

A simple theoretical model for estimating the frequency characteristics of black hole mergers

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Abstract

Gravitational waves, despite predicted within the Theory of General Relativity about a century ago, were observed experimentally only in the last years, thanks to the recent advances of the LIGO and Virgo detectors. In this work, we introduce a simple theoretical model to study the gravitational irradiation of binary systems and derive the characteristics of the emitted gravitational waves, e.g., in terms of frequency. The simplicity of the model relies firstly on the consideration of the two-body problem and circular orbits. Then, we follow an energetic approach and make us of only one result of the Theory of General Relativity, in order to make the procedure clear and suitable for undergraduate education. The numerical solution is finally compared with the experimental data available for a few black hole mergers.

Keywords

Binary systems, gravitational irradiation, gravitational waves, theory of general relativity, black hole mergers.

1. Introduction

Gravitational waves were first theorised within the frame of the Einstein's Theory of General Relativity [1,2] and they appear following a time variation of the quadrupole moment of a mass distribution. The effects deriving from the gravitational irradiation are extremely small. For instance, an attempt to quantify this effect can be obtained by assuming that the Earth, in its revolution around the Sun, follows a circular orbit of radius $R = 1.5 \cdot 10^{11}$ m. In order that the radius of the orbit reduces of only one metre due to the emission of gravitational waves, as we will see later in this paper, the time necessary would be around $9 \cdot 10^{19}$ s, i.e., roughly 2800 billions of years, which is much longer than the age of the Universe.

The first indirect proof of the existence of gravitational waves belongs to Hulse and Taylor, who discovered in 1975 the binary system PSR B1913+16 [3]. It was made of two pulsars rotating around their centre of mass and the dynamics of the system was in agreement with the gravitational radiation predicted by the Theory of General Relativity.

However, due to the huge technical challenges in the measurement of extremely small quantities, the existence of gravitational wave has been confirmed only very recently. In fact, only in some particular cosmological situations, e.g., when the involved masses are exceptionally high and the distances relatively small, experimental measurements are able to extract reliable values. It was within this frame that the Advanced LIGO detectors [4] have measured, for the first time, gravitational waves from the collision of two black holes (event known as GW150914), with masses $m_1 = 36M_s$ and $m_2 = 29M_s$, being M_s the mass of the Sun [5]. The discovery was first published in February 2016 and has been embraced with great enthusiasm from the international scientific community.

The objective of this work is to present a simple theoretical model for studying the gravitational irradiation of black hole mergers and determining their frequency characteristics. For this purpose, we start from the two-body problem, which is well known and widely dealt with in education of classical mechanics. Then, we employ simple derivations and finite integrals, together with only one result taken from the Theory of General Relativity, to compute the frequency of the gravitational waves. The whole approach is based uniquely on the energy conservation law, and specifically on the balance between mechanical and irradiated energy. The model is finally applied to a few black hole merger events and the results compared with available experimental data.

The way in which the model is presented makes it particularly suitable for undergraduate students, who can follow smoothly the proposed procedure, even without any previous knowledge on Relativity or gravitational physics. These subjects have recently gained a lot of interest in physics education, thanks to the experimental observation of gravitational waves. However, teaching General Relativity and gravitation to undergraduate students is extremely challenging and new pedagogical approaches are being developed for this purpose [6,7,8]. Recently, some authors have made use of electromagnetic [9] or acoustic [10] analogies to describe gravitational waves. We base our model on Newtonian gravity (and not entirely on General Relativity) because it is generally within the knowledge domain of an undergraduate audience. The presented model inevitably introduces some approximations: a detailed analysis of gravitational irradiation phenomena is beyond the scope of the present work and specialised literature is available. An example is the use of the post-Newtonian approximation, which is based on the concept that many astrophysical systems can evolve from a Newtonian gravity-dominant condition to situations in which relativistic effects can also play a role [11], e.g., in the presence of increasing gravitational fields. The post-Newtonian approximation has been applied, e.g., to the study of binary pulsars

[12,13,14], and a quite complex and iterative mathematical methodology is usually needed (see, e.g., Ref. [15] and references therein).

2. Model

2.1 Introduction to the classical two-body problem

Let us consider a system made of two bodies, with masses m_1 and m_2 . It is well known that the gravitational interaction between the two bodies can be reduced to a one-body problem, i.e., to a material point with reduced mass [2]:

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad (1)$$

that moves within the potential:

$$V(\rho) = -G \frac{\mu M}{\rho} \quad (2)$$

where ρ is the absolute value of the distance between the masses, $G = 6.67 \cdot 10^{-11} \text{ Nm}^2 / \text{kg}^2$ is the universal gravitational constant and $M = m_1 + m_2$ is the total mass.

2.2 Evolution of a perfect binary system due to gravitational irradiation

For simplicity, let us consider circular orbits, with $t = 0$ the time instant in which the mass μ has an orbit of radius R . The gravitational irradiation causes an energy loss that produce a decrease in the radius as function of time, i.e., the mass m_1 tends to collide onto the mass m_2 .

If $\rho \leq R$ is the radius of a generic circular orbit, the total mechanical energy of the system is given by:

$$E_{mech} = E_{kin} + V = \frac{1}{2} \mu v^2 - G \frac{\mu M}{\rho} \quad (3)$$

being E_{kin} the kinetic energy and v the velocity. This latter quantity can be expressed as function of the radius of the circular orbit. In fact, considering that the modulus of the force acting on the reduced mass μ is $F = G\mu M / \rho^2$, the Newton's equation of motion applied in the radial direction gives the acceleration as:

$$G \frac{\mu M}{\rho^2} = \mu a \Rightarrow a = G \frac{M}{\rho^2} \quad (4)$$

Thus, since in a circular motion $a = v^2 / \rho$, we get:

$$v = \sqrt{G \frac{M}{\rho}} \quad (5)$$

Therefore, by inserting Equation (5) into Equation (3), we obtain:

$$E_{mech} = -\frac{1}{2} G \frac{\mu M}{\rho} \quad (6)$$

If the radius of the orbit changes of an infinitesimal quantity $d\rho$, the corresponding variation in the total mechanical energy, by differentiating Equation (6), becomes:

$$dE_{mech} = \frac{1}{2} G \frac{\mu M}{\rho^2} d\rho \quad (7)$$

Now, in order to estimate the gravitational energy irradiated, we need a result from the Theory of General Relativity, since an exact estimation cannot be provided by using only the classical mechanics. Despite being two conceptually different and alternative frameworks, the classical and the relativistic approaches are here employed to have a correct estimate of the energies needed into the energy conservation law. The objective is only to have some mathematically “comfortable” quantities to deal with. Thus, from the Theory of General Relativity, the power of the gravitational irradiation (deriving from the time variation of the quadrupole moment [2]) along a circumference of radius ρ is given by [1,2]:

$$P_{irr} = \frac{32G^4 M^3 \mu^2}{5c^5 \rho^5} \quad (8)$$

being $c \approx 3 \cdot 10^8$ m/s is the speed of light in vacuum. The complete derivation of Equation (8) is not trivial and is reported in the Appendix.

Thus, the irradiated energy in a small amount of time $d\tau$ is:

$$dE_{irr} = \frac{32G^4 M^3 \mu^2}{5c^5 \rho^5} d\tau \quad (9)$$

The conservation of energy requires that:

$$dE_{mech} + dE_{irr} = 0 \quad (10)$$

Thus, by inserting Equations (7) and (9) into Equation (10), we get:

$$\rho^3 d\rho = -\frac{64G^3 M^2 \mu}{5c^5} d\tau \quad (11)$$

Now we can integrate Equation (11) by considering that the variable ρ can vary from R to r , i.e.:

$$\int_R^r \rho^3 d\rho = -\frac{64G^3 M^2 \mu}{5c^5} \int_0^t d\tau \Rightarrow \frac{1}{4}(r^4 - R^4) = -\frac{64G^3 M^2 \mu}{5c^5} t \quad (12)$$

The characteristic time T_0 , needed by the binary system to collapse in a single body due to gravitational irradiation, can be found directly from Equation (12) by posing $r = 0$, i.e.:

$$T_0 = \frac{5c^5 R^4}{256G^3 M^2 \mu} \quad (13)$$

Therefore, by making use of Equation (13), we can write explicitly the time evolution of the distance between the two masses m_1 and m_2 as:

$$r(t) = R \left(1 - \frac{t}{T_0} \right)^{\frac{1}{4}} \quad (14)$$

A similar approach is followed by Misner *et al.* [16], who start from electrostatics considerations and denote T_0 as the “spiral time”. Here we make a step forward to compute the frequency of the emitted irradiation.

By introducing the third Kepler’s law [17], i.e., $T^2 = \frac{4\pi^2}{GM} r^3$, we can compute the period T of the circular orbit. From Equation (14), we get:

$$T(t)^2 = \frac{4\pi^2}{GM} R^3 \left(1 - \frac{t}{T_0}\right)^{\frac{3}{4}} \Rightarrow T(t) = 2\pi \sqrt{\frac{R^3}{GM}} \left(1 - \frac{t}{T_0}\right)^{\frac{3}{8}} \quad (15)$$

Consequently, it is straightforward to write the time dependence of the frequency as:

$$f(t) = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{GM}{R^3}} \left(1 - \frac{t}{T_0}\right)^{-\frac{3}{8}} \quad (16)$$

Finally, the time-averaged value of the frequency in the interval $[0, T_0]$ is obtained by integrating Equation (16), i.e.:

$$\langle f \rangle = \frac{1}{T_0} \int_0^{T_0} \frac{1}{2\pi} \sqrt{\frac{GM}{R^3}} \left(1 - \frac{t}{T_0}\right)^{-\frac{3}{8}} dt = \frac{4}{5\pi} \sqrt{\frac{GM}{R^3}} \quad (17)$$

which represents the sought estimation of the frequency of the emitted gravitational irradiation.

Note that the obtained result derives directly from the conservation of energy, as defined by Equation (10). Therefore, we expect to get a correct order of magnitude despite the introduced approximations.

3. Results

3.1 Application to the Earth-Sun binary system

The Earth follows an elliptical orbit around the Sun, which can be well approximated through a circumference of average radius $R = 1.5 \cdot 10^{11}$ m. The masses of the Earth and the Sun are, respectively, $m_1 = M_T = 6 \cdot 10^{24}$ kg and $m_2 = M_S = 2 \cdot 10^{30}$ kg. Since $m_1 \ll m_2$, we get $M = m_1 + m_2 \approx M_S$ and the reduced mass, from Equation (1), becomes simply $\mu \approx M_T$.

Therefore, Equation (13) gives a period $T_0 = 3.4 \cdot 10^{30}$ s $= 10^{23}$ years. This value is extremely large and is much larger than the age of the Universe itself, which is in the order of 10^{10} years. Thus, our model predicts that, if the future age of the Universe would be enough long, the Earth is going to collapse onto the Sun due to the effects of gravitational irradiation. However, these effects are extremely small that cannot be detected, for the considered binary system, with the instrumentation nowadays available. In fact, we can estimate the time needed for the orbit of the Earth to reduce its radius of only 1 m, by inserting $\Delta\rho = -1$ m in Equation (11), obtaining $\Delta\tau \approx 9 \cdot 10^{19}$ s, which corresponds roughly to 2800 billions of years.

More realistic quantities emerge when larger masses and/or shorter distances are considered, e.g., in the case of black hole mergers, as we are discussing in the next section.

3.2 Application to black hole mergers

Let us now consider the binary mergers events observed by LIGO and Virgo [18] during the first observation runs, going from September 2015 to August 2017. After the first experimental evidence of gravitational irradiation (i.e., GW150914 [5]), several more black hole mergers and neutron star inspirals have been detected.

Estimation of the radius at the time $t = 0$

When considering a black hole merger, the result of the coalescence is a body with a total mass smaller than the sum of the two initial masses m_1 and m_2 , i.e., smaller than M . The difference between M and the residual mass of the black hole is the equivalent in energy that has been irradiated in the form of gravitational waves. We denote this equivalent “irradiated” mass as m_{irr} .

From Equations (8) and (14) above, the power of the gravitational irradiation is given by:

$$P(t) = \frac{32G^4 M^3 \mu^2}{5c^5 R^5} \left(1 - \frac{t}{T_0}\right)^{-5/4} \quad (18)$$

Therefore, by integration, the total energy irradiated during the whole process is given by:

$$E = \int_0^{T_0} P(t) dt = \frac{32G^4 M^3 \mu^2}{5c^5 R^5} \int_0^{T_0} \left(1 - \frac{t}{T_0}\right)^{-5/4} dt = \frac{128G^4 M^3 \mu^2 T_0}{5c^5 R^5} \quad (19)$$

By substituting the value of the characteristic time T_0 from Equation (13), we get:

$$E = \frac{GM\mu}{2R} = \frac{Gm_1 m_2}{2R} \quad (20)$$

Note that the same result can be obtained considering that, in the time interval $[0, T_0]$, the separation decreases from R to 0, i.e., all the irradiated energy is equal to the initial energy, as expected from the energy conservation.

We can now equal this value to the energy corresponding to the equivalent “irradiated” mass m_{irr} introduced above, i.e., $m_{irr}c^2$, obtaining:

$$R = \frac{Gm_1 m_2}{2m_{irr}c^2} \quad (21)$$

Now, for convenience, we can make the substitutions:

$$m_1 = m_1^* M_S \quad , \quad m_2 = m_2^* M_S \quad , \quad M = M^* M_S \quad , \quad m_{irr} = m_{irr}^* M_S \quad (22)$$

where the masses with the symbol $*$ are dimensionless quantities. Therefore, after simple algebraic rearrangements in Equation (21), we get:

$$R = \frac{m_1^* m_2^*}{2m_{irr}^*} \frac{GM_s}{c^2} = \frac{m_1^* m_2^*}{4m_{irr}^* M^*} \frac{2GM}{c^2} = \frac{m_1^* m_2^*}{4m_{irr}^* M^*} R_s = \frac{m_1^* m_2^*}{4m_{irr}^*} R_{s,s} \quad (23)$$

being $R_s = 2GM / c^2$ the Schwarzschild's radius of the mass M and $R_{s,s}$ the Scharzschild's radius of the Sun.

Frequency of the emitted gravitational irradiation

Table 1 reports a list of a few selected black hole mergers: for each of them, we have estimated the average frequency $\langle f \rangle$ of the emitted gravitational irradiation through Equation (17).

Moreover, we have used Equation (23) for a realistic estimation of the radius R .

Except for the event GW151226, for which we get an overestimated average frequency, the other results are in very good agreement with the corresponding experimental observations. The computed values of the average frequency, in fact, fall within or slightly outside the measured range. The disagreement between experimental and computed values is likely due to: (i) the fact that we have neglected the relativistic corrections needed in presence of large gravitational fields, i.e., above all when the two bodies come to small distances; and (ii) the sensitivity of the measurement instruments, which provide frequency values representative of complex frequency spectra. Note that in Table 1 we refer only to the average frequency at peak strain.

However, if considering the introduced approximations and the simplicity of the proposed approach, this is an important result that confirms the good potential of our method for an easy and quick estimation of the main characteristics of gravitational wave events.

Table 1. List of the selected black hole mergers, with corresponding main characteristics, and estimated values of the average frequency compared with the experimental ranges. All the masses are in units of solar mass M_s , and are taken from the literature. The reader is referred to two Refs: the first one is the original scientific publication, the second one is the Factsheet, which reports schematically all the relevant information used in this study.

Black hole merger	Refs.	$m_1^* = m_1 / M_s$	$m_2^* = m_2 / M_s$	$m_{irr}^* = m_{irr} / M_s$	Experimental frequency (range)	Estimated average frequency
GW150914	[5,19]	35.6	30.6	3.0	150 Hz	171.4 Hz
GW151226	[20,21]	13.7	7.7	0.95	420 Hz	576.3 Hz
GW170104	[22,23]	31.0	20.1	1.95	160÷199 Hz	182.4 Hz
GW170608	[24,25]	10.9	7.6	0.795	453÷610 Hz	589.5 Hz
GW170814	[26,27]	30.7	25.3	2.75	155÷203 Hz	229.8 Hz

An improved version of the present model could be obtained by considering that R is, in general, not constant but slightly decreasing with time (as in the same Figure 2 of Ref. [5] mentioned above). Therefore, considering also that the frequency itself contains a time dependence as $\sim t^{-3/8}$, as highlighted in Equation (16), the time variation of R leads to a certain dependence of the frequency on t . However, taking into account this effect would not change significantly the estimations of the average frequency reported above, and it is therefore beyond the scope of the present study.

4. Conclusions

We have presented some simple theoretical estimations on the gravitational irradiation of binary systems. Starting from an energetic approach applied to the two-body problem, we have employed a classical mechanics approach, and used only one result from the Einstein's Theory of General Relativity, to predict the characteristic frequency associated to the emission of gravitational waves. The application of the proposed model to some recently-observed black hole mergers, provides quantitative results in close agreement with the experimental measurements by the Advanced LIGO and the Advanced Virgo detectors. Our approach, therefore, presents a good potential for a quick and simple estimation of gravitational wave characteristics and thus is extremely suitable for undergraduate educational purposes. As future work, it would be interesting to test this approach on students and to collect feedbacks on the most difficult points of the proposed method.

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Competing interests

The authors declare there are no competing interests.

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Appendix – Derivation of the irradiated power

Let us consider a system of N point masses m_1, m_2, \dots, m_N positioned following the vectors $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N$. For large distances, the gravitational potential can be written in series giving the sum of a monopole term, a dipole term, a quadrupole term, an octupole term, etc. The monopole term depends on the total mass M of the considered system and the corresponding emitted radiation (in terms of power per unit surface) depends on the time derivative of M . Since M is constant in time, there is no gravitational irradiation due to the monopole term.

When analysing the dipole term, considering an electric analogy, it can be written as $\vec{P} = m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots + m_N \vec{r}_N = M \vec{r}_{\text{cm}}$, where \vec{r}_{cm} is the position vector of the centre of mass of the system. The emitted radiation depends on the second time derivative of \vec{P} . Since the system is considered to be isolated, its centre of mass is moving with a constant velocity, thus $\vec{a}_{\text{cm}} \equiv \ddot{\vec{r}} = 0$ and we get $\ddot{\vec{P}} = 0$. This means that no gravitational irradiation takes place due to the contribution of the dipole moment.

Let us now consider the quadrupole moment of a system of N point masses. It can be written via a 3-by-3 matrix whose elements, as function of the masses m_k and of the corresponding coordinates (x_k, y_k, z_k) , are given by [16]:

$$\left\{ \begin{array}{ll} Q_{11} = \sum_{k=1}^N m_k (3x_k^2 - r_k^2) & Q_{12} = Q_{21} = \sum_{k=1}^N m_k (3x_k y_k) \\ Q_{22} = \sum_{k=1}^N m_k (3y_k^2 - r_k^2) & Q_{13} = Q_{31} = \sum_{k=1}^N m_k (3x_k z_k) \\ Q_{33} = \sum_{k=1}^N m_k (3z_k^2 - r_k^2) & Q_{23} = Q_{32} = \sum_{k=1}^N m_k (3y_k z_k) \end{array} \right. \quad (\text{A1})$$

The gravitational irradiation, in this case, is given by the third time derivative of the quadrupole moment. Specifically, from Einstein's equations within the Theory of General Relativity, it can be demonstrated that the irradiated power is given by [16]:

$$P = \frac{G}{45c^5} \left(\ddot{Q}_{11}^2 + \ddot{Q}_{22}^2 + \ddot{Q}_{33}^2 + 2\ddot{Q}_{12}^2 + 2\ddot{Q}_{13}^2 + 2\ddot{Q}_{23}^2 \right) \quad (A2)$$

This latter Equation can be verified for a binary system comprising the masses m_1 and m_2 . Since the angular momentum is conserved, the motion of this binary system takes place in a plane Σ perpendicular to the angular momentum vector. In the reference system of the centre of mass, the binary system can be observed as a single point mass with reduced mass $\mu = m_1 m_2 / M$, where $M = m_1 + m_2$. Let us assume, for simplicity, that this point mass follows a circular orbit with radius R , centre O and has an angular velocity ω , on the plane Σ . By introducing a Cartesian reference system $Oxyz$, with x and y in-plane and z out-of-plane coordinates, respectively, the coordinates of the point mass μ are: $x = R \cos(\omega t)$, $y = R \sin(\omega t)$, $z = 0$, where t is the time.

Therefore, the non-zero components of the quadrupole moment matrix are:

$$\begin{cases} Q_{11} = \mu R^2 (3 \cos^2(\omega t) - 1) = \frac{\mu R^2}{2} (2 + 3 \cos(2\omega t)) \\ Q_{22} = \mu R^2 (3 \sin^2(\omega t) - 1) = \frac{\mu R^2}{2} (2 - 3 \cos(2\omega t)) \\ Q_{33} = -\mu R^2 \\ Q_{12} = Q_{21} = 3\mu R^2 \sin(\omega t) \cos(\omega t) = \frac{3}{2} \mu R^2 \sin(2\omega t) \end{cases} \quad (A3)$$

whose third time derivatives are given by:

$$\begin{cases} \ddot{\ddot{Q}}_{11} = 12\mu R^2 \omega^3 \sin(2\omega t) \\ \ddot{\ddot{Q}}_{22} = -12\mu R^2 \omega^3 \sin(2\omega t) \\ \ddot{\ddot{Q}}_{33} = 0 \\ \ddot{\ddot{Q}}_{12} = \ddot{\ddot{Q}}_{21} = -12\mu R^2 \omega^3 \cos(2\omega t) \end{cases} \quad (\text{A4})$$

Finally, from Equation (A2), we get the irradiated power for a binary system:

$$P = \frac{G}{45c^5} (\ddot{\ddot{Q}}_{11}^2 + \ddot{\ddot{Q}}_{22}^2 + 2\ddot{\ddot{Q}}_{12}^2) = \frac{32}{5} \frac{G\mu^2 R^4 \omega^6}{c^5} \quad (\text{A5})$$

and by substituting the third Kepler's law [17], i.e., $\omega^2 = GM/R^3$, Equation (8) introduced in the text is easily obtained.