A convex set of robust $\mathcal{D}-$stabilizing controllers using Cauchy's argument principle

Philippe Schuchert and Alireza Karimi

Laboratoire d'Automatique
Ecole Polytechinique Fédérale de Lausanne (EPFL), Switzerland

Abstract: A new approach is presented to obtain a convex set of robust $\mathcal{D}-$stabilizing fixed structure controllers, relying on Cauchy's argument principle. A convex set of $\mathcal{D}-$stabilizing controllers around an initial $\mathcal{D}-$stabilizing controller for a multi-model set is represented by an infinite set of Linear Matrix Inequalities (LMIs). By appropriate sampling of the $\mathcal{D}-$stability boundary, a Semi-Definite Programming (SDP) is proposed that can be integrated in other synthesis approaches to ensure $\mathcal{D}-$stability along other design specifications. To showcase utility of the proposed approach, two different examples are given: a boost converter with multi-model uncertainty and a laser-beam system modeled by an identified finite impulse response.

Keywords: Controller constraints and structure, Convex optimization, Robust controller synthesis, Data-based control

1. INTRODUCTION

Poles of Linear Time-Invariant (LTI) systems can be directly linked to their stability and transient dynamics (Ackermann et al., 1993). Time-domain performance, such as damping or overshoot corresponds for many systems to a domain in the complex plane where the poles are located. Guaranteeing such specifications on a closed-loop system is commonly referred as pole clustering or $\mathcal{D}-$stability, where $\mathcal{D}$ is the desired closed-loop pole region.

The $\mathscr{H}_2$ and $\mathscr{H}_\infty$ full-order controller synthesis problem can be formulated using linear matrix inequalities (LMIs) (Gahinet and Apkarian, 1994; Scherer et al., 1997), and can be augmented with $\mathcal{D}-$stability constraints (Chilali and Gahinet, 1996; Chilali et al., 1999). This requires the region $\mathcal{D}$ to be described as intersections of LMI regions, where regions such as half-planes, disks, cones and parabolas can be readily described (Henrion et al., 2001). Any convex region symmetric w.r.t. the real axis can be approximated arbitrary well using intersection of the previously mentioned LMI regions, at the cost of using possibly a large number of LMI regions. Non-convex regions can be approximated by computing a convex inner approximation (e.g. Rosinová and Holič, 2014; Wisniewski et al., 2019) at the cost of additional conservatism. Finding the correct LMI regions is a challenging task, and requires the existence of a single shared Lyapunov matrix between the different LMI regions, which adds again some conservatism (Chilali et al., 1999). For non-LMI approaches, such as the one in Doyle et al. (1988), where a controller is obtained by solving algebraic Riccati equations, only limited pole clustering techniques are available. The algebraic Riccati equations used to derive the optimal controller can be modified (Furuta and Kim, 1987; Haddad and Bernstein, 1992; Garcia and Bernussou, 1995; Hench et al., 1998) to ensure $\mathcal{D}-$stability, but often accommodating only a single LMI region.

Recently, a methods to solve the $\mathscr{H}_2$ or $\mathscr{H}_\infty$ control problem using only the frequency response data of a multi-variable system have been proposed in Karimi and Kammer (2017). In this method, an inner-approximation of the performance criterion is derived around a given initial stabilizing controller, and the optimal controller is obtained by solving iteratively a sequence of convex optimization problems. Since the method is based only on the frequency response data, the stability of the closed-loop system is guaranteed via the Nyquist stability criterion. A semi-infinite programming is proposed that guarantees the number of encirclement of the Nyquist plot around the critical point is equal to the number of unstable poles of the open-loop transfer function.

The main contribution of this paper is to extend the results of Karimi and Kammer (2017) to the design of robust $\mathcal{D}-$stabilizing controllers for multiple-input multiple-output (MIMO) systems where the domain $\mathcal{D}$ is not necessarily a convex region. A second contribution is the mitigation of numerical problems arising when sampling the stability boundary in the single-input single-output (SISO) case. The results are given for the case that the transfer function of the system is known as well as in a data-driven setting when only a set of data is available. All the developments are given in this paper for discrete-time systems, but can be applied to continuous-time systems with only minor modifications.

This paper is organized as follows: preliminaries and notions are given in Section 2. Theoretical developments are presented in Section 3. Implementation issues are discussed in Section 4. Simulation and experimental results are given in Section 5 and conclusions in Section 6.

⋆ This work is supported by INNOSUISSE grant 48289.1 IP-ENG
⋆⋆Corresponding author: alireza.karimi@epfl.ch

2. PRELIMINARIES

Notation: The imaginary unit is denoted j. The real and the imaginary parts of a complex valued matrix are denoted $\Re\{\cdot\}$ and $\Im\{\cdot\}$, respectively. The Hermitian transpose is shown by $(\cdot)^*$. The argument of a complex scalar is $\angle\{\cdot\}$. A simply connected set in complex plane is denoted by $\mathcal{D}$, and $\partial\mathcal{D}$ is its boundary. The function composition $f(g)$ is denoted $f \circ g$. The positive definiteness of a symmetric matrix A is shown by $A \succ 0$.

Cauchy's argument principle and winding number: A fundamental result in complex analysis is Cauchy's argument principle (Howie, 2003) that relates the number of poles and zeros of a meromorphic function f to a contour integral. If P and Z denotes the number of poles and zeros inside $\mathcal{D}$ respectively, then the argument principle states:

$$\frac{1}{2\pi j}\oint_{\mathscr{B}} \frac{f'(z)}{f(z)} dz = Z - P \tag{1}$$

where $\mathscr{B}$ is a counterclockwise (CCW) parametrization of $\partial\mathcal{D}$, and f'/f the logarithmic derivative of f. It is assumed that f has no poles or zeros on $\partial\mathcal{D}$. Similarly, define the winding number around the origin of a closed curve C in the complex plane not passing through the origin as:

$$\text{wno}\{C\} = \frac{1}{2\pi j}\oint_{C} \frac{dz}{z} \tag{2}$$

If C is the image of f evaluated on $\mathscr{B}$, then the CCW winding number of C can be directly related to the number of zeros and poles of f inside $\mathcal{D}$:

$$\text{wno}\{C\} = \text{wno}\{f \circ \mathscr{B}\} = Z - P \tag{3}$$

Two properties of the winding number function are:

$$\text{wno}\{(f \cdot g) \circ \mathscr{B}\} = \text{wno}\{f \circ \mathscr{B}\} + \text{wno}\{g \circ \mathscr{B}\} \tag{4}$$

$$\text{wno}\{g^* \circ \mathscr{B}\} = -\,\text{wno}\{g \circ \mathscr{B}\} \tag{5}$$

where f and g are non-zero meromorphic functions and $\mathscr{B}$ is a closed curve not passing through poles or zeros of both functions.

Nyquist stability criterion: The Cauchy's argument principle has direct application in control theory via the Nyquist stability criterion (Skogestad and Postlethwaite, 2001). The Nyquist stability criterion is a graphical test which can be used to check the stability of a closed-loop system by counting the winding number of a specific curve. Although usually framed as counting the number of unstable poles of the closed-loop, we will present it as counting the number of stable poles. The Nyquist contour, denoted $\mathscr{B}_{\mathcal{N}}$, is defined for discrete-time systems as the CCW oriented boundary of the unit disk $\mathcal{N} = \{z \in \mathbb{C} \mid |z| < 1\}$.

Let G be a proper MIMO system with n_y outputs and n_u inputs, K a proper controller with n_u outputs and n_y inputs, and define $L = GK$ as the loop transfer function and assume that it has no unstable hidden modes. If some poles of L lies exactly on the unit circle, the contour must be locally deformed with small detours to avoid these poles. Under these conditions, the closed-loop stability is given by the following theorem:

Theorem 1. (Nyquist stability criterion) The closed-loop system with loop transfer function L and negative unity feedback is stable if and only if the plot of $\det(I + L) \circ \mathscr{B}_{\mathcal{N}}$ does not pass through the origin and makes P_u CCW winding around the origin, where P_u is the number of unstable poles of L.

Proof. The poles of the closed-loop are the zeros of $\det(I + L)$. The Nyquist contour is the CCW oriented boundary of the unit circle. Since $\mathscr{B}_{\mathcal{N}}$ is a simply-connected curve and $\det(I + L)$ meromorphic, then the Cauchy's argument principle can be directly applied :

$$\text{wno}\{\det(I + L) \circ \mathscr{B}_{\mathcal{N}}\} = Z - P \tag{6}$$

where Z and P are, respectively, the number of stable zeros and poles of $\det(I + L)$. If we define η_L as the number of zeros (stable and unstable) of $\det(I+L)$, then $P_u = \eta_L - P$. Therefore $\text{wno}\{\det(I + L) \circ \mathscr{B}_{\mathcal{N}}\} = Z + P_u - \eta_L$ and the closed-loop system is stable if and only if $Z = \eta_L$ that leads to $\text{wno}\{\det(I + L) \circ \mathscr{B}_{\mathcal{N}}\} = P_u$. ■

$\mathcal{D}-$stability: The simply-connected open set of desired pole location will be denoted hereafter $\mathcal{D}$, and $\mathscr{B}$ a CCW oriented parametrization of its boundary $\partial\mathcal{D}$. All the results are presented for the discrete-time case, but it is directly applicable to continuous-time systems simply by changing the set $\mathcal{D}$. For example, for the discrete-time systems, poles with magnitude less than 0.925 are given by the set $\mathcal{D} = \{z \in \mathbb{C} \mid |z| < 0.925\}$.

3. DEVELOPMENTS

The Nyquist stability criterion requires knowledge of the unstable poles from the loop transfer function. If a dynamic controller $K = K(z)$ is used, $P_u = P_u^G + P_u^K$ includes both unstable poles of the controller P_u^K and unstable poles of the system P_u^G. Assume the controller can be expressed as a matrix transfer function $K(z) = X(z)Y^{-1}(z)$, where $X(z)$ and $Y(z)$ are polynomial matrices in the z-transform variable. The loop transfer function is $L = GXY^{-1}$ and the closed-loop poles are the zeros of $I + GXY^{-1} = (Y + GX)Y^{-1}$. Let $\tilde{L}(z)$, linear in controller parameters, be defined as:

$$\tilde{L}(z) = Y(z) + G(z)X(z) \tag{7}$$

We propose a modified version of the Nyquist stability criterion based only on the linearly parameterized function $\tilde{L}$ and without any information about the number of unstable poles of K. Denote η the number of zeros (order) of $\det(Y)$, then $\text{wno}\{\det(Y) \circ \mathscr{B}_{\mathcal{N}}\} = \eta - P_u^K$ corresponds to the number of stable zeros of $\det(Y)$. Applying the Cauchy's argument principle to $\det(I + L)$ results in:

$$\begin{aligned}
\text{wno}\{\det(I + L) \circ \mathscr{B}_{\mathcal{N}}\} &= \text{wno}\{\det(I + GXY^{-1}) \circ \mathscr{B}_{\mathcal{N}}\} \\
&= \text{wno}\{\det(Y + GX) \circ \mathscr{B}_{\mathcal{N}}\} \\
&\quad - \text{wno}\{\det(Y) \circ \mathscr{B}_{\mathcal{N}}\} \\
&= \text{wno}\{\det(\tilde{L}) \circ \mathscr{B}_{\mathcal{N}}\} - (\eta - P_u^K)
\end{aligned}$$

Therefore, taking into account $P_u = P_u^G + P_u^K$, if

$$\text{wno}\{\det(\tilde{L}) \circ \mathscr{B}_{\mathcal{N}}\} = \eta + P_u^G \tag{8}$$

holds, the closed-loop system is stable.

The stability criterion in (8) does not require any information about the number of unstable poles of controller P_u^K but it depends on η, the order of the controller. The Nyquist stability criterion can be generalized to check for $\mathcal{D}-$stability by replacing $\mathscr{B}_{\mathcal{N}}$ with any other CCW oriented contour $\mathscr{B}$.

Let $K_c = X_c Y_c^{-1}$ be a known $\mathcal{D}-$stabilizing controller, with the same order as K, and $\tilde{L}_c$ defined as

$$\tilde{L}_c(z) = Y_c(z) + G(z)X_c(z) \tag{9}$$

Then the following theorem represents a convex set of $\mathcal{D}-$stabilizing controllers around K_c.

Theorem 2. Given a discrete-time LTI system $G(z)$ and a $\mathcal{D}-$stabilizing controller $K_c(z)$, then following set of LMIs

$$\tilde{L}(z)\,\tilde{L}_c^*(z) + \tilde{L}_c(z)\,\tilde{L}^*(z) \succ 0 \quad \forall z \in \partial\mathcal{D} \tag{10}$$

represents a convex set of $\mathcal{D}-$stabilizing controllers if the following two assumptions hold:

(A1) $\det(Y)$ and $\det(Y_c)$ have the same order,
(A2) $\partial\mathcal{D}$ does not pass through any poles of G and K.

Proof. If (10) holds, then $\tilde{L}(z)\tilde{L}_c^*(z)$ is a non-Hermitian positive definite matrix in the sense that

$$\Re\{v^*\tilde{L}(z)\tilde{L}_c^*(z)v\} > 0 \;\forall v \neq 0 \in \mathbb{C}^{n_y}$$

As a result, all eigenvalues $\lambda_i(z), i = 1, \dots, n_y$ of $\tilde{L}(z)\tilde{L}_c^*(z)$ have positive real part for all $z \in \partial\mathcal{D}$. The determinant of a matrix is the product of its eigenvalues, and the winding number of $\det(\tilde{L}(z)\tilde{L}_c^*(z))$ when evaluated on $\mathscr{B}$ is :

$$\text{wno}\{\det(\tilde{L}\tilde{L}_c^*) \circ \mathscr{B}\} = \sum_{i=1}^{n_y} \text{wno}\,\{\lambda_i \circ \mathscr{B}\} \tag{11}$$

As every $\lambda_i \circ \mathscr{B}$ has positive real part, it resides entirely in the open-right half plane and cannot wind around the origin. Therefore

$$\text{wno}\,\{\lambda_i \circ \mathscr{B}\} = 0 \tag{12}$$

must hold. The determinant in (11) is distributive over the matrix multiplication, and $\det(\tilde{L}_c^*) = \det(\tilde{L}_c)^*$. Using this in combination with the mentioned winding number properties and (12) results ultimately in:

$$\text{wno}(\det(\tilde{L}) \circ \mathscr{B}) = \text{wno}(\det(\tilde{L}_c) \circ \mathscr{B}) \tag{13}$$

On the other hand, K_c is assumed to be a $\mathcal{D}-$stabilizing controller, therefore we have $\text{wno}(\det(\tilde{L}_c) \circ \mathscr{B}) = \eta + P_u^G$, where η is the order of $\det(Y_c)$ and P_u^G the number of poles of G outside $\mathcal{D}$. By assumption (A1), the order of $\det(Y)$ is also equal to η and $\text{wno}\{\det(Y) \circ \mathscr{B}\} = \eta - P_u^K$, where P_u^K is the number of zeros of $\det(Y)$ outside $\mathcal{D}$. Therefore, the winding number of $\det(I + L) \circ \mathscr{B}$ is given by:

$$\begin{aligned}\text{wno}\{\det(I + L) \circ \mathscr{B}\} = &\; \text{wno}\{\det(\tilde{L}) \circ \mathscr{B}\} \\ &- \text{wno}\{\det(Y) \circ \mathscr{B}\} \qquad (14)\\ = &\; \eta + P_u^G - (\eta - P_u^K) = P_u\end{aligned}$$

where $P_u = P_u^G + P_u^K$ is the number of poles of L outside $\mathcal{D}$. As a result the closed-loop system is $\mathcal{D}-$stable. ■

The following remarks are in order:

Remark 1: One of the main assumptions of the Nyquist stability criterion is that there are no unstable pole-zero cancellations in L. Note that the controllers leading to unstable pole-zero cancellation in L do not belong to the interior of the convex set represented by (10). Because by a small variation of the controller parameters the number of unstable poles of L changes. The only possible case is pole-zero cancellation on the stability boundary, which is avoided by Assumption (A2).

Remark 2: Appropriate indentations in $\partial\mathcal{D}$ must be made if G has poles on the boundary to satisfy (A2). Note that (10) must be evaluated on these indentations as well.

Remark 3: Appropriate indentations in $\partial\mathcal{D}$ must be made if $\det(Y)$ has zeros on the boundary to satisfy (A2). However it is not necessary to evaluate (10) on these additional indentations because the variation of $\tilde{L} = Y + GX$ around the zeros of Y is negligible.

If multiple models $G \in \{G_1, \dots, G_q\}$ are given and simultaneous $\mathcal{D}-$stabilization must be achieved, the constraint (10) should be added for every different model considered:

$$\tilde{L_i}(z)\,\tilde{L_i}_c^{\,*}(z) + \tilde{L_i}^{\,*}(z)\,\tilde{L}_{ic}(z) \succ 0 \quad \forall z \in \partial\mathcal{D}$$

for all $i = 1, \dots, q$, where $\tilde{L_i} = Y + G_iX$, and $\tilde{L}_{ic} = Y_c + G_iX_c$. A feasible solution always exists if an appropriate K_c is given, namely $K = K_c$.

4. PRACTICAL CONSIDERATIONS

Gridding the $\mathcal{D}-$stability boundary: The number of constraints in (10) is infinity due to the continuum of $z \in \partial\mathcal{D}$, and cannot be implemented nor solved using numerical optimization software packages. A common approach to handle such constraints is to pick a large collection $\mathcal{Z} = \{z_1, \dots, z_M\} \subset \partial\mathcal{D}$ and solve (10) only at $z \in \mathcal{Z}$ using off-the-shelf convex numerical solvers. The gridding should be dense around poles close to the boundary. Using a high-order controller will require more points than a low-order controller for the same system, as there are possibly more poles near the boundary. The SIP is solved at only a finite number of points, and it should be verified that (10) holds between points. If this is not the case, additional points are added where (10) does not hold.

For systems with real coefficients and symmetrical contours w.r.t. the real axis, only the positive imaginary part of the boundary must be constrained.

Improved conditions for SISO systems: The previously presented SIP is valid for MIMO systems. For SISO systems, better guarantees can be given in the sampled contour case, when sampling $\partial\mathcal{D}$ at only a finite number of points $\mathcal{Z}$. One can define two polygonal chains $\underline{\tilde{L}}$ and $\underline{\tilde{L}_c}$. A polygonal chain is defined as a series of line segments, connected end to end. Given M samples $z_m \in \mathcal{Z}$, each segment of the polygonal chains are defined as:

$$\underline{\tilde{L}}^m(\delta) = (1-\delta)\tilde{L}(z_m) + \delta\tilde{L}(z_{m+1}) \qquad m = 1, \dots, M$$
$$\underline{\tilde{L}_c}^m(\delta) = (1-\delta)\tilde{L}_c(z_m) + \delta\tilde{L}_c(z_{m+1}) \qquad m = 1, \dots, M$$

where $\delta \in [0, 1]$. Both chains are closed and therefore $z_{M+1} = z_1$. It is assumed that the boundary is sampled such that the polygonal chain approximates $\partial\mathcal{D}$. Moreover, it is assumed that both polygonal chains do not intersect the origin, as otherwise the winding number is not defined. If one can find a complex scalar r_m such that

$$\Re\{\underline{\tilde{L}}^m(\delta)r_m\} > 0 \quad \forall\delta \in [0, 1] \tag{15}$$
$$\Re\{\underline{\tilde{L}_c}^m(\delta)r_m\} > 0 \quad \forall\delta \in [0, 1] \tag{16}$$

for every segment m, then both polygonal chains have the same winding number. Because if (15) and (16) holds, given any $0 \le \delta \le 1$, the angle between $\underline{\tilde{L}}^m(\delta)$ and $\underline{\tilde{L}_c^m}(\delta)$ can be upper bounded by π:

$$\begin{aligned}\angle\left\{\underline{\tilde{L}^m}\left(\underline{\tilde{L}_c^m}\right)^*\right\} &= \angle\left\{\left(\underline{\tilde{L}^m}r_m\right)\left(\underline{\tilde{L}_c^m}r_m\right)^*\right\}\\ &= \angle\left\{\underline{\tilde{L}^m}r_m\right\} - \angle\left\{\underline{\tilde{L}_c^m}r_m\right\}\\ &\le \left|\angle\left\{\underline{\tilde{L}^m}r_m\right\}\right| + \left|\angle\left\{\underline{\tilde{L}_c^m}r_m\right\}\right|\\ &< \frac{\pi}{2} + \frac{\pi}{2} = \pi.\end{aligned}$$

Since the angle of $\underline{\tilde{L}^m}(\underline{\tilde{L}_c^m})^*$ is always less than π, $\underline{\tilde{L}}(\underline{\tilde{L}_c})^*$ cannot wind around the origin and therefore $\underline{\tilde{L}}$ and $\underline{\tilde{L}_c}$ must have the same winding number.

A possible choice for r_m is the closest point of $\underline{\tilde{L}_c^m}(\delta)$ to the origin:

$$r_m = \underset{p_m \in \underline{\tilde{L}_c^m}(\delta)}{\arg\min}\ |p_m| \tag{17}$$

With this choice, (16) is by construction always satisfied, and only (15) must be considered. It is also sufficient to constraint only the end-points of each segment, resulting in

$$\begin{aligned}\Re\{\underline{\tilde{L}}(z_m)r_m\} > 0 \quad \forall m = 1,\dots,M\\ \Re\{\underline{\tilde{L}}(z_{m+1})r_m\} > 0 \quad \forall m = 1,\dots,M\end{aligned} \tag{18}$$

This ensures both polygonal chains have the same winding number. Note (18) does not guarantee $\mathcal{D}-$stability, as the polygonal chain assumes a straight line between two consecutive samples, but this is a good approximation if $\partial\mathcal{D}$ is sampled appropriately. The approximation error decreases asymptotically with the square of the number of sampled points: when sampled sufficiently densely, $\tilde{L}$ is well approximated by an arc between two consecutive samples, and the corresponding segment of $\underline{\tilde{L}}$ corresponds to the chord of this arc. When doubling the number of points, and choosing the new points such that the mid-point of every arc is sampled, the maximal distance between the chord and the arc is reduced by a factor 4. Thus only using a moderate number of samples is usually sufficient to ensure the poles of the closed-loop are within a small distance of the stability region.

Initial controller: An initial $\mathcal{D}-$stabilizing controller K_c is needed to derive (10). We argue that such controller is, in many cases, easy to obtain using exact pole placement, as long as the chosen poles reside inside $\mathcal{D}$. If a reduced order controller is desired, an initial controller can be obtained using the procedure proposed in Apkarian and Noll (2006), solving a non-linear program, or any other techniques resulting in a $\mathcal{D}-$stabilizing controller.

It is also important to note that the choice of initial controller will have an impact on the convex set of $\mathcal{D}-$stabilizing controllers. Assume $J(K)$ corresponds to a closed-loop cost function to be minimized under the $\mathcal{D}-$stability constraint. It is clear that if the initial controller satisfies the $\mathcal{D}-$stability constraint, the optimal controller K_o, satisfies $J(K_o) \le J(K_c)$. Then the optimal controller can be used as a new starting point for another optimization. This iterative optimization procedure leads to a monotonically decreasing $J(K)$ that converges to a local minimum or a saddle point.

5. NUMERICAL EXAMPLES

Theorem 2 gives a convex set of $\mathcal{D}-$stabilizing controllers and it should be combined with other design requirements. The scope of this paper is not to dwell into the selection of those requirements, but the focus of the following examples is on the $\mathcal{D}-$stability property of the closed-loop system.

5.1 Application to a boost-converter

This example is taken from Wisniewski et al. (2019), where a controller is designed to regulate a boost-converter. A second order state-space model is derived with multimodel uncertainty. It is assumed the internal states are available, and therefore the system has two outputs. Eight models, $G_1,\dots,G_8$, can be obtained from the aforementioned paper. The controller is $X = K = [k_1,\, k_2]$, $Y = I$, and k_1, k_2 are optimization variables. The closed-loop poles must have a damping factor $0 < \zeta \le 1$, resulting in a non-convex carotid-shaped domain. To obtain an overshoot less than 10%, $\zeta = 0.5912$ is chosen. An additional constraint is added to obtain an upper bound on the settling time that corresponds to a disk centered at the origin with radius 0.852. The desired $\mathcal{D}-$stability set describing the pole location is given by

$$\mathcal{D} = \left\{z \in \mathbb{C} \mid z = r(\theta)e^{j\theta},\right.$$
$$\left. 0 \le\ r(\theta) < \min\left(e^{\frac{-\zeta|\theta|}{\sqrt{1-\zeta^2}}}, 0.852\right), \theta \in (-\pi,\pi]\right\} \tag{19}$$

and shown in Fig. 1. The proposed approach in Wisniewski et al. (2019) results in the following robust $\mathcal{D}-$stabilizing controller: $K_c = [-0.01707,\, 0.00493]$, which can be used as the initial controller in our approach. The two-norm of the controller gains is minimized while preserving robust $\mathcal{D}-$stability. This objective is chosen as it is easy to visualize: given the feasible set in the control parameter space, the optimal solution is the closest point to the origin. Since $\mathcal{D}$ is symmetrical w.r.t the real axis, only one side of $\partial\mathcal{D}$ must be constrained. To implement the synthesis method, $\partial\mathcal{D}$ is sampled at 501 different points, with equidistant angle between two consecutive points.

The full optimization problem to solve is:

$$\underset{K}{\arg\min}\ \|K\|_2 \tag{20}$$

subject to:

$$\tilde{L}_i(z_m)\tilde{L}_{ic}^*(z_m) + \tilde{L}_i(z_m)\tilde{L}_{ic}^*(z_m))^* \succeq I \cdot 10^{-5} \tag{21}$$
$$\forall m = 1,\dots,501,\ \forall i = 1,\dots,8$$

where $\tilde{L}_i = I + G_iK$ and $\tilde{L}_{ic} = I + G_iK_c$. A small amount of additional conservatism is added in (21), with intent to ensure positive-definiteness of $\tilde{L}_i(z)\tilde{L}_{ic}(z)$ between two sampled points of the contour. As the convex set of $\mathcal{D}-$stabilizing controllers depends on K_c, the optimisation problem is solved multiple times. The optimal controller from the previous optimization problem is used as K_c in the next iteration until convergence to a final controller. After 64 iterations, the final controller gains are:

$$K = [-0.0076,\, 0.0075]$$

The norm of the controller gains is approximately 40% lower than that of the initial one while still preserving the $\mathcal{D}-$stability. The closed-loop poles using K and K_c are shown in Fig. 1.

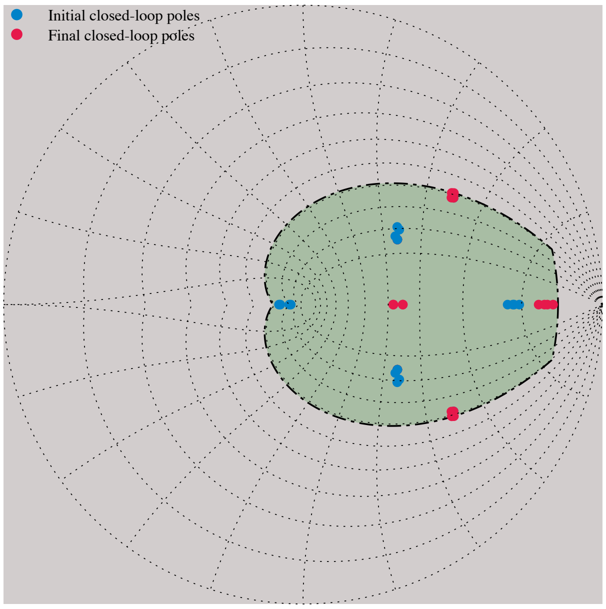

Fig. 1. Closed-loop poles of boost-converters using K and K_c. $\partial\mathcal{D}$ is indicated using the dot-dashed line.

Since the controller has only two variables, the feasible sets corresponding to the $\mathcal{D}-$stability can be drawn in the controller parameter space. The set of all $\mathcal{D}-$stabilizing controller is found using a grid search, and corresponds to the yellow shape shown in Fig. 2. For this example, the complete set of $\mathcal{D}-$stabilizing controller is a non-convex set resembling a triangle, where the bottom edge curves slightly inwards. The global optimal solution corresponds to the rightmost vertex of the triangle. The feasible sets described by LMIs in (21) are shown for the five first iterations in the same figure. Different colors indicate the sets at different iterations, where the initial feasible set is the left-most red set, and subsequent sets each move more towards the global optimum.

The controllers gains $K = [k_1,\, k_2]$ are plotted for each of the 64 iterations, and correspond to the black dots. Note that the controller ultimately does not converge to the global optimum but to a point close to it, as conservatism in (21) was added. This conservatism can be reduced when increasing the number of sampled points on the boundary.

5.2 Application to finite impulse response systems

For the second example, a laser-beam system from Quanser is used. The set-up consists of a laser-diode, a mirror actuated by a voice-coil reflecting the laser, and a position sensing device (PSD). The objective is to track a reference position of the laser beam on the PSD. A second order controller is used to improve the tracking performance and disturbance rejection. The step response is obtained by applying a unit step-change in the input and measuring the output y_k. Therefore, the impulse response of the system is given by

$$G(z) = \sum_{k\geq 0}(y_k - y_{k-1})z^{-k}$$

with $y_{-1} := 0$. After 60 samples, the transient is indistinguishable from noise, and therefore the impulse response is truncated to 60 samples. This finite impulse response is used as model for the system dynamics and is shown in Fig. 4.

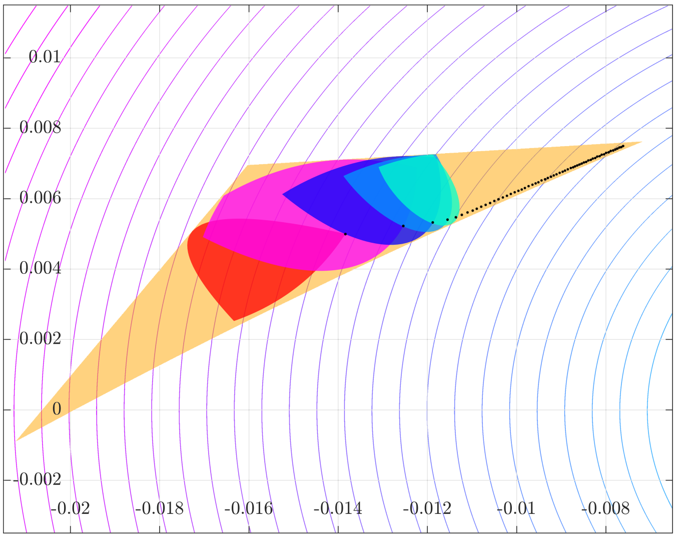

Fig. 2. $\mathcal{D}-$stabilizing sets plotted in the controller $K = [k_1, k_2]$ parameters space. x-axis corresponds to k_1, and the y-axis to k_2.

An initial second order stabilizing controller $K_c = X_c/Y_c$ is found by solving a non-linear optimization problem which minimizes the spectral radius of the closed-loop poles, resulting in closed-loop poles with magnitude $|z| < 0.925$. The stability set $\mathcal{D}$ and is chosen the same as presented in Sec. 2.

The control objective is formulated as minimizing the two-norm of the tracking error given a step reference, and corresponds to minimizing the following system norm:

$$\min \left\| W_1(z)\frac{1}{1+G(z)K(z)} \right\|_2^2$$

where $W_1(z) = \frac{1}{z-1}$ is the z-transform of a step signal. To solve the $\mathscr{H}_2$ problem, it is proposed in Karimi and Kammer (2017) to minimize an upper-bound of the $\mathscr{H}_2$ norm, resulting in:

$$\underset{X,Y}{\arg\min} \quad \int_0^{2\pi} \mu(\omega)d\omega \tag{22}$$

subject to:

$$\left(W_1(e^{j\omega})Y(e^{j\omega})\right)^* \left(W_1(e^{j\omega})Y(e^{j\omega})\right) \leq \mu(\omega)\Phi(e^{j\omega}) \tag{23}$$
$$\mu(\omega) \geq 0$$

for all $\omega \in [0,\, 2\pi]$, where

$$\Phi(e^{j\omega}) = 2\Re\{\tilde{L}(e^{j\omega})\tilde{L}_c^*(e^{j\omega})\} - \tilde{L}_c(e^{j\omega})\tilde{L}_c^*(e^{j\omega})$$

and $\tilde{L}$, $\tilde{L}_c$ as defined in (7) and (9), respectively. This is a semi-infinite but convex optimization program, depending on a continuous variable ω. It is proposed to solve the sampled-frequency problem by choosing discrete values of $\omega = \{\omega_1, \ldots, \omega_N\}$, and approximate the integral (22) with a Riemann sum. The controller is obtained by solving (22)-(23) at 1000 linearly spaced point $\omega_n \in [10^{-2}, 2\pi - 10^{-2}]$, with the added polygonal winding number constraint (18) implemented using $M = 1001$ equidistant points on the boundary $z_m \in \{0.925, 0.925e^{j\pi/1000}, \ldots, 0.925e^{j\pi}\}$, and r_m computed using (17). As $\mathcal{D}$ is symmetrical w.r.t the real axis, only one side must be constrained.

The full problem to be solved is

$$\underset{X,Y}{\arg\min} \quad \omega_1\mu_1 + \sum_{n=2}^{1000}(\omega_n - \omega_{n-1})\mu_n$$

subject to:

$$\left(W_1(e^{j\omega_n})Y(e^{j\omega_n})\right)^* \left(W_1(e^{j\omega_n})Y(e^{j\omega_n})\right) \le \mu_n \Phi_n(e^{j\omega_n})$$
$$\mu_n \ge 0, \quad \Re\{\tilde{L}(z_m)r_m\} > 0, \quad \Re\{\tilde{L}(z_{m+1})r_m\} > 0$$
$$\forall n = 1, \dots, 1000, \ \forall m = 1, \dots, 1001$$

The optimisation problem is solved multiple times, at each iteration using K_c as the optimal controller from the previous optimization problem, until convergence to a final controller. The final closed-loop poles are shown in Fig. 3. The tracking performance of the final controller is shown in Fig. 4 along with the the step reference.

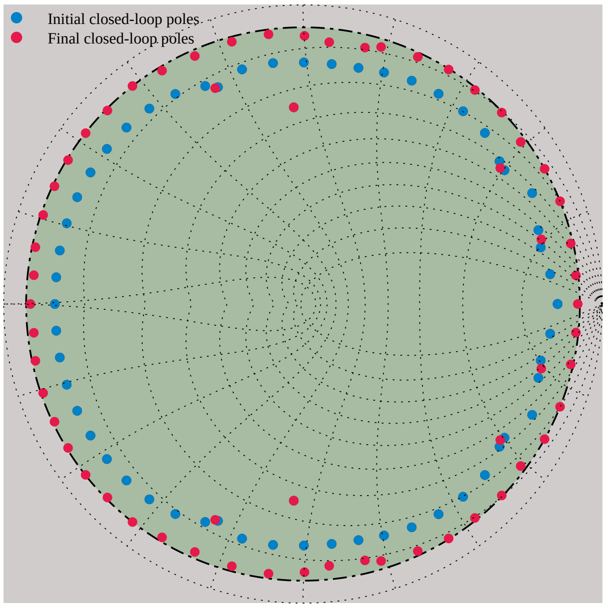

Fig. 3. Closed-loop poles marked as red dots. Stability boundary indicated using the dot-dashed line.

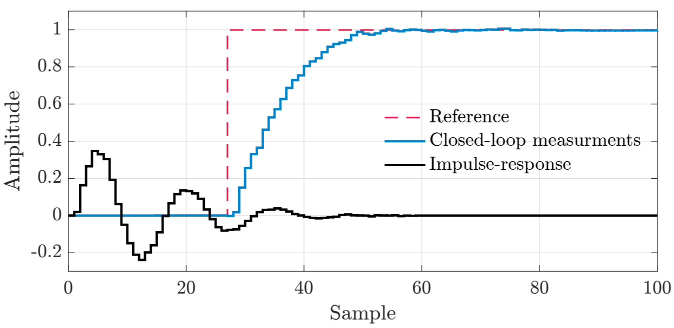

Fig. 4. Measurements from the laser-beam system.

6. CONCLUSION

We have presented a new approach to obtain a semi-infinite convex in the controller parameters constraint for robust $\mathcal{D}-$stabilizing controllers. This constraint can be used in conjunction with other synthesis approaches to guarantee a robust $\mathcal{D}-$stable closed-loop while minimizing a desired objective or maintaining other design requirements. This approach has been showcased on two control-relevant examples.

Future research directions include 1) better stability guarantees in the sampled contour case: derive improved stability MIMO conditions, similar to the polygonal chain in the SISO case or 2) extension to controllers parametrized using a state-space formulation.

REFERENCES

Ackermann, J., Kaesbauer, D., Sienel, W., Steinhauser, R., and Bartlett, A. (1993). *Robust Control: Systems with uncertain physical parameters*. Springer London.

Apkarian, P. and Noll, D. (2006). Nonsmooth $\mathscr{H}_\infty$ synthesis. *IEEE Transactions on Automatic Control*, 51, 71–86.

Chilali, M. and Gahinet, P. (1996). $\mathscr{H}_\infty$ design with pole placement constraints: An LMI approach. *IEEE Transactions on Automatic Control*, 41, 358–367.

Chilali, M., Gahinet, P., and Apkarian, P. (1999). Robust pole placement in LMI regions. *IEEE Transactions on Automatic Control*, 44, 2257–2270.

Doyle, J., Glover, K., Khargonekar, P., and Francis, B. (1988). State-space solutions to standard $\mathscr{H}_2$ and $\mathscr{H}_\infty$ control problems. *Proceedings of the American Control Conference*, 88 pt 1-3, 1691–1696.

Furuta, K. and Kim, S.B. (1987). Pole assignment in a specified disk. *IEEE Transactions on Automatic Control*, 32, 423–427.

Gahinet, P. and Apkarian, P. (1994). A linear matrix inequality approach to $\mathscr{H}_\infty$ control. *International Journal of Robust and Nonlinear Control*, 4, 421–448.

Garcia, G. and Bernussou, J. (1995). Pole assignment for uncertain systems in a specified disk by state feedback. *IEEE Transactions on Automatic Control*, 40, 184–190.

Haddad, W.M. and Bernstein, D.S. (1992). Controller design with regional pole constraints. *IEEE Transactions on Automatic Control*, 37, 54–69.

Hench, J.J., He, C., Kučera, V., and Mehrmann, V. (1998). Dampening controllers via a Riccati equation approach. *IEEE Transactions on Automatic Control*, 43, 1280–1284.

Henrion, D., Bachelier, O., and Šebek, M. (2001). $\mathcal{D}$-stability of polynomial matrices. *International Journal of Control*, 74, 845–856.

Howie, J.M. (2003). *Complex Analysis*. Springer London.

Karimi, A. and Kammer, C. (2017). A data-driven approach to robust control of multivariable systems by convex optimization. *Automatica*, 85, 227–233.

Rosinová, D. and Holič, I. (2014). LMI approximation of pole-region for discrete-time linear dynamic systems. *Proceedings of the 2014 15th International Carpathian Control Conference, ICCC 2014*, 497–502.

Scherer, C., Gahinet, P., and Chilali, M. (1997). Multiobjective output-feedback control via LMI optimization. *IEEE Transactions on Automatic Control*, 42, 896–911.

Skogestad, S. and Postlethwaite, I. (2001). *Multivariable Feedback Control - Analysis and design*, volume 1. John Wiley & Sons, Ltd.

Wisniewski, V.L., Maddalena, E.T., and Godoy, R.B. (2019). Discrete-time regional pole-placement using convex approximations: Theory and application to a boost converter. *Control Engineering Practice*, 91, 104–102.