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A Hopf algebra model for Dwyer's tame spaces

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万卷古今消永日，一窗昏晓送流年。

To freedom. . .
致自由

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Abstract

In this thesis, we give a modern treatment of Dwyer's tame homotopy theory using the language of ∞-categories. We introduce the notion of tame spectra and show it has a concrete algebraic description. We then carry out a study of ∞-operads and define tame spectral Lie algebras and tame spectral Hopf algebras. Finally, we prove that the homotopy theory of tame spectral Hopf algebras is equivalent to that of tame spaces. To recover Dwyer's Lie algebra model for tame spaces, we use Koszul duality to construct a universal enveloping algebra functor, and show it is an equivalence from the ∞-category of tame spectral Lie algebras to the ∞-category of tame spectral Hopf algebras.

Key words: ∞-category, tame homotopy theory, spectral Lie algebra, Koszul duality

Résumé

À travers cette thèse, nous étudions la théorie moderne de l'homotopie modérée de Dwyer en utilisant le langage des ∞-catégories. Nous introduisons la notion de spectres modérée et montrons qu'elle a une description algébrique concrète. Nous effectuons ensuite une étude des ∞-opérades et définissons les algèbres de Lie spectrales modérée et les algèbres de Hopf spectrales modérée. Enfin, nous prouvons que la théorie de l'homotopie des algèbres de Hopf spectrales modérée est équivalente à celle des espaces modérée. Pour retrouver le modèle d'algèbre de Lie de Dwyer pour les espaces modérée, nous utilisons la dualité de Koszul pour construire un foncteur universel d'algèbre enveloppante et montrons qu'il s'agit d'une équivalence entre la ∞-catégorie des algèbres de Lie spectrales modérée et la ∞-catégorie des algèbres de Hopf spectrales modérée.

Mots clefs : ∞-catégorie, théorie de l'homotopie modérée, algèbre de Lie spectrale, dualité de Koszul.

Contents

Chapter 1

Introduction

1.1 A Brief History of Algebraic Models for Spaces

Algebraic topology is the study of classification of topological spaces up to certain equivalence relations via algebraic invariants. The homotopy groups $\pi_* X$ of a space X are among those invariants. For $n = 1$, $\pi_1(X)$ is called the *fundamental group* of X. For $n \geq 2$, $\pi_n(X)$ is an abelian group.

Spheres are the building blocks in topology, since the category of spaces is generated under (homotopy) colimits by S^n for $n \geq 0$. Perhaps surprisingly, we only know very little about the homotopy groups of spheres.

To motivate the study of algebraic models of spaces, we now survey some important results about the homotopy groups of spheres $\pi_m(S^n)$ for $m, n \geq 1$. From our first course of topology, we know that the fundamental group of the circle S^1 is isomorphic to the group of integers $\mathbb{Z}$. Moreover, $\pi_1(S^1)$ is the only non-vanishing homotopy group of S^1, i.e., S^1 is an Eilenberg-Maclane space of type $K(\mathbb{Z}, 1)$.

In 1930's, Hurewicz showed that, if X is simply-connected, then the lowest non-vanishing homology group of X is isomorphic to its lowest non-vanishing homotopy group. Hurewicz's theorem then implies the following result, which states that any map from a low dimensional sphere to a higher dimensional sphere is null-homotopic.

Theorem 1.1.1 (Hurewicz). *The homotopy group $\pi_m(S^n)$ is zero for $m < n$ and is isomorphic to $\mathbb{Z}$ if $m = n$.*

Shortly after Hurewicz, Freudenthal discovered the statbility of the groups $\pi_m(S^n)$.

Theorem 1.1.2 (Freudenthal's Suspension Theorem). *The suspension morphism*

$$\sigma : S^n \to \Omega S^{n+1}$$

induces isomorphisms:

$$\pi_{n+k}(S^n) \xrightarrow{\sigma_*} \pi_{n+k+1}(S^{n+1})$$

for $k < n - 1$.

Freudenthal's theorem motivates the definition of the stable homotopy groups of spheres

$$\pi_k^s := \operatorname{colim}_n \pi_{n+k}(S^n), \quad \forall k \in \mathbb{Z}$$

which led to the study of a whole new subject, the *stable homotopy theory*. The stable homotopy groups of spheres are slightly easier to compute than the unstable ones, but still largely unknown.

In the 1950's, Serre used his spectral sequence to compute $\pi_m(S^n)$ modulo torsion.

Theorem 1.1.3 (Serre). *For each $n \geq 1$, the homotopy groups of the sphere S^n are finitely generated abelian groups. Moreover, after tensoring with $\mathbb{Q}$,*

$$\pi_m(S^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } m = n, \\ \mathbb{Q} & \text{if } m = 2n-1 \text{ and } n \text{ is even}, \\ 0 & \text{otherwise.} \end{cases}$$

Serre's theorem initiated the study of the homotopy theory of spaces modulo torsion, which is the subject called *rational homotopy theory*. A simply-connected space X is *rational* if its homotopy groups $\pi_* X$ are vector spaces over $\mathbb{Q}$. Quillen [Qui69] showed that rational homotopy theory has a complete algebraic description. We shall now use the modern language of ∞-categories from [Lur09] to explain Quillen's theorem in more detail.

Let $\mathcal{S}_*^{\geq 2}$ denote the ∞-category of pointed, simply-connected spaces and let $\mathcal{S}_{\mathbb{Q}}^{\geq 2}$ denote the full subcategory of $\mathcal{S}_*^{\geq 2}$ spanned by rational spaces. A map $f : X \to Y$ in $\mathcal{S}_*^{\geq 2}$ is a *rational equivalence* if it induces isomorphisms on rational homotopy groups

$$\pi_*(f) : \pi_* X \otimes \mathbb{Q} \to \pi_* Y \otimes \mathbb{Q};$$

or equivalently, if it induces isomorphisms on rational homology groups

$$H_*(f) : H_*(X;\mathbb{Q}) \to H_*(Y;\mathbb{Q}).$$

For any space $X \in \mathcal{S}_*^{\geq 2}$, there exists a map $\eta : X \to X_{\mathbb{Q}}$ such that

1. $X_{\mathbb{Q}}$ is a rational space;
2. η is a rational equivalence.

The ∞-category $\mathcal{S}_{\mathbb{Q}}^{\geq 2}$ of simply-connected rational spaces can be constructed by formally inverting rational equivalences in $\mathcal{S}_*^{\geq 2}$.

Let $\mathrm{Ch}_{\mathbb{Q}}$ denote the ∞-category of rational chain complexes. The ∞-categorical version of Quillen's theorem can be stated as follows.

Theorem 1.1.4. *[Qui69] There are equivalences of ∞-categories*

$$\mathrm{coCAlg}(Ch_{\mathbb{Q}})^{\geq 2} \simeq \mathcal{S}_{\mathbb{Q}}^{\geq 2} \simeq \mathrm{Lie}(Ch_{\mathbb{Q}})^{\geq 1},$$

where $\mathrm{Lie}(Ch_{\mathbb{Q}})^{\geq 1}$ denotes the ∞-category of connected rational dg Lie algebras, and $\mathrm{coCAlg}(Ch_{\mathbb{Q}})^{\geq 2}$ denotes the ∞-category of simply-connected rational commutative dg coalgebras.

Quillen's model for rational homotopy theory is not only conceptual, but also allows computation using algebraic gadgets. Later Sullivan [Sul77] defined the *polynomial de Rham complex functor* $\mathcal{A}_{PL} : (\mathcal{S}_{\mathbb{Q}}^{\geq 2})^{op} \to \mathrm{CAlg}(\mathrm{Ch}_{\mathbb{Q}})$, where the target is the category of commutative differential graded algebras (CDGAs) over $\mathbb{Q}$. The functor $\mathcal{A}_{PL}$ is a complete homotopy invariant, in the sense that two rational spaces X and Y of finite type (i.e. their homology groups are finitely generated) are equivalent if and only if $\mathcal{A}_{PL}(X)$ and $\mathcal{A}_{PL}(Y)$ are equivalent (i.e. connected by a zig-zag of quasi-isomorphisms) as CDGAs over $\mathbb{Q}$. Moreover, every $\mathcal{A}_{PL}(X)$ is equivalent to a *minimal model* Λ_X, which is generally quite computable. The rational homotopy groups of X can be directly computed from the minimal model Λ_X.

From a modern prospective, Sullivan's functor $\mathcal{A}_{PL}$ is essentially equivalent to the rational cochain functor $C^*(-;\mathbb{Q})$, and $C^*(X;\mathbb{Q})$ has a E_∞-algebra structure coming from the chain-level cup products. Sullivan's theorem can then be rephrased in the language of ∞-categories as follows.

Theorem 1.1.5. *[Sul77] Let $\mathcal{S}_{\mathbb{Q}}^{\geq 2,\mathrm{fin}}$ be the ∞-category of simply-connected rational spaces of finite type, and let $\mathrm{CAlg}(\mathrm{Ch}_{\mathbb{Q}})$ be the ∞-category of E_∞-algebras over $\mathbb{Q}$. The cochain functor*

$$C^*(-;\mathbb{Q}) : \mathcal{S}_{\mathbb{Q}}^{\geq 2,\mathrm{fin}} \to \mathrm{CAlg}(\mathrm{Ch}_{\mathbb{Q}})$$

is fully faithful.

One natural question to ask is what other subcategories of spaces admit concrete algebraic descriptions as in the case of rational spaces. We will now survey some results that answer this question. Recall that we obtain the ∞-category $\mathcal{S}_{\mathbb{Q}}^{\geq 2}$ by formally inverting rational equivalences in $\mathcal{S}_*^{\geq 2}$. This technique is called *localization*, and it can be formulated in a general setting as follows.

Definition 1.1.6. Let $\mathcal{C}$ be an ∞-category and let S be a (small) collection of morphisms in $\mathcal{C}$. An object X in $\mathcal{C}$ is *S-local* if the induced map on mapping spaces

$$\mathrm{Map}_{\mathcal{C}}(B, X) \to \mathrm{Map}_{\mathcal{C}}(A, X)$$

is a weak equivalence for all $f : A \to B$ in S. A map $g : Y \to Z$ is an *S-equivalence* if the induced map on mapping spaces

$$\mathrm{Map}_{\mathcal{C}}(Z, X) \to \mathrm{Map}_{\mathcal{C}}(Y, X)$$

is an equivalence for every S-local object X.

Let $\mathcal{C}'$ be the full subcategory of $\mathcal{C}$ spanned by S-local objects. The ∞-category $\mathcal{C}'$ is called a *localization* of $\mathcal{C}$ if the embedding functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ admits a left adjoint.

Proposition 1.1.7. *[Lur09, Proposition 5.5.4.15.] If $\mathcal{C}$ is a presentable ∞-category and S is a collection of morphisms in $\mathcal{C}$, then there exists a localization functor*

$$L : \mathcal{C} \to \mathcal{C}'$$

so that a map $f : A \to B$ in $\mathcal{C}$ is an S-equivalence if and only if Lf is an equivalence in $\mathcal{C}'$.

Using Proposition 1.1.7, one can show that the ∞-category $\mathcal{S}_{\mathbb{Q}}^{\geq 2}$ of simply-connected rational spaces is obtained by inverting rational homology equivalences. A next step to consider is to take S to be the collection of morphisms in $\mathcal{S}_*^{\geq 2}$ that induce isomorphisms on mod-p homology groups, i.e., $f : X \to Y$ is in S if

$$H_*(f) : H_*(X; \mathbb{F}_p) \xrightarrow{\cong} H_*(Y; \mathbb{F}_p).$$

An S-local space is called a *p-complete space*. By Proposition 1.1.7, any simply-connected space X admits a *p-completion* $X \to X_p^\wedge$, i.e., $X_p^\wedge$ is p-complete and the map $X \to X_p^\wedge$ induces isomorphisms on mod-p homology groups.

Let $\overline{\mathbb{F}}_p$ denote the algebraic closure of $\mathbb{F}_p$. Mandell constructed an algebraic model for the ∞-category $\mathcal{S}_p^{\geq 2,\mathrm{fin}}$ of simply-connected p-complete spaces of finite type.

Theorem 1.1.8. *[Man01] The cochain functor*

$$C^*(-; \overline{\mathbb{F}}_p) : (\mathcal{S}_p^{\geq 2,\mathrm{fin}})^{op} \to \mathrm{CAlg}(\mathrm{Ch}_{\overline{\mathbb{F}}_p})$$

is fully faithful.

One may then wonder whether the integral homotopy type might be characterized by the integral cochain functor

$$C^*(-; \mathbb{Z}) : (\mathcal{S}^{\geq 2,\mathrm{fin}})^{op} \to \mathrm{CAlg}(\mathrm{Ch}_{\mathbb{Z}}).$$

Unfortunately, this is not the case due to the following theorem proved again by Mandell.

Theorem 1.1.9. *[Man06] The integral cochain functor $C^*(-; \mathbb{Z})$ (as a functor between ordinary categories) is faithful but not full.*

The next best hope is to find a way to assemble information from rationalization and p-completions. In number theory, one can recover the ring of integers $\mathbb{Z}$ by the following pullback square:

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & \prod_p \mathbb{Z}_p^\wedge \arrow[d] \\ \mathbb{Q} \arrow[r] & (\prod_p \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}, \end{tikzcd}$$

where $\mathbb{Z}_p^\wedge$ denotes the ring of p-adic integers. In homotopy theory, Bousfield–Kan [BK72] and Sullivan [Sul05] proved that there is pullback square for any nilpotent space X:

$$\begin{array}{ccc} X & \longrightarrow & \prod_p X_p^\wedge \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & (\prod_p X_p^\wedge) \otimes \mathbb{Q}. \end{array}$$

One might then hope to construct the integral model for nilpotent spaces by assembling Sullivan's cochain model for rational spaces and Mandell's cochain model for p-complete spaces. However, the difficulty lies in how to assemble information from E_∞-rings over fields of different characteristics.

Mandell's theorem suggests that we need structures beyond just commutativity to capture all the information of integral homotopy. We now introduce recent results of Yuan [Yua19] concerning the integral homotopy type.

For each prime p, Nikolaus-Scholze [NS18] showed that any E_∞-ring A admits a *Frobenius action* $\varphi_A : A \to A^{tC_p}$, where $(-)^{tC_p}$ denotes the Tate construction of a C_p-spectrum. Yuan [Yua19] then defines a p-complete E_∞-ring to be *p-perfect* by imposing conditions on the Frobenius action and showed that the ∞-category $\mathrm{CAlg}_p^{\mathrm{perf}}$ of p-perfect E_∞-rings admits an S^1-action. A canonical example of a p-perfect E_∞-ring is the p-complete sphere spectrum $\mathbb{S}_p^\wedge$.

Yuan defined the ∞-category $\mathrm{CAlg}_p^{\varphi=1}$ of *p-Frobenius fixed E_∞-rings* as the S^1-fixed points of $\mathrm{CAlg}_p^{\mathrm{perf}}$, and referred to a lift $A_{\varphi=1} \in \mathrm{CAlg}_p^{\varphi=1}$ of a p-perfect E_∞-ring A as the *F_p-trivialization* of A. The p-complete sphere spectrum $\mathbb{S}_p^\wedge$ admits an F_p-trivialization and hence the cochain $(\mathbb{S}_p^\wedge)^X_{\varphi=1}$ lies in the ∞-category $\mathrm{CAlg}_p^{\varphi=1}$. Yuan then constructed a new algebraic model for simply-connected p-complete spaces of finite type.

Theorem 1.1.10. *[Yua19, Theorem B] The functor $(\mathcal{S}_p^{\geq 2,\mathrm{fin}})^{op} \to \mathrm{CAlg}_p^{\varphi=1}$ that sends X to $(\mathbb{S}_p^\wedge)^X_{\varphi=1}$ is fully faithful.*

It turns out that the cochains of X with coefficient in the p-Frobenius fixed E_∞-ring $(\mathbb{S}_p^\wedge)_{\varphi=1}$ can be assembled for different primes p and give rise to an algebraic model for the integral homotopy type.

An E_∞-ring A is called *Frobenius fixed* if its p-completion $A_p^\wedge$ admits F_p-trivialization for each prime p. Let $\mathrm{CAlg}^{\varphi=1}$ denote the ∞-category of Frobenius fixed E_∞-rings.

Theorem 1.1.11. *[Yua19, Theorem C] The functor $(\mathcal{S}_*^{\geq 2,\mathrm{fin}})^{op} \to \mathrm{CAlg}^{\varphi=1}$ that sends X to $\mathbb{S}^X_{\varphi=1}$ is fully faithful.*

So far we have only seen the (co)algebra analogues of Quillen's rational homotopy theory. We now state a result of Heuts [Heu21b], which concerns a Lie algebra model for v_n-periodic spaces. Since the prerequisites to understand the exact statement are substantial, we give only a rough explaination here and refer the readers to [Heu21b] and the survey paper [Heu20] for more details.

For each prime p, there is a sequence of homology thoeries

$$K(0), K(1), K(2), \dots, K(\infty)$$

on p-local spaces, called the *Morava K-theories*. For certain n, these homology theories are well-known. For instance, $K(0)$ is rational homology theory, $K(1)$ is mod-p complex K-theory and $K(\infty)$ is mod-p homology theory. These homology theories are the fundamental objects in *chromatic homotopy theory*.

A p-local finite complex V is of *type n* if $K_*(m)V$ vanishes for $m < n$ and is non-zero for $m = n$. By the famous periodicity theorem of Hopkins-Smith [HS98], any p-local finite complex V of type n admits a *v_n-self map* $v : \Sigma^d V \to V$ so that $K_*(m)v$ is an isomorphism for $m = n$ and $K_*(m)v$ is the zero map for $m \neq n$. Hence, for any p-local space X, the v_n-self map v acts invertibly on the homotopy groups of the mapping space $\mathrm{Map}_*(V, X)$ and one can define the *v_n-periodic homotopy groups* of X as

$$v^{-1}\pi_*(X; V) := \mathbb{Z}[v^{\pm 1}] \otimes_{\mathbb{Z}[v]} \pi_* \mathrm{Map}_*(V, X).$$

A map $f : X \to Y$ between two p-local spaces X, Y is called a *v_n-periodic equivalence* if it induces isomorphisms on the v_n-periodic homotopy groups. We remark that the v_n-periodic homotopy groups depend on the choice of the type n complex V, but the class of v_n-periodic equivalences does not. The ∞-category $\mathcal{S}_{v_n}$ of *v_n-periodic spaces* is then obtained by formally inverting the v_n-periodic equivalences. In a similar manner, we can also define the ∞-category Sp_{v_n} of v_n-periodic spectra.

Heuts proved that the ∞-category of v_n-periodic spaces admits a Lie algebra model.

Theorem 1.1.12. *[Heu21b] The ∞-category of v_n-periodic spaces is equivalent to the ∞-category $\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}_{v_n})$ of spectral Lie algebras in Sp_{v_n}.*

We will give the definition of the ∞-category of spectral Lie algebras in Chapter 3.

We now introduce the algebraic model for Dwyer's tame homotopy theory [Dwy79], which is also the main topic of this thesis. All the localizations we have seen so far are obtained by first inverting a set of primes for all spaces. For rational spaces, we invert all the primes at all degrees of the homotopy groups. What if we progressively invert more primes as the degree of the homotopy groups increases? The motivation for doing so was inspired by the following theorem of Serre.

Theorem 1.1.13 (Serre). *For $n \geq 3$, the first p-torsion in the homotopy group $\pi_k(S^n)$ appears in degree $n + 2p - 3$.*

A *ring system* R_* is a sequence of subrings R_j of $\mathbb{Q}$ for $j \geq 0$ such that R_m is a subring of R_n if $m < n$. In this thesis, we will be interested in only one ring system, defined as follows.

Definition 1.1.14. The *tame ring system* $\{R_j\}_{j\geq 0}$ is defined as

$$R_j := \mathbb{Z}[\frac{1}{k}|k \leq \frac{j+3}{2}].$$

In other words, R_j is the smallest subring of $\mathbb{Q}$ in which p is inverted for all primes $p \leq \frac{j+3}{2}$.

Let $r \geq 3$ be an integer. Dwyer [Dwy79] defined the tame model structure on the category $\mathcal{S}_*^{\geq r}$ of pointed r-connective spaces, in which a map f is a weak equivalence if $\pi_{r+j}(f) \otimes R_j$ is an isomorphism for all $j \geq 0$. On the algebraic side, he defined the tame model structure on the category $\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Ch}_{\mathbb{Z}})^{\geq r-1}$ of $(r-1)$-connective dg Lie algebras over the integers $\mathbb{Z}$, in which a map f is a weak equivalence if the induced maps on homology groups with coefficients in the tame ring system are isomorphisms, that is, $H_{r+j-1}(f) \otimes R_j$ is an isomorphism for all $j \geq 0$. There seems to exist no direct homotopy functor that connects these two categories on the model category level. Dwyer then defined the notion of *Lazard algebras*, which are Lie algebras with just enough extra structure so that the Baker-Campbell-Hausdorff formula makes sense. He equipped the category of simplicial Lazard algebras with a model structure and proved that it is the intermediate category in a zig-zag of Quillen equivalences between $\mathcal{S}_*^{\geq r}$ and $\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Ch}_{\mathbb{Z}})^{\geq r-1}$.

Theorem 1.1.15. *[Dwy79] With the model structures described above, there is a zig-zag of Quillen equivalences between $\mathcal{S}_*^{\geq r}$ and $\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Ch}(\mathbb{Z}))^{\geq r-1}$.*

We end this section by stating a result of Anick [Ani89] which motivated us to consider the Hopf algebra model for tame spaces. Let p be a prime number, and let $R = \mathbb{Z}[\frac{1}{(p-1)!}]$. A free differential graded (dg) Lie algebra is *r-mild* if it is generated in the range of dimension from r to $rp-1$. Denote the category of free r-mild dg Lie algebras over R by $\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Ch}_r(R))$. Anick introduced a notion of *Hopf algebra up to homotopy* (Hah) [Ani89, Definition 4.1], which is a generalization of dg cocommutative Hopf algebras with the usual structure diagrams commuting up to homotopy. Let $\mathrm{Hah}_r(R)$ denote the category of r-mild Hah over R.

Theorem 1.1.16 ([Ani89] Theorem 4.8). *The universal enveloping algebra functor $U : \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Ch}_r(R)) \to \mathrm{Hah}_r(R)$ induces an equivalence on their homotopy categories:*

$$\mathrm{Ho}(\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Ch}_r(R))) \simeq \mathrm{Ho}(\mathrm{Hah}_r(R)).$$

1.2 Outline of the Thesis

The main objectives of the thesis are four-fold. In chapter 2, we will streamline the discussion of tame homotopy theory using the modern language of ∞-categories. A space X is *r-tame* if its homotopy groups are modules over the tame ring system, i.e. $\pi_{r+j}X$ is an R_j-module for all $j \geq 0$. We show that the ∞-category of r-tame spaces can be obtained by inverting maps that induce isomorphisms on homotopy groups with coefficients in the tame ring system. We will refer to such maps as *tame equivalences*. More concretely, we prove that the ∞-category $\mathcal{S}^{\geq r}_{\mathrm{tame}}$ of r-tame spaces is a localization of the ∞-category $\mathcal{S}^{\geq r}_{*}$ of pointed r-connective spaces at the class of tame equivalences.

Secondly, we define the ∞-category $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ of *r-tame spectra* and tame equivalences between them in a similar manner. We show that $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ is a localization of the ∞-category $\mathrm{Sp}^{\geq r}$ of r-connective spectra. The ∞-category $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ of tame spectra appears to have a nice algebraic description. Let $(\mathrm{Mod}^{\geq r}_{H\mathbb{Z}})_{\mathrm{tame}}$ be the ∞-category of r-connective $H\mathbb{Z}$-modules whose underlying spectra are tame.

Theorem A. *There is an symmetric monoidal equivalence of ∞-categories*

$$\mathrm{Sp}^{\geq r}_{\mathrm{tame}} \simeq (\mathrm{Mod}^{\geq r}_{H\mathbb{Z}})_{\mathrm{tame}}.$$

In Chapter 3, we define the ∞-category $\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}})$ of Lie algebras in tame spectra , which we will refer as the *tame spectral Lie algebras*. We apply Koszul duality to produce a universal enveloping algebra functor

$$U : \mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \to \mathrm{HopfAlg}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}),$$

where the target is the ∞-category of Hopf algebras in $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$.

In chapter 4, we establish a new Hopf algebra model for tame spaces.

Theorem B. *There is an equivalence of ∞-categories*

$$\mathcal{S}^{\geq r}_{\mathrm{tame}} \simeq \mathrm{HopfAlg}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}}).$$

Using the Hopf algebra model, we prove a Milnor-Moore theorem for tame spectra which has already been observed by Hess [Hes93, Theorem 6.6].

Theorem C. *The universal enveloping algebra functor U is an equivalence of ∞-categories.*

Assembling these three theorems, we then recover Dwyer's Lie algebra model for tame spaces.

1.3 Conventions and Notation

Throughout this paper, we will freely use the language of ∞-categories (i.e. quasi-categories) developed in [Lur09] and higher algebra from [Lur17]. We will try our best to provide explicit references to the relevant results in these books.

1.3.1 Conventions

- If $\mathcal{D}$ is an ordinary category, then we won't distinguish $\mathcal{D}$ and its nerve $N\mathcal{D}$ (when viewed as an ∞-category).
- We say a morphism $f : X \to Y$ in an ∞-category $\mathcal{D}$ is an *equivalence* if it is an isomorphism after passing to the homotopy category $\mathrm{h}\,\mathcal{D}$.
- If $\mathcal{D}$ is an ∞-category, we denote by $\mathcal{D}^{\simeq}$ the *core* of $\mathcal{D}$, i.e. the largest Kan subcomplex contained in $\mathcal{D}$.
- Let n be a non-negative integer. We will call a space (or spectrum) X *n-connective* if $\pi_i(X) = 0$ for $i < n$. Dually, we will call a space X (or spectrum) is n-truncated if $\pi_i(X) = 0$ for $i > n$.

- Let X be a space or spectrum, we denote $\tau_{\leq k}X$ (resp. $\tau_{\geq k}X$) for the k-truncation (resp. k-connective cover) of X.

- A map $f : X \to Y$ between spaces or spectra is n-connective (resp. n-truncated) if the (homotopy) fibers of f are n-connective (resp. n-truncated).

- We always assume spaces are pointed in this thesis.

1.3.2 Notation

- Δ denotes the category of non-empty finite linearly ordered sets.

- Δ_+ denotes the category of (possibly empty) finite linearly ordered sets. We will abuse notation by denoting the empty set by $[-1]$.

- $\Delta_+^{\leq n}$ is the full subcategory of Δ_+ spanned by the objects $\{[k]\}_{-1\leq k\leq n}$.

- Fin^{nu} is the category of non-empty finite sets.

- Fin is the category of (possibly empty) finite sets.

- Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. We let $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ denote the ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$.

- Let $f : K \to \mathcal{C}$ be a map of simplicial sets. We denote $\mathcal{C}_{/f}$ the slice category defined below [Lur09, Proposition 1.2.9.2.].

- $\mathcal{S}_*$ is the ∞-category of pointed spaces and Sp is the ∞-category of spectra.

- Cat_∞ is the ∞-category of (small) ∞-categories.

- Pr^L is the ∞-category of presentable ∞-categories with colimit-preserving functors as morphisms [Lur09, Definition 5.5.3.1.].

- Let $\mathcal{C}, \mathcal{D} \in \mathrm{Pr}^L$. We let $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ (resp. $\mathrm{Fun}^R(\mathcal{C}, \mathcal{D})$) denote the ∞-category of colimit-preserving functors from $\mathcal{C}$ to $\mathcal{D}$ that are left adjoints (resp. right adjoints).

- For every pair of ∞-categories $\mathcal{C}$ and $\mathcal{D}$, we let $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ denote the ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$.

- If $F : \mathcal{C} \to \mathcal{C}'$ is a functor between ∞-categories, then we denote $F_* : \mathrm{Fun}(\mathcal{D}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{D}, \mathcal{C}')$ and $F^* : \mathrm{Fun}(\mathcal{C}', \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the functor induced by post-composing and pre-composing with F, respectively.

- $\mathbb{1}$ denotes the trivial ∞-operad, which is the unit object in the monoidal category of ∞-operads with respect to the composition product.

- If $\mathcal{C}$ is a monoidal ∞-category. We let $\mathrm{Alg}(\mathcal{C})$ denote the ∞-category of associative algebras in $\mathcal{C}$. If $\mathcal{C}$ is a symmetric monoidal ∞-category, we let $\mathrm{CAlg}(\mathcal{C})$ denote the ∞-category of commutative algebras in $\mathcal{C}$.

- Informally, a monad (resp. comonad) ([Lur17, Definition 4.7.3.2.]) T on $\mathcal{C}$ is an associative (resp. coassociative) algebra object in the ∞-category of endofunctors on $\mathcal{C}$. $\mathrm{LMod}_T(\mathcal{C})$ denotes the ∞-category of left modules over the monad T as defined in [Lur17, Section 4.2]. Dually, $\mathrm{LcoMod}_{\mathcal{Q}}(\mathcal{C})$ denotes the ∞-category of left comodules over the comonad $\mathcal{Q}$.

Chapter 2

Tame Spaces and Tame Spectra

In this chapter, we give a modern treatment of Dwyer's tame spaces in §2.1 using the theory of localizations. In §2.2, we give a concrete characterization of tame equivalences. We then extend the definition of tameness to spectra in §2.3. We prove that the ∞-category of r-tame spaces (resp. r-tame spectra) is a localization of the ∞-category of r-connective spaces (resp. r-connective spectra). We then discuss properties of these two categories and establish an algebraic characterization of r-tame spectra.

2.1 Tame Spaces

In this section, we first define the notion of a tame space. Then we explain how to obtain the ∞-category of r-tame spaces as a localization of the ∞-category $\mathcal{S}_*^{\geq r}$ of pointed r-connective spaces. Recall that we have defined the tame ring system $\{R_j\}_{j\leq 0}$ in Definition 1.1.14.

Definition 2.1.1. Let $X \in \mathcal{S}_*^{\geq r}$ be a pointed r-connective space. We say X is *r-tame* if for all $j \geq 0$, the $(r+j)$-th homotopy group $\pi_{r+j}(X)$ is uniquely p-divisible for all $p \leq \frac{j+3}{2}$. This is equivalent to saying that $\pi_{r+j}(X)$ is a R_j-module for each j, that is,

$$\pi_{r+j}(X) \cong \pi_{r+j}(X) \otimes R_j.$$

Remark 2.1.2. Note that the definition of tame spaces depends on an integer r. It is clear from the definiton that, for two integers $s \geq r \geq 3$, if a s-connective space X is r-tame, then it is also s-tame.

Notation 2.1.3. We let $\mathcal{S}^{\geq r}_{\text{tame}}$ denote the full subcategory of $\mathcal{S}^{\geq r}_{*}$ spanned by r-tame spaces. We decide to omit the r from the subscript since the level of tameness will be clear from the superscript. Moreover, we will simply call a space *tame* if the level of tameness is clear from the context.

Given a space $X \in \mathcal{S}^{\geq r}_{*}$, we want to produce a space X_{tame} that is the universal tame space receiving a map from X. Moreover, we want this assignment to be functorial. The theory of localization, well-studied in Bousfield [Bou75], [Bou79] and Farjoun [Far96], provides powerful machinery for doing so in the context of model categories. For localization of ∞-categories, we will follow [Lur09].

Definition 2.1.4. [Lur09, Definition 5.2.7.2.] A functor $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories is a *localization* if F has a fully faithful right adjoint.

In our case, we will construct a *tame localization functor*

$$L_{\text{tame}} : \mathcal{S}^{\geq r}_{*} \to \mathcal{S}^{\geq r}_{\text{tame}}$$

so that the inclusion functor is its right adjoint. Our strategy for proving the existence of such a localization functor is to show the effect of inverting primes in homotopy groups is equivalent to inverting some maps in $\mathcal{S}^{\geq r}_{*}$.

To explain this idea, we now recall some terminology from Definition 1.1.6. Let $\mathcal{C}$ be an ∞-category and S a collection of morphisms in $\mathcal{C}$. An object Z is said to be *S-local* if for every morphism $f : X \to Y$ in S, restricting along f

$$\mathrm{Map}_{\mathcal{C}}(Y, Z) \to \mathrm{Map}_{\mathcal{C}}(X, Z)$$

is a weak equivalence. A morphism $f : X \to Y$ in $\mathcal{C}$ is an *S-equivalence* if, for every S-local object Z, composition with f induces a weak equivalence

$$\mathrm{Map}_{\mathcal{C}}(Y, Z) \to \mathrm{Map}_{\mathcal{C}}(X, Z).$$

Remark 2.1.5. Let $L : \mathcal{C} \to L\mathcal{C} \hookrightarrow \mathcal{C}$ be a localization of an ∞-category $\mathcal{C}$, and let S be the class of morphisms f in $\mathcal{C}$ such that Lf is an equivalence. By [Lur09, Proposition 5.5.4.2.], the full subcategory spanned by S-local objects is precisely the essential image $L\mathcal{C}$ under L, and every S-equivalence in $\mathcal{C}$ belongs to S.

To understand the effect of inverting primes in homotopy groups, let's first consider the following simple case. Let $m_p : S^n \xrightarrow{p} S^n$ be a map of degree p. Observe that a space X is m_p-local if and only if the multiplication-by-p map on the homotopy groups

$$\pi_*(X) \xrightarrow{\cdot p} \pi_*(X)$$

is an isomorphism in degrees $* \geq n$. In other words, a space X is m_p-local if and only if its homotopy groups $\pi_*(X)$ are uniquely p-divisible in degrees large or equal to n.

Let P denote the set of all primes. For each prime q, we consider the following composition

$$m_q : S^{r+2q-3} \xrightarrow{\cdot q} S^{r+2q-3} \xrightarrow{\text{inclusion}} \bigvee_{p\in P} S^{r+2p-3},$$

which induces a canonical map

$$\vee_{p\in P}\, m_p : \bigvee_{p\in P} S^{r+2p-3} \to \bigvee_{p\in P} S^{r+2p-3}. \tag{2.1}$$

To simplify notation, we will denote the map $\vee_{p\in P} m_p$ by f.

Lemma 2.1.6. *A space $X \in \mathcal{S}_*^{\geq r}$ is tame if and only if it is f-local.*

Proof. A space X is f-local if and only if the induced map on mapping spaces

$$\mathrm{Map}_*(\bigvee_{p\in P} S^{r+2p-3}, X) \xrightarrow{f^*} \mathrm{Map}_*(\bigvee_{p\in P} S^{r+2p-3}, X)$$

is a weak equivalence. This is equivalent to asking that the homotopy groups $\pi_{r+j}(X)$ be uniquely p-divisible for all primes $p \leq \frac{j+3}{2}$ for every fixed $j \geq 0$. The lemma now follows from the following commuting diagram

$$\begin{tikzcd}
\pi_{r+j}(X) \arrow[r, "\cdot p"] \arrow[d] & \pi_{r+j}(X) \arrow[d] \\
\pi_{j-(2p-3)}(\mathrm{Map}_*(S^{r+2p-3}, X)) \arrow[r, "\cdot p"] & \pi_{j-(2p-3)}(\mathrm{Map}_*(S^{r+2p-3}, X)),
\end{tikzcd}$$

where the vertical arrows are isomorphisms. Hence the top horizontal map is an isomorphism if and only if the bottom horizontal map is an isomorphism.

□

The following proposition guarantees the existence of the localization of presentable ∞-categories.

Proposition 2.1.7. *[Lur09, Proposition 5.5.4.15.] Let $\mathcal{C}$ be a presentable ∞-category, and let S be a (small) set of morphisms of $\mathcal{C}$. Let $\overline{S}$ denote the strongly saturated (cf. [Lur09, Definition 5.5.4.5.]) class of morphisms generated by S. If $\mathcal{C}' \subseteq \mathcal{C}$ denotes the full subcategory of $\mathcal{C}$ spanned by S-local objects, then*

1. *For each object $C \in \mathcal{C}$, there exists a morphism $s : C \to C'$ such that C' is S-local and s belongs to $\overline{S}$.*
2. *The ∞-category $\mathcal{C}'$ is presentable.*
3. *The inclusion $\mathcal{C}' \subseteq \mathcal{C}$ has a left adjoint L.*
4. *For every morphism f of $\mathcal{C}$, the following are equivalent:*
   - *(i) The morphism f is an S-equivalence.*
   - *(ii) The morphism f belongs to $\overline{S}$.*
   - *(iii) The induced morphism Lf is an equivalence.*

If $\mathcal{C} = \mathcal{S}_*^{\geq r}$ and $S = \{f\}$ in (2.1), we will also call an S-local space (resp. S-equivalence) L_f-local (resp. L_f-equivalence). The proposition above guarantees the existence of the tame localization functor.

Corollary 2.1.8. *There exists a localization functor*

$$L_{\mathrm{tame}} : \mathcal{S}_*^{\geq r} \to \mathcal{S}_{\mathrm{tame}}^{\geq r}.$$

Moreover, the ∞-category $\mathcal{S}_{\mathrm{tame}}^{\geq r}$ of tame spaces is a presentable ∞-category.

Definition 2.1.9. A morphism $g : X \to Y$ in $\mathcal{S}_*^{\geq r}$ is said to be a *tame equivalence* if it is an L_f-equivalence in the sense of Definition 1.1.6.

2.2 Characterization of Tame Equivalences

In this section, we give an explicit characterization of tame equivalences by showing that the functor $L_{tame} : \mathcal{S}_*^{\geq r} \to \mathcal{S}_{tame}^{\geq r}$ is an infinite composite of localization functors.

We first consider the effect of localizing with respect to the multiplication-by-p map

$$m_p : S^{r+2p-3} \to S^{r+2p-3}$$

for a fixed prime p. Note that a space $X \in \mathcal{S}_*^{\geq r}$ is L_{m_p}-local if and only if $\pi_* X$ is uniquely p-divisible for $* \geq r+2p-3$. Let $\mathcal{S}_{m_p}^{\geq r}$ denote the full subcategory of $\mathcal{S}_*^{\geq r}$ spanned by spaces whose homotopy groups are uniquely p-divisible in degree larger or equal to $r+2p-3$. Proposition 2.1.7 guarantees the existence of a localization functor

$$L_{m_p} : \mathcal{S}_*^{\geq r} \to \mathcal{S}_{m_p}^{\geq r}.$$

A map $g : X \to Y$ is a L_{m_p}-equivalence if and only if $L_{m_p} g : L_{m_p} X \to L_{m_p} Y$ is an equivalence. We first show that L_{m_p}-localization does not change the homotopy groups below degree $r+2p-3$.

Proposition 2.2.1. *If $g : X \to Y$ is a n-connective map in $\mathcal{S}_*$, then every L_g-equivalence is n-connective.*

Proof. Take $S = \{g\}$ and denote by $\overline{S}$ the strongly saturated class of morphisms generated by S. Proposition 2.1.7 implies that $\overline{S} = \{L_g\text{-equivalences}\}$, hence it suffices to show that $\overline{S}$ is contained in the class of n-connective maps. Note that the class of n-connective maps is saturated in the sense of [Lur09, Definition 5.5.5.1]. Since every strongly saturated class of morphisms in a presentable ∞-category is also saturated [Lur09, Example 5.5.5.5.], we have

$$\{L_g\text{-equivalences}\} = \overline{S} \subseteq \tilde{S} \subseteq \{n\text{-connective maps}\}$$

where $\tilde{S}$ denotes the saturated class of morphisms generated by S. □

Corollary 2.2.2. *If $g : A \to B$ is n-connective, then the localization $L_g : \mathcal{S}_* \to \mathcal{S}_*$ doesn't change the homotopy groups of a space X below degree n, i.e.*

$$\pi_i(L_g X) \cong \pi_i(X)$$

for $i < n$.

We now compare L_{m_p}-localization with the Bousfield localization with respect to the homology theory $H_*(-; \mathbb{Z}[\frac{1}{p}])$.

In this case, we let S' be the collection of morphisms that induce isomorphisms on homology with coefficients in $\mathbb{Z}[\frac{1}{p}]$, or equivalently on $\pi_*(-) \otimes \mathbb{Z}[\frac{1}{p}]$

by [Bou96, Proposition 4.3], as we are working in $\mathcal{S}_*^{\geq r}$. We will call an S'-local space $\mathbb{Z}[\frac{1}{p}]$-*local*. By [Bou96, Theorem 5.5], a space whose homotopy groups are uniquely p-divisible is $\mathbb{Z}[\frac{1}{p}]$-local. Hence, any $\mathbb{Z}[\frac{1}{p}]$-local space is also L_{m_p}-local, and any L_{m_p}-equivalence also induces isomorphisms on $\pi_*(-) \otimes \mathbb{Z}[\frac{1}{p}]$.

Lemma 2.2.3. *If $f : X \to Y$ is a L_{m_p}-equivalence of spaces, then it induces isomorphisms*

$$\pi_*(X) \otimes \mathbb{Z}[\tfrac{1}{p}] \to \pi_*(Y) \otimes \mathbb{Z}[\tfrac{1}{p}].$$

Moreover, if $f : X \to Y$ is a map between $(r + 2p - 3)$-connective spaces, then f is a L_{m_p}-equivalence if and only if it is a $\mathbb{Z}[\frac{1}{p}]$-equivalence.

Proof. The first part is clear. The second part follows from the fact that a $(r + 2p - 3)$-connective space is L_{m_p}-local if and only if it is $\mathbb{Z}[\frac{1}{p}]$-local. □

Let $\tau_{\geq r+2p-3}X$ denote the $(r+2p-3)$-connective cover and $\tau_{\leq r+2p-4}X$ the $(r + 2p - 4)$-truncation of X. We claim that L_{m_p}-localization commutes with taking $(r + 2p - 3)$-connective cover.

Lemma 2.2.4. *The localization functor L_{m_p} commutes with taking $(r+2p-3)$-th connective cover, i.e., there is an equivalence*

$$L_{m_p}(\tau_{\geq r+2p-3}X) \simeq \tau_{\geq r+2p-3}(L_{m_p}X)$$

for any $X \in \mathcal{S}_*^{\geq r}$.

Proof. It suffices to show the map $\tau_{\geq r+2p-3}X \to \tau_{\geq r+2p-3}L_{m_p}X$ is a L_{m_p}-equivalence, since $\tau_{\geq r+2p-3}L_{m_p}X$ is L_{m_p}-local. Consider the following commutative diagram of fiber sequences

$$\begin{tikzcd}
\tau_{\geq r+2p-3}X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \tau_{\leq r+2p-4}X \arrow[d] \\
\tau_{\geq r+2p-3}L_{m_p}X \arrow[r] & L_{m_p}X \arrow[r] & \tau_{\leq r+2p-4}L_{m_p}X \simeq \tau_{\leq r+2p-4}X,
\end{tikzcd}$$

where the equivalence at the lower right corner follows from Corollary 2.2.2. Note that the diagram above induces a commutative diagram of two long exact sequences of homotopy groups. For each $n \geq r+2p-3$, there is a commutative diagram

$$\begin{tikzcd}
\pi_n(\tau_{\geq r+2p-3}X) \otimes \mathbb{Z}[\frac{1}{p}] \arrow[r] \arrow[d] & \pi_n(X) \otimes \mathbb{Z}[\frac{1}{p}] \arrow[d] \\
\pi_n(\tau_{\geq r+2p-3}L_{m_p}X) \otimes \mathbb{Z}[\frac{1}{p}] \arrow[r] & \pi_n(L_{m_p}X) \otimes \mathbb{Z}[\frac{1}{p}],
\end{tikzcd}$$

where the horizontal maps are isomorphisms and the right vertical map is also an isomorphism by Lemma 2.2.3, hence the left map is also an isomorphism. The lemma then follows from the second part of Lemma 2.2.3.

□

We can now compute the homotopy groups of the L_{m_p}-localization of a space.

Corollary 2.2.5. *The homotopy groups of $L_{m_p}X$ are given by*

$$\pi_* L_{m_p}X = \begin{cases} \pi_* X & \text{if } * < r+2p-3 \\ \pi_* X \otimes \mathbb{Z}[\frac{1}{p}] & \text{if } * \geq r+2p-3. \end{cases}$$

Proof. Since the map m_p is $(r+2p-3)$-connective, the map $X \to L_{m_p}X$ is also $(r+2p-3)$-connective. By Proposition 2.2.1. Hence there are isomorphisms in homotopy groups

$$\pi_* L_{m_p}X \cong \pi_* X$$

for $* < r+2p-3$. Since L_{m_p} commutes with taking $(r+2p-3)$-connective cover by Lemma 2.2.4, we can assume X is $(r+2p-3)$-connective. We claim that $L_{m_p}X \simeq L_{\mathbb{Z}[\frac{1}{p}]}X$, that is, $L_{m_p}X$ is the localization of X with respect to the homology theory $H_*(-;\mathbb{Z}[\frac{1}{p}])$.

Indeed, the homotopy groups of $L_{m_p}X$ are uniquely p-divisible, so $L_{m_p}X$ is $L_{\mathbb{Z}[\frac{1}{p}]}$-local and we have a unique factorization

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & L_{m_p}X \\
& L_{\mathbb{Z}[\frac{1}{p}]}X \arrow[ur, dashed] &
\end{tikzcd}.$$

By Lemma 2.2.3, the horizontal map is an $L_{\mathbb{Z}[\frac{1}{p}]}X$-equivalence. Hence the dashed diagonal map is also an $L_{\mathbb{Z}[\frac{1}{p}]}$-equivalence, which implies that $L_{m_p}X \simeq L_{\mathbb{Z}[\frac{1}{p}]}X$.

Therefore, we conclude that $\pi_* L_{m_p} X = \pi_* X \otimes \mathbb{Z}[\frac{1}{p}]$ for $* \geq r + 2p - 3$.

□

Corollary 2.2.6. *A map $g : X \to Y$ is an L_{m_p}-equivalence if and only if the induced maps on homotopy groups satisfy:*

1. *$\pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i < r + 2p - 3$;*
2. *$\pi_i(X) \otimes \mathbb{Z}[\frac{1}{p}] \to \pi_i(Y) \otimes \mathbb{Z}[\frac{1}{p}]$ is an isomorphism for $i \geq r + 2p - 3$.*

Proof. Note that f is a L_{m_p}-equivalence if and only if

$$L_{m_p} X \to L_{m_p} Y$$

is an equivalence. By our computation of the homotopy groups in Corollary 2.2.5, this is equivalent to requiring that f is $(r+2p-3)$-connective and induces isomorphisms on $\pi_{n+r+2p-3} X \otimes \mathbb{Z}[\frac{1}{p}] \to \pi_{n+r+2p-3} Y \otimes \mathbb{Z}[\frac{1}{p}]$ for any $n \geq 0$.

□

We conclude this section by showing that localization with respect to the map f is equivalent to localization with respect to an infinite composite of localizations.

Proposition 2.2.7. *If f is the map in (2.1), then there is an equivalence*

$$L_f X \simeq \operatorname{colim}(X \to L_{m_2} X \to L_{m_3} L_{m_2} X \to \cdots)$$

in $\mathcal{S}_*^{\geq r}$.

Proof. Let $X_\infty := \operatorname{colim}(X \to L_{m_2} X \to L_{m_3} L_{m_2} X \to \cdots)$. The space X_∞ is tame; indeed, for fixed j, we have

$$\pi_{r+j}(X_\infty) \cong \pi_{r+j}(L_{m_q} \cdots L_{m_2} X) \cong \pi_{r+j}(X) \otimes R_j,$$

where q is the largest prime number less than or equal to $\frac{j+3}{2}$. Since there is also an isomorphism $\pi_{r+j}(L_f X) \cong \pi_{r+j} X \otimes R_j$, we conclude that the canonical map

$$L_f X \to X_\infty$$

is an equivalence.

□

As a consequence, a map $g : X \to Y$ in $\mathcal{S}_*^{\geq r}$ is a tame equivalence if and only if it is a L_{m_p}-equivalence for every prime p, which implies the following corollary.

Corollary 2.2.8. *A map $g : X \to Y$ between r-connective spaces is a tame equivalence if and only if the induced maps*

$$\pi_{r+j}X \otimes R_j \to \pi_{r+j}Y \otimes R_j$$

are isomorphisms for all $j \geq 0$.

Remark 2.2.9. Let $s \leq r$ be non-negative integers. If $X \to Y$ is an r-tame equivalence, then it's also an s-tame equivalence.

We end this section with a basic example of tame equivalences.

Example 2.2.10. For r an odd number, the canonical map

$$S^r \to K(\mathbb{Z}, r)$$

is a tame equivalence since it is a rational equivalence.

2.3 Tame Spectra

In this section, we introduce the notion of tame spectra. These spectra are defined analogously to tame spaces, that is, we impose divisibility conditions on their homotopy groups. Then we establish functors which connect tame spaces and tame spectra. Finally, we establish an algebraic characterization of the ∞-category of tame spectra.

As opposed to the case of spaces, we allow r to be any non-negative integer, and we denote the ∞-category of r-connective spectra by $\mathrm{Sp}^{\geq r}$.

Definition 2.3.1. A r-connective spectrum X is *r-tame* if the $(r+j)$-th homotopy group $\pi_{r+j}(X)$ is an R_j-module for $j \geq 0$, or equivalently, there are isomorphisms

$$\pi_{r+j}(X) \cong \pi_{r+j}(X) \otimes R_j$$

for all $j \leq 0$.

Notation 2.3.2. We let $\mathrm{Sp}^{\geq r}_{tame}$ denote the full subcategory of $\mathrm{Sp}^{\geq r}$ spanned by r-tame spectra.

Let f be the map

$$f : \bigvee_{p \in P} \mathbb{S}^{r+2p-3} \to \bigvee_{p \in P} \mathbb{S}^{r+2p-3} \tag{2.2}$$

defined by assembling the multiplication by different primes as we did for spaces in (2.1).

Definition 2.3.3. A map $g : X \to Y$ between r-connective spectra is a *tame equivalence* if it is an L_f-equivalence.

We summarize the results regarding tame localization of spectra and tame equivalences below.

Proposition 2.3.4. *1. A spectrum $X \in \mathrm{Sp}^{\geq r}$ is tame if and only if it is L_f-local.*

2. The tame localization

$$L_{\mathrm{tame}} : \mathrm{Sp}^{\geq r} \to \mathrm{Sp}^{\geq r}_{\mathrm{tame}}$$

exists. Moreover, the ∞-category $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ of tame spectra is presentable.

3. A map $g : X \to Y$ between r-connective spectra is a tame equivalence if and only if the induced maps

$$\pi_{r+j} X \otimes R_j \to \pi_{r+j} Y \otimes R_j$$

are isomorphisms for all $j \geq 0$.

Proof. The proofs of these statements are completely analogous to those of Lemma 2.1.6, Corollary 2.1.8 and Proposition 2.2.8.

□

Remark 2.3.5. We remark that tame localization of $\mathrm{Sp}^{\geq r}$ is no longer prestable [Lur18b, Definition C.1.2.1.], since the tame localization is no longer compatible with the suspension Σ in Sp. This is contrast to many usual localizations of the ∞-category of spectra.

Let $\mathcal{C}$ be a presentable ∞-category and let S be a collection of morphisms in $\mathcal{C}$. We now state a closure property S-equivalences in $\mathcal{C}$. In particular, the lemma below implies that the collection of tame equivalences is closed under colimits.

Lemma 2.3.6. *Let K be a simplicial set and let $f : X \to Y$ be a pointwise S-equivalence in* $\mathrm{Fun}(K, \mathcal{C})$*. Then the induced map on colimits*

$$\mathrm{colim}_{k \in K} X_k \to \mathrm{colim}_{k \in K} Y_k$$

is an S-equivalence.

Proof. This follows immediately from Proposition 2.1.7, since the collection of S-equivalences is strongly saturated, and hence is closed under colimits in $\mathrm{Fun}(\Delta^1, \mathcal{C})$. □

We now discuss the relation between the ∞-category $\mathcal{S}^{\geq r}_{\mathrm{tame}}$ of tame spaces and the ∞-category $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ of tame spectra. Since there are isomorphisms

$$\pi_*(\Omega^\infty X) \cong \pi_*(X)$$

for any spectrum X, the infinite loop space of a tame spectrum is a tame space. Therefore, the functor $\Omega^\infty : \mathrm{Sp}^{\geq r} \to \mathcal{S}^{\geq r}$ restricts to a functor

$$\Omega^\infty : \mathrm{Sp}^{\geq r}_{\mathrm{tame}} \to \mathcal{S}^{\geq r}_{\mathrm{tame}}.$$

Lemma 2.3.7. *The suspension functor $\Sigma^\infty : \mathcal{S}^{\geq r}_* \to \mathrm{Sp}^{\geq r}$ sends tame equivalences of spaces to tame equivalences of spectra.*

Proof. Let $f : X \to Y$ be a tame equivalence in $\mathcal{S}^{\geq r}_*$. By definition, $\Sigma^\infty f : \Sigma^\infty X \to \Sigma^\infty Y$ is a tame equivalence if, for every tame spectrum Z, the induced map

$$\mathrm{Map}_{\mathrm{Sp}^{\geq r}}(\Sigma^\infty Y, Z) \to \mathrm{Map}_{\mathrm{Sp}^{\geq r}}(\Sigma^\infty X, Z)$$

is a weak equivalence. Note that the map above is equivalent to the map

$$\mathrm{Map}_{\mathcal{S}^{\geq r}_*}(Y, \Omega^\infty Z) \to \mathrm{Map}_{\mathcal{S}^{\geq r}_*}(X, \Omega^\infty Z)$$

which is a weak equivalence as $\Omega^\infty Z$ is a tame space and f is a tame equivalence by assumption. □

We now discuss some basic examples of tame equivalences in $\mathrm{Sp}^{\geq r}$.

Example 2.3.8. 1. For any integer $r \geq 0$, the r-truncation map

$$\mathbb{S}^r \to \Sigma^r H\mathbb{Z}$$

is an r-tame equivalence. Indeed, the first p-torsion in the homotopy groups of the shifted sphere spectrum $\mathbb{S}^r$ appears in degree $r+2p-3$, hence its r-tame localization is the shifted Eilenberg-MacLane spectrum $\Sigma^r H\mathbb{Z}$. This example indicates that $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ is generated by $\Sigma^r H\mathbb{Z}$ under colimits, since $\mathrm{Sp}^{\geq r}$ is generated by $\mathbb{S}^r$ under colimits.

2. Let $r \geq 3$ be an odd integer, then the canonical map

$$S^r \to K(\mathbb{Z}, r)$$

is an r-tame equivalence by Example 2.2.10. Then Lemma 2.3.7 implies that

$$\mathbb{S}^r \to \Sigma^\infty K(\mathbb{Z}, r)$$

is a tame equivalence. Combining with the previous example, we conclude that the r-truncation map

$$\Sigma^\infty K(\mathbb{Z}, r) \to \Sigma^r H\mathbb{Z}$$

is also a tame equivalence. Since $\Sigma^r H\mathbb{Z}$ is r-tame, hence we have

$$L_{\mathrm{tame}}\Sigma^\infty K(\mathbb{Z}, r) \simeq \Sigma^r H\mathbb{Z}.$$

We now show that the class of r-tame equivalences is closed under smashing with a connective spectrum.

Lemma 2.3.9. *Let $r \geq 0$ and let E be a connective spectrum. If $f : X \to Y$ is an r-tame equivalence in $\mathrm{Sp}^{\geq r}$, then $E \otimes X \to E \otimes Y$ is an r-tame equivalence in $\mathrm{Sp}^{\geq r}$.*

Proof. Let $\mathcal{E}$ be the full subcategory of $\mathrm{Sp}^{\geq 0}$ spanned by spectra F satisfying the condition that

$$F \otimes X \to F \otimes Y$$

is an r-tame equivalence whenever $X \to Y$ is an r-tame equivalence. Clearly, the sphere spectrum $\mathbb{S}$ is in $\mathcal{E}$. Since $\mathrm{Sp}^{\geq 0}$ is generated by $\mathbb{S}$ under colimits, the claim will follow if we can show $\mathcal{E}$ is closed under colimits. Let $Z : K \to \mathcal{E}$ be a diagram in $\mathcal{E}$. Note that by Lemma 2.3.6 and the fact that smash product commutes with colimits in $\mathrm{Sp}^{\geq 0}$, we conclude that

$$(\mathrm{colim}_K Z_k) \otimes X \to (\mathrm{colim}_K Z_k) \otimes Y$$

is an r-tame equivalence.

□

Remark 2.3.10. 1. As a direct consequence of Lemma 2.3.9, if $X \to Y$ is an r-tame equivalence, then $X \otimes H\mathbb{Z} \to Y \otimes H\mathbb{Z}$ is also an r-tame equivalence. Hence, for every $k \geq 0$, the induced map on homology

$$H_{r+k}(X) \otimes R_k \to H_{r+k}(Y) \otimes R_k$$

is an isomorphism.

2. Lemma 2.3.9 above implies that r-tame localization is compatible with the smash product in $\mathrm{Sp}^{\geq r}$ in the sense of [Lur17, Definition 2.2.1.6.]. By [Lur17, Proposition 2.2.1.9.], the ∞-category $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ of tame spectra admits a (non-unital) symmetric monoidal structure given by

$$X \hat{\otimes} Y := L_{\mathrm{tame}}(X \otimes Y).$$

Moreover, the tame localization

$$L_{\mathrm{tame}} : \mathrm{Sp}^{\geq r} \to \mathrm{Sp}^{\geq r}_{\mathrm{tame}}$$

is symmetric monoidal, hence the tensor product $\hat{\otimes}$ preserves colimits in each variable.

Corollary 2.3.11. *Let $E \in \mathrm{Sp}^{\geq r}$. Then the map*

$$E \simeq E \otimes \mathbb{S} \to E \otimes H\mathbb{Z}$$

is an r-tame equivalence.

Proof. This is immediate from Lemma 2.3.9, since $\mathbb{S} \to H\mathbb{Z}$ is a 0-tame equivalence, and E is r-connective. □

Remark 2.3.12. One might ask whether $H\mathbb{Z}$-module spectra are r-tame for all r, but this is false. For instance, $\Sigma H\mathbb{Z}$ is a 1-connective $H\mathbb{Z}$-module but it's not 0-tame, as $\pi_1 \Sigma H\mathbb{Z} \cong \mathbb{Z}$ is not uniquely 2-divisible.

We now give an algebraic characterization of tame spectra. Let $\mathrm{Mod}_{H\mathbb{Z}}$ denote the ∞-category of $H\mathbb{Z}$-modules. Note that $\mathrm{Mod}_{H\mathbb{Z}}$ can be identified with the derived ∞-category $D(\mathbb{Z})$ by [Lur17, Remark 7.1.1.16.]. We let $(\mathrm{Mod}^{\geq r}_{H\mathbb{Z}})_{\mathrm{tame}}$

denote the full subcategory of $\mathrm{Mod}_{H\mathbb{Z}}^{\geq r}$ spanned by r-connective $H\mathbb{Z}$-modules whose underlying spectra are tame.

Construction 2.3.13. We now explain how to realize $(\mathrm{Mod}_{H\mathbb{Z}}^{\geq r})_{\mathrm{tame}}$ as a localization of $\mathrm{Mod}_{H\mathbb{Z}}^{\geq r}$. We first remark that this procedure is completely analogous to the case of spaces and spectra. Similar to (2.2), we assemble the multiplication-by-p maps $m_p : \Sigma^{r+2p-3}H\mathbb{Z} \to \Sigma^{r+2p-3}H\mathbb{Z}$ to a map

$$f : \bigvee_{p\in P} \Sigma^{r+2p-3}H\mathbb{Z} \to \bigvee_{p\in P} \Sigma^{r+2p-3}H\mathbb{Z}. \tag{2.3}$$

By definition, $X \in \mathrm{Mod}_{H\mathbb{Z}}^{\geq r}$ is f-local if

$$\mathrm{Map}_{\mathrm{Mod}_{H\mathbb{Z}}}(\bigvee_{p\in P} \Sigma^{r+2p-3}H\mathbb{Z}, X) \to \mathrm{Map}_{\mathrm{Mod}_{H\mathbb{Z}}}(\bigvee_{p\in P} \Sigma^{r+2p-3}H\mathbb{Z}, X)$$

is a weak equivalence. Since we have an equivalence

$$\mathrm{Map}_{\mathrm{Mod}_{H\mathbb{Z}}}(\Sigma^{r+2p-3}H\mathbb{Z}, X) \simeq \mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}^{r+2p-3}, X),$$

X is f-local if and only if its underlying spectrum is tame, i.e., $X \in (\mathrm{Mod}_{H\mathbb{Z}}^{\geq r})_{\mathrm{tame}}$. By Proposition 2.1.7, there is a localization functor

$$\mathrm{Mod}_{H\mathbb{Z}}^{\geq r} \to (\mathrm{Mod}_{H\mathbb{Z}}^{\geq r})_{\mathrm{tame}}$$

which we will again denote by L_{tame}. We will refer to L_{tame}-equivalences in $\mathrm{Mod}_{H\mathbb{Z}}^{\geq r}$ as *tame $H\mathbb{Z}$-equivalences*.

It is clear from the definition of the ∞-category $(\mathrm{Mod}_{H\mathbb{Z}}^{\geq r})_{\mathrm{tame}}$ that a map $f : X \to Y$ in $\mathrm{Mod}_{H\mathbb{Z}}^{\geq r}$ is a tame $H\mathbb{Z}$-equivalence if it is a tame equivalence in $\mathrm{Sp}^{\geq r}$.

Remark 2.3.14. Let $f : M \to M'$ be a tame $H\mathbb{Z}$-equivalence, then

$$f \underset{\mathbb{Z}}{\otimes} \mathrm{id} : M \underset{\mathbb{Z}}{\otimes} \Sigma^r H\mathbb{Z} \to M' \underset{\mathbb{Z}}{\otimes} \Sigma^r H\mathbb{Z}$$

is a tame $H\mathbb{Z}$-equivalence by Lemma 2.3.6. Since $\mathrm{Mod}_{H\mathbb{Z}}^{\geq r}$ is generated under colimits by $\Sigma^r H\mathbb{Z}$, the map

$$f \underset{\mathbb{Z}}{\otimes} N : M \underset{\mathbb{Z}}{\otimes} N \to M' \underset{\mathbb{Z}}{\otimes} N$$

is a tame $H\mathbb{Z}$-equivalence for any $N \in \mathrm{Mod}_{H\mathbb{Z}}^{\geq r}$.

Therefore, tame $H\mathbb{Z}$-equivalences are compatible with the symmetric monoidal structure on $\mathrm{Mod}_{H\mathbb{Z}}^{\geq r}$ and we can equip $(\mathrm{Mod}_{H\mathbb{Z}}^{\geq r})_{\mathrm{tame}}$ with a symmetric monoidal structure given by

$$M \hat{\otimes} N \simeq L_{\mathrm{tame}}(M \underset{\mathbb{Z}}{\otimes} N).$$

Note that the functor

$$L_{\mathrm{tame}}(- \otimes H\mathbb{Z}) : \mathrm{Sp}^{\geq r} \to (\mathrm{Mod}_{H\mathbb{Z}}^{\geq r})_{\mathrm{tame}}$$

is left adjoint to the forgetful functor $U : (\mathrm{Mod}_{H\mathbb{Z}}^{\geq r})_{\mathrm{tame}} \to \mathrm{Mod}_{H\mathbb{Z}}^{\geq r} \to \mathrm{Sp}^{\geq r}$. Moreover, the forgetful functor has essential images in $\mathrm{Sp}_{\mathrm{tame}}^{\geq r}$, hence by Proposition A.1.6, we obtain a pair of adjoint functors

$$L_{\mathrm{tame}}(- \otimes H\mathbb{Z})\colon \mathrm{Sp}_{\mathrm{tame}}^{\geq r} \rightleftarrows (\mathrm{Mod}_{H\mathbb{Z}}^{\geq r})_{\mathrm{tame}} : U.$$

We can now state and prove Theorem A stated in the introduction.

Theorem 2.3.15. *The functor*

$$L_{\mathrm{tame}}(- \otimes H\mathbb{Z}) : \mathrm{Sp}_{\mathrm{tame}}^{\geq r} \to (\mathrm{Mod}_{H\mathbb{Z}}^{\geq r})_{\mathrm{tame}}$$

is a symmetric monoidal equivalence of ∞-categories.

Proof. We first show $L_{\mathrm{tame}}(- \otimes H\mathbb{Z})$ is an equivalence of ∞-categories. Since the forgetful functor U is conservative, it suffices to show $L_{\mathrm{tame}}(- \otimes H\mathbb{Z})$ is fully faithful. Let $X \in \mathrm{Sp}_{\mathrm{tame}}^{\geq r}$ be a tame spectrum. The map

$$X \otimes \mathbb{S} \to X \otimes H\mathbb{Z}$$

is an r-tame equivalence by Corollary 2.3.11 and

$$H\mathbb{Z} \to L_{tame}(X \otimes H\mathbb{Z})$$

is a tame equivalence since it's a tame $H\mathbb{Z}$-equivalence . Hence the composite

$$X \to X \otimes H\mathbb{Z} \to L_{tame}(X \otimes H\mathbb{Z})$$

is an equivalence as both X and $L_{\mathrm{tame}}(X \otimes H\mathbb{Z})$ are tame.

To show $L_{\mathrm{tame}}(- \otimes H\mathbb{Z})$ is symmetric monoidal, by [Lur17, Proposition

2.2.1.9.], it remains to check tame equivalences is compatible with the tensor product on $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$, which is the content of Lemma 2.3.9.

□

As a direct corollary, we see that the tame homotopy type of a spectrum is determined by its homology with coefficients in the tame ring system.

Corollary 2.3.16. *If $X \in \mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ is a tame spectrum, then*

$$\pi_{r+k}X \cong H_{r+k}X \otimes R_k$$

for every $k \geq 0$. Therefore, a map $g : X \to Y$ in $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ between tame spectra is an equivalence if and only if the induced map on homology with coefficients in the tame ring system

$$H_{r+k}(X) \otimes R_k \to H_{r+k}(Y) \otimes R_k$$

is an isomorphism for all $k \geq 0$.

Proof. Using Theorem 2.3.15, we compute

$$\begin{aligned}\pi_{r+k}X &\cong \pi_{r+k}L_{\mathrm{tame}}(X \otimes H\mathbb{Z}) \\ &\cong \pi_{r+k}(X \otimes H\mathbb{Z}) \otimes R_k \\ &\cong H_{r+k}(X) \otimes R_k\end{aligned}$$

□

Corollary 2.3.17. *For any spectrum $X \in \mathrm{Sp}^{\geq r}$,*

$$\begin{aligned}\pi_{r+k}L_{tame}X &\cong H_{r+k}L_{tame}X \otimes R_k \\ &\cong H_{r+k}X \otimes R_k\end{aligned}$$

for any $k \geq 0$.

Proof. The first isomorphism follows from corollary 2.3.16 and the second isomorphism follows from the fact that tensoring with $H\mathbb{Z}$ preserves r-tame equivalences.

□

Remark 2.3.18. As an immediate consequence of Corollary 2.3.17, the functor $L_{\mathrm{tame}}\Sigma^{\infty} : \mathcal{S}_*^{\geq r} \to \mathrm{Sp}_{tame}^{\geq r}$ sends a Moore space $M(V, r+k)$ for V an R_k-module to a shifted Eilenberg-Maclane spectrum $\Sigma^{r+k}HV$, i.e.,

$$L_{\mathrm{tame}}\Sigma^{\infty}M(V, r+k) \simeq \Sigma^{r+k}HV.$$

Notation 2.3.19. Since $\Sigma^{\infty} : \mathcal{S}_*^{\geq r} \to \mathrm{Sp}_{\mathrm{tame}}^{\geq r}$ preserves tame equivalences by Lemma 2.3.7, we obtain a canonical lift by the universal property of the tame localization

$$\begin{tikzcd} \mathcal{S}_*^{\geq r} \arrow[rr, "L_{\mathrm{tame}}\Sigma^{\infty}"] \arrow[dr, "L_{\mathrm{tame}}"'] & & \mathrm{Sp}_{\mathrm{tame}}^{\geq r} \\ & \mathcal{S}_{\mathrm{tame}}^{\geq r} \arrow[ur, dashed] & \end{tikzcd} .$$

We let $\Sigma_{\mathrm{tame}}^{\infty}$ denote the resulting functor

$$\Sigma_{\mathrm{tame}}^{\infty} : \mathcal{S}_{\mathrm{tame}}^{\geq r} \to \mathrm{Sp}_{\mathrm{tame}}^{\geq r} .$$

We now establish a pair of adjoint functors between tame spaces and tame spectra.

Proposition 2.3.20. *There is an adjoint pair*

$$\Sigma_{\mathrm{tame}}^{\infty} \colon \mathcal{S}_{\mathrm{tame}}^{\geq r} \rightleftarrows \mathrm{Sp}_{\mathrm{tame}}^{\geq r} : \Omega^{\infty}$$

Proof. We first note that $L_{\mathrm{tame}}\Sigma^{\infty} \colon \mathcal{S}_*^{\geq r} \rightleftarrows \mathrm{Sp}_{tame}^{\geq r} : \Omega^{\infty}$ is an adjoint pair, where we abuse notation by writing Ω^{∞} for the composite

$$\mathrm{Sp}_{\mathrm{tame}}^{\geq r} \hookrightarrow \mathrm{Sp}^{\geq r} \xrightarrow{\Omega^{\infty}} \mathcal{S}_*^{\geq r} .$$

Indeed, $\Sigma_{\mathrm{tame}}^{\infty}$ is equivalent to the composition of L_{tame} and Σ^{∞}, which are left adjoint to the inclusion functor $\mathrm{Sp}_{\mathrm{tame}}^{\geq r} \hookrightarrow \mathrm{Sp}^{\geq r}$ and Ω^{∞}, respectively. The statement then follows from Proposition A.1.6 and the fact that $\mathcal{S}_{\mathrm{tame}}^{\geq r}$ is a full subcategory of $\mathcal{S}_*^{\geq r}$.

□

Let G be a finite group. For any preadditive ∞-category $\mathcal{C}$ (see Definition A.1.7) with finite limits and colimits, there is a norm natural transformation constructed in [Lur17, §6.1.6] and [NS18, Definition I.1.10]:

$$\mathrm{Nm} : (-)_{hG} \to (-)^{hG}$$

from the homotopy orbits functor $(-)_{hG} : \mathrm{Fun}(BG, \mathcal{C}) \to \mathcal{C}$ to the homotopy fixed points functor $(-)^{hG} : \mathrm{Fun}(BG, \mathcal{C}) \to \mathcal{C}$. The *Tate construction* of an object $X \in \mathrm{Fun}(BG, \mathcal{C})$ is defined as the cofiber of the norm map:

$$X^{tG} := \mathrm{cofib}(\mathrm{Nm}_X : (X)_{hG} \to (X)^{hG}).$$

If $G = \Sigma_n$ and $\mathcal{C} = \mathrm{Sp}$, then the norm map of a Σ_n-spectrum X

$$\mathrm{Nm}_X : X_{h\Sigma_n} \to X^{h\Sigma_n}$$

is an equivalence if $n!$ is invertible in the homotopy groups of X.

For any $X \in \mathrm{Sp}^{\geq r}$, we can identify $L_{\mathrm{tame}} X^{\otimes n}$ as an object in $\mathrm{Fun}(B\Sigma_n, \mathrm{Sp}^{\geq r}_{\mathrm{tame}})$. We end this chapter with the Tate vanishing property of tame spectra.

Lemma 2.3.21. *The Tate construction of a tame spectrum vanishes, i.e., for all $X \in \mathrm{Sp}^{\geq r}$*

$$\left(L_{\mathrm{tame}}(X^{\otimes n})\right)^{t\Sigma_n} \simeq *$$

for all $n \geq 2$. Therefore,

$$\left(L_{\mathrm{tame}}(X^{\otimes n})\right)_{h\Sigma_n} \simeq \left(L_{\mathrm{tame}}(X^{\otimes n})\right)^{h\Sigma_n}.$$

Proof. If X is r-connective, then its n-th tensor power $X^{\otimes n}$ is nr-connective. We claim that the homotopy groups of the tame spectrum $L_{\mathrm{tame}}(X^{\otimes n})$ are uniquely $n!$-divisible, from which the conclusion follows. Indeed, since

$$nr - 2n - r + 3 = (r-2)(n-1) + 1 \geq 0$$

for $r \geq 3$, it follows that $\pi_* L_{\mathrm{tame}}(X^{\otimes n})$ is uniquely k-divisible for all $k \leq n$ in all degrees. $\square$

Chapter 3

Koszul Duality

In this chapter, we provide the necessary background on ∞-operads (resp. ∞-cooperads) and algebras (resp. coalgebras) over them. For convenience, we will assume $\mathcal{C}$ is a pointed, presentably symmetric monoidal ∞-category throughout this chapter.

Our main references will be [Bra17], [Heu22], [FG12], [Hei19] and [Lur17].

In §3.1, we define ∞-operads and ∞-cooperads in a symmetric monoidal ∞-category $\mathcal{C}$, as associative algebras and associative coalgebras in the ∞-category of symmetric sequences in $\mathcal{C}$, respectively.

In §3.2, we discuss algebras over an ∞-operad and some relevant constructions which will be useful in the proof of the main theorems.

In §3.3, we explain the bar-cobar duality between connected ∞-operads and cooperads. We then define divided power, conilpotent coalgebras over a ∞-cooperad and define the Koszul duality functor $\mathrm{indec}_{\mathcal{O}}$.

In §3.4, we define the spectral Lie operad and introduce shifting of an ∞-operad and relevant properties.

In §3.5, we explain how to produce monads and comonads on $\mathrm{Sp}^{\geq r}$ and $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ from certain monads and comonads on Sp.

In §3.6, we define the ∞-category of commutative coalgebras in an ∞-category. In the case of tame spectra, we prove that it is equivalent to the ∞-category of divided power, conilpotent coalgebras.

In §3.7, we define spectral Lie algebras and tame spectral Lie algebras. We then use Koszul duality between the spectral Lie operad and commutative operad to define the Chevalley-Eilenberg functor as a functor from the ∞-category of tame spectral Lie algebras to the ∞-category of divided power, conilpotent commuative coalgebras in tame spectra.

3.1 (co)Operads in Infinity-Categories

In this section, we collect and extend some results from [Bra17] and [Hei19].

Let $\mathsf{Fin}^{\simeq}$ denote the ∞-category of finite sets with bijections between them, i.e., the core of Fin. Note that $\mathsf{Fin}^{\simeq}$ carries a natural symmetric monoidal structure with the tensor product given by the disjoint union. Moreover, it's the free symmetric monoidal ∞-category generated by the one-object category $\{*\}$. The ∞-category $\mathcal{P}(\mathsf{Fin}^{\simeq})$ of presheaves admits a symmetric monoidal structure given by Day convolution [Lur17, Example 2.2.6.9.], hence we can also consider it as a presentable symmetric monoidal ∞-category. This suggests an alternative description of $\mathcal{C}$:

$$\mathsf{Fun}_{\mathsf{CAlg}(\mathsf{Pr}^L)}(\mathcal{P}(\mathsf{Fin}^{\simeq}), \mathcal{C}) \simeq \mathsf{Fun}_{\mathsf{CAlg}(\mathsf{Cat}_\infty)}(\mathsf{Fin}^{\simeq}, \mathcal{C}) \simeq \mathsf{Fun}(\{*\}, \mathcal{C}) \simeq \mathcal{C}.$$

Definition 3.1.1. A *symmetric sequence* in $\mathcal{C}$ is a functor

$$A : \mathsf{Fin}^{\simeq} \to \mathcal{C}.$$

We denote the ∞-category of symmetric sequences in $\mathcal{C}$ by $\mathsf{SSeq}(\mathcal{C})$.

Remark 3.1.2. Informally, one can think of a symmetric sequence A in $\mathcal{C}$ as a sequence of objects $\{A(n)\}_{n\geq 0}$ in $\mathcal{C}$ where $A(n) := A(\langle n\rangle)$ carries an action of Σ_n for each n. We will refer to the number n as the *arity* of the symmetric sequence A. We will sometimes describe an ∞-operad informally by giving a sequence $\{A(0), A(1), \dots\}$.

Recall the ∞-category of presentable ∞-categories admits a symmetric monoidal structure. The tensor product $\otimes$ in Pr^L admits an explicit formula as in [Lur17, Proposition 4.8.1.17.]. The following lemma points out that the ∞-category $\mathsf{SSeq}(\mathcal{C})$ of symmetric sequences in $\mathcal{C}$ is tensored over the ∞-category $\mathsf{SSeq}(\mathcal{S})$ of symmetric sequences in the ∞-category $\mathcal{S}$ of spaces.

Lemma 3.1.3. *Let $\mathcal{C}$ be a presentably symmetric moniodal ∞-category. Then*

$$\mathsf{SSeq}(\mathcal{C}) \simeq \mathsf{SSeq}(\mathcal{S}) \otimes \mathcal{C} \simeq \mathcal{P}(\mathsf{Fin}^{\simeq}) \otimes \mathcal{C}.$$

Proof. First note that $\mathsf{SSeq}(\mathcal{S}) = \mathsf{Fun}(\mathsf{Fin}^{\simeq}, \mathcal{S}) \simeq \mathcal{P}(\mathsf{Fin}^{\simeq})$, since $\mathsf{Fin}^{\simeq}$ is isomorphic to its opposite category. This proves the second equivalence. Using

[Lur17, Proposition 4.8.1.17.], we deduce that there are equivalences

$$\begin{aligned}
\mathsf{SSeq}(\mathcal{S}) \otimes \mathcal{C} &\simeq \mathsf{Fun}^R(\mathcal{P}(\mathsf{Fin}^{\simeq})^{op}, \mathcal{C}) \\
&\simeq (\mathsf{Fun}^L(\mathcal{P}(\mathsf{Fin}^{\simeq}), \mathcal{C}^{op}))^{op} \\
&\simeq \mathsf{Fun}(\mathsf{Fin}^{\simeq}, \mathcal{C}^{op})^{op} \\
&\simeq \mathsf{SSeq}(\mathcal{C}).
\end{aligned}$$

□

We now explain a monoidal structure on $\mathsf{SSeq}(\mathcal{C})$ that will be used to define ∞-operads. Note that we can view $\mathcal{C}$ as the full subcategory of $\mathsf{SSeq}(\mathcal{C})$ spanned by symmetric sequences that evaluates to the zero object on all non-empty finite sets in $\mathsf{Fin}^{\simeq}$. The ∞-category $\mathsf{SSeq}(\mathcal{C})$ of symmetric sequences is equipped with a monoidal structure that corresponds to the composition of functors in the ∞-category $\mathsf{Fun}_{\mathsf{CAlg}(\mathsf{Pr}^L)_{\mathcal{C}/}}(\mathsf{SSeq}(\mathcal{C}), \mathsf{SSeq}(\mathcal{C}))$. More precisely, there are equivalences of ∞-categories

$$\begin{aligned}
\mathsf{Fun}_{\mathsf{CAlg}(\mathsf{Pr}^L)_{\mathcal{C}/}}(\mathsf{SSeq}(\mathcal{C}), \mathsf{SSeq}(\mathcal{C})) &\simeq \mathsf{Fun}_{\mathsf{CAlg}(\mathsf{Pr}^L)}(\mathcal{P}(\mathsf{Fin}^{\simeq}), \mathsf{SSeq}(\mathcal{C})) \\
&\simeq \mathsf{Fun}_{\mathsf{CAlg}(\mathsf{Cat}_\infty)}(\mathsf{Fin}^{\simeq}, \mathsf{SSeq}(\mathcal{C})) \\
&\simeq \mathsf{Fun}(\{*\}, \mathsf{SSeq}(\mathcal{C})) \\
&\simeq \mathsf{SSeq}(\mathcal{C}),
\end{aligned}$$

allowing the transfer of the monoidal structure on $\mathsf{Fun}_{\mathsf{CAlg}(\mathsf{Pr}^L)_{\mathcal{C}/}}(\mathsf{SSeq}(\mathcal{C}), \mathsf{SSeq}(\mathcal{C}))$ to $\mathsf{SSeq}(\mathcal{C})$. Given two symmetric sequences X and Y in $\mathcal{C}$, we will write $X \circ Y$ for this monoidal product on $\mathsf{SSeq}(\mathcal{C})$ and refer to it as the *composition product*.

Remark 3.1.4. For some categories $\mathcal{C}$, e.g., $\mathcal{C} = \mathsf{Set}, \mathsf{Sp}$ etc. with a point-set model, there is a concrete formula for the composition product (cf. [Bra17, Section 4.1.2.])

$$A \circ B(J) \cong \coprod_{n \geq 0} \Big(\coprod_{J = J_1 \coprod \cdots \coprod J_n} A(\underline{n}) \otimes B(J_1) \otimes \cdots \otimes B(J_n) \Big)_{h\Sigma_n},$$

where $\underline{n}$ denotes the finite set with n elements and the second coproduct runs over all partitions of J.

Remark 3.1.5. Let $X \in \mathcal{C}$ be an object in $\mathcal{C}$, which we can identify as a symmetric sequence. By the equivalence

$$\mathrm{Fun}_{\mathrm{CAlg}(\mathrm{Pr}^L)_{\mathcal{C}/}}(\mathrm{SSeq}(\mathcal{C}), \mathrm{SSeq}(\mathcal{C})) \simeq \mathrm{SSeq}(\mathcal{C}),$$

we can identify X as a functor $\mathrm{SSeq}(\mathcal{C}) \to \mathrm{SSeq}(\mathcal{C})$ that factors through $\mathcal{C}$. To any symmetric sequence A in $\mathcal{C}$, we can then identify $A \circ X$ as an object $A(X)$ in $\mathcal{C}$. By Remark 3.1.4, we obtain a functor which has an explicit description as follows

$$\begin{aligned} F_{(-)} : \mathrm{SSeq}(\mathcal{C}) &\to \mathrm{Fun}(\mathcal{C}, \mathcal{C}) \\ A &\mapsto \big(X \mapsto A(X) := \coprod_{n \geq 0} (A(n) \otimes X^{\otimes n})_{h\Sigma_n}\big). \end{aligned}$$

where Σ_n acts on the n-th tensor power of X via permutation. By our definition of the composition products, this assignment is monoidal with respect to the composition product in $\mathrm{SSeq}(\mathcal{C})$ (see [FG12, §3.1]) and composition of functors in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$. We call the the endofunctor F_A the *Schur functor associated to the symmetric sequence A*.

We can now define ∞-operads (resp. ∞-cooperads) as associative algebras (resp. coalgebras) with respect to the composition products in $\mathrm{SSeq}\,\mathcal{C}$.

Definition 3.1.6. The *∞-category* $\mathrm{Op}(\mathcal{C})$ *of ∞-operads in $\mathcal{C}$* (resp. ∞-category $\mathrm{coOp}(\mathcal{C})$ of ∞-cooperads in $\mathcal{C}$) is defined as the ∞-category $\mathrm{Alg}(\mathrm{SSeq}(\mathcal{C}))$ (resp.$\mathrm{coAlg}(\mathrm{SSeq}(\mathcal{C}))$) of associative algebras (resp. coalgebras) in $\mathrm{SSeq}(\mathcal{C})$ with respect to the composition product.

Remark 3.1.7. Although Lurie's approach to ∞-operads [Lur17, Chapter 2] is seemingly different from the one we give here, Haugseng [Hau17] and Heine [Hei19] showed that Lurie's model is equivalent to the definition of ∞-operads in $\mathcal{S}$. Hence all the results in [Lur17] transfer smoothly to the setting of ∞-operads in spaces.

In this paper, we will work exclusively with *non-unital ∞-operads*. Roughly speaking, a non-unital ∞-operad $\mathcal{O}$ is an operad without nullary operations.

Definition 3.1.8. An ∞-operad $\mathcal{O}$ in $\mathcal{C}$ is *non-unital* if $\mathcal{O}(0)$ is equivalent to the zero object of $\mathcal{C}$.

Remark 3.1.9. As a consequence of Remark 3.1.5, the Schur functor $F_{\mathcal{O}}$ associated to an operad $\mathcal{O}$ is a monad on $\mathcal{C}$. If additionally we assume $\mathcal{C}$ is preadditive

(Definition A.1.7) and $\mathcal{O}$ is a non-unital ∞-operad, then the Schur functor $F_{\mathcal{Q}}$ associated to a cooperad $\mathcal{Q}$ is a comonad on $\mathcal{C}$ by [Hau17, Remark 2.21].

Definition 3.1.10. A (non-unital) ∞-operad $\mathcal{O}$ is *connected* [1] if there is an equivalence $\mathcal{O}(1) \simeq 1_{\mathcal{C}}$.

Definition 3.1.11. An *augmentation* of an ∞-operad $\mathcal{O}$ is a map of ∞-operads $\epsilon : \mathcal{O} \to \mathbb{1}_{\mathcal{C}}$ such that $\epsilon \circ \eta \simeq id_{\mathbb{1}_{\mathcal{C}}}$. We call an ∞-operad $\mathcal{O}$ in $\mathcal{C}$ together with an augmentation ϵ an *augmented* ∞-operad.

Remark 3.1.12. A connected ∞-operad is augmented with the canonical augmentation $\epsilon : \mathcal{O} \to \mathbb{1}_{\mathcal{C}}$.

Remark 3.1.13. From now on, whenever we say ∞-operads (resp. ∞-cooperads) we mean non-unital connected ∞-operads (resp. ∞-cooperads).

We now introduce two simple examples.

Example 3.1.14. We denote by $\mathbb{1}_{\mathcal{C}}$ the symmetric sequence that takes value the tensor unit $1_{\mathcal{C}}$ of $\mathcal{C}$ at $\{*\}$ and the zero object otherwise. One can check that Schur functor $F_{\mathbb{1}_{\mathcal{C}}}$ associated to $\mathbb{1}_{\mathcal{C}}$ is the identity endofunctor on $\mathrm{SSeq}(\mathcal{C})$. Hence $\mathbb{1}_{\mathcal{C}}$ serves as a monoidal unit for the composition product in $\mathrm{SSeq}(\mathcal{C})$. We will refer to $\mathbb{1}_{\mathcal{C}}$ as the *trivial ∞-operad* in $\mathcal{C}$.

Example 3.1.15. Consider the *symmetric algebra functor* on $\mathcal{C}$ defined as

$$\begin{aligned} \mathrm{Sym} : \mathcal{C} &\to \mathcal{C} \\ X &\mapsto \amalg_{n \geq 1} (X^{\otimes n})_{h\Sigma_n}, \end{aligned}$$

which is the Schur functor of the symmetric sequence $\mathrm{Com} := (1_{\mathcal{C}}, 1_{\mathcal{C}}, \dots)$, where $1_{\mathcal{C}}$ is the tensor unit of $\mathcal{C}$. Since Com is clearly an associative algebra in $\mathrm{SSeq}(\mathcal{C})$ with respect to the composition product, Com is called the *commutative operad* in $\mathcal{C}$, and Sym is a monad on $\mathcal{C}$. If we additionally assume $\mathcal{C}$ is preadditive, then Sym associated to the symmetric sequence Com defines a comonad on $\mathcal{C}$, and we will abuse notation by writing Com for the commutative cooperad in $\mathcal{C}$.

3.1.1 Truncations of ∞-operads

The goal of this subsection is to set up the prerequisites to state Proposition 3.3.12 [Heu22, Theorem 4.12], which we will use in the proof of Theorem 4.0.2.

[1] In [Chi05], operads with this property are called reduced. However, the term reduced operads sometimes has another meaning in the literature.

We discuss the notion of truncations of ∞-(co)operads, for which we follow the upcoming paper by Heuts [Heu22] closely. The upshot of these truncations is to produce natural filtrations on algebras (resp. coalgebras) over operads (resp. cooperads).

We fix a pointed presentably symmetric monoidal ∞-category $\mathcal{C}$ with tensor product compatible with colimits. Let $\mathsf{Fin}^{nu}_{\leq n}$ denote the full subcategory of Fin^{nu} spanned by (non-empty) finite sets with cardinality less or equal to n. We will refer to

$$\mathsf{SSeq}_{\leq n}(\mathcal{C}) := \mathsf{Fun}(\mathsf{Fin}^{nu}_{\leq n}, \mathcal{C})$$

as the ∞-category $\mathsf{SSeq}_{\leq n}(\mathcal{C})$ of *n-truncated symmetric sequences in $\mathcal{C}$*. Restriction along the inclusion $\mathsf{Fin}^{nu}_{\leq n} \to \mathsf{Fin}^{nu}$ induces a functor

$$(-)_{\leq n} : \mathsf{SSeq}(\mathcal{C}) \to \mathsf{SSeq}_{\leq n}(\mathcal{C}).$$

which preserves both limits and colimits.

Lemma 3.1.16. *[Hei19, Lemma 2.16] If $\mathcal{C}$ is a pointed symmetric monoidal ∞-category with colimits, then the restriction functor*

$$(-)_{\leq n} : \mathsf{SSeq}(\mathcal{C}) \to \mathsf{SSeq}_{\leq n}(\mathcal{C})$$

is compatible with the composition products. Hence it equips a monoidal structure on $\mathsf{SSeq}_{\leq n}(\mathcal{C})$ and it lifts to functors between algebras and coalgebras.

We can now define n-truncated operads and cooperads in $\mathcal{C}$.

Definition 3.1.17. We define the ∞-category of *n-truncated operads in $\mathcal{C}$* as

$$\mathsf{Op}_{\leq n}(\mathcal{C}) := \mathsf{Alg}(\mathsf{SSeq}_{\leq n}(\mathcal{C})).$$

Dually, we define

$$\mathsf{coOp}_{\leq n}(\mathcal{C}) := \mathsf{coAlg}(\mathsf{SSeq}_{\leq n}(\mathcal{C}))$$

to be the ∞-category of *n-truncated cooperads in $\mathcal{C}$*.

Remark 3.1.18. As a consequence of Lemma 3.1.16, there are commutative diagrams

$$\begin{tikzcd} \mathsf{Op}(\mathcal{C}) \arrow[r, "\rho_n"] \arrow[d, "\mathsf{oblv}"'] & \mathsf{Op}_{\leq n}(\mathcal{C}) \arrow[d, "\mathsf{oblv}'"] \\ \mathsf{SSeq}(\mathcal{C}) \arrow[r, "(-)_{\leq n}"] & \mathsf{SSeq}_{\leq n}(\mathcal{C}) \end{tikzcd}$$

and

$$\begin{tikzcd}
\mathsf{coOp}(\mathcal{C}) \arrow[r, "\rho^n"] \arrow[d, "\mathsf{oblv}"'] & \mathsf{coOp}_{\leq n}(\mathcal{C}) \arrow[d, "\mathsf{oblv}'"] \\
\mathsf{SSeq}(\mathcal{C}) \arrow[r, "(-)_{\leq n}"] & \mathsf{SSeq}_{\leq n}(\mathcal{C}).
\end{tikzcd}$$

where the vertical arrows are forgetful functors.

We now explain the relations between operads and their truncations, following [Heu22]. Consider the functor

$$\zeta_n : \mathsf{SSeq}_{\leq n}(\mathcal{C}) \to \mathsf{SSeq}(\mathcal{C})$$

given by extending by zero objects in arities above n. One can check that it is both left and right adjoint to the restriction functor $(-)_{\leq n}$.

The lemma above has an immediate corollary.

Corollary 3.1.19. *The functor*

$$\zeta_n : \mathsf{SSeq}_{\leq n}(\mathcal{C}) \to \mathsf{SSeq}(\mathcal{C})$$

admits both a lax monoidal and an oplax monoidal strucutre.

The lax monoidal structure on ζ_n induces a functor on algebras

$$\tau_n : \mathsf{Op}_{\leq n}(\mathcal{C}) \to \mathsf{Op}(\mathcal{C})$$

which is right adjoint to ρ_n. On the other hand, the functor $\rho_n : \mathsf{Op}(\mathcal{C}) \to \mathsf{Op}_{\leq n}(\mathcal{C})$ preserves limits and filtered colimits (since they are computed in the underlying ∞-category of symmetric sequences), hence by the adjoint functor theorem [Lur09, Corollary 5.5.2.9.], it admits a left adjoint

$$\varphi_n : \mathsf{Op}_{\leq n}(\mathcal{C}) \to \mathsf{Op}(\mathcal{C}).$$

To sum up, we have a diagram consisting of functors described above

$$\begin{tikzcd}
\mathsf{Op}(\mathcal{C}) \arrow[r, "\rho_n"] & \mathsf{Op}_{\leq n}(\mathcal{C}), \arrow[l, bend right=40, "\varphi_n"'] \arrow[l, bend left=40, "\tau_n"]
\end{tikzcd} \tag{3.1}$$

where the left adjoints are above the right adjoints.

Dually, the oplax structure on ζ_n induces a functor on coalgebras

$$\tau^n : \mathrm{coOp}_{\leq n}(\mathcal{C}) \to \mathrm{coOp}(\mathcal{C}),$$

which is left adjoint to the restriction ρ^n. The restriction ρ^n preserves colimits, hence admits a right adjoint

$$\varphi^n : \mathrm{coOp}_{\leq n}(\mathcal{C}) \to \mathrm{coOp}(\mathcal{C}),$$

and we have the following diagram which summarizes the situation for cooperads

$$\begin{tikzcd} \mathrm{coOp}(\mathcal{C}) \arrow[r, "\rho^n"] & \mathrm{coOp}_{\leq n}(\mathcal{C}). \arrow[l, bend right, "\tau^n"'] \arrow[l, bend left, "\varphi^n"] \end{tikzcd} \tag{3.2}$$

Notation 3.1.20. We will abuse notation by writing

- for any operad $\mathcal{O}$ in $\mathcal{C}$, $\mathcal{O} \to \tau_n\mathcal{O}$ for the unit of the bottom adjunction in (3.1) ;
- for any operad $\mathcal{O}$ in $\mathcal{C}$, $\varphi_n\mathcal{O} \to \mathcal{O}$ for the counit of the top adjunction in (3.1) ;
- for any cooperad $\mathcal{Q}$ in $\mathcal{C}$, $\tau^n\mathcal{Q} \to \mathcal{Q}$ for the counit of the top adjunction in (3.2) ;
- for any cooperad $\mathcal{Q}$ in $\mathcal{C}$, $\mathcal{Q} \to \varphi^n\mathcal{Q}$ for the unit of the bottom adjunction in (3.2).

We state the following facts from [Heu22] without proof.

Fact 3.1.21. *[Heu22]*

- *For any ∞-operad $\mathcal{O}$ in $\mathcal{C}$, there is a sequence of operad maps*

$$\varphi_n\mathcal{O} \to \mathcal{O} \to \tau_n\mathcal{O}.$$

The operad $\tau_n\mathcal{O}$ is the n-truncation of $\mathcal{O}$, i.e., operations of arity higher than n are set to zero. The map $\mathcal{O} \to \tau_n\mathcal{O}$ is terminal among those operad maps from $\mathcal{O}$ that are equivalences in arities up to n. The operad

$\varphi_n\mathcal{O}$ is equivalent to $\mathcal{O}$ in arities up to n, while higher arity operations are "freely generated" by operations of arity less or equal to n. Hence, the map $\varphi_n\mathcal{O} \to \mathcal{O}$ is initial among those operad maps to $\mathcal{O}$ that are equivalences in arities up to n.

- *Similarly, for any ∞-cooperad $\mathcal{Q}$ in $\mathcal{C}$, there is a sequence of cooperad maps*
$$\tau^n\mathcal{Q} \to \mathcal{Q} \to \varphi^n\mathcal{Q}.$$
The cooperad $\tau^n\mathcal{Q}$ is the n-truncation of $\mathcal{Q}$, i.e., cooperations of arity higher than n are set to zero. The map $\mathcal{Q} \to \varphi^n\mathcal{Q}$ is terminal among those cooperad maps from $\mathcal{O}$ that are equivalences in arities up to n. The cooperad $\varphi^n\mathcal{Q}$ is "cofreely generated" by cooperations of arities up to n. The map $\tau^n\mathcal{Q} \to \mathcal{Q}$ is initial among those cooperad maps from $\mathcal{Q}$ that are equivalences in arities up to n.

Remark 3.1.22. Using the fact above, we see that there is a direct system for an ∞-operad $\mathcal{O}$,
$$\varphi_1\mathcal{O} \to \varphi_2\mathcal{O} \to \cdots \to \mathcal{O}. \tag{3.3}$$
Similarly, there is an inverse system for the n-truncations of ∞-operads.
$$\cdots \to \tau_2\mathcal{O} \to \tau_1\mathcal{O} \to \mathcal{O} \tag{3.4}$$

Proposition 3.1.23. *[Heu22] If $\mathcal{O}$ is an ∞-operad in $\mathcal{C}$, then there is an equivalence*
$$\mathcal{O} \simeq \mathrm{colim}_n\, \varphi_n\mathcal{O}$$
in $\mathrm{Op}(\mathcal{C})$.

Sketch of proof: It suffices to check the equivalence aritywise. For any arity k, the direct system of objects
$$\varphi_1\mathcal{O}(k) \to \varphi_2\mathcal{O}(k) \to \cdots$$
stabilizes for $n \geq k$. □

3.2 Algebras over Operads

In this section, we will review the definition of algebras over an ∞-operad. Recall that a symmetric sequence $\mathcal{O} \in \mathrm{SSeq}(\mathcal{C})$ acts on $\mathcal{C}$ via its Schur functor $F_{\mathcal{O}}$. If

$\mathcal{O}$ is an ∞-operad, then its Schur functor $F_{\mathcal{O}}$ is a monad, hence we can consider the ∞-category $\mathrm{LMod}_{F_{\mathcal{O}}}(\mathcal{C})$ of left $F_{\mathcal{O}}$-modules in $\mathcal{C}$.

Definition 3.2.1. Let $\mathcal{O}$ be an ∞-operad in $\mathcal{C}$. The ∞-category of $\mathcal{O}$-algebras is $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) := \mathrm{LMod}_{F_{\mathcal{O}}}(\mathcal{C})$.

For a symmetric sequence A in $\mathcal{C}$, we consider the following *extended power functors*

$$D_n^A(X) := (A(n) \otimes X^{\otimes n})_{h\Sigma}, \quad D_A^n(X) := (A(n) \otimes X^{\otimes n})^{h\Sigma}. \tag{3.5}$$

Informally, an $\mathcal{O}$-algebra X in $\mathcal{C}$ is equipped with maps

$$D_n^{\mathcal{O}}(X) \to X$$

for each $n \geq 1$ and homotopy coherent data that keeps track of the associativity.

Definition 3.2.2. Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor. We define the ∞-category $\mathrm{Alg}_F(\mathcal{C})$ of *F-algebras in $\mathcal{C}$* to be the pullback of the following diagram in Cat_∞

$$\begin{tikzcd}
\mathrm{Alg}_F(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d, "(ev_0, ev_1)"] \\
\mathcal{C} \arrow[r, "(F, id)"] & \mathcal{C} \times \mathcal{C}.
\end{tikzcd}$$

That is, $\mathrm{Alg}_F(\mathcal{C})$ has objects X of $\mathcal{C}$ equipped with a morphism $F(X) \to X$.

Remark 3.2.3. Our notation might cause potential confusion when F is a monad T, since we write $\mathrm{Alg}_T(\mathcal{C})$ for the ∞-category of left modules over T. Note that this is different from the pullback definition in Definition 3.2.2.

Proposition 3.1.23 motivates us to ask the following question: Can we write an $\mathcal{O}$-algebra, as the limit of $\varphi_k\mathcal{O}$-algebras as in the case of Postnikov decomposition of a simply-connected space?

Heuts answers this question in the following theorem.

Theorem 3.2.4. *[Heu22, Theorem 4.1] For each $n \geq 2$, the commutative square of ∞-categories*

$$\begin{tikzcd}
\mathrm{Alg}_{\varphi_n\mathcal{O}}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Alg}_{D_n^{\mathcal{O}}}(\mathcal{C}) \arrow[d] \\
\mathrm{Alg}_{\varphi_{n-1}\mathcal{O}}(\mathcal{C}) \arrow[r] & \mathrm{Alg}_{D_n^{\varphi_{n-1}\mathcal{O}}}(\mathcal{C}).
\end{tikzcd}$$

is a pullback square. Furthermore, the natural map

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{C}) \to \lim_{n} \mathrm{Alg}_{\varphi_n \mathcal{O}}(\mathcal{C})$$

is an equivalence of ∞-categories.

Remark 3.2.5. Heuts stated Theorem 3.2.4 for a stable ∞-category, but a careful examination of the proof shows that the theorem holds for a presentably pointed symmetric monoidal ∞-category $\mathcal{C}$, which is our assumption throughout this chapter.

Remark 3.2.6. Theorem 3.2.4 has the following informal interpretation: suppose X is a $\varphi_{n-1}\mathcal{O}$-algebra, then to specify a $\varphi_n\mathcal{O}$-algebra structure on X, it suffices to equip X with a multiplication map $\mu_n : D_n^{\mathcal{O}}(X) \to X$ that is compatible with the $\varphi_{n-1}\mathcal{O}$-algebra structure maps. We refer the readers to [Heu22] for further details.

3.3 Koszul Duality and Divided Power Conilpotent Coalgebras

In this section, we discuss Koszul duality for operads in the sense of [GK94] and define divided power, conilpotent coalgebras over an ∞-cooperad. We will assume $\mathcal{C}$ is a presentably stable, symmetric monoidal ∞-category in this section.

We first discuss bar-cobar duality between ∞-operads and ∞-cooperads. The general form of bar-cobar duality is exhibited in the form of associative algebras and coassociative coalgebras in a nice ∞-category in [Lur17].

Proposition 3.3.1. *[Lur17, Remark 5.2.2.19.] Let $\mathcal{D}$ be a pointed monoidal ∞-category admitting geometric realizations of simplicial objects and totalizations of cosimplicial objects. Then there is an adjunction*

$$B\colon \mathrm{Alg}^{\mathrm{aug}}(\mathcal{D}) \rightleftarrows \mathrm{coAlg}^{\mathrm{aug}}(\mathcal{D}) : C.$$

If $\mathcal{D} = \mathrm{SSeq}(\mathcal{C})$, then we obtain an adjunction between (augmented) operads and (coaugmented) cooperads.

$$B\colon \mathrm{Op}^{aug}(\mathcal{C}) \rightleftarrows \mathrm{coOp}^{aug}(\mathcal{C}) : C.$$

Explicitly, the bar construction $B\mathcal{O}$ of an ∞-operad $\mathcal{O}$ is computed as the geometric realization of the simplicial bar construction with respect to the composition products [Lur17, Section 4.4.2]

$$\mathrm{Bar}(\mathbb{1}, \mathcal{O}, \mathbb{1})_\bullet := \mathbb{1} \leftleftarrows \mathcal{O} \leftleftarrows \mathcal{O} \circ \mathcal{O} \cdots,$$

and we will write

$$B\mathcal{O} := |\,\mathrm{Bar}(\mathbb{1}, \mathcal{O}, \mathbb{1})_\bullet|.$$

Similarly, the cobar construction $C\mathcal{Q}$ of a cooperad $\mathcal{Q}$ is computed as the totalization of the cosimplicial object

$$\mathrm{Cobar}(\mathbb{1}, \mathcal{Q}, \mathbb{1})^\bullet := \mathbb{1} \rightrightarrows \mathcal{Q} \rightrightarrows \mathcal{Q} \circ \mathcal{Q} \cdots.$$

and we will write

$$C\mathcal{Q} := \mathrm{Tot}\,\mathrm{Cobar}(\mathbb{1}, \mathcal{Q}, \mathbb{1})^\bullet.$$

Definition 3.3.2. An ∞-operad $\mathcal{O} \in \mathrm{Op}^{aug}(\mathcal{C})$ is *Koszul* if the unit map

$$\mathcal{O} \to CB(\mathcal{O})$$

is an equivalence.

If we restrict to ∞-operads in a stable ∞-category $\mathcal{C}$, then every connected ∞-operad is Koszul.

Proposition 3.3.3. *[Heu22, Proposition 3.4] Let $\mathcal{C}$ be a presentably stable symmetric monoidal ∞-category. Then the bar-cobar adjunction*

$$B\colon \mathrm{Op}^{aug}(\mathcal{C}) \rightleftarrows \mathrm{coOp}^{aug}(\mathcal{C}) : C.$$

restricts to an equivalence on the ∞-categories of connected operads and cooperads.

We now introduce functors that will play important roles throughout the rest of the thesis. Let $\mathcal{O}$ be a connected ∞-operad with its canonical augmentation $\epsilon : \mathcal{O} \to \mathbb{1}$. Restriction along ϵ induces a functor

$$\mathrm{triv}_{\mathcal{O}} : \mathcal{C} \simeq \mathrm{Alg}_{\mathbb{1}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}),$$

which we will call the *trivial $\mathcal{O}$-algebra functor*. The trivial $\mathcal{O}$-algebra functor admits a left adjoint $\mathrm{cot}_{\mathcal{O}} : \mathcal{C} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ given by the relative tensor product

$\mathbb{1} \circ_{\mathcal{O}} (-)$, whose existence can also be deduced from the adjoint functor theorem [Lur09, Corollary 5.5.2.9.]. The functor $\mathrm{cot}_{\mathcal{O}}$ is called the *cotangent fiber functor* in [Heu22].

Remark 3.3.4. Informally, a trivial $\mathcal{O}$-algebra X is an $\mathcal{O}$-algebra whose structure map $(\mathcal{O}(n) \otimes X^{\otimes n})_{h\Sigma_n} \to X$ factors through 0 when $n > 1$.

Remark 3.3.5. If $\mathcal{O}$ is the commutative operad, the functor $\mathrm{cot}_{\mathcal{O}}$ is often called *Topological André-Quillen homology* in the literature. The choice of the terminology from [Heu22] comes from the following fact. Let $\mathcal{C}$ be the category of chain complexes over a commutative ring k and $\mathcal{O}$ the non-unital commutative operad. If A is a commutative k-algebra, then $\mathrm{cot}_{\mathcal{O}}(A)$ computes the fiber of the cotangent complex $L_{A/k}$ at the k-point of $\mathrm{Spec}(A)$ determined by the augmentation of A.

Similarly, restriction along the unit map $\eta : \mathbb{1} \to \mathcal{O}$ induces a functor

$$\mathrm{oblv}_{\mathcal{O}} : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C} \simeq \mathrm{Alg}_{\mathbb{1}}(\mathcal{C}),$$

which we will refer as the *$\mathcal{O}$-forgetful functor*. The $\mathcal{O}$-forgetful functor admits a left adjoint $\mathrm{free}_{\mathcal{O}} : \mathcal{C} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ given by $\mathcal{O} \circ (-)$ which we will call the *free $\mathcal{O}$-algebra functor*.

The composite

$$\mathbb{1} \xrightarrow{\eta} \mathcal{O} \xrightarrow{\epsilon} \mathbb{1}$$

is equivalent to the identity, hence we have equivalences

$$\mathrm{cot}_{\mathcal{O}} \circ \mathrm{free}_{\mathcal{O}} \simeq \mathrm{id}$$

and

$$\mathrm{oblv}_{\mathcal{O}} \circ \mathrm{triv}_{\mathcal{O}} \simeq \mathrm{id}\,.$$

The situation can be summarized in the following diagram of adjunctions

$$\mathcal{C} \underset{\mathrm{oblv}\,\mathcal{O}}{\overset{\mathrm{free}_{\mathcal{O}}}{\rightleftarrows}} \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \underset{\mathrm{triv}_{\mathcal{O}}}{\overset{\mathrm{cot}_{\mathcal{O}}}{\rightleftarrows}} \mathcal{C}$$

in which the left adjoints are above their right adjoints and both horizontal composites are equivalent to the identity functor.

The following corollary of the Barr-Beck-Lurie theorem [Lur17, Theorem 4.7.3.5.] says that any $\mathcal{O}$-algebra X is canonically equivalent to the geomectric realization of a simplicial object of $\mathcal{O}$-algebras.

Proposition 3.3.6. *If X is an $\mathcal{O}$-algebra, then it can be resolved as the geometric realization of the simplicial objects* $(\mathrm{free}_{\mathcal{O}} \circ \mathrm{oblv}_{\mathcal{O}})_{\bullet+1}X$*, i.e.,*

$$X \simeq |(\mathrm{free}_{\mathcal{O}} \circ \mathrm{oblv}_{\mathcal{O}})_{\bullet+1}X| \simeq |\operatorname{Bar}(\mathcal{O}, \mathcal{O}, X)_{\bullet}|. \tag{3.6}$$

Proof. This follows from the fact that the adjunction $(\mathrm{free}_{\mathcal{O}}, \mathrm{oblv}_{\mathcal{O}})$ is monadic and Corollary A.3.2.

□

Applying $\mathrm{cot}_{\mathcal{O}}$ to both sides of (3.6) yields a formula for computing the cotangent fiber of X.

Corollary 3.3.7. *[Heu22, Proposition 4.4] The cotangent fiber of an $\mathcal{O}$-algebra X can be computed as*

$$\mathrm{cot}_{\mathcal{O}} X \simeq |(\mathrm{oblv}_{\mathcal{O}} \circ \mathrm{free}_{\mathcal{O}})_{\bullet}\, \mathrm{oblv}_{\mathcal{O}} X|,$$

where the geometric realization is computed in the underlying ∞-category $\mathcal{C}$.

As a consequence, we can identify the comonad $\mathrm{cot}_{\mathcal{O}} \circ \mathrm{triv}_{\mathcal{O}}$ in terms of the bar construction of $\mathcal{O}$.

Proposition 3.3.8. *[Heu22, Proposition 4.5] The comonad* $\mathrm{cot}_{\mathcal{O}} \circ \mathrm{triv}_{\mathcal{O}}$ *is naturally equivalent to the comonad $F_{B\mathcal{O}}$ associated with the cooperad $B\mathcal{O}$.*

We are now ready to define the ∞-category of divided power, conilpotent coalgebras over an ∞-cooperad.

Definition 3.3.9. Let $\mathcal{Q}$ be an ∞-cooperad in $\mathcal{C}$. We define the ∞-category of divided power, conilpotent coalgebras over a cooperad $\mathcal{Q}$ as the ∞-category of left comodules over $F_{\mathcal{Q}}$,

$$\mathrm{coAlg}_{\mathcal{Q}}^{\mathrm{dp,nil}}(\mathcal{C}) := \mathrm{LcoMod}_{F_{\mathcal{Q}}}(\mathcal{C}).$$

Analogous to the case of algebras over an operad, we have two pairs of adjunctions

$$\mathcal{C} \underset{\mathrm{oblv}_{\mathcal{Q}}^{\mathrm{nil}}}{\overset{\mathrm{cofree}_{\mathcal{Q}}^{\mathrm{nil}}}{\rightleftarrows}} \mathrm{coAlg}_{\mathcal{Q}}^{\mathrm{dp,nil}}(\mathcal{C}) \underset{\mathrm{triv}_{\mathcal{Q}}^{\mathrm{nil}}}{\overset{\mathrm{Prim}_{\mathcal{Q}}^{\mathrm{nil}}}{\rightleftarrows}} \mathcal{C}$$

where the right adjoints are above the left adjoints, and compositions of the horizontal functors are equivalent to the identity functor on $\mathcal{C}$.

Moreover, by an argument dual to the proof of Proposition 3.3.6, we can resolve any divided power, conilpotent coalgebra by a totalization of cosimplicial cofree coalgebras.

Proposition 3.3.10. *Let X be a divided power, conilpotent $\mathcal{Q}$-coalgebra, then it can be resolved as the totalization of the cosimplicial object* $(\mathrm{cofree}_{\mathcal{Q}} \circ \mathrm{oblv}_{\mathcal{Q}})^{\bullet+1}X$*, i.e.*

$$X \simeq \mathrm{Tot}(\mathrm{cofree}_{\mathcal{Q}} \circ \mathrm{oblv}_{\mathcal{Q}})^{\bullet+1}X \simeq \mathrm{Tot}\,\mathrm{Cobar}(\mathcal{Q}, \mathcal{Q}, X)^{\bullet}. \tag{3.7}$$

The functor $\mathrm{cot}_{\mathcal{O}}$ factors through the ∞-category $\mathrm{LcoMod}_{\mathrm{cot}_{\mathcal{O}} \circ \mathrm{triv}_{\mathcal{O}}}(\mathcal{C})$ of left comodules over the comonad $(\mathrm{cot}_{\mathcal{O}} \circ \mathrm{triv}_{\mathcal{O}})$, which can be identified with the ∞-category of left comodules over the comonad $F_{B\mathcal{O}}$ by Proposition 3.3.8, hence we can write the resulting factorization as

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[rr, "\mathrm{cot}_{\mathcal{O}}"] \arrow[dr, "\mathrm{indec}_{\mathcal{O}}"'] & & \mathcal{C} \\
& \mathrm{coAlg}^{\mathrm{dp,nil}}_{B\mathcal{O}}(\mathcal{C}) \arrow[ur, "\mathrm{oblv}^{\mathrm{nil}}_{B\mathcal{O}}"'] &
\end{tikzcd}.$$

The functor $\mathrm{indec}_{\mathcal{O}}$ preserves colimits, hence it admits a right adjoint $\mathrm{prim}^{\mathrm{nil}}_{B\mathcal{O}}$ by the adjoint functor theorem [Lur09, Corollary 5.5.2.9.], which we will call the *primitives functor*; for $Y \in \mathrm{coAlg}^{\mathrm{dp,nil}}_{B\mathcal{O}}(\mathcal{C})$, the *primitives* $\mathrm{prim}^{\mathrm{nil}}_{B\mathcal{O}}(Y)$ of Y can be computed explicitly as [Heu22, Lemma 4.7]

$$\mathrm{prim}^{\mathrm{nil}}_{B\mathcal{O}}(Y) \simeq \mathrm{Tot}\,\mathrm{triv}_{\mathcal{O}}(F_{B\mathcal{O}})^{\bullet}\,\mathrm{oblv}^{nil}_{B\mathcal{O}}(Y).$$

For future application, we record the following lemma.

Lemma 3.3.11. *[FG12, (3.4) and (3.5)] There are equivalences*

$$\mathrm{indec}_{\mathcal{O}} \circ \mathrm{triv}_{\mathcal{O}} \simeq \mathrm{cofree}^{\mathrm{nil}}_{B\mathcal{O}} \;\; \textit{and} \;\; \mathrm{prim}^{\mathrm{nil}}_{B\mathcal{O}} \circ \mathrm{cofree}^{\mathrm{nil}}_{B\mathcal{O}} \simeq \mathrm{triv}_{\mathcal{O}}\,.$$

Proof. The first equivalence is a direct consequence of Proposition 3.3.8. The composite $\mathrm{prim}^{\mathrm{nil}}_{B\mathcal{O}} \circ \mathrm{cofree}^{\mathrm{nil}}_{B\mathcal{O}}$ is right adjoint to $\mathrm{oblv}_{\,\mathcal{O}} \circ \mathrm{indec}_{\mathcal{O}} \simeq \mathrm{cot}_{\mathcal{O}}$, hence $\mathrm{prim}^{\mathrm{nil}}_{B\mathcal{O}} \circ \mathrm{cofree}^{\mathrm{nil}}_{B\mathcal{O}} \simeq \mathrm{triv}_{\mathcal{O}}$. $\square$

We also have the following decomposition result for divided power, conilpotent $\mathcal{Q}$-coalgebras, which will be used in the proof of the essential surjectivity of the functor C_{tame}. Let $\mathcal{Q}$ be a connected ∞-cooperad.

Proposition 3.3.12. *[Heu22, Theorem 4.13] For $n \geq 2$, the following commutative square of ∞-categories*

$$\begin{tikzcd} \mathrm{coAlg}^{\mathrm{nil,dp}}_{\varphi^n\mathcal{Q}}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{coAlg}^{\mathrm{nil,dp}}_{D^n_{\mathcal{Q}}}(\mathcal{C}) \arrow[d] \\ \mathrm{coAlg}_{\varphi^{n-1}\mathcal{Q}}(\mathcal{C}) \arrow[r] & \mathrm{coAlg}_{D^n_{\varphi^{n-1}\mathcal{Q}}}(\mathcal{C}) \end{tikzcd}$$

is a pullback square. Moreover, the natural map

$$\mathrm{coAlg}_{\mathcal{Q}}(\mathcal{C}) \to \lim_n \mathrm{coAlg}_{\varphi^n\mathcal{Q}}(\mathcal{C})$$

is an equivalence.

3.4 The spectral Lie operad

In this section, we introduce an operad in Sp which will be central to this thesis. We first need to provide some background on partition posets. Consider the set $\mathbf{P}(n)$ of partitions (i.e., equivalence relations) on the finite set $\{1, \cdots, n\}$. $\mathbf{P}(n)$ is a poset with minimal element the trivial partition and maximal element the discrete partition. Write $\mathbf{P}^+(n)$ (resp. $\mathbf{P}^-(n)$) for the subposet of $\mathbf{P}(n)$ obtained by deleting the minimal (resp. maximal) partition.

Goodwillie calculus is an important tool in homotopy theory, we will assume basic familiarity of its theory and applications. The survey paper [AC19] is an excellent source on this subject. The Goodwillie derivative $\partial_* \mathrm{id}$ of the identity functor $\mathrm{id} : \mathcal{S}_* \to \mathcal{S}_*$ forms a symmetric sequence in Sp, i.e., each $\partial_n \mathrm{id}$ is a Σ_n-spectrum in the naive sense. In [Joh95] and [AM99], it was shown that

$$\partial_n \mathrm{id} \simeq \mathbb{D}(\Sigma^\infty K_n),$$

where $K_n := |\mathbf{P}(n)|/\left(|\mathbf{P}^+(n)| \cup |\mathbf{P}^-(n)|\right)$ and $\mathbb{D}$ denotes the Spanier-Whitehead dual. Let Com denote the commutative cooperad in Sp. Ching [Chi05] identified the derivatives of the identity functor as the cobar construction of the commutative cooperad and hence admits an operad structure.

Remark 3.4.1. We remark that Ching works with point-set models of operads in spectra. However, one can choose a simplicial model for the operads in [Chi12] and apply operadic nerve [Lur17, Definition 2.1.1.23.] to obtain relevant results for ∞-operads. The operad $\partial_* \mathrm{id}$ is the cobar construction of the commutative cooperad in Sp ([Chi05, Remark 8.9])

$$\mathrm{Cobar}(\mathrm{Com}) \simeq \partial_* \mathrm{id}\,.$$

Remark 3.4.2. Classical Koszul duality between the Lie operad and the commutative cooperad Com in (graded) abelian groups is studied in [GK94] and [LV12]; there is an equivalence of operads in (graded) abelian groups

$$\mathrm{Lie}[-1] \simeq \mathrm{Com}^\vee$$

where $\mathrm{Com}^\vee$ denotes the linear dual of the commutative cooperad and $\mathrm{Lie}[-1](n)_p := \mathrm{Lie}(n)_{p+n-1}$.

Definition 3.4.3. We define the *shifted spectral Lie operad* to be

$$\mathbb{L} := \partial_* \mathrm{id}\,.$$

The name comes from the following computation in [Chi05, Example 9.50],

$$H_*\mathbb{L}(n) \cong \begin{cases} \mathrm{Lie}(n) \otimes \mathrm{sgn} & \text{if } * = 1-n, \\ 0 & \text{otherwise} \end{cases}$$

where sgn denotes the sign representation.

The mismatch of degrees between the spectral Lie operad $\mathbb{L}$ and the Lie operad Lie in abelian groups is rather inconvenient for applications. To remedy this, we recall a shift operation for ∞-operads in Sp introduced in [Hei19, §2.2.4.] and [Cam16, Section 3].

Definition 3.4.4. Suppose there is an adjunction

$$L\colon \mathrm{Sp} \rightleftarrows \mathcal{C} : R$$

for some ∞-category $\mathcal{C}$. Let T be the monad arising from this adjunction, then the *desuspended monad* $\Omega T \Sigma$ is a monad on Sp and we denote it by $\boldsymbol{\Sigma}^{-1} T$. If the monad T is the Schur functor $F_{\mathcal{O}}$ of an ∞-operad $\mathcal{O}$, then there is a *desuspended operad* $\Sigma^{-1}\mathcal{O}$ corresponding to the monad $\boldsymbol{\Sigma}^{-1} F_{\mathcal{O}}$, which has underlying symmetric sequence

$$(\boldsymbol{\Sigma}^{-1}\mathcal{O})(n) = \mathbb{S}^n \otimes \Sigma^{-1}\mathcal{O}(n)$$

where Σ_n acts on $\mathbb{S}^n$ by permuting the n factors of $\mathbb{S}^1$, and Σ_n acts on $\mathbf{\Sigma}^{-1}\mathcal{O}(n)$ via the its action on $\mathcal{O}(n)$.

We now state some results concerning shifts of monads and ∞-operads without proof. We refer the reader to [Hei19, Section 2.2.4.] and [Cam16] for more details. The following lemma indicates that shifting an ∞-operad is indeed harmless to the category of algebras we want to study.

Lemma 3.4.5. *[Hei19, Section 2.2.4.] Let $\mathcal{C}$ be a stable ∞-category. There is a pullback diagram*

$$\begin{tikzcd} \mathrm{Alg}_{\mathbf{\Sigma}^{-1}\mathcal{O}}(\mathcal{C}) \arrow[r, "\Sigma'", "\simeq"'] \arrow[d] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[d] \\ \mathcal{C} \arrow[r, "\Sigma", "\simeq"'] & \mathcal{C}. \end{tikzcd}$$

of ∞-categories.

As a consequence, we write

$$\Omega' : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbf{\Sigma}^{-1}\mathcal{O}}(\mathcal{C})$$

for the inverse of Σ', which fits into a commutative diagram

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[r, "\Omega'", "\simeq"'] \arrow[d] & \mathrm{Alg}_{\mathbf{\Sigma}^{-1}\mathcal{O}}(\mathcal{C}) \arrow[d] \\ \mathcal{C} \arrow[r, "\Omega", "\simeq"'] & \mathcal{C}. \end{tikzcd}$$

Moreover, we have two commutative diagrams of right adjoints

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[r, "\Omega'", "\simeq"'] & \mathrm{Alg}_{\mathbf{\Sigma}^{-1}\mathcal{O}}(\mathcal{C}) \\ \mathcal{C} \arrow[u, "\mathrm{triv}_{\mathcal{O}}"] \arrow[r, "\Omega", "\simeq"'] & \mathcal{C}, \arrow[u, "\mathrm{triv}_{\mathbf{\Sigma}^{-1}\mathcal{O}}"'] \end{tikzcd} \tag{3.8}$$

and

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[r, "\Omega'", "\simeq"'] \arrow[d, "\Omega_{\mathcal{O}}"'] & \mathrm{Alg}_{\mathbf{\Sigma}^{-1}\mathcal{O}}(\mathcal{C}) \arrow[d, "\Omega_{\mathbf{\Sigma}^{-1}\mathcal{O}}"] \\ \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[r, "\Omega'", "\simeq"'] & \mathrm{Alg}_{\mathbf{\Sigma}^{-1}\mathcal{O}}(\mathcal{C}), \end{tikzcd} \tag{3.9}$$

where $\Omega_{\mathcal{O}}$ and $\Omega_{\mathbf{\Sigma}^{-1}\mathcal{O}}$ denote the loop functor in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ and $\mathrm{Alg}_{\mathbf{\Sigma}^{-1}\mathcal{O}}(\mathcal{C})$, respectively. The commutativity of (3.8) follows from the fact that $\Omega' \circ \mathrm{triv}_{\mathcal{O}}$

and $\mathrm{triv}_{\mathbf{\Sigma}^{-1}\mathcal{O}} \circ \Omega$ are right adjoint to $\mathrm{cot}_{\mathcal{O}} \circ \Sigma'$ and $\Sigma \circ \mathrm{cot}_{\mathbf{\Sigma}^{-1}\mathcal{O}}$, respectively; and there are equivalences

$$\begin{aligned}\mathrm{cot}_{\mathcal{O}} \circ (\Sigma' X) &\simeq |(F_{\mathcal{O}})_\bullet(\Sigma X)| \\ &\simeq |\Sigma(\Omega F_{\mathcal{O}} \Sigma)_\bullet(X)| \\ &\simeq \Sigma \,\mathrm{cot}_{\mathbf{\Sigma}^{-1}\mathcal{O}}\end{aligned}$$

where we used that there is an equivalence of simplicial objects

$$\Sigma(\Omega F_{\mathcal{O}}\Sigma)_\bullet(X) \to (F_{\mathcal{O}})_\bullet(\Sigma X)$$

in the second equivalence.

The commutativity of (3.8) says that applying Ω' to a trivial $\mathcal{O}$-algebra gives rise to a trivial $\mathbf{\Sigma}^{-1}\mathcal{O}$-algebra, and the commutativity of (3.9) says that there is an equivalence $\Omega_{\mathbf{\Sigma}^{-1}\mathcal{O}} X \simeq \Omega' \circ \Omega_{\mathcal{O}}(\Sigma' X)$ for $X \in \mathrm{Alg}_{\mathbf{\Sigma}^{-1}\mathcal{O}}(\mathcal{C})$. Therefore, we obtain the following lemma.

Lemma 3.4.6. *Let $Y \in \mathrm{Alg}_{\mathbf{\Sigma}^{-1}\mathcal{O}}(\mathcal{C})$. If $\Omega_{\mathcal{O}}(\Sigma' Y)$ is a trivial $\mathcal{O}$-algebra, then $\Omega_{\mathbf{\Sigma}^{-1}\mathcal{O}} Y$ is a trivial $\mathbf{\Sigma}^{-1}\mathcal{O}$-algebra.*

Proof. By the assumption, $\Omega_{\mathcal{O}}(\Sigma' Y)$ is a trivial $\mathcal{O}$-algebra, i.e. $\Omega_{\mathcal{O}}(\Sigma' Y) \simeq \mathrm{triv}_{\mathcal{O}} Y$. Moreover, by the commutativity of (3.8) and (3.9), there are equivalences

$$\Omega_{\mathbf{\Sigma}^{-1}\mathcal{O}} Y \simeq \Omega' \Omega_{\mathcal{O}}(\Sigma' Y) \simeq \Omega' \,\mathrm{triv}_{\mathcal{O}}(Y) \simeq \mathrm{triv}_{\mathbf{\Sigma}^{-1}\mathcal{O}}(\Omega Y).$$

□

Construction 3.4.7 (Suspension morphism)**.** Consider the commutative diagram

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[rr, "\mathrm{oblv}_{\mathcal{O}} \circ \Omega_{\mathcal{O}}"] \arrow[dr, "\Omega_{\mathcal{O}}"'] & & \mathcal{C} \\ & \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[ur, "\mathrm{oblv}_{\mathcal{O}}"'] & \end{tikzcd}.$$

The forgetful functor $\mathrm{oblv}_{\mathcal{O}}$ is monadic and it corresponds to the monad $F_{\mathcal{O}}$. Since the functor $\mathrm{oblv}_{\mathcal{O}} \circ \Omega_{\mathcal{O}}$ is conservative and preserves sifted colimits, it is also monadic by the Barr-Beck-Lurie theorem [Lur17, Theorem 4.7.3.5.]. Moreover, the monad induced from the right adjoint $\mathrm{oblv}_{\mathcal{O}} \circ \Omega_{\mathcal{O}}$ and its left adjoint $\Sigma_{\mathcal{O}} \circ \mathrm{free}_{\mathcal{O}}$ is exactly $F_{\mathbf{\Sigma}^{-1}\mathcal{O}}$; indeed, the monad arising from this adjunction is given by

$$\mathrm{oblv}_{\mathcal{O}}\,\Omega_{\mathcal{O}}\Sigma_{\mathcal{O}}\,\mathrm{free}_{\mathcal{O}} \simeq \Omega\,\mathrm{oblv}_{\mathcal{O}}\,\mathrm{free}_{\mathcal{O}}\,\Sigma \simeq \Omega F_{\mathcal{O}}\Sigma,$$

where we use that $\mathrm{free}_{\mathcal{O}}$ preserves colimits so $\Sigma_{\mathcal{O}}\,\mathrm{free}_{\mathcal{CO}} \simeq \mathrm{free}_{\mathcal{O}}\,\Sigma$. Therefore, we obtain a functor

$$\sigma^* : \mathrm{Alg}_{\mathbf{\Sigma}^{-1}\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$$

which is the dashed arrow in the following commutative diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[rr, "\mathrm{oblv}_{\mathcal{O}} \circ \Omega_{\mathcal{O}}"] \arrow[dr, "\Omega'"] \arrow[ddr, bend right, "\Omega_{\mathcal{O}}"'] & & \mathcal{C} \\
& \mathrm{Alg}_{\mathbf{\Sigma}^{-1}\mathcal{O}}(\mathcal{C}) \arrow[ur, "\mathrm{oblv}_{\mathbf{\Sigma}^{-1}\mathcal{O}}"] \arrow[d, dashed, "\sigma^*"] & \\
& \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}). \arrow[uur, bend right, "\mathrm{oblv}_{\mathcal{O}}"'] &
\end{tikzcd}$$

By the dual statement of Theorem A.4.4, the functor

$$\sigma^* : \mathrm{Alg}_{\mathbf{\Sigma}^{-1}\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$$

induces a map of monads

$$\sigma : F_{\mathcal{O}} \to \Omega F_{\mathcal{O}}\Sigma, \tag{3.10}$$

which we will refer as the *suspension morphism* associated to the operad $\mathcal{O}$.

Definition 3.4.8. We define the *spectral Lie operad* as

$$\mathbf{Lie} := \mathbf{\Sigma}^{-1}\mathbb{L}.$$

Remark 3.4.9. The underlying symmetric sequence of the spectral Lie operad $\mathbf{Lie} := \mathbf{\Sigma}^{-1}\mathbb{L}$ has the form

$$\mathbf{Lie}(n) = \mathbb{S}^n \otimes \Sigma^{-1}\mathbb{L}(n),$$

whose homology is exactly $\mathrm{Lie}(n)$ concentrated in degree 0. Moreover, the homology of the spectral Lie operad $H_*\mathbf{Lie}$ is the Lie operad Lie in graded abelian groups. On the other hand, $\mathbf{Lie}$ is the Goodwillie derivative of the functor $\Omega\Sigma$ on $\mathcal{S}_*$, see [Goo03, Section 8]. Moreover, one can check the associated Schur functor $F_{\mathbf{Lie}}$ of $F_{\mathbf{Lie}}$ is indeed given by $\Omega\mathbb{L}\Sigma$,

$$\begin{aligned} F_{\mathbf{Lie}}(X) &\simeq \coprod_{n\geq 1} \left(\mathbb{S}^n \otimes \Sigma^{-1}\mathbb{L}(n) \otimes X^{\otimes n}\right)_{h\Sigma_n} \\ &\simeq \Sigma^{-1} \coprod_{n\geq 1} \left(\mathbb{L}(n) \otimes (\Sigma X)^{\otimes n}\right)_{h\Sigma_n} \\ &\simeq \Omega F_{\mathbb{L}}\Sigma(X). \end{aligned}$$

3.5 Induced Monads and Comonads

The goal for this section is to explain how to produce monads on $\mathrm{Sp}^{\geq r}$ and $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ from certain monads on Sp.

We start by proving a general fact regarding how localizations interact with endofunctors. Suppose $L : \mathcal{C} \to \mathcal{D}$ is a symmetric monoidal localization and, we let $j : \mathcal{D} \to \mathcal{C}$ denote the embedding that is right adjoint to L. We then have an adjunction on the ∞-category of endofunctors.

Proposition 3.5.1. *There is an adjunction*

$$j^*L_*\colon \mathrm{Fun}(\mathcal{C},\mathcal{C}) \rightleftarrows \mathrm{Fun}(\mathcal{D},\mathcal{D}) : j_*L^*. \tag{3.11}$$

Moreover, the left adjoint j^*L_* is a localization functor.

Proof. We need to show there are two pairs of adjunctions

$$L_*\colon \mathrm{Fun}(\mathcal{C},\mathcal{C}) \rightleftarrows \mathrm{Fun}(\mathcal{C},\mathcal{D}) : j_*$$

and

$$j^*\colon \mathrm{Fun}(\mathcal{C},\mathcal{D}) \rightleftarrows \mathrm{Fun}(\mathcal{D},\mathcal{D}) : L^*.$$

We only prove the first adjunction, as the proof of the second is similar. Let $\eta : \mathrm{id}_{\mathcal{C}} \to j \circ L$ be the unit natural transformation arising from the localization. By post-composition, η lifts to a natural transformation

$$\eta' : \mathrm{id}_{\mathrm{Fun}(\mathcal{C},\mathcal{C})} \to j_*L_*.$$

It suffices to check the following composite is an equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(L_*F, G) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{C})}(j_*L_*F, j_*G) \xrightarrow{\eta'^*} \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{C})}(F, j_*G)$$

for $F \in \mathrm{Fun}(\mathcal{C},\mathcal{C})$ and $G \in \mathrm{Fun}(\mathcal{C},\mathcal{D})$. But this follows from the fact that the composite

$$\mathrm{Map}_{\mathcal{C}}(LF(X), G(X)) \to \mathrm{Map}_{\mathcal{D}}(jLF(X), jG(X)) \xrightarrow{\eta^*} \mathrm{Map}_{\mathcal{D}}(F(X), jG(X))$$

is an equivalence for every $X \in \mathcal{C}$.

Note that the counit of the adjunction in (3.11)

$$j^* L_* j_* L^* \to \mathrm{id}_{\mathrm{Fun}(\mathcal{D},\mathcal{D})}$$

is an equivalence, hence the right adjoint $j_* L^*$ is fully faithful and the proposition is proved. □

The following corollary is a direct consequence of the proposition above and [Lur17, Proposition 2.2.1.1.].

Corollary 3.5.2. *The functor $j_* L^* : \mathrm{Fun}(\mathcal{D},\mathcal{D}) \to \mathrm{Fun}(\mathcal{C},\mathcal{C})$ is monoidal. Therefore, the left adjoint $j^* L_*$ is oplax monoidal.*

Let $\mathrm{Fun}_L(\mathcal{C},\mathcal{C})$ be the full subcategory of $\mathrm{Fun}(\mathcal{C},\mathcal{C})$ spanned by functors that preserve L-equivalences. The goal of the rest of this section is to show $j^* L_*$ is monoidal when restricted to $\mathrm{Fun}_L(\mathcal{C},\mathcal{C})$. Note first that for any $F \in \mathrm{Fun}_L(\mathcal{C},\mathcal{C})$, we have a natural equivalence

$$L \circ F \simeq L \circ F \circ L \tag{3.12}$$

since F preserves L-equivalences.

Remark 3.5.3. Let $\mathrm{Fun}_L(\mathcal{D},\mathcal{D})$ denote the essential image of $\mathrm{Fun}_L(\mathcal{C},\mathcal{C})$ under the functor $j^* L_*$. For any pair of functors $F, G \in \mathrm{Fun}_L(\mathcal{C},\mathcal{C})$, there is an natural equivalence

$$L \circ (F \circ G) \simeq L \circ (F \circ LG)$$

by (3.12). If $f : X \to Y$ is an L-equivalence in $\mathcal{C}$, then $FG(X) \to FG(Y) \to F \circ LG(Y)$ is an L-equivalence since both F and G preserve L-equivalences. Hence, we conclude that both $\mathrm{Fun}_L(\mathcal{C},\mathcal{C})$ and $\mathrm{Fun}_L(\mathcal{D},\mathcal{D})$ are closed under composition.

For $F \in \mathrm{Fun}_L(\mathcal{C},\mathcal{C})$, the right adjoint $j_* L^*$ sends any functor of the form $j^* L_* F$ to a functor in $\mathrm{Fun}_L(\mathcal{C},\mathcal{C})$; indeed, this follows from the following equivalence

$$j_* L^* j^* L_* F \simeq j_* L_* F$$

by (3.12). Since the functor $j_* L_* F$ clearly preserves L-equivalences, the functor $j^* L_*$ restricts to a localization

$$\mathrm{Fun}_L(\mathcal{C},\mathcal{C}) \to \mathrm{Fun}_L(\mathcal{D},\mathcal{D}).$$

Lemma 3.5.4. *The functor*

$$j^* L_* : \mathrm{Fun}_L(\mathcal{C},\mathcal{C}) \to \mathrm{Fun}_L(\mathcal{D},\mathcal{D})$$

is a monoidal.

Proof. By [Lur17, Proposition 2.2.1.9.], it suffices to check the monoidal structure is compatible with the localization functor $j^* L_*$ in the sense of [Lur17, Definition 2.2.1.6.]. Suppose for F, F' *in* $\mathrm{Fun}_L(\mathcal{C},\mathcal{C})$ there is a natural transformation

$$T : F \to F'$$

that is an equivalence after applying $j^* L_*$. It suffices to check

$$j^* L_*(T \circ \mathrm{id}) : L(F \circ G) \to L(F' \circ G)$$

is an equivalence for any $G \in \mathrm{Fun}_L(\mathcal{C},\mathcal{C})$. Consider the commutative diagram

$$\begin{tikzcd} L(F \circ G) \arrow[r] \arrow[d] & LF \circ j \circ LG \arrow[d] \\ L(F' \circ G) \arrow[r] & LF' \circ j \circ LG. \end{tikzcd}$$

The horizontal arrows are equivalences since both F and F' preserve L-equivalences, and the right vertical arrow is an equivalence by the assumption. Therefore, the left arrow is an equivalence and the proof is now complete. □

Remark 3.5.5. As a consequence, if $F \in \mathrm{Fun}_L(\mathcal{C},\mathcal{C})$ is a monad, then $j^* L_*(F)$ is a monad on $\mathcal{D}$. For simplicity, we will abuse notation by writing LF for the monad $j^* L_*(F)$. Similarly, if $G \in \mathrm{Fun}_L(\mathcal{C},\mathcal{C})$ is a comonad, then $j^* L_*(G)$ is a comonad on $\mathcal{D}$.

Let $\{\mathcal{O}(k)\}_{k\geq 1}$ be a symmetric sequence in Sp such that the connectivity of $\mathcal{O}(k)$ is $1-k$ for each k. For $r \geq 1$, let X be an r-connective spectrum. Since

$(\mathcal{O}(k) \otimes X^{\otimes k})_{h\Sigma_k}$ is still r-connective, the Schur functor associated to $\mathcal{O}$

$$F_{\mathcal{O}}(X) \simeq \coprod_{k \geq 1} (\mathcal{O}(k) \otimes X^{\otimes k})_{h\Sigma_k}$$

induces an endofunctor on the ∞-category $\mathrm{Sp}^{\geq r}$ of r-connective spectra.

Let $\mathrm{SSeq}(\mathrm{Sp})^{\geq 1}$ be the full subcategory of $\mathrm{SSeq}(\mathrm{Sp})$ spanned by symmetric sequences $\{\mathcal{O}(k)\}_{k\geq 1}$ such that $\mathcal{O}(k)$ is $(1-k)$-connective.

Lemma 3.5.6. *The ∞-category $\mathrm{SSeq}(\mathrm{Sp})^{\geq 1}$ is closed under the composition product.*

Proof. For any $\mathcal{O}, \mathcal{P} \in \mathrm{SSeq}(\mathrm{Sp})^{\geq 1}$, it suffices to check $(\mathcal{O} \circ \mathcal{P})(n)$ is still $(1-n)$-connective for every $n \geq 1$. By the formula in Remark 3.1.4, a component in $(\mathcal{O} \circ \mathcal{P})(n)$ is of the form

$$\big(\mathcal{O}(k) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k)\big)_{h\Sigma_k}$$

where $\sum_i n_i = n$. The lemma then follows from the calculation of the connectivity of $\mathcal{O}(k) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k)$, which equals to

$$1 - k + \sum_i (1 - n_i) = 1 - \sum_i n_i = 1 - n.$$

$\square$

We extend the definition of coanlytic functors in [Heu21b, Definition 4.3].

Definition 3.5.7. We say an endofunctor $F : \mathrm{Sp}^{\geq r} \to \mathrm{Sp}^{\geq r}$ is *coanalytic* over Sp if there is a natural equivalence

$$F(X) \simeq \coprod_{k \geq 1} (\mathcal{O}(k) \otimes X^{\otimes k})_{h\Sigma_k}$$

for $\{\mathcal{O}(k)\}_{k\geq 1}$ a symmetric sequence in $\mathrm{SSeq}(\mathrm{Sp})^{\geq 1}$. We write $\mathrm{coAn}_{\mathrm{Sp}}(\mathrm{Sp}^{\geq r})$ for the full subcategory of $\mathrm{Fun}(\mathrm{Sp}^{\geq r}, \mathrm{Sp}^{\geq r})$ spanned by coanalytic functors over Sp.

As a consequence of Lemma 3.5.6, the composition of two coanalytic functors is again a coanalytic functor, hence the monoidal structure on $\mathrm{Fun}(\mathrm{Sp}^{\geq r}, \mathrm{Sp}^{\geq r})$ restricts to one on $\mathrm{coAn}_{\mathrm{Sp}}(\mathrm{Sp}^{\geq r})$. Moreover, if $\mathcal{O}$ is an operad (resp. cooperad) in $\mathrm{SSeq}(\mathrm{Sp})^{\geq 1}$, then $F_{\mathcal{O}}$ restricts to a monad (resp. comonad) on $\mathrm{Sp}^{\geq r}$.

Example 3.5.8. The shifted free Lie algebra monad $F_{\mathbb{L}}$ is a coanalytic functor on $\mathrm{Sp}^{\geq r}$ over Sp, since $\mathbb{L}(n)$ is a wedge of $\mathbb{S}^{1-n}$. Similarly, $F_{\mathbf{Lie}}$ is a coanalytic functor on $\mathrm{Sp}^{\geq r}$.

Example 3.5.9. The free commutative comonad F_{Com} is a coanalytic functor $\mathrm{Sp}^{\geq r}$ over Sp, since $\mathrm{Com}(n)$ is $\mathbb{S}$.

Remark 3.5.10. Since tame equivalences are closed under tensor products and colimits, we conclude that coanalytic functors on $\mathrm{Sp}^{\geq r}$ over Sp preserve tame equivalences.

Combining Lemma 3.5.4 and Remark 3.5.10, we end this section with the following proposition.

Proposition 3.5.11. *If $\mathcal{O}$ (resp. $\mathcal{Q}$) is an ∞-operad (resp. ∞-cooperad) whose underlying symmetric sequence is in $\mathrm{SSeq}(\mathrm{Sp})^{\geq 1}$, then we obtain an induced monad $L_{\mathrm{tame}}F_{\mathcal{O}}$ (resp. comonad $L_{\mathrm{tame}}F_{\mathcal{Q}}$) on the ∞-category $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ of tame spectra.*

3.6 Commutative Coalgebras in Tame Spectra

In this section, we define and study the ∞-category of commutative coalgebras in the category of r-tame spectra. The main aim is to show the ∞-category of divided power, conilpotent commutative coalgebras is equivalent to the ∞-category of commutative coalgebras in $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$. We start by collecting some results from [Lur18a].

Definition 3.6.1. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. The ∞-category $\mathrm{coCAlg}(\mathcal{C})$ of commutative coalgebras in $\mathcal{C}$ is defined to be $\mathrm{coCAlg}(\mathcal{C}) := \mathrm{CAlg}(\mathcal{C}^{op})^{op}$.

Proposition 3.6.2. *[Lur18a, Corollary 3.1.5] Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Suppose that the ∞-category $\mathcal{C}$ is presentable and that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves colimits separately in each variable. Then the forgetful functor $\mathrm{coCAlg}(\mathcal{C}) \to \mathcal{C}$ admits a right adjoint $\mathrm{cofree} : \mathcal{C} \to \mathrm{coCAlg}(\mathcal{C})$.*

Corollary 3.6.3. *The forgetful functor*

$$\mathrm{oblv}_{\mathrm{coCAlg}} : \mathrm{coCAlg}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \to \mathrm{Sp}^{\geq r}_{\mathrm{tame}}$$

admits a right adjoint functor, which we will denote by $\mathrm{cofree}_{\mathrm{tame}}$.

Proof. Note that $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ is presentable as it's an accessible localization of the presentable ∞-category $\mathrm{Sp}^{\geq r}$. Since the tensor product $\hat{\otimes}$ in $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ preserves colimits separately in each variable by Remark 2.3.10, the existence of the cofree commutative coalgebra functor is ensured by Proposition 3.6.2. □

Since the monad F_{Com} associated to the commutative operad has underlying endofunctor

$$F_{\mathrm{Com}}(X) := \coprod_{n\geq 1}(X^{\otimes n})_{h\Sigma_n},$$

hence $F(X) := \prod_{n\geq 1}(X^{\otimes n})^{h\Sigma_n}$ defines a comonad on $\mathcal{C}$. For a general symmetric monoidal ∞-category $\mathcal{C}$, we do not know any explicit identification of the cofree comonad $Q := \mathrm{oblv}_{\mathrm{coCAlg}} \circ \mathrm{cofree}$. However, we do know an explicit formula for the comonad $Q_{\mathrm{tame}} := \mathrm{oblv}_{\mathrm{coCAlg}} \circ \mathrm{cofree}_{\mathrm{tame}}$ in the ∞-category of tame spectra. To explain this, we need the following lemma which can be deduced from [Lur17, Proposition 3.1.3.3] and [Lur17, Example 3.1.1.17].

Lemma 3.6.4. *Let $\mathcal{C}$ be a presentable symmetric monoidal ∞-category. Define F to be the endofunctor on $\mathcal{C}$*

$$F(X) := \prod_{n\geq 1}(X^{\otimes n})^{h\Sigma_n}.$$

If the canonical map

$$\gamma : \Big(\prod_{n\geq 1}(X^{\otimes n})^{h\Sigma_n}\Big) \otimes Y \to \prod_{n\geq 1}(X^{\otimes n} \otimes Y)^{h\Sigma_n}$$

is an equivalence for any $Y \in \mathcal{C}$ with trivial Σ_n-action, for all n, then the cofree comonad Q is given by F.

Proof. By [Lur17, Definition 3.1.3.1.] the cofree coalgebra of an obejct X in $\mathcal{C}$ is defined as a operadic limit diagram (cf. [Lur17, Definition 3.1.1.2.]) $p : \mathrm{Fin}_* \to \mathcal{C}$. By a dual statement of [Lur17, Proposition 3.1.3.3] in the special case where $\mathcal{A}$ is the trivial ∞-operad, $\mathcal{B}$ and $\mathcal{O}$ are the commutative ∞-operad Fin_*, such a operadic limit exists. The lemma then follows from the dual version of [Lur17, Example 3.1.1.17], which says that a diagram $p : \mathrm{Fin}_* \to \mathcal{C}$ is an operadic limit diagram if it remains as a limit diagram in $\mathcal{C}$ after tensoring with any object $Y \in \mathcal{C}$. In our case, this means that $\prod_{n\geq 1}(X^{\otimes n})^{h\Sigma_n}$ is an operadic limit diagram if $\big(\prod_{n\geq 1}(X^{\otimes n})^{h\Sigma_n}\big) \otimes Y$ is a limit diagram for any $Y \in \mathcal{C}$, which is the condition required for γ being an equivalence.

□

We now prove that the ∞-category of tame spectra satisfies the conditions of Lemma 3.6.4, therefore we obtain a simple description of the comonad Q_{tame}. We write $\mathrm{Sym}_{\mathrm{tame}}$ for the symmetric algebra functor on $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$, which is given explicitly by

$$\mathrm{Sym}_{\mathrm{tame}}(X) := \bigoplus_{n\geq 1} L_{\mathrm{tame}}(X^{\otimes n})_{h\Sigma_n}.$$

Proposition 3.6.5. *The cofree commutative comonad on the ∞-category $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ of tame spectra is given by*

$$Q_{\mathrm{tame}}(X) \simeq \bigoplus_{n\geq 1} L_{\mathrm{tame}}(X^{\otimes n})_{h\Sigma_n} \simeq \mathrm{Sym}_{\mathrm{tame}}(X)$$

for any $X \in \mathrm{Sp}^{\geq r}_{\mathrm{tame}}$.

Proof. We first claim that the canonical map

$$\gamma : \bigoplus_{n\geq 1}(L_{\mathrm{tame}}X^{\otimes n})_{h\Sigma_n} \to \prod_{n\geq 1}(L_{\mathrm{tame}}X^{\otimes n})^{h\Sigma_n}$$

is an equivalence for any $X \in \mathrm{Sp}^{\geq r}_{\mathrm{tame}}$. Note that the target is equivalent to

$$\prod_{n\geq 1}(L_{\mathrm{tame}}X^{\otimes n})_{h\Sigma_n},$$

by Proposition 2.3.21.

We now show that γ is an equivalence after applying k-truncation functor $\tau_{\leq k}$ for each integer $k \geq r$. There exists a maximum integer $l(k)$ such that $X^{\otimes(l+1)}$ is $(k+1)$-connective. Hence the k-truncation of γ can be identified as a map

$$\tau_{\leq k}\gamma : \bigoplus_{n\geq 1}^{l(k)} \tau_{\leq k}(L_{\mathrm{tame}}X^{\otimes n})_{h\Sigma_n} \to \bigoplus_{n\geq 1}^{l(k)} \tau_{\leq k}(L_{\mathrm{tame}}X^{\otimes n})_{h\Sigma_n}$$

which is an equivalence for every k. Hence, γ is an equivalence. Since tensor product commutes with colimits in $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$, $F(X) := \prod_{n\geq 1}(L_{\mathrm{tame}}X^{\otimes n})^{h\Sigma_n}$ satisfies the condition of Lemma 3.6.4, and thus the comonad Q_{tame} is given by

$$Q_{\mathrm{tame}}(X) \simeq \mathrm{Sym}_{\mathrm{tame}}(X).$$

□

We now define the ∞-category of divided power, conilpotent coalgebras in $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$. Note that $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ is a non-unital symmetric monoidal ∞-category, so there is no commutative cooperad in $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$. However, we can define a comonad which comes from the commutative cooperad in Sp via tame localization.

By example 3.1.15, the Schur functor F_{Com} associated to the commutative cooperad in Sp is a comonad and is given by

$$F_{\mathrm{Com}}(X) := \coprod_{n\geq 1} (X^{\otimes n})_{h\Sigma_n}.$$

Note that F_{Com} restricts to a comonad on the ∞-category $\mathrm{Sp}^{\geq r}$ of r-connective spectra, since colimits in $\mathrm{Sp}^{\geq r}$ are computed in Sp. By Proposition 3.5.11, we obtain a comonad $L_{\mathrm{tame}}F_{\mathrm{Com}}$ on $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$, given by

$$L_{\mathrm{tame}}F_{\mathrm{Com}}(X) := \coprod_{n\geq 1} L_{\mathrm{tame}}(X^{\otimes n})_{h\Sigma_n}.$$

as L_{tame} is colimit-preserving and symmetric monoidal.

Definition 3.6.6. The ∞-category of *divided power, conilpotent coalgebras in tame spectra* is defined to be the ∞-category of left comodules over the comonad $L_{\mathrm{tame}}F_{\mathrm{Com}}$

$$\mathrm{coCAlg}^{\mathrm{dp,nil}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) := \mathrm{LcoMod}_{L_{\mathrm{tame}}F_{\mathrm{Com}}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}).$$

We now want to show these two ∞-categories of coalgebras in tame spectra are actually equivalent. Since the underlying endofunctors of Q_{tame} and $L_{\mathrm{tame}}F_{\mathrm{Com}}$ are equivalent, it suffices to show there is a natural transformation of comonads

$$\Gamma : L_{\mathrm{tame}}F_{\mathrm{Com}} \to Q_{\mathrm{tame}},$$

which gives rise to an equivalence $L_{\mathrm{tame}}F_{\mathrm{Com}} \simeq Q_{\mathrm{tame}}$. By the discussion in Appendix A.4, there is a map of comonads on $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$

$$\Gamma : (L_{\mathrm{tame}})_* j^* F_{\mathrm{Com}} \to Q_{\mathrm{tame}}.$$

which is the unique map that makes the diagram in Remark A.4.5 commute.

The map of comonads Γ then induces a comparison functor

$$\zeta : \mathrm{coCAlg}^{\mathrm{dp,nil}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \to \mathrm{coCAlg}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}).$$

The following proposition now follows immediately from Proposition 3.6.5.

Proposition 3.6.7. *The comparison functor ζ is an equivalence of ∞-categories.*

Proof. Since the forgetful functor $\mathrm{Comonad}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \to \mathrm{End}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}})$ is conservative, it suffices to show the Γ induces an equivalence between the underlying endofunctors of $L_{\mathrm{tame}} F_{\mathrm{Com}}$ and Q_{tame}. For $X \in \mathrm{Sp}^{\geq r}_{\mathrm{tame}}$, the map Γ is obtained from the projection $L_{\mathrm{tame}} F_{\mathrm{Com}} X \to X$ to the first factor (see Appendix A.4), which agrees with the canonical projection $Q_{\mathrm{tame}} X \to X$ by 3.6.5. Hence Γ is an equivalence since it is an equivalence between the underlying endorfunctors. □

3.7 Lie Algebras in Tame Spectra

In this section, we discuss spectral Lie algebras in Sp and $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$. We will prove that the ∞-category of tame spectral Lie algebras can be identified as the full subcategory of spectral Lie algebras whose underlying spectra are tame. We then define the Chevalley-Eilenberg functor that goes from the ∞-category of spectral Lie algebras to the ∞-category of divided power, conilpotent commutative coalgebras.

Definition 3.7.1. We define the ∞-category $\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp})$ of *spectral Lie algebras* to be the ∞-category of algebras over the spectral Lie operad **Lie**.

Remark 3.7.2. Note that our definition of spectral Lie algebras might be different from the one given in the literature, e.g. [Cam16] and [Chi05], where $\mathbb{L}$ is referred as the spectral Lie operad. Our definition of the spectral Lie operad **Lie** differs by a shift. The advantage of this definition is that the homology $H_* \mathbf{Lie}$ is precisely the Lie operad in abelian groups, hence the homology of a spectral Lie algebra is a graded Lie algebra.

By Proposition 3.5.11, $L_{\mathrm{tame}} F_{\mathbf{Lie}}$ defines a monad on $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$. The underlying endofunctor of the monad $L_{\mathrm{tame}} F_{\mathbf{Lie}}$ on $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ is simply given by the tame localization of the free spectral Lie algebra monad on $\mathrm{Sp}^{\geq r}$:

$$X \mapsto \coprod_{n \geq 1} L_{\mathrm{tame}} \big(\mathbf{Lie}(n) \otimes (L_{\mathrm{tame}} X)^{\otimes n} \big)_{h\Sigma_n}.$$

We are now ready to define the ∞-category of tame spectral Lie algebras.

Definition 3.7.3. We define the ∞-category $\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}})$ of *tame spectral Lie algebras* to be

$$\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) := \mathrm{LMod}_{L_{\mathrm{tame}} F_{\mathbf{Lie}}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}).$$

Similarly, we define the ∞-category of *shifted tame spectral Lie algebras*

$$\mathrm{Alg}_{\mathbb{L}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) := \mathrm{LMod}_{L_{\mathrm{tame}} F_{\mathbb{L}}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}).$$

On the level of underlying spectra, the induced functor

$$\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp})^{\geq r} \to \mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}).$$

is simply given by sending an r-connective spectrum X to its tame localization $L_{\mathrm{tame}} X$.

Remark 3.7.4. Although the monad $L_{\mathrm{tame}} F_{\mathrm{Lie}}$ does not come from an operad in $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$, it does have all the associated functors we discussed in Section 3.3. Indeed, the spectral Lie monad on $\mathrm{Sp}^{\geq r}$ has unit η and augmentation ϵ natural transformations so that the composition

$$\mathrm{id}_{\mathrm{Sp}^{\geq r}} \xrightarrow{\eta} F_{\mathrm{Lie}} \xrightarrow{\epsilon} \mathrm{id}_{\mathrm{Sp}^{\geq r}}$$

is the identity natural transformation. Applying tame localization, there is again a composite of monads on $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$

$$\mathrm{id}_{\mathrm{Sp}^{\geq r}_{\mathrm{tame}}} \xrightarrow{\eta} L_{\mathrm{tame}} F_{\mathrm{Lie}} \xrightarrow{\epsilon} \mathrm{id}_{\mathrm{Sp}^{\geq r}_{\mathrm{tame}}}$$

which is equivalent to the identity natural transformation on $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$. Therefore, we have the adjoint pairs between tame spectral Lie algebras and tame spectra

$$\mathrm{Sp}^{\geq r}_{\mathrm{tame}} \underset{\mathrm{oblv}\,\mathbf{Lie}}{\overset{\mathrm{free}_{\mathbf{Lie}}}{\rightleftarrows}} \mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \underset{\mathrm{triv}_{\mathbf{Lie}}}{\overset{\mathrm{cot}_{\mathbf{Lie}}}{\rightleftarrows}} \mathrm{Sp}^{\geq r}_{\mathrm{tame}}.$$

Remark 3.7.5. Dually, recall we have defined the ∞-category of divided power, conilpotent tame commutative coalgebras in Definition 3.6.6. As in the case of spectral Lie algebras, we have a pair of adjunctions

$$\mathrm{Sp}^{\geq r}_{\mathrm{tame}} \underset{\mathrm{oblv}^{\mathrm{nil}}_{\mathrm{Com}}}{\overset{\mathrm{cofree}^{\mathrm{nil}}_{\mathrm{Com}}}{\rightleftarrows}} \mathrm{coAlg}^{\mathrm{dp,nil}}_{\mathrm{Com}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \underset{\mathrm{triv}^{\mathrm{nil}}_{\mathrm{Com}}}{\overset{\mathrm{Prim}^{\mathrm{nil}}_{\mathrm{Com}}}{\rightleftarrows}} \mathrm{Sp}^{\geq r}_{\mathrm{tame}}.$$

We claim the ∞-category of tame spectral Lie algebras is a full subcategory of the ∞-category of r-connective spectral Lie algebras whose underlying spectra are tame. It turns out that this is a consequence of the following more general result.

Proposition 3.7.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories whose tensor products are compatible with colimits. Suppose $L : \mathcal{C} \to \mathcal{D}$ is a symmetric monoidal localization functor. If $F_{\mathcal{O}}$ is a monad in $\mathcal{C}$, then the induced functor*

$$L' : \mathrm{Alg}_{F_{\mathcal{O}}}(\mathcal{C}) \to \mathrm{Alg}_{LF_{\mathcal{O}}}(\mathcal{D}).$$

is a localization.

Proof. The existence of the functor L' follows from Remark 3.5.5. It remains to show L' is a localization functor. Let g be the fully faithful right adjoint of L. We first claim that g lifts to a functor

$$g' : \mathrm{Alg}_{LF_{\mathcal{O}}}(\mathcal{D}) \to \mathrm{Alg}_{F_{\mathcal{O}}}(\mathcal{C}),$$

which is right adjoint to L'. Indeed, since L is symmetric monoidal, it follows that g is lax symmetric monoidal; hence g induces a functor $g' : \mathrm{Alg}_{LF_{\mathcal{O}}}(\mathcal{D}) \to \mathrm{Alg}_{F_{\mathcal{O}}}(\mathcal{C})$. To see that g' is right adjoint to L', let $\eta : id_{\mathcal{C}} \to gL$ be the unit natural transformation. Note that η is a $\mathrm{End}(\mathcal{C})$-linear natural transformation in the sense of [Lur17, Definition 4.6.2.7.], hence it induces a natural transformation $\eta' : \mathrm{id}_{\mathrm{Alg}_{F_{\mathcal{O}}}(\mathcal{C})} \to g' \circ L'$. We claim the map

$$\rho : \mathrm{Map}_{\mathrm{Alg}_{LF_{\mathcal{O}}}(\mathcal{D})}(L'M, N) \to \mathrm{Map}_{\mathrm{Alg}_{F_{\mathcal{O}}}(\mathcal{C})}(M, g'N),$$

induced from η', is an equivalence for any $M \in \mathrm{Alg}_{F_{\mathcal{O}}}(\mathcal{C})$ and $N \in \mathrm{Alg}_{LF_{\mathcal{O}}}(\mathcal{D})$.

First observe that L' sends free $F_{\mathcal{O}}$-algebras to free $LF_{\mathcal{O}}$-algebras, i.e.

$$L'(\mathrm{free}_{F_{\mathcal{O}}} X) \simeq \mathrm{free}_{LF_{\mathcal{O}}} f(X)$$

for $X \in \mathcal{C}$, as L is symmetric monoidal and colimit-preserving. Hence,

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Alg}_{LF_{\mathcal{O}}}(\mathcal{D})}(L' \,\mathrm{free}_{F_{\mathcal{O}}} X, N) &\simeq \mathrm{Map}_{\mathrm{Alg}_{LF_{\mathcal{O}}}(\mathcal{D})}(\mathrm{free}_{LF_{\mathcal{O}}} L(X), N) \\
&\simeq \mathrm{Map}_{\mathcal{D}}(L(X), \mathrm{oblv}_{LF_{\mathcal{O}}} N) \\
&\simeq \mathrm{Map}_{\mathcal{C}}(X, g \circ \mathrm{oblv}_{LF_{\mathcal{O}}} N) \\
&\simeq \mathrm{Map}_{\mathcal{C}}(X, \mathrm{oblv}_{F_{\mathcal{O}}} \circ g' N) \\
&\simeq \mathrm{Map}_{\mathcal{C}}(\mathrm{free}_{F_{\mathcal{O}}} X, g' N)
\end{aligned}$$

and we conclude that ρ is an equivalence whenever M is a free $F_{\mathcal{O}}$-algebra. For a general $F_{\mathcal{O}}$-algebra, we use Proposition 3.3.6 to write M as a geometric realization of a $\mathrm{oblv}_{F_{\mathcal{O}}}$-split simplicial free $F_{\mathcal{O}}$-algebras. Then the claim follows from the Barr-Beck-Lurie theorem [Lur17, Theorem 4.7.3.5.] and the fact that L' preserves geometric realizations of $\mathrm{oblv}_{F_{\mathcal{O}}}$-split simplicial objects.

For the fully-faithfulness of g', it suffices to show the counit map $L'g'N \to N$ is an equivalence for any $N \in \mathrm{Alg}_{LF_{\mathcal{O}}}(\mathcal{D})$. This follows from the fact that the forgetful functor $\mathrm{oblv}_{LF_{\mathcal{O}}}$ is conservative and g is fully faithful. □

Corollary 3.7.7. *The ∞-category $\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}})$ can be identified with the full subcategory of $\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp})^{\geq r}$ spanned by spectral Lie algebras whose underlying spectra are tame.*

Proof. Proposition 3.7.6 provides a localization functor

$$L' : \mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp})^{\geq r} \to \mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}).$$

The corollary then follows from the commuting diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp})^{\geq r} \arrow[d, "\mathrm{oblv}"'] & \mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \arrow[l] \arrow[d, "\mathrm{oblv}'"] \\
\mathrm{Sp}^{\geq r} & \mathrm{Sp}^{\geq r}_{\mathrm{tame}} \arrow[l]
\end{tikzcd}$$

where the horizontal functors are fully faithful.

□

We now give an algebraic characterization of tame spectral Lie algebras and tame commutative coalgebras. We first define Lie algebras in $\mathrm{Mod}_{H\mathbb{Z}}$, for which we need the following result from Haugseng.

Lemma 3.7.8. *[Hau17, Corollary 4.2.9.] Let $\mathcal{C}$ and $\mathcal{D}$ be preadditive symmetric monoidal ∞-categories whose tensor products are compatible with colimits. Let*

$$f : \mathcal{C} \to \mathcal{D}$$

be a colimit-preserving, symmetric monoidal functor. Postcomposing with f gives a monoidal functor

$$\mathrm{SSeq}(\mathcal{C}) \to \mathrm{SSeq}(\mathcal{D}).$$

Since the functor $(-\otimes H\mathbb{Z}) : \mathrm{Sp} \to \mathrm{Mod}_{H\mathbb{Z}}$ satisfies the condition of Lemma 3.7.8, we conclude that the symmetric sequence

$$(\mathbf{Lie} \otimes H\mathbb{Z})(n) := \mathbf{Lie}(n) \otimes H\mathbb{Z}$$

defines an operad in $\mathrm{Mod}_{H\mathbb{Z}}$.

Definition 3.7.9. We define the ∞-category of Lie algebras in $H\mathbb{Z}$-modules by

$$\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Mod}_{H\mathbb{Z}}) := \mathrm{LMod}_{\mathbf{Lie}\otimes H\mathbb{Z}}(\mathrm{Mod}_{H\mathbb{Z}}).$$

Remark 3.7.10. The homotopy groups of $\mathbf{Lie}(n)\otimes H\mathbb{Z}$ are the homology groups of $\mathbf{Lie}(n)$ (concentrated in degree 0), hence $\pi_*\mathbf{Lie}(n) \otimes H\mathbb{Z}$ is the Lie operad in graded abelian groups by our discussion in §3.4. Since $\mathrm{Mod}_{H\mathbb{Z}}$ models the derived ∞-category $D(\mathbb{Z})$ of chain complexes, we can identify $\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Mod}_{H\mathbb{Z}})$ as the ∞-category of dg Lie algebras over $\mathbb{Z}$.

Let L_{tame} be the localization $\mathrm{Mod}_{H\mathbb{Z}}^{\geq r} \to (\mathrm{Mod}_{H\mathbb{Z}}^{\geq r})_{\mathrm{tame}}$. Since the associated Schur functor for the operad $\mathbf{Lie}\otimes H\mathbb{Z}$ also preserves tame $H\mathbb{Z}$-equivalences, by Lemma 3.5.4, we obtain a monad $L_{\mathrm{tame}}F_{\mathbf{Lie}\otimes H\mathbb{Z}}$ on $(\mathrm{Mod}_{H\mathbb{Z}}^{\geq r})_{\mathrm{tame}}$.

Definition 3.7.11. We define the ∞-category of tame Lie algebras in $(\mathrm{Mod}_{H\mathbb{Z}}^{\geq r})_{\mathrm{tame}}$ by

$$\mathrm{Alg}_{\mathbf{Lie}}((\mathrm{Mod}_{H\mathbb{Z}}^{\geq r})_{\mathrm{tame}}) := \mathrm{LMod}_{L_{\mathrm{tame}}F_{\mathbf{Lie}\otimes H\mathbb{Z}}}((\mathrm{Mod}_{H\mathbb{Z}}^{\geq r})_{\mathrm{tame}}).$$

Remark 3.7.12. Since the underlying free Lie monad on $(\mathrm{Mod}_{H\mathbb{Z}}^{\geq r})_{\mathrm{tame}}$ is given by

$$L_{\mathrm{tame}}F_{\mathbf{Lie}\otimes H\mathbb{Z}}(M) \simeq \coprod_{k\geq 1} L_{\mathrm{tame}}(\mathbf{Lie}(n) \otimes H\mathbb{Z} \underset{\mathbb{Z}}{\otimes} M^{\otimes_{\mathbb{Z}}^{n}})_{h\Sigma_n},$$

by Proposition 3.7.6, we conclude that $\mathrm{Alg}_{\mathbf{Lie}}((\mathrm{Mod}_{H\mathbb{Z}}^{\geq r})_{\mathrm{tame}})$ is the full subcategory of $\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Mod}_{H\mathbb{Z}}^{\geq r})$ consisting of Lie algebras in $\mathrm{Mod}_{H\mathbb{Z}}^{\geq r}$ whose underlying spectra are tame. Therefore, we can identify $\mathsf{Alg}_{\mathbf{Lie}}((\mathsf{Mod}_{H\mathbb{Z}}^{\geq r})_{\mathsf{tame}})$ as Dwyer's tame Lie algebras in [Dwy79].

Recall from Theorem 2.3.15 that we have a symmetric monoidal equivalence

$$L_{\mathsf{tame}}(- \otimes H\mathbb{Z}) : \mathsf{Sp}_{\mathsf{tame}}^{\geq r} \simeq (\mathsf{Mod}_{H\mathbb{Z}}^{\geq r})_{\mathsf{tame}}.$$

Apply the discussion above, we obtain the following lemma.

Lemma 3.7.13. *There is an equivalence of ∞-categories*

$$\mathsf{Alg}_{\mathbf{Lie}}(\mathsf{Sp}_{\mathsf{tame}}^{\geq r}) \simeq \mathsf{Alg}_{\mathbf{Lie}}\left((\mathsf{Mod}_{H\mathbb{Z}}^{\geq r})_{\mathsf{tame}}\right).$$

By an argument dual to the proof of Corollary 3.7.7, we obtain the analogous result for tame commutative coalgebras in $\mathsf{Sp}_{\mathsf{tame}}^{\geq r}$.

Corollary 3.7.14. *The ∞-category $\mathsf{coCAlg}(\mathsf{Sp}_{\mathsf{tame}}^{\geq r})$ can be identified with the full subcategory of $\mathsf{coCAlg}^{\mathsf{dp,nil}}(\mathsf{Sp})^{\geq r}$ spanned by commutative divided power, conilpotent coalgebras whose underlying spectra are tame.*

Remark 3.7.15. The ∞-category $\mathsf{coCAlg}(\mathsf{Sp}_{\mathsf{tame}}^{\geq r})$ of tame commutative coalgebras can be identified with the full subcategory of $\mathsf{coCAlg}^{\mathsf{dp,nil}}(\mathsf{Mod}_{H\mathbb{Z}})^{\geq r}$ spanned by commutative divided power, conilpotent coalgebras whose underlying $H\mathbb{Z}$-module spectra are tame.

To conclude this section, we discuss Koszul duality between Lie algebras and divided power, conilpotent commutative coalgebras. That is, we consider the functor

$$\mathsf{indec}_{\mathcal{O}} : \mathsf{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathsf{coAlg}_{\mathsf{Bar}(\mathcal{O})}(\mathcal{C})$$

for $\mathcal{O}$ the Lie operad. Classically, if $\mathcal{C}$ is the ∞-category $\mathsf{Sp}_{\mathbb{Q}}$ of rational spectra, or equivalently, the ∞-category of rational chain complexes, then the functor

$$\mathsf{cot}_{\mathbb{L}} : \mathsf{Alg}_{\mathbb{L}}(\mathsf{Sp}_{\mathbb{Q}}) \to \mathsf{Sp}_{\mathbb{Q}}$$

computes the Chevalley-Eilenberg cohomology of a dg Lie algebra over $\mathbb{Q}$.

Definition 3.7.16. We define two versions of the *Chevalley-Eilenberg* functor on a stable ∞-category Sp:

- $\mathsf{CE} : \mathsf{Alg}_{\mathbf{Lie}}(\mathsf{Sp}) \xrightarrow{\Sigma'} \mathsf{Alg}_{\mathbb{L}}(\mathsf{Sp}) \xrightarrow{\mathsf{cot}_{\mathbb{L}}} \mathsf{Sp}$ where Σ' is the equivalence in Lemma 3.4.5;

- $\widetilde{\mathrm{CE}} : \mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}) \xrightarrow{\Sigma'} \mathrm{Alg}_{\mathbb{L}}(\mathrm{Sp}) \xrightarrow{\mathrm{indec}_{\mathbb{L}}} \mathrm{coAlg}^{\mathrm{dp,nil}}_{\mathrm{Com}}(\mathrm{Sp})$.

For later application, we record the following theorem from [CH19].

Theorem 3.7.17. *[CH19] Let R be an E_∞-ring spectrum and $\mathcal{O}$ an operad in the ∞-category Mod_R of R-module spectra with $\mathcal{O}(1) = R$. Suppose R and all $\mathcal{O}(n)$ are connective. Then the functor*

$$\mathrm{indec}_{\mathcal{O}} : \mathrm{Alg}_{\mathcal{O}}(\mathrm{Mod}_R)^{\geq 1} \to \mathrm{coAlg}^{nil,dp}_{\mathrm{Bar}(\mathcal{O})}(\mathrm{Mod}_R)^{\geq 1}$$

is an equivalence.

Taking $R = H\mathbb{Z}$ and let Lie denote the Lie operad in $\mathrm{Mod}_{H\mathbb{Z}}$, we obtain the following corollary.

Corollary 3.7.18. *For $r \geq 1$, the Chevalley-Eilenberg functor*

$$\widetilde{\mathrm{CE}} : \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{H\mathbb{Z}})^{\geq r} \to \mathrm{coAlg}^{nil,dp}_{\mathrm{Com}}(\mathrm{Mod}_{H\mathbb{Z}})^{\geq r}$$

is an equivalence of ∞-categories.

We now define the Chevalley-Eilenberg functor for tame spectral Lie algebras. First we need the following lemma regarding the bar construction of the monad $(L_{\mathrm{tame}})_! F_{\mathcal{O}}$.

Lemma 3.7.19. *Let $\mathcal{O}$ be a connected ∞-operad in Sp and let $F_{\mathcal{O}}$ be the induced monad on $\mathrm{Sp}^{\geq r}$, then*

$$\mathrm{Bar}(L_{\mathrm{tame}} F_{\mathcal{O}}) \simeq L_{\mathrm{tame}} \mathrm{Bar}(F_{\mathcal{O}}).$$

Proof. Unravelling the definition of bar construction, we need to show there is an equivalence

$$|\mathrm{Bar}(\mathrm{id}, L_{\mathrm{tame}}\mathcal{O}, \mathrm{id})_\bullet| \simeq L_{\mathrm{tame}} |\mathrm{Bar}(\mathrm{id}, \mathcal{O}, \mathrm{id})_\bullet|.$$

Since L_{tame} preserves colimits, the proof is reduced to checking that there is a tame equivalence

$$\mathrm{Bar}(\mathrm{id}, \mathcal{O}, \mathrm{id})_\bullet \to \mathrm{Bar}(\mathrm{id}, L_{\mathrm{tame}}\mathcal{O}, \mathrm{id})_\bullet \tag{3.13}$$

of simplicial objects in $\mathrm{Monad}(\mathrm{Sp}^{\geq r})$. We claim that there is an equivalence

$$L_{\mathrm{tame}}(\underbrace{\mathcal{O} \circ \cdots \circ \mathcal{O}}_{n\text{-fold}}) \simeq \underbrace{L_{\mathrm{tame}}\mathcal{O} \circ \cdots \circ L_{\mathrm{tame}}\mathcal{O}}_{n\text{-fold}}$$

for each $n \geq 1$. It suffices to check this for two-fold composition, which can be done directly on the composition of the associated Schur functors

$$F_{\mathcal{O}}(X) := \coprod_{n \geq 1} (\mathcal{O}(n) \otimes X^{\otimes n})_{h\Sigma_n}.$$

The desired equivalence follows from the fact that L_{tame} is symmetric monoidal and preserves colimits. Therefore, the map (3.13) is a point-wise tame equivalence, which is a tame equivalence after taking geometric realization. □

As a consequence, taking $\mathcal{O} = \mathbf{Lie}$ allows us to identify the bar construction of the monad $L_{\mathrm{tame}}F_{\mathbf{Lie}}$ as the comonad $L_{\mathrm{tame}}F_{\mathrm{Com}}$. Explicitly, by Proposition 3.6.5, the underlying endofunctor of the comonad $L_{\mathrm{tame}}F_{\mathrm{Com}}$ on a tame spectrum X is given by

$$X \mapsto \coprod_{n \geq 1} L_{\mathrm{tame}}(X^{\otimes n})_{h\Sigma_n}.$$

Therefore, we obtain a functor

$$L_{\mathrm{tame}}\,\mathrm{indec}_{\mathbb{L}} : \mathrm{Alg}_{\mathbb{L}}(\mathrm{Sp}^{\geq r+1}_{\mathrm{tame}}) \to \mathrm{coAlg}^{\mathrm{dp,nil}}_{\mathrm{Com}}(\mathrm{Sp}^{\geq r+1}_{\mathrm{tame}}).$$

Note that there is an equivalence of ∞-categories between r-tame spectra and $(r+1)$-tame spectra by Lemma A.1.13, hence we obtain a pullback diagram

$$\begin{tikzcd}
\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \arrow[r, "\Sigma'", "\simeq"'] \arrow[d] & \mathrm{Alg}_{\mathbb{L}}(\mathrm{Sp}^{\geq r+1}_{\mathrm{tame}}) \arrow[d] \\
\mathrm{Sp}^{\geq r}_{\mathrm{tame}} \arrow[r, "\Sigma", "\simeq"'] & \mathrm{Sp}^{\geq r+1}_{\mathrm{tame}}
\end{tikzcd}$$

as in Lemma 3.4.5.

Definition 3.7.20. We define the Chevalley-Eilenberg functor for tame spectra to be the following composite

$$\widetilde{\mathrm{CE}}_{\mathrm{tame}} : \mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \xrightarrow[\simeq]{\Sigma'} \mathrm{Alg}_{\mathbb{L}}(\mathrm{Sp}^{\geq r+1}_{\mathrm{tame}}) \xrightarrow{L_{\mathrm{tame}}\,\mathrm{indec}_{\mathbb{L}}} \mathrm{coAlg}^{\mathrm{dp,nil}}_{\mathrm{Com}}(\mathrm{Sp}^{\geq r+1}_{\mathrm{tame}}).$$

Definition 3.7.21. The functor $\widetilde{\mathsf{CE}}_{\mathsf{tame}}$ admits a right adjoint by the adjoint functor theorem, which we will denote by $\widetilde{\mathsf{prim}}$.

Remark 3.7.22. The underlying spectrum of $\widetilde{\mathsf{CE}}_{\mathsf{tame}}(L)$ can be computed as the geometric realization of the bar construction

$$|B(\mathrm{id}_{\mathsf{Sp}^{\geq r+1}_{\mathsf{tame}}}, L_{\mathsf{tame}}F_{\mathbb{L}} \circ \Sigma', \mathrm{id}_{\mathsf{Sp}^{\geq r+1}_{\mathsf{tame}}})|$$

which is equivalent to $L_{\mathsf{tame}}\widetilde{\mathsf{CE}}(L)$ as L_{tame} is colimit-preserving and symmetric monoidal.

Lemma 3.7.23. *There are equivalences of functors*

$$\widetilde{\mathsf{CE}}_{\mathsf{tame}} \circ \mathrm{triv}_{\mathbf{Lie}} \simeq \mathrm{Sym}_{\mathsf{tame}} \circ \Sigma$$

and

$$\widetilde{\mathsf{prim}} \circ \mathrm{Sym}_{\mathsf{tame}} \simeq \mathrm{triv}_{\mathbf{Lie}} \circ \Omega.$$

Proof. The proof is analogous to that of Lemma 3.3.11, plus the equivalences $\mathrm{triv}_{\mathbb{L}} \circ \Sigma \simeq \Sigma' \circ \mathrm{triv}_{\mathbf{Lie}}$ and $\Omega' \circ \mathrm{triv}_{\mathbb{L}} \simeq \mathrm{triv}_{\mathbf{Lie}} \circ \Omega$.

□

Proposition 3.6.7 allows us to identify the codomain of the Chevalley-Eilenberg functor $\widetilde{\mathsf{CE}}_{\mathsf{tame}}$ on tame spectra as $\mathsf{coCAlg}(\mathsf{Sp}^{\geq r}_{\mathsf{tame}})$, whose categorical product is given by the tensor product $\hat{\otimes}$ in $\mathsf{Sp}^{\geq}_{\mathsf{tame}}$. We now show that the Chevalley-Eilenberg functor $\widetilde{\mathsf{CE}}$ preserves finite products, hence it induces a functor on the categories of group objects.

Lemma 3.7.24. *The functor $\widetilde{\mathsf{CE}}_{\mathsf{tame}}$ preserves finite products.*

Proof. Since Σ' is an equivalence, it suffices to show the natural morphism

$$L_{\mathsf{tame}}\,\mathrm{indec}_{\mathbb{L}}(L \times L') \to L_{\mathsf{tame}}\,\mathrm{indec}_{\mathbb{L}}(L) \hat{\otimes} L_{\mathsf{tame}}\,\mathrm{indec}_{\mathbb{L}}(L')$$

is an equivalence for $L, L' \in \mathsf{Alg}_{\mathbb{L}}(\mathsf{Sp}^{\geq r}_{\mathsf{tame}})$. By Remark 3.7.22, the proof reduces to showing that the natural map

$$c : \mathrm{indec}_{\mathbb{L}}(L \times L') \to \mathrm{indec}_{\mathbb{L}}(L) \otimes \mathrm{indec}_{\mathbb{L}}(L')$$

is a tame equivalence when we regard L, L' as objects in $\mathsf{Alg}_{\mathbb{L}}(\mathsf{Sp})$.

For a spectral Lie algebra X, we let X_* denote its canonical filtration from Theorem A.2.7. Note that $\mathrm{indec}_{\mathbb{L}}$ induces a functor

$$(\mathrm{indec})_* : \mathrm{Fil}^+(\mathrm{Alg}_{\mathbb{L}}(\mathrm{Sp})) \to \mathrm{Fil}^+(\mathrm{coCAlg}^{\mathrm{dp,nil}}(\mathrm{Sp}))$$

by postcomposing with $\mathrm{indec}_{\mathbb{L}}$. Since $X \simeq \mathrm{colim}\, X_*$ and $\mathrm{indec}_{\mathbb{L}}$ preserves colimits, it suffices to show c induces an equivalence on the canonical filtrations, i.e.,

$$(\mathrm{indec}_{\mathbb{L}})_*(L_* \times L'_*) \to (\mathrm{indec}_{\mathbb{L}})_*(L_*) \otimes (\mathrm{indec}_{\mathbb{L}})_*(L'_*).$$

Note also that the associated graded functor Gr is conservative by Lemma A.2.4, hence the proof further reduces to proving that the induced map on associated gradeds

$$\nu : \mathrm{Gr}(\mathrm{indec}_{\mathbb{L}})_*((L \times L')_*) \to \mathrm{Gr}((\mathrm{indec}_{\mathbb{L}})_*(L_*) \otimes (\mathrm{indec}_{\mathbb{L}})_*(L'_*))$$

is an equivalence.

Since Gr is symmetric monoidal by Remark A.2.6, and it commutes with $(\mathrm{indec}_{\mathbb{L}})_*$ (since Gr is defined using colimits), there are equivalences

$$\mathrm{Gr}(\mathrm{indec}_{\mathbb{L}})_*((L \times L')_*) \simeq \mathrm{indec}_{\mathbb{L}}^{\mathrm{Gr}}(\mathrm{Gr}(L \times L')_*)$$

and

$$\mathrm{Gr}((\mathrm{indec}_{\mathbb{L}})_*(L_*) \otimes (\mathrm{indec}_{\mathbb{L}})_*(L'_*)) \simeq \mathrm{indec}_{\mathbb{L}}^{\mathrm{Gr}}(\mathrm{Gr}(L_*)) \otimes \mathrm{indec}_{\mathbb{L}}^{\mathrm{Gr}}(\mathrm{Gr}(L'_*))$$

where $\mathrm{indec}_{\mathbb{L}}^{\mathrm{Gr}}$ denotes the induced functor on associated graded. Hence we can identify ν as the map

$$\mathrm{indec}_{\mathbb{L}}^{\mathrm{Gr}}(\mathrm{Gr}(L \times L')_*) \to \mathrm{indec}_{\mathbb{L}}^{\mathrm{Gr}}(\mathrm{Gr}(L_*)) \otimes \mathrm{indec}_{\mathbb{L}}^{\mathrm{Gr}}(\mathrm{Gr}(L'_*)).$$

By theorem A.2.7, the associated graded of the canonical filtration of a spectral Lie algebra X has the form

$$\mathrm{free}_{\mathbb{L}}((B\mathbb{L}(n) \otimes X^{\otimes n})_{h\Sigma_n} \simeq \mathrm{free}_{\mathbb{L}}(X^{\otimes n})_{h\Sigma_n}$$

where the latter equivalence comes from the fact that $B\mathbb{L} \simeq \mathrm{Com}$. Hence, there are equivalences

$$\begin{aligned}\mathsf{oblv}_{\mathbb{L}}\,\mathsf{indec}_{\mathbb{L}}\,\mathsf{free}_{\mathbb{L}}(X^{\otimes n})_{h\Sigma_n} &\simeq \mathsf{cot}_{\mathbb{L}}\circ\mathsf{free}_{\mathbb{L}}(X^{\otimes n})_{h\Sigma_n}\\ &\simeq (X^{\otimes n})_{h\Sigma_n}\\ &= \mathsf{Sym}^n(X).\end{aligned}$$

Therefore, the map $\mathsf{oblv}(\nu)$ is equivalent to the natural map

$$\mathsf{Sym}(L\times L')\to\mathsf{Sym}(L)\otimes\mathsf{Sym}(L'),$$

which is a tame equivalence since the symmetric coalgebra functor Sym is the cofree coalgebra comonad which preserves products and the tensor product in $\mathsf{Sp}^{\geq r}_{\mathsf{tame}}$ is the product in $\mathsf{coCAlg}^{\mathsf{dp,nil}}(\mathsf{Sp}^{\geq r}_{\mathsf{tame}})\simeq\mathsf{coCAlg}(\mathsf{Sp}^{\geq r}_{\mathsf{tame}})$.

□

Chapter 4

A Hopf Algebra Model for Tame spaces

In this last chapter, we prove Theorem B and Theorem C stated in the introduction, concerning a new Hopf algebra model for r-tame spaces. Note that every pointed space X is a commutative coalgebra in $\mathcal{S}_*$ via the diagonal $X \to X \times X \to X \wedge X$. Since the functors Σ^∞ and L_{tame} are symmetric monoidal, the functor

$$\Sigma^\infty_{\mathsf{tame}} : \mathcal{S}^{\geq r}_{\mathsf{tame}} \to \mathsf{Sp}^{\geq r}_{\mathsf{tame}}$$

factors through the ∞-category $\mathsf{coCAlg}(\mathsf{Sp}^{\geq r}_{\mathsf{tame}})$ of commutative coalgebras in $\mathsf{Sp}^{\geq r}_{\mathsf{tame}}$, and we denote the resulting functor as

$$C_{\mathsf{tame}} : \mathcal{S}^{\geq r}_{\mathsf{tame}} \to \mathsf{coCAlg}(\mathsf{Sp}^{\geq r}_{\mathsf{tame}}).$$

The funcor C_{tame} is colimit-preserving, hence it admits a right adjoint R by the adjoint functor theorem [Lur09, Corollary 5.5.2.9.]. We summarize the situation in the following diagram

$$\begin{tikzcd}
\mathcal{S}^{\geq r}_{\mathsf{tame}} \arrow[rr, shift left, "\Sigma^\infty_{\mathsf{tame}}"] \arrow[dr, shift left, "C_{\mathsf{tame}}"] & & \mathsf{Sp}^{\geq r}_{\mathsf{tame}} \arrow[ll, shift left, "\Omega^\infty"] \arrow[dl, shift left, "\mathsf{Sym}_{\mathsf{tame}}"] \\
& \mathsf{coCAlg}(\mathsf{Sp}^{\geq r}_{\mathsf{tame}}) \arrow[ul, shift left, "R"] \arrow[ur, shift left, "\mathsf{oblv}"] &
\end{tikzcd},$$

where left adjoints sit above right adjoint.

Classically, cocommutative Hopf algebras can be identified as group objects in the Cartesian category of cocommutative counital coalgebras [MM65]. This motivates us to give the following definition of the ∞-category of Hopf algebras in tame spectra.

Definition 4.0.1. We define the ∞-category of *Hopf algebras in r-tame spectra* to be the category of group objects in $\mathrm{coCAlg}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}})$, i.e.,

$$\mathrm{HopfAlg}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) := \mathrm{Grp}(\mathrm{coCAlg}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}})).$$

Our first goal is to establish a Hopf algebra model for tame spaces. The following is the precise statement of Theorem B.

Theorem 4.0.2. *The composite*

$$\mathcal{S}^{\geq r}_{\mathrm{tame}} \xrightarrow{\Omega} \mathrm{Grp}(\mathcal{S}^{\geq r-1}_{\mathrm{tame}}) \xrightarrow{\mathrm{Grp}(C_{\mathrm{tame}})} \mathrm{HopfAlg}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}})$$

is an equivalence of ∞-categories.

We now sketch our strategy for the proof of Theorem 4.0.2. We originally thought that there should be a coalgebra model for tame spaces, i.e., we suspected that the functor

$$C_{\mathrm{tame}} : \mathcal{S}^{\geq r}_{tame} \to \mathrm{coCAlg}(\mathrm{Sp}_{tame})$$

was an equivalence, as this is indeed the case in Quillen's rational homotopy theory and Mandell's p-adic homotopy thoery. However, this functor fails to be fully faithful; indeed, if C_{tame} is fully faithful, then the composition

$$\mathrm{oblv}_{\mathrm{Com}} \circ C_{\mathrm{tame}} \simeq \Sigma^{\infty}_{\mathrm{tame}}$$

would be conservative. Since the homotopy groups of $\Sigma^{\infty}_{\mathrm{tame}} X$ are computed as the homology groups of X with coefficients in the tame ring system by Corollary 2.3.16, the statement that $\Sigma^{\infty}_{\mathrm{tame}}$ is conservative is equivalent to a Whitehead theorem for tame spaces. That is, a map $f : X \to Y$ of tame spaces is an equivalence if and only if the induced map on homology groups with coefficients in the tame ring system is an isomorphism

$$H_{r+j} f \otimes R_j : H_{r+j} X \otimes R_j \to H_{r+j} Y \otimes R_j$$

for all $j \geq 0$.

However, we do not have a Whitehead theorem for tame spaces, as we need more conditions on the cokernel $\mathrm{coker}(h_{r+j,X})$ of the Hurewicz map

$$h_{r+j,X} : \pi_{r+j}X \otimes R_j \to H_{r+j}X \otimes R_j.$$

Proposition 4.0.3. *[FL88, Proposition 1.1] If $f : X \to Y$ is a map between r-connective spaces, then the following are equivalent:*

1. *f is a tame equivalence.*
2. *For all $j \geq 0$,*

$$H_{r+j}f \otimes R_j : H_{r+j}X \otimes R_j \to H_{r+j}Y \otimes R_j$$

is an isomorphism and

$$\mathrm{coker}(h_{r+j,X}) \to \mathrm{coker}(h_{r+j,Y})$$

is surjective.

Taking a step back, we prove that C_{tame} is fully faithful when restricted to the full subcategory of tame Eilenberg-Maclane spaces. We then show that the functors Ω and $\mathrm{Grp}(C_{\mathrm{tame}})$ are both equivalences of ∞-categories. The crux of the latter is that the loop of a r-tame space splits into a product of tame Eilenberg-Maclane spaces, and therefore we can conclude that $\mathrm{Grp}(C_{\mathrm{tame}})$ is fully faithful.

The second goal of this final chapter is to connect the Hopf algebra model for tame spaces with tame spectral Lie algebras defined in §3.7.

Since Lemma 3.7.24 ensures the functor $\widetilde{\mathrm{CE}}_{\mathrm{tame}}$ preserves group objects, there is a functor on the categories of groups

$$\mathrm{Grp}(\widetilde{\mathrm{CE}}_{\mathrm{tame}}) : \mathrm{Grp}(\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}})) \to \mathrm{Grp}(\mathrm{coCAlg}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}})).$$

Let $\Omega_{\mathbf{Lie}}$ denote the loop functor in $\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}})$. We prove in Proposition A.1.14 that $\Omega_{\mathbf{Lie}}$ factors through the ∞-category of group objects $\mathrm{Grp}(\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}}))$ in tame spectral Lie algebras. Moreover, we prove that the factorization

$$\Omega_{\mathbf{Lie}} : \mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \to \mathrm{Grp}(\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}}))$$

is an equivalence of ∞-categories.

In [Knu18], it is shown that the universal enveloping algebra $U(L)$ of a spectral Lie algebra defined there satisfies a Poincare-Birkhoff-Witt theorem [Knu18, Theorem B]. Motivated by this, we define the universal enveloping algebra functor as follows.

Definition 4.0.4. The *universal enveloping algebra functor* is defined as the following composite

$$U : \mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \xrightarrow[\simeq]{\Omega_{\mathbf{Lie}}} \mathrm{Grp}(\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}})) \xrightarrow{\mathrm{Grp}(\widetilde{\mathrm{CE}}_{\mathrm{tame}})} \mathrm{HopfAlg}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}).$$

The following is the precise statement of Theorem C.

Theorem 4.0.5. *The universal enveloping algebra functor*

$$U : \mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \to \mathrm{HopfAlg}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}})$$

is an equivalence of ∞-categories.

Combining Theorem 4.0.5 and Theorem 4.0.2, we establish an ∞-categorical version of Dwyer's Lie algebra model for tame spaces in [Dwy79].

Theorem 4.0.6. *[Dwy79] There is an equivalence of ∞-categories*

$$\mathcal{S}^{\geq r}_{\mathrm{tame}} \to \mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}}).$$

In §4.1, we prove the functor C_{tame} is fully faithful on the full subcategory of Eilenberg-Maclane spaces by computing the cohomology of Eilenberg-Maclane spaces with coefficients in the tame ring system. In §4.2, we finalize our proof of Theorem 4.0.2. In §4.3, we show that there is an equivalence from the ∞-category of tame Lie algebras to the ∞-category of tame Hopf algebras.

4.1 Fully faithfulness on Eilenberg-MacLane spaces

Our goal in this section is to prove the following proposition.

Proposition 4.1.1. *The functor*

$$C_{\mathrm{tame}} : \mathcal{S}^{\geq r}_{\mathrm{tame}} \to \mathrm{coCAlg}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}})$$

is fully faithful on the full subcategory spanned by Eilenberg-MacLane spaces.

To simplify notation, we let $n = r + k$ in this section, where k is a non-negative integer. Fix an R_k-module V. We first show that $C_{\mathrm{tame}}K(V, n)$ is equivalent to the cofree commutative coalgebra generated by $\Sigma^n HV$. By Proposition 3.6.7, the cofree commutative coalgebra generated by $\Sigma^n HV$ is given by the symmetric coalgebra $\mathrm{Sym}_{\mathrm{tame}}\, \Sigma^n HV$ in $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$. There is a canonical map

$$\Sigma^{\infty}_{\mathrm{tame}} K(V, n) \to \Sigma^n HV \tag{4.1}$$

given by n-truncation. By the forgetful-cofree adjunction, the map (4.1) corresponds to a map

$$C_{\mathrm{tame}}K(V, n) \to \mathrm{Sym}_{\mathrm{tame}}\, \Sigma^n HV. \tag{4.2}$$

Since the forgetful functor

$$\mathrm{oblv} : \mathrm{coCAlg}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \to \mathrm{Sp}^{\geq r}_{\mathrm{tame}}\,.$$

is conservative, it suffices to check that the map obtained by applying forgetful functor to (4.2)

$$\gamma : \Sigma^{\infty}_{\mathrm{tame}} K(V, n) \to \mathrm{Sym}_{\mathrm{tame}}\, \Sigma^n HV \tag{4.3}$$

is an equivalence. Here we abuse notation by writing $\mathrm{Sym}_{\mathrm{tame}}\, \Sigma^n HV$ also for the underlying spectrum of the cofree coalgebra generated by $\Sigma^n HV$. Since both $\Sigma^{\infty}_{\mathrm{tame}}\Omega^{\infty}$ and $\mathrm{Sym}_{\mathrm{tame}}$ preserves filtered colimits, we can reduce to the case of V being a finitely generated R_k-module.

Let $M(V, n)$ be the Moore space of type (V, n). By Remark 2.3.18, $L_{\mathrm{tame}}\Sigma^{\infty}$ sends $M(V, n)$ to a tame Eilenberg-Maclane spectrum, i.e.,

$$L_{\mathrm{tame}}\Sigma^{\infty} M(V, n) \simeq \Sigma^n HV.$$

Let X be a pointed connected space. The underlying space of the free E_∞-space generated by X is given by the symmetric algebra (see Example 3.1.15)

$$\mathrm{Sym}(X) = \bigvee_{m \geq 1} (X^{\wedge m})_{h\Sigma_m};$$

If X is tame, then the free tame E_∞-space generated by X is

$$\mathrm{Sym}_{\mathrm{tame}}(X) = \bigvee_{m \geq 1} L_{\mathrm{tame}}(X^{\wedge m})_{h\Sigma_m};$$

We claim the functor $\mathrm{Sym}_{\mathrm{tame}}$ on tame spaces is analogous to the infinite symmetric product construction, i.e., it sends the tame localization of Moore spaces to Eilenberg-MacLane spaces.

Lemma 4.1.2. *Let V be a finitely generated R_k-module. Then there is an equivalence of tame spaces*

$$\mathrm{Sym}_{\mathrm{tame}}\, L_{\mathrm{tame}} M(V, n) \simeq K(V, n).$$

Proof. By May's theorem [May72, 6.3], there is an equivalence

$$\mathrm{Sym}\, X \simeq \Omega^\infty \Sigma^\infty X$$

for any connected space X. Hence, the space $\mathrm{Sym}_{\mathrm{tame}}\, L_{\mathrm{tame}} M(V, n)$ is equivalent to the tame localization of $\Omega^\infty \Sigma^\infty L_{\mathrm{tame}} M(V, n)$. Since both Σ^∞ and Ω^∞ preserve tame equivalences,

$$L_{\mathrm{tame}} \Omega^\infty \Sigma^\infty L_{\mathrm{tame}} M(V, n) \simeq L_{\mathrm{tame}} \Omega^\infty \Sigma^\infty M(V, n)$$

the homotopy groups of which are given by

$$H_* M(V, n) \otimes R_{*-r} = \begin{cases} V & \text{for } * = n; \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

Since both the functors $L_{\mathrm{tame}} : \mathrm{Sp}^{\geq r} \to \mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ and $\Sigma^\infty_{\mathrm{tame}} : \mathcal{S}^{\geq r}_{\mathrm{tame}} \to \mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ are colimit-preserving and symmetric monoidal, we have the following lemma.

Lemma 4.1.3. *There is a commutative diagram*

$$\begin{tikzcd} \mathcal{S}^{\geq r}_{\mathrm{tame}} \arrow[r, "\mathrm{Sym}_{\mathrm{tame}}"] \arrow[d, "\Sigma^\infty_{\mathrm{tame}}"'] & \mathcal{S}^{\geq r}_{\mathrm{tame}} \arrow[d, "\Sigma^\infty_{\mathrm{tame}}"] \\ \mathrm{Sp}^{\geq r}_{\mathrm{tame}} \arrow[r, "\mathrm{Sym}_{\mathrm{tame}}"'] & \mathrm{Sp}^{\geq r}_{\mathrm{tame}} . \end{tikzcd}$$

Proposition 4.1.4. *The map*

$$\gamma : \Sigma^\infty_{\mathrm{tame}} K(V, n) \to \mathrm{Sym}_{\mathrm{tame}}\, \Sigma^n HV$$

of (4.3) is an equivalence.

Proof. Combining Lemma 4.1.2 and Lemma 4.1.3, we have equivalences

$$\begin{aligned}\Sigma^\infty_{\mathrm{tame}} K(V,n) &\simeq \Sigma^\infty_{\mathrm{tame}} \operatorname{Sym}_{\mathrm{tame}} L_{\mathrm{tame}} M(V,n)\\ &\simeq \operatorname{Sym}_{\mathrm{tame}} \Sigma^\infty_{\mathrm{tame}} L_{\mathrm{tame}} M(V,n)\\ &\simeq \operatorname{Sym}_{\mathrm{tame}} L_{\mathrm{tame}} \Sigma^\infty M(V,n)\\ &\simeq \operatorname{Sym}_{\mathrm{tame}} \Sigma^n HV,\end{aligned}$$

where the last equivalence follows from Remark 2.3.18. Hence γ can be identified as the unique map that lifts $\Sigma^\infty_{\mathrm{tame}} K(V,n) \to \Sigma^n HV$ through the projection

$$\operatorname{Sym}_{\mathrm{tame}} \Sigma^n HV \to \Sigma^n HV,$$

which is an equivalence. □

Now we can prove Proposition 4.1.1. We denote the right adjoint of C_{tame} by R as above.

Proof of Proposition 4.1.1: Using Proposition 4.1.4, we see that

$$\begin{aligned}RC_{\mathrm{tame}} K(V,n) &\simeq R \operatorname{Sym}_{\mathrm{tame}} \Sigma^n HV\\ &\simeq \Omega^\infty \Sigma^n HV\\ &\simeq K(V,n),\end{aligned}$$

so C_{tame} is fully faithful on the full subcategory of tame Eilenberg-Maclane spaces.

□

4.2 A Hopf algebra model for tame spaces

First we establish some preliminary results.

Proposition 4.2.1. *For $r \geq 4$, the loop functor $\Omega : \mathcal{S}^{\geq r}_{\mathrm{tame}} \to \operatorname{Grp}(\mathcal{S}^{\geq r-1}_{\mathrm{tame}})$ is an equivalence.*

Proof. We have a commutative diagram

$$\begin{tikzcd} \mathcal{S}^{\geq r}_{\mathrm{tame}} \arrow[r, "\Omega"] \arrow[d, hook] & \operatorname{Grp}(\mathcal{S}^{\geq r-1}_{\mathrm{tame}}) \arrow[d, hook] \\ \mathcal{S}^{\geq r}_{*} \arrow[r, "\Omega", "\simeq"'] & \operatorname{Grp}(\mathcal{S}^{\geq r-1}_{*}) \end{tikzcd}$$

where the left vertical functor is fully faithful as is right vertical functor by Proposition A.1.5. The bottom arrow is an equivalence by May's recognition theorem (cf. see [Lur17, Theorem 5.2.6.10.]). Hence the top loop functor $\mathcal{S}_{\mathrm{tame}}^{\geq r} \to \mathrm{Grp}(\mathcal{S}_{\mathrm{tame}}^{\geq r-1})$ is fully faithful. For the essential surjectivity, we know that any $X \in \mathrm{Grp}(\mathcal{S}_{\mathrm{tame}}^{\geq r-1})$ is equivalent to ΩY for some $Y \in \mathcal{S}_*^{\geq r}$, while Y has to be r-tame since X is $(r-1)$-tame.

□

Lemma 4.2.2. *Let $\mathcal{C}$ be a Cartesian symmetric monoidal ∞-category and let $L : \mathcal{C} \to \mathcal{C}'$ be a localization functor that preserves finite products. Then both forgetful functors*

$$\mathrm{Grp}(\mathcal{C}) \to \mathcal{C}$$

and

$$\mathrm{Grp}(\mathcal{C}') \to \mathcal{C}'$$

are conservative.

Proof. Since the category of groups is a full subcategory of monoids, it suffices to prove the forgetful functor $\mathrm{Mon}(\mathcal{C}) \to \mathcal{C}$ is conservative. The statements then follows from [Lur17, Proposition 2.4.2.5] and [Lur17, Lemma 3.2.2.6.].

□

Applying Lemma 4.2.2 to the case of tame spaces, we have the following corollary.

Corollary 4.2.3. *The forgetful functor*

$$\mathrm{Grp}(\mathcal{S}_{\mathrm{tame}}^{\geq r-1}) \xrightarrow{\mathrm{oblv}_{\mathrm{Grp}}} \mathcal{S}_{\mathrm{tame}}^{\geq r-1}$$

is conservative.

Since the categorical product of $\mathrm{coCAlg}(\mathrm{Sp}_{\mathrm{tame}}^{\geq r-1})$ is given by the smash product in $\mathrm{Sp}_{\mathrm{tame}}^{\geq r-1}$, the smash product equips $\mathrm{coCAlg}(\mathrm{Sp}_{\mathrm{tame}}^{\geq r-1})$ with the structure of a Cartesian symmetric monoidal ∞-category. Moreover, the functor

$$C_{tame} : \mathcal{S}_{\mathrm{tame}}^{\geq r-1} \to \mathrm{coCAlg}(\mathrm{Sp}_{\mathrm{tame}}^{\geq r-1})$$

sends products to smash products, and therefore induces a functor

$$G_{tame} : \mathrm{Grp}(\mathcal{S}_{\mathrm{tame}}^{\geq r-1}) \to \mathrm{Grp}(\mathrm{coCAlg}(\mathrm{Sp}_{\mathrm{tame}}^{\geq r-1})) \tag{4.4}$$

where the latter ∞-category is the ∞-category of tame Hopf algebras (Definition 4.0.1).

Applying Lemma 4.2.2 again, we have the following result.

Corollary 4.2.4. *The forgetful functor*

$$\mathsf{HopfAlg}(\mathsf{Sp}^{\geq r-1}_{\mathsf{tame}}) \xrightarrow{\mathsf{oblv}_{\mathsf{Grp}}} \mathsf{coCAlg}(\mathsf{Sp}^{\geq r-1}_{\mathsf{tame}})$$

is conservative.

Proof. This follows from the fact that $\mathsf{coCAlg}(\mathsf{Sp}^{\geq r-1}_{(r-1)\text{-}\mathsf{tame}})$ is a Cartesian symmetric monoidal ∞-category and Lemma 4.2.2.

□

Recall that in rational homotopy theory, any rational H-space splits as a product of Eilenberg-Maclane spaces. Our proof of the Hopf algebra model of tame spaces will be based on an analogous splitting result for tame H-spaces. The following proposition was already proved in [ST91, Proposition 1.7] using Dwyer's Lie algebra model for tame spaces.

Proposition 4.2.5. *Let X be an r-tame E_1-space. Then X is equivalent to a product of Eilenberg-Maclane spaces, that is,*

$$X \simeq \prod_i K(\pi_i X, i).$$

Proof. We first assume X is of finite type, i.e., all its homotopy groups are finitely generated abelian groups. If X is rational, then the result follows from rational homotopy theory. Otherwise, we can assume the homotopy groups of X is p-torsion for some prime p.

Observe that if X is both tame and p-torsion, then X is $(r+2p-4)$-truncated; since X is p-torsion and $\pi_n X$ is uniquely p-divisible for $n \geq r+2p-3$. Note also that if $p=2$, then X is an Eilenberg-MacLane space $K(\pi_r X, r)$ (as 2 is inverted at degree $r+1$), hence we can assume p is an odd prime.

We proceed by induction on the Postnikov tower of X. The base case is obvious. For the inductive step, consider the principal fiber sequence

$$K(\pi_{n+1}X, n+1) \to \tau_{\leq n+1}X \to \tau_{\leq n}X \xrightarrow{k_n} K(\pi_{n+1}X, n+2).$$

We want to show that the k-invariant $[k_n] \in H^{n+2}(\tau_{\leq n}X; \pi_{n+1}X)$ vanishes. Note that k_n represents a primitive element in the p-local Hopf algebra $H^*(\tau_{\leq n}X; \pi_{n+1}X)$ (cf. [Kah63, Theorem 3.2]). We claim $[k_n]$ is also indecomposable. A primitive element in a p-local commutative Hopf algebra is decomposable if and only if it's a p-th power [MM65, Proposition 4.21]. However, $[k_n]$ cannot be a p-th power for degree reasons. As $n < r+2p-3$, whence

$$n - rp < r + 2p - 3 - rp < (2-r)(p-1) < 0.$$

By the inductive hypothesis, the cohomology of $\tau_{\leq n}X$ is a product of Eilenberg-Maclane spaces. By the Künneth formula, it suffices to show

$$H^{n+2}(K(\pi_i X, i); \pi_{n+1}X) = 0$$

for $r \leq i \leq n$. Cartan [Car54] shows that if R is a p-local ring, all cohomology classes $[\alpha] \in H^*(K(A,k); R)$ are decomposable in degree $k < * < k+2(p-1)$. Since $n+2 < i+2(p-1)$ for any $r \leq i \leq n$, $[k_n] = 0$ for degree reasons.

Now a general tame E_1-space X can be written as a filtered colimits of tame E_1-spaces of finite type

$$X \simeq \operatorname{colim}_\alpha X_\alpha.$$

Since the n-truncation commutes with colimits, there are equivalences:

$$\begin{aligned}
\tau_{\leq n}X &\simeq \operatorname{colim}_\alpha \tau_{\leq n}X_\alpha \\
&\simeq \operatorname{colim}_\alpha \prod_{i=r}^{n} K(\pi_i X_\alpha, i) \\
&\simeq \prod_{i=r}^{n} \operatorname{colim}_\alpha K(\pi_i X_\alpha, i) \\
&\simeq \prod_{i=r}^{n} K(\operatorname{colim}_\alpha \pi_i X_\alpha, i) \\
&\simeq \prod_{i=r}^{n} K(\pi_i X, i)
\end{aligned}$$

where we also used that filtered colimits commute with finite limits and filtered colimits commutes with π_*. Therefore, the proposition is proved. $\square$

Remark 4.2.6. We learned part of the proof of Proposition 4.2.5 from Soulé [Sou85][Proposition 3], where the author further attributed the idea to L.Smith.

We need the following technical lemma for the proof of Theorem 4.0.2.

Lemma 4.2.7. *The monad $RC_{\text{tame}} : \mathcal{S}^{\geq r}_{\text{tame}} \to \mathcal{S}^{\geq r}_{\text{tame}}$ preserves connectivity.*

Proof. Suppose X is a k-connective tame space. By Corollary A.3.2, $RC_{\text{tame}}X$ is given by the totalization

$$RC_{\text{tame}}X \simeq \operatorname{Tot}(\Omega^\infty\Sigma^\infty_{\text{tame}})^{\bullet+1}X.$$

Let $Q := \Omega^\infty\Sigma^\infty_{\text{tame}}$. Recall that $\operatorname{Tot}(\Omega^\infty\Sigma^\infty_{\text{tame}})^{\bullet+1}X$ can be written as the inverse limit

$$\lim_n \operatorname{Tot}^n Q^{\bullet+1}X,$$

where $\operatorname{Tot}^n Q^{\bullet+1}X$ denotes the limit of the diagram over $Q^{\bullet+1}X|_{\Delta_{\leq n}}$. Let T be a set with n elements and $\mathcal{P}(T)$ be the poset of subsets of T. We define a diagram $\mathcal{F}$ over $\mathcal{P}(T)$ by

$$\mathcal{F}(S) := Q^{|T-S|+1}X.$$

By the appendix of [AK98],

$$\mathrm{fib}(\mathrm{Tot}^n\, Q^{\bullet+1}X \to \mathrm{Tot}^{n-1}\, Q^{\bullet+1}X) \simeq \Omega^{n-1}F_n(X)$$

is nk-connective. Hence, the lemma follows from the Milnor exact sequence

$$0 \to \lim{}^1_n \pi_{m+1}\,\mathrm{Tot}^n\, Q^{\bullet+1}X \to \pi_m\,\mathrm{Tot}\, Q^{\bullet+1}X \to \lim_n \pi_m\,\mathrm{Tot}^n\, Q^{\bullet+1}X \to 0.$$

□

Lemma 4.2.8. *If $F : \mathcal{S}_*^{\geq r} \to \mathcal{S}_*^{\geq r}$ preserves connectivity and finite products, then*

$$F(\prod_n K(A_n, n)) \simeq \prod_n F(K(A_n, n))$$

where $\{A_n\}_{n\leq r}$ is a sequence of abelian groups.

Proof. It suffices to check the canonical map

$$F(\prod_n K(A_n, n)) \to \prod_n F(K(A_n, n))$$

is an equivalence after k-truncation $\tau_{\leq k}$ for every $k \geq r$. Since F preserves connectivity and finite products, there is an equivalence after k-truncation

$$F(\prod_{n=r}^{k} K(A_n, n)) \xrightarrow{\simeq} \prod_{n=r}^{k} F(K(A_n, n)).$$

□

Theorem 4.2.9. *The functor $G_{\mathrm{tame}} : \mathrm{Grp}(\mathcal{S}_{\mathrm{tame}}^{\geq r-1}) \to \mathrm{HopfAlg}(\mathrm{Sp}_{\mathrm{tame}}^{\geq r-1})$ (cf. (4.4)) is an equivalence of ∞-categories.*

Proof. First, we prove the functor G_{tame} is fully faithful. Since both C_{tame} and R preserve finite products, they lift to a pair of adjunction

$$G_{tame} \colon \mathrm{Grp}(\mathcal{S}_{\mathrm{tame}}^{\geq r-1}) \rightleftarrows \mathrm{HopfAlg}(\mathrm{Sp}_{\mathrm{tame}}^{\geq r-1}) : R'. \tag{4.5}$$

We observe that there are commutative diagrams of ∞-categories

$$\begin{tikzcd}
\mathrm{Grp}(\mathcal{S}^{\geq r-1}_{\mathrm{tame}}) \arrow[r, "G_{tame}"] \arrow[d, "\mathrm{oblv}"'] & \mathrm{HopfAlg}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}}) \arrow[d, "\mathrm{oblv}'"] \\
\mathcal{S}^{\geq r-1}_{\mathrm{tame}} \arrow[r, "C_{tame}"] & \mathrm{coCAlg}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Grp}(\mathcal{S}^{\geq r-1}_{\mathrm{tame}}) \arrow[d, "\mathrm{oblv}"'] & \mathrm{HopfAlg}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}}) \arrow[l, "R'"'] \arrow[d, "\mathrm{oblv}'"] \\
\mathcal{S}^{\geq r-1}_{\mathrm{tame}} & \mathrm{coCAlg}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}}) \arrow[l, "R"']
\end{tikzcd}$$

where oblv and oblv$'$ denote the forgetful functors from the category of groups in $\mathcal{S}^{\geq r-1}_{\mathrm{tame}}$ and from $\mathrm{HopfAlg}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}})$ to the underlying categories of coalgebras.

We want to show the unit map $X \to R' G_{\mathrm{tame}} X$ is an equivalence for any $X \in \mathrm{Grp}(\mathcal{S}^{\geq r-1}_{\mathrm{tame}})$. Since the functor oblv is conservative by Lemma 4.2.3, it suffices to show the map $\mathrm{oblv}(X) \to \mathrm{oblv}(R' G_{\mathrm{tame}} X)$ is an equivalence. By the commutativity of the diagrams above,

$$\mathrm{oblv}(R' G_{\mathrm{tame}} X) \simeq R \circ \mathrm{oblv}'(G_{\mathrm{tame}} X) \simeq R C_{\mathrm{tame}}(\mathrm{oblv}(X)).$$

Note that Proposition 4.2.1 implies that any $(r-1)$-tame group X is equivalent to the loop space of a r-tame space Y. Since $X \simeq \Omega Y$ is in particular an E_1-space, it splits into a product of Eilenberg-Maclane spaces by Proposition 4.2.5. Hence,

$$\begin{aligned}
R C_{\mathrm{tame}}(\mathrm{oblv}(X)) &\simeq R C_{\mathrm{tame}}(\prod_i K(\pi_i X, i))) \\
&\simeq \prod_i R C_{\mathrm{tame}}(K(\pi_i X, i)) \\
&\simeq \prod K(\pi_i X, i) \\
&\simeq X
\end{aligned}$$

where the second equivalence follows from Lemma 4.2.8. Hence, we conclude that the functor G_{tame} is fully faithful.

To finish the proof, it suffices to show the right adjoint R' is conservative. Since oblv$'$ is conservative by Corollary 4.2.4, we are reduced to showing that

the functor
$$R : \mathrm{coCAlg}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}}) \to \mathcal{S}^{\geq r-1}_{\mathrm{tame}}$$
is conservative.

Let $f : U \to V$ be a morphism in $\mathrm{coCAlg}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}})$ and suppose that $R(f) : RU \to RV$ is an equivalence. For any coalgebra $X \in \mathrm{coCAlg}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}})$, we claim that the induced map on the mapping space

$$\mathrm{Map}_{\mathrm{coCAlg}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}})}(X, U) \to \mathrm{Map}_{\mathrm{coCAlg}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}})}(X, V)$$

is an equivalence, which implies f is an equivalence. Note that if X lies in the essential image of C_{tame}, then this certainly holds. Hence, it suffices to show that the functor
$$C_{tame} : \mathcal{S}^{\geq r-1}_{\mathrm{tame}} \to \mathrm{coCAlg}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}})$$
is essentially surjective, which we will prove in Lemma 4.2.12.

□

The crux of the essential surjectivity argument for C_{tame} is the following lemma.

Lemma 4.2.10. *$C_{\mathrm{tame}} L_{\mathrm{tame}} S^{r-1}$ is equivalent to the trivial tame coalgebra on $\Sigma^{r-1} H\mathbb{Z}$.*

Proof. Since the coalgebra $C_{\mathrm{tame}} L_{\mathrm{tame}} S^{r-1}$ is obtained by applying tame localization to the coalgebra $Y := \Sigma^\infty L_{\mathrm{tame}} S^{r-1}$ in $\mathrm{coCAlg}(\mathrm{Sp})$, it suffices to prove Y is a trivial coalgebra after tame localization.

By Proposition 3.3.12, we can build a commutative coalgebra X in Sp by assembling compatible coalgebra structures of X in $\mathrm{coAlg}_{\varphi^n \mathrm{Com}}(\mathrm{Sp})$ for each n. Moreover, by Proposition 3.1.23 there is an equivalence

$$\mathrm{colim}_n\, L_{\mathrm{tame}} F_{\varphi^n \mathrm{Com}} \simeq L_{\mathrm{tame}} F_{\mathrm{Com}}.$$

It suffices to prove by induction that, $C_{\mathrm{tame}} L_{\mathrm{tame}} S^{r-1}$ is a trivial $L_{\mathrm{tame}} F_{\varphi^n \mathrm{Com}}$-coalgebra for each $n \geq 1$. The case for $n = 1$ is obvious since $\varphi^1 \mathrm{Com}$ is the trivial operad.

Assume $L_{\mathrm{tame}} Y$ is a trivial $(L_{\mathrm{tame}} F_{\varphi^{n-1} \mathrm{Com}})$-coalgebra, i.e., the structure map of Y as an $(\varphi^{n-1} \mathrm{Com})$-coalgebra becomes trivial after apply L_{tame}. By Proposition 3.3.12 and the vanishing of the Tate construction in tame spectra, specifying a $L_{\mathrm{tame}} F_{\varphi^n \mathrm{Com}}$-coalgebra structure on $C_{\mathrm{tame}} L_{\mathrm{tame}} S^{r-1}$ is equivalent to a lift in the following diagram

$$\begin{tikzcd} & L_{\mathrm{tame}}(Y \otimes \cdots \otimes Y)_{h\Sigma_n} \arrow[d] \\ L_{\mathrm{tame}} Y \arrow[r, "0"'] \arrow[ur, dashed] & L_{\mathrm{tame}}(\varphi^{n-1}\,\mathrm{Com}(n) \otimes Y \otimes \cdots \otimes Y)_{h\Sigma_n} \end{tikzcd}$$

where the vertical map is induced from the map of operads $\mathrm{Com} \to \varphi^{n-1}\,\mathrm{Com}$. Let F denote the fiber of the vertical map. We claim that the connectivity of F is at least r, hence any lift is null-homotopic and has to be the trivial map. The connectivity of $L_{\mathrm{tame}}(Y \otimes \cdots \otimes Y)_{h\Sigma_n}$ is at least $n(r-1)$, which is larger than $r+1$ (recall $r \geq 4$). The connectivity of

$$L_{\mathrm{tame}}(\varphi^{n-1}\,\mathrm{Com}(n) \otimes Y \otimes \cdots \otimes Y)_{h\Sigma_n}$$

is at least $n(r-1)-(n-3) > r+1$ (cf. [Heu21a, Proposition 4.10 and Example 4.7]). Hence the connectivity of F is larger than r, and the lemma is proved.

□

Remark 4.2.11. The proof of Lemma 4.2.10 is almost identical to the proof of [Heu21a, Lemma 6.17]. There Heuts shows that $\Sigma^\infty S^{r-1}$ is a trivial coalgebra in the ∞-category $\mathrm{coAlg}^{\mathrm{nil,dp}}_{\mathrm{Com}}(\tau_{p-1}\,\mathrm{Sp}^{\geq r-1})$. Informally, $\mathrm{coAlg}^{\mathrm{nil,dp}}_{\mathrm{Com}}(\tau_{p-1}\,\mathrm{Sp}^{\geq r-1})$ consists of conilpotent, divided power coalgebras X in Sp with $(p-1)!$ inverted in Sp and with coherent structure maps $X \to (X^{\otimes k})_{h\Sigma_k}$ for $1 \leq k \leq p-1$. The important ingredients of both proofs are the inductive construction of coalgebras and the vanishing of Tate construction.

Using Lemma 4.2.10, we can now prove the essential surjectiveness of the functor C_{tame}.

Lemma 4.2.12. *The functor*

$$C_{tame} : \mathcal{S}^{\geq r-1}_{\mathrm{tame}} \to \mathrm{coCAlg}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}})$$

is essentially surjective.

Proof. Recall that Theorem 3.7.17 gives an equivalence

$$\widetilde{\mathrm{CE}} : \mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Mod}_{H\mathbb{Z}})^{\geq r-1} \to \mathrm{coCAlg}^{\mathrm{dp,nil}}(\mathrm{Mod}_{H\mathbb{Z}})^{\geq r-1}$$

of ∞-categories. The ∞-category $\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Mod}_{H\mathbb{Z}})^{\geq r-1}$ is generated by the free Lie algebra $\mathrm{free}_{\mathbf{Lie}}(\Sigma^{r-1}H\mathbb{Z})$ (in $\mathrm{Mod}_{H\mathbb{Z}}$) under colimits. It follows that the ∞-category $\mathrm{coCAlg}^{\mathrm{dp,nil}}(\mathrm{Mod}_{H\mathbb{Z}})^{\geq r-1}$ is generated under colimits by the trivial commutative coalgebra $\mathrm{triv}_{\mathrm{Com}}(\Sigma^{r-1}H\mathbb{Z})$, since we have an equivalence of functors

$$\widetilde{\mathrm{CE}} \circ \mathrm{free}_{\mathbf{Lie}} \simeq \mathrm{triv}_{\mathrm{Com}}\,.$$

Since $\mathrm{coCAlg}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}})$ is a localization of $\mathrm{coCAlg}^{\mathrm{dp,nil}}(\mathrm{Mod}^{\geq r-1}_{H\mathbb{Z}})$ by Remark 3.7.15, it is generated under colimits by the trivial tame coalgebra

$$\mathrm{triv}_{\mathrm{Com,tame}}(\Sigma^{r-1}H\mathbb{Z}),$$

where $\mathrm{triv}_{\mathrm{Com,tame}}$ denotes the composite $L_{\mathrm{tame}} \circ \mathrm{triv}_{\mathrm{Com}}$. Hence it suffices to show

$$C_{\mathrm{tame}}L_{\mathrm{tame}}S^{r-1} \simeq \mathrm{triv}_{\mathrm{Com},tame}(\Sigma^{r-1}H\mathbb{Z}),$$

which is the content of Lemma 4.2.10.

$\square$

4.3 The equivalence of tame Lie algebras and tame Hopf algebras

In the last section of this chapter, we prove Theorem 4.0.5. We first prove an important proposition about the loop of a tame spectral Lie algebra. Over $\mathbb{Q}$, if L is a Lie algebra in $\mathrm{Ch}_{\mathbb{Q}}$, then $\Omega_{\mathrm{Lie}}L$ is a trivial Lie algebra for degree reasons. We claim the same phenomenon happens in the case of tame spectral Lie algebras.

Proposition 4.3.1. *Let $L \in \mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}})$ be a tame spectral Lie algebra. Then there is an equivalence*

$$\Omega_{\mathbf{Lie}}L \simeq \mathrm{triv}_{\mathbf{Lie}}(\Omega L).$$

Assuming Proposition 4.3.1, we can now prove Theorem 4.0.5.

Proof of Theorem 4.0.5: First we claim the universal enveloping algebra functor

$$U : \mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \xrightarrow{\Omega_{\mathbf{Lie}}} \mathrm{Grp}(\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}})) \xrightarrow{\mathrm{Grp}(\widetilde{\mathrm{CE}}_{\mathrm{tame}})} \mathrm{HopfAlg}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}})$$

is fully faithful. Let $\mathrm{Grp}(\widetilde{\mathrm{prim}})$ denote the right adjoint of $\mathrm{Grp}(\widetilde{\mathrm{CE}}_{\mathrm{tame}})$. Note that $\Omega_{\mathbf{Lie}}$ is an equivalence, hence it suffices to show the unit map

$$\eta : \Omega_{\mathbf{Lie}} X \to \mathrm{Grp}(\widetilde{\mathrm{prim}}) \circ \mathrm{Grp}(\widetilde{\mathrm{CE}}_{\mathrm{tame}})(\Omega_{\mathbf{Lie}} X)$$

is an equivalence. By Proposition 4.3.1, $\Omega_{\mathbf{Lie}} X \simeq \mathrm{triv}_{\mathbf{Lie}}\, \Omega X$. Hence η can be written as

$$\eta : \mathrm{triv}_{\mathbf{Lie}}\, \Omega X \to \mathrm{Grp}(\widetilde{\mathrm{prim}}) \circ \mathrm{Grp}(\widetilde{\mathrm{CE}}_{\mathrm{tame}})(\mathrm{triv}_{\mathbf{Lie}}\, \Omega X).$$

The claim then follows from the fact that the unit from the Koszul duality adjunction $\widetilde{\mathrm{CE}}_{\mathrm{tame}}$-$\widetilde{\mathrm{prim}}$ is an equivalence on trivial algebras, which is due to the following formal equivalences

$$\widetilde{\mathrm{CE}}_{\mathrm{tame}} \circ \mathrm{triv}_{\mathbf{Lie}} \simeq \mathrm{Sym}_{\mathrm{tame}}\, \Sigma \quad \widetilde{\mathrm{prim}} \circ \mathrm{Sym}_{\mathrm{tame}} \simeq \mathrm{triv}_{\mathbf{Lie}}\, \Omega$$

by Lemma 3.7.23. This completes the proof of the fully faithfulness of U.

For essential surjectivity, note first that U preserves colimits, as it is a composition of an equivalence $\Omega_{\mathbf{Lie}}$ and a left adjoint $\mathrm{Grp}(\widetilde{\mathrm{CE}}_{\mathrm{tame}})$. Since we have an equivalence of ∞-categories

$$\mathrm{HopfAlg}(\mathrm{Sp}_{\mathrm{tame}}^{\geq r-1}) \simeq \mathcal{S}_{tame}^{\geq r}$$

by Theorem 4.0.2, it suffices to show that the Hopf algebra H corresponding to the generator $K(\mathbb{Z}, r)$ of $\mathcal{S}_{tame}^{\geq r}$, lies in the essential image of U. Observe that the underlying $(r-1)$-tame spectrum of H is

$$\Sigma_{\mathrm{tame}}^{\infty} \Omega K(\mathbb{Z}, r) \simeq \Sigma_{\mathrm{tame}}^{\infty} K(\mathbb{Z}, r-1) \simeq \mathrm{Sym}_{\mathrm{tame}}(\Sigma^{r-1} H\mathbb{Z}),$$

where the second equivalence follows from Proposition 4.1.4. Moreover, for the trivial r-tame Lie algebra $\mathrm{triv}_{\mathbf{Lie}}(\Sigma^r H\mathbb{Z})$,

$$\begin{aligned}\widetilde{\mathrm{CE}}_{\mathrm{tame}} \circ \Omega_{\mathbf{Lie}}(\mathrm{triv}_{\mathbf{Lie}}(\Sigma^r H\mathbb{Z})) &\simeq \widetilde{\mathrm{CE}}_{\mathrm{tame}} \circ \mathrm{triv}_{\mathbf{Lie}}(\Sigma^{r-1} H\mathbb{Z}) \\ &\simeq \mathrm{Sym}_{\mathrm{tame}}(\Sigma^{r-1} H\mathbb{Z}),\end{aligned}$$

hence we see that H is the image of $\mathrm{triv}_{\mathbf{Lie}}(\Sigma^r H\mathbb{Z})$ under U.

□

We conclude this chapter with the proof of Proposition 4.3.1. We learned this proof from Heuts. Recall that there is a pair of adjoint equivalences of

∞-categories by Lemma 3.4.5

$$\Sigma' \colon \mathsf{Alg}_{\Omega\mathbb{L}\Sigma}(\mathsf{Sp}) \rightleftarrows \mathsf{Alg}_{\mathbb{L}}(\mathsf{Sp}) : \Omega'.$$

By Lemma 3.4.6, it suffices to show $\Omega_{\mathbb{L}} X$ is a trivial $\mathbb{L}$-algebra for any $X \in \mathsf{Alg}_{\mathbb{L}}(\mathsf{Sp}^{\geq r})$ after tame localization.

Let σ denote the *suspension morphism* in (3.10)

$$F_{\mathbb{L}} \xrightarrow{\sigma} F_{\Omega\mathbb{L}\Sigma},$$

or equivalently, this can be obtained via the induced map on Goodwillie derivatives $\partial_* \mathrm{id} \to \partial_* \Omega\Sigma$. The restriction along σ fits in a factorization

$$\begin{tikzcd} \mathsf{Alg}_{\mathbb{L}}(\mathsf{Sp}) \arrow[rr, "\Omega_{\mathbb{L}}"] \arrow[dr, "\Omega'"'] & & \mathsf{Alg}_{\mathbb{L}}(\mathsf{Sp}) \\ & \mathsf{Alg}_{\Omega\mathbb{L}\Sigma}(\mathsf{Sp}) \arrow[ur, "\sigma^*"'] & \end{tikzcd}.$$

We claim $\sigma : F_{\mathbb{L}} \to F_{\Omega\mathbb{L}\Sigma}$ factors through the identity monad as the augmentation of $F_{\mathbb{L}}$ followed by the unit of $F_{\Omega\mathbb{L}\Sigma}$, after tame localization, which would complete the proof of Proposition 4.3.1, since the restriction along $F_{\mathbb{L}} \to \mathrm{id}$ is indeed the trivial Lie algebra functor.

The ∞-category Sp is generated under sifted colimits by wedge sum of shifted spheres $\mathbb{S}^k$. Therefore, it suffices to prove the claim when evaluating σ at a wedge of spheres $\mathbb{S}^{k_1} \oplus \cdots \oplus \mathbb{S}^{k_n}$. That is, we want to show

$$\sigma_{\mathbb{S}^{k_1} \oplus \cdots \oplus \mathbb{S}^{k_n}} : \mathbb{L}(\mathbb{S}^{k_1} \oplus \cdots \oplus \mathbb{S}^{k_n}) \to \Omega\mathbb{L}\Sigma(\mathbb{S}^{k_1} \oplus \cdots \oplus \mathbb{S}^{k_n}).$$

factors as the coaugmentation of $\mathbb{L}$ follows by the augmentation of $\Omega\mathbb{L}\Sigma$ on $\mathbb{S}^{k_1} \oplus \cdots \oplus \mathbb{S}^{k_n}$ after tame localization.

Theorem 4.3.2 (Hilton-Milnor Theorem, [BH20], [AK98])**.** *For any collection of spheres $\mathbb{S}^{k_1}, \ldots, \mathbb{S}^{k_n}$, there is an equivalence*

$$\Omega\mathbb{L}\Sigma(\mathbb{S}^{k_1} \oplus \cdots \oplus \mathbb{S}^{k_n}) \to \prod_{\omega \in \mathsf{Lie}_n}{}' \, \Omega\mathbb{L}(\Sigma\omega(\mathbb{S}^{k_1}, \cdots, \mathbb{S}^{k_n})),$$

where $\prod'$ denotes the weak product (i.e., filtered colimits of finite products) and Lie_n *denotes the ordered set of Lie words with n generators, i.e., every $\omega \in \mathsf{Lie}_n$ is a basis element of the free Lie algebra on n generators.*

Using the Hilton-Milnor theorem, we can describe the suspension morphism σ on the underlying spectra as

$$\sigma : \prod_{w\in \mathrm{Lie}_n}{}' \mathbb{L}(\Sigma\omega(\mathbb{S}^{k_1-1}, \cdots, \mathbb{S}^{k_n-1})) \to \prod_{w\in \mathrm{Lie}_n}{}' \Omega\mathbb{L}(\Sigma\omega(\mathbb{S}^{k_1}, \cdots, \mathbb{S}^{k_n})).$$

Since σ comes from a map of spectral Lie algebras, it should preserve components indexed by the same Lie word. We claim that σ restricts to a null map on components corresponding to those Lie words ω with length $n \geq 2$. Indeed, write $F : \mathrm{Sp}^n \to \mathrm{Sp}$ for the multivariable functor $\mathbb{L}(\Sigma\omega(-, \cdots, -))$. Then we can identify the suspension morphism σ as

$$F(\mathbb{S}^{k_1-1}, \cdots, \mathbb{S}^{k_n-1}) \to \Omega F(\Sigma\mathbb{S}^{k_1-1}, \cdots, \Sigma\mathbb{S}^{k_n-1}).$$

Observe that this map can be further factored by first applying the suspension morphism component wise, then applying the "diagonal embedding" $\mathbb{S}^{-n} \to \mathbb{S}^{-1}$. This can be summarized as the diagram below

$$\begin{tikzcd}
F(\mathbb{S}^{k_1-1}, \cdots, \mathbb{S}^{k_n-1}) \arrow[r, "\sigma"] \arrow[dr] & \Omega F(\Sigma\mathbb{S}^{k_1-1}, \cdots, \Sigma\mathbb{S}^{k_n-1}) \\
 & \Omega^n F(\Sigma\mathbb{S}^{k_1-1}, \cdots, \Sigma\mathbb{S}^{k_n-1}). \arrow[u, "0"]
\end{tikzcd}$$

We remind the reader that the map $\mathbb{S}^{-n} \to \mathbb{S}^{-1}$ is obtained from the Spainer-Whitehead dual of the diagonal embedding $\mathbb{S}^1 \to \mathbb{S}^n$, which is null if $n \geq 2$.

Therefore σ factors as a map

$$\bigoplus_{i=1}^{n} \mathbb{L}(\mathbb{S}^{k_i}) \to \bigoplus_{i=1}^{n} \Omega\mathbb{L}\Sigma(\mathbb{S}^{k_i}),$$

which sends $\mathbb{L}(S^{k_i})$ to $\Omega\mathbb{L}\Sigma(\mathbb{S}^{k_i})$ for every i. It now suffices to show that for each k, the suspension map

$$\sigma' : \mathbb{L}(\mathbb{S}^k) \to \Omega\mathbb{L}\Sigma(\mathbb{S}^k)$$

factors through $\mathbb{S}^k$ after tame localization, for which we need the following theorem by Arone-Mahowald.

Theorem 4.3.3. *[AM99, Theorem 3.13, Theorem 4.4] Let X denote the p-localization of the sphere $\mathbb{S}^k$ at a prime p, then*

1. *if k is odd, then*
$$\left(\mathbb{L}(n) \otimes X^{\otimes n}\right)_{h\Sigma_n} \simeq *.$$
if $n \neq p^l$ for some l. If $n = p^l$ then $\left(\mathbb{L}(n) \otimes X^{\otimes n}\right)_{h\Sigma_n}$ has only p-primary torsion.

2. *if k is even, then*
$$\left(\mathbb{L}(n) \otimes X^{\otimes n}\right)_{h\Sigma_n} \simeq *.$$
if n is not equal to p^l or $2p^j$ for $l, j > 0$. If $n = p^l$ or $n = 2p^l$ then $\left(\mathbb{L}(n) \otimes X^{\otimes n}\right)_{h\Sigma_n}$ has only p-primary torsion.

Lemma 4.3.4. *The suspension morphism evaluated on a sphere admits a factorization*

$$\begin{tikzcd}
\mathbb{L}(\mathbb{S}^k) \arrow[rr, "\sigma"] \arrow[dr] & & \Omega\mathbb{L}(\Sigma\mathbb{S}^k) \\
& \mathbb{S}^k \arrow[ur] &
\end{tikzcd}$$

after tame localization.

Proof. The lemma follows from Theorem 4.3.3; indeed, for $n = p^l$ or $n = 2p^l$ and $l > 0$, the connectivity of

$$\left(\mathbb{L}(n) \otimes (\mathbb{S}^k)^{\otimes n}\right)_{h\Sigma_n}$$

is at least $kn - n + 1$, which is larger than $r + 2p - 3$ (note that $k \leq r$), hence $\left(\mathbb{L}(n) \otimes (\mathbb{S}^k)^{\otimes n}\right)_{h\Sigma_n}$ has only p-primary torsion and is contractible after tame localization for $n > 1$. $\square$

The proof of Theorem 4.0.5 is now complete.

Appendix A

Higher Algebra Preliminaries

In this appendix, we recall some notions and results from [Lur09] and [Lur17] that are used repeatedly in this thesis.

A.1 Monoids and Groups in ∞-Categories

Let $\mathcal{C}$ be a pointed ∞-category with finite limits. We recall that a *monoid object* in an ∞-category $\mathcal{C}$ [Lur17, Definition 4.1.2.5.] is a simplicial object $X : \Delta^{op} \to \mathcal{C}$ satisfying the Segal condition: the collection of face maps $X([n]) \to X(\{i-1, i\})$ for $1 \leq i \leq n$ exhibits $X([n])$ as a product of $\{X(\{i-1, i\})\}_{1\leq i \leq n}$. We will denote the full subcategory of monoids in $\mathcal{C}$ by $\mathrm{Mon}(\mathcal{C})$.

Under the identification $\mathrm{Mon}(\mathcal{C}) \simeq \mathrm{Mon}_{\mathrm{Assoc}}(\mathcal{C})$ (see [Lur17, Proposition 4.1.2.10.]), a monoid X is equipped with a (homotopy coherently) associative multiplication:

$$m : X \times X \to X;$$

we say that X is a *group* in $\mathcal{C}$ [Lur17, Definition 5.2.6.2.] if both shearing maps

$$(p_1, m) : X \times X \to X \times X$$
$$(m, p_2) : X \times X \to X \times X$$

are equivalences. We will denote the ∞-category of groups in $\mathcal{C}$ as $\mathrm{Grp}(\mathcal{C})$.

Definition A.1.1. [Lur09, Definition 6.1.2.7.] A simplicial object $U_\bullet$ in $\mathcal{D}$ is a *groupoid object* if for every $n \geq 0$ and every partition $[n] = S \cup S'$ such that

$S \cap S' = \{s\}$, the diagram

$$\begin{tikzcd} U([n]) \arrow[r] \arrow[d] & U(S) \arrow[d] \\ U(S') \arrow[r] & U(\{s\}) \end{tikzcd}$$

is a pullback square in the ∞-category $\mathcal{C}$.

Remark A.1.2. By [Lur17, Remark 5.2.6.5.], the notion of group in an ∞-category is equivalent to the definition of *groupoid objects* in [Lur09, Definition 6.1.2.7.]. For a groupoid object $X_\bullet$, we write X_1 for the corresponding group object.

We now give another characterization of group objects in terms of presheaves on $\mathcal{C}$. Since the Yoneda embedding preserves limits, it induces a functor

$$\mathrm{Grp}(\mathcal{C}) \to \mathrm{Grp}(\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})) \simeq \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Grp}(\mathcal{S})).$$

Hence we can identify group objects in $\mathcal{C}$ as those representable presheaves that factor through $\mathrm{Grp}(\mathcal{S})$.

The *loop object* ΩY of an object $Y \in \mathcal{C}$ is defined as

$$\Omega Y := 0 \underset{Y}{\times} 0.$$

We want to show that ΩY is a group object of $\mathcal{C}$.

Lemma A.1.3. *Let $\mathcal{C}$ be a pointed ∞-category with finite limits, then ΩY is a group object for any $Y \in \mathcal{C}$.*

Proof. The image of the Yoneda embedding of ΩY is given by

$$X \mapsto \mathrm{Map}_{\mathcal{C}}(X, \Omega Y) \simeq \Omega \, \mathrm{Map}_{\mathcal{C}}(X, Y),$$

where latter is in $\mathrm{Grp}(\mathcal{S})$ for any $X \in \mathcal{C}$. $\square$

Remark A.1.4. In particular, ΩY is a monoid in $\mathcal{C}$. The multiplication map is given by the "concatenation" map

$$0 \underset{Y}{\times} 0 \underset{Y}{\times} 0 \simeq \Omega Y \times \Omega Y \to \Omega Y \simeq 0 \underset{Y}{\times} Y \underset{Y}{\times} 0$$

which is unique up to contractible ambiguity.

Proposition A.1.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories with finite products. Suppose $F : \mathcal{C} \hookrightarrow \mathcal{D}$ is a fully faithful, product-preserving functor, then*

(1) the functor F induces a fully faithful embedding $\mathrm{Mon}(F) : \mathrm{Mon}(\mathcal{C}) \to \mathrm{Mon}(\mathcal{D})$ *between the category of associative monoids.*

(2) the functor F induces a fully faithful embedding $\mathrm{Grp}(F) : \mathrm{Grp}(\mathcal{C}) \to \mathrm{Grp}(\mathcal{D})$ *between the category of group objects.*

Proof. Observe first that the functor F induces a fully faithful embedding

$$\mathrm{Fun}(\Delta^{op}, \mathcal{C}) \to \mathrm{Fun}(\Delta^{op}, \mathcal{D})$$

between the ∞-categories of simplicial objects. Then (1) follows from the fact that F send monoids to monoids and $\mathrm{Mon}(\mathcal{C})$ is a full subcategory of $\mathrm{Fun}(\Delta^{op}, \mathcal{C})$; and (2) follows from the fact that $\mathrm{Grp}(\mathcal{C})$ is a full subcategory of $\mathrm{Mon}(\mathcal{C})$. □

Proposition A.1.6. *Let $F \colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a pair of adjoint functors between ∞-categories. Let $\mathcal{C}^0$ be a full subcategory of $\mathcal{C}$ so that the essential image of G is contained in $\mathcal{C}^0$. Then $F^0 := F|_{\mathcal{C}^0}$ restricts to an adjoint pair*

$$F^0 \colon \mathcal{C}^0 \rightleftarrows \mathcal{D} : G.$$

Definition A.1.7. [GGN15, Definition 2.1] An ∞-category is *preadditive* if the canonical morphism

$$X \coprod Y \to X \times Y$$

is an equivalence for any pair of objects $X, Y \in \mathcal{C}$.

Example A.1.8.

1. Any stable ∞-category is preadditive.
2. The ∞-category $\mathrm{Sp}^{\geq r}$ of r-connective spectra is preadditive; indeed, the coproduct of $X, Y \in \mathrm{Sp}^r$ is computed in Sp and the product of $X, Y \in \mathrm{Sp}^r$ is computed by $\tau_{\geq r}(X \oplus Y) \simeq X \oplus Y$, where $\oplus$ denotes the direct sum in Sp.
3. Any product-preserving localization of a stable ∞-category is preadditive. Therefore, the ∞-category $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ of r-tame spectra is preadditive; the product of $X, Y \in \mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ is computed in $\mathrm{Sp}^{\geq r}$ and the coproduct is computed by $L_{\mathrm{tame}}(X \oplus Y) \simeq X \oplus Y$.

Let $\mathrm{CMon}(\mathcal{C})$ [Lur17, Definition 2.4.2.1.] denote the ∞-category of commutative monoids in $\mathcal{C}$.

Proposition A.1.9. *[GGN15, Proposition 2.3] Let $\mathcal{C}$ be an ∞-category with finite products and finite coproducts, then the following are equivalent:*

1. *The ∞-category $\mathcal{C}$ is preadditive.*
2. *The homotopy category $h\mathcal{C}$ is preadditive.*
3. *The forgetful functor $\mathrm{CMon}(\mathcal{C}) \to \mathcal{C}$ is an equivalence.*

If $M \in \mathcal{C}$ is an object in a preadditive ∞-category $\mathcal{C}$, then M can be equipped with a commutative monoid structure by Proposition A.1.9. We will call the monoid structure map $\nabla : M \oplus M \to M$ the *fold map*. The *shearing map* defined as

$$s : M \oplus M \xrightarrow{(pr_1, \nabla)} M \oplus M,$$

where pr_1 denotes the projection to the first factor.

Definition A.1.10. [GGN15, Definition 2.6] A preadditive ∞-category $\mathcal{C}$ is *additive* if the shearing map s is an equivalence.

Proposition A.1.11. *[GGN15, Proposition 2.8] Let $\mathcal{C}$ be an ∞-category with finite products and finite coproducts, then the following are equivalent:*

1. *The ∞-category $\mathcal{C}$ is additive.*
2. *The homotopy category $h\mathcal{C}$ is additive.*
3. *The forgetful functor $\mathrm{Grp}(\mathcal{C}) \to \mathcal{C}$ is an equivalence.*

Proposition A.1.9 and Proposition A.1.11 then imply that there are equivalences

$$\mathrm{Grp}(\mathcal{C}) \simeq \mathrm{CMon}(\mathcal{C}) \simeq \mathcal{C}$$

if $\mathcal{C}$ is additive. Moreover, if $\mathcal{C}$ is stable, we have the following corollary regarding the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ of $\mathcal{O}$-algebras in $\mathcal{C}$.

Corollary A.1.12. *Let $\mathcal{C}$ be a presentably stable ∞-category. Let $B_{\mathcal{O}}(M)$ denote the bar construction of a monoid M (with respect to the monoid structure) in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. Then the pair of adjoint functors*

$$B_{\mathcal{O}} \colon \mathrm{Grp}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})) \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) : \Omega_{\mathcal{O}}$$

is an equivalence.

Proof. Since both of the functors $B_{\mathcal{O}}$ and $\mathrm{oblv}_{\mathcal{O}}$ preserves sifted colimits, the bar construction $B_{\mathcal{O}}(M)$ of $M \in \mathrm{Grp}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}))$ is computed in $\mathcal{C}$. Since $\mathcal{C}$ is stable, the bar complex $\mathrm{Bar}(0, M, 0)_\bullet$ comes from the Čech nerve of $0 \to \Sigma M$, which implies that $B_{\mathcal{O}} M \simeq \Sigma M$. Therefore we obtain two commuting diagrams

$$\begin{tikzcd}
\mathrm{Grp}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})) \arrow[r, "B_{\mathcal{O}}"] \arrow[d, "\mathrm{oblv}_{\mathcal{O}} \circ \mathrm{oblv}_{\mathrm{Grp}}"'] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[d, "\mathrm{oblv}_{\mathcal{O}}"] \\
\mathcal{C} \arrow[r, "\Sigma"] & \mathcal{C};
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Grp}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})) \arrow[d, "\mathrm{oblv}_{\mathcal{O}} \circ \mathrm{oblv}_{\mathrm{Grp}}"'] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[l, "\Omega_{\mathcal{O}}"] \arrow[d, "\mathrm{oblv}_{\mathcal{O}}"] \\
\mathcal{C} & \mathcal{C}. \arrow[l, "\Omega"]
\end{tikzcd}$$

Since the vertical forgetful functors are conservative, there are equivalences

$$\mathrm{oblv}_{\mathcal{O}} \circ B_{\mathcal{O}} \circ \Omega_{\mathcal{O}}(X) \simeq \Sigma\Omega(\mathrm{oblv}_{\mathcal{O}} X) \simeq \mathrm{oblv}_{\mathcal{O}} X$$

for any $X \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$, and

$$\mathrm{oblv}_{\mathcal{O}} \circ \mathrm{oblv}_{\mathrm{Grp}} \circ \Omega_{\mathcal{O}} \circ B_{\mathcal{O}} Y \simeq \Omega\Sigma(\mathrm{oblv}_{\mathcal{O}} \circ \mathrm{oblv}_{\mathrm{Grp}}\, Y) \simeq (\mathrm{oblv}_{\mathcal{O}} \circ \mathrm{oblv}_{\mathrm{Grp}}\, Y)$$

for any $Y \in \mathrm{Grp}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}))$. □

We consider now the ∞-category $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ of r-tame spectra. Let Σ and Ω denote the suspension and loop functor in Sp respectively. Note that the category $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ is not closed under Σ, so it's not an inverse of Ω. However, it's easy to see we have the following lemma.

Lemma A.1.13. *The adjunction*

$$\Sigma\colon \mathrm{Sp}^{\geq r-1}_{\mathrm{tame}} \rightleftarrows \mathrm{Sp}^{\geq r}_{\mathrm{tame}} :\Omega$$

is an equivalence.

Moreover, if we let $L_{r-1,\mathrm{tame}}$ and $L_{r,\mathrm{tame}}$ be $(r-1)$-localization and r-localization functors, respectively. It's easy to see there are equivalences

$$L_{r-1,\mathrm{tame}}\Omega X \simeq \Omega L_{r,\mathrm{tame}} X$$

for an r-connective spectrum X, and

$$L_{r,\mathrm{tame}}\Sigma Y \simeq \Sigma L_{r-1,\mathrm{tame}} Y$$

for an $(r-1)$-connective spectrum Y.

Since $\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{tame})$ is Cartesian symmetric monoidal, the loop functor

$$\Omega : \mathrm{Sp}^{\geq r}_{\mathrm{tame}} \to \mathrm{Sp}^{\geq r-1}_{\mathrm{tame}}$$

induces a functor

$$\Omega_{\mathbf{Lie}} : \mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \to \mathrm{Grp}(\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}}))$$

which takes a r-tame Lie algebra X to a $(r-1)$-tame Lie algebra $\Omega_{\mathbf{Lie}}X$ whose underlying spectrum is ΩX.

Proposition A.1.14. *The functor*

$$\Omega_{\mathbf{Lie}} : \mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \to \mathrm{Grp}(\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}}))$$

is an equivalence.

Proof. Let $B_{\mathbf{Lie}}$ denote the bar construction functor that is left adjoint to $\Omega_{\mathbf{Lie}}$. Since the forgetful functor $\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \to \mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ preserves sifted colimits, the bar construction is computed in the underlying category $\mathrm{Sp}^{\geq r}_{\mathrm{tame}}$ of r-tame spectra.

Let $X_\bullet$ be a groupoid object in $\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}}$, the geometric realization of $X_\bullet$ is computed by first taking the geometric realization of $X_\bullet$ in Sp, then applying the r-tame localization functor L_{tame}, but $|X_\bullet| \simeq \Sigma X$ which is already r-tame, hence we have $B_{\mathbf{Lie}}X \simeq \Sigma X$. Therefore we have a commuting diagram

$$\begin{tikzcd}
\mathrm{Grp}(\mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}})) \arrow[r, shift left, "B_{\mathbf{Lie}}"] \arrow[d] & \mathrm{Alg}_{\mathbf{Lie}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \arrow[l, shift left, "\Omega_{\mathbf{Lie}}"] \arrow[d] \\
\mathrm{Sp}^{\geq r-1}_{\mathrm{tame}} \arrow[r, shift left, "\Sigma"] & \mathrm{Sp}^{\geq r}_{\mathrm{tame}}; \arrow[l, shift left, "\Omega"]
\end{tikzcd}$$

Hence $B_{\mathbf{Lie}}$ and $\Omega_{\mathbf{Lie}}$ are mutually inverses because Σ and Ω are mutually inverses. □

A.2 Filtered and Graded Objects in Infinity-Categories

In this section, we introduce the notion of filtered and graded objects in a symmetric monoidal ∞-category $\mathcal{C}$. The main reference for this section is [BM19].

Definition A.2.1. Let $\mathbb{Z}$ denote the poset of the integers and let $\mathcal{C}$ be an ∞-category. The ∞-category of *filtered objects in* $\mathcal{C}$ is defined as the functor category

$$\mathsf{Fil}(\mathcal{C}) := \mathsf{Fun}(\mathbb{Z}, \mathcal{C}).$$

Let $\mathbb{Z}^{\mathsf{disc}}$ denote the groupoid with objects the integers and identity morphisms.

Definition A.2.2. The ∞-category of *graded objects in* $\mathcal{C}$ is defined as the functor category

$$\mathsf{Gr}(\mathcal{C}) := \mathsf{Fun}(\mathbb{Z}^{\mathsf{disc}}, \mathcal{C}).$$

Remark A.2.3. We can also extend the definitions above to the category $\mathsf{Fil}^{+}(\mathcal{C})$ (resp. $\mathsf{Gr}^{+}(\mathcal{C})$) of *non-negatively filtered* (resp. *non-negatively graded*) objects by restricting to the category $\mathbb{Z}_{\geq 0}$ (resp. $\mathbb{Z}^{\mathsf{disc}}_{\geq 0}$) of non-negative integers.

The natural inclusion $\mathbb{Z}^{\mathsf{disc}} \hookrightarrow \mathbb{Z}$ induces a forgetful functor $\mathsf{Fil}(\mathcal{C}) \to \mathsf{Gr}(\mathcal{C})$. We denote by

$$U : \mathsf{Gr}(\mathcal{C}) \to \mathsf{Fil}(\mathcal{C})$$

the left Kan extension along the inclusion $\mathbb{Z}^{\mathsf{disc}} \hookrightarrow \mathbb{Z}$; explicitly, the filtered object X_* evaluated at n is given by $\bigoplus_{k \leq n} X_k$.

We can also define the *associated graded* of a filtered object

$$\mathsf{Gr}(-) : \mathsf{Fil}(\mathcal{C}) \to \mathsf{Gr}(\mathcal{C})$$

to be $X_* \mapsto (n \mapsto \mathsf{cofib}(X_{n-1} \to X_n))$. The following lemma follows immediately from an inductive argument.

Lemma A.2.4. *The associated graded functor*

$$\mathsf{Gr}(-) : \mathsf{Fil}^{+}(\mathcal{C}) \to \mathsf{Gr}^{+}(\mathcal{C})$$

is conservative.

We also have the following obvious observation.

Lemma A.2.5. *The composite*

$$\mathrm{Gr}(\mathcal{C}) \xrightarrow{U} \mathrm{Fil}(\mathcal{C}) \xrightarrow{\mathrm{Gr}(-)} \mathrm{Gr}(\mathcal{C})$$

is equivalent to the identity functor.

The following remark from [BM19] equips the categories of both filtered objects and graded objects with symmetric monoidal structure.

Remark A.2.6. [BM19, Definition 2.5] Suppose that $\mathcal{C}$ is (nonunital) presentably symmetric monoidal ∞-category in which tensor product commutes with colimits. Using Day convolution, one can equip both $\mathrm{Fil}(\mathcal{C})$ and $\mathrm{Gr}(\mathcal{C})$ with the structure of presentably (nonunital) symmetric monoidal ∞-categories. Furthermore, the associated graded functor $\mathrm{Gr} : \mathrm{Fil}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$ is (nonunital) symmetric monoidal (cf. [Gla13, Sec. 2.23]).

Consider now an ∞-operad $\mathcal{O}$ in $\mathcal{C}$. We can now state a theorem in [Heu22] which says that every $\mathcal{O}$-algebra admits a *canonical filtration* so that its associated graded is free.

Theorem A.2.7. *[Heu22, Theorem 5.2 (2)] For an $\mathcal{O}$-algebra X, there exists a canonical filtered object X_* so that*

1. *The filtration is* exhaustive, *i.e., there is an equivalence*

$$\operatorname{colim} X_* \to X.$$

2. *The filtered $\mathcal{O}$-algebra has associated graded*

$$\mathrm{Gr}(X_*) \simeq \mathrm{free}_{\mathcal{O}}(B\mathcal{O}(n) \otimes X^{\otimes n})_{h\Sigma_n}.$$

A.3 The Barr-Beck-Lurie Theorem

Theorem A.3.1. *[Lur17, Theorem 4.7.3.5.] Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjoint pair of ∞-categories. Then G is monadic if and only if*

1. *G is conservative, and*
2. *If $X_\bullet$ is a G-split simplicial object in $\mathcal{D}$, then its geometric realization exists in $\mathcal{D}$ and G preserves geometric realization of $X_\bullet$.*

In practice, the category $\mathcal{D}$ often admits all geometric realizations for simplicial objects. In this case, we have a technically convenient criteria for determining monadicity of a functor. We learned the proof of the following corollary from Heuts.

Corollary A.3.2 ([Heu20]). *If $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ an adjoint pair and that $\mathcal{D}$ admits colimits of G-split simplicial objects. Then this pair is monadic if and only if for every object X of $\mathcal{D}$, the map*

$$|(FG)^{\bullet+1}X| \to X$$

is an equivalence.

Proof. Suppose this pair is monadic, then it satisfies the conditions of Theorem A.3.1. For an object X in $\mathcal{D}$, the simplicial object $(FG)^{\bullet+1}X$ is G-split. Indeed, the simplicial object $G((FG)^{\bullet+1}X)$ admits a contracting homotopy via the unit natural transformation $X \to GF(X)$. Applying G to the map

$$|(FG)^{\bullet+1}X| \to X$$

one obtains

$$\begin{aligned} G(|(FG)^{\bullet+1}X|) &\simeq |G(FG)^{\bullet+1}X| \\ &\simeq G(X) \end{aligned}$$

where the first equivalence is due to the fact that G preserves geometric realization of G-split objects and the second equivalence is due to the assumption that $(FG)^{\bullet+1}X$ is G-split. Since G is conservative, we conclude that $|(FG)^{\bullet+1}X| \to X$ is an equivalence.

Suppose now $|(FG)^{\bullet+1}X| \to X$ is an equivalence for every X in $\mathcal{D}$. If $G(f) : G(X) \to G(Y)$ is an equivalence in $\mathcal{C}$ for some morphism $f : X \to Y$, then

$(FG)^{\bullet+1}X \to (FG)^{\bullet+1}Y$ is an equivalence of simplicial objects in $\mathcal{D}$. Therefore,

$$|(FG)^{\bullet+1}X| \to |(FG)^{\bullet+1}Y|$$

is an equivalence and hence so is $f : X \to Y$. So G is conservative. We now claim that G preserves geometric realization of G-split objects. Let $X_\bullet$ be a G-split simplicial object and consider the following commuting diagram

$$\begin{tikzcd} |(FG)^{\bullet+1}|X_\bullet|| \arrow[r] \arrow[d] & |(FG)^{\bullet+1}X_{-1}| \arrow[d] \\ |X_\bullet| \arrow[r] & X_{-1}. \end{tikzcd}$$

We claim the bottom horizontal arrow is an equivalence. Note that two vertical morphisms are equivalences by our assumption. We claim that the top horizontal map is an equivalence as well. Indeed, if we view $(FG)^{p+1}X_q$ as a bisimplicial object, then

$$|(FG)^{\bullet+1}|X_\bullet|| \simeq \operatorname{colim}_p \operatorname{colim}_q (FG)^{q+1}X_p.$$

For fixed q, the simplicial object $(FG)^{q+1}X_\bullet$ is split since it's a composite of functors starting with G, hence one has

$$\operatorname{colim}_p (FG)^{q+1}X_p \simeq (FG)^{q+1}X_{-1}$$

and $\operatorname{colim}_q (FG)^{q+1}X_{-1} \to X_{-1}$ is an equivalence by the assumption. Therefore, we conclude that

$$G(|X_\bullet|) \simeq G(X_{-1}) \simeq |G(X_\bullet)|$$

where the last equivalence follows from the fact that $X_\bullet$ is G-split. □

As an application of the Corollary A.3.2 in homotopy theory, we prove the following folklore proposition that is well-known among seasoned homotopy theorists.

Proposition A.3.3. *For $r \geq 2$, the functor $\Sigma^\infty : \mathcal{S}_*^{\geq r} \to \mathrm{Sp}^{\geq r}$ is comonadic. In other words, there is an equivalence of ∞-categories:*

$$\phi : \mathcal{S}_*^{\geq r} \to \mathrm{coAlg}_{\Sigma^\infty\Omega^\infty}(\mathrm{Sp}^{\geq r}).$$

between the ∞-category of r-connective spaces and the ∞-category of r-connective

$\Sigma^\infty\Omega^\infty$*-coalgebras.*

Proof. By the dual of Corollary A.3.2, it suffices to show there is an equivalence

$$X \to \mathrm{Tot}(\Omega^\infty\Sigma^\infty)^{\bullet+1}X$$

for every $X \in \mathcal{S}_*^{\geq r}$. We prove this by induction on the Postnikov tower. For X an Eilenberg-Maclane space $K(A,n) \simeq \Omega^{\infty-n}HA$, its associated augmented cosimplicial object $(\Omega^\infty\Sigma^\infty)^{\bullet+1}X$ splits, with the contracting homotopy induced by the counit $\Sigma^\infty\Omega^\infty\Omega^n HA \to \Omega^n HA$.

For the inductive step, we have a principal fibration sequence

$$K(\pi_n X, n) \to \tau_{\leq n}X \to \tau_{\leq n-1}X \to K(\pi_n X, n+1).$$

By the principal fibration lemma [BK72], the functor $\mathrm{Tot}(\Omega^\infty\Sigma^\infty)^{\bullet+1}$ preserves principal fibrations. Hence the vertical sequences in the following diagram are fiber sequences.

$$\begin{tikzcd}
\tau_{\leq n}X \arrow[r] \arrow[d] & \mathrm{Tot}(\Omega^\infty\Sigma^\infty)^{\bullet+1}(\tau_{\leq n}X) \arrow[d] \\
\tau_{\leq n-1}X \arrow[r] \arrow[d] & \mathrm{Tot}(\Omega^\infty\Sigma^\infty)^{\bullet+1}(\tau_{\leq n-1}X) \arrow[d] \\
K(\pi_n X, n+1) \arrow[r] & \mathrm{Tot}(\Omega^\infty\Sigma^\infty)^{\bullet+1}K(\pi_n X, n+1)
\end{tikzcd}$$

Observe that the bottom two horizontal arrows are equivalences by the inductive hypothesis, hence the induced map on the fibers is an equivalence. This completes the inductive step of the proof. □

A.4 Construction of the Comparison Functor

In this section, we construct the comparison functor from divided power conilpotent commutative coalgebras in tame spectra to commutative coalgebras in tame spectra

$$\zeta : \mathrm{coCAlg}^{\mathrm{dp,nil}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \to \mathrm{coCAlg}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}).$$

We fix a pre-additive, presentable, symmetric monoidal ∞-category $\mathcal{C}$ in which the tensor product is compatible with colimits. Consider the following lax monoidal functor

$$\begin{aligned} \mathrm{SSeq}(\mathcal{C}) &\to \mathrm{Fun}(\mathcal{C},\mathcal{C}) \\ \mathcal{O} &\mapsto (X \mapsto \prod_{n\geq 1}(\mathcal{O}(n)\otimes X^{\otimes n})_{h\Sigma_n}). \end{aligned}$$

Let K be the image of the commutative cooperad under the above functor, one can consider the K-comodules in $\mathcal{C}$ ([FG12, §3.5]).

Definition A.4.1. We define the *∞-category of divided power commutative coalgebras* in $\mathcal{C}$ as

$$\mathrm{coCAlg}^{\mathrm{dp}}(\mathcal{C}) := \mathrm{LcoMod}_K(\mathcal{C}).$$

Remark A.4.2. If $\mathcal{C} = \mathrm{Sp}^{\geq r}_{\mathrm{tame}}$, then there is an equivalence of ∞-categories

$$\mathrm{coCAlg}^{\mathrm{dp}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \to \mathrm{coCAlg}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}})$$

by Lemma 2.3.21. Moreover, K defines a cooperad on $\mathrm{coCAlg}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}})$.

Remark A.4.3. When $\mathcal{C} = \mathrm{Sp}$, there is a comparison functor

$$\mathrm{coAlg}^{\mathrm{dp,nil}}_{\mathrm{Com}}(\mathrm{Sp}) \to \mathrm{coCAlg}^{\mathrm{dp}}(\mathrm{Sp})$$

(see [FG12, Section 3.5] or [Heu22]), which induces a comparison functor

$$\mathrm{coCAlg}^{\mathrm{dp,nil}}(\mathrm{Sp}^{\geq r}) \to \mathrm{coCAlg}^{\mathrm{dp}}(\mathrm{Sp}^{\geq r}) \tag{A.1}$$

since colimits in $\mathrm{Sp}^{\geq r}$ are computed in Sp.

By Remark A.4.2, it suffices to construct a comparison functor

$$\zeta' : \mathrm{coCAlg}^{\mathrm{dp,nil}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \to \mathrm{coCAlg}^{\mathrm{dp}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}).$$

For the rest of this section, we explain how to obtain ζ' given (A.1). The crux for this construction is the following theorem by Heine, which allows us to identify maps between comonads with functors between comodules

Theorem A.4.4. *[Hei17, Theorem 5.1] Let $\mathcal{C}$ be a presentable ∞-category, then there is a localization*

$$\mathrm{Comonad}(\mathcal{C}) \to \mathrm{Pr}^L_{/\mathcal{C}}$$

that sends Q to $\mathrm{LcoMod}_Q(\mathcal{C})$, where $\mathrm{Pr}^L_{/\mathcal{C}}$ denotes the ∞-category of presentable ∞-categories over $\mathcal{C}$.

As a consequence of Corollary 3.5.2, the functor $j_* L^*$ preserves monads and comonads. Furthermore, the left adjoint $j^* L_*$ is oplax monoidal, so it also preserves comonads. The counit map of adjunction (A.2) evaluated on any comonad Q on $\mathcal{D}$

$$j^* L_* j_* L^* Q \to Q \tag{A.2}$$

is an equivalence of comonads. Therefore, the functor $j_* L^* : \mathrm{Comonad}(\mathcal{D}) \to \mathrm{Comonad}(\mathcal{C})$ is fully faithful.

Let $Q_{\mathcal{C}}$ and $Q_{\mathcal{D}}$ denote the comonads arising from the forgetful-cofree adjunction

$$\mathrm{oblv}_{\mathcal{C}} \colon \mathrm{coCAlg}^{\mathrm{dp}}(\mathcal{C}) \rightleftarrows \mathcal{C} : \mathrm{cofree}_{\mathcal{C}}$$

and

$$\mathrm{oblv}_{\mathcal{D}} \colon \mathrm{coCAlg}^{\mathrm{dp}}(\mathcal{D}) \rightleftarrows \mathcal{D} : \mathrm{cofree}_{\mathcal{D}} .$$

For every $X \in \mathcal{D}$, there is a natural map

$$Q_{\mathcal{C}}(X) = \prod_n (X^{\otimes n})_{h\Sigma_n,\mathcal{C}} \to Q_{\mathcal{D}}(X) = \prod_n (X^{\otimes n})_{h\Sigma_n,\mathcal{D}}$$

where we use $(X^{\otimes n})_{h\Sigma_n,\mathcal{C}}$ (resp. $(X^{\otimes n})_{h\Sigma_n,\mathcal{D}}$) to indicate the homotopy orbits is computed in $\mathcal{C}$ (resp. $\mathcal{D}$). Therefore, we obtain a map of comonads

$$L Q_{\mathcal{C}} j \to Q_{\mathcal{D}}. \tag{A.3}$$

Consider now the composite of comonads

$$Q_{\mathcal{C}} \to j \circ L \circ Q_{\mathcal{C}} \circ j \circ L \to j \circ Q_{\mathcal{D}} \circ L \tag{A.4}$$

where the first map is the unit of adjunction (3.11), and the second map (A.3).

Since $\mathcal{C}$ is preadditive, the comonad F_{Com} is equivalent to the symmetric algebra functor $\mathrm{Sym}_{\mathcal{C}}$, hence there is a natural transformation

$$\mathrm{Sym}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$$

induced from the coaugmentation.

Remark A.4.5. Suppose there is a map of comonads

$$\theta : F_{\mathrm{Com}} \to Q_{\mathcal{C}},$$

then θ must be the canonical map obtained by the universal property of $Q_{\mathcal{C}}$. More concretely, if $X \in \mathcal{C}$, then the map $F_{\mathrm{Com}}(X) \to Q_{\mathcal{C}}(X)$ is the unique map (up to contractible ambiguity) that makes the following diagram commute

$$\begin{tikzcd} F_{\mathrm{Com}}(X) \arrow[rr] \arrow[dr] & & X \\ & Q_{\mathcal{C}}(X). \arrow[ur] & \end{tikzcd}$$

Postcomposing with (A.4) then induces a map

$$F_{\mathrm{Com}} \to Q_{\mathcal{C}} \to j \circ Q_{\mathcal{D}} \circ L,$$

which corresponds to

$$\Gamma : L \circ F_{\mathrm{Com}} \circ j \to L \circ Q_{\mathcal{C}} \circ j \to Q_{\mathcal{D}}$$

by the adjunction (3.11).

Applying Theorem A.4.4, we define the comparison functor as follows.

Definition A.4.6. The comparison functor

$$\zeta' : \mathrm{coCAlg}^{\mathrm{dp,nil}}(\mathcal{D}) := \mathrm{LcoMod}_{L \circ F_{\mathrm{Com}}}(\mathcal{D}) \to \mathrm{coCAlg}^{\mathrm{dp}}(\mathcal{D}) := \mathrm{LcoMod}_{Q_{\mathcal{D}}}$$

is the functor corresponding to Γ under the correspondence of Theorem A.4.4.

To construct the comparison functor in the case of tame spectra, we let $\mathcal{C} = \mathrm{Sp}^{\geq r}$ and $\mathcal{D} = \mathrm{Sp}^{\geq r}_{\mathrm{tame}}$. There is a comparison functor on coalgebras in r-connective spectra

$$\mathrm{coCAlg}^{\mathrm{dp,nil}}(\mathrm{Sp}^{\geq r}) \to \mathrm{coCAlg}^{\mathrm{dp}}(\mathrm{Sp}^{\geq r})$$

by Remark A.4.3. Combining with the discussion above, we obtain the comparison functor

$$\zeta : \mathrm{coCAlg}^{\mathrm{dp,nil}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \to \mathrm{coCAlg}^{\mathrm{dp}}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}) \simeq \mathrm{coCAlg}(\mathrm{Sp}^{\geq r}_{\mathrm{tame}}).$$

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EDUCATION

École Polytechnique Fédérale de Lausanne , Switzerland

- Ph.D.in Pure Mathematics — Sep 2017 – Aug 2022
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  - Thesis: Operadic Bimodules and Dwyer-Hess Conjecture Advisor: Prof. Kathryn Hess

Eidgenössische Technische Hochschule Zürich , Switzerland

- M.S. in Pure Mathematics — Sep 2014 – Aug 2017
  - Thesis: Operadic Bimodules and Dwyer-Hess Conjecture
  - Advisor: Prof. Thomas Willwacher

East China University of Science and Technology, China

- B.S. in Applied Mathematics and Statistics — Sep 2010 – Jun 2014
  - Thesis: Approximation Bayesian Computation and its Application
  - Advisor: Prof. Xiyuan Qian

RESEARCH INTEREST

Homotopy Theory, Algebraic models for spaces

RESEARCH VISITS

Utrecht University , The Netherlands

- Conduct research supervised by Prof. Gijs Heuts — Oct 2021 – Nov 2021

TEACHING EXPERIENCES

- Metric spaces and topology — Fall 2017
- Homology and cohomology — Spring 2018
- Metric spaces and topology — Fall 2018
- Advanced linear algebra II — Spring 2019
- Group theory — Fall 2019
- Topology — Spring 2020

CONFERENCES CO-RGANISED

- **The Young Topologists Meeting 2019**, Lausanne, Switzerland — 2019

LANGUAGES

- English: Fluent
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SKILLS

LaTeX, Python, Rust, MATLAB, R