\dot{\pi} + 2\mu r. \tag{2.112}$$

The quantized fields (and the spectrum) are obtained by considering the linearized equations of motion

$$\begin{pmatrix} \partial_\tau^2 + \Delta_{S^{d-1}} - M^2 & 2i\mu\partial_\tau \\ -2i\mu\partial_\tau & \partial_\tau^2 + \Delta_{S^{d-1}} \end{pmatrix}\begin{pmatrix} r \\ \pi \end{pmatrix} = 0, \tag{2.113}$$

and by finding the complete set of harmonic mode solutions of the form

$$\begin{pmatrix} r_{\ell\vec{m}}(\tau) \\ \pi_{\ell\vec{m}}(\tau) \end{pmatrix} Y_{\ell\vec{m}}(\vec{n}) = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} e^{-\omega\tau} Y_{\ell\vec{m}}(\vec{n})\,. \tag{2.114}$$

For $\ell \neq 0$ the following is a solution for arbitrary coefficients $C_A(\ell,\vec{m})$ and $C_B(\ell,\vec{m})$ (we impose reality condition $r(\tau)^\dagger = r(-\tau)$ and $\pi(\tau)^\dagger = \pi(-\tau)$)

$$\begin{aligned}
\begin{pmatrix} r \\ \pi \end{pmatrix}_{\ell\vec{m}} &= C_A(\ell,\vec{m})\begin{pmatrix} J_\ell^2 - \omega_A^2(\ell) \\ 2i\mu\omega_A(\ell) \end{pmatrix} e^{-\omega_A(\ell)\tau} Y_{\ell\vec{m}} + C_A^*(\ell,\vec{m})\begin{pmatrix} J_\ell^2 - \omega_A^2(\ell) \\ -2i\mu\omega_A(\ell) \end{pmatrix} e^{\omega_A(\ell)\tau} Y^*_{\ell\vec{m}} \\
&\quad + C_B(\ell,\vec{m})\begin{pmatrix} \omega_B^2(\ell) - J_\ell^2 \\ -2i\mu\omega_B(\ell) \end{pmatrix} e^{-\omega_B(\ell)\tau} Y_{\ell\vec{m}} + C_B^*(\ell,\vec{m})\begin{pmatrix} \omega_B^2(\ell) - J_\ell^2 \\ 2i\mu\omega_B(\ell) \end{pmatrix} e^{\omega_B(\ell)\tau} Y^*_{\ell\vec{m}},
\end{aligned}$$

while for $\ell = 0$ we get a solution for arbitrary r_0 and $\hat{\pi}$

$$\begin{aligned}\begin{pmatrix} r \\ \pi \end{pmatrix}_{0,\vec{0}} &= \begin{pmatrix} r_0 \\ \hat{\pi} - i\frac{M^2 r_0}{2\mu}\tau \end{pmatrix} Y_{0,\vec{0}} \\ &+ C_B(0,\vec{0}) \begin{pmatrix} \omega_B^2(0) \\ -2i\mu\omega_B(0) \end{pmatrix} e^{-\omega_B(0)\tau} Y_{0,\vec{0}} + C_B^*(0,\vec{0}) \begin{pmatrix} \omega_B^2(0) \\ 2i\mu\omega_B(0) \end{pmatrix} e^{\omega_B(0)\tau} Y_{0,\vec{0}}.\end{aligned}$$

Computing also the corresponding momenta

$$\begin{pmatrix} P \\ \Pi \end{pmatrix} = \begin{pmatrix} \dot{r} \\ \dot{\pi} + 2\mu r \end{pmatrix}, \tag{2.115}$$

and imposing canonical commutation relations (2.109) we get

$$\begin{aligned}\begin{pmatrix} r \\ \pi \end{pmatrix} &= \begin{pmatrix} \dfrac{2\mu}{\omega_B^2(0)} p_\pi \\ \hat{\pi} - i p_\pi \tau \left(1 - \dfrac{4\mu^2}{\omega_B^2(0)}\right) \end{pmatrix} Y_{0\vec{0}} \qquad (2.116) \\ &+ \sum_{\ell=1}^{\infty} \sum_{\vec{m}} \sqrt{\frac{\omega_A(\ell)}{2\left[\omega_B^2(\ell) - \omega_A^2(\ell)\right]}} \left[\begin{pmatrix} \sqrt{\dfrac{J_\ell^2}{\omega_A^2(\ell)} - 1} \\ i\sqrt{\dfrac{\omega_+^2(\ell)}{J_\ell^2} - 1} \end{pmatrix} A_{\ell\vec{m}} Y_{\ell\vec{m}} e^{-\omega_A(\ell)\tau} + h.c. \right] \\ &+ \sum_{\ell=0}^{\infty} \sum_{\vec{m}} \sqrt{\frac{\omega_B(\ell)}{2\left[\omega_B^2(\ell) - \omega_A^2(\ell)\right]}} \left[\begin{pmatrix} \sqrt{1 - \dfrac{J_\ell^2}{\omega_B^2(\ell)}} \\ -i\sqrt{1 - \dfrac{\omega_A^2(\ell)}{J_\ell^2}} \end{pmatrix} B_{\ell\vec{m}} Y_{\ell\vec{m}} e^{-\omega_B(\ell)\tau} + h.c. \right],\end{aligned}$$

where operators $(A_{\ell\vec{m}}, A^\dagger_{\ell\vec{m}})$, $(B_{\ell\vec{m}}, B^\dagger_{\ell\vec{m}})$ and $(\hat{\pi}, p_\pi)$ are canonically conjugated pairs:

$$[A_{\ell\vec{m}}, A^\dagger_{\ell'\vec{m}'}] = \delta_{\ell\ell'}\delta_{\vec{m}\vec{m}'}, \quad [B_{\ell\vec{m}}, B^\dagger_{\ell'\vec{m}'}] = \delta_{\ell\ell'}\delta_{\vec{m}\vec{m}'}, \quad [\hat{\pi}, p_\pi] = i, \tag{2.117}$$

with all other commutators vanishing. Notice that the $A_{\ell\vec{m}}$ are defined for $\ell \geq 1$ and have frequency $\omega_A(\ell)$, while the $B_{\ell\vec{m}}$ are defined for $\ell \geq 0$ and have frequency $\omega_B(\ell)$. The role of the $\ell = 0$ mode in the A sector is played by $\hat{\pi}$. The frequencies are given by

$$\begin{aligned}\omega_A^2(\ell) &= J_\ell^2 + \frac{V''_{int}(f) + 3\mu^2 + m^2}{2} - \sqrt{\left(\frac{V''_{int}(f) + 3\mu^2 + m^2}{2}\right)^2 + 4\mu^2 J_\ell^2}\,, \\ \omega_B^2(\ell) &= J_\ell^2 + \frac{V''_{int}(f) + 3\mu^2 + m^2}{2} + \sqrt{\left(\frac{V''_{int}(f) + 3\mu^2 + m^2}{2}\right)^2 + 4\mu^2 J_\ell^2}\,.\end{aligned} \tag{2.118}$$

Note that in the last sum's $\ell = 0$ term of (2.117), one has to use the limit

$$\lim_{\ell \to 0} \frac{\omega_A^2(\ell)}{J_\ell^2} = 1 - \frac{4\mu^2}{\omega_B^2(0)}. \tag{2.119}$$

The $\omega_{A,B}^2(\ell)$ determine the energy spectrum of fluctuations around the semiclassical saddle. We will see in chapter 4 that these excitations correspond to higher-energy charge-n states, or equivalently higher-dimensional charge-n operators.

Several features of (2.118) are worth remarking. The first is

$$\omega_A(0) = 0\,. \tag{2.120}$$

This is the manifestation of a Goldstone boson associated with $U(1)$ symmetry breaking around the saddle. The $U(1)$ acts as a constant shift of π, while ρ is invariant. Therefore $A_{\ell\vec{m}}$ and $B_{\ell\vec{m}}$ are all neutral while $\hat{\pi}$ transforms by a constant shift. Notice that the conjugated momentum p_π precisely generates these transformations. Indeed, applying Noether's theorem to the quadratic Lagrangian (2.99) and comparing the result to the generator Q of χ shifts in (2.86), we find

$$p_\pi = (Q - n)\frac{Y_{0\vec{0}}}{f}. \tag{2.121}$$

Up to a factor, the zero mode $\hat{\pi}$ is the phase χ_i that exactly parametrizes the family of solutions at the full non-linear level. It therefore makes sense to treat this mode fully non-linearly, singling it out when expressing ϕ in terms of the harmonic modes

$$\pi(\tau, \vec{n}) = \hat{\pi} Y_{0\vec{0}} + \tilde{\pi}(\tau, \vec{n}), \tag{2.122}$$

and factoring it out from ϕ[18]

$$\hat{\phi}(\tau, \vec{n}) = \frac{f + r}{\sqrt{2}} e^{\mu\tau} e^{i\frac{\hat{\pi} Y_{00}}{f}} e^{i\frac{\tilde{\pi}}{f}} \simeq \frac{f + r}{\sqrt{2}} e^{\mu\tau} e^{i\frac{\hat{\pi} Y_{0\vec{0}}}{f}} (1 + i\frac{\tilde{\pi}}{f}). \tag{2.123}$$

As dictated by the commutation relations and by the definition of the modes, the factor $e^{i\frac{\hat{\pi} Y_{00}}{f}}$ has charge 1 while the fields $r, \tilde{\pi}$ are neutral, which is consistent with the transformation property of $\hat{\phi}$. Thus $\hat{\pi}$ is a cyclic coordinate with periodicity $2\pi f / Y_{0\vec{0}}$ and the canonical pair $(\hat{\pi}, p_\pi)$ does not correspond to a harmonic oscillator with an associated Fock space. The Hamiltonian for this pair is

$$H_{\hat{\pi}} = \frac{p_\pi^2}{2} \left[1 - \left(\frac{4\mu^2}{\omega_B^2(0)} \right)^2 \right]. \tag{2.124}$$

[18] Here we have also absorbed the $i\mu\tau_i$ in (2.91) into $\hat{\pi}/f$ or equivalently set $\tau_i = 0$.

The second important feature is that the B-mode is gapped

$$\omega_B^2(\ell) \geq \omega_B^2(0) = V_{int}''(f) + 3\mu^2 + m^2 > 0\,. \tag{2.125}$$

For large μ, or equivalently large λn (see (2.95)), we could then integrate this mode out and derive an effective field theory description for the Goldstone mode [74, 75], which consists of the $A_{\ell\vec{m}}$ and $\hat{\pi}$.

The third property is that, for the classically scale invariant case, $(\bar{\phi}\phi)^2$ in $d=4$ exactly, we have

$$\omega_A(1) = 1. \tag{2.126}$$

Meaning the excitation created by $A_{1,\vec{m}}^\dagger$ corresponds to a descendent operator. We will come back to the question of operators corresponding to this spectrum, and the classification into primaries and descendants, in chapter 4. There, we will see that this is associated with the fact that $A_{1\vec{m}}$ and $A_{1\vec{m}}^\dagger$ are respectively the $K_{\vec{m}}$ and $P_{\vec{m}}$ generators (see for instance (4.111)). As such they have scaling dimension -1 and 1. Acting with $A_{1\vec{m}}^\dagger$ on a state therefore produces a descendant. Finally, we have that in the free limit, $\lambda = 0$, the two modes become

$$\omega_A(\ell) = \ell, \quad \omega_B(\ell) = \ell + d - 2, \tag{2.127}$$

and at finite coupling their asymptotic behavior is given by

$$\omega_A(\ell) \underset{\ell\to\infty}{=} \omega_B(\ell) \underset{\ell\to\infty}{=} \ell. \tag{2.128}$$

The formalism being fully set up, we can finally turn to the computation of the scaling dimension Δ_{ϕ^n}.

2.4.4 Leading order: Δ_{-1}

As explained above, the action (2.86) evaluated on the semiclassical configuration (2.91) provides the leading order value for the energy (2.70):

$$\frac{1}{\lambda_0}\frac{e_{-1}(\lambda_0 n, d)}{R} = S_{eff}/T = \frac{n}{2}\left(\frac{3}{2}\mu + \frac{1}{2}\frac{m^2}{\mu}\right). \tag{2.129}$$

Had we chosen $\rho_i, \rho_f \neq f$, $\rho(\tau)$ would have approached exponentially fast the value $\rho = f$ away from the boundaries. As a result, in the $T \to \infty$ limit the contribution of the action growing linearly in time is independent of the precise value of the boundary conditions for ρ.

To obtain the leading order Δ_{-1} in (2.71), we consider the classical value for the chemical potential obtained from (2.95) setting $\lambda_0 = \lambda_*$ and $d = 4$ everywhere else:

$$R\mu_* = \frac{3^{1/3} + \left[9\frac{\lambda_* n}{(4\pi)^2} - \sqrt{81\frac{(\lambda_* n)^2}{(4\pi)^4} - 3}\right]^{2/3}}{3^{2/3}\left[9\frac{\lambda_* n}{(4\pi)^2} - \sqrt{81\frac{(\lambda_* n)^2}{(4\pi)^4} - 3}\right]^{1/3}}. \tag{2.130}$$

Plugging in (2.129) and taking $m = 1/R$ we conclude that the classical contribution to the scaling dimension is

$$\frac{1}{\lambda_*}\Delta_{-1} = nF_0\left(\lambda_* n\right), \tag{2.131}$$

where the function F_0 reads:

$$\begin{aligned} F_0(16\pi^2 x) \quad &= \quad \frac{3\left[9x - \sqrt{81x^2-3}\right]^{1/3} + 3^{2/3}\left[9x - \sqrt{81x^2-3}\right]}{\left[\left(9x - \sqrt{81x^2-3}\right)^{2/3} + 3^{1/3}\right]^2} \\ &\quad + \frac{9\times 3^{1/3}x\left[9x - \sqrt{81x^2-3}\right]^{2/3}}{2\left[\left(9x - \sqrt{81x^2-3}\right)^{2/3} + 3^{1/3}\right]^2}. \end{aligned} \tag{2.132}$$

Though not obvious, for $x > 0$ this is a real and positive function, which grows monotonically with x. Remarkably, eq. (2.131) explicitly resums the contribution of infinitely many Feynman diagrams.

The form of the result becomes particularly simple (and interesting) in the two extreme regimes, $\lambda_* n \ll (4\pi)^2$ and $\lambda_* n \gg (4\pi)^2$, where eq. (2.131) reads

$$\frac{\Delta_{-1}}{\lambda_*} = \begin{cases} n\left[1 + \frac{1}{2}\left(\frac{\lambda_* n}{16\pi^2}\right) - \frac{1}{2}\left(\frac{\lambda_* n}{16\pi^2}\right)^2 + O\left(\frac{(\lambda_* n)^3}{(4\pi)^6}\right)\right], & \text{for} \quad \lambda_* n \ll (4\pi)^2, \\ \frac{8\pi^2}{\lambda_*}\left[\frac{3}{4}\left(\frac{\lambda_* n}{8\pi^2}\right)^{4/3} + \frac{1}{2}\left(\frac{\lambda_* n}{8\pi^2}\right)^{2/3} + O\left(1\right)\right], & \text{for} \quad \lambda_* n \gg (4\pi)^2. \end{cases} \tag{2.133}$$

The first line of (2.133) reproduces the result (2.30) up to higher orders and thus provides a non trivial check of our approach. Notice that the agreement is independent of the precise value of λ_*, since at tree-level the Lagrangian (2.1) is Weyl invariant for every value of the coupling and the theory can be safely mapped to the cylinder through a change of coordinates and a field redefinition. In the opposite regime, the result is organized as an expansion in powers of $(\lambda_* n)^{2/3}$, in agreement with the predictions of the large charge expansion in CFT [74, 75].

The parameter which marks the difference between the two regimes is the chemical potential μ_*, since, as we have seen explicitly in section 2.4.3, the latter controls the gap of the radial mode. For small $\lambda_* n$ the chemical potential, is of order of R^{-1}, while in the opposite regime its value is proportional to $j_0^{1/3} \gg R^{-1} \sim (\lambda n)^{1/3}/R$. Henceforth, in this regime we can integrate out this mode and the lightest states at charge n are described by an effective theory for the Goldstone mode only. The form of the effective theory was used in [74, 75] to study the spectrum at large charge in a generic $U(1)$ invariant CFT and derive the form of the expansion in the second line of (2.133), The power law of the first term following just from dimensional analysis. In this regime, the squared sound speed of the Goldstone mode, given by

$$\left(\frac{d\omega_A^2}{dJ_\ell^2}\right)_{\ell=0} = \frac{\mu^2 - m^2}{3\mu^2 - m^2}, \tag{2.134}$$

approaches the value 1/3 dictated by scale invariance in a fluid.

2.4.5 One-loop correction: Δ_0

Let us now compute the first subleading correction Δ_0. We consider the one-loop expression for the path-integral (2.79):

$$\begin{aligned} \langle\psi_n|e^{-HT}|\psi_n\rangle &= e^{-\frac{e_{-1}(\lambda_0 n,d)T}{\lambda_0 R}} \frac{\int \mathcal{D}r\mathcal{D}\pi \,\exp\left[-S^{(2)}\right]}{\int \mathcal{D}\phi\mathcal{D}\bar{\phi}\,\exp\left[-\int_{-T/2}^{T/2}\left(\partial\phi\partial\bar{\phi}+m^2\phi\bar{\phi}\right)\right]} \\ &= \tilde{\mathcal{N}}\exp\left\{-\left[\frac{1}{\lambda_0}e_{-1}(\lambda_0 n,d)+e_0(\lambda_0 n,d)\right]\frac{T}{R}\right\}, \end{aligned} \tag{2.135}$$

where the normalization factor $\tilde{\mathcal{N}}$ is T-independent. The latter contains a factor $\lambda_0^{-1/2}$ coming form the integration over the zero mode (see the comments below (2.37)). Since this factor does not scale with T, it does not contribute to the scaling dimension. The denominator in the first line of (2.135) arises from the normalization factor (2.85). In the second line, the correction to the energy arises from the fluctuation determinant of the Gaussian integrals in the numerator and the denominator. It can be written explicitly in terms of the expressions (2.118) and the formula for the free dispersion relation $\omega_0^2(\ell) = J_\ell^2 + m^2 = \left(\ell + \frac{d-2}{2}\right)^2/R^2$:

$$\begin{aligned} T\frac{e_0}{R} = \log\frac{\sqrt{\det S^{(2)}}}{\det\left(-\partial_\tau^2 - \Delta_{S^{d-1}} + m^2\right)} &= \frac{T}{2}\sum_{\ell=0}^{\infty} n_\ell \int\frac{d\omega}{2\pi}\log\frac{\left[\omega^2+\omega_A^2(\ell)\right]\left[\omega^2+\omega_B^2(\ell)\right]}{\left[\omega^2+\omega_0^2(\ell)\right]^2} \\ &= \frac{T}{2}\sum_{\ell=0}^{\infty} n_\ell\left[\omega_B(\ell)+\omega_A(\ell)-2\omega_0(\ell)\right], \end{aligned} \tag{2.136}$$

where n_ℓ is the multiplicity of the Laplacian on the $(d-1)$-dimensional sphere:

$$n_\ell = \frac{(2\ell+d-2)\Gamma(\ell+d-2)}{\Gamma(\ell+1)\Gamma(d-1)}. \tag{2.137}$$

In $d=4$ the multiplicity is $n_\ell = (1+\ell)^2$. In dimensional regularization, we can use the following identities which hold for sufficiently negative d

$$\sum_{\ell=0}^{\infty} n_\ell = \sum_{\ell=0}^{\infty} n_\ell \, \ell = 0 \qquad \Longrightarrow \qquad \sum_{\ell=0}^{\infty} n_\ell \, \omega_0(\ell) = 0. \tag{2.138}$$

Finally we formally find the second term in the expansion (2.70) as a sum of zero point energies, as it could have been intuitively expected:

$$e_0(\lambda_0 n, d) = \frac{R}{2} \sum_{\ell=0}^{\infty} n_\ell \left[\omega_B(\ell) + \omega_A(\ell)\right]. \tag{2.139}$$

We can now compute the leading correction to the scaling dimension (2.71). The details of the calculation are given in the appendix A. The result is formally written in terms of the classical value of the chemical potential (2.130) and a convergent infinite sum:

$$\Delta_0 = -\frac{15\mu_*^4 R^4 + 6\mu_*^2 R^2 - 5}{16} + \frac{1}{2} \sum_{\ell=1}^{\infty} \sigma(\ell) + \frac{\sqrt{3\mu_*^2 R^2 - 1}}{\sqrt{2}}, \tag{2.140}$$

where $\sigma(\ell)$ is obtained by subtracting the divergent piece from the summand in (2.139)

$$\sigma(\ell) = (1+\ell)^2 R \left[\omega_B^*(\ell) + \omega_A^*(\ell)\right] - 2\ell^3 - 6\ell^2 - \left(2\mu_*^2 R^2 + 4\right)\ell - 2R^2\mu_*^2 + \frac{5\left(\mu_*^2 R^2 - 1\right)^2}{4\ell}. \tag{2.141}$$

As in equation (2.130), the star stresses that all quantities are evaluated setting $\lambda_0 = \lambda_*$ and $d=4$ everywhere else. The infinite sum in (2.140), albeit convergent, cannot be performed analytically. However, it is possible to compute it numerically for any given $\lambda_* n$ at arbitrary precision.

In the small $\lambda_* n$ limit, we can expand the summand and compute the sum in (2.140) analytically and we find

$$\Delta_0 = -\frac{3\lambda_* n}{(4\pi)^2} + \frac{\lambda_*^2 n^2}{2(4\pi)^4} + O\left(\frac{\lambda_*^3 n^3}{(4\pi)^6}\right). \tag{2.142}$$

Summing this to the leading order result (2.133) and recalling the relation between the coupling and the number of space dimensions (2.8), we determine Δ_{ϕ^n} as:

$$\Delta_{\phi^n} = n\left(\frac{d}{2} - 1\right) + \frac{\varepsilon}{10} n(n-1) - \frac{\varepsilon^2}{50} n(n^2 - 4n) + O\left(\varepsilon^2 n, \varepsilon^3 n^4\right). \tag{2.143}$$

This is in perfect agreement with the diagrammatic calculation in eq. (2.31). This agreement has since been checked further, up to 5-loops [104, 105].

In the large $\lambda_* n$ limit the result (2.140) develops a contribution proportional to $\log(\lambda_* n)$, which arises from the divergent tail of the sum in (2.139). As in (2.133), the result can be expanded in powers of $(\lambda_* n)^{2/3}$ and reads:

$$\Delta_0 = \left[\alpha + \frac{5}{24}\log\left(\frac{\lambda_* n}{8\pi^2}\right)\right]\left(\frac{\lambda_* n}{8\pi^2}\right)^{4/3} + \left[\beta - \frac{5}{36}\log\left(\frac{\lambda_* n}{8\pi^2}\right)\right]\left(\frac{\lambda_* n}{8\pi^2}\right)^{2/3} + O(1), \quad (2.144)$$

where the coefficients α and β are

$$\alpha = -0.5753315(3), \qquad \beta = -0.93715(9). \quad (2.145)$$

The logarithmic terms are computed analytically, while the coefficients α and β follow from a numerical fit. Details of the calculation are given in the appendix A.2. The structure of the result (2.144) is in agreement with the expected form of the large charge expansion in d dimensions. This is evident summing (2.144) to the leading order in (2.133) and writing the result in the form

$$\begin{aligned}\Delta_{\phi^n} = &\frac{1}{\varepsilon}\left(\frac{2}{5}\varepsilon n\right)^{\frac{4-\varepsilon}{3-\varepsilon}}\left[\frac{15}{8} + \varepsilon\left(\alpha + \frac{3}{8}\right) + O\left(\varepsilon^2\right)\right] \\ &+ \frac{1}{\varepsilon}\left(\frac{2}{5}\varepsilon n\right)^{\frac{2-\varepsilon}{3-\varepsilon}}\left[\frac{5}{4} + \varepsilon\left(\beta - \frac{1}{4}\right) + O\left(\varepsilon^2\right)\right] + O\left((\varepsilon n)^0\right). \quad (2.146)\end{aligned}$$

The change in the exponents of the (εn) terms with respect to the leading order (2.133) account for the logarithms in (2.144). Recalling that $d = 4 - \varepsilon$, eq. (2.146) is clearly in agreement with the structure predicted in [74, 75], which is:

$$\Delta_n = n^{\frac{d}{d-1}}\left[c_0(d) + c_1(d) n^{-\frac{2}{d-1}} + c_2(d) n^{-\frac{4}{d-1}} + \ldots\right] + n^0\left[b_0(d) + b_1(d) n^{-\frac{2}{d-1}} + \ldots\right]. \quad (2.147)$$

From the point of view of the large charge EFT, the first term is a purely classical contribution, while the second term is the one-loop Casimir energy of the Goldstone mode. In non-even dimensions, the latter term is independent of the Wilson coefficients of the EFT and is hence universal [74]; the numerical value $b_0(3) \simeq -0.937$ will be checked in the next section. We have checked that the coefficients of the logarithms multiplied by subleading powers of $(\lambda_* n)$ ensure the agreement between our result and the predicted structure (2.147) also in the subleading orders in n. The large $\lambda_* n$ expansion of the classical result determines the coefficients $c_i(d)$ at leading order, while eq. (2.146) determines $c_0(d)$ and $c_1(d)$ to order $O(\varepsilon)$. Even though we computed also the coefficient of the $(\lambda_* n)^0$ term in (2.144) (see eq. (A.16)), in the expansion of (2.147) for $d = 4 - \varepsilon$ to first order, we cannot disentangle the first correction in ε to $c_2(d)$ and the leading order value of $b_0(d)$ (which is zero at tree-level).

2.4.6 Comments on the result

Large order behavior

Expanding all functions Δ_ℓ in a power series in $\lambda_* n$

$$\Delta_\ell = \sum_k f_{\ell,k}(\lambda_* n)^k, \tag{2.148}$$

it naively seems that the anomalous dimension (2.71) has, at fixed order in the semiclassical expansion, contributions from arbitrarily large powers of n. This, however, does not match the diagrammatic computation which is valid for small $\lambda_* n$ but virtually large n. Indeed, beyond order $\lfloor n/2 \rfloor$ in the ordinary loop expansion the operator ϕ^n does not have enough free legs to form "daisy" diagrams and provide terms with higher and higher powers of n.

To understand what happens from the semiclassical perspective, we can compare contributions to the anomalous dimension that are of the same order in λ_* but which come from different orders in the semiclassical expansion. For instance we can consider Δ_ℓ and $\Delta_{\ell+1}$. The contributions of the same order in λ_* are controlled by $\lambda_*^{\ell+k} f_{\ell,k} n^k$ and $\lambda_*^{\ell+k} f_{\ell+1,k-1} n^{k-1}$ respectively. Therefore, if

$$\frac{f_{\ell+1,k-1}}{f_{\ell,k}} \sim k, \tag{2.149}$$

there can be a potential cancellation at order $k \sim n$, thus resulting in the correct behavior of the anomalous dimensions for k beyond roughly $\lfloor n/2 \rfloor$. The authors of [106], based on resurgence arguments, have argued that (2.149) is incorrect for Δ_0 and Δ_{-1}. Indeed, based on their remarks, we checked that what happens for $f_{-1,k}$ and $f_{0,k-1}$ is more accurately described by

$$\frac{f_{0,k-1}}{f_{-1,k}} \sim k^{5/4}. \tag{2.150}$$

Hence, it may be the case that the small-λn expansion of the semiclassical result does not match the Feynman diagrams computation at orders larger that $\lfloor n/2 \rfloor$.

Boosting diagrammatic loop calculations

At the Wilson-Fisher fixed point, the expansion in (2.32) for the anomalous dimension of ϕ^n, valid for small εn, is written as

$$\gamma_{\phi^n} = n \sum_{\ell=1} \varepsilon^\ell P_\ell(n), \tag{2.151}$$

Hence, at any fixed order ℓ in (2.151) there are ℓ independent coefficients to be determined. We can thus take advantage of existing results in the literature, as well as of the small

$\lambda_* n$ expansion of our results (2.131) and (2.140), to fix some or all of them. The anomalous dimensions of ϕ, ϕ^2 and ϕ^4 are known to order ε^5 with analytical coefficients [98, 107], while the anomalous dimension of ϕ^3 is known to the same order with numerical coefficients [108]. These results then provide four constraints on each of the first five orders in (2.151) and are enough to fix all the cofficients in $P_1(n), P_2(n)$ and $P_3(n)$. Furthermore, expanding the results (2.131) and (2.140) derived in this chapter to order $O(\varepsilon^5 n^5)$, we have a total of six constraints on each of the first five orders in (2.151). This clearly fully fixes the form of the five polynomials $P_1(n), P_2(n), \ldots, P_5(n)$. The form of the first two was given in (2.30), while the others read

$$P_3(n) = \frac{n^3}{125} + \frac{n^2\,[16\zeta(3) - 29]}{500} + \frac{n\,[599 - 672\zeta(3)]}{5000} + \frac{[1024\zeta(3) - 603]}{10000}, \tag{2.152}$$

$$\begin{aligned} P_4(n) = &-\frac{21n^4}{5000} + \frac{n^3\,[214 - 77\zeta(3) - 80\zeta(5)]}{5000} + \frac{n^2\,[66336\zeta(3) + 160\pi^4 - 89491]}{600000} \\ &+ \frac{n\,[41073 - 45864\zeta(3) + 46720\zeta(5) - 224\pi^4]}{200000} \\ &+ \frac{75888\zeta(3) - 130560\zeta(5) + 512\pi^4 - 53717}{600000}, \end{aligned} \tag{2.153}$$

$$\begin{aligned} P_5(n) = &\frac{n^5 8}{3125} + \frac{n^4\,[476\zeta(3) + 480\zeta(5) + 448\zeta(7) - 1683]}{50000} \\ &+ 0.00093n^3 - 0.01067n^2 - 0.2460n + 0.2680. \end{aligned} \tag{2.154}$$

We checked that $P_3(n)$ agrees both with the previous literature and our results, providing another non trivial check of our approach. The polynomial $P_4(n)$ was determined using our results and those in the literature for ϕ, ϕ^2 and ϕ^4; we checked that it agrees numerically within 10% level with the coefficient reported in [108] for ϕ^3. We do not know if this discrepancy is due to the numerical uncertainty of this result, as the latter is not reported in [108]. For the same reason, we cannot quote the error on the last four coefficients of $P_5(n)$.

Comparison with Monte-Carlo results at large charge

We can compare our result in the large $(\lambda_* n)$ limit, given by (2.146) in the first two leading orders, with the recent results of Monte-Carlo lattice simulations of the three-dimensional $O(2)$ model [109]. There, the authors computed the scaling dimensions of the lightest charge n operator for various values of n and compared their result with the predicted form (2.147), which in $d = 3$ reads:

$$\Delta_n \simeq c_{3/2} n^{3/2} + c_{1/2} n^{1/2} - 0.0937 + c_{-1/2} n^{-1/2} + O\left(n^{-1}\right). \tag{2.155}$$

The authors there determined the coefficients $c_{3/2}$ and $c_{1/2}$ fitting the result of the lattice computation.

	$c_{3/2}$	$c_{1/2}$
Monte-Carlo [109]	0.337(3)	0.27(4)
ε-expansion: LO	0.47	0.79
ε-expansion: NLO	0.42	0.04

Table 2.1: Comparison of the Monte-Carlo result in [109] with the ε-expansion; we display both the leading order (LO) result as well as the next to leading order (NLO).

We compared the coefficients they obtained with those which follow from (2.146) putting $\varepsilon = 1$. The results are displayed in the table 2.1. Using the next to leading order contribution as an estimate of the error, the result for $c_{3/2}$ is roughly within two standard deviations from the Monte-Carlo result, while for $c_{1/2}$ the error is as big as the leading order, making a quantitative analysis impossible. It is however interesting to notice that for both coefficients the next to leading order values are closer than the leading order ones to the results obtained by the Monte-Carlo. It would be interesting to compute the two-loop order result to explore the convergence properties of the expansion.

2.5 Sextic interaction in $d = 3 - \varepsilon$

In this section we apply the same methodology to compute the scaling dimension of ϕ^n in the theory (2.9) at its conformally invariant point in $3 - \varepsilon$ dimensions. Within this convention for the Lagrangian, one can easily realize that λ_0 is again the loop counting parameter by rescaling $\phi \to \phi/\sqrt{\lambda_0}$ similarly as in (2.35). The β-function was given in (2.10) and its Wilson-Fisher fixed point in (2.11). Notice that the β-function starts at two-loop order at $\varepsilon = 0$. Hence the model is conformally invariant up to $O(\lambda)$ in exactly $d = 3$ for any value of λ. This observation will be important for what follows. The field wave-function renormalization starts at four loops and does not contribute to the following.

In complete analogy with the $(\phi\bar{\phi})^2$ case discussed in section 2.2.2, the diagrammatic calculation for the anomalous dimension takes the form

$$\gamma_{\phi^n} = n \sum_{\ell=1} \lambda^\ell P_\ell(n), \tag{2.156}$$

where P_ℓ is a polynomial of degree ℓ for $\ell \leq n$, and of degree n for $\ell > n$. Thus, the loop order ℓ contribution grows as $\lambda^\ell n^{\ell+1}$ for $\ell \leq n$, implying that the diagrammatic expansion breaks down for sufficiently large λn. Re-organizing the series in (2.156), the scaling dimension can also be expanded as

$$\Delta_{\phi^n} = n\left(\frac{d}{2} - 1\right) + \gamma_{\phi^n} = \sum_{\kappa=-1} \lambda^\kappa \Delta_\kappa(\lambda n). \tag{2.157}$$

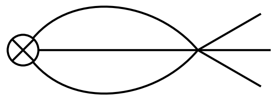


Figure 2.3: Two-loop diagram contributing to the ϕ^n anomalous dimension. The crossed circle denotes the ϕ^n insertion.

As in the previous case the scaling dimension (2.157) is a physical (scheme-independent) quantity only at the fixed-point (2.11). However, in light of the observation above, working at order $O(\lambda)$ we can take $\varepsilon \to 0$ without affecting the conformal invariance of the theory[19]. The leading order term $\Delta_{-1}(\lambda n)$ and the one-loop correction $\Delta_0(\lambda n)$ are hence scheme-independent for generic λ.

Working at fixed n, at leading order in λ, the anomalous dimension of $\phi^n(x)$ is determined by the diagram in Fig. 2.3 and it is given by

$$\gamma_{\phi^n} = \frac{\lambda^2 n(n-1)(n-2)}{36(4\pi)^2} + O\left(\frac{\lambda^4 n^5}{(4\pi)^4}\right). \tag{2.158}$$

2.5.1 Semiclassical computation

To compute the scaling dimension Δ_{ϕ^n} for arbitrary λn we proceed as previously in section 2.4. Most of the steps are exactly the same, hence we only report the differences due to the different interaction term. The lagrangian on the cylinder is given by

$$\mathcal{L}_{cyl} = \partial\bar{\phi}\partial\phi + m^2\bar{\phi}\phi + \frac{\lambda^2}{36}\left(\bar{\phi}\phi\right)^3, \tag{2.159}$$

where $m^2 = \left(\frac{d-2}{2R}\right)^2 \overset{d=3}{=} \frac{1}{4R^2}$ again is the $\mathcal{R}(g)\bar{\phi}\phi$ coupling to the Ricci scalar which is enforced by conformal invariance. Working at $O(\lambda)$, we neglect the difference between bare and renormalized coupling, as that arises at $O(\lambda^2)$. We define the polar field coordinates as in (2.80), obtaining the same action (2.86), but this time with potential

$$V_{\text{int}}(\rho) = \frac{\lambda^2}{288}\rho^6. \tag{2.160}$$

The derivation of the saddle goes through in the same way, giving in particular instead of (2.92)

$$\mu^2 - m^2 = \frac{1}{f}\frac{\partial V_{int}(f)}{\partial f} = \frac{\lambda^2}{48}f^4. \tag{2.161}$$

[19]Dimensional regularization is still used in the intermediate steps.

Given the constraint $f^2 \geq 0$, the equations admit a unique solution. In particular for $n > 0$, μ reads:

$$\mu = \mu_6(\lambda n, d) = \frac{(d-2)}{2R} \frac{\sqrt{1 + \sqrt{1 + \frac{\lambda^2 n^2 \Gamma(d/2)^2}{3\pi^d (d-2)^4}}}}{\sqrt{2}}. \tag{2.162}$$

Plugging the solution into the classical action we extract the leading order contribution to the scaling dimension:

$$S_{eff}/T = \frac{n}{3}\left(2\mu + \frac{m^2}{\mu}\right) \overset{d=3}{=} \frac{1}{R}\frac{\Delta_{-1}(\lambda n)}{\lambda}\,. \tag{2.163}$$

Explicitly, the result reads

$$\Delta_{-1}(\lambda n) = \lambda n \, F_{-1}\left(\frac{\lambda^2 n^2}{12\pi^2}\right)\,, \tag{2.164}$$

where

$$F_{-1}(x) = \frac{1 + \sqrt{1+x} + x/3}{\sqrt{2}\,(1 + \sqrt{1+x})^{3/2}}\,. \tag{2.165}$$

As previously the one-loop correction Δ_0 is determined by the fluctuation determinant around the leading trajectory (2.136). The expression for the spectrum (2.118) is also modified with the potential (2.160). The infinite sum is regularized as previously; the final result is finite in the limit $d \to 3$, consistently with the coupling not being renormalized at one-loop. Eventually, Δ_0 can be written in terms of an infinite convergent sum as

$$\Delta_0(\lambda n) = \frac{1}{4} - 3(R\mu)^2 + \frac{\sqrt{8R^2\mu^2 - 1}}{2} + \frac{1}{2}\sum_{\ell=1}^{\infty} \sigma(\ell)\,, \tag{2.166}$$

where $\sigma(\ell)$ is obtained from the summand in (2.136) by subtracting the divergent piece:

$$\sigma(\ell) = (1+2\ell) R\left[\omega_B(\ell) + \omega_A(\ell)\right] - 4\ell\,(\ell+1) - \left(6R^2\mu^2 - \frac{1}{2}\right)\,. \tag{2.167}$$

In (2.166) all quantities are evaluated in $d = 3$, hence μ is given by

$$\mu = \mu_6(\lambda n, 3) = \frac{1}{R}\frac{\sqrt{1 + \sqrt{1 + \frac{\lambda^2 n^2}{12\pi^2}}}}{2\sqrt{2}} \tag{2.168}$$

and $m = \frac{1}{2R}$.

2.5.2 Comments on the result

Equations (2.164) and (2.166) provide the first two terms of the expansion (2.157) for the scaling dimension of the operator ϕ^n, Δ_{ϕ^n}. The result holds for arbitrary values of λn. The result follows the same pattern observed in the previous section.

Let us consider first the small λn regime. From eq. (2.162) it follows that the chemical potential, and consequently all the functions Δ_κ, can be expanded in powers of $\lambda^2 n^2$. Explicitly neglecting terms of order $O\left(\frac{\lambda^6 n^7}{(4\pi)^6}\right)$, we get:

$$\Delta_{\phi^n} = \frac{n}{2} + \frac{\lambda^2}{(4\pi)^2}\left[\frac{n^3-3n^2}{36} + O\left(n\right)\right] - \frac{\lambda^4}{(4\pi)^4}\left[\frac{n^5}{144} - \frac{n^4(64-9\pi^2)}{1152} + O\left(n^3\right)\right] + \ldots . \tag{2.169}$$

In this regime we can compare eq. (2.169) with the diagrammatic result (2.158), finding perfect agreement. This check was extended to 6-loop level in [110].

Besides confirming the generality of the method, the main interest of $(\bar{\phi}\phi)^3$ in $d = 3-\varepsilon$ lies in the possibility of non-trivially comparing to the universal predictions of the large charge EFT of 3D CFT [74]. Indeed since the β-function arises only at 2-loops, at the 1-loop level the theory is conformally invariant at $d = 3$ for any value of λ. Thus the anomalous dimension is expected in this regime to take the form

$$\begin{aligned}\Delta_{\phi^n} = \;& t^{3/2}\;\left[c_{3/2} + c_{1/2}t^{-1} + c_{-1/2}t^{-2} + \ldots\right] \\ + \;& \left[d_0 + d_{-1}t^{-1} + \ldots\right] ,\end{aligned} \tag{2.170}$$

Again, computing the one-loop contribution (2.166) at large λn can be achieved by evaluating numerically for large $\mu \sim (\lambda n)^{1/2}$ and then fitting[20] to the functional form (2.170). When doing this we also verified that the coefficients of terms which might modify the form of the expansion, such as a term linear in λn, are compatible with zero within the numerical uncertainty. We get the following numerical coefficients

$$\begin{aligned}
c_{3/2} &= \frac{\sqrt{3}\pi}{6\lambda} - 0.0653 + O\left(\frac{\lambda}{\sqrt{3}\pi}\right) , \\
c_{1/2} &= \frac{\sqrt{3}\pi}{2\lambda} + 0.2088 + O\left(\frac{\lambda}{\sqrt{3}\pi}\right) , \\
c_{-1/2} &= -\frac{\sqrt{3}\pi}{4\lambda} - 0.2627 + O\left(\frac{\lambda}{\sqrt{3}\pi}\right) , \\
d_0 &= -0.0937255(3) , \\
d_{-1} &= 0.096(1) + O\left(\frac{\lambda}{\sqrt{3}\pi}\right) .
\end{aligned} \tag{2.171}$$

[20]We computed Δ_0 numerically for $R\mu = 10, 11, \ldots 210$ to perform the fit; the final results are obtained using six fitting parameters in the expansion (2.170).

The parentheses show the numerical error on the last digit, when the latter is not negligible at the reported precision.

This result nicely matches the universal predictions of the large charge EFT. Within the general EFT construction the c_k's are model dependent Wilson coefficients, but the d's are universally calculable effects associated to the 1-loop Casimir energy. Our result thus matches the general theory. In particular we find

$$d_0 = -0.0937255(3) \tag{2.172}$$

in agreement with [75] to almost seven digit accuracy.

3 Computation of 3- and 4-point functions

In this chapter, we further explore the semiclassical methodology described in section 2.4. Focusing on the Wilson-Fisher fixed point in $4-\varepsilon$ dimensions, corresponding to the theory in eq. (2.1), we will derive new results by studying 3- and 4-point functions involving two operators with large charge n at next to leading order in ε (or equivalently in n^{-1}) [3, 111]. More precisely, we will compute correlators of the class presented in Eq. (2.72) involving one or two additional operators $\mathcal{O}_i$, i.e. N equals 1 or 2. For simplicity we will focus on insertions of just one specific type of neutral operators

$$\mathcal{O}(x) = (\bar{\phi}\phi)^k(x). \tag{3.1}$$

These computations are performed at next-to-leading order in the $1/n$ expansion. From these correlators, we are able to derive more elements of the CFT data of the theory, namely coefficients of the $\mathcal{O} \times \phi^n$ operator product expansion (OPE), also at next-to-leading order in the $1/n$ expansion. In this chapter and the next, we will ignore the powers of R, the radius of the cylinder, in all computations; in other words we set $R = 1$.

We start with the 3-point function of $\bar{\phi}\phi$, which, up to the normalization, is fully determined by the scaling dimensions and a single fusion coefficient. The scaling dimension of ϕ^n is given by (2.71, 2.131, 2.140), while that of $\bar{\phi}\phi$ can be easily computed using standard perturbation theory through Feynman diagrams as we will see shortly. As a result the only parameter to compute is the fusion coefficient $\lambda_{\bar{\phi}\phi}$, which appears in the 3-pt function of canonically (re-)normalized operators $[\mathcal{O}_i]$ as

$$\left\langle [\bar{\phi}^n](x_f)[\bar{\phi}\phi](x)[\phi^n](x_i) \right\rangle = \frac{\lambda_{\bar{\phi}\phi}}{(x_f - x_i)^{2\Delta_{\phi^n} - \Delta_{\mathcal{O}}}(x - x_i)^{\Delta_{\mathcal{O}}}(x_f - x)^{\Delta_{\mathcal{O}}}}. \tag{3.2}$$

On the cylinder, using (2.79), one can more simply write

$$\lambda_{\bar{\phi}\phi} = \lim_{\substack{\tau_f \to \infty \\ \tau_i \to -\infty}} \frac{\langle 0|\widehat{[\bar{\phi}^n]}(\tau_f, \vec{n}_f)\widehat{[\bar{\phi}\phi]}(\tau, \vec{n})\widehat{[\phi^n]}(\tau_i, \vec{n}_i)|0\rangle}{\langle 0|\widehat{[\bar{\phi}^n]}(\tau_f, \vec{n}_f)\widehat{[\phi^n]}(\tau_i, \vec{n}_i)|0\rangle} \equiv \langle n|\widehat{[\bar{\phi}\phi]}(\tau, \vec{n})|n\rangle. \tag{3.3}$$

For the theory and the operators at hand, renormalization is multiplicative, so that canonically normalized and bare operators are related by $[\mathcal{O}_i] = \mathcal{O}_i/Z_i$, with Z_i generally UV divergent. For instance, the 2-point function of $\bar{\phi}\phi$ is given by

$$\langle(\bar{\phi}\phi)(x)(\bar{\phi}\phi)(y)\rangle = \frac{Z^2_{\bar{\phi}\phi}}{(x-y)^{2\Delta_{\bar{\phi}\phi}}}, \tag{3.4}$$

where at one-loop order, i.e. just the diagram in Fig. 3.1,

$$Z_{\bar{\phi}\phi} = \Omega^{-1}_{d-1}(d-2)^{-1}\left[1 - \frac{\lambda}{8\pi^2}\frac{1}{4-d}\right]\left[1 - \frac{\lambda}{16\pi^2}\left(1+\gamma+\log\pi\right)\right], \tag{3.5}$$

which implies the scaling dimension is

$$\Delta_{\bar{\phi}\phi} \equiv (d-2) + \gamma_{\bar{\phi}\phi} = (d-2) + \frac{\lambda}{8\pi^2}. \tag{3.6}$$

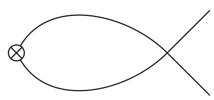


Figure 3.1: One-loop renormalization of $\bar{\phi}\phi$.

3.1 3-pt function

For large n, (3.3) can be computed semiclassically by expanding around the saddle point (2.91). Equation (2.98) yields in this case

$$\lambda_{\bar{\phi}\phi} = Z^{-1}_{\bar{\phi}\phi}\,\frac{\displaystyle\int \mathcal{D}r\mathcal{D}\pi\,(\widehat{\bar{\phi}\phi})(\tau,\vec{n})e^{-\hat{S}[r,\pi]}}{\displaystyle\int \mathcal{D}r\mathcal{D}\pi\, e^{-\hat{S}[r,\pi]}}, \tag{3.7}$$

where the path integrals have the boundary conditions specified by (2.84) and the action for the r,π fields is (2.99).

Leading order: the computation boils down to evaluating the integrands on the saddle, leading to

$$\lambda_{\bar{\phi}\phi} = f^2\Omega_3 = \frac{n}{\mu_*}, \tag{3.8}$$

where we used (2.93) and the leading order result $Z^{-1}_{\bar{\phi}\phi} = 2\Omega_3$. For small λn we have $\mu_* = 1$, see (2.95), and the result, $\lambda_{\bar{\phi}\phi} = n$, coincides with the tree level computation using Feynman diagrams. In this chapter the symbol μ refers to $\mu_4(\lambda n, d)$ while μ_* refers

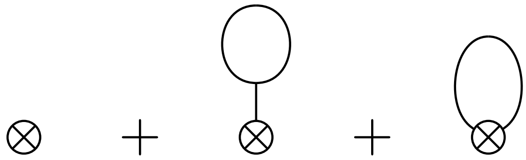


Figure 3.2: Topology of diagrams entering $\langle n|(\bar{\phi}\phi)|n\rangle$ at NLO

to $\mu_4(\lambda_* n, 4)$. Notice this is the chemical potential of the 4D theory evaluated at the critical coupling of the theory in $d = 4 - \varepsilon$, given in (2.8).

Next to leading order. The result is independent of the choice of $(\tau, \vec{n})$ in (3.7), therefore, we make the convenient choice $(\tau, \vec{n}) = (0, \vec{n}_d)$, with

$$\vec{n}_d = (\underbrace{0, 0, \ldots, 0, 1}_{d}). \tag{3.9}$$

By expanding around the saddle (2.97), the expectation value of the bare operator is then

$$\left\langle n\right| \left(\bar{\phi}\phi\right)(0, \vec{n}_d)\left|n\right\rangle = \frac{1}{2}\left\langle n\right|f^2 + 2fr(0, \vec{n}_d) + r^2(0, \vec{n}_d)\left|n\right\rangle, \tag{3.10}$$

which at NLO, i.e. 1-loop, gives

$$\left\langle n\right| \left(\bar{\phi}\phi\right)(0, \vec{n}_d)\left|n\right\rangle = \frac{f^2}{2} - \left\langle r(0, \vec{n}_d)\int d\tau d\Omega_{d-1}\left[r(\partial\pi)^2 - i\mu r^2\dot{\pi} + \frac{\lambda f^2 r^3}{4}\right]\right\rangle + \frac{1}{2}\left\langle r^2(0, \vec{n}_d)\right\rangle, \tag{3.11}$$

where we inserted interaction terms from (2.101) within the $\langle \ldots \rangle$ and the fields r, π are free fields (2.116) propagating according to the quadratic action expanded around the background (2.100). The resulting Feynman diagrams are depicted in Fig. 3.2. The first step is to find the propagator of (r, π). In matrix form this can be written as

$$D(\tau - \tau', \vec{n}\cdot\vec{n}') = \sum_{\ell} F^{(\ell)}(\tau - \tau')C_{\ell}^{(d/2-1)}(\vec{n}\cdot\vec{n}'), \tag{3.12}$$

where $C_{\ell}^{(d/2-1)}(\cos\theta)$ are Gegenbauer polynomials and $F^{(\ell)}(\tau)$ is a 2×2 matrix whose exact expression is given in appendix B.1.

The details of the computation can be found in appendix B.2. The result is

$$\lambda_{\bar{\phi}\phi} = \frac{n}{\mu_*} + \frac{2(3\mu_*^2 + 1)}{\left[2(3\mu_*^2 - 1)\right]^{3/2}} - \frac{3 - 2\mu_*^2 + 3\mu_*^4}{2(3\mu_*^2 - 1)} + \sum_{\ell=1}^{\infty}\left[S_{\ell}(\mu_*) - c_{-1}(\mu_*)\ell - c_0(\mu_*) - \frac{c_1(\mu_*)}{\ell}\right], \tag{3.13}$$

with

$$S_{\ell}(\mu) \equiv S_{\ell}(\mu, 1, 4), \tag{3.14}$$

where

$$S_\ell(\mu, m, d) = \frac{2\ell + d - 2}{\omega_B^2(0)} \frac{\omega_B(\ell)\omega_A(\ell)(3\mu^2 + m^2) - J_\ell^2(\mu^2 - m^2)}{\omega_B(\ell)\omega_A(\ell)\left[\omega_B(\ell) + \omega_A(\ell)\right]} C_\ell^{(d/2-1)}(1), \quad (3.15)$$

while the coefficients $c_{-1,0,1}(\mu)$ are defined by the asymptotic behavior of the summand

$$S_\ell(\mu) \underset{\ell\to\infty}{\equiv} c_{-1}(\mu)\ell + c_0(\mu) + \frac{c_1(\mu)}{\ell} + \dots, \quad (3.16)$$

so as to render the sum in (3.13) finite. Their exact values are

$$c_{-1}(\mu) = c_0(\mu) = \frac{\mu^2 + 1}{3\mu^2 - 1}, \quad c_1(\mu) = -\frac{\mu^4 + 2\mu^2 - 3}{2(3\mu^2 - 1)}. \quad (3.17)$$

As discussed in the appendix, the transcendental terms proportional to the Euler constant γ, and to $\ln\pi$, which normally appear in 1-loop expressions, cancel out as expected once we fix $\lambda = \lambda_*$.

As usual the result can be expanded in both regimes of large or small $\lambda_* n$. For small $\lambda_* n$, expanding μ_* in a power series in $\lambda_* n$

$$\mu_* = 1 + \frac{\lambda_* n}{16\pi^2} - \frac{3}{2}\left(\frac{\lambda_* n}{16\pi^2}\right)^2 + O(\lambda_*^3 n^3), \quad (3.18)$$

we get

$$\lambda_{\bar\phi\phi} \underset{\lambda_* n\to 0}{=} n\left[1 - \frac{\lambda_* n}{16\pi^2} + \frac{5}{2}\left(\frac{\lambda_* n}{16\pi^2}\right)^2 + O(\lambda_* n)^3\right] + \left[6\zeta^2(3) - \frac{13}{2}\right]\left(\frac{\lambda_* n}{16\pi^2}\right)^2 + O(\lambda_* n)^3 + O(n^{-1}). \quad (3.19)$$

While for large $\lambda_* n$, and therefore $\mu_* \gg 1$, the sum in (3.13) approximately satisfies

$$\sum_{\ell=1}^{\infty}\left[S_\ell(\mu_*, 1, 4) - c_{-1}(\mu_*)\ell - c_0(\mu_*) - \frac{c_1(\mu_*)}{\ell}\right] \underset{\mu\to\infty}{=} \frac{1}{6}\mu_*^2 \log\mu_*. \quad (3.20)$$

Combining that with the leading contribution n/μ_* and using the relation $\mu_*^3 = \lambda_* n/4\Omega_3$, which applies in the large $\lambda_* n$ regime, we get

$$\begin{aligned}\lambda_{\bar\phi\phi} &\underset{\lambda n\to\infty}{=} \frac{8\pi^2}{\lambda_*}\left(\frac{\lambda_* n}{8\pi^2}\right)^{2/3}\left(1 + \frac{\lambda_*}{144\pi^2}\log\frac{\lambda_* n}{8\pi^2}\right) \\ &\simeq \frac{8\pi^2}{\lambda_*}\left(\frac{\lambda_* n}{8\pi^2}\right)^{\frac{2}{3}+\frac{\lambda_*}{144\pi^2}} = \frac{5}{2\varepsilon}\left(\frac{2\varepsilon n}{5}\right)^{\frac{2}{3}+\frac{\varepsilon}{45}} \sim n^{\frac{\Delta_{\bar\phi\phi}}{d-1}}, \end{aligned} \quad (3.21)$$

where $\Delta_{\bar\phi\phi}$ is given by eq. (3.6) with $\lambda \to \lambda_*$. The scaling with n is once more precisely as predicted by the large charge EFT description [75].

3.2 4-pt function

We will now study, by the same methodology, the four point function with two insertions of $(\bar{\phi}\phi)$.

Let us recall that a general 4-point correlator in a CFT can be written using s- and t-channel representations

$$\begin{aligned} \langle \mathcal{O}_4(x_4)\mathcal{O}_3(x_3)\mathcal{O}_2(x_2)\mathcal{O}_1(x_1)\rangle &= \frac{g_{12,34}(z,\bar{z})}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \left(\frac{x_{24}}{x_{14}}\right)^{\Delta_1-\Delta_2} \left(\frac{x_{14}}{x_{13}}\right)^{\Delta_3-\Delta_4} && (3.22) \\ &= \frac{g_{32,14}(1-z,1-\bar{z})}{x_{32}^{\Delta_3+\Delta_2} x_{14}^{\Delta_1+\Delta_4}} \left(\frac{x_{24}}{x_{34}}\right)^{\Delta_3-\Delta_2} \left(\frac{x_{34}}{x_{13}}\right)^{\Delta_1-\Delta_4} && (3.23) \end{aligned}$$

where z and $\bar{z}$ are defined by the conformal ratios according to

$$u = \bar{z}z = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \tag{3.24}$$

Modulo the kinematic factors, fixed by conformal symmetry, the relevant information is encapsulated in the $g_{ij,kl}(z,\bar{z})$.

For Euclidean signature, the two variables $z \equiv e^{\tau+i\theta}$ and $\bar{z} \equiv e^{\tau-i\theta}$ are related by complex conjugation. Using conformal transformations to map $x_1 \to 0$, $x_4 \to \infty$ and

$$x_3 = \vec{n} \equiv (0,0,\ldots,0,1) \qquad x_2 = \vec{n}(\theta)e^{\tau} \equiv (0,0,\ldots,\sin\theta,\cos\theta)e^{\tau}, \tag{3.25}$$

we can rewrite

$$\begin{aligned} g_s(z,\bar{z}) &\equiv g_{12,34}(z,\bar{z}) = |z|^{\Delta_1} \langle \mathcal{O}_4|\hat{\mathcal{O}}_3(0,\vec{n})\hat{\mathcal{O}}_2(\tau,\vec{n}(\theta))|\mathcal{O}_1\rangle, && (3.26) \\ g_t(z,\bar{z}) &\equiv g_{32,14}(1-z,1-\bar{z}) = \frac{|1-z|^{\Delta_2+\Delta_3}}{|z|^{\Delta_2}} \langle \mathcal{O}_4|\hat{\mathcal{O}}_3(0,\vec{n})\hat{\mathcal{O}}_2(\tau,\vec{n}(\theta))|\mathcal{O}_1\rangle. && (3.27) \end{aligned}$$

The $g_{ij,kl}(z,\bar{z})$ can be decomposed as a sum over the primary operators that appear in both the operator product expansions (OPEs) of $\mathcal{O}_i \times \mathcal{O}_j$ and $\mathcal{O}_k \times \mathcal{O}_l$

$$g_{ij,kl}(z,\bar{z}) = \sum_\alpha \lambda_{ij\alpha}\bar{\lambda}_{kl\alpha} g_{\Delta_\alpha,\ell_\alpha}^{\Delta_{ji},\Delta_{kl}}(z,\bar{z}), \quad \Delta_{ij} = \Delta_i - \Delta_j, \tag{3.28}$$

where α labels the primaries while Δ_α, ℓ_α and $\lambda_{ij\alpha}$ respectively represent their dimensions, spins and fusion coefficients. The conformal blocks $g_{\Delta,\ell}^{\Delta_{ji},\Delta_{kl}}(z,\bar{z})$ are completely fixed functions: their functional form is fixed by the conformal group and their normalization by (3.28). Their explicit expressions in $d = 2,4$ can be found in [112]. What matters for our discussion is that in any dimension they admit a power series expansion in $|z|$

[112, 113]

$$g^{\Delta_{21},\Delta_{34}}_{\Delta,\ell}(z,\bar z) = |z|^{\Delta}\sum_{k=0}^{\infty}|z|^k \sum_{j=j_0(\ell,k)}^{\ell+k} A^{\Delta_{21},\Delta_{34}}_{k,j}(\Delta,\ell)C_j^{(d/2-1)}(\cos\theta), \quad z=|z|e^{i\theta} \tag{3.29}$$

with $j_0(\ell,k)=\max(\ell-k, k-\ell \mod 2)$, where the term proportional to $|z|^kC_j(\cos\theta)$ corresponds to the level k descendant with spin j. The dimension and spin of the intermediate primaries is directly read from this expansion. The $A^{\Delta_{21},\Delta_{34}}_{k,j}(\Delta,\ell)$ are calculable coefficients, in particular $A^{\Delta_{21},\Delta_{34}}_{0,0}(\Delta,0)=1$.

We will here study the specific correlator

$$\left\langle[\bar\phi^n](x_4)[\bar\phi\phi](x_3)[\bar\phi\phi](x_2)[\phi^n](x_1)\right\rangle, \tag{3.30}$$

so that equations (3.26) and (3.27) reduce to

$$g_s(z,\bar z) \equiv g_{\phi^n,\bar\phi\phi;\bar\phi\phi,\bar\phi^n}(z,\bar z) = Z^{-2}_{\bar\phi\phi}|z|^{\Delta_{\phi^n}}\frac{\langle n|(\widehat{\bar\phi\phi})(0,\vec n)(\widehat{\bar\phi\phi})(\tau,\vec n(\theta))|n\rangle}{\langle n|n\rangle}, \tag{3.31}$$

$$g_t(z,\bar z) \equiv g_{\bar\phi\phi,\bar\phi\phi;\phi^n,\bar\phi^n}(1-z,1-\bar z) = Z^{-2}_{\bar\phi\phi}\frac{|1-z|^{2\Delta_{\bar\phi\phi}}}{|z|^{\Delta_{\bar\phi\phi}}}\frac{\langle n|(\widehat{\bar\phi\phi})(0,\vec n)(\widehat{\bar\phi\phi})(\tau,\vec n(\theta))|n\rangle}{\langle n|n\rangle}. \tag{3.32}$$

In the regime $\Delta_{\phi^n}\gg\Delta_{\bar\phi\phi}$, the s-channel is controlled by the "Heavy-Light" OPE, while the t-channel is controlled by the "Heavy-Heavy" and the "Light-Light" OPEs.

3.2.1 Leading order

As before, the leading order contribution corresponds to evaluating the path integral on the saddle and gives

$$\frac{\langle n|(\hat{\bar\phi}\hat\phi)(0,\vec n_d)(\hat{\bar\phi}\hat\phi)(\tau,\vec n)|n\rangle}{\langle n|n\rangle} = \frac{f^4}{4}. \tag{3.33}$$

The implications of this result, when considering the s- and t-channels are as follows.

s-channel. From (3.31) and (3.5) we obtain

$$g_s(z,\bar z) = \left(f^2\Omega_3\right)^2|z|^{\Delta_{\phi^n}}. \tag{3.34}$$

Therefore, the only operator appearing in the $\phi^n\times\bar\phi\phi$ OPE is $\phi^n(x)$ itself with the fusion coefficient (3.8). Moreover, we see that at this order the descendants of ϕ^n do not contribute. This is to be expected, because the contribution of descendants is suppressed by powers of the ratio $\frac{\Delta_{\bar\phi\phi}}{\Delta_{\phi^n}}$, and thus by an inverse power of n, just as a consequence of conformal symmetry (see also [114]). For instance, the first descendant term in the conformal block has coefficient

$$A_{1,1}^{\Delta_{21},\Delta_{34}}(\Delta,0)=\frac{(\Delta_{21}+\Delta)(\Delta_{34}+\Delta)}{4\Delta}, \tag{3.35}$$

which, for the case at hand, equals

$$\frac{\Delta_{\bar\phi\phi}^2}{4\Delta_{\phi^n}}, \tag{3.36}$$

and is suppressed in the limit $n\gg 1$.

t-channel. From eqs. (3.32,3.33) we obtain

$$g_t(y,\bar y)=\left(\frac{n}{\mu}\right)^2\frac{|y|^{2\Delta_{\bar\phi\phi}}}{|1-y|^{\Delta_{\bar\phi\phi}}}. \tag{3.37}$$

Expanding in powers of y

$$g_t(y,\bar y)=\left(\frac{n}{\mu}\right)^2|y|^4\left[1+|y|C_1^{(1)}(\cos\theta)+|y|^2C_2^{(1)}(\cos\theta)+|y|^3C_3^{(1)}(\cos\theta)+\dots\right], \tag{3.38}$$

and comparing with the expansion in conformal blocks, (3.28), we deduce that in this channel there appears a tower of primary operators labelled by their spin ℓ and by an integer k, with dimension

$$\Delta_{(k,\ell)}=4+2k+2\ell,\qquad \ell,k=0,1,2,\dots, \tag{3.39}$$

and with fusion coefficients satisfying

$$\lambda_{(k,\ell)}^{n,n}\bar\lambda_{(k,\ell)}^{\bar\phi\phi,\bar\phi\phi}=\frac{f^4}{4}(-1)^k\frac{(k!)^2(k+2\ell)!(k+2\ell+1)!}{(2k)!(2k+4\ell+1)!}. \tag{3.40}$$

Determining whose operators those are the dimensions will be the subject of the last chapter of the thesis. For the moment, let us just note that at weak coupling these operators are

$$\mathcal{O}_{(k,\ell)}(x)=\left(\bar\phi\phi\,\partial^{2k}\partial_{\{\mu_1}\dots\partial_{\mu_{2\ell}\}}\bar\phi\phi\right)(x), \tag{3.41}$$

where {} indicates the traceless symmetric component.

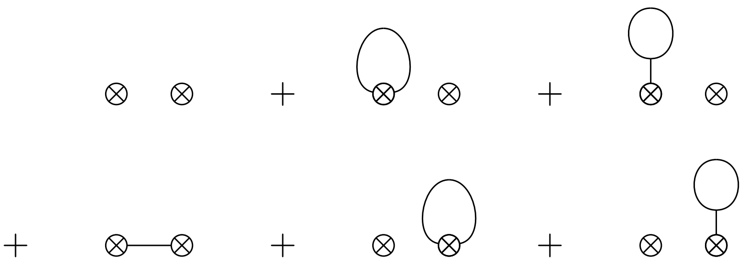


Figure 3.3: Topology of diagrams entering $\langle n|(\bar\phi\phi)(\bar\phi\phi)|n\rangle$

3.2.2 NLO

At next to leading order we must consider, in full analogy with (3.10),

$$\langle n|(\bar\phi\phi)(0,\vec n)(\bar\phi\phi)(\tau,\vec n(\theta))|n\rangle = \langle n|\left[\frac{f^2}{2}+fr(0,\vec n)+\frac{r^2(0,\vec n)}{2}\right]\left[\frac{f^2}{2}+fr(\tau,\vec n(\theta))+\frac{r^2(\tau,\vec n(\theta))}{2}\right]|n\rangle\,, \tag{3.42}$$

from which both connected and disconnected diagrams arise at NLO, see Fig. 3.3.

Disconnected diagrams just correspond to factorized 3-point functions, which we computed before. Therefore, what is left to compute is the "one-phonon" exchange connected diagram, which leads to

$$Z^{-2}_{\bar\phi\phi}\frac{\langle n|(\hat{\bar\phi}\hat\phi)(0,\vec n_d)(\hat{\bar\phi}\hat\phi)(\tau,\vec n)|n\rangle}{\langle n|n\rangle} = \lambda^2_{\bar\phi\phi}\left[1+\frac{4\mu\Omega_3}{n}D_{rr}(z,\bar z)\right], \tag{3.43}$$

with $\lambda_{\bar\phi\phi}$ the fusion coefficient in (3.13) and with the propagator for the radial mode given by (see appendix B.1)

$$D_{rr}(z,\bar z) = \sum_{\ell=0}^{\infty}\frac{\ell+1}{\Omega_3}\left(|z|^{\omega_A(\ell)}\frac{J_\ell^2-\omega_A^2(\ell)}{2\omega_A(\ell)}+|z|^{\omega_B(\ell)}\frac{\omega_B^2(\ell)-J_\ell^2}{2\omega_B(\ell)}\right)\frac{C_\ell^{(1)}(\cos\theta)}{\omega_B^2(\ell)-\omega_A^2(\ell)}. \tag{3.44}$$

Having secured the four-point function at NLO, we can now turn our attention to the spectrum of operators appearing in the different channels.

s-channel. The analysis is straightforward. Indeed using (3.31) and (3.44) we see that the four-point function

$$g_s(z,\bar z) = \lambda^2_{\bar\phi\phi}|z|^{\Delta_{\phi^n}}\left[1+\frac{4\mu}{n}\sum_{\ell=0}^{\infty}\left(|z|^{\omega_A(\ell)}\frac{J_\ell^2-\omega_A^2(\ell)}{2\omega_A(\ell)}+|z|^{\omega_B(\ell)}\frac{\omega_B^2(\ell)-J_\ell^2}{2\omega_B(\ell)}\right)\frac{(\ell+1)C_\ell^{(1)}(\cos\theta)}{\omega_B^2(\ell)-\omega_A^2(\ell)}\right] \tag{3.45}$$

is already in the form (3.29). Therefore, we can identify the primary operators by simply looking at the powers of $|z|$ in the expansion. These are in one-to one correspondence with the A- and B-type single phonon states found in section (2.4.3) and result in two separated towers of primaries with dimension

$$\Delta_A = \Delta_{\phi^n} + \omega_A(\ell), \quad \ell \geq 2 \qquad\qquad \Delta_B = \Delta_{\phi^n} + \omega_B(\ell), \quad \ell \geq 0\,. \tag{3.46}$$

Notice that the tower of A-type primaries starts at $\ell = 2$. Indeed, the $\ell = 1$ A-phonon does appear in (3.45) but it corresponds to the descendant $\partial_\mu \phi^n$. Instead $\ell = 0$ corresponds to the “Goldstone mode", which controls the global fluctuations of the phase of ϕ and, as such, is not excited by neutral operators like $\bar{\phi}\phi$. The corresponding fusion coefficients can be read off from the coefficients in front of $|z|^\Delta$

$$\begin{aligned}\lambda^{\ell}_{\bar{\phi}\phi,A} &= \lambda_{\bar{\phi}\phi}\sqrt{\frac{4\mu}{n}(\ell+1)\frac{J_\ell^2 - \omega_A^2(\ell)}{2\omega_A(\ell)}}, \quad \ell \geq 1,\\ \lambda^{\ell}_{\bar{\phi}\phi,B} &= \lambda_{\bar{\phi}\phi}\sqrt{\frac{4\mu}{n}(\ell+1)\frac{\omega_B^2(\ell) - J_\ell^2}{2\omega_B(\ell)}}, \quad \ell \geq 0.\end{aligned} \tag{3.47}$$

We see that these are n suppressed by $\sim \sqrt{\mu/n}$ with respect to $\lambda_{\bar{\phi}\phi}$. It should also be noted that these operators enter the OPE without their descendants, similarly to ϕ^n at leading order.

t-channel. The analysis is somewhat more complicated. The reason is that the corresponding expression of the four-point function

$$\begin{aligned} g_t(1-z, 1-\bar{z}) \quad &= \quad \lambda^2_{\bar{\phi}\phi}\frac{|1-z|^{2\Delta_{\bar{\phi}\phi}}}{|z|^{\Delta_{\bar{\phi}\phi}}} \\ &\left[1 + \frac{4\mu}{n}\sum_{\ell=0}^{\infty}\left(|z|^{\omega_A(\ell)}\,\frac{J_\ell^2 - \omega_A^2(\ell)}{2\omega_A(\ell)} + |z|^{\omega_B(\ell)}\,\frac{\omega_B^2(\ell) - J_\ell^2}{2\omega_B(\ell)}\right)\frac{(\ell+1)C_\ell^{(1)}(\cos\theta)}{\omega_B^2(\ell) - \omega_A^2(\ell)}\right]\end{aligned} \tag{3.48}$$

is written as a power series in $|z|$, and not in $|1-z|$. In order to get the latter, we have to analytically continue the four-point function to the region $z = 1$. That would allow to analyze the spectrum of operators appearing in the t-channel at next to leading order.

Unfortunately, we do not know how to perform the analytic continuation in closed form[1], and instead, we will illustrate the principle with an example. For that, let us consider a simplified situation, $z = \bar{z} \in \mathbb{R}$, in other words $\theta = 0$. Introducing the following notation for the summand in (3.48)

$$G(z;\ell) = 2\left(z^{\omega_A(\ell)}\,\frac{J_\ell^2 - \omega_A^2(\ell)}{2\omega_A(\ell)} + z^{\omega_B(\ell)}\,\frac{\omega_B^2(\ell) - J_\ell^2}{2\omega_B(\ell)}\right)\frac{(\ell+1)C_\ell^{(1)}(1)}{\omega_B^2(\ell) - \omega_A^2(\ell)}, \tag{3.49}$$

[1] In other words, we do not possess the propagator in closed form for $z \sim \bar{z} \sim 1$.

and using its asymptotic behavior (see also (2.128))

$$G(z;\ell) \underset{\substack{\ell\to\infty \\ z\to 1}}{=} z^\ell \left(\ell + 1 - \frac{3}{2}\frac{\mu^2-1}{\ell} - (1-z)\ell + \dots\right) \tag{3.50}$$

we can find leading asymptotic of the four point function for $z \to 1$

$$g_t(1-z,1-z) \underset{z\to 1}{=} \lambda^2_{\bar\phi\phi}(1-z)^{2\Delta_{\bar\phi\phi}} \left\{1 + \frac{2\mu}{n}\left[\frac{1}{(1-z)^2} + \frac{\Delta_{\bar\phi\phi}-1}{1-z} + \frac{3}{2}(\mu^2-1)\log(1-z)\right] + \mathcal{R}(z)\right\} \tag{3.51}$$

with the remainder

$$\mathcal{R}(z) = \frac{2\mu}{n}G(z;0) + \frac{2\mu}{n}\sum_{\ell=1}^{\infty}\left[G(z;\ell) - z^\ell\left(\ell+1-\frac{3}{2}\frac{\mu^2-1}{\ell}\right)\right] \underset{z\to 1}{=} O((1-z)\log(1-z)) \tag{3.52}$$

a less singular function.

Singular terms in (3.51) correspond to different operators. The term proportional to $(1-z)^{-2}$ corresponds to an operator with scaling dimension $\Delta = 2$, which is nothing else but $\bar\phi\phi$ (its anomalous dimension is invisible at this order).[2] The second term corresponds to its descendant, whose coefficient is fixed by the conformal symmetry (compare with (3.35)). Lastly, the term with $\log(1-z)$ can be exponentiated, leading to a modified prefactor

$$g_t(1-z,1-z) \underset{z\to 1}{\supset} \lambda^2_{\bar\phi\phi}(1-z)^{2\Delta_{\bar\phi\phi}}(1-z)^{\frac{3\mu}{n}(\mu^2-1)}. \tag{3.54}$$

The resulting exponent should correspond to the scaling dimension of $\Delta_{(\bar\phi\phi)^2}$ at NLO (which we already computed as the $k=\ell=0$ case of (3.39) and (3.41)). Indeed, using (2.92), (2.93), and (3.6) we can write

$$\Delta_{(\bar\phi\phi)^2} = 2\Delta_{\bar\phi\phi} + \frac{3\mu}{n}(\mu^2-1) = 4 + O(\varepsilon^2), \tag{3.55}$$

which coincides with the computation using Feynman diagrams, see appendix C. The last result can also be directly derived using the general relation (see e.g. [115])

$$\Delta_{(\bar\phi\phi)^2} = d + \beta'(\lambda_*) \tag{3.56}$$

between the dimension of the interaction term $(\bar\phi\phi)^2$ and $\beta' \equiv \partial_\lambda\beta$. Using (2.8) and (2.6) immediately gives $\Delta_{(\bar\phi\phi)^2} = 4 + O(\varepsilon^2)$.

[2] In general we expect not only $\bar\phi\phi$ but its spin-ℓ analogues of the form

$$\bar\phi\partial_{\mu_1}\dots\partial_{\mu_\ell}\phi, \tag{3.53}$$

to appear in the $\bar\phi\phi \times \bar\phi\phi$ OPE.

The relation (3.56) can also be derived from the requirement that

$$\frac{\partial}{\partial\lambda}\langle[\phi](x_1)\dots[\phi](x_N)[\bar{\phi}](y_1)\dots[\bar{\phi}](y_N)\rangle \tag{3.57}$$

is free of UV-divergence. Using the relation between bare and renormalized coupling $\lambda_0=\mu^\varepsilon\lambda Z_\lambda$, and the path integral picture, one can show [101] the above is equal to

$$\frac{1}{4}\frac{\varepsilon\mu^\varepsilon\lambda_0}{\beta(\lambda)}Z_{(\bar{\phi}\phi)^2}\int \mathrm{d}^dx\,\langle[\phi](x_1)\dots[\bar{\phi}](y_N)[(\bar{\phi}\phi)^2](x)\rangle\;+\;\dots \tag{3.58}$$

where the dots stand for other terms related to the mixing with lower-dimensional operators, which can be ignored in this discussion of divergences (moreover knowing that ultimately the mixing cancels at the critical coupling because $(\bar{\phi}\phi)^2$ is a primary operator). The term we kept is a zero-momentum insertion of the renormalized operator into a renormalized correlator, hence is free of UV-divergence[3]. Thus its coefficient must be finite as well. Note that for this purpose one must consider the inverse of the beta function

$$\frac{1}{\beta(\lambda)}=\frac{1}{-\varepsilon\lambda+\beta_4(\lambda)}$$

as expanded in a series of poles in ε. Since the renormalization factor $Z_{(\bar{\phi}\phi)^2}$ must also have the form of (1+ a series of poles), we conclude the coefficient is finite if

$$Z_{(\bar{\phi}\phi)^2}=-\frac{\beta(\lambda)\mu^\varepsilon}{\varepsilon\lambda_0}, \tag{3.59}$$

from which we derive

$$\Delta_{(\bar{\phi}\phi)^2}=4\left(\frac{d-2}{2}\right)+\left.\frac{\mathrm{d}\log Z_{(\bar{\phi}\phi)^2}}{\mathrm{d}\log\mu}\right|_{\lambda=\lambda_*}=2d-4+\beta'(\lambda_*)+\varepsilon=d+\beta'(\lambda_*)=4+O(\varepsilon^2)\,. \tag{3.60}$$

3.2.3 Comments

We conclude this section with two comments. First, it is straightforward to extend the computation presented above to the case when the two 'light' operators are $(\bar{\phi}\phi)^k$. (3.51) is just minimally modified to [4]

$$g_t(1-z,1-z)\underset{z\to1}{=}\lambda^2_{(\bar{\phi}\phi)^k}(1-z)^{2\Delta_{(\bar{\phi}\phi)^k}}\left\{1+\frac{2\mu k^2}{n}\left[\frac{1}{(1-z)^2}+\frac{\Delta_{(\bar{\phi}\phi)^k}-1}{1-z}+\frac{3}{2}(\mu^2-1)\log(1-z)\right]+\dots\right\} \tag{3.61}$$

which implies that the two leading contributions are associated to $(\bar{\phi}\phi)^{2k}$ and $(\bar{\phi}\phi)^{2k-1}$. Moreover, by exponentiating the term with $\log(1-z)$ we obtain, at 1-loop accuracy, a

[3] The integral is IR-divergent, but that can be regulated by introducing a regulator mass which does not affect the derivation of the anomalous dimension.

[4] The fusion coefficient $\lambda_{(\bar{\phi}\phi)^k}$ can be computed by repeating the steps of section 3.1.

relation between scaling dimensions

$$\Delta_{(\bar{\phi}\phi)^{2k}} - 2\Delta_{(\bar{\phi}\phi)^k} = k^2 \left(\Delta_{(\bar{\phi}\phi)^2} - 2\Delta_{(\bar{\phi}\phi)}\right), \tag{3.62}$$

which can be checked perturbatively using the results of appendix C. This provides an additional cross-check.

The second comment concerns the computation of similar correlators in a general CFT using the universal EFT superfluid description, as done in [74, 75]. Even though the EFT description can be trusted only for sufficiently large separations between the two 'light' operators, we can try and use the results of [75] for the 4-point function to formally analyze what operators appear in t-channel. Repeating almost verbatim (albeit unjustifiably) the computation leading to (3.39) we conclude that the spectrum of operators in this case is given by

$$\Delta = \delta_1 + \delta_2 + 2k + \ell, \tag{3.63}$$

which for large $\ell \gg 1$ coincides with the predictions of the analytic bootstrap [84, 83]. This fact indicates there should be a way to frame the statement, which is purely within the reach of EFT. But we do not know how.

4 Identifying large charge operators

4.1 Introduction

4.1.1 Motivation and goals

In this chapter, we turn to a question which was raised by some of the results of the previous chapter. Indeed, in section 3.2.2, we identified by their scaling dimension a tower of primary operators appearing in the s-channel conformal block decomposition, but this approach did not furnish any direct information about the identity of these operators. Otherwise stated, the semiclassical approach delivers the operator spectrum, but it does so somewhat formally, without telling concretely what these operators look like. It is the goal of this chapter to investigate this issue, as we now explain in more detail.

Generally, the large charge expansion of CFT correlators relies on a universal description in terms of a finite density superfluid state, described by an effective field theory EFT for the excitations of the superfluid (which for convenience we will call "hydrodynamic excitations" in this chapter). In particular that implies that there exists a non trivial correspondence between large charge operators and the hydrodynamic excitations in a superfluid. That motivated exploring large charge operators using instead the conformal bootstrap [114]. Perfect agreement was found, thus remarkably showing that the superfluid phase dynamics is somewhat encapsulated in the bootstrap constraints at large charge n.

As we have seen in chapter 2, for specific CFTs that admit a definition within perturbation theory, through the ε-expansion, the semiclassical approach is also very powerful. There, the superfluid description was elucidated by considering the properties of the simplest charge n operator ϕ^n. CFT correlators (or any observable for that matter, see for example chapter 3) involving this large charge operator, could be computed in a double scaling limit

$$n \gg 1, \quad \lambda \ll 1, \quad \lambda n = \text{fixed}, \tag{4.1}$$

by finding a saddle explicitly and expanding around it. Computations were simplified due to the enhanced symmetry of the problem. Given the coupling λ, it was found it is the combination λn that controls the convergence of the standard Fenynman diagram approach: only for $\lambda n \ll 1$ is perturbation theory applicable. Amusingly the parameter λn shares some features with the 't Hooft coupling in AdS/CFT [48]. In particular, $\lambda n \gg 1$ corresponds to the regime where all the modes beside the hydrodynamic ones are gapped and can be integrated out, very much like in AdS/CFT the large 't Hooft coupling allows to integrate out the string modes to obtain the supergravity limit. It is in this regime that the semiclassical result matches the general EFT treatment of the large charge sector of CFTs.

This clearly invites to see how the hydrodynamic Fock space structure emerges in our ε-expansion models based on the elementary fields and their derivatives. Indeed, an interesting aspect of Wilson-Fisher models is that, at least for $\lambda n \ll 1$, the operator spectrum can be explicitly constructed both in terms of fields and derivatives and in terms of hydrodynamic modes around the semiclassical saddle.

4.1.2 Spectrum of superfluid excitations

Let us review the information we have on the spectrum of superfluid excitations from our semiclassical expansion. As we have seen, the operator-state correspondence allows to map the theory on the cylinder. For instance, the scaling dimension of ϕ^n, which is the lightest operator in the sector of charge n, was given by the energy of the charge-n ground state $|n\rangle$ (2.74). To simplify the computation, we introduced the state $|\psi_n\rangle$ which is the superfluid with homogeneous charge density (2.83). It spontaneously breaks time translation invariance and the $U(1)$ group. The two states are related, since the ground-state $|n\rangle$ is the least-suppressed component of any overlapping charge-n state at the infinite past on the cylinder (for $|\psi_n\rangle$ see (2.77)).

Similarly, we found in section (2.4.3) that the excitations of the superfluid are given by phonons of spin ℓ and energies (2.118). The excited states of the superfluid are obtained by acting on the ground state $|n\rangle$ with the associated creation operators $A^\dagger_{\ell,\vec{m}}, B^\dagger_{\ell,\vec{m}}$ introduced in (2.116), thus generating a Fock space. The Fock space of phonon excitations corresponds to the space of operators with charge n, whose spectrum of scaling dimensions at next to leading order (NLO) is then given by [1]

$$\Delta\left(\{k^A\},\{k^B\}\right) = \Delta_{\phi^n} + \sum_{\ell=1}^{\infty} k^A_\ell \omega_A(\ell) + \sum_{\ell'=0}^{\infty} k^B_{\ell'} \omega_B(\ell'), \tag{4.2}$$

with k^A_ℓ and k^B_ℓ non-negative integers. The above result applies for states with a finite

[1]As Δ_{ϕ^n} is $O(n)$ and the $\omega_{A,B}(\ell)$ are $O(1)$, the tree level frequencies are sufficient to compute the dimension $\Delta\left(\{k^A\},\{k^B\}\right)$ at NLO. On the other hand, in order to compute the splittings at NLO, one would need to perform a full 1-loop computation.

number of phonons and finite spin as $n \to \infty$. For large enough total spin, one expects a non-homogeneous configuration to dominate the path integral (see for instance [116]).

We call A- and B-type the phonons with energy ω_A and ω_B respectively. Notice that primary operators correspond to states with $k_1^A = 0$, and that descendants are obtained by adding spin-1 A-type phonons. Compatibly with that, and with the accuracy of (4.2), one indeed has

$$\omega_A(1) = 1 + O(\varepsilon)\,. \tag{4.3}$$

The approach outlined above provides the spectrum of the operators but it does not say anything about their explicit form in terms of elementary fields and derivatives. Establishing such form is one of the goals of this chapter. Notice though that the explicit form of composite operators depends on the renormalization procedure and that, moreover, for large enough λn we do not possess such a procedure. We will thus content ourselves with the construction of the operators in the free field theory limit $\lambda \to 0$ and with their correspondence to superfluid excitations.

4.1.3 Why free theory ?

As we shall see, the tree-level result is already structurally informative. Indeed, the properties of the operator spectrum vary continuously with λ (in truth with ε): by varying λ we obtain operator *families* $\mathcal{O}_\lambda^{(n,\ell,\alpha)}(x)$, with α a discrete label characterizing the phonon composition (the k^A and k^B mentioned in the previous section). As qualitatively depicted in Fig. 4.1, the dimensions Δ, and OPE coefficients, of the $\mathcal{O}_\lambda^{(n,\ell,\alpha)}(x)$ are continuous functions of λ. Our tree level construction will thus correspond to the starting point at $\lambda = 0$ of each trajectory. Such endpoints, however, fully characterize the families non-perturbatively, even if indirectly[2].

Therefore, in this chapter we are going to work in free theory $\lambda = 0$, which corresponds to the Wilson-Fisher fixed point at $\varepsilon = 0$, hence in an exactly integer number of dimensions. For simplicity we work in three dimensions, thus in the free limit of theory (2.9), but the conclusions of this chapter can be generalized to higher dimensions as well.

This chapter is organized as follows. In section 4.2, we discuss the classification of operators with charge n in 3D free field theory quantized around $\phi = 0$. We explain how to explicitly use the state-operator correspondence to identify primary operators, and provide a systematic construction for a sub-class of them. Section 4.3 constructs the mapping between superfluid Fock states and operators. We also discuss the identification of primary operators in this picture. In section 4.4, we discuss the breakdown of the

[2] The identification of these trajectories is ambiguous in case of level crossing at some value of λ. Since the publication of [3], a more cautious study of the spectrum (4.2) has revealed that there is indeed level-crossing. Thus the identification of operators proposed in this chapter is actually only valid at weak coupling, and we still lack a procedure to extend it to large λn.

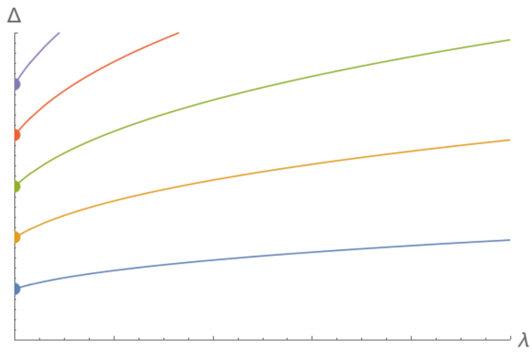


Figure 4.1: Scaling dimension of families of operators as a function of λ. Each family can be labeled with the corresponding "seed" operator in free theory (at $\lambda = 0$).

homogeneous superfluid description as the spin gets large.

4.2 Operators corresponding to vacuum fluctuations in free theory

Our goal is to classify the families of large charge operators by focusing on their representatives in the free limit, as sketched in figure 4.1. As explained in section 4.1.3, the first step is the classification of the operators of free field theory in terms of conformal multiplets. This amounts to identifying the conformal primaries.

In a CFT every local operator corresponds to a state and vice versa (operator-state correspondence). In particular primary states, i.e. those annihilated by the special conformal generators, correspond to primary operators. The goal of this section is to set up the methodology for identifying these states. To make things explicit we will fully construct a subclass of the operators.

Working in radial quantization we will now, in turn, construct the Fock space of vacuum fluctuations, derive the state-operator correspondence and write the conformal group generators. We will then write down in closed form a subset of primary states, also showing by a combinatoric argument that it forms a complete basis of the subspace of primary operators with a number of derivatives smaller than the charge.

4.2.1 Fock space of vacuum fluctuations

Let us consider a free complex scalar field in $d = 3$ Euclidean dimensions

$$\mathcal{L} = \partial\bar{\phi}\partial\phi. \tag{4.4}$$

As usual it is beneficial to put the theory (4.4) on the cylinder $\mathbb{R}\times\mathbb{S}^2$ by redefining the coordinates

$$x^\mu = rn^\mu, \quad r = e^\tau, \quad \vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \tag{4.5}$$

and the field having scaling dimension $\Delta_\phi = \frac{d-2}{2} = \frac{1}{2}$

$$\hat{\phi}(\tau,\theta,\varphi) = e^{\tau/2}\phi(x). \tag{4.6}$$

As a result we have the following action on the cylinder

$$S = \int d\tau d\Omega_2 \left[g^{\mu\nu}\partial_\mu\hat{\bar{\phi}}\partial_\nu\hat{\phi} + \frac{1}{4}\hat{\bar{\phi}}\hat{\phi}\right], \quad g_{\mu\nu} = \mathrm{diag}(1,1,\sin^2\theta). \tag{4.7}$$

Time translations on the cylinder are generated by the corresponding Hamiltonian H in the following way

$$\hat{\phi}(\tau,\theta,\phi) = e^{H\tau}\hat{\phi}(0,\theta,\phi)e^{-H\tau}, \tag{4.8}$$

and are related to dilatations on the plane, which are generated by D

$$e^{D\lambda}\phi(x)e^{-D\lambda} = e^{\lambda/2}\phi(e^\lambda x). \tag{4.9}$$

This implies

$$H = D, \tag{4.10}$$

so that operator dimensions are in one to one correspondence with energy levels on the cylinder.

Hermitian conjugation in radial quantization of the parent Euclidean field theory implies $\hat{\bar{\phi}}(0,\theta,\varphi) = \hat{\phi}(0,\theta,\varphi)^\dagger$, which at arbitary τ on the cylinder and arbitrary x on the plane implies respectively

$$\hat{\bar{\phi}}(\tau,\theta,\varphi) = \hat{\phi}(-\tau,\theta,\varphi)^\dagger \qquad \text{and} \qquad \bar{\phi}(x) = |x|^{-1}\phi(x^{-1})^\dagger\,. \tag{4.11}$$

Quantization proceeds by expanding the fields in spherical harmonics $Y_{\ell m}$, to solve the quadratic equations of motion. The process is similar to what was done in section 2.4.3, but made much simpler by the absence of mixing of the fields, yielding

$$\hat{\phi}(\tau,\theta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{\sqrt{2\omega_0(\ell)}}\left[a^\dagger_{\ell m}e^{\omega_0(\ell)\tau}Y^*_{\ell m}(\theta,\varphi) + b_{\ell m}e^{-\omega_0(\ell)\tau}Y_{\ell m}(\theta,\varphi)\right], \tag{4.12}$$

and[3]

$$\hat{\bar{\phi}}(\tau,\theta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{\sqrt{2\omega_0(\ell)}}\left[b^\dagger_{\ell m}e^{\omega_0(\ell)\tau}Y^*_{\ell m}(\theta,\varphi) + a_{\ell m}e^{-\omega_0(\ell)\tau}Y_{\ell m}(\theta,\varphi)\right], \tag{4.13}$$

[3] Notice $\hat{\bar{\phi}}(\tau,\theta,\varphi) = \hat{\phi}(-\tau,\theta,\varphi)^\dagger$ in accordance with (4.11).

with energies (using the same convention as in (2.136))

$$\omega_0(\ell) = \ell + \frac{1}{2}. \tag{4.14}$$

The corresponding canonically conjugated momenta are given by[4]

$$p_{\hat{\phi}}(\tau,\theta,\varphi) = i\partial_\tau \hat{\bar{\phi}} = i \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{\frac{\omega_0(\ell)}{2}} \left[b_{\ell m}^{\dagger} e^{\omega_0(\ell)\tau} Y_{\ell m}^{*}(\theta,\varphi) - a_{\ell m} e^{-\omega_0(\ell)\tau} Y_{\ell m}(\theta,\varphi) \right], \tag{4.15}$$

and

$$p_{\hat{\bar{\phi}}}(\tau,\theta,\varphi) = i\partial_\tau \hat{\phi} = i \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{\frac{\omega_0(\ell)}{2}} \left[a_{\ell m}^{\dagger} e^{\omega_0(\ell)\tau} Y_{\ell m}^{*}(\theta,\varphi) - b_{\ell m} e^{-\omega_0(\ell)\tau} Y_{\ell m}(\theta,\varphi) \right]. \tag{4.16}$$

Creation and annihilation operators, satisfying the usual commutation relations

$$[a_{\ell m}, a_{\ell' m'}^{\dagger}] = [b_{\ell m}, b_{\ell' m'}^{\dagger}] = \delta_{\ell\ell'}\delta_{mm'}, \tag{4.17}$$

allow us to build the Hilbert space. Defining the vacuum state $|0\rangle$ as

$$a_{\ell m}|0\rangle = b_{\ell m}|0\rangle = 0, \qquad \forall\, \ell, m \tag{4.18}$$

states featuring a string of creation operators acting on the vacuum

$$\prod_{i=1}^{n_a} a_{\ell_i m_i}^{\dagger} \prod_{j=1}^{n_b} b_{\ell'_j m'_j}^{\dagger} |0\rangle \tag{4.19}$$

provide a basis of the Hilbert space, and give it the standard Fock space structure. The $U(1)$ charge of these states is determined by the charge operator

$$Q = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(a_{\ell m}^{\dagger} a_{\ell m} - b_{\ell m}^{\dagger} b_{\ell m} \right). \tag{4.20}$$

4.2.2 Operator-state correspondence

Combining (4.12) with (4.16) and using the orthonormality of the $Y_{\ell m}$ (2.107) we get

$$a_{\ell m}^{\dagger} = \frac{e^{-\omega_0(\ell)\tau}}{\sqrt{2\omega_0(\ell)}} \int d\Omega_2\, Y_{\ell m} \left(\partial_\tau \hat{\phi}(\tau) + \omega_0(\ell)\hat{\phi}(\tau) \right), \tag{4.21}$$

which is valid at any finite τ. Remembering the change of coordinates (4.5) and the relation between fields on the plane and on the cylinder (4.6), this expression can be

[4]There appears an "i" in front of the time derivatives because we work in Euclidean time.

rewritten as

$$a^\dagger_{\ell m} = \frac{r^{-\ell}}{\sqrt{2\omega_0(\ell)}} \int d\Omega_2\, Y_{\ell m} \Big(x^\mu \partial_\mu \phi(x) + (\ell+1)\phi(x) \Big), \tag{4.22}$$

where the integral is over the unit sphere and $x^\mu = rn^\mu$.

Acting on the vacuum and Taylor expanding around the origin[5] we get

$$\phi(x)|0\rangle = \sum_{\ell'=0}^{\infty} \frac{1}{\ell'!} x^{\{\mu_1} \cdots x^{\mu_{\ell'}\}} \partial_{\mu_1} \cdots \partial_{\mu_{\ell'}} \phi(0)|0\rangle \,, \tag{4.23}$$

where by $\{\ldots\}$ we indicate the traceless symmetric combination, which arises because of the equation of motion $\partial^2\phi(x) = 0$. Noting that

$$\int d\Omega_2\, Y_{\ell m} x^{\{\mu_1} \cdots x^{\mu_{\ell'}\}} = 0, \quad \ell' \neq \ell, \tag{4.24}$$

the expansion results in

$$a^\dagger_{\ell m}|0\rangle = \frac{\sqrt{2\ell+1}}{\ell!} \int d\Omega_2\, Y_{\ell m} n^{\mu_1} \cdots n^{\mu_\ell} \partial_{\mu_1} \cdots \partial_{\mu_\ell} \phi(0)|0\rangle. \tag{4.25}$$

This can be explicitly computed, and is most easily performed in a different basis for the coordinates

$$x_\pm = \frac{x_1 \pm i x_2}{\sqrt{2}}, \quad x_0 = x_3. \quad \text{and} \quad n_\pm = \frac{n_1 \pm i n_2}{\sqrt{2}}, \quad n_0 = n_3, \tag{4.26}$$

in which the spherical harmonics have polynomial expressions

$$\begin{aligned} Y_{\ell m} &= [-\mathrm{sign}(m)]^m \sqrt{\frac{(2\ell+1)(\ell+m)!(\ell-m)!}{2^{|m|}\, 4\pi}} \sum_{\substack{\alpha_+ + \alpha_- + \alpha_0 = \ell \\ \alpha_+ - \alpha_- = m}} \frac{n_+^{\alpha_+} n_0^{\alpha_0} n_-^{\alpha_-}}{(-2)^{\min(\alpha_+,\alpha_-)} \alpha_+! \alpha_0! \alpha_-!} \\ &= [-\mathrm{sign}(m)]^m \sqrt{\frac{(2\ell+1)(\ell+m)!(\ell-m)!}{2^{|m|}\, 4\pi}} \sum_{k \text{ step } 2}^{\ell-|m|} \frac{n_+^{\frac{\ell+m-k}{2}} n_0^k n_-^{\frac{\ell-m-k}{2}}}{(-2)^{\frac{\ell-|m|-k}{2}} \left(\frac{\ell+m-k}{2}\right)! k! \left(\frac{\ell-m-k}{2}\right)!}, \end{aligned} \tag{4.27}$$

where the sum over k is taken in steps of 2, starting form $\ell - |m|$ mod 2.

The integral in (4.25) yields

$$a^\dagger_{\ell m}|0\rangle = \mathcal{Y}^{\mu_1 \ldots \mu_\ell}_{\ell m} \partial_{\mu_1} \cdots \partial_{\mu_\ell} \phi(0)|0\rangle, \tag{4.28}$$

[5] As can be seen in (4.12), the field is singular at the origin, $r \to 0$ or $\tau \to -\infty$ due to negative-frequency $b_{\ell m}$ modes. However, in $\phi(x)|0\rangle$ the singular terms drop and Taylor expansion is legitimate.

where we denote

$$\mathcal{Y}_{\ell m}^{\mu_1\ldots\mu_\ell} = \frac{\sqrt{2\ell+1}}{\ell!}\int d\Omega_2\, Y_{\ell m} n^{\mu_1}\cdots n^{\mu_\ell}. \tag{4.29}$$

given explicitly as

$$\begin{aligned}\mathcal{Y}_{\ell m}^{\overbrace{+\ldots+}^{\alpha_+}\overbrace{0\ldots0}^{\alpha_0}\overbrace{-\ldots-}^{\alpha_-}} &= \delta_{\alpha_++\alpha_0+\alpha_-,\ell}\delta_{\alpha_--\alpha_+,m}\frac{[-\mathrm{sign}(m)]^m\sqrt{\pi}\,(2\ell+1)\sqrt{(\ell+m)!(\ell-m)!}}{\ell!}\\ &\times\sum_{k\text{ step }2}^{\ell-|m|}\frac{(-1)^{\frac{\ell-k-|m|}{2}}}{2^{\frac{3}{2}\ell-k-\frac{\alpha_0}{2}}}\frac{\Gamma\left(\frac{k+\alpha_0+1}{2}\right)}{\Gamma\left(\ell+\frac{3}{2}\right)}\frac{\left(\ell-\frac{k+\alpha_0}{2}\right)!}{\left(\frac{\ell+m-k}{2}\right)!k!\left(\frac{\ell-m-k}{2}\right)!}.\end{aligned} \tag{4.30}$$

In particular, we obtain as a simple example

$$\mathcal{Y}_{\ell\ell}^{\mu_1\ldots\mu_\ell} = \frac{(-1)^\ell 2^{\frac{\ell}{2}+1}\sqrt{\pi}}{\sqrt{(2\ell)!}}\delta_-^{\mu_1}\cdots\delta_-^{\mu_\ell}, \tag{4.31}$$

$$a_{\ell\ell}^\dagger|0\rangle = \frac{(-1)^\ell 2^{\frac{\ell}{2}+1}\sqrt{\pi}}{\sqrt{(2\ell)!}}\left(\partial_-\right)^\ell\phi(0)|0\rangle. \tag{4.32}$$

Repeating the same steps starting with (4.13) and (4.15), we get similarly

$$b_{\ell m}^\dagger|0\rangle = \mathcal{Y}_{\ell m}^{\mu_1\ldots\mu_\ell}\partial_{\mu_1}\cdots\partial_{\mu_\ell}\bar{\phi}(0)|0\rangle. \tag{4.33}$$

This generalizes to multi-particle Fock states (4.19). For example[6],

$$a_{\ell_1 m_1}^\dagger b_{\ell_2 m_2}^\dagger|0\rangle = \mathcal{Y}_{\ell_1 m_1}^{\mu_1\ldots\mu_{\ell_1}}\mathcal{Y}_{\ell_2 m_2}^{\nu_1\ldots\nu_{\ell_2}} :\partial_{\mu_1}\cdots\partial_{\mu_{\ell_1}}\phi(0)\,\partial_{\nu_1}\cdots\partial_{\nu_{\ell_2}}\bar{\phi}(0): |0\rangle. \tag{4.34}$$

Finally, note that hermitian conjugation of (4.25), together with (4.11), implies

$$\langle 0|a_{\ell m} = \frac{\sqrt{2\ell+1}}{\ell!}\int d\Omega_2\, Y_{\ell m}^* n^{\mu_1}\cdots n^{\mu_\ell}\lim_{x\to\infty}\langle 0|\partial_{\mu_1}^{(1/x)}\cdots\partial_{\mu_\ell}^{(1/x)}\Big(|x|\bar{\phi}(x)\Big), \tag{4.35}$$

where we defined

$$\partial_\mu^{(1/x)} = \left(x^2\delta_{\mu\nu} - 2x^\mu x^\nu\right)\partial_\nu^{(x)}. \tag{4.36}$$

Notice, this time the field is evaluated at infinity, because hermitian conjugation in radial quantization involves a space inversion.

[6] In field products acting on the vacum the singular terms at the origin are eliminated by normal-ordering. In the rest of the chapter, normal ordering will always be intended and we will drop the ":" symbol.

4.2.3 Conformal generators

In order to proceed with the classification and construction of the operators we first need the explicit expression of the conformal group generators in terms of the ladder operators. We provide them in this section.

In $d = 3$, defining

$$J_i = \frac{1}{2}\varepsilon_{ijk}J_{jk}, \quad J_\pm = J_1 \pm iJ_2 \tag{4.37}$$

and

$$P_\pm = \frac{1}{\sqrt{2}}\left(P_1 \pm iP_2\right), \quad K_\pm = \frac{1}{\sqrt{2}}\left(K_1 \pm iK_2\right), \quad P_0 \equiv P_3, \quad K_0 = K_3, \tag{4.38}$$

such that

$$P_\pm^\dagger = K_\mp, \quad P_0^\dagger = K_0, \tag{4.39}$$

the commutation relations of the conformal algebra take the form (with $X_\bullet = P_\bullet, K_\bullet$)

$$\begin{aligned}
&[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_3,\\
&[J_3, X_\pm] = \pm X_\pm, \quad [J_3, X_0] = 0,\\
&[J_+, X_+] = 0, \quad [J_+, X_0] = -\sqrt{2}X_+, \quad [J_+, X_-] = \sqrt{2}X_0,\\
&[J_-, X_+] = -\sqrt{2}X_0, \quad [J_-, X_0] = \sqrt{2}X_-, \quad [J_-, X_-] = 0\\
&[D, K_i] = -K_i, \quad [D, P_i] = P_i,\\
&[K_-, P_+] = 2\left(D + J_3\right), \quad [K_+, P_-] = 2\left(D - J_3\right), \quad [K_0, P_0] = 2D\\
&[K_0, P_+] = -\sqrt{2}J_+, \quad [K_+, P_0] = \sqrt{2}J_+, \quad [K_-, P_0] = -\sqrt{2}J_-, \quad [K_0, P_-] = \sqrt{2}J_-,
\end{aligned} \tag{4.40}$$

and all generators (P_i, K_i, J_i, D) commute with the charge generator Q (4.20).

The generators, as computed from the Noether currents of the theory, read

$$D = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \omega_0(\ell)\left(a_{\ell m}^\dagger a_{\ell m} + b_{\ell m}^\dagger b_{\ell m}\right), \tag{4.41}$$

$$J_3 = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} m\left(a_{\ell m}^\dagger a_{\ell m} + b_{\ell m}^\dagger b_{\ell m}\right), \tag{4.42}$$

$$P_0 = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \sqrt{(\ell+1)^2 - m^2}\left(a_{\ell+1,m}^\dagger a_{\ell m} + b_{\ell+1,m}^\dagger b_{\ell m}\right), \tag{4.43}$$

$$K_0 = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \sqrt{(\ell+1)^2 - m^2}\left(a_{\ell m}^\dagger a_{\ell+1,m} + b_{\ell m}^\dagger b_{\ell+1,m}\right), \tag{4.44}$$

$$
\begin{aligned}
J_+ &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sqrt{\ell(\ell+1)-m(m+1)}\left(a^\dagger_{\ell,m+1}a_{\ell m}+b^\dagger_{\ell,m+1}b_{\ell m}\right), \quad (4.45)\\
&= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sqrt{\ell(\ell+1)-m(m-1)}\left(a^\dagger_{\ell m}a_{\ell,m-1}+b^\dagger_{\ell m}b_{\ell,m-1}\right),\\
J_- &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sqrt{\ell(\ell+1)-m(m+1)}\left(a^\dagger_{\ell m}a_{\ell,m+1}+b^\dagger_{\ell m}b_{\ell,m+1}\right) \quad (4.46)\\
&= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sqrt{\ell(\ell+1)-m(m-1)}\left(a^\dagger_{\ell,m-1}a_{\ell m}+b^\dagger_{\ell,m-1}b_{\ell m}\right),
\end{aligned}
$$

$$
\begin{aligned}
P_+ &= -\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sqrt{\frac{(\ell+m+1)(\ell+m+2)}{2}}\left(a^\dagger_{\ell+1,m+1}a_{\ell m}+b^\dagger_{\ell+1,m+1}b_{\ell m}\right), \quad (4.47)\\
K_- &= -\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sqrt{\frac{(\ell+m+1)(\ell+m+2)}{2}}\left(a^\dagger_{\ell m}a_{\ell+1,m+1}+b^\dagger_{\ell m}b_{\ell+1,m+1}\right), \quad (4.48)\\
P_- &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sqrt{\frac{(\ell-m+1)(\ell-m+2)}{2}}\left(a^\dagger_{\ell+1,m-1}a_{\ell m}+b^\dagger_{\ell+1,m-1}b_{\ell m}\right), \quad (4.49)\\
K_+ &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sqrt{\frac{(\ell-m+1)(\ell-m+2)}{2}}\left(a^\dagger_{\ell m}a_{\ell+1,m-1}+b^\dagger_{\ell m}b_{\ell+1,m-1}\right). \quad (4.50)
\end{aligned}
$$

4.2.4 Primary states and operators

Besides the quantum numbers associated with the conformal group, states can be characterized by their charge and by their parity. Charge is quickly dealt with. Any state of the form (4.19) is an eigenstate of the charge operator Q.

Consider now parity. At fixed charge n and spin ℓ, states divide into two *polarity* classes: polar states with parity $(-1)^\ell$ and axial states with parity $(-1)^{\ell+1}$. The two classes can schematically be written as

$$
\begin{array}{lllll}
\text{polar} & P=(-1)^\ell & \Rightarrow & \partial^{\ell+2k}\phi^{n_a}\bar\phi^{n_b}\delta^k & k\geq 0 \quad (4.51)\\
\text{axial} & P=(-1)^{\ell+1} & \Rightarrow & \partial^{\ell+2k+1}\phi^{n_a}\bar\phi^{n_b}\epsilon\delta^k & k\geq 0 \quad (4.52)
\end{array}
$$

where ∂, δ and ϵ represent respectively a spacetime derivative ∂_i, the Kronecker delta δ_{ij} and the Levi-Civita tensor ϵ_{ijk}. The δ's and the ϵ are all contracted with a pair of derivatives, while the remaining ℓ indices are symmetrized and trace-subtracted.

As the $U(1)$ charge Q commutes with the conformal group, the conformal multiplets have definite charge. On the other hand, by considering that $\partial\to-\partial$ under parity and the standard rule for adding angular momenta, one is easily convinced that the descendants of an operator with given polarity (polar or axial) can have either polarity. One can nonetheless label a conformal multiplet by the polarity of its primary state.

In this chapter, we will provide a systematic construction of all primaries whose number of derivatives is bounded by the charge n (see [117, 118, 119] for a different but less explicit procedure to construct primaries of given spin and charge). In a first time we describe this construction, before proving via a combinatorics argument that our procedure indeed generates all such primaries. This will enable us to concretely illustrate the emergence of the superfluid Fock space structure within the operator spectrum at large charge.

Construction of primaries

States of the form

$$a^\dagger_{\ell_1,m_1}\dots a^\dagger_{\ell_{n_a},m_{n_a}} b^\dagger_{j_1,m_1}\dots b^\dagger_{j_{n_b},m_{n_b}}|0\rangle, \tag{4.53}$$

can be decomposed into irreducible representations of $SO(3)$

$$\ell_1\otimes\dots\otimes\ell_{n_a}\otimes j_1\dots\otimes j_{n_b} = (\ell_1+\dots+\ell_{n_a}+j_1\dots+j_{n_b})\oplus\dots \tag{4.54}$$

Let us first consider the states with the highest total spin $\ell=\ell_1+\dots+j_{n_b}$ in the tensor product (4.54), indicating them by

$$|n;\ell,m\rangle \tag{4.55}$$

where $n=n_a-n_b$ and m are respectively the Q and J_3 eigenvalues. The highest weight state from which all the multiplet is constructed by repeatedly acting with J_- is

$$|n;\ell,\ell\rangle = a^\dagger_{\ell_1,\ell_1}\dots a^\dagger_{\ell_{n_a},\ell_{n_a}} b^\dagger_{j_1,j_1}\dots b^\dagger_{j_{n_b},j_{n_b}}|0\rangle. \tag{4.56}$$

By eqs. (4.28), (4.33) and by the discussion at the beginning of this section, the corresponding operators are polar and have the schematic form $\phi^{n_a}\bar\phi^{n_b}\partial^\ell$. In the basis (4.26), the operator corresponding to (4.56) involves only ∂_- derivatives, as is made clear by (4.32). We can now search for combinations of states of the form (4.56) that correspond to primaries.

Let us first consider states involving creation operators of only one sort, say $a^\dagger$. A first obvious example is the state of charge n with lowest dimension, which is given by

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger_{00})^n|0\rangle = \frac{(4\pi)^{n/2}}{\sqrt{n!}}\phi^n(0)|0\rangle \tag{4.57}$$

This state has spin 0 and is a primary as it is annihilated by the K_i's.

To find a spin-ℓ primary we start with the ansatz

$$|n;\ell,\ell\rangle_A^{(0)} = (a_{00}^\dagger)^{n-1} a_{\ell\ell}^\dagger |0\rangle. \tag{4.58}$$

Acting on it with K_- we get[7]

$$K_-|n;\ell,\ell\rangle_A^{(0)} = -\sqrt{\frac{2\ell(2\ell-1)}{2}}(a_{00}^\dagger)^{n-1} a_{\ell-1,\ell-1}^\dagger |0\rangle, \quad \ell \neq 1. \tag{4.60}$$

In order to cancel this contribution we modify the vector

$$|n;\ell,\ell\rangle_A^{(1)} = (a_{00}^\dagger)^{n-1} a_{\ell\ell}^\dagger |0\rangle - \sqrt{\frac{2\ell(2\ell-1)}{2}}(a_{00}^\dagger)^{n-2} a_{\ell-1,\ell-1}^\dagger a_{1,1}^\dagger |0\rangle. \tag{4.61}$$

Acting with K_- on the new state we find

$$K_-|n;\ell,\ell\rangle_A^{(1)} = \sqrt{\frac{2\ell(2\ell-1)}{2}}\sqrt{\frac{(2\ell-2)(2\ell-3)}{2}}(a_{00}^\dagger)^{n-2} a_{\ell-2,\ell-2}^\dagger a_{1,1}^\dagger |0\rangle\,. \tag{4.62}$$

Again, to cancel this contribution we add an extra term to (4.61) and we continue further until we finally arrive at an exact primary

$$|n;\ell,\ell\rangle_A = \alpha_0 \sum_{k=0}^{\ell} \gamma_{k,\ell} (a_{00}^\dagger)^{n-k-1} (a_{1,1}^\dagger)^k a_{\ell-k,\ell-k}^\dagger |0\rangle, \tag{4.63}$$

with

$$\gamma_{k,\ell} = \frac{(-1)^k}{k!}\sqrt{\frac{(2\ell)!}{2^k(2\ell-2k)!}}, \tag{4.64}$$

with the overall coefficient α_0 fixed by the normalization condition $\| \, |n;\ell,\ell\rangle_A \, \|^2 = 1$

$$\alpha_0^2 \left[\sum_{k=0}^{\ell-2} \gamma_{k,\ell}^2 (n-k-1)!k! + (\gamma_{\ell-1,\ell} + \gamma_{\ell,\ell})^2 (n-\ell)!\ell!\right] = 1. \tag{4.65}$$

As can be verified, this construction works if $1 < \ell \leq n$. Explicit constructions of these states and of the corresponding operators for $\ell = 0, 1, 2, 3$ can be found in appendix D.1. By using (4.41) one can also check that the energy of this state, or equivalently the dimension of the corresponding operator, is given by

$$\Delta_A(n,\ell) = \frac{n}{2} + \ell, \tag{4.66}$$

[7]As can be derived from (4.44), (4.50), $[K_0, a_{\ell\ell}^\dagger] = [K_+, a_{\ell\ell}^\dagger] = 0$ for all ℓ, so the state is annihilated by both K_0 and K_+. Moreover (4.48) yields

$$[K_-, a_{\ell\ell}^\dagger] = -\sqrt{\frac{(2\ell)(2\ell-1)}{2}} a_{\ell-1,\ell-1}^\dagger. \tag{4.59}$$

as expected in free theory.

Similarly we can consider states that involve one creation operator $b^\dagger$. Repeating the construction it is straightforward to construct, for $\ell \leq n+1$, a primary

$$|n;\ell,\ell\rangle_B = \beta_0 \sum_{k=0}^{\ell} \gamma_{k,\ell} (a^\dagger_{00})^{n-k+1} (a^\dagger_{1,1})^k b^\dagger_{\ell-k,\ell-k}|0\rangle, \tag{4.67}$$

with

$$\beta_0^2 \sum_{k=0}^{\ell} \gamma^2_{k,\ell}(n-k-1)!k! = 1. \tag{4.68}$$

These special cases can be combined to generate more primaries. Indeed, one can define spin ℓ multiplets of operators $\left\{\mathcal{A}^\dagger_{\ell,m}\right\}$, $\left\{\mathcal{B}^\dagger_{\ell,m}\right\}$ with $m=-\ell,\dots,\ell$ whose highest weight elements are

$$\begin{aligned}
\mathcal{A}^\dagger_{\ell,\ell} &= \sum_{k=0}^{\ell} \gamma_{k,\ell} (a^\dagger_{00})^{\ell-k-1} (a^\dagger_{1,1})^k a^\dagger_{\ell-k,\ell-k}, \qquad \ell \geq 2 &(4.69)\\
\mathcal{B}^\dagger_{\ell,\ell} &= \sum_{k=0}^{\ell} \gamma_{k,\ell} (a^\dagger_{00})^{\ell-k+1} (a^\dagger_{1,1})^k b^\dagger_{\ell-k,\ell-k}, \qquad \ell \geq 0. &(4.70)
\end{aligned}$$

$\mathcal{A}^\dagger_{\ell,m}$ and $\mathcal{B}^\dagger_{\ell,m}$ are polar primaries, because they commute with all K_i, and they have charge ℓ. Notice, that $\mathcal{A}_{0,0}$ is not defined and $\mathcal{A}_{1,m}=0$, while $\mathcal{B}^\dagger_{0,0} = a^\dagger_{00} b^\dagger_{00}$. The primary states we constructed are then given by

$$\begin{aligned}
|n;\ell,\ell\rangle_A &= \alpha_0 (a^\dagger_{00})^{n-\ell} \mathcal{A}^\dagger_{\ell,\ell}|0\rangle, &(4.71)\\
|n;\ell,\ell\rangle_B &= \beta_0 (a^\dagger_{00})^{n-\ell} \mathcal{B}^\dagger_{\ell,\ell}|0\rangle. &(4.72)
\end{aligned}$$

Since the $\mathcal{A}^\dagger_{\ell,m}$'s, $\mathcal{B}^\dagger_{\ell,m}$'s, as well as $a^\dagger_{00}$, are all primaries, any product of them is a primary as well. This lets us generate primaries of various spins and charges by acting on the vacuum with these operators

$$\left(a^\dagger_{00}\right)^{n-\sum_\alpha \ell_\alpha - \sum_\beta \tilde{\ell}_\beta} \prod_\alpha \mathcal{A}^\dagger_{\ell_\alpha,m_\alpha} \prod_\beta \mathcal{B}^\dagger_{\tilde{\ell}_\beta,\tilde{m}_\beta}|0\rangle, \tag{4.73}$$

where the number of derivatives of the corresponding operator is $P \equiv \sum_\alpha \ell_\alpha + \sum_\beta \tilde{\ell}_\beta$. Notice this state is an eigenstate of J^2 only for maximal spin states ($m_\alpha = \ell_\alpha, \tilde{m}_\beta = \tilde{\ell}_\beta$ or $m_\alpha = -\ell_\alpha, \tilde{m}_\beta = -\tilde{\ell}_\beta$). Otherwise, one must take linear combinations of these terms to build spin multiplets. By inspecting their definition, one can be convinced that $\mathcal{A}^\dagger_{\ell,\ell}$ and $\mathcal{B}^\dagger_{\ell,\ell}$ (and hence the corresponding spin multiplets) can not be written as products of $\mathcal{A}^\dagger$'s and $\mathcal{B}^\dagger$'s with lower-spin – they are in a sense "irreducible". Thus the above representation of a primary is unique.

Indeed, as it turns out, the above representation generates all the primaries with number of derivatives bounded by n. In the next section we will offer a combinatoric proof of that. Before moving to that, and to ease the counting, it is convenient to note that the dimensionality of the space generated by (4.73) is the same as that of space generated by (4.19) barring the spin 1 ladder operators $a^\dagger_{1,m}$. This can be seen by picking only the $k = 0$ terms in the series (4.69) for the $\mathcal{A}$'s and $\mathcal{B}$'s. This remark will be used in the next section to prove that (4.73) provide a complete basis for primaries.

Combinatorics: counting primaries

As a warmup, we will first consider different subclasses of operators for which we can provide explicit expressions for the number of primaries. After having done that, we will prove that the set (4.73) is indeed a complete basis for the primaries.

No $\bar{\phi}$, spin $\ell \leq n$, number of derivatives equal to ℓ
Consider the polar operators with $k = 0$ and no $\bar{\phi}$ fields in eq. (4.51). They correspond to symmetric traceless tensors with schematic form $\partial^\ell \phi^n$. Using coordinates (4.26), the highest weight elements of the corresponding $SO(3)$ multiplets have the schematic form $\partial_-^\ell \phi^n$. The counting is now straightforward: there are as many operators as there are inequivalent ways of distributing ℓ derivatives ∂_- among n fields ϕ. That is given by the number of partitions of ℓ into at most n integers, which we denote by $p(\ell, n)$. In the case $\ell \leq n$, the partition cannot contain more than n elements, and so $p(\ell, n)$ reduces to the number $p(\ell)$ of unconstrained partitions of ℓ.

For example, for $\ell = 5$ we get the following partitions.

$$5: \quad (5),\ (4,1),\ (3,2),\ (3,1,1),\ (2,2,1),\ (2,1,1,1),\ (1,1,1,1,1), \tag{4.74}$$

Thus, there are

$$p(5, n) = 7 \tag{4.75}$$

operators with spin $\ell = 5$ and charge $n \geq 5$, while for charge $n = 3$ there are only

$$p(5, 3) = 5 \tag{4.76}$$

operators in total, counted by the first five partitions in (4.74).

We can now count primary operators. Obviously, at spin ℓ, primaries will be in one to one correspondence with operators that cannot be obtained by acting with derivatives on all operators with spin $\ell - 1$. Therefore the number of primaries is given by

$$\mathrm{Prim}(\ell, n) \equiv p(\ell, n) - p(\ell - 1, n). \tag{4.77}$$

For $\ell \leq n$ this number has a simple interpretation. Namely, it corresponds to the number of partitions of ℓ, except those that can be obtained from partitions of $\ell-1$ by adding 1, in other words partitions of ℓ containing 1 should be eliminated [8]. As an example, for $\ell = 5$ and $\ell = 4$ we have respectively

$$\begin{array}{llllllll} 5: & (5) & (4,1) & (3,2) & (3,1,1) & (2,2,1) & (2,1,1,1) & (1,1,1,1,1) \\ 4: & & (4) & & (3,1) & (2,2) & (2,1,1) & (1,1,1,1). \end{array} \tag{4.78}$$

Clearly, the $\ell = 5$ primaries are counted by the partitions without 1, so that for $n \geq 5$

$$\mathrm{Prim}(5, n) = 2\,. \tag{4.79}$$

In the previous subsection we found that any string $(a^\dagger_{00})^{n-\ell}\Pi_\alpha \mathcal{A}^\dagger_{\ell_\alpha \ell_\alpha}$ forms a primary with total spin $\ell = \sum_\alpha \ell_\alpha \leq n$. Since there is no $\mathcal{A}_{1,1}$, it is clear that these primary states correspond to partitions of ℓ without 1's. Our counting argument then shows these are all the primaries of our class (polar with $k = 0$ and no $\bar{\phi}$'s).

No $\bar{\phi}$, arbitrary spin ℓ, number of derivatives equal to ℓ

For arbitrary ℓ, the number of primaries (4.77) is given by the number of partitions of ℓ with each part bigger than 1 and not larger than n, i.e. by the number of solutions of the equation

$$\sum_i \ell_i = \ell, \quad 1 < \ell_i \leq n. \tag{4.80}$$

That can be proven as follows. Every partition t can be associated with a Young tableau. For instance, the partition $t = (4,3,2)$ of 9 corresponds to

$$t = \ydiagram{4,3,2} \tag{4.81}$$

A conjugated Young tableau t^* is defined by interchanging columns and rows, meaning that for the example above $t^* = (3,3,2,1)$

$$t^* = \ydiagram{3,3,2,1} \tag{4.82}$$

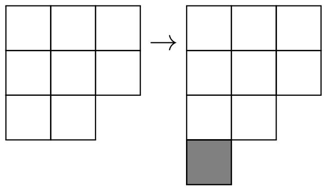
This map obviously establishes an equality between the number $p(\ell, n)$ of partitions of ℓ into at most n parts – i.e. the number of Young tableaux with at most n rows – and the number of partitions $p^*(\ell, n)$ with parts bounded by n – i.e. the number of Young

[8] That is because when acting with a derivative on an operator involving less than n derivatives, among many terms, there will always arise one involving a single derivative on ϕ.

tableaux with at most n columns. Therefore, the number of primaries can also be written as

$$\mathrm{Prim}(\ell, n) = p(\ell, n) - p(\ell - 1, n) = p^*(\ell, n) - p^*(\ell - 1, n). \tag{4.83}$$

As before, we observe that every tableau counted by $p^*(\ell - 1, n)$ can be promoted to a tableau counted by $p^*(\ell, n)$ by adding a row with just one box,



(4.84)

therefore, as claimed the number of primaries is given by the number $p^*(\ell, n, 2)$ of Young tableaux with each row bounded by $2 \leq \ell_i \leq n$ (see appendix D.2 for examples). Clearly, this is equal to the number of products of operators $\mathcal{A}^\dagger_{\ell_\alpha,\ell_\alpha}$ defined in (4.69) such that $2 \leq \ell_\alpha \leq n$ and $\sum_\alpha \ell_\alpha = \ell$. Notice that, while the counting is still valid, the construction does not work for $\ell > n$, since the $\mathcal{A}^\dagger_{\ell,\ell}$ operators have charge equal to spin, and thus cannot be used to generate operators with spin higher than the charge.

ϕ and $\bar{\phi}$, arbitrary spin ℓ, number of derivatives equal to ℓ

Consider now polar operators with $k = 0$ but involving $\bar{\phi}$ fields. In this case the highest weight elements have the schematic form $\partial_-^\ell \phi^{n_a} \bar{\phi}^{n_b}$. To count the number of such operators, one can first distribute the derivatives as $\partial_-^{\ell-\ell'} \phi^{n_a} \times \partial_-^{\ell'} \bar{\phi}^{n_b}$, and compute the total number of operators as

$$\sum_{\ell'=0}^{\ell} p(\ell - \ell', n_a) p(\ell', n_b). \tag{4.85}$$

This implies the number of primaries is given by

$$\begin{aligned} \mathrm{Prim}(\ell, n_a, n_b) &= \sum_{\ell'=0}^{\ell} p(\ell - \ell', n_a) p(\ell', n_b) - \sum_{\ell'=0}^{\ell-1} p(\ell - 1 - \ell', n_a) p(\ell', n_b) \\ &= \sum_{\ell'=0}^{\ell-1} p^*(\ell - \ell', n_a, 2) p^*(\ell', n_b), \end{aligned} \tag{4.86}$$

where we have used the equalities deduced above from Young tableaux. This number is easy to interpret as the number of products of the form

$$(a^\dagger_{00})^{n-\ell} \prod_\alpha \mathcal{A}^\dagger_{\ell_\alpha,\ell_\alpha} \prod_\beta \mathcal{B}^\dagger_{\tilde{\ell}_\beta,\tilde{\ell}_\beta} \tag{4.87}$$

with $2 \leq \ell_\alpha \leq n$, $0 \leq \tilde{\ell}_\beta \leq n$ and $\sum_\alpha \ell_\alpha + \sum_\beta \tilde{\ell}_\beta = \ell$. Thus, these products of operators are all the highest-weight polar primaries with $k = 0$ and $\ell \leq n$. Again, the counting

(4.86) is valid for $\ell > n$, but the explicit construction does not apply in this regime.

All operators with number of derivatives bounded by n

We finally consider operators made of both ϕ and $\bar{\phi}$ fields and a number of derivatives $P \leq n$, with eventually contracted indices. We will not provide an explicit formula for the number of primaries in the general case, but will show that primaries are in one-to-one correspondence with operators of the form (4.73). The following argument is valid in any dimension.

A basis of the linear space of charge-n operators is obtained by considering the set of monomials of the schematic form $\partial_{\mu_1} \dots \partial_{\mu_P} \phi^{n_a} \bar{\phi}^{n_b}$ with $n_a - n_b = n$ and with the P derivatives distributed on the fields in all possible ways (removing the operators which are made redundant by the equations of motion $\partial^2 \phi = \partial^2 \bar{\phi} = 0$). We focus on a finite-dimensional subspace $H_{n_a, n_b, P}$ of fixed n_a, n_b and P. The counting argument that we will provide works for each of those subspaces individually, and thus extends to the full space of operators. For each subspace, we construct a different basis, in which part of the elements are manifestly descendant states. The remaining elements of the basis span a subspace of same dimensionality as the subspace of explicitly known primary operators. This means that we have successfully identified complete basis of primaries and descendants.

The construction is the following. Monomials in the basis can be organized by factoring out all powers of ϕ carrying either 0 or 1 derivative

$$\mathbb{B}_{n_a, n_b, P} = \left\{ \phi^q (\partial_{\mu_1} \phi)(\partial_{\mu_2} \phi) \cdots (\partial_{\mu_p} \phi)\, O^{(P-p)}_{n_a - p - q, n_b} \ , p \leq P \right\}, \tag{4.88}$$

with $O^{(p)}_{n,m}$ any monomial involving n ϕ's, m $\bar{\phi}$'s and p derivatives, such that each ϕ is derived at least twice. Notice that all ∂_{μ_i} factors commute with each other, hence without loss of generality we can assume they are ordered $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_p$. For any $p \leq P$ we will also consider the sub-basis of operators where p ϕ's have a single derivative

$$\mathbb{B}^p_{n_a, n_b, P} = \left\{ \phi^q (\partial_{\mu_1} \phi)(\partial_{\mu_2} \phi) \cdots (\partial_{\mu_p} \phi)\, O^{(P-p)}_{n_a - p - q, n_b} \ \right\}. \tag{4.89}$$

Now, for $p \geq 1$, we can rewrite the elements of $\mathbb{B}^p_{n_a, n_b, P}$ as (for simplicity we write O

instead of $O^{(P-p)}_{n_a-p-q,n_b}$)

$$
\begin{aligned}
&\phi^q(\partial_{\mu_1}\phi)(\partial_{\mu_2}\phi)\cdots(\partial_{\mu_p}\phi)\,O \\
&\quad = \frac{1}{q+1}\Bigg(\partial_{\mu_1}\Big(\phi^{q+1}(\partial_{\mu_2}\phi)\cdots(\partial_{\mu_p}\phi)\,O\Big) \\
&\qquad\qquad - \sum_{k=2}^{p}\phi^{q+1}(\partial_{\mu_2}\phi)\cdots(\partial_{\mu_{k-1}}\phi)(\partial_{\mu_{k+1}}\phi)\cdots(\partial_{\mu_p}\phi)\partial_{\mu_1}\partial_{\mu_k}\phi\,O \\
&\qquad\qquad - \phi^{q+1}(\partial_{\mu_2}\phi)\cdots(\partial_{\mu_p}\phi)\partial_{\mu_1}O\Bigg).
\end{aligned}
\tag{4.90}
$$

The term in the first line of the right-hand side is obviously a descendant operator, while the two other lines contain monomials belonging to $\mathbb{B}^{p-2}_{n_a,n_b,P}$ and $\mathbb{B}^{p-1}_{n_a,n_b,P}$. This process can be repeated, rewriting the operators of the two last lines in the same way, as linear combinations of descendants and members of the lower sub-bases. The process can be iterated until the right hand side is written as a linear combination of descendants and monomials in $\mathbb{B}^{0}_{n_a,n_b,P}$. The latter involve no single-derivative ϕ fields and cannot be further rewritten. Our result implies that the subspace generated by $\mathbb{B}^{p}_{n_a,n_b,P}$ has the same primary content as the subspace generated by $\mathbb{B}^{0}_{n_a,n_b,P}$. Indeed, as this holds for any p, the very space generated by $\mathbb{B}_{n_a,n_b,P}$ has the same primary content as the subspace generated by $\mathbb{B}^{0}_{n_a,n_b,P}$. We therefore conclude that the subspace of primaries within $H_{n_a,n_b,P}$ is $\leq$ than the number of elements in $\mathbb{B}^{0}_{n_a,n_b,P}$.

Our proof can now be completed by comparing the elements in $\mathbb{B}^{0}_{n_a,n_b,P}$ to the linearly independent primary states provided in (4.73). The latter, as we already remarked, are in a one-to-one correspondence with the set (4.19), barring states involving $a^\dagger_{\ell=1,m}$. By the operator-state correspondence the traceless symmetric derivatives $\partial_{\mu_1}\ldots\partial_{\mu_r}\phi$ and $\partial_{\mu_1}\ldots\partial_{\mu_s}\bar\phi$ match respectively $a^\dagger_{r,m_r}$ and $b^\dagger_{s,m_s}$. It is then manifest that the elements in $\mathbb{B}^{0}_{n_a,n_b,P}$ and in (4.19) are in a one-to-one correspondence. In particular the exclusion of $\partial_i\phi$ factors in $\mathbb{B}^{0}_{n_a,n_b,P}$ crucially matches the exclusion of $a^\dagger_{1,m}$ in (4.19), which is mandated in turn to match the building blocks (4.73). As the cardinality of the basis $\mathbb{B}^{0}_{n_a,n_b,P}$ sets an upper bound to then dimension of the sub-space of primaries, it must be that the states (4.73) with the same n_a, n_b and P are a complete basis for the corresponding space of primaries.

4.3 Fock space of superfluid excitations

Free field theory can be successfully studied around both the trivial $\phi = 0$ and the non-trivial superfluid (2.91) saddles. The latter is the basis of the semiclassical description, which applies for sufficiently large charge, and was presented in detail in section 2.4. The development includes the definition of the fluctuations around the non-trivial saddle

(2.97) and their quantization (2.116), with spectrum (2.118). These formulas were derived in the interacting theory, but they can be safely taken in the free theory limit of $\lambda = 0$ and $\lambda n = 0$ and in $d = 3$ dimensions. In particular, the spectrum takes a simple expression in this limit (2.127).

Excitations around the charge n ground state $|n\rangle$ are obtained by acting with the neutral creation operators $A^\dagger_{\ell\vec{m}}$ and $B^\dagger_{\ell\vec{m}}$

$$(A^\dagger_{\ell_1 m_1})^{n_1^A} \dots (B^\dagger_{j_1 k_1})^{n_1^B} \dots |n\rangle. \tag{4.91}$$

These states all have charge n, so that their Fock space is the charge-n subspace of the full Hilbert space. As explained below equation (2.120), the zero-mode described by the canonical pair $\hat{\pi}, p_\pi$ does not belong to that Fock space. Instead $e^{i\frac{\hat{\pi} Y_{00}}{f}}$ and $e^{-i\frac{\hat{\pi} Y_{00}}{f}}$ respectively raise and decrease the charge by one unit, thus mapping to the corresponding fixed-charge Fock spaces. The energy of the states (4.91), equivalently the dimensions of corresponding operators, are given by (4.2) using the free-theory spectrum (2.127). This indicates that states involving at least one $A^\dagger_{1\vec{m}}$ are descendants at leading order, since these creation operators increment the dimension exactly by 1 at that order. This is also consistent with the counting established in section 4.2.4.

The goal of the present section is to find explicitly the mapping between these two Fock spaces, the vacuum Fock space (4.19) and the hydrodynamic one (4.91). Using the results of the previous section, that allows to associate operators to excitations around the superfluid state. Finally we will be able to identify primaries among them, and verify they are the states that do not contain $A^\dagger_{1,m}$ creation operators.

4.3.1 Relation between different Fock spaces in free theory

The map between the two spaces corresponds to a canonical transformation resulting from equations (4.15), (2.80), (2.97) and (2.102)

$$(\hat{\phi}, p_{\hat{\phi}}), \quad (\hat{\bar{\phi}}, p_{\hat{\bar{\phi}}}) \qquad \Rightarrow \qquad (r, P), \quad (\pi, \Pi). \tag{4.92}$$

We use the decomposition of the fields in harmonic components (2.103). In three dimensions $m \in \{-\ell, -\ell+1, \dots, \ell\}$ is a simple index. The fields clearly satisfy

$$\begin{aligned} r_{\ell m}(\tau) &= (-1)^m \Big(r_{\ell,-m}(-\tau)\Big)^\dagger, \quad \pi_{\ell m}(\tau) = (-1)^m \Big(\pi_{\ell,-m}(-\tau)\Big)^\dagger, \\ P_{\ell m}(\tau) &= (-1)^m \Big(P_{\ell,-m}(-\tau)\Big)^\dagger, \quad \Pi_{\ell m}(\tau) = (-1)^m \Big(\Pi_{\ell,-m}(-\tau)\Big)^\dagger. \end{aligned} \tag{4.93}$$

These components are written in terms of the zero mode and creation and annihilation operators yielding

$$\begin{aligned}
r_{00}(\tau) &= p_\pi + \frac{1}{\sqrt{2}}\left(B_{00}(\tau) + B_{00}^\dagger(-\tau)\right), \\
\pi_{00}(\tau) &= \hat{\pi} + \frac{i}{\sqrt{2}}\left(B_{00}^\dagger(-\tau) - B_{00}(\tau)\right), \\
P_{00}(\tau) &= \frac{i}{\sqrt{2}}\left(B_{00}^\dagger(-\tau) - B_{00}(\tau)\right), \\
\Pi_{00}(\tau) &= p_\pi,
\end{aligned} \tag{4.94}$$

and (for $\ell > 0$)

$$\begin{aligned}
r_{\ell m}(\tau) &= \frac{1}{2\sqrt{\omega_0(\ell)}}\left[A_{\ell m}(\tau) + (-1)^m A_{\ell,-m}^\dagger(-\tau) + B_{\ell m}(\tau) + (-1)^m B_{\ell,-m}^\dagger(-\tau)\right], \\
\pi_{\ell m}(\tau) &= \frac{i}{2\sqrt{\omega_0(\ell)}}\left[A_{\ell m}(\tau) - (-1)^m A_{\ell,-m}^\dagger(-\tau) - B_{\ell m}(\tau) + (-1)^m B_{\ell,-m}^\dagger(-\tau)\right], \\
P_{\ell m}(\tau) &= \frac{i}{2\sqrt{\omega_0(\ell)}}\left[-(-1)^m \ell A_{\ell,-m}(\tau) + \ell A_{\ell,m}^\dagger(-\tau) - (-1)^m(\ell+1)B_{\ell,-m}(\tau) + (\ell+1)B_{\ell,m}^\dagger(-\tau)\right], \\
\Pi_{\ell m}(\tau) &= \frac{1}{2\sqrt{\omega_0(\ell)}}\left[(-1)^m(\ell+1)A_{\ell,-m}(\tau) + (\ell+1)A_{\ell,m}^\dagger(-\tau) - (-1)^m \ell B_{\ell,-m}(\tau) - \ell B_{\ell,m}^\dagger(-\tau)\right].
\end{aligned} \tag{4.95}$$

Here we do not consider fields evolved only with the quadratic Hamiltonian for fluctuations around the saddle (2.100) but take into account the exact solutions of the full action (2.99). Thus operators $A_{\ell m}(\tau), B_{\ell m}(\tau)$ have complicated time dependence, not just a simple phase rotation. However, they satisfy the commutation relations (2.109), (2.117) and hermiticity (4.93) at all τ. At $\tau = 0$ they coincide with the τ-independent creation-annihilation operators introduced in section 2.4.3 for quadratic fluctuations. Getting the full time dependence is a consequence of the fact that we are mapping from the vacuum quantization, where the time dependence was simple (4.12).

Our goal is to express these operators in terms of the ladder operators of vacuum fluctuations. The form of (2.80) makes the mapping non-linear, which makes it difficult to find a closed form solution. However at large n the solution can be reliably expressed as a systematic expansion in inverse powers of n.

We will be studying fluctuations around the lowest energy state with charge n, for which $\langle a_{00}^\dagger a_{00}\rangle \sim n$. The large charge expansion can then be organized by assigning to operators a scaling with n

$$a_{00} \sim O(\sqrt{n}), \qquad a_{\ell\neq 0,m} \sim b_{\ell m} \sim O(1). \tag{4.96}$$

For instance, by singling out $a_{00}^\dagger a_{00}$ in the expression for Q (4.20) we can write

$$a_{00}^\dagger a_{00} = n\left\{1 + \frac{1}{n}\left[Q - n + b_{00}^\dagger b_{00} - \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\left(a_{\ell m}^\dagger a_{\ell m} - b_{\ell m}^\dagger b_{\ell m}\right)\right]\right\}, \tag{4.97}$$

where the term in square brackets represents an $O(n^0)$ perturbation. In what follows we treat the fields as classical variables, disregarding issues of ordering. Expressions for quantum operators can be restored, in principle, by finding an appropriate ordering such that the commutation relations are satisfied.

Our goal can be achieved through the following steps:

1. Remembering that for free theory in $d = 3$ we have

$$\mu_3(0,3) = \frac{1}{2}, \quad f = \sqrt{\frac{n}{2\pi}}, \quad \omega_0(\ell) = \ell + \frac{1}{2}, \tag{4.98}$$

   and combining equations (4.12), (2.80), (2.91) and (2.97) we can write

$$\frac{f+r}{\sqrt{2}} e^{\frac{i\pi}{f}} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{\sqrt{2\omega_0(\ell)}} \left(a_{\ell m}^\dagger e^{\ell\tau} Y_{\ell m}^*(\vec{n}) + b_{\ell m} e^{-(\ell+1)\tau} Y_{\ell m}(\vec{n}) \right) \equiv h(\tau, \vec{n}) \,. \tag{4.99}$$

   It is also convenient to write

$$r(\tau, \vec{n}) = \sqrt{2 h(\tau, \vec{n}) h(-\tau, \vec{n})^\dagger} - f, \tag{4.100}$$

$$e^{\frac{i\pi(\tau,\vec{n})}{f}} = \frac{h(\tau, \vec{n})}{\sqrt{h(\tau, \vec{n}) h(-\tau, \vec{n})^\dagger}}. \tag{4.101}$$

   Notice that, here and later, we formally treat $a_{0,0}$ and $a_{0,0}^\dagger$ as invertible as we are working in a subspace with large charge. For example, we can write

$$\frac{1}{\sqrt{h(\tau,\vec{n}) h(-\tau,\vec{n})^\dagger}} = \frac{1}{\sqrt{\frac{n}{2\pi} + s(\tau,\vec{n})}} \approx \sqrt{\frac{2\pi}{n}} - \sqrt{\frac{2\pi^3}{n^3}} s(\tau,\vec{n}) + 3\sqrt{\frac{\pi^5}{2n^5}} (s(\tau,\vec{n}))^2 + \dots \tag{4.102}$$

   where we used $a_{00}^\dagger a_{00} = n + \dots$ and parametrized all subleading effects by $s(\tau, \vec{n})$.

2. Using the orthonormality of spherical harmonics (2.107), we extract the harmonic components $r_{\ell m}, \pi_{\ell m}, P_{\ell m}, \Pi_{\ell m}$ from (2.103).

3. We finally solve (4.94) and (4.95), for $A_{\ell m}, B_{\ell m}$, $\hat{\pi}$ and p_π.

Leading order

At leading order in the n^{-1} expansion, we get

$$p_\pi = 0, \quad \exp\left[i \frac{\hat{\pi}}{\sqrt{2n}} \right] = \frac{a_{00}^\dagger}{\sqrt{n}}, \quad B_{\ell m}(\tau) = \frac{a_{00} b_{\ell m}}{\sqrt{n}} e^{-(\ell+1)\tau}, \quad A_{\ell m}(\tau) = \frac{a_{00}^\dagger a_{\ell m}}{\sqrt{n}} e^{-\ell\tau}. \tag{4.103}$$

As explained, the zero-mode $\hat{\pi}$ is kept in the exponential. This also ensures that the expressions are polynomial (monomial at this order) in the vacuum ladder operators. One further justification of the exponential notation was given at the end of the computation of the propagator in appendix B.1.

The commutation relations have the form

$$\left[A_{\ell m}, A^\dagger_{\ell' m'}\right] = \frac{1}{n}\left(a^\dagger_{00} a_{00}\,\delta_{\ell\ell'}\delta_{mm'} - a_{\ell m} a^\dagger_{\ell' m'}\right) = \delta_{\ell\ell'}\delta_{mm'} + O\left(n^{-1}\right), \tag{4.104}$$

$$\left[B_{\ell m}, B^\dagger_{\ell' m'}\right] = \frac{1}{n}\left(a_{00} a^\dagger_{00}\,\delta_{\ell\ell'}\delta_{mm'} + b_{\ell m} b^\dagger_{\ell' m'}\right) = \delta_{\ell\ell'}\delta_{mm'} + O\left(n^{-1}\right), \tag{4.105}$$

which are canonical at the required accuracy (see (2.117)).

Next to leading order

We find that $\exp\left[i\frac{\hat{\pi}}{\sqrt{2n}}\right]$ is still given by (4.103) while p_π is given by its exact result (2.121). For $A_{\ell m}$ we find

$$\begin{aligned}
A_{\ell m} = \frac{a^\dagger_{00} a_{\ell m}}{\sqrt{n}} + \frac{1}{4(1+2\ell)n^{3/2}}\Big(&(1+4\ell)\left(n b_{00} - b^\dagger_{00}(a^\dagger_{00})^2\right)a_{\ell m} - 2n b^\dagger_{00} b_{\ell m} \\
&+ (-1)^m\left((-1+2\ell)n b^\dagger_{00} + (1+2\ell) b_{00} a^2_{00}\right)a^\dagger_{\ell,-m} \\
&- (-1)^m\left(2(1+\ell)n b_{00} + 2\ell b^\dagger_{00}(a^\dagger_{00})^2\right)b^\dagger_{\ell,-m}\Big) \\
+ \sum_{\substack{\ell_1,\ell_2>0 \\ \text{all } m_1,m_2}} \frac{(-1)^m\sqrt{\pi}C^{\ell,\ell_1,\ell_2}_{-m,m_1,m_2}}{8\sqrt{2\omega_0(\ell)\omega_0(\ell_1)\omega_0(\ell_2)}n^{3/2}}\Big(&-(2+3\ell+\ell_1+\ell_2)(a^\dagger_{00})^2 a_{\ell_1,m_1}a_{\ell_2,m_2} \\
&+ 2(1+\ell-\ell_1+3\ell_2)n b_{\ell_1,m_1}a_{\ell_2,m_2} \\
&+ (\ell-\ell_1-\ell_2)a^2_{00} b_{\ell_1,m_1} b_{\ell_2,m_2} \\
&+ 2(-1)^{m_2}(2+\ell+3\ell_1+\ell_2)n a_{\ell_1,m_1} a^\dagger_{\ell_2,-m_2} \\
&- 2(-1)^{m_1}(1+3\ell-\ell_1+\ell_2)(a^\dagger_{00})^2 b^\dagger_{\ell_1,-m_1} a_{\ell_2,m_2} \\
&+ 2(-1)^{m_2}(1+\ell-\ell_1+\ell_2)a^2_{00} b_{\ell_1,m_1} a^\dagger_{\ell_2,-m_2} \\
&- 2(-1)^{m_2}(2-\ell+\ell_1+3\ell_2)n b_{\ell_1,m_1} b^\dagger_{\ell_2,-m_2} \\
&+ (-1)^{m_1+m_2}(2+\ell+\ell_1+\ell_2)a^2_{00} a^\dagger_{\ell_1,-m_1} a^\dagger_{\ell_2,-m_2} \\
&- 2(-1)^{m_1+m_2}(1-\ell+3\ell_1-\ell_2)n b^\dagger_{\ell_1,-m_1} a^\dagger_{\ell_2,-m_2} \\
&- (-1)^{m_1+m_2}(3\ell-\ell_1-\ell_2)(a^\dagger_{00})^2 b^\dagger_{\ell_1,-m_1} b^\dagger_{\ell_2,-m_2}\Big).
\end{aligned} \tag{4.106}$$

For $B_{\ell,m}$ we have to treat separately the cases $\ell = 0$:

$$
\begin{aligned}
B_{00} &= \frac{a_{00}b_{00}}{\sqrt{n}} + \frac{6nb_{00}b_{00}^{\dagger} - 3a_{00}^{2}b_{00}^{2} + (a_{00}^{\dagger})^{2}(b_{00}^{\dagger})^{2}}{8n^{3/2}} \\
&\quad + \sum_{\substack{\ell>0 \\ \text{all } m}} \frac{1}{8(1+2\ell)n^{3/2}} \Big((-1)^{m}4n(1+\ell)a_{\ell m}b_{\ell,-m} - (-1)^{m}(3+2\ell)a_{00}^{2}b_{\ell m}b_{\ell,-m} \\
&\qquad - (-1)^{m}(1+2\ell)(a_{00}^{\dagger})^{2}a_{\ell m}a_{\ell,-m} - (1+4\ell)2na_{\ell m}a_{\ell,m}^{\dagger} \\
&\qquad - 4a_{00}^{2}b_{\ell m}a_{\ell m}^{\dagger} + (-1)^{m}(-1+2\ell)a_{00}^{2}a_{\ell m}^{\dagger}a_{\ell,-m}^{\dagger} \\
&\qquad - (-1)^{m}4n\ell a_{\ell m}^{\dagger}b_{\ell,-m}^{\dagger} + 2n(3+4\ell)b_{\ell m}b_{\ell m}^{\dagger} \\
&\qquad + (-1)^{m}(1+2\ell)(a_{00}^{\dagger})^{2}b_{\ell m}^{\dagger}b_{\ell,-m}^{\dagger} \Big)
\end{aligned} \tag{4.107}
$$

and $\ell > 0$:

$$
\begin{aligned}
B_{\ell m} = \frac{a_{00}b_{\ell m}}{\sqrt{n}} &+ \frac{1}{4(1+2\ell)n^{3/2}} \Big((3+4\ell)\Big(nb_{00}^{\dagger} - b_{00}a_{00}^{2}\Big)b_{\ell m} + 2nb_{00}a_{\ell m} \\
&\qquad + (-1)^{m}\Big((3+2\ell)nb_{00} + (1+2\ell)b_{00}^{\dagger}(a_{00}^{\dagger})^{2}\Big)b_{\ell,-m}^{\dagger} \\
&\qquad - (-1)^{m}\Big(2\ell nb_{00}^{\dagger} + 2(1+\ell)b_{00}a_{00}^{2}\Big)a_{\ell,-m}^{\dagger} \Big) \\
+ \sum_{\substack{\ell_1,\ell_2>0 \\ \text{all } m_1,m_2}} & \frac{(-1)^{m}\sqrt{\pi}C_{-m,m_1,m_2}^{\ell,\ell_1,\ell_2}}{8\sqrt{2\omega_0(\ell)\omega_0(\ell_1)\omega_0(\ell_2)}n^{3/2}} \Big(-(3+3\ell+\ell_1+\ell_2)a_{00}^{2}b_{\ell_1,m_1}b_{\ell_2,m_2} \\
&\qquad + 2(2+\ell+3\ell_1-\ell_2)nb_{\ell_1,m_1}a_{\ell_2,m_2} \\
&\qquad + 2(-1)^{m_2}(3+\ell+3\ell_1+\ell_2)nb_{\ell_1,m_1}b_{\ell_2,-m_2}^{\dagger} \\
&\qquad - (1-\ell+\ell_1+\ell_2)(a_{00}^{\dagger})^{2}a_{\ell_1,m_1}a_{\ell_2,m_2} \\
&\qquad + 2(-1)^{m_1}(\ell+\ell_1-\ell_2)(a_{00}^{\dagger})^{2}b_{\ell_1,-m_1}^{\dagger}a_{\ell_2,m_2} \\
&\qquad + (-1)^{m_1+m_2}(1+\ell+\ell_1+\ell_2)(a_{00}^{\dagger})^{2}b_{\ell_1,-m_1}^{\dagger}b_{\ell_2,-m_2}^{\dagger} \\
&\qquad - 2(-1)^{m_2}(2+3\ell+\ell_1-\ell_2)a_{00}^{2}b_{\ell_1,m_1}a_{\ell_2,-m_2}^{\dagger} \\
&\qquad - 2(-1)^{m_2}(1-\ell+\ell_1+3\ell_2)na_{\ell_1,m_1}a_{\ell_2,-m_2}^{\dagger} \\
&\qquad + 2(-1)^{m_1+m_2}(\ell+\ell_1-3\ell_2)nb_{\ell_1,-m_1}^{\dagger}a_{\ell_2,-m_2}^{\dagger} \\
&\qquad - (-1)^{m_1+m_2}(1+3\ell-\ell_1-\ell_2)a_{00}^{2}a_{\ell_1,-m_1}^{\dagger}a_{\ell_2,-m_2}^{\dagger} \Big) .
\end{aligned} \tag{4.108}
$$

Here we introduced the Gaunt coefficients

$$
\begin{aligned}
C_{m_1 m_2 m_3}^{\ell_1\ \ell_2\ \ell_3} &= \int Y_{\ell_1 m_1}Y_{\ell_2 m_2}Y_{\ell_3 m_3}d\Omega_2 \\
&= \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} ,
\end{aligned} \tag{4.109}
$$

given in terms of Wigner $3j$ symbols. These coefficients vanish unless the spins satisfy the triangle inequality

$$|\ell_1 - \ell_2| \leq \ell_3 \leq \ell_1 + \ell_2, \tag{4.110}$$

meaning each spin has to be in the tensor product of the other two. Moreover Gaunt coefficients vanish unless $m_1 + m_2 + m_3 = 0$ and $\ell_1 + \ell_2 + \ell_3$ is even.

Some remarks on (4.106 – 4.108) are in order. First, notice that the NLO corrections have relative size $n^{-1/2}$. Higher orders behave similarly, resulting in an expansion in powers of $n^{-1/2}$. However, when computing observables, the NLO terms do not interfere with the leading order terms, resulting in an expansion in powers of $1/n$, as expected in the semiclassical framework. This will be exemplified in section 4.4.2.

When considering even higher orders, $A_{\ell m}$ will contain sums over $4, 6, \ldots$ spins with coefficients that, like (4.109), are integrals of products of respectively $5, 7, \ldots$ spherical harmonics. As these coefficients go like powers of ℓ one would expect the parameter controlling the convergence of the expansion to go like $\frac{\ell^\kappa}{n}$ for some κ. We will discuss this in detail in section 4.4.

Finally, notice that some of the NLO terms don't annihilate the state $|n\rangle$, so that $A_{\ell m}|n\rangle \neq 0$. The reason is that $|n\rangle$ is the lowest energy state of charge n for the full hamiltonian (the one associated with (2.99)), while $A_{\ell m}$ and $B_{\ell m}$ are the ladder operators for the quadratic hamiltonian (associated with (2.100)). The vacuum $|\Omega\rangle$, which is annihilated by $A_{\ell m}$ and $B_{\ell m}$, coincides with $|n\rangle$ only at leading order, hence our result.

4.3.2 Mapping superfluid excitations to operators

With the tools presented in the previous sections, we are now ready to identify operators and map them to superfluid excitations. The latter, as defined in (4.91), can be expressed as a power series in $n^{-1/2}$ of polynomials of $a_{\ell m}, a^\dagger_{\ell m}, b_{\ell m}, b^\dagger_{\ell m}$ acting on the free Fock vacuum $|0\rangle$. These, by the operator state correspondence, can in turn be written in terms of operators involving $\bar{\phi}, \phi$ and their derivatives.

To identify primary states, we must express the special conformal generators in terms of $A_{\ell m}, B_{\ell m}$ and $A^\dagger_{\ell m}, B^\dagger_{\ell m}$. This is done by inverting (4.106) and the other formulae relating ladder operators in the two frames, plugging the result in (4.44), (4.48), and (4.50). For instance, at leading order, using (4.103), we get

$$K_0 = \sqrt{n} A_{1,0}, \quad K_- = -\sqrt{n} A_{1,-1}, \quad K_+ = \sqrt{n} A_{1,1}. \tag{4.111}$$

Thus, as was already discussed, at leading order only strings of creation operators not containing $A^\dagger_{1,m}$ are primaries. There is a clear parallel with the conclusion of section 4.2.4. This is due to the fact that, at leading order, states generated by creation operators $A^\dagger_{\ell m}, B^\dagger_{\ell m}$ correspond to the states generated by $\mathcal{A}^\dagger_{\ell,m}, \mathcal{B}^\dagger_{\ell,m}$.

As a result, due to the following identities

$$A^{\dagger}_{\ell m}|n\rangle = \frac{a_{00}a^{\dagger}_{\ell m}}{\sqrt{n}}\frac{(a^{\dagger}_{00})^n}{\sqrt{n!}}|0\rangle = \frac{a^{\dagger}_{\ell m}(a^{\dagger}_{00})^{n-1}}{\sqrt{(n-1)!}}|0\rangle = \frac{(4\pi)^{\frac{n-1}{2}}}{\sqrt{(n-1)!}}\mathcal{Y}^{\mu_1\ldots\mu_\ell}_{\ell m}\phi^{n-1}\partial_{\mu_1}\cdots\partial_{\mu_\ell}\phi|0\rangle, \tag{4.112}$$

the state $A^{\dagger}_{\ell m}|n\rangle$ corresponds at leading order to an operator with ℓ derivatives all acting on the same field

$$\phi^{n-1}\partial_{\mu_1\ldots\mu_\ell}\phi. \tag{4.113}$$

4.4 How large is “large spin” ?

Quantization around the saddle offers a systematic computation of observables for states with charge n as a power series in n^{-1}. Clearly, as $n\to\infty$ the procedure works for states with finite spin ℓ, for the ground state $|n\rangle$ in particular. In this section we will study the convergence of the expansion when both ℓ and n become large.

4.4.1 Matrix elements for excited states

On general grounds we expect the expansion to be controlled by the ratio ℓ^κ/n for some κ. One way to find out what κ is, would be to perform NLO computations around the non-trivial saddle. However, we'll make use of the fact that we know a class of primary states in free theory in exact form and not just as an expansion in inverse powers of the charge. That will give us full control of the computation, allowing to successfully trace any transition between different regimes (see section 4.4.3).

Intuitively we expect the radial component of ϕ to be a good parameter to control the validity of the semiclassical approximation. The smallness of the size of its quantum fluctuation relative to its expectation value is a necessary condition for the semi-classicality of a state [9]. Fluctuations comparable to the expectation value, and thus consistent with the vanishing of ϕ (at least somewhere), signal the breakdown of the semiclassical approximantion.

We will thus study the large n behavior of the following matrix elements

$$\Phi(\theta;\ell,n,p) = {}_A\langle n;\ell,\ell| : \partial^p_\tau\hat{\bar{\phi}}(\tau,\vec{n})\partial^p_\tau\hat{\phi}(\tau,\vec{n}) : |n;\ell,\ell\rangle_A. \tag{4.114}$$

for arbitrary integer p, where $|n,\ell,\ell\rangle_A$ is the primary state found in (4.63).

[9] For an illustrative example based on the spinning top see [75].

Rewriting the fields in terms of ladder operators (4.12) and (4.13) yields[10]

$$\Phi(\tau;\ell,n,p)=\alpha_0^2\sum_{k,k'=0}^{\ell}\langle\psi_k|\sum_{\substack{\ell',m'\\ \ell'',m''}}(-1)^p\omega_{\ell'}^p\omega_{\ell''}^p a^\dagger_{\ell',m'}a_{\ell'',m''}Y^*_{\ell',m'}Y_{\ell'',m''}\frac{e^{(\omega_{\ell'}-\omega_{\ell''})\tau}}{\sqrt{4\omega_{\ell'}\omega_{\ell''}}}|\psi_{k'}\rangle, \tag{4.115}$$

where we introduced the following notation

$$|\psi_k\rangle=\gamma_{k,\ell}(a^\dagger_{00})^{n-k-1}(a^\dagger_{1,1})^k a^\dagger_{\ell-k,\ell-k}|0\rangle. \tag{4.116}$$

Using that for $k,k'\neq\ell-1,\ell$ we have

$$\begin{aligned}&\langle 0|a_{00}^{n-k-1}a_{11}^k a_{\ell-k,\ell-k}a^\dagger_{\ell',m'}a_{\ell'',m''}(a^\dagger_{00})^{n-k'-1}(a^\dagger_{11})^{k'}a^\dagger_{\ell-k',\ell-k'}|0\rangle\\ &=(n-k-1)!k!\left[(n-k-1)\delta_{\ell'0}+k\delta_{\ell'1}+\delta_{\ell',\ell-k}\right]\delta_{\ell'\ell''}\delta_{\ell'm'}\delta_{\ell'm''}\delta_{kk'},\end{aligned} \tag{4.117}$$

and neglecting the terms with $k,k'=\ell-1,\ell$, which are subleading, we get

$$\Phi(\tau;\ell,n,p)=(-1)^p\alpha_0^2\sum_{k=0}^{\ell-2}\frac{(2\ell)!(n-k-1)!}{2^{k+1}k!(2\ell-2k)!}\left[(n-k-1)|Y_{00}|^2\omega_0^{2p-1}+k|Y_{11}|^2\omega_1^{2p-1}+|Y_{\ell-k,\ell-k}|^2\omega_{\ell-k}^{2p-1}\right]. \tag{4.118}$$

We then use

$$|Y_{\ell\ell}(\varphi,\theta)|=\frac{1}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\sin^\ell\theta\,, \tag{4.119}$$

which means $|Y_{\ell-k,\ell-k}|^2$ is maximal at $\theta=\pi/2$. Approximating factorials by Stirling's formula, we finally find

$$\begin{aligned}\Phi(\pi/2;\ell,n,p)=\frac{n}{4^{p+1}\pi}\Bigg[&Q_0(\ell,p)+\frac{Q_1(\ell,p)}{n}+\frac{Q_2(\ell,p)}{n^2}+\dots\\ &+\frac{\ell^\xi}{n}\left(P_0(\ell,p)+\frac{P_1(\ell,p)}{n}+\frac{P_2(\ell,p)}{n^2}+\dots\right)\Bigg],\end{aligned} \tag{4.120}$$

where $P_k(\ell)$ and $Q_k(\ell)$ are n-independent functions which at large ℓ scale as ℓ^k, and $\xi=2p-\frac{1}{2}$.

It can be concluded that for the case at hand $\kappa=1$. In other words, the semiclassical expansion can be trusted as long as $\ell\ll n$. We expect that for a wide class of observables, even for theories with interaction, computations around the non-trivial saddle can be organized in a systematic series in powers of ℓ/n. We now examine another instance.

[10] To lighten the notation, in this chapter we denote $\omega_\ell\equiv\omega_0(\ell)$.

4.4.2 Norm of an excited state

To furnish one more example of the perturbative expansion of quantities involving spinning charged states discussed in 4.4.1, we now consider the computation of $\langle n|A_{\ell\ell}A^{\dagger}_{\ell\ell}|n\rangle$. This is equivalent to the norm of the state $A^{\dagger}_{\ell\ell}|n\rangle$. Writing the state as a power series in $n^{-1/2}$

$$A^{\dagger}_{\ell\ell}|n\rangle = |\Psi_0\rangle + \frac{1}{\sqrt{n}}|\Psi_1\rangle + \frac{1}{n}|\Psi_2\rangle + \dots \tag{4.121}$$

we have from (4.106)

$$\begin{aligned}
|\Psi_0\rangle &= \frac{a_{00}a^{\dagger}_{\ell\ell}}{\sqrt{n}}|n\rangle \\
|\Psi_1\rangle &= \Bigg(\frac{(1+4\ell)b^{\dagger}_{00}a^{\dagger}_{\ell\ell}}{4(1+2\ell)} + \sum_{\substack{\ell_1,\ell_2>0 \\ \text{all } m_1,m_2}} \frac{(-1)^m\sqrt{\pi}C^{\ell,\ell_1,\ell_2}_{\ell,m_1,m_2}}{8\sqrt{2\omega_\ell\omega_{\ell_1}\omega_{\ell_2}n}}\Big(-(2+3\ell+\ell_1+\ell_2)a^2_{00}a^{\dagger}_{\ell_1,-m_1}a^{\dagger}_{\ell_2,-m_2} \\
&\qquad + 2(1+\ell-\ell_1+3\ell_2)nb^{\dagger}_{\ell_1,-m_1}a^{\dagger}_{\ell_2,-m_2} \\
&\qquad + (\ell-\ell_1-\ell_2)(a^{\dagger}_{00})^2b^{\dagger}_{\ell_1,-m_1}b^{\dagger}_{\ell_2,-m_2}\Big)\Bigg)|n\rangle
\end{aligned} \tag{4.122}$$

since many terms vanish when applied to $|n\rangle$. We have not computed $|\Psi_2\rangle$ as this would require the NNLO expression for $A^{\dagger}_{\ell\ell}$. It is easy to see that $\langle\Psi_0|\Psi_0\rangle = 1$ and $\langle\Psi_0|\Psi_1\rangle = 0$. Thus order n^{-1} correction to the norm is given by

$$||A^{\dagger}_{\ell\ell}|n\rangle||^2 = 1 + \frac{1}{n}\Big(\langle\Psi_1|\Psi_1\rangle + \langle\Psi_0|\Psi_2\rangle + \langle\Psi_2|\Psi_0\rangle\Big). \tag{4.123}$$

We cannot directly evaluate the last two terms, but we can analyze the term $\langle\Psi_1|\Psi_1\rangle$. This will be given by an infinite sum over spins such as ℓ_1, ℓ_2, which we have no reason to expect will converge. Hence $\langle\Psi_0|\Psi_2\rangle$ has to be an infinite sum as well, such that its divergent part cancels with that of $\langle\Psi_1|\Psi_1\rangle$. The order of magnitude of the spins for which the cancellation starts taking effect can only be the only characteristic spin of the problem : ℓ. We can thus estimate that the tails of both sums will cancel when summed spins are greater than ℓ. In other words, the behavior of both sums can be approximated, barring some unexpected cancellations, by estimating the behavior of the sum in $\langle\Psi_1|\Psi_1\rangle$ with a cutoff of order ℓ. We observe all terms of $|\Psi_1\rangle$ in (4.122) are orthogonal to each other, so we must estimate the norm of these individual terms in the limit of large summed spins ℓ_1, ℓ_2.

First, we consider $\ell_1 \sim \ell_2 \sim \ell$. We estimate the contribution of such terms to the norm as

$$\sum_{\substack{\ell_1\sim\ell_2\sim\ell \\ m_1,m_2}} \frac{\left|C^{\ell,\ell_1,\ell_2}_{\ell,m_1,m_2}\right|^2}{\omega_\ell\omega_{\ell_1}\omega_{\ell_2}}(\ell^2) \sim \sum_{\ell_1\sim\ell_2\sim\ell} \left(\ell^{-7/4}\right)^2 (\ell^2) \sim \ell^2 \times \ell^{-3/2} \sim \ell^{1/2}, \tag{4.124}$$

Let us briefly explain this estimation. We start with a single sum because of the orthogonality of terms in (4.122), and the summand is the square of the coefficient in that equation. Then, as observed in the appendix D.3, in this regime there is only one choice of m_1, m_2 which give a non-suppressed term (D.24). Finally, the double sum over ℓ_1, ℓ_2 yields an additional ℓ^2 factor.

Secondly, we consider the case $\ell_1 - \ell \sim \ell_2 \sim 1$. Again, there is only one choice of m_1, m_2 that yields the dominant term (D.27). Neglecting other terms, we estimate the contribution to the norm as

$$\sum_{\substack{\ell_1\sim\ell,\ell_2\sim1 \\ m_1,m_2}} \frac{\left|C^{\ell,\ell_1,\ell_2}_{\ell,m_1,m_2}\right|^2}{\ell\ell_1\ell_2}(\ell^2) \sim \sum_{\ell_1\sim\ell,\ell_2\sim1} \left(\ell^{-1}\right)^2 (\ell^2) \sim \ell \times 1 \sim \ell, \tag{4.125}$$

where in the second estimation the sum yields a single ℓ factor since only ℓ_1 is summed up to order ℓ. We notice this contribution is dominating that of (4.124). Evidently the case $\ell_2 - \ell \sim \ell_1 \sim 1$ gives an equal contribution.

Therefore, the series expansion of the norm is estimated schematically as

$$||A^\dagger_{\ell\ell}|n\rangle||^2 \sim 1 + \frac{\ell + \dots}{n} + O(n^{-2}). \tag{4.126}$$

where the dots represent terms which are subdominant at large ℓ.

We see the result is again expressed as a series in $\frac{\ell}{n}$. However, not all quantities have this type of expansion as we now discuss.

4.4.3 Primary states

Let us consider $1/n$ corrections to the operator whose leading term is given by (4.113) and whose associated state is given in exact form by (4.63) [11]. By the notation (4.116) we can write the state succinctly as

$$|n;\ell,\ell\rangle_A = \alpha_0 \sum_{k=0}^{\ell} |\psi_k\rangle\,. \tag{4.127}$$

[11] The spin ℓ is bounded by $2 \leq \ell < n$.

The vectors $|\psi_k\rangle$ are mutually orthogonal, but they are not normalized. Comparing their relative norms we find

$$\frac{\langle\psi_k|\psi_k\rangle}{\langle\psi_{k-1}|\psi_{k-1}\rangle} = \frac{(\ell-k+1)(2\ell-2k+1)}{(n-k)k} \sim \frac{\ell^2}{nk}\,, \tag{4.128}$$

where in the last equation we used $k \leq \ell \ll n$. This equation implies the norms $\langle\psi_k|\psi_k\rangle \propto (\ell^2/n)^k/k!$ approximate the coefficients in the expansion of the exponential $\exp(\ell^2/n)$. We then have two regimes depending on whether $\ell^2/n \ll 1$ or $\ell^2/n \gtrsim 1$. In the first case the succession $\langle\psi_k|\psi_k\rangle$ is peaked at $k = 0$. Instead, for $\ell^2/n \gtrsim 1$ the succession is peaked at

$$k_{max} = \frac{2\ell^2}{n}, \tag{4.129}$$

and has a width of order $\sqrt{k_{max}} = \ell/\sqrt{n}$ (see Figure 4.2). Thus, the primary state (4.63) is dominated by the sum of $|\psi_k\rangle$ roughly in the range $k_{\max} - \sqrt{k_{\max}} \lesssim k \lesssim k_{\max} - \sqrt{k_{\max}}$.

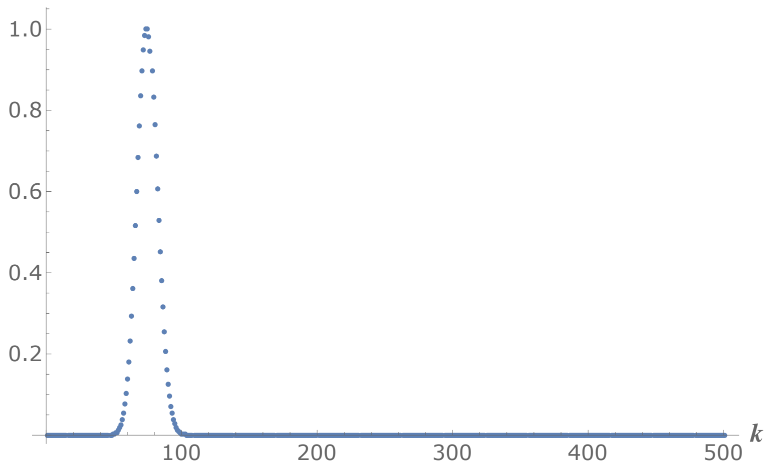


Figure 4.2: Normalized $\langle\psi_k|\psi_k\rangle$ as function of k for $n = 10^3$ and $\ell = 500$.

This result seems to suggest that, for primary states, the $1/n$ expansion (4.63), or equivalently (4.127), breaks down at $\ell \sim \sqrt{n}$. However, the expressions for (primary) operators are coordinate dependent. What we have shown here is that, when expressed in terms of creation-annihilation operators $a_{\ell,m}$, $a^\dagger_{\ell,m}$, primary operators are written as power series in $\ell/\sqrt{n}$. There may exist other coordinates that partially resum the series leading to a manifest expansion in powers of ℓ/n. The mere fact that the expectation value (4.114), which is coordinate-independent, is presented as a power series in ℓ/n, speaks in favor of that possibility.

Unfortunately, those coordinates are certainly not the creation-annihilation operators corresponding to phonons $A_{\ell,m}$, $A^\dagger_{\ell,m}$. Indeed, rewriting the first two terms in (4.63) using the leading order relation (4.103), gives

$$|n,\ell,\ell\rangle_A \underset{\ell\gg 1}{=} \alpha_0\sqrt{(n-1)!}\left(A^\dagger_{\ell\ell} - \frac{\sqrt{2}\ell}{\sqrt{n}}A^\dagger_{\ell-1,\ell-1}A^\dagger_{1,1}\right)|n\rangle. \tag{4.130}$$

One may hope that the second term in parenthesis is cancelled by NLO corrections (4.106), however, it is straightforward to show that it is not the case. We can show using (D.27) that the only potentially relevant term in (4.106)

$$\frac{(-1)^{\ell-1}\sqrt{\pi}C^{\ell,\ell_1,\ell_2}_{-\ell,m_1,m_2}}{8\sqrt{2\omega_\ell\omega_{\ell_1}\omega_{\ell_2}n}}(2+3\ell+\ell_1+\ell_2)A^\dagger_{\ell_1,m_1}A^\dagger_{\ell_2,m_2}, \tag{4.131}$$

scales as $O(\ell^0)/\sqrt{n}$, for $\ell_1 = m_1 = \ell-1$, $\ell_2 = m_2 = 1$, so it cannot cancel the term scaling as $\ell/\sqrt{n}$.

Our conclusion of this section and chapter is that the semiclassical expansion can be trusted for spins as large as the $U(1)$ charge, $\ell \sim n$, as long as we are dealing with coordinate-independent quantities. On the other hand, if we want to identify primary states, using creation-annihilation operators corresponding to phonons, perturbative expansion breaks down much earlier, for $\ell \sim \sqrt{n}$. We expect that for spins in the window $\sqrt{n} < \ell \ll n$ there should exist different semiclassical backgrounds, expanding around which would allow to describe primary states perturbatively[12].

[12]Expanding the summand in (4.65) for large ℓ and n and computing the sum via saddle-point approximation leads to

$$\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{2\ell^2}{n}\right)^k \exp\left(-\frac{k^2}{\ell}\right) = \exp\left\{\frac{2\ell^2}{n}\left[1-\frac{2\ell}{n}+O\left(\frac{\ell^2}{n^2}\right)\right]\right\}, \tag{4.132}$$

which suggests that this result can be obtained perturbatively in a double scaling limit $n \gg 1$, $\ell \gg 1$, $\ell/n = $ fixed.

Conclusion

To conclude this thesis, we provide a summary of the original results that were presented, and of the avenues they open for future research. This includes applications of our methodology, or exploration of related ideas, that have been performed by other authors following the publication of [1, 2].

Summary of this work

Quantum Field Theories represent a challenge when it comes to computing their strongly coupled observables. There is no all-powerful method able to handle every situation, and we have to take advantage of any feature that can give us more control. Recently, such a technique has been developed for correlators of operators with large quantum numbers in Conformal Field Theories. This general method exploits the fact that the theory can be approximated by an Effective Field Theory in terms of fluctuations around a classical background, which corresponds to a superfluid state of finite charge density.

In this thesis, we investigated a special case of a CFT, the Wilson-Fisher fixed point scalar field theories in the ε-expansion in euclidean space. The advantage of using such a UV-complete theory is that all steps of the semiclassical method are completely explicit. The methodology was described in detail in section 2.4, where it was applied to the computation of correlators involving two large-charge operators (ϕ^n and $\bar{\phi}^n$) and N arbitrary operators with small quantum numbers. The steps were illustrated with one precise model in mind, the $U(1)$-invariant complex scalar with $\lambda(\bar{\phi}\phi)^2$ interaction in $4-\varepsilon$ dimensions, but are easily generalised to other theories (see the next section for examples). For example in subsection 2.5 it was readily adapted to the model with $\lambda^2(\bar{\phi}\phi)^3$ interaction in $3-\varepsilon$ dimensions.

The thesis presented several computations made possible with our technique. Chapter 2 presented the computation of the scaling dimension of the operator ϕ^n. We demonstrated the semiclassical expansion to the next-to-leading order in the coupling λ, and for arbitrary values of λn (2.13). This was performed in both theories mentioned above: with either quartic (see (2.131) and (2.140)) or sextic interaction (see (2.164) and (2.166)). In both cases, our results nicely interpolate between the small λn regime ((2.143) and

(2.169)), where it agrees with diagrammatic calculations, and the large λn regime ((2.146) and (2.170)), where it agrees with the expectation for the universal conformal superfluid phase of CFTs at large charge (2.147). In fact, the parameter λn controls the size of the gap of the radial mode, which grows with large-λn, in which limit this mode can thus be integrated out, leaving us with the universal effective theory of the Goldstone mode. Thus in this particular example, it is possible to explicitly compute the Wilson coefficients of the EFT explicitly in terms of the microscopic parameters. Moreover, in the case of an odd number of dimensions, a component of the next-to-leading order contribution is completely universal and does not depend on the Wilson coefficients. When using our method in $3 - \varepsilon$ dimensions, we have been able to compute this component (2.172) and verify the matching with the universal prediction to seven digits of accuracy. This remarkable agreement provides a nontrivial check of the validity of our methodology. Our results have other interesting consequences, which were discussed in section 2.4.6. Especially, we were able to combine the small-εn limit of our NLO result to preexisting 5-loop results for $n = 1, 2, 3, 4$, allowing us to get the low-εn answer at $\mathrm{N}^4\mathrm{LO}$.

In section 3, we showed a second application of the semiclassical expansion by computing some 3- and 4-point correlators, from which we extracted some of the OPE coefficients of $\phi^n \times \bar{\phi}\phi$ (e.g. (3.19), (3.21), (3.40), (3.47)).

Scaling dimensions and OPE coefficients are in principle all that is needed to completely characterize a conformal theory. Using the constraints from conformal symmetry, all observables can be expressed using this “CFT data”. Thus our method is quite powerful, but so far we have only applied it in cases where ϕ^n and $\bar{\phi}^n$ are the large-charge operators. What about the data concerning other large-charge operators ? Chapter 4 aimed at making a first step in this direction, by discussing how one can interpret these operators in the first place. Indeed, although the spectrum of dimensions of those operators is given by our method (at leading order in their splitting), at large λn there is no known way to associate them with local expressions combining the elementary fields and their derivatives. In that chapter, we discussed how operators can indirectly by labeled by their free-theory equivalents, and how those can be classified into conformal multiplets (section 4.2). One interesting byproduct is a complete classification of the free theory primaries with number of derivatives smaller that the charge in vacuum quantization (section 4.2.4). We also compared the two Fock spaces corresponding to either the trivial quantization of the theory or the quantization around the large-charge saddle, described the mapping between them, and how to find primary operators (section 4.3).

Extensions and outlook

The most obvious possible extension is the application of our methodology to different models. Here we give some examples, in a non-exhaustive list, of works which consider a larger, non-abelian internal symmetry group. That allows to study the patterns of symmetry breaking induced by the choice of the Cartan charges, again illuminating the more general, but abstract, work in [75, 120, 121].

- the quartic $O(N)$ model in $4-\varepsilon$ dimensions [104, 122],
- the sextic $O(N)$ model in $3-\varepsilon$ dimensions [110],
- the cubic $O(N)$ model in $6-\varepsilon$ dimensions, where the result has provided a nontrivial check of the conjectured duality [123, 124] between that theory and the UV fixed point of the quartic $O(N)$ model [125, 126],
- the $U(N)\times U(N)$ matrix models in $4-\varepsilon$ dimensions [127, 128], and similarly $U(N)\times U(M)$ [129],
- Boundary Conformal Field Theories, including the $U(1)$ model in a $4-\varepsilon$ dimensions half-space [130],
- the quartic $U(1)$ or $O(N)$ model in $4+\varepsilon$ dimension, the difference with the previous case being that the theory has been proved to be non-unitary in the UV, and to flow to a complex CFT as a fixed point, where scaling dimensions have an imaginary part [131, 132].

It has also been applied to supersymmetric (SUSY) theories, for example the $\mathcal{N}=2$ SUSY Wess-Zumino model with a cubic superpotential in $d=3$ [133].

A cousin of the ε-expansion is found in the large-N expansion, where $1/N$ is also a small parameter, playing the same role as ε, in the sense that observables with charge n are computed in the double scaling limit $n, N\to\infty, n/N=$ fixed. This has been applied to the $\mathbb{C}P^{N-1}$ model in $2+1$ dimensions [134], the $O(2N)$ Landau-Ginzburg model in $2+1$ dimensions [135], and theories with fermions, gauge fields and Yukawa interaction at the Veneziano limit [136, 137].

An other application our results have found is in the analysis of the resurgence properties of the large charge expansion. Such an analysis was first performed in this context in an example of large-N expansion [138], and since has also been done for ε-expansion $O(N)$ models [106].

Our results also turned out to be useful by providing nontrivial checks to a recently proposed reformulation of the Weak Gravity Conjecture in the context of the AdS/CFT correspondence, where instead of a bound on the mass-to-charge ratio of particles in a gravity theory, the conjecture translates to a convexity constraint on the dimension $\Delta(n)$ of the lowest charge-n operator in a CFT [139]. This constraint was checked to work in many ε-expansion models cited above [139, 140].

Besides further potential applications in line with the above, let us briefly mention a few other possible directions for future research which could profit from the ε-expansion point of view. First, the large charge expansion has already been applied to investigate the AdS/CFT correspondence, studying numerically boson star solutions in an AdS model that are dual to a superfluid in the CFT, with an operator dimension verifying the expected behaviour [141]. In the reverse sense, it could be interesting to study the holographic dual of a weakly coupled CFT, either a large-N or an ε-expansion model.

An other pending question which warrants further study is the convergence properties of the ε-expansion. Indeed, as discussed in section 2.4.6, we tried comparing the results of the $d = 4 - \varepsilon$ dimensional computation evaluated at $\varepsilon = 1$ to a numerical Monte-Carlo computation in $d = 3$. The large uncertainty made it difficult to jusge if the result is converging towards the numerical result or not. In that regard, it would be very interesting to see the result of the NNLO computation, of the Δ_1 contribution in (2.13). The building blocks for that computation are already present: what is needed is to compute vacuum bubble diagrams of the superfluid fluctuations action (2.99), using interaction vertices (2.101) and the propagators (B.3). Thus the computation should not pose any conceptual difficulty, but can be rather involve and necessitate the use of numerics. The NNLO result would also yield more information for the study of resurgence properties of the ε-expansion at large charge [106].

In the introduction of this thesis, we motivated the study of the ϕ^n operator, by noting a similarity with the breakdown of perturbation theory when computing amplitudes with a large number of particles. Even if there is no conformal symmetry in that case, and we would have to work in Lorentzian spacetime, these results motivate further investigations into the more difficult problem of particle production, to see if some features of our method can be transposed there.

Let us finally come back to the question of operators with large charge and non-zero spin, which we studied in chapter 4. As we said, the main result of that section is the mapping between the vacuum quantization Fock space and the superfluid excitations Fock space, detailed in section 4.3. A second remarkable result that was derived in this chapter is a complete characterization of free theory primary operators that have a number of derivatives lower that their charge, described in section 4.2.4. This result complements other approaches to the problem of counting and writing down primaries using Hilbert Series [117, 118, 119].

In section 4.4, we discussed the limitation of the semiclassical method when it comes to describing states with spin ℓ and charge n. We explained, based on a few example computations, that the primary states we identified were well-described by a single-phonon state only in the limit $\ell \ll \sqrt{n}$, beyond which many-phonon states components take over. However, the computation of observables with these states, staying in the vacuum quantization picture, remains valid and reliable up to $\ell \ll n$. We believe this is an indication that the semiclassical expansion breaks down at $\ell \ll \sqrt{n}$, and that the physically-relevant states are not well approximated anymore beyond this limit. There might however exist a workaround for this breakdown, and the intuition behind it again comes from the superfluid picture. Indeed, it is known both experimentally and from the EFT of superfluids, that giving a low spin to a superfluid only excites a few phonons, but above a given spin threshold the superfluid start forming vortices [116, 142].

Therefore, we conjecture that there exists a different solution to the equations of motion, with large charge and large spin $\ell \geq \sqrt{n}$, which can serve as a valid saddle for the semiclassical expansion in this regime. The correct way to reliably characterize states of higher spins would then be by expanding around that saddle. We have started investigating this issue, and numerically found solutions of the equations of motion for spins given by integer multiples of the charge $\ell = n, 2n, 3n, \dots$. In the large λn limit the $\ell = n$ solution describes a pair of vortex-antivortex living at the poles of the sphere, as expected from the EFT [142]. However, the interesting question is the existence and behaviour of solutions with $\ell < n$. The EFT indicates a pair of vortex-antivortex closer to the equator, looking for these solutions will be the next step in our investigation of the large charge and large spin regime of conformal field theories.

A Details of the one-loop computation of Δ_{ϕ^n} on the cylinder

A.1 Next to leading order corrections for generic λn

Here we discuss the derivation of (2.140) from (2.129). To this aim, we first compute $\bar{e}_0$ expanding the first line in (2.70) from the expression of the bare coupling (2.4):

$$\bar{e}_0(\lambda n, RM, d) = e_0(\lambda n, d) + \left\{ \frac{5}{8}(\mu^2 R^2 - 1)^2 \left[\frac{1}{\varepsilon} - \log(M\tilde{R}) \right] \right. \\ \left. + \frac{1}{16}(\mu^2 R^2 + 3)(\mu^2 R^2 - 1) + O(\varepsilon) \right\}_{\lambda_0 = \lambda}, \quad \text{(A.1)}$$

where we defined $\tilde{R} \equiv \sqrt{\pi} e^{\gamma/2} R$ and we used the equations of motion (2.92) to expand the leading order in the coupling:

$$\frac{\partial}{\partial \lambda_0} \left[\frac{e_{-1}(\lambda_0 n, d)}{\lambda_0 R} \right] = \frac{R^{d-1} \Omega_{d-1} f^4}{16}. \quad \text{(A.2)}$$

To compute Δ_0 in (2.71), we need to evaluate (A.1) in $d = 4$ and add the expansion of the leading order $\bar{e}_{-1}/\lambda$ to first order in ε (at fixed coupling)

$$\begin{aligned} \Delta_0 &= \left\{ \bar{e}_0(\lambda n, RM, 4) + \frac{\partial}{\partial \varepsilon} \left[\frac{1}{\lambda} \bar{e}_{-1}(\lambda n, RM, 4 - \varepsilon) \right]_{\varepsilon = 0} \right\}_{\lambda = \lambda_*} \\ &= \left\{ \lim_{\varepsilon \to 0} \left[\frac{R}{2} \sum_{\ell = 0}^{\infty} n_\ell \left[\omega_B(\ell) + \omega_A(\ell) \right] + \frac{5}{8\varepsilon}(\mu^2 R^2 - 1)^2 \right] \right\}_{\lambda_0 = \lambda_*} \end{aligned} \quad \text{(A.3)}$$

where the limit $\varepsilon \to 0$ is taken at λ_0 fixed, we used eq. (2.8) and

$$\frac{1}{\lambda} \bar{e}_{-1}(\lambda n, RM, 4 - \varepsilon) = \frac{1}{\lambda M^\varepsilon} e_{-1}(\lambda n M^\varepsilon, 4 - \varepsilon). \quad \text{(A.4)}$$

As anticipated, at the fixed point the dependence on the sliding scale drops.

To proceed, we need to isolate the divergent contribution in the sum in eq. (A.3). We use the $\ell \to \infty$ expansion of the summand

$$n_\ell \left[\omega_B(\ell) + \omega_A(\ell)\right] \sim \sum_{n=1}^{\infty} c_n \ell^{d-n}. \tag{A.5}$$

The first five terms provide a divergent contribution in $d = 4$. The expansion in $4 - \varepsilon$ dimensions of the coefficients is

$$\begin{gathered} c_1 = \frac{2}{R} + O\left(\varepsilon\right), \quad c_2 = \frac{6}{R} + O\left(\varepsilon\right), \quad c_3 = 2\mu^2 R + \frac{4}{R} + O\left(\varepsilon\right), \quad c_4 = 2\mu^2 R + O\left(\varepsilon\right), \\ c_5 = -\frac{5\left(\mu^2R^2-1\right)^2}{4R} + \varepsilon\frac{\left[-225\mu^4R^4 + 50\mu^2R^2 + 150\gamma\left(\mu^2R^2-1\right)^2 + 113\right]}{120R} + O\left(\varepsilon^2\right). \end{gathered} \tag{A.6}$$

We can now rewrite the sum isolating explicitly the divergent contribution as

$$\frac{1}{2}\sum_{\ell=0}^{\infty} n_\ell \left[\omega_B(\ell) + \omega_A(\ell)\right] = \frac{1}{2}\sum_{n=1}^{5} c_n \sum_{\ell=1}^{\infty} \ell^{d-n} + \frac{1}{2}\sum_{\ell=1}^{\infty} \bar{\sigma}(\ell) + \frac{1}{2}\omega_B(0), \tag{A.7}$$

where $\bar{\sigma}(\ell)$ is defined subtracting the first five terms in (A.5) from the original summand,

$$\bar{\sigma}(\ell) = n_\ell \left[\omega_B(\ell) + \omega_A(\ell)\right] - \sum_{n=1}^{5} c_n \ell^{d-n}, \tag{A.8}$$

and we used that $\omega_A(0) = 0$. From (A.5) we see that the sum over $\bar{\sigma}(\ell)$ is convergent and can be evaluated directly in $d = 4$. The first terms provide a divergent contribution which can be computed using $\sum_{\ell=1}^{\infty} \ell^x = \zeta(-x)$ and recalling $\zeta(1-\varepsilon) \sim -1/\varepsilon$:

$$\frac{1}{2}\sum_{n=1}^{5} c_n \sum_{\ell=1}^{\infty} \ell^{d-n} = -\frac{5\left(\mu^2R^2-1\right)^2}{8R\,\varepsilon} - \frac{15\mu^4R^4 - 6\mu^2R^2 + 7}{16R}. \tag{A.9}$$

Using equations (A.7) and (A.9) in (A.3), we obtain the result in the main text (2.140).

A.2 Next to leading order corrections for large λn

Here we explain the calculation of the result (2.144). To this aim, it is convenient to start from eq. (A.3), derived in the previous appendix. We denote the summand in (2.139) with the bare coupling replaced by the renormalized one as

$$s(\ell, d) \equiv n_\ell R \left[\omega_B(\ell) + \omega_A(\ell)\right]_{\lambda_0 = \lambda}. \tag{A.10}$$

We then separate the sum over $s(\ell, d)$ into two terms introducing a cutoff $AR\mu$, where $A \gtrsim 1$ is an arbitrary number such that $AR\mu_*$ is an integer:

$$\frac{1}{2}\sum_{\ell=0}^{\infty} s(\ell,d) = \frac{1}{2}\sum_{\ell=0}^{AR\mu} s(\ell,d) + \frac{1}{2}\sum_{AR\mu+1}^{\infty} s(\ell,d). \tag{A.11}$$

We can approximate the second sum using the Euler-Maclaurin formula:

$$\sum_{AR\mu+1}^{\infty} s(\ell,d) \simeq \int_{AR\mu}^{\infty} d\ell s(\ell,d) - \frac{s(AR\mu,4)}{2} - \sum_{k=1}^{N_1} \frac{B_{2k}}{(2k)!} s^{(2k+1)}(AR\mu,4) + O(\varepsilon), \tag{A.12}$$

where B_{2k} are the Bernoulli numbers and N_1 is an integer. As $s^{(k)}(AR\mu) \sim (AR\mu)^{1-k}$ and $\frac{B_{2k}}{(2k)!}$ approaches zero exponentially fast as k grows, the error we make in (A.12) can be made arbitrarily small incresing N_1. The integral in (A.12) is approximately evaluated using the expansion (A.5) truncated after N_2 terms, giving

$$\begin{aligned}\frac{1}{2}\int_{AR\mu}^{\infty} d\ell s(\ell,d) &\simeq \frac{1}{2}(AR\mu)^d \sum_{n=1}^{N_2} \frac{Rc_n}{(AR\mu)^{n-1}(n-1-d)} \\ &\equiv -\frac{5\left(\mu^2R^2-1\right)^2}{8\,\varepsilon} + \frac{5}{8}\left(R^2\mu^2-1\right)^2 \log(AR\mu) + f_{N_2,A}(R\mu) + O(\varepsilon),\end{aligned} \tag{A.13}$$

where f is a regular function of $R\mu$. As before, increasing N_2 we can improve at will the precision of our calculation for $A \gtrsim 1$. Using (A.3) we then conclude

$$\Delta_0 = \frac{5}{8}\left(R^2\mu_*^2-1\right)^2 \log(R\mu_*) + F(R\mu_*), \tag{A.14}$$

where the function $F(R\mu_*)$ can be computed from

$$F(R\mu_*) \simeq f_{N_2,A}(R\mu_*) - \frac{s(AR\mu_*)}{2} + \left[\frac{1}{2}\sum_{\ell=0}^{AR\mu_*} s(\ell,4) - \sum_{k=1}^{N_1}\frac{B_{2k}}{(2k)!} s^{(2k+1)}(AR\mu_*)\right]_{\mu=\mu_*}. \tag{A.15}$$

The function $F(R\mu_*)$ can now be evaluated numerically and then fitted to the expected functional form, estimating the error from the first subleading terms neglected in the sums in (A.12) and (A.13). Using $N_1 = 4$, $N_2 = 10$ and $A = 10$, we evaluated (A.15) for $R\mu_* = 11, 12, \ldots 210$. The result was fitted with an expansion in $(R\mu_*)^{-2}$, starting from $(R\mu_*)^4$, with four parameters[1]. The first three terms read:

$$F(R\mu_*) = -2.01444683(3)(R\mu_*)^4 + 2.49986(9)(R\mu_*)^2 - 0.55(4) + O\left((R\mu_*)^{-2}\right). \tag{A.16}$$

We have also verfied that the coefficients of $(R\mu_*)$, $(R\mu_*)^3$, $(R\mu_*)^4\log(R\mu_*)$ and $(R\mu_*)^2\log(R\mu_*)$

[1] A fit with three parameter produces the same results with smaller standard errors.

are compatible with zero if included, individually or in combination, in the fit of the function in (A.15). Notice that the functional form (A.16) agrees with (2.147) for $d = 4$ after expansing $R\mu_*$ in terms of $(\lambda_* n)^{2/3}$.

The expansion of the first term in (A.14) produces logarithms of $\lambda_* n$:

$$\begin{aligned} \frac{5}{8}\left(R^2\mu_*^2 - 1\right)^2 \log(R\mu_*) =& 5\left(\frac{(\lambda_* n)^{4/3}}{384\pi^{8/3}} - \frac{(\lambda_* n)^{2/3}}{144\pi^{4/3}} + \frac{1}{72}\right)\log\left(\frac{\lambda_* n}{8\pi^2}\right) \\ &+ \frac{5}{288}\left(\frac{3(\lambda_* n)^{2/3}}{\pi^{4/3}} - 10\right) + O\left(\left(\frac{\lambda_* n}{16\pi^2}\right)^{-2/3}\right). \end{aligned} \tag{A.17}$$

As explained in the main text, the coefficients of the logarithms ensure that the one-loop result takes the form predicted by the large charge CFT predictions. Assuming that $F(R\mu_*)$ contains only powers of $R\mu_*$ (as we checked in (A.16)), one can verify that this is true for all the subleading orders in $(\lambda_* n)$ as well. Summing (A.16) and (A.17) and expanding $(R\mu_*)^2$ in powers of $(\lambda_* n)^{2/3}$, we obtain the result stated in the main text.

B Quantum corrections on the cylinder

In this chapter we provide details regarding computations of correlators in the semiclassical expansion, which are the subject of chapter 3. To perform those, we need to quantize the field theory of fluctuations (r, π) around the semiclassical saddle.

B.1 Propagator on the cylinder

In this section we show how to construct propagators corresponding to fluctuations of fields r, π in (2.116). From time translation and rotation symmetry we know the propagator can be written as

$$\langle x(\tau_1, \vec{n}_1) y(\tau_2, \vec{n}_2) \rangle = D_{xy}(\tau_1 - \tau_2, \vec{n}_1 \cdot \vec{n}_2), \tag{B.1}$$

where $x, y \in \{r, \pi\}$ are fields and $\langle \ldots \rangle$ is the τ-ordered Wick contraction. The quadratic Lagrangian yields a matrix equation similar to (2.113) for the propagator (note that it is not diagonal due to mixing between $\dot{\pi}$ and r)

$$-\begin{pmatrix} \partial_\tau^2 + \Delta_{S^{d-1}} - M^2 & 2i\mu\partial_\tau \\ -2i\mu\partial_\tau & \partial_\tau^2 + \Delta_{S^{d-1}} \end{pmatrix} \begin{pmatrix} D_{rr} & D_{r\pi} \\ D_{\pi r} & D_{\pi\pi} \end{pmatrix} = \delta(\tau_1 - \tau_2)\delta^{(S^{d-1})}(\vec{n}_1 \cdot \vec{n}_2). \tag{B.2}$$

Expanding in spherical harmonics (in this case only with $\vec{m} = \vec{0}$, which corresponds to Gegenbauer polynomials)

$$D(\tau, \vec{n}_1 \cdot \vec{n}_2) = \sum_\ell F^{(\ell)}(\tau) C_\ell^{(d/2-1)}(\cos(\vec{n}_1 \cdot \vec{n}_2)), \tag{B.3}$$

we obtain

$$-N_\ell \Omega_{d-2} \begin{pmatrix} \partial_\tau^2 - J_\ell^2 - M^2 & 2i\mu\partial_\tau \\ -2i\mu\partial_\tau & \partial_\tau^2 - J_\ell^2 \end{pmatrix} \begin{pmatrix} F_{rr}^{(\ell)} & F_{r\pi}^{(\ell)} \\ F_{\pi r}^{(\ell)} & F_{\pi\pi}^{(\ell)} \end{pmatrix} = C_\ell^{(d/2-1)}(1)\delta(\tau), \tag{B.4}$$

where the normalization factor of Gegenbauer polynomials is given by

$$N_\ell \int_{-1}^{1} C_\ell^{(d/2-1)}(x) C_\ell^{(d/2-1)}(x)(1-x^2)^{\frac{d-3}{2}} dx = \frac{2^{4-d}\,\pi\,\Gamma(\ell+d-2)}{(2\ell+d-2)\,\ell!\,\Gamma^2\left(\frac{d}{2}-1\right)}. \tag{B.5}$$

We can look for solutions of this equations for $\tau < 0$ and $\tau > 0$, which will be given by expressions similar to (2.116), and then find the propagator by matching this solutions at $\tau = 0$ with a specific discontinuity of derivatives. Alternatively, we can Fourier transform (B.4) and use (see [143], table 18.6.1)

$$C_\ell^{(d/2-1)}(1) = \frac{\Gamma(\ell+d-2)}{\ell!\,\Gamma(d-2)}, \tag{B.6}$$

to obtain

$$F^{(\ell)}(\tau) = \frac{2\ell+d-2}{(d-2)\Omega_{d-1}} \int \frac{d\omega}{2\pi} e^{-i\omega\tau} \frac{M^{(\ell)}(\omega)}{(\omega^2+\omega_B^2(\ell))(\omega^2+\omega_A^2(\ell))}, \tag{B.7}$$

with

$$M^{(\ell)}(\omega) = \begin{pmatrix} \omega^2 + J_\ell^2 & 2\mu\omega \\ -2\mu\omega & \omega^2 + J_\ell^2 + M^2 \end{pmatrix}. \tag{B.8}$$

For $\ell \neq 0$ integration in (B.7) can be easily done using Cauchy's theorem, resulting in

$$F^{(\ell)}(\tau) = \frac{2\ell+d-2}{(d-2)\Omega_{d-1}} \left(\frac{M^{(\ell)}(-i\omega_A(\ell))e^{-\omega_A(\ell)\tau}}{2\omega_A(\ell)} - \frac{M^{(\ell)}(-i\omega_B(\ell))e^{-\omega_B(\ell)\tau}}{2\omega_B(\ell)} \right) \frac{1}{\omega_B^2(\ell)-\omega_A^2(\ell)} \tag{B.9}$$

for $\tau > 0$, and

$$F^{(\ell)}(\tau) = \frac{2\ell+d-2}{(d-2)\Omega_{d-1}} \left(\frac{M^{(\ell)}(i\omega_A(\ell))e^{\omega_A(\ell)\tau}}{2\omega_A(\ell)} - \frac{M^{(\ell)}(i\omega_B(\ell))e^{\omega_B(\ell)\tau}}{2\omega_B(\ell)} \right) \frac{1}{\omega_B^2(\ell)-\omega_A^2(\ell)} \tag{B.10}$$

for $\tau < 0$. The same result can obviously be obtained directly from (2.116). Indeed, say for $\tau_1 < \tau_2$ computing non-zero spin contribution to time ordered correlator we get

$$\langle n|r(\tau_2)r(\tau_1)|n\rangle_\ell = \left(\frac{J_\ell^2 - \omega_A(\ell)^2}{2\omega_A(\ell)} e^{-\omega_A(\ell)|\tau_2-\tau_1|} + \frac{\omega_B(\ell)^2 - J_\ell^2}{2\omega_B(\ell)} e^{-\omega_B(\ell)|\tau_2-\tau_1|} \right) \frac{1}{\omega_B^2(\ell)-\omega_A^2(\ell)} \sum_{\vec{m}} Y_{\ell\vec{m}} Y_{\ell\vec{m}}^*, \tag{B.11}$$

which upon using (B.6) and (see [144])

$$\sum_{\vec{m}} Y_{\ell\vec{m}}(\vec{n}_1) Y_{\ell\vec{m}}^*(\vec{n}_2) = \frac{2\ell+d-2}{(d-2)\Omega_{d-1}} C_\ell^{(d/2-1)}(\vec{n}_1\cdot\vec{n}_2) \tag{B.12}$$

reproduces (B.9). Similarly, we can compute $r\pi$ and $\pi\pi$ components of the propagator.

Dealing with $\ell = 0$ modes is somewhat more subtle. The difficulty is that apart from the gapped mode corresponding to $(B_{0\vec{0}}, B^\dagger_{0\vec{0}})$ there is also the gapless mode $\hat{\pi}, p_\pi$, for which

$$p_\pi |n\rangle = 0, \tag{B.13}$$

and which does not have the Fock space structure. It does not present a problem for $\langle rr \rangle$, indeed using (2.116) we get

$$\langle n| r(\tau_2) r(\tau_1) |n\rangle_0 = \frac{1}{2\omega_B(0)\Omega_{d-1}} e^{-\omega_B(0)|\tau_2-\tau_1|}, \tag{B.14}$$

which is consistent with (B.9) and (B.10). On the other hand considering correlators linear in π is problematic. However, that is not an issue, for in all instances the field π appears only in the exponent[1], hence, we need only to worry about correlators involving $e^{i\pi(\tau)/f}$. For instance, using Baker-Campbell-Hausdorff formula we obtain (for $\tau < 0$)

$$\langle e^{-i\pi(0)/f} e^{i\pi(\tau)/f} \rangle_0 = \exp\left[-\frac{1-\frac{4\mu^2}{\omega_B^2(0)}}{2\Omega_{d-1} f^2}\tau \right] \exp\left[\frac{1}{f^2}\frac{4\mu^2}{\omega_B^2(0)}\frac{e^{\omega_B(0)\tau}-1}{2\omega_B(0)\Omega_{d-1}} \right]. \tag{B.16}$$

Comparing with the naive expectation

$$\langle e^{-i\pi(0)/f} e^{i\pi(\tau)/f} \rangle_0 = 1 + \frac{D^{(0)}_{\pi\pi}(|\tau|) - D^{(0)}_{\pi\pi}(0)}{f^2} + O(f^{-4}), \tag{B.17}$$

it is consistent to define (compare with (B.9) and (B.10))

$$F^{(0)}_{\pi\pi}(\tau) = -\frac{1-\frac{4\mu^2}{\omega_B^2(0)}}{2\Omega_{d-1}}|\tau| + \frac{4\mu^2}{\omega_B^2(0)}\frac{e^{-\omega_B(0)|\tau|}}{2\omega_B(0)\Omega_{d-1}} + \text{const.} \tag{B.18}$$

Similarly, computing $\langle e^{-i\pi(0)/f} r(0) e^{i\pi(\tau)} \rangle_0$ allows to define

$$F^{(0)}_{r\pi}(\tau) = \text{sign}(\tau)\frac{i\mu}{\omega_B^2(0)}\frac{e^{-\omega_B(0)|\tau|}}{\Omega_{d-1}} + \text{const.} \tag{B.19}$$

B.2 3-pt function computation

We now detail the computation of the next-to-leading order contribution to the 3-point function (3.10), which means evaluating (3.11) using Wick contractions. Here as in

[1] Bear in mind that $\hat{\pi}$ is defined on a compact space (circle), since charge is quantized. As such, the corresponding canonical momentum p_π is defined only on the space of periodic functions. Otherwise p_π is not Hermitian. Indeed, the following relation holds

$$\int_0^{2\pi} d\hat{\pi}\psi_2^*(\hat{\pi})\left[-i\partial_{\hat{\pi}}\psi_1(\hat{\pi})\right] = \int_0^{2\pi} d\hat{\pi}(\hat{\pi})\left[-i\partial_{\hat{\pi}}\psi_2\right]^* \psi_1(\hat{\pi}), \tag{B.15}$$

only if $\psi_2(\hat{\pi})\psi_1(\hat{\pi})\Big|_0^{2\pi} = 0$, i.e. for periodic functions $\psi_i(\hat{\pi})$.

section 3.1, the symbol μ refers to $\mu_4(\lambda n, d)$. One must keep in mind that m, f, J_ℓ, $\omega_{A,B}(\ell)$ are functions of d, μ and n. The symbol μ_* will refer to $\mu_4(\lambda_* n, 4)$. Using the propagator (B.3) we compute contractions for terms without spatial derivatives

$$\int \langle r_1 r_2^3 \rangle \quad \to \quad 3\Omega_{d-1} \left[\int d\tau F_{rr}^{(0)}(\tau)\right] \left[\sum_{\ell=0}^{\infty} F_{rr}^{(\ell)}(0) C_\ell^{(d/2-1)}(1)\right], \tag{B.20}$$

$$\begin{aligned}\int \langle r_1 \dot{\pi}_2 r_2^2 \rangle \quad &\to \quad 2\Omega_{d-1} \left[\int d\tau F_{rr}^{(0)}(\tau)\right] \left[\sum_{\ell=0}^{\infty} \dot{F}_{\pi r}^{(\ell)}(0) C_\ell^{(d/2-1)}(1)\right] \\ &\quad + \Omega_{d-1} \left[\int d\tau \dot{F}_{\pi r}^{(0)}(\tau)\right] \left[\sum_{\ell=0}^{\infty} F_{rr}^{(\ell)}(0) C_\ell^{(d/2-1)}(1)\right],\end{aligned} \tag{B.21}$$

$$\begin{aligned}\int \langle r_1 \dot{\pi}_2^2 r_2 \rangle \quad &\to \quad 2\Omega_{d-1} \left[\int d\tau \dot{F}_{\pi r}^{(0)}(\tau)\right] \left[\sum_{\ell=0}^{\infty} \dot{F}_{\pi r}^{(\ell)}(0) C_\ell^{(d/2-1)}(1)\right] \\ &\quad - \Omega_{d-1} \left[\int d\tau F_{rr}^{(0)}(\tau)\right] \left[\sum_{\ell=0}^{\infty} \ddot{F}_{\pi\pi}^{(\ell)}(0) C_\ell^{(d/2-1)}(1)\right],\end{aligned} \tag{B.22}$$

$$\int \langle r^2 \rangle \quad \to \quad \left[\sum_{\ell=0}^{\infty} F_{rr}^{(\ell)}(0) C_\ell^{(d/2-1)}(1)\right], \tag{B.23}$$

where indices indicate evaluation point, for example $r_1 = r(\tau_1, \vec{n}_1)$. For the remaining term

$$\int g_2^{ij} \langle r_1 \partial_i \pi_2 \partial_j \pi_2 r_2 \rangle \tag{B.24}$$

in order to find contraction of two fields at the same point it is necessary to introduce a splitting, compute derivative(s) and then consider the limit. For example one of the Wick contractions yields

$$\langle r_2 \partial_i \pi_2 \rangle = \lim_{\vec{n}_2' \to \vec{n}_2} \partial_i' D_{\pi r}(0, \vec{n}_2' \cdot \vec{n}_2) = 0, \tag{B.25}$$

which vanishes since $\vec{n}_2' \cdot \vec{n}_2$ is maximal for $\vec{n}_2' = \vec{n}_2$. Similarly, using the chain rule and the same argument we show

$$\langle \partial_i \pi_2 \partial_j \pi_2 \rangle = \lim_{\vec{n}_2' \to \vec{n}_2} \partial_i' \partial_j D_{\pi\pi}(0, \vec{n}_2' \cdot \vec{n}_2) = \frac{d}{dx} D_{\pi\pi}(0, x)\Big|_{x=1} (\partial_i \vec{n}_2) \cdot (\partial_j \vec{n}_2) \tag{B.26}$$

and one can show, for example by choosing specific coordinates on the sphere, that

$$g_2^{ij} (\partial_i \vec{n}_2) \cdot (\partial_j \vec{n}_2) = d - 1. \tag{B.27}$$

Using (see [143], eq. (18.9.19))

$$\frac{d}{dx} C_n^{(\lambda)}(x) = 2\lambda C_{n-1}^{(\lambda+1)}(x), \tag{B.28}$$

and (see (B.6))

$$C_{\ell-1}^{(d/2)}(1) = C_\ell^{(d/2-1)}(1)\,\frac{J_\ell^2}{(d-1)(d-2)}, \tag{B.29}$$

we obtain

$$\int g_2^{ij}\langle r_1\partial_i\pi_2\partial_j\pi_2 r_2\rangle \to \Omega_{d-1}J_\ell^2\left[\int d\tau F_{rr}^{(0)}(\tau)\right]\left[\sum_{\ell=0}^{\infty}F_{\pi\pi}^{(\ell)}(0)C_\ell^{(d/2-1)}(1)\right]. \tag{B.30}$$

For what follows we will need explicit expressions

$$\begin{aligned}
F_{rr}^{(0)}(\tau) &= \frac{e^{-\omega_B(0)|\tau|}}{2\Omega_{d-1}\,\omega_B(0)}, && \text{(B.31)}\\
\dot F_{\pi r}^{(0)}(\tau) &= \frac{i\mu\, e^{-\omega_B(0)|\tau|}}{\Omega_{d-1}\,\omega_B(0)}, && \text{(B.32)}\\
F_{rr}^{(0)}(0) &= \frac{1}{2\Omega_{d-1}\,\omega_B(0)}, && \text{(B.33)}\\
F_{rr}^{(\ell)}(0) &= \frac{2\ell+d-2}{\Omega_{d-1}(d-2)}\frac{\omega_B(\ell)\omega_A(\ell)+J_\ell^2}{2\omega_B(\ell)\omega_A(\ell)\left[\omega_B(\ell)+\omega_A(\ell)\right]}, \quad \ell\neq 0, && \text{(B.34)}\\
F_{\pi\pi}^{(\ell)}(0) &= \frac{2\ell+d-2}{\Omega_{d-1}(d-2)}\frac{\omega_B(\ell)\omega_A(\ell)+J_\ell^2+2(\mu^2-m^2)}{2\omega_B(\ell)\omega_A(\ell)\left[\omega_B(\ell)+\omega_A(\ell)\right]}, \quad \ell\neq 0, && \text{(B.35)}\\
\dot F_{\pi r}^{(0)}(0) &= \frac{i\mu}{\Omega_{d-1}\,\omega_B(0)}, && \text{(B.36)}\\
\dot F_{\pi r}^{(\ell)}(0) &= \frac{2\ell+d-2}{\Omega_{d-1}(d-2)}\frac{i\mu}{\omega_B(\ell)+\omega_A(\ell)}, \quad \ell\neq 0, && \text{(B.37)}\\
\ddot F_{\pi\pi}^{(0)}(0) &= \frac{2\mu^2}{\Omega_{d-1}\,\omega_B(0)}, && \text{(B.38)}\\
\ddot F_{\pi\pi}^{(\ell)}(0) &= \frac{2\ell+d-2}{\Omega_{d-1}(d-2)}\frac{\omega_+^2(\ell)+\omega_A^2(\ell)+\omega_B(\ell)\omega_A(\ell)-J_\ell^2-2(\mu^2-m^2)}{2\left[\omega_B(\ell)+\omega_A(\ell)\right]}, \quad \ell\neq 0, && \text{(B.39)}
\end{aligned}$$

and integrals

$$\begin{aligned}
\int d\tau F_{rr}^{(0)}(\tau) &= \frac{1}{\Omega_{d-1}\,\omega_B^2(0)}, && \text{(B.40)}\\
\int d\tau \dot F_{\pi r}^{(0)}(\tau) &= \frac{2i\mu}{\Omega_{d-1}\,\omega_B^2(0)}. && \text{(B.41)}
\end{aligned}$$

We obtain

$$\begin{aligned}
\left\langle n|\bar\phi\phi(0,\vec n_d)|n\right\rangle &= \frac{n}{2\mu\Omega_{d-1}} - \frac{2(\mu^2-m^2)}{\omega_B^2(0)}\sum_{\ell=0}^{\infty}F_{rr}^{(\ell)}(0)C_\ell^{(d/2-1)}(1) - \frac{2i\mu}{\omega_B^2(0)}\sum_{\ell=0}^{\infty}\dot F_{\pi r}^{(\ell)}(0)C_\ell^{(d/2-1)}(1)\\
&\quad + \frac{1}{\omega_B^2(0)}\sum_{\ell=0}^{\infty}\left[\ddot F_{\pi\pi}^{(\ell)}(0) - J_\ell^2 F_{\pi\pi}^{(\ell)}(0)\right]C_\ell^{(d/2-1)}(1). && \text{(B.42)}
\end{aligned}$$

Which can be further simplified with

$$\omega_B^2(\ell)\omega_A^2(\ell) = J_\ell^4 + 2J_\ell^2(\mu^2 - m^2), \quad \omega_B^2(\ell) + \omega_A^2(\ell) = 2(J_\ell^2 + 3\mu^2 - m^2) \tag{B.43}$$

leading to

$$\left\langle n|(\bar{\phi}\phi)(0,\vec{n}_d)|n\right\rangle = \frac{n}{2\mu\Omega_{d-1}} + \sum_{\ell=0}^{\infty}\frac{1}{\omega_B^2(0)}\frac{2\ell+d-2}{\Omega_{d-1}(d-2)}C_\ell^{(d/2-1)}(1)\frac{\omega_B(\ell)\omega_A(\ell)(3\mu^2+m^2) - J_\ell^2(\mu^2-m^2)}{\omega_B(\ell)\omega_A(\ell)\left[\omega_B(\ell)+\omega_A(\ell)\right]}. \tag{B.44}$$

Denoting the summand in (B.44) by

$$\frac{S_\ell(\mu,m,d)}{(d-2)\Omega_{d-1}} \tag{B.45}$$

and considering its asymptotics at $\ell \to \infty$

$$S_\ell(\mu,m,d) \underset{\ell\to\infty}{\equiv} c_{-1}(\mu,m,d)\ell^{d-3} + c_0(\mu,m,d)\ell^{d-4} + c_1(\mu,m,d)\ell^{d-5} + \dots, \tag{B.46}$$

we get

$$\begin{aligned}\left\langle n|(\bar{\phi}\phi)(0,\vec{n}_d)|n\right\rangle &= \frac{n}{2\Omega_{d-1}\mu_4((\lambda_*+\delta\lambda)n,d)} \qquad\qquad (\text{B.47})\\ &+ \frac{1}{(d-2)\Omega_{d-1}}\Bigg\{S_0(\mu,m,d) + \sum_{\ell=1}^{\infty}\Big[S_\ell(\mu,m,d) - c_{-1}(\mu,m,d)\ell^{d-3}\\ &- c_0(\mu,m,d)\ell^{d-4} - c_1(\mu,m,d)\ell^{d-5}\Big]\\ &+ c_{-1}(\mu,m,d)\zeta(3-d) + c_0(\mu,m,d)\zeta(4-d) + c_1(\mu,m,d)\zeta(5-d)\Bigg\}_{\lambda_*}\end{aligned}$$

where we put explicit dependence of μ_4 on the 1-loop coupling counterterm, with

$$\delta\lambda = \frac{5(\lambda_*)^2}{16\pi^2}\frac{1}{4-d}, \tag{B.48}$$

and 1-loop terms don't need any counterterm corrections at this order. The renormalized coupling is denoted by λ_* which at this stage is considered as independent of the dimension. Expanding the first term in λ and other terms in $4-d$, keeping only $O(1)$ terms (these are the only ones that are needed at this order), leads to

$$\begin{aligned}\left\langle n|(\bar{\phi}\phi)(0,\vec{n}_d)|n\right\rangle &= \Bigg\{\frac{n}{2\Omega_{d-1}\mu} - \frac{5\lambda^2}{16\pi^2}\frac{1}{4-d}\frac{n}{2\Omega_{d-1}\mu^2}\frac{\partial\mu}{\partial\lambda} \qquad (\text{B.49})\\ &+ \frac{1}{(d-2)\Omega_{d-1}}\left(R(\mu_*) + \frac{c_{1P}(\mu,m)}{4-d} + c_{1F}(\mu_*,1)\right)\Bigg\}_{\lambda_*}\end{aligned}$$

where we introduced

$$\begin{aligned} R(\mu) &= S_0(\mu,1,4)+\sum_{\ell=1}^{\infty}\left[S_\ell(\mu,1,4)-c_{-1}(\mu,1,4)\ell-c_0(\mu,1,4)-\frac{c_1(\mu,1,4)}{\ell}\right] \\ &\quad + c_{-1}(\mu,1,4)\zeta(-1)+c_0(\mu,1,4)\zeta(0), \end{aligned} \tag{B.50}$$

and

$$c_1(\mu,m,d)\zeta(5-d) \underset{d\to 4}{=} \frac{c_{1P}(\mu,m)}{4-d}+c_{1F}(\mu,m), \tag{B.51}$$

with

$$c_{1P}(\mu,m) = \frac{m^2+2m^4+\mu^2-3m^2\mu^2-\mu^4}{2(3\mu^2-m^2)}, \tag{B.52}$$

$$c_{1F}(\mu,m) = \frac{12m^4-5\mu^2-6\mu^4-m^2(18\mu^2+5)}{12(3\mu^2-m^2)}. \tag{B.53}$$

In the theory at hand, equation (2.94) takes the form

$$\mu^2-m^2=\frac{n\lambda}{4\mu\Omega_{d-1}}. \tag{B.54}$$

This implies

$$\frac{\partial\mu}{\partial\lambda}=\frac{n}{4\Omega_{d-1}(3\mu^2-m^2)}, \tag{B.55}$$

which yields

$$\begin{aligned} \left\langle n|(\bar\phi\phi)(0,\vec n_d)|n\right\rangle &= \Bigg\{\frac{n}{2\mu\Omega_{d-1}}-\frac{5}{16\pi^2}\frac{1}{4-d}\frac{\lambda^2n^2}{8\Omega_{d-1}^2\mu^2(3\mu^2-m^2)} \\ &\quad +\frac{1}{(d-2)\Omega_{d-1}}\left(R(\mu_*)+\frac{c_{1P}(\mu,m)}{4-d}+c_{1F}(\mu_*,1)\right)\Bigg\}_{\lambda_*}. \end{aligned} \tag{B.56}$$

Taking into account normalization (3.5) and expanding in λ we get

$$\begin{aligned} \lambda_{\bar\phi\phi} &= Z^{-1}_{\bar\phi\phi}\left\langle n|(\bar\phi\phi)(0,\vec n_d)|n\right\rangle \\ &= \Bigg\{\frac{n(d-2)}{2\mu}+\frac{\lambda n(d-2)}{16\pi^2\mu(4-d)}+\frac{\lambda n(d-2)}{32\pi^2\mu}(1+\gamma+\log\pi) \\ &\quad -\frac{5}{16\pi^2}\frac{1}{4-d}\frac{\lambda^2n^2}{8\Omega_{d-1}^2\mu^2(3\mu^2-m^2)}+R(\mu_*)+\frac{c_{1P}(\mu,m)}{4-d}+c_{1F}(\mu_*,1)\Bigg\}_{\lambda_*}. \end{aligned} \tag{B.57}$$

Using (B.54) to substitute λn, we can gather the three order $\frac{1}{4-d}$ terms, then expand μ,

m and Ω_{d-1} in $4-d$ to show cancellation of divergences and get a finite part

$$\frac{1}{4-d}\left(\frac{(d-2)\Omega_{d-1}(\mu^2-m^2)}{4\pi^2}-\frac{5\Omega_{d-1}(d-2)(\mu^2-m^2)^2}{8\pi^2(3\mu^2-m^2)}+c_{1P}(\mu,m)\right)$$
$$\xrightarrow{d\to 4}\quad\frac{2(\mu_*^2+1)-(\gamma+\log\pi)(\mu_*^4+2\mu_*^2-3)}{4(3\mu_*^2-1)}. \tag{B.58}$$

Some $\gamma+\log\pi$ appeared from

$$\frac{1}{\Omega_3}\left.\frac{\partial\Omega_{d-1}}{\partial d}\right|_{d=4}=\frac{1}{2}\left(\gamma+\log\pi-1\right). \tag{B.59}$$

We can as well substitute λn in the term

$$\frac{\lambda n(d-2)}{32\pi^2\mu}(1+\gamma+\log\pi)\ \xrightarrow{d\to 4}\ \frac{\mu_*^2-1}{2}(1+\gamma+\log\pi). \tag{B.60}$$

We see there is not yet full cancellation of $\gamma+\log\pi$ terms. The reason is that the leading order term (3.8), which is enhanced by n, contains $\mu(\lambda_* n,d)$ which we still have to expand in $4-d$, bringing $n(4-d)$ contributions at NLO. To this end, we can express the derivative of μ with respect to d from (2.95)

$$\left.\frac{\partial\mu}{\partial d}\right|_{d=4}=\frac{\mu}{3\mu^2-1}\left.\left[1-\frac{1}{\Omega_3}(\mu^2-1)\frac{\partial\Omega_{d-1}}{\partial d}\right]\right|_{d=4} \tag{B.61}$$

and use (B.59). This yields

$$\frac{n(d-2)}{2\mu}=\frac{n}{\mu_*}-\frac{(4-d)n\left(\mu_*^2-1\right)\left(2+\gamma+\log\pi\right)}{2\mu_*\left(3\mu_*^2-1\right)}. \tag{B.62}$$

Now that $\frac{1}{4-d}$ poles have cancelled and everything has been expanded to relevant order in $4-d$, we can take the theory at the fixed point (2.8). This yields for the previous equation

$$\frac{n(d-2)}{2\mu}=\frac{n}{\mu_*}-\frac{5\left(\mu_*^2-1\right)^2\left(2+\gamma+\log\pi\right)}{4(3\mu_*^2-1)}. \tag{B.63}$$

Putting everything together, we notice all $\gamma+\log\pi$ terms cancel, and we get the final result

$$\lambda_{\bar\phi\phi}=\frac{n}{\mu_*}-\frac{2\mu_*^4-7\mu_*^2+3}{2(3\mu_*^2-1)}+R(\mu_*)+c_{1F}(\mu_*,1). \tag{B.64}$$

Plugging explicit expressions from (B.50) and (B.53) results in

$$\lambda_{\bar\phi\phi}=\frac{n}{\mu_*}+\frac{2(3\mu_*^2+1)}{\left[2(3\mu_*^2-1)\right]^{3/2}}-\frac{3\mu_*^4-2\mu_*^2+3}{2(3\mu_*^2-1)}+\sum_{\ell=1}^{\infty}\left[S_\ell(\mu_*)-c_{-1}(\mu_*)\ell-c_0(\mu_*)-\frac{c_1(\mu_*)}{\ell}\right], \tag{B.65}$$

where we noted

$$S_\ell(\mu) = S_\ell(\mu, 1, 4), \qquad c_i(\mu) = c_i(\mu, 1, 4). \tag{B.66}$$

C Feynman diagram computation of $\Delta_{(\bar{\phi}\phi)^k}$

We compute diagrammatically the one-loop anomalous dimension of $(\bar{\phi}\phi)^k$ in theory (2.1) in $d = 4 - \varepsilon$. As always we consider the MS renormalization of operators, in the following momentum-space correlator[1]:

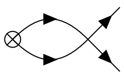
$$\langle (\bar{\phi}\phi)^k \bar{\phi}(p) \cdots \phi(p) \cdots \rangle = Z_{(\bar{\phi}\phi)^k} Z_\phi^{2k} \langle \left[(\bar{\phi}\phi)^k \right] [\bar{\phi}](p) \cdots [\phi](p) \cdots \rangle \,, \tag{C.1}$$

where there are k insertions of field $\phi(p)$ and $\bar{\phi}(p)$. The field renormalization factor Z_ϕ has no one-loop contribution so we consider it equal to 1. The bare $(\bar{\phi}\phi)^k$ operator is normalized :

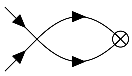
$$\otimes \;=\; 1 \,. \tag{C.2}$$

We do not draw exterior lines in the diagrams.

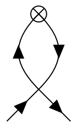
There are three diagrams at one-loop level, of which we compute the divergent part:



$=$



$$= \frac{k(k-1)}{4}(-\lambda)\frac{1}{8\pi^2\varepsilon} + O(\varepsilon^0)$$



$$= k^2(-\lambda)\frac{1}{8\pi^2\varepsilon} + O(\varepsilon^0) \,. \tag{C.3}$$

Summing all diagrams yields

$$\langle (\bar{\phi}\phi)^k \bar{\phi}(p) \cdots \phi(p) \cdots \rangle = 1 - \frac{k(3k-1)\lambda}{16\pi^2\varepsilon} + O(\lambda\varepsilon^0, \lambda^2) \,, \tag{C.4}$$

from which we get

$$Z_{(\bar{\phi}\phi)^k} = 1 - \frac{k(3k-1)\lambda}{16\pi^2\varepsilon} + O(\lambda^2) \,. \tag{C.5}$$

[1] To be more precise, the operator $(\bar{\phi}\phi)^k$ mixes with other operators [101]. However, since this operator is a primary of the critical theory, the mixing cancels in that case. This means we can neglect the mixing when diagramatically computing the anomalous dimension off-criticality.

The one-loop anomalous dimension is then given by

$$\gamma_{(\bar{\phi}\phi)^k} = -\lambda\varepsilon\frac{\partial \log Z_{(\bar{\phi}\phi)^2}}{\partial\lambda} = \frac{k(3k-1)\lambda}{16\pi^2} + O(\lambda^2)\,. \tag{C.6}$$

At the Wilson-Fisher fixed point (2.8) the dimension is

$$\Delta_{(\bar{\phi}\phi)^k} = 2k\left(\frac{d}{2}-1\right) + \gamma_{(\bar{\phi}\phi)^k} = 2k + \frac{3k(k-2)}{5}\varepsilon + O(\varepsilon^2)\,. \tag{C.7}$$

D Free Theory construction and counting of primaries

D.1 Explicit expressions for spin 2 and 3 primaries

In this appendix we give explicit expression for spin-ℓ primary operators associated to the primary states (4.63) using operator-state correspondence (4.25). As discussed in section 4.2.2, it is assumed products of (derivatives of) ϕ are normal-ordered and evaluated at the origin.

The counting of primaries given in (4.77) indicates there is one primary of spin $\ell = 0, 2, 3$ but none of spin 1.

Spin 0is trivial, for we have only one state given by (4.57).

Spin 1Explicitly, we have three states

$$\left(a_{00}^{\dagger}\right)^{n-1} a_{1,m}^{\dagger}|0\rangle, \tag{D.1}$$

which correspond to

$$\begin{aligned}
\left(a_{00}^{\dagger}\right)^{n-1} a_{1,1}^{\dagger}|0\rangle &= -(4\pi)^{n/2}\phi^{n-1}\partial_{-}\phi|0\rangle, & \text{(D.2)}\\
\left(a_{00}^{\dagger}\right)^{n-1} a_{1,0}^{\dagger}|0\rangle &= (4\pi)^{n/2}\phi^{n-1}\partial_{0}\phi|0\rangle, & \text{(D.3)}\\
\left(a_{00}^{\dagger}\right)^{n-1} a_{1,-1}^{\dagger}|0\rangle &= (4\pi)^{n/2}\phi^{n-1}\partial_{+}\phi|0\rangle. & \text{(D.4)}
\end{aligned}$$

It is straightforward to show using (4.44), (4.48) and (4.50) that those states, hence operators, are descendants, as we would expect since these operators can be written as derivatives of ϕ^n.

Spin 2We can write two spin-2 operators by combining n fields ϕ and two derivatives in a traceless and symmetric way

$$O^{(2,1)}_{\mu\nu} = \phi^{n-1}\left(\partial_\mu\partial_\nu\phi - \frac{\delta_{\mu\nu}}{3}\partial^2\phi\right), \quad O^{(2,2)}_{\mu\nu} = \phi^{n-2}\left(\partial_\mu\phi\partial_\nu\phi - \frac{\delta_{\mu\nu}}{3}(\partial\phi)^2\right). \tag{D.5}$$

One linear combination of these is the spin-two primary.

To give examples of primary states with non-maximal J_3 eigenvalue, let us repeat the method of section 4.2.4 in this simple case. We consider a state

$$|n;2,0\rangle_A = \alpha_1\left(a^\dagger_{00}\right)^{n-1} a^\dagger_{2,0}|0\rangle + \alpha_2\left(a^\dagger_{00}\right)^{n-2}\left(a^\dagger_{1,0}\right)^2|0\rangle + \beta_2\left(a^\dagger_{00}\right)^{n-2} a^\dagger_{1,-1}a^\dagger_{1,1}|0\rangle. \tag{D.6}$$

Acting with $K_\pm$ and K_0 we see that this state is primary provided $\alpha_2 = \beta_2 = -\alpha_1$. It follows from (4.25) that

$$\begin{aligned}
|n;2,0\rangle_A &= (4\pi)^{n/2}\alpha_1\left[\frac{1}{3}\phi^{n-1}\left(\partial_0^2\phi - \partial_+\partial_-\phi\right) - \phi^{n-2}\left(\partial_0\phi\partial_0\phi - \partial_+\phi\partial_-\phi\right)\right]|0\rangle && \text{(D.7)}\\
&= (4\pi)^{n/2}\frac{\alpha_1}{2}\left[\frac{1}{3}\phi^{n-1}\left(3\partial_0^2\phi - \partial^2\phi\right) - \phi^{n-2}\left(3\partial_0\phi\partial_0\phi - (\partial\phi)^2\right)\right]|0\rangle && \text{(D.8)}\\
&= (4\pi)^{n/2}\frac{3\alpha_1}{2}\left[\frac{1}{3}\phi^{n-1}\left(\partial_0^2\phi - \frac{1}{3}\partial^2\phi\right) - \phi^{n-2}\left(\partial_0\phi\partial_0\phi - \frac{1}{3}(\partial\phi)^2\right)\right]|0\rangle && \text{(D.9)}\\
&= (4\pi)^{n/2}\frac{3\alpha_1}{2}\left(\frac{1}{3}O^{(2,1)}_{00} - O^{(2,2)}_{00}\right)|0\rangle. && \text{(D.10)}
\end{aligned}$$

Hence, we conclude the operator $O^{(2,1)}_{\mu\nu} - 3O^{(2,2)}_{\mu\nu}$ is primary.

Spin 3In this case we have an ansatz

$$\begin{aligned}
|n;3,0\rangle_A &= \alpha_1\left(a^\dagger_{00}\right)^{n-1} a^\dagger_{3,0}|0\rangle && \text{(D.11)}\\
&+ \alpha_2\left(a^\dagger_{00}\right)^{n-2} a^\dagger_{2,0}a^\dagger_{1,0}|0\rangle + \beta_2\left(a^\dagger_{00}\right)^{n-2} a^\dagger_{2,1}a^\dagger_{1,-1}|0\rangle + \gamma_2\left(a^\dagger_{00}\right)^{n-2} a^\dagger_{2,-1}a^\dagger_{1,1}|0\rangle\\
&+ \alpha_3\left(a^\dagger_{00}\right)^{n-3}\left(a^\dagger_{1,0}\right)^3|0\rangle + \beta_3\left(a^\dagger_{00}\right)^{n-3} a^\dagger_{1,-1}a^\dagger_{1,1}a^\dagger_{1,0}|0\rangle.
\end{aligned}$$

As before, acting with K and imposing that the state be primary we get

$$\begin{aligned}
|n;3,0\rangle_A &= (4\pi)^{n/2}\alpha_1\left[\left(a^\dagger_{00}\right)^{n-1} a^\dagger_{3,0}\right. && \text{(D.12)}\\
&- 3\left(a^\dagger_{00}\right)^{n-2} a^\dagger_{2,0}a^\dagger_{1,0} - \sqrt{2}\left(a^\dagger_{00}\right)^{n-2} a^\dagger_{2,1}a^\dagger_{1,-1} - \sqrt{2}\left(a^\dagger_{00}\right)^{n-2} a^\dagger_{2,-1}a^\dagger_{1,1}\\
&+ \left. 2\left(a^\dagger_{00}\right)^{n-3}\left(a^\dagger_{1,0}\right)^3 + 6\left(a^\dagger_{00}\right)^{n-3} a^\dagger_{1,-1}a^\dagger_{1,1}a^\dagger_{1,0}\right]|0\rangle,
\end{aligned}$$

which corresponds to

$$|n;3,0\rangle_A = (4\pi)^{n/2}\alpha_1\left(\frac{1}{6}O^{(3,1)}_{000} - \frac{5}{6}O^{(3,2)}_{000} + 5O^{(3,3)}_{000}\right)|0\rangle = (4\pi)^{n/2}\frac{\alpha_1}{6}\left(O^{(3,1)}_{000} - 5O^{(3,2)}_{000} + 30O^{(3,3)}_{000}\right)|0\rangle, \tag{D.13}$$

with

$$O^{(3,1)}_{\mu\nu\lambda} = \phi^{n-1}\partial_\mu\partial_\nu\partial_\lambda\phi - \frac{1}{5}\left(\delta_{\mu\nu}\partial_\lambda + \delta_{\mu\lambda}\partial_\nu + \delta_{\lambda\nu}\partial_\mu\right)\partial^2\phi, \tag{D.14}$$

$$\begin{aligned} O^{(3,2)}_{\mu\nu\lambda} = {} & \phi^{n-2}\partial_\mu\partial_\nu\phi\partial_\lambda\phi + \phi^{n-2}\partial_\mu\phi\partial_\nu\partial_\lambda\phi + \phi^{n-2}\partial_\nu\phi\partial_\mu\partial_\lambda\phi \\ & -\frac{\delta_{\mu\nu}}{5}\left(\partial_\lambda\phi\partial^2\phi + \partial_\lambda(\partial\phi)^2\right) - \frac{\delta_{\mu\lambda}}{5}\left(\partial_\nu\phi\partial^2\phi + \partial_\nu(\partial\phi)^2\right) - \frac{\delta_{\lambda\nu}}{5}\left(\partial_\mu\phi\partial^2\phi + \partial_\mu(\partial\phi)^2\right), \end{aligned} \tag{D.15}$$

$$O^{(3,3)}_{\mu\nu\lambda} = \phi^{n-3}\partial_\mu\phi\partial_\nu\phi\partial_\lambda\phi - \frac{1}{5}\left(\delta_{\mu\nu}\partial_\lambda\phi + \delta_{\mu\lambda}\partial_\nu\phi + \delta_{\lambda\nu}\partial_\mu\phi\right)(\partial\phi)^2. \tag{D.16}$$

the three spin-3 operators. We conclude operator $O^{(3,1)}_{\mu\nu\lambda} - 5O^{(3,2)}_{\mu\nu\lambda} + 30O^{(3,3)}_{\mu\nu\lambda}$ is primary.

D.2 Counting primaries with $\ell > n$

Here we give several examples of the formula (4.77) presented in main text.

First, let us consider the case of charge 2. We detail the partitions mentioned in the argument of the main text. We do this for the examples of spin 4 and 5:

- $\mathrm{Prim}(4,2) = 1$

$p(4,2)$	$p(3,2)$	$p^*(4,2)$	$p^*(3,2)$	$\mathrm{Prim}(4,2)$
(4)	(3)	$(1,1,1,1)$	$(1,1,1)$	$\times$
$(3,1)$	$(2,1)$	$(2,1,1)$	$(2,1)$	$\times$
$(2,2)$		$(\mathbf{2},\mathbf{2})$		$\checkmark$

(D.17)

- $\mathrm{Prim}(5,2) = 0$

$p(5,2)$	$p(4,2)$	$p^*(5,2)$	$p^*(4,2)$	$\mathrm{Prim}(5,2)$
(5)	(4)	$(1,1,1,1,1)$	$(1,1,1,1)$	$\times$
$(4,1)$	$(3,1)$	$(2,1,1,1)$	$(2,1,1)$	$\times$
$(3,2)$	$(2,2)$	$(2,2,1)$	$(2,2)$	$\times$

(D.18)

In general, we find there is one primary operator for even spins and none for odd spins.

Let us now give a more involved example with charge 3 and spin 8, resulting in

$\text{Prim}(8,3) = 2$.

$$
\begin{array}{l|l||l|l||c}
p(8,3) & p(7,3) & p^*(8,3) & p^*(7,3) & \text{Prim}(8,3) \\
\hline
(8) & (7) & (1,1,1,1,1,1,1,1) & (1,1,1,1,1,1,1) & \times \\
(7,1) & (6,1) & (2,1,1,1,1,1,1) & (2,1,1,1,1,1) & \times \\
(6,2) & (5,2) & (2,2,1,1,1,1) & (2,2,1,1,1) & \times \\
(6,1,1) & (5,1,1) & (3,1,1,1,1,1) & (3,1,1,1,1) & \times \\
(5,3) & (4,3) & (2,2,2,1,1) & (2,2,2,1) & \times \\
(5,2,1) & (4,2,1) & (3,2,1,1,1) & (3,2,1,1) & \times \\
(4,4) & & (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2}) & & \checkmark \\
(4,3,1) & (3,3,1) & (3,2,2,1) & (3,2,2) & \times \\
(4,2,2) & (3,2,2) & (3,3,1,1) & (3,3,1) & \times \\
(3,3,2) & & (\mathbf{3},\mathbf{3},\mathbf{2}) & & \checkmark
\end{array}
\tag{D.19}
$$

For arbitrary spin and charge $n = 3$ an explicit expression is given by

$$
\text{Prim}(\ell,3) = \begin{cases} \left\lfloor \frac{\ell}{6} \right\rfloor, & \text{if } \ell = 6p+1 \text{ for some } p \in \mathbb{N}, \\ \left\lfloor \frac{\ell}{6} \right\rfloor + 1, & \text{if } \ell \neq 6p+1 \text{ for all } p \in \mathbb{N}. \end{cases} \tag{D.20}
$$

In general, the number of primaries can be found from

$$
\sum_{\ell=0}^{\infty} \text{Prim}(\ell,n) x^{\ell} = \prod_{k=2}^{n} \frac{1}{(1-x^k)}. \tag{D.21}
$$

D.3 Some asymptotics of Gaunt coefficients

In the main text we are interested in the expansion at large ℓ of $A^{\dagger}_{\ell\ell}$. Thus we provide here some formulas for the asymptotics of relevant Gaunt coefficients. We use special cases of $3j$ symbols [145]

$$
\begin{pmatrix} \ell & \ell_1 & \ell_2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{cases} 0 & L \text{ odd}, \\ (-1)^{\frac{L}{2}} \sqrt{\frac{(L-2\ell)!(L-2\ell_1)!(L-2\ell_2)!}{(L+1)!}} \frac{\left(\frac{L}{2}\right)!}{\left(\frac{L-2\ell}{2}\right)!\left(\frac{L-2\ell_1}{2}\right)!\left(\frac{L-2\ell_2}{2}\right)!} & L \text{ even}, \end{cases} \tag{D.22}
$$

$$
\begin{pmatrix} \ell & \ell_1 & \ell_2 \\ \ell & -\ell-m_2 & m_2 \end{pmatrix} = (-1)^{\ell-\ell_1-m_2} \sqrt{\frac{(2\ell)!(L-2\ell)!(\ell+\ell_1+m_2)!(\ell_2-m_2)!}{(L+1)!(L-2\ell_1)!(L-2\ell_2)!(-\ell+\ell_1-m_2)!(\ell_2+m_2)!}}, \tag{D.23}
$$

where $L = \ell + \ell_1 + \ell_2$. We can use Stirling formula to estimate these at large spin. If we consider ℓ to be large, due to triangle inequality (4.110), at least one of ℓ_1, ℓ_2 has to be of order ℓ.

If we assume $\ell_1 \sim \ell_2 \sim \ell$, we have

$$\frac{C^{\ell,\ell_1,\ell_2}_{\ell,-\ell-m_2,m_2}}{\sqrt{\omega_\ell \omega_{\ell_1} \omega_{\ell_2}}} \overset{\ell_1\sim\ell_2\sim\ell}{\underset{\ell\to\infty}{\longrightarrow}} g_1\!\left(\frac{\ell_1}{\ell}, \frac{\ell_2}{\ell}, \frac{m_2}{\ell}\right)^{\ell/2} h_1\!\left(\frac{\ell_1}{\ell}, \frac{\ell_2}{\ell}, \frac{m_2}{\ell}\right) \left(\ell^{-7/4} + O(\ell^{-11/4})\right), \tag{D.24}$$

where

$$g_1(x,y,z) = \frac{4(-1)^{1-x+y-2z}(x+y-1)^{x+y-1}(x+z+1)^{x+z+1}(y-z)^{y-z}}{(x-y+1)^{x-y+1}(y-x+1)^{y-x+1}(x+y+1)^{x+y+1}(x-z-1)^{x-z-1}(y+z)^{y+z}}, \tag{D.25}$$

whose absolute value is bounded by 1 and for each pair x, y there is one unique z such that $|g_1(x,y,z)| = 1$, namely $z = \frac{x^2-y^2-1}{2}$, and

$$h_1(x,y,z) = \frac{2}{\pi^{5/4}} \frac{(y-z)^{1/4}(1+x+z)^{1/4}}{(x-y+1)^{1/2}(y-x+1)^{1/2}(x+y+1)(x-z-1)^{1/4}(y+z)^{1/4}}. \tag{D.26}$$

Hence, for each ℓ_1, ℓ_2 there is only one m_1, m_2 for which the coefficient is not exponentially suppressed, and for that choice (D.24) is of order $\ell^{-7/4}$.

On the other hand if we assume $(\ell_1 - \ell) \sim \ell_2 \sim 1$ (the case $\ell_2 \sim \ell, \ell_1 \sim 1$ will of course give similar result)

$$\frac{C^{\ell,\ell_1,\ell_2}_{\ell,-\ell-m_2,m_2}}{\sqrt{\omega_\ell \omega_{\ell_1} \omega_{\ell_2}}} \overset{\ell_1\sim\ell}{\underset{\ell\to\infty}{\longrightarrow}} (-1)^\ell h_2(\ell_1-\ell, \ell_2, m_2)\, \ell^{\frac{\ell-\ell_1+m_2}{2}} \left(\ell^{-1} + O(\ell^{-2})\right), \tag{D.27}$$

with

$$h_2(x,y,z) = \frac{(-1)^{\frac{x+y-2z}{2}} 2^{\frac{x-2y+z-1}{2}} (y-x)! \sqrt{(y-z)!}}{\sqrt{\pi} \left(\frac{x+y}{2}\right)! \left(\frac{y-x}{2}\right)! \sqrt{(y+z)!(-x-z)!}}. \tag{D.28}$$

Hence (D.27) is least suppressed in case $m_2 = \ell_1 - \ell$ which is its maximum allowed value since $|m_1| = |-\ell - m_2| \leq \ell_1$, and in that case (D.27) is of order ℓ^{-1}.

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