

Hopf algebras and Hopf–Galois extensions in ∞ -categories

Présentée le 24 août 2022

Faculté des sciences de la vie
Laboratoire pour la topologie et les neurosciences
Programme doctoral en mathématiques

pour l'obtention du grade de Docteur ès Sciences

par

Aras ERGUS

Acceptée sur proposition du jury

Prof. C. Hongler, président du jury
Prof. K. Hess Bellwald, directrice de thèse
Prof. A. M. Bohmann, rapporteuse
Prof. J. Beardsley, rapporteur
Prof. Z. Patakfalvi, rapporteur

Abstract

In this thesis, we study interactions between algebraic and coalgebraic structures in ∞ -categories (more precisely, in the quasicategorical model of $(\infty, 1)$ -categories). We define a notion of a *Hopf algebra* H in an \mathbb{E}_2 -monoidal ∞ -category and lift some results about ordinary Hopf algebras, such as the fundamental theorem of Hopf modules, to this setting. We also study *Hopf–Galois extensions* in this context. Given a candidate Hopf–Galois extension, i.e., a map $\varphi: A \rightarrow B$ of H -comodule algebras where H coacts on A trivially, we construct a structured version of the comparison map $B \otimes_A B \rightarrow H \otimes B$ that allows us to compare the category of descent data for φ with a category of “ B -modules equipped with a semilinear coaction of H ”. We provide further insights into the case of commutative (i.e., \mathbb{E}_∞) comodule algebras over a commutative Hopf algebra, for instance a description of the aforementioned category of modules equipped with a semilinear coaction as the limit of a “categorified cobar construction”. Moreover, we provide a simple description of comodules over a space in slice categories of the ∞ -category of spaces, which enables us to realize multiplicative Thom objects as comodule algebras and thus incorporate them into the aforementioned framework.

Keywords: ∞ -category, Hopf algebra, comodule, Hopf–Galois extension, descent, Thom object

Résumé

Dans cette thèse, on étudie les interactions entre les structures algébriques et coalgébriques dans les ∞ -catégories (plus précisément, dans le modèle quasi-catégoriel des $(\infty, 1)$ -catégories). On définit une notion d’une *algèbre de Hopf* H dans une ∞ -catégorie \mathbb{E}_2 -monoïdale et adapte quelques résultats sur les algèbres de Hopf ordinaires, comme le théorème fondamental des modules de Hopf, à ce cadre. On étudie également les *extensions de Hopf–Galois* dans ce contexte. Étant donné un candidat pour une extension de Hopf–Galois, c.-à-d. un morphisme $\varphi: A \rightarrow B$ d’algèbres dans la catégorie des H -comodules, où la coaction de H sur A est triviale, on construit une version plus élaborée du morphisme de comparaison $B \otimes_A B \rightarrow H \otimes B$ qui permet de comparer la catégorie des données de descente pour φ avec une catégorie de « B -modules munies par une coaction semilinéaire de H ». Dans le cas des algèbres commutatives (c.-à-d. \mathbb{E}_∞) dans la catégorie des comodules sur une algèbre de Hopf commutative, on démontre des résultats plus explicites, par exemple, on décrit la catégorie susmentionnée des modules munies par une coaction semilinéaire comme la limite d’une « construction de bar catégorielle ». De plus, on donne une description simple des comodules sur un espace dans les tranches de la ∞ -catégorie des espaces, qui permet de réaliser les objets de Thom multiplicatifs comme algèbres dans une catégorie des comodules et donc les intégrer dans le cadre susmentionné.

Mots-clés: ∞ -catégorie, algèbre de Hopf, comodule, extension de Hopf–Galois, descente, objet de Thom

Acknowledgements

This thesis couldn't have been written without the persistent support and encouragement of my advisor Kathryn Hess. I would like to thank her for her guidance in finding and executing this project.

Moreover, I would like to thank the other members of my PhD Jury; Jonathan Beardsley, Anna Marie Bohmann, Clément Hongler and Zsolt Patakfalvi for evaluating this thesis and making suggestions for improvement.

I am also grateful to my colleagues at the Laboratory for Topology and Neuroscience for accompanying me through this journey, both professionally and on a personal level. I shared many transforming experiences with my fellow PhD students Stefania Ebli, Adélie Garin, Celia Hacker, Samuel Lavenir, Kelly Maggs, Olivia Monjon, Lyne Moser and Haoqing Wu. I learned a lot about math (and life) from Jérôme Scherer and the postdocs interested in “pure” topology; Daniela Egas Santander, Hadrian Heine, Jens Jakob Kjaer, Inbar Klang, Anibal Maximiliano Medina-Mardones, Nima Rasekh, Beren Sanders and Kürşat Sözer. I also enjoyed spending time with the postdocs from the “applied side”; Nicolas Berkouk, Darrick Lee, Nicolas Ninin, Raphael Reinauer, Martina Scolamiero, Gard Spreemann and Guillaume Tauzin. Maroussia Schaffner Portillo made sure that the administrative aspects went smoothly.

Several interactions were especially fruitful for this thesis. Input from Hadrian Heine was crucial for the results of Subsection 3.2 and Appendix A. I would also like to thank Nima Rasekh for listening to my ramblings for hours and answering my questions. Moreover, I am indebted to Malte Leip for giving me access to drafts of his thesis, answering my related (and unrelated) questions, and providing feedback on drafts of this thesis. Conversations with many other mathematicians, most notably Jonathan Beardsley and Maximilien Péroux have also left their marks on this thesis.

I would also like to thank my friends who helped me cope with the challenges of doing a PhD; especially the community of QED e.V. for providing an escape from the daily struggles, Malte for chats about anything and everything, and Yuşa for all the refreshing calls.

Finally, I would like to thank my parents and my brother for their enduring patience and support throughout the years.

Contents

Abstract & Résumé	3
Acknowledgements	5
Introduction	9
1. Preliminaries	13
1.1. (Co)algebras and (co)modules	13
1.2. Bialgebras	22
1.3. (Co)monads	28
1.4. Linear functors	37
2. Comodules over spaces	41
3. Hopf–Galois extensions	49
3.1. Hopf–Galois contexts	49
3.2. The structured shear map	53
4. Hopf algebras	57
4.1. The antipode	57
4.2. Hopf modules and descent	60
4.3. Comodules over dualizable coalgebras	63
5. Hopf–Galois extensions of commutative algebras	71
5.1. Overview of cosimplicial models	71
5.2. Relative Hopf–Galois contexts	80
5.3. Another structured shear map	83
5.4. Semilinear coactions of a commutative Hopf algebra	88
5.5. Tensors with spaces	100
A. Algebras in coalgebras versus coalgebras in algebras	103
B. Lifting adjunctions	113
References	119
Curriculum vitae	123

Introduction

In algebraic topology, one studies spaces by associating algebraic invariants such as homology groups to them and reducing topological questions to algebraic questions about these invariants. These invariants often come equipped with several algebraic and coalgebraic structures whose interplay can be very revealing.

For example, consider a space X equipped with a multiplication $\mu: X \times X \rightarrow X$ that is associative and unital up to homotopy. Then the homology $H_*(X; k)$ of X with coefficients in a field k admits a k -algebra structure given by the *Pontryagin product*, which is defined as the composite

$$H_*(X; k) \otimes_k H_*(X; k) \cong H_*(X \times X; k) \xrightarrow{\mu_*} H_*(X; k).$$

Moreover, $H_*(X; k)$ (in fact, the homology of any space) also admits a k -coalgebra structure, i.e., a comultiplication

$$\Delta: H_*(X; k) \xrightarrow{(\text{Id}_X, \text{Id}_X)_*} H_*(X \times X; k) \cong H_*(X; k) \otimes_k H_*(X; k)$$

and a counit

$$\epsilon: H_*(X; k) \xrightarrow{(X \rightarrow \{*\})_*} H_*(\{*\}; k) \cong k$$

which satisfy coassociativity and counitality conditions dual to the associativity and unitality of algebras. These two structures are compatible with each other in the sense that Δ and ϵ are homomorphisms of algebras (or equivalently, the Pontryagin product and the associated unit map are maps of coalgebras) and thus define a *bialgebra*. The interactions between these operations pose restrictions on the structure of the (co)homology of X , whose study goes back to [Hop41].

One of the main insights of homotopy theory is that one should think of algebraic structures of invariants of spaces as truncations of “homotopy coherent” algebraic structures in a higher category. For instance, many spaces such as loop spaces admit not only a multiplication that is associative and unital up to homotopy, but in fact an algebra structure over the little intervals operad \mathbb{E}_1 , which encodes homotopies between associativity and unitality homotopies, homotopies between those, and so on, whose systematic analysis goes back to [Sta63]. The aforementioned algebra structure on the homology is then obtained by transferring the \mathbb{E}_1 -algebra structure of X along the functor $H_*(-; k)$, under which homotopies between maps become equalities.

In this thesis, we study analogues of the aforementioned interplay between algebraic and coalgebraic structures in the higher categorical setting. An important class of examples that can be viewed through this lens is *Thom spectra*, whose study goes back to [Tho54, Chapitre IV] as a means to classify certain classes of manifolds up to bordism. Consider an \mathbb{E}_1 -map $f: X \rightarrow \text{BGL}_1(\mathbb{S})$ into the classifying space of the automorphisms of the sphere spectrum and its Thom spectrum $M(f)$. We can view the space X as a bialgebra in the (higher) category of spaces and thus its suspension spectrum $\Sigma_+^\infty X$ as a bialgebra in the (higher) category of spectra. The Thom diagonal $M(f) \rightarrow (\Sigma_+^\infty X) \otimes M(f)$ defines a coaction of $\Sigma_+^\infty X$ on $M(f)$, which is also a map of \mathbb{E}_1 -ring spectra.

One of the main concepts we study here is that of a *Hopf–Galois extension*, which is a generalization of the concept of a Galois extension, as we explain now. Let $K \subseteq L$ be fields and G a finite group acting on L via K -linear field automorphisms. Then L is a G -Galois extension of K if and only if the following hold (cf. [Gre92, Section 0.1]).

- K agrees with the fixed field L^G .
- The map $\gamma: L \otimes_K L \rightarrow \prod_{g \in G} L$ given by $\gamma(l_1 \otimes l_2) = ((g.l_1) \cdot l_2)_{g \in G}$ is an isomorphism.

These conditions can be described in terms of the group algebra $\mathbb{Z}[G]$, or rather its dual $\mathbb{Z}[G]^\vee$, as follows. The group algebra is in fact a bialgebra whose comultiplication is given by $\Delta(g) = g \otimes g$. This bialgebra structure induces a dual bialgebra structure on $\mathbb{Z}[G]^\vee$ such that the action of G on L can be transposed to a coaction $\rho: L \rightarrow \mathbb{Z}[G]^\vee \otimes L$ given by $\rho(l) = \sum_{g \in G} g^\vee \otimes (g.l)$, where g^\vee is the dual basis element corresponding to the basis element $g \in \mathbb{Z}[G]$. Now the fixed point field L^G can be alternatively described as the *primitives*

$$\text{Prim}_{\mathbb{Z}[G]^\vee}(L) := \left\{ l \in L : \rho(l) = 1_{\mathbb{Z}[G]^\vee} \otimes l \left(= \sum_{g \in G} g^\vee \otimes l \right) \right\}$$

of this coaction. Moreover, there is an isomorphism $\prod_{g \in G} L \cong \mathbb{Z}[G]^\vee \otimes L$ under which $\gamma(l_1 \otimes l_2)$ can be identified with

$$\sum_{g \in G} g^\vee \otimes ((g.l_1) \cdot l_2) = \rho(l_1) \cdot (1 \otimes l_2).$$

Motivated by this description, one can define “Galois extensions” over an arbitrary bialgebra H as follows. Given a ring map $\varphi: A \rightarrow B$ and a coaction $\rho: B \rightarrow H \otimes B$ that is a ring map and such that φ factors through $\text{Prim}_H(B)$, φ is called an *H -Hopf–Galois extension* if the following maps are isomorphisms.

- The restriction $A \rightarrow \text{Prim}_H(B)$
- The map $\gamma: B \otimes_A B \rightarrow H \otimes B$ given by $\gamma(b_1 \otimes b_2) = \rho(b_1) \cdot (1 \otimes b_2)$

The definition of Hopf–Galois extensions in this form in the algebraic setting goes back to [KT81]. In the homotopical setting, Galois and Hopf–Galois extensions of ring spectra were introduced in [Rog08, Part I]. Variations of the notion of a Hopf–Galois extensions in the context of model categories were also studied in [Rot09], [Hes09], [Kar14] and [BH18].

While we draw a lot of inspiration from these works in point-set models, we use the framework of quasicategories (which we refer to simply as ∞ -categories) developed in [Lur09] and [Lur17]. Using non-strict models for higher categories appears to be necessary in this context because, as shown in [PS18], all coalgebras in common point-set models of spectra are cocommutative, whereas one would expect to have non-cocommutative examples, such as dual group algebras of non-abelian groups. In the ∞ -categorical setting, bialgebras and comodule algebras over them were studied in [Bea16] and [Bea21], results of which we use and build upon.

Structure of this thesis

In Section 1, we review certain aspects of algebraic and coalgebraic structures in ∞ -categories that we need later.

In Section 2, we consider coalgebraic structures in the ∞ -category \mathcal{S} of spaces. The main result of this section (Proposition 2.0.6) states that a coaction of a space X on an object $f: Y \rightarrow T$ of the slice category $\mathcal{S}_{/T}$ is uniquely determined by a coaction of X on Y , which allows us to view every such f as a comodule over its source Y (cf. Construction 2.0.11). A monoidal version of this construction is used later in Example 3.1.3 to realize multiplicative Thom objects as comodule algebras.

In Section 3, we discuss a precise definition of Hopf–Galois extensions in ∞ -categories and relate them to questions of descent. The crucial ingredient for this discussion is a lift of the map $B \otimes_A B \rightarrow H \otimes B$ sketched above to a map of comonads on the ∞ -category of B -modules (cf. Corollary 3.2.8).

In Section 4, we define a notion of Hopf algebras in ∞ -categories and lift some results about Hopf algebras in the 1-categorical setting to our setup. In Corollary 4.2.9, we obtain an equivalence between the base category and an ∞ -category of objects equipped with an action and a coaction of a (certain type of) Hopf algebra that are compatible in an appropriate sense. In Proposition 4.3.10, we identify the ∞ -category of comodules over a dualizable coalgebra with the ∞ -category of modules over its dual.

In Section 5, we discuss the case where all the algebra structures in question are commutative, which allows us to work with a simplified description of coalgebraic structures. In Proposition 5.4.24, we show that for a commutative comodule algebra B over a commutative Hopf algebra H , the ∞ -category of B -modules in the ∞ -category of H -comodules is equivalent to the limit of a “categorified cobar construction” $\mathrm{LComod}_{\Omega_H^\bullet(B)}(\mathcal{C})$. In Construction 5.5.6 we extend the map $B \otimes_A B \rightarrow B \otimes H$ to more general “multiplicative tensors with spaces”, which in particular yields an equivalence $\mathrm{THH}^A(B) \simeq (\mathbf{1}_{\mathcal{C}} \otimes_H \mathbf{1}_{\mathcal{C}}) \otimes B$ for an H -Hopf–Galois extension $A \rightarrow B$ (cf. Example 5.5.9).

There are two appendices dedicated to rather long proofs that do not fit well into other sections. In Appendix A, we show that defining bialgebras as “algebras in a monoidal ∞ -category of coalgebras” and “coalgebras in a monoidal ∞ -category of algebras” yield equivalent results (cf. Corollary A.0.17). In Appendix B, we show that an adjunction $F: \mathcal{D} \rightleftarrows \mathcal{E} : G$ of lax monoidal functors can be lifted to an adjunction between ∞ -categories of R - and $F(R)$ -modules for every algebra R in \mathcal{D} (cf. Proposition B.0.6).

General conventions

As mentioned above, we work with quasicategories, which we refer to as ∞ -categories. We freely employ the terminology and some results from [Lur09]. In particular, when we speak of (co)limits, we mean them to be invariant under homotopy equivalences, hence *homotopy* (co)limits in the classical terminology.

Our notation mostly agrees with that of [Lur09] and [Lur17]. One significant difference is that we implicitly identify a 1-category with its nerve in simplicial sets. For mapping objects, we use the following notation.

$\mathrm{Hom}_{\mathcal{C}}(-, -)$	morphism set in the 1-category \mathcal{C}
$\mathrm{Map}_{\mathcal{C}}(-, -)$	mapping space in the ∞ -category \mathcal{C}
$\mathrm{Fun}(-, -)$	functor (∞ -)category

For the simplex category Δ and its variations, we employ the following conventions.

Category	... of	... with objects	... describing
Δ_+	finite ordinals	$[n] = \{0, \dots, n\},$ $n \in \mathbb{Z}_{\geq -1}$	(co)augmented (co)simplicial objects
Δ	<i>non-empty</i> finite ordinals	$[n], n \in \mathbb{N} = \mathbb{Z}_{\geq 0}$	(co)simplicial objects
Δ_{\perp}	finite ordinals with a distinguished <i>bottom</i> element	$[n]_{\perp} = \{\perp, 0, \dots, n\},$ $n \in \mathbb{Z}_{\geq -1}$	(left) split (co)simplicial objects
Δ_{\top}	finite ordinals with a distinguished <i>top</i> element	$[n]_{\top} = \{0, \dots, n, \top\},$ $n \in \mathbb{Z}_{\geq -1}$	(right) split (co)simplicial objects
Fin_*	pointed finite sets	$\langle n \rangle = \{*, 1, \dots, n\},$ $n \in \mathbb{N} = \mathbb{Z}_{\geq 0}$	N/A

Recall that Δ is generated by coface maps $\delta^i: [n] \rightarrow [n+1]$ and codegeneracy maps $\sigma^j: [n+1] \rightarrow [n]$ for varying $n \in \mathbb{N}$, $i \in \{0, \dots, n+1\}$ and $j \in \{0, \dots, n\}$. Moreover, Δ_+ can be obtained from Δ by adding an extra coface map $\delta^0: [-1] \rightarrow [0]$ subject to certain relations. We sometimes abbreviate the image of δ^i under a functor as d^i and the image of σ^j as s^j . Besides the full inclusion $\Delta \subseteq \Delta_+$, we have non-full inclusions

$$\Delta_+ \xrightarrow{\text{add disjoint } \perp} \Delta_{\perp} \xrightarrow{\text{forget distinction of } \perp} \Delta$$

and

$$\Delta_+ \xrightarrow{\text{add disjoint } \top} \Delta_{\top} \xrightarrow{\text{forget distinction of } \top} \Delta,$$

which we use, for instance, to restrict split (co)simplicial objects to (co)augmented (co)simplicial objects.

Unless more precision is necessary or preferable, we employ the usual abuse of notation of using “underlying objects” to refer to more structured objects. Moreover, we use the following notation for certain maps and functors induced by (co)algebraic structures, often without explicitly introducing them in the specific case in question.

$\mu_?$	multiplication of an algebra	$\Delta_?$	comultiplication of a coalgebra
$\eta_?$	unit map of an algebra	$\epsilon_?$	counit map of a coalgebra
$\alpha_?$	action map of a module	$\rho_?$	coaction map of a comodule
$U_?$	forgetful functor of a module category	$V_?$	forgetful functor of a comodule category
$F_?$	free module functor	$C_?$	cofree comodule functor

1. Preliminaries

In this section, we discuss certain properties of ∞ -categories of (co)algebras and (co)modules that we will need later. In addition to stating precise mathematical results, we try to establish some intuition about these categories. Readers familiar with the algebraic theory of ∞ -operads of [Lur17] and its extension to coalgebras and bialgebras as discussed in, for instance, [Pér20], [Bea21] and [Lei22] could skip this section and look up specific statements later as needed.

1.1. (Co)algebras and (co)modules

In [Lur17, Definition 2.1.1.10], ∞ -operads (which are rather a generalization of *colored symmetric operads*) are defined as functors $p: \mathcal{O}^\otimes \rightarrow \text{Fin}_*$ satisfying certain properties. We will not unpack the full definition here, but the following description of the fibers of p will be useful, especially in Appendix A.

Recall that a map $f: \langle m \rangle \rightarrow \langle n \rangle$ in Fin_* is called *inert* if it restricts to a bijection from a subset of $\{1, \dots, m\} \subseteq \langle m \rangle$ to $\{1, \dots, n\} \subseteq \langle n \rangle$ and maps the rest of $\langle m \rangle$ to the base point. The definition of an ∞ -operad guarantees that

- inert maps admit p -cocartesian lifts, which are called *inert maps in \mathcal{O}^\otimes* , and
- for every partition $n_1 + \dots + n_k = n \in \mathbb{N}$, the functor $\mathcal{O}_{\langle n \rangle}^\otimes \rightarrow \prod_{i=1}^k \mathcal{O}_{\langle n_i \rangle}^\otimes$ induced by cocartesian lifts of the inert maps $(f_i: \langle n \rangle \rightarrow \langle n_i \rangle)_{1 \leq i \leq k}$, where f_i maps the i -th part of the partition bijectively to $\{1, \dots, n_i\}$ and collapses the rest of $\langle n \rangle$ to the base point, is an equivalence.

Given $v_i \in \mathcal{O}_{\langle n_i \rangle}^\otimes$ for $1 \leq i \leq k$, the essentially unique object of $\mathcal{O}_{\langle n \rangle}^\otimes$ that corresponds to $(v_1, \dots, v_k) \in \prod_{i=1}^k \mathcal{O}_{\langle n_i \rangle}^\otimes$ under the equivalence above is denoted by $v_1 \oplus \dots \oplus v_k$ (cf. [Lur17, Remark 2.1.1.15]). Note that it comes equipped with inert morphisms $\pi_{v_i}: v_1 \oplus \dots \oplus v_k \rightarrow v_i$ for $1 \leq i \leq k$.

We think of $\mathcal{O} := \mathcal{O}_{\langle 1 \rangle}^\otimes$ as the “underlying ∞ -category” or “ ∞ -category of colors” of \mathcal{O}^\otimes . Pushing the colored operad analogy further, given $\mathfrak{o}_1, \dots, \mathfrak{o}_n, \mathfrak{o}' \in \mathcal{O}$, multimorphisms from $(\mathfrak{o}_1, \dots, \mathfrak{o}_n)$ to \mathfrak{o}' are encoded by a certain subspace of $\text{Map}_{\mathcal{O}^\otimes}(\mathfrak{o}_1 \oplus \dots \oplus \mathfrak{o}_n, \mathfrak{o}')$ (cf. [Lur17, Notation 2.1.1.16]).

In this framework, monoidal categories and algebras are defined as follows.

- [Lur17, Definition 2.1.2.13] Given an ∞ -operad \mathcal{O} , an \mathcal{O} -monoidal ∞ -category is a cocartesian fibration $\mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes$ such that the composite $\mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes \rightarrow \text{Fin}_*$ exhibits \mathcal{C}^\otimes as an ∞ -operad. In this case, we also informally speak of an \mathcal{O} -monoidal structure on $\mathcal{C} := \mathcal{C}_{\langle 1 \rangle}^\otimes$. The idea is that cocartesian lifts of a multimorphism $\phi: \mathfrak{o}_1 \oplus \dots \oplus \mathfrak{o}_n \rightarrow \mathfrak{o}'$ define an associated operation $\otimes^\phi: \mathcal{C}_{\mathfrak{o}_1 \oplus \dots \oplus \mathfrak{o}_n}^\otimes \simeq \prod_{i=1}^n \mathcal{C}_{\mathfrak{o}_i}^\otimes \rightarrow \mathcal{C}_{\mathfrak{o}'}$.
- [Lur17, Definition 2.1.2.7] A map $\mathcal{Q}^\otimes \rightarrow \mathcal{O}^\otimes$ of ∞ -operads is a map over Fin_* that preserves inert edges. The idea is that such a map is compatible with the equivalences $(-)_\langle n \rangle \simeq ((-)_\langle 1 \rangle)^n$ and thus induces maps on spaces of multimorphisms.

- [Lur17, Definition 2.1.3.1] Given a map $f: \mathcal{Q}^\otimes \rightarrow \mathcal{O}^\otimes$ of ∞ -operads and an \mathcal{O} -monoidal ∞ -category $\mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes$, the ∞ -category $\text{Alg}_{\mathcal{Q}/\mathcal{O}}(\mathcal{C})$ of \mathcal{Q} -algebras in \mathcal{C} is defined as the ∞ -category of ∞ -operad maps $a: \mathcal{Q}^\otimes \rightarrow \mathcal{C}^\otimes$ over \mathcal{O}^\otimes . In this case, the conditions guarantee that multimorphism $\phi: \mathbf{q}_1 \oplus \dots \oplus \mathbf{q}_n \rightarrow \mathbf{q}'$ of \mathcal{Q} induces a map $\otimes^{f(\phi)}(a(\mathbf{q}_1), \dots, a(\mathbf{q}_n)) \rightarrow a(\mathbf{q}')$ encoding an associated operation.

When $\mathcal{Q}^\otimes \rightarrow \mathcal{O}^\otimes$ is the identity map of \mathcal{O}^\otimes , we also write $\text{Alg}_{/\mathcal{O}}(\mathcal{C})$ instead of $\text{Alg}_{\mathcal{O}/\mathcal{O}}(\mathcal{C})$. Note that the pullback $\mathcal{C}^\otimes \times_{\mathcal{O}^\otimes} \mathcal{Q}^\otimes \rightarrow \mathcal{Q}^\otimes$ exhibits $\mathcal{C} \times_{\mathcal{O}} \mathcal{Q}$ as a \mathcal{Q} -monoidal ∞ -category such that $\text{Alg}_{\mathcal{Q}/\mathcal{O}}(\mathcal{C}) \simeq \text{Alg}_{/\mathcal{Q}}(\mathcal{C} \times_{\mathcal{O}} \mathcal{Q})$.

For a \mathcal{Q} -coalgebra, we would like to have “cooperations” $a(\mathbf{q}') \rightarrow \otimes^{f(\phi)}(a(\mathbf{q}_1), \dots, a(\mathbf{q}_n))$ instead. We can achieve this by considering algebras in an appropriate “opposite \mathcal{O} -monoidal ∞ -category”.

Construction 1.1.1 (cf. [Lur17, Remark 2.4.2.7]). Let \mathcal{O} be an ∞ -operad and $p: \mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes$ an \mathcal{O} -monoidal ∞ -category.

Recall that under the straightening-unstraightening equivalence of [Lur09, Theorem 3.2.0.1], $p: \mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes$ corresponds to a functor $\hat{p}: \mathcal{O}^\otimes \rightarrow \text{Cat}_\infty$ to the ∞ -category of ∞ -categories. Composing \hat{p} with $(-)^{\text{op}}$ and unstraightening back, we obtain a fiberwise opposite cocartesian fibration $p^{\text{fop}}: (\mathcal{C}^\otimes)^{\text{fop}} \rightarrow \mathcal{O}^\otimes$ satisfying $(\mathcal{C}^\otimes)_v^{\text{fop}} \simeq (\mathcal{C}_v^\otimes)^{\text{op}}$ for all $v \in \mathcal{O}^\otimes$.¹

Using that $(-)^{\text{op}}$ commutes with products and the criterion of [Lur17, Proposition 2.1.2.12], we see that p^{fop} exhibits the fiberwise opposite \mathcal{C}^{fop} of $p_{\langle 1 \rangle}: \mathcal{C} = \mathcal{C}_{\langle 1 \rangle}^\otimes \rightarrow \mathcal{O}_{\langle 1 \rangle}^\otimes = \mathcal{O}$ as an \mathcal{O} -monoidal ∞ -category, which we call the (fiberwise) opposite \mathcal{O} -monoidal ∞ -category of \mathcal{C} .

Note that when $\mathcal{O} = \mathcal{O}_{\langle 1 \rangle}^\otimes$ is a discrete ∞ -category, \mathcal{C}^{fop} can be identified with \mathcal{C}^{op} , so in that case, we obtain an \mathcal{O} -monoidal structure on \mathcal{C}^{op} . This will in fact be the case for all the concrete operads we consider.

Definition 1.1.2. Let $\mathcal{Q}^\otimes \rightarrow \mathcal{O}^\otimes$ be a map of ∞ -operads and \mathcal{C} an \mathcal{O} -monoidal ∞ -category. Using the opposite \mathcal{O} -monoidal structure of Construction 1.1.1, we define the ∞ -category of \mathcal{Q} -coalgebras in \mathcal{C} as

$$\text{Coalg}_{\mathcal{Q}/\mathcal{O}}(\mathcal{C}) := \text{Alg}_{\mathcal{Q}/\mathcal{O}}(\mathcal{C}^{\text{fop}})^{\text{op}}$$

and abbreviate $\text{Coalg}_{\mathcal{O}/\mathcal{O}}$ as $\text{Coalg}_{/\mathcal{O}}$.

Convention 1.1.3. Defining coalgebraic structures as algebraic structures in an opposite category allows us to use formally dualize the theory of algebras to coalgebras to a certain extent.² When working with coalgebraic structures, we will often refer to algebraic results and expect the reader to appropriately dualize them. This in particular applies to the theory of monads developed in [Lur17, Section 4.7], which we apply extensively to comonads (cf. Subsection 1.3).

¹There is also a more direct construction of $(-)^{\text{fop}}$ carried out in [BGN18], which we will make use of in particular in Appendix A.

²Note, however, that in many monoidal ∞ -categories we are interested in (such as that of spectra), the monoidal product commutes with colimits but not necessarily with limits, which breaks the symmetry between the algebraic and the coalgebraic theory (cf. Remark 1.1.14).

Next, we discuss different notions of functors between \mathcal{O} -monoidal ∞ -categories.

Definition 1.1.4. Let \mathcal{O} be an ∞ -operad, and $p: \mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes$ and $q: \mathcal{D}^\otimes \rightarrow \mathcal{O}^\otimes$ \mathcal{O} -monoidal ∞ -categories.

A *lax \mathcal{O} -monoidal functor* $F^\otimes: \mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ is a map of ∞ -operads over \mathcal{O}^\otimes (i.e., a functor over \mathcal{O}^\otimes that preserves inert morphisms). We view $F^\otimes: \mathcal{C}^\otimes_{\langle 1 \rangle} = \mathcal{C} \rightarrow \mathcal{D} = \mathcal{D}^\otimes_{\langle 1 \rangle}$ as the underlying functor of F^\otimes .

An *oplax \mathcal{O} -monoidal functor* is a lax monoidal functor $\bar{F}^\otimes: (\mathcal{C}^\otimes)^{\text{fop}} \rightarrow (\mathcal{D}^\otimes)^{\text{fop}}$ between the opposite \mathcal{O} -monoidal ∞ -categories. If $\bar{F}^\otimes: (\mathcal{C}^\otimes)^{\text{fop}}_{\langle 1 \rangle} = \mathcal{C}^{\text{fop}} \rightarrow \mathcal{D}^{\text{fop}} = (\mathcal{D}^\otimes)^{\text{fop}}_{\langle 1 \rangle}$ maps p^{fop} -cocartesian morphisms to q^{fop} -cocartesian morphisms, then it induces a functor $(\bar{F}^\otimes)^{\text{fop}}: (\mathcal{C}^{\text{fop}})^{\text{fop}} \simeq \mathcal{C} \rightarrow \mathcal{D} \simeq (\mathcal{D}^{\text{fop}})^{\text{fop}}$, which we view as the underlying functor of \bar{F}^\otimes (viewed as an oplax \mathcal{O} -monoidal functor). Note that if $\mathcal{O}^\otimes_{\langle 1 \rangle}$ is discrete, then $\bar{F}^\otimes_{\langle 1 \rangle}$ always preserves cocartesian morphisms because cocartesian morphisms with respect to a cocartesian fibration over a discrete ∞ -category are given by equivalences.

A *strongly \mathcal{O} -monoidal functor* $F^\otimes: \mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ is a functor over \mathcal{O}^\otimes that maps *all* p -cocartesian morphisms to q -cocartesian morphisms. We denote the ∞ -category of such functors by $\text{Fun}_{\mathcal{O}}^\otimes(\mathcal{C}, \mathcal{D})$ (cf. [Lur17, Definition 2.1.3.7]). Note that a strongly \mathcal{O} -monoidal functor is in particular lax \mathcal{O} -monoidal. Moreover, a strongly \mathcal{O} -monoidal functor F^\otimes induces a strongly \mathcal{O} -monoidal functor $(F^\otimes)^{\text{fop}}: (\mathcal{C}^\otimes)^{\text{fop}} \rightarrow (\mathcal{D}^\otimes)^{\text{fop}}$ between the opposite \mathcal{O} -monoidal ∞ -categories, implying in particular that its underlying functor can be viewed also as the underlying functor of an oplax monoidal functor.

We will often not notationally distinguish between a lax, oplax or strongly \mathcal{O} -monoidal functor and its underlying functor. In particular, we will speak of a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between the underlying ∞ -categories being lax, oplax or strongly \mathcal{O} -monoidal.

In order to relate the notions of Definition 1.1.4 to 1-categorical ones, consider a multimorphism $\phi: \mathfrak{o}_1 \oplus \mathfrak{o}_2 \rightarrow \mathfrak{o}$ of \mathcal{O}^\otimes and $c_1 \in \mathcal{C}_{\mathfrak{o}_1}^\otimes$, $c_2 \in \mathcal{C}_{\mathfrak{o}_2}^\otimes$. Let $F^\otimes: \mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ be a functor over Fin_* preserving inert morphisms. Then we obtain a commutative diagram

$$\begin{array}{ccc} F^\otimes(c_1) \oplus F^\otimes(c_2) & \xrightarrow{\quad} & \otimes_{\mathcal{D}}^\phi(F(c_1), F(c_2)) \\ \wr & \nearrow & \downarrow u \\ F^\otimes(c_1 \oplus c_2) & \xrightarrow{\quad} & F^\otimes(\otimes_{\mathcal{C}}^\phi(c_1, c_2)) \end{array},$$

where the upper horizontal arrow is a q -cocartesian lift of ϕ , the lower horizontal arrow is the image under F^\otimes of a p -cocartesian lift of ϕ , and u is induced by the universal property of the diagonal composite as a q -cocartesian arrow. This induced map is analogous to the comparison transformation $F(-) \otimes F(-) \rightarrow F(- \otimes -)$ of lax monoidal functors in the 1-categorical setting. Moreover, if F^\otimes maps all p -cocartesian morphisms to q -cocartesian morphisms, then the lower map is also q -cocartesian and hence u an equivalence, which is the ∞ -categorical analogue of the fact that “strongly monoidal functors commute with tensor products”.

Remark 1.1.5. Let $\mathcal{Q}^\otimes \rightarrow \mathcal{O}^\otimes$ be a map of ∞ -operads, and $\mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes$ and $\mathcal{D}^\otimes \rightarrow \mathcal{O}^\otimes$ \mathcal{O} -monoidal ∞ -categories. Then a lax \mathcal{O} -monoidal functor $F^\otimes: \mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$ induces a

functor $\text{Alg}_{\mathcal{Q}/\mathcal{O}}(F^\otimes): \text{Alg}_{\mathcal{Q}/\mathcal{O}}(\mathcal{C}) \rightarrow \text{Alg}_{\mathcal{Q}/\mathcal{O}}(\mathcal{D})$ via postcomposition because a composite of maps preserving inert morphisms also preserves inert morphisms. Dually, an oplax \mathcal{O} -monoidal functor induces a functor between coalgebra categories. We will often not notationally distinguish such induced functors from their underlying functors.

Recall that in the 1-categorical setting, a lax monoidal structure on a right adjoint functor G gives rise to an oplax monoidal structure on its left adjoint F and vice versa. For example, a natural transformation $\beta: G(-) \otimes G(-) \rightarrow G(- \otimes -)$ gives rise to a natural transformation

$$F(- \otimes -) \xrightarrow{\text{unit}} F(G(F(-)) \otimes G(F(-))) \xrightarrow{\beta} F(G(F(-) \otimes F(-))) \xrightarrow{\text{counit}} F(-) \otimes F(-).$$

The following is an analogue of this phenomenon in the ∞ -categorical setting.

Fact 1.1.6 ([HHLN21, Thorem 3.4.7]). *Let \mathcal{O} be an ∞ -operad, $\mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes$ and $\mathcal{D}^\otimes \rightarrow \mathcal{O}^\otimes$ \mathcal{O} -monoidal ∞ -categories, and $G^\otimes: \mathcal{D}^\otimes \rightarrow \mathcal{C}^\otimes$ a lax \mathcal{O} -monoidal functor. Assume that for all $\mathfrak{o} \in \mathcal{O}$, $G^\otimes_{\mathfrak{o}}: \mathcal{D}^\otimes_{\mathfrak{o}} \rightarrow \mathcal{C}^\otimes_{\mathfrak{o}}$ admits a left adjoint $F_{\mathfrak{o}}$. Then there exists an oplax \mathcal{O} -monoidal functor $\overline{F}^\otimes: (\mathcal{C}^\otimes)^{\text{fop}} \rightarrow (\mathcal{D}^\otimes)^{\text{fop}}$ such that the fiber of \overline{F}^\otimes over $\mathfrak{o} \in \mathcal{O}$ is given by $(F_{\mathfrak{o}}^\otimes)^{\text{op}}: (\mathcal{C}^\otimes_{\mathfrak{o}})^{\text{op}} \rightarrow (\mathcal{D}^\otimes_{\mathfrak{o}})^{\text{op}}$.*

In particular, if $\mathcal{O} = \mathcal{O}_{\langle 1 \rangle}^\otimes$ is discrete, the left adjoint of the underlying functor of a lax monoidal functor admits an oplax monoidal structure. Dually, the right adjoint of the underlying functor of an oplax \mathcal{O} -monoidal functor can be extended to a lax \mathcal{O} -monoidal functor.

Our most common application of Fact 1.1.6 will be constructing a lax \mathcal{O} -monoidal structure on the right adjoint of the underlying functor of a strongly \mathcal{O} -monoidal functor. Such a pair of functors induces further adjunctions of interest, to which Appendix B is dedicated.

We will be mainly interested in (co)algebras over the ∞ -categorical version \mathbb{E}_n^\otimes of the little n -cubes operad for some $n \in \mathbb{N}$ (see [Lur17, Definition 5.1.0.2]), whose main features can be summarized as follows.

- It has a single color $*$, so we can think of multimorphisms $*^{\oplus m} \rightarrow *$ as m -ary operations, which are given by rectilinear configurations of m n -dimensional open cubes in an n -dimensional open cube.
- Its composition maps are given by grafting such configurations.
- $(\mathbb{E}_n^\otimes)_{\langle 1 \rangle}$ is contractible because the endomorphism space of its single object, i.e., the space of rectilinear configurations of an n -cube in an n -cube, is contractible.
- For $n \geq 1$ and $m > 1$, m -ary operations of \mathbb{E}_n are generated by a single binary operation (which is encoded by a disjoint embedding of 2 n -cubes in an n -cube) under homotopies, permutations of inputs and grafting.

Notation 1.1.7. Let $n \in \mathbb{N}$. When working with an \mathbb{E}_n -monoidal ∞ -category \mathcal{C} , we will denote its tensor product (induced by the generating binary operation of \mathbb{E}_n) by $\otimes_{\mathcal{C}}$ (or simply \otimes if there is no room for confusion) and its unit (induced by the unique 0-ary operation of \mathbb{E}_n) by $\mathbf{1}_{\mathcal{C}}$.

Convention 1.1.8. In light of [Lur17, Example 5.1.0.7], \mathbb{E}_1^\otimes is one of the many incarnations of the operad that encodes associative algebras. We will freely use results proven for different (but equivalent) manifestations of associative algebras, including the planar version of [Lur17, Section 4.1.3].

Moreover, when we speak of “algebras”, “coalgebras” etc. without further qualifiers, we will mean algebras, coalgebras etc. *over* \mathbb{E}_1 . In particular, we will employ the following simplifications.

- By a “monoidal ∞ -category” we will mean an \mathbb{E}_1 -monoidal ∞ -category.
- When a map $\mathbb{E}_1^\otimes \rightarrow \mathcal{O}^\otimes$ of operads is clear from context, we will simply write Alg instead of $\text{Alg}_{\mathbb{E}_1/\mathcal{O}}$, and similarly for Coalg .

More generally, different \mathbb{E}_n ’s can be thought of encoding different levels of commutativity of the generating binary operation. For instance, as any two embeddings of two squares in a square are homotopic via embeddings, the binary operation of \mathbb{E}_2 is commutative up to homotopy, which in particular yields a *braiding* $(-) \otimes (?) \simeq (?) \otimes (-)$ for every \mathbb{E}_2 -monoidal ∞ -category. \mathbb{E}_3 -monoidality is sufficient to obtain a *symmetric* monoidal structure on the homotopy category of an ∞ -category (cf. [Lur17, Corollary 5.1.1.7]), but on the level of ∞ -categories, there are higher coherences to consider.

Convention 1.1.9. Consider the maps $\mathbb{E}_0^\otimes \rightarrow \mathbb{E}_1^\otimes \rightarrow \mathbb{E}_2^\otimes \rightarrow \dots$ of ∞ -operads given by taking the product of configurations with an open interval. We will usually make restrictions along these maps implicit, e.g., view an \mathbb{E}_n -monoidal ∞ -category as an \mathbb{E}_m -monoidal ∞ -category for all $m \leq n$.

By [Lur17, Corollary 5.1.1.5], the colimit of this sequence is equivalent to the terminal ∞ -operad Fin_* . When we want to refer to Fin_* as this colimit, i.e., an ∞ -operad encoding a “highly coherent” commutative multiplication, we will use the notation $\mathbb{E}_\infty^\otimes$. Moreover, we will use the following terms for structures over \mathbb{E}_∞ .

- By a “symmetric monoidal ∞ -category” we will mean an \mathbb{E}_∞ -monoidal category.
- By a “commutative algebra” we will mean an \mathbb{E}_∞ -algebra. We will write CAlg instead of $\text{Alg}_{/\mathbb{E}_\infty}$.
- Dually, we will use the term “cocommutative” for \mathbb{E}_∞ -coalgebras.

Next, we informally recall the operad \mathcal{LM} of [Lur17, Section 4.2], which encodes a left action of an algebra on an object.

- [Lur17, Remarks 4.2.1.8 and 4.2.1.10] $\mathcal{LM} = \mathcal{LM}_{(1)}^\otimes$ is a discrete category with two objects \mathfrak{a} and \mathfrak{m} , encoding the algebra and the object it acts on, respectively. There is a map $\mathbb{E}_1^\otimes \rightarrow \mathcal{LM}^\otimes$ of ∞ -operads which, intuitively speaking, maps the universal algebra in \mathbb{E}_1 to the algebra \mathfrak{a} in \mathcal{LM} . We will always mean this map when we speak of *the* map $\mathbb{E}_1^\otimes \rightarrow \mathcal{LM}^\otimes$.

- [Lur17, Remarks 4.2.1.20 and 4.2.1.21] An \mathcal{LM} -monoidal ∞ -category $p: \mathcal{D}^\otimes \rightarrow \mathcal{LM}^\otimes$ encodes a monoidal structure on \mathcal{D}_a given by the pulling back p along the map $\mathbb{E}_1^\otimes \rightarrow \mathcal{LM}^\otimes$, and a *left-tensoring* of \mathcal{D}_m over \mathcal{D}_a , in particular “an action map” $\mathcal{D}_a \times \mathcal{D}_m \rightarrow \mathcal{D}_m$, which is informally also simply denoted by \otimes . When we informally speak of an ∞ -category left-tensored over a monoidal ∞ -category, we will implicitly have an \mathcal{LM} -monoidal category with appropriate fibers in mind.
- [Lur17, Definition 4.2.1.13] Given an \mathcal{LM} -monoidal ∞ -category \mathcal{D} , $\text{Alg}_{/\mathcal{LM}}(\mathcal{D})$ is called *∞ -category of left module objects of \mathcal{D}_m* and denoted by $\text{LMod}^{\mathcal{D}_a}(\mathcal{D}_m)$ or $\text{LMod}(\mathcal{D}_m)$ when there is no room for confusion. Intuitively, it can be thought of as a category of pairs (A, M) where A is an algebra in \mathcal{D}_a acting on an object M in \mathcal{D}_m . The map $\mathbb{E}_1^\otimes \rightarrow \mathcal{LM}^\otimes$ induces a functor $\theta_{\mathcal{D}}: \text{LMod}(\mathcal{D}_m) \rightarrow \text{Alg}(\mathcal{D}_a)$ which “forgets the module”. Its fiber over an algebra $A \in \text{Alg}(\mathcal{D}_a)$ is called the *∞ -category of left A -modules in \mathcal{D}_m* and is denoted by $\text{LMod}_A(\mathcal{D}_m)$. Similarly, evaluation at \mathbf{m} induces a functor $\varsigma_{\mathcal{D}}: \text{LMod}(\mathcal{D}_m) \rightarrow \mathcal{D}_m$ that “forgets the action of the algebra”.
- [Lur17, Remarks 4.2.1.5 and 4.2.1.9] The map $\mathbb{E}_1^\otimes \rightarrow \mathcal{LM}^\otimes$ admits a “retraction” $\mathcal{LM}^\otimes \rightarrow \mathbb{E}_1^\otimes$, which, intuitively speaking, classifies the universal algebra in \mathbb{E}_1 as a left module over itself. Given a monoidal ∞ -category $\mathcal{C}^\otimes \rightarrow \mathbb{E}_1^\otimes$, pulling back along this map yields an \mathcal{LM} -monoidal ∞ -category, which encodes the left action of \mathcal{C} on itself. When we speak of modules in a monoidal category, we implicitly employ this construction.

Similarly, there is an ∞ -operad \mathcal{RM} encoding *right* modules and *right*-tensorings (cf. [Lur17, Variant 4.2.1.36]), and an ∞ -operad \mathcal{BM} encoding *bimodules* (cf. [Lur17, Definition 4.3.1.6]). Provided that the monoidal ∞ -category in question admits geometric realizations and the tensor product commutes with them in each variable, there is a theory of *relative tensor products* developed in [Lur17, Section 4.4], which is similar to the 1-categorical case in the sense that it is associative, gives rise to restriction-extension of scalars adjunctions etc. In particular, ∞ -categories of modules have functoriality properties similar to their 1-categorical analogues, which we record below.

Fact 1.1.10. *Let \mathcal{M} be an ∞ -category left-tensored over a monoidal ∞ -category \mathcal{C} . Then, by [Lur17, Corollary 4.2.4.8] for every algebra R in \mathcal{C} , we have a free-forgetful adjunction $F_R: \mathcal{M} \rightleftarrows \text{LMod}_R(\mathcal{M}) : U_R$ such that $U_R \circ F_R \simeq R \otimes (-)$.*

Moreover the forgetful functor $\theta_{\mathcal{C}}: \text{LMod}(\mathcal{M}) \rightarrow \text{Alg}(\mathcal{C})$ is a cartesian fibration, and a map is $\theta_{\mathcal{C}}$ -cartesian if and only if its underlying map in \mathcal{M} is an equivalence (cf. [Lur17, Corollary 4.2.3.2]). This means that to each map $\psi: R \rightarrow S$ of algebras in \mathcal{C} , one can functorially associate a restriction of scalars functor $\psi^: \text{LMod}_S(\mathcal{M}) \rightarrow \text{LMod}_R(\mathcal{M})$ that does not change the underlying objects in \mathcal{M} .*

If we further assume that \mathcal{C} and \mathcal{M} admit geometric realizations, and that both the tensor product in \mathcal{C} and the action of \mathcal{C} on \mathcal{M} preserve them in each variable, then each such ψ^ admits a left adjoint $\psi_!: \text{LMod}_R(\mathcal{M}) \rightarrow \text{LMod}_S(\mathcal{M})$, called extension of scalars along ψ (cf. [Lur17, Proposition 4.6.2.17]). On the level of underlying objects, $\psi_!$ sends $M \in \text{LMod}_R(\mathcal{M})$ to the relative tensor product $S \otimes_R M$. Moreover, as discussed in*

[Lur17, Lemma 4.5.3.6], this implies that $\theta_{\mathcal{C}}$ is also a cocartesian fibration, i.e., extensions of scalars are also functorial.

Remark 1.1.11. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a lax monoidal functor between monoidal ∞ -categories. Then the functors induced on ∞ -categories of \mathcal{LM} - and \mathbb{E}_1 -algebras discussed in Remark 1.1.5 fit into a commutative diagram

$$\begin{array}{ccc} \mathrm{LMod}(\mathcal{C}) & \xrightarrow{\mathrm{LMod}(F)} & \mathrm{LMod}(\mathcal{D}) \\ \theta_{\mathcal{C}} \downarrow & & \downarrow \theta_{\mathcal{D}} \\ \mathrm{Alg}(\mathcal{C}) & \xrightarrow{\mathrm{Alg}(F)} & \mathrm{Alg}(\mathcal{D}). \end{array}$$

In particular, for an algebra $R \in \mathrm{Alg}(\mathcal{C})$, F induces a functor $\mathrm{LMod}_R(\mathcal{C}) \rightarrow \mathrm{LMod}_{F(R)}(\mathcal{D})$, which we will often abbreviate as F_R or F^R .

Moreover, $\mathrm{LMod}(F)$ maps $\theta_{\mathcal{C}}$ -cocartesian morphism to $\theta_{\mathcal{D}}$ -cocartesian edges. Indeed, a morphism f is $\theta_{\mathcal{C}}$ -cocartesian if and only if its image under the functor $\varsigma_{\mathcal{C}}: \mathrm{LMod}(\mathcal{C}) \rightarrow \mathcal{C}$ that picks the the object being acted on is an equivalence, and similarly for $\theta_{\mathcal{D}}$. Now $\varsigma_{\mathcal{D}}(\mathrm{LMod}(F)(f)) \simeq F(\varsigma_{\mathcal{C}}(f))$, which implies that $\mathrm{LMod}(F)(f)$ is $\theta_{\mathcal{D}}$ -cocartesian if f is $\theta_{\mathcal{C}}$ -cocartesian. This means that F is compatible with restrictions of scalars in the sense that for every map $\psi: R \rightarrow S$ of algebras in \mathcal{C} , we have a commutative diagram

$$\begin{array}{ccc} \mathrm{LMod}_S(\mathcal{C}) & \xrightarrow{\psi^*} & \mathrm{LMod}_R(\mathcal{C}) \\ F_S \downarrow & & \downarrow F_R \\ \mathrm{LMod}_{F(S)}(\mathcal{D}) & \xrightarrow{F(\psi)^*} & \mathrm{LMod}_{F(R)}(\mathcal{D}) \end{array} \quad .$$

Now assume that \mathcal{C} and \mathcal{D} admit geometric realizations and their tensor products preserve geometric realizations in each variable. Then, for a map $\psi: R \rightarrow S$ of algebras in \mathcal{C} , we obtain a comparison map $F(\psi)_! \circ F_R \rightarrow F_S \circ \psi_!$ (cf. [Lur09, Remark 7.3.1.3]). Unpacking the construction of relative tensor products as geometric realizations of bar constructions ([Lur17, Theorem 4.4.2.8]), this map can be factored as

$$|F(S) \otimes F(R)^{\otimes \bullet} \otimes F(-)| \rightarrow |F(S \otimes R^{\otimes \bullet} \otimes (-))| \rightarrow F(|S \otimes R^{\otimes \bullet} \otimes (-)|),$$

where the former map is induced by the lax monoidality of F and the latter map is a colimit comparison map. Therefore, it is an equivalence if F is *strongly* monoidal and preserves geometric realizations. In summary, geometric-realization-preserving strongly monoidal functors are also compatible with extensions of scalars.

We now move on to comodules.

Definition 1.1.12. Let \mathcal{D} be an \mathcal{LM} -monoidal ∞ -category exhibiting $\mathcal{M} := \mathcal{D}_{\mathfrak{m}}$ as left-tensored over $\mathcal{C} := \mathcal{D}_{\mathfrak{a}}$.

We define the ∞ -category of left comodule objects in \mathcal{M} as

$$\mathrm{LComod}(\mathcal{M}) := \mathrm{Coalg}_{/\mathcal{LM}}(\mathcal{D}) = \mathrm{Alg}_{/\mathcal{LM}}(\mathcal{D}^{\mathrm{op}})^{\mathrm{op}},$$

which is equipped with a forgetful functor

$$\mathrm{LComod}(\mathcal{M}) = \mathrm{Alg}_{/\mathcal{L}\mathcal{M}}(\mathcal{D}^{\mathrm{op}})^{\mathrm{op}} \rightarrow \mathrm{Alg}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathrm{Coalg}(\mathcal{C}). \quad (1.1.13)$$

We will employ similar notational conventions for LComod as for LMod , such as writing $\mathrm{LComod}^{\mathcal{C}}(\mathcal{M})$ to disambiguate the monoidal category and denoting the functor (1.1.13) by θ with appropriate indices.

For a coalgebra D in \mathcal{D} , we define the ∞ -category of left D -comodules in \mathcal{M} as³

$$\mathrm{LComod}_D(\mathcal{M}) := \mathrm{LComod}(\mathcal{M}) \times_{\mathrm{Coalg}(\mathcal{C})} \{D\}.$$

Dualizing the first part of Fact 1.1.10, we obtain a cofree-forgetful adjunction

$$V_D: \mathrm{LComod}_D(\mathcal{M}) \rightleftarrows \mathcal{M} : C_D.$$

Similarly, we see that the functor (1.1.13) is a cocartesian fibration, meaning that a map $\zeta: D \rightarrow E$ a map of coalgebras in \mathcal{C} induces a *corestriction of scalars functor*

$$\zeta_*: \mathrm{LComod}_D(\mathcal{M}) \rightarrow \mathrm{LComod}_E(\mathcal{M}).$$

While these definitions are formally dual to the algebraic case, let us unpack them a bit for the sake of clarity. Let D be a coalgebra with comultiplication $\Delta_D: D \rightarrow D \otimes D$ and counit $\epsilon_D: D \rightarrow \mathbf{1}_{\mathcal{C}}$. A left D -comodule is an object M equipped with a coaction map $\rho_M: M \rightarrow D \otimes M$ satisfying duals of associativity and unitality conditions for an action. The cofree left D -comodule on an object $X \in \mathcal{C}$ is $D \otimes X$ equipped with the coaction map $\Delta_D \otimes X: D \otimes X \rightarrow D \otimes (D \otimes X)$. This defines a *right* adjoint of the forgetful functor $\mathrm{LComod}_D(\mathcal{C}) \rightarrow \mathcal{C}$ with unit transformation $\rho_M: M \rightarrow D \otimes M \simeq C_D(V_D(M))$ and counit transformation $\epsilon_D \otimes M: V_D(C_D(M)) \simeq D \otimes M \rightarrow M$. Given a coalgebra map $\zeta: D \rightarrow E$ and a left D -comodule M with coaction map $\rho_M: M \rightarrow D \otimes M$, $\zeta_* M$ has the same underlying object and its coaction map is the composite $M \xrightarrow{\rho_M} D \otimes M \xrightarrow{\zeta \otimes D} E \otimes M$. Note that corestrictions of scalars depend *covariantly* on coalgebra maps.

Remark 1.1.14. One might consider constructing a right adjoint $\zeta^\#$ of the corestriction of scalars functor ζ_* by dualizing the construction of the extension of scalars functor of Fact 1.1.10. However, recall that that construction relies on the theory of relative tensor products, which in turn requires that the tensor product preserves geometric realizations in each variable. Since the dual condition for limits of cosimplicial objects is not satisfied in important examples such as the category of spectra, we will need a construction that does not rely on “relative cotensor products” directly, even though it will be of similar flavor (cf. Corollary 1.3.15).

Convention 1.1.15. When we speak of “modules” or “comodules” without further qualifiers, we implicitly mean *left* modules and *left* comodules, respectively.

³Informally speaking, $\mathrm{LComod}_D(\mathcal{M})$ is just “ $\mathrm{LMod}_D(\mathcal{M}^{\mathrm{op}})^{\mathrm{op}}$ ”, but we would like to emphasize that we are taking the fiberwise opposite of $\mathcal{D}^{\otimes} \rightarrow \mathcal{L}\mathcal{M}^{\otimes}$, not of $\mathcal{C}^{\otimes} \rightarrow \mathbb{E}_1^{\otimes}$ and $\mathcal{M} \rightarrow \{*\}$ separately.

Next, we recall some facts about how we can interpret monoidal ∞ -categories as algebras in the ∞ -category of ∞ -categories.

Fact 1.1.16 ([Lur17, Example 2.4.2.4 and Proposition 2.4.2.5]). *Let \mathcal{O} be an ∞ -operad. Consider the ∞ -category Cat_∞ of ∞ -categories equipped with the symmetric monoidal structure given by the cartesian product.*

Then \mathcal{O} -algebras in Cat_∞ correspond to \mathcal{O} -monoidal ∞ -categories via an unstraightening construction. In particular, a monoidal structure on an ∞ -category \mathcal{C} corresponds to an algebra structure on \mathcal{C} in Cat_∞ and a left-tensoring of an ∞ -category \mathcal{M} over a monoidal ∞ -category corresponds to a left module structure over the corresponding algebra. Under this correspondence, maps of \mathcal{O} -algebras in Cat_∞ correspond to strongly \mathcal{O} -monoidal functors.

We will also use variants of this correspondence for presentable ∞ -categories.⁴

Fact 1.1.17. *Let Pr^L denote the ∞ -category of presentable ∞ -categories with colimit-preserving functors between them (cf. [Lur09, Definition 5.5.3.1]). Then, by [Lur17, Proposition 4.8.1.15], Pr^L admits a symmetric monoidal structure and, by [Lur17, Corollary 4.8.1.4], the inclusion $\text{Pr}^L \rightarrow \text{Cat}_\infty$ is lax symmetric monoidal. For an ∞ -operad \mathcal{O} , we will call an \mathcal{O} -algebra in Pr^L a presentably \mathcal{O} -monoidal ∞ -category.*

Fact 1.1.18. *By [Lur17, Example 4.8.1.20], the ∞ -category \mathcal{S} of spaces is the unit of Pr^L . In particular, it admits a commutative algebra structure in Pr^L (given by the product of spaces) and every presentable ∞ -category \mathcal{C} is a left module over it. We will denote the corresponding action map by $\odot_{\mathcal{C}}: \mathcal{S} \times \mathcal{C} \rightarrow \mathcal{C}$. By virtue of [Lur09, Corollary 4.4.4.9], for a space X and an object C of \mathcal{C} , $X \odot_{\mathcal{C}} C$ can be computed as the colimit of the constant diagram $X \rightarrow \mathcal{C}$ at C .*

Moreover, by the initiality of the algebra structure of the monoidal unit \mathcal{S} , for every $n \in \mathbb{N} \cup \{\infty\}$ and every presentably \mathbb{E}_n -monoidal ∞ -category \mathcal{C} , the functor $(-)\odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}: \mathcal{S} \rightarrow \mathcal{C}$ lifts to a strongly \mathbb{E}_n -monoidal functor.

Fact 1.1.19. *By [Lur17, Proposition 4.8.2.1], the ∞ -category \mathcal{S}_* of pointed spaces admits a presentably symmetric monoidal structure (given by the smash product). Moreover, the forgetful functor $\text{LMod}_{\mathcal{S}_*}(\text{Pr}^L) \rightarrow \text{Pr}^L$ is fully faithful and its image consists of presentable pointed ∞ -categories. Given a presentable pointed ∞ -category \mathcal{C} , we will denote the action map of its \mathcal{S}_* -module structure by $\otimes_{\mathcal{C}}: \mathcal{S}_* \times \mathcal{C} \rightarrow \mathcal{C}$. By virtue of [RSV19, Corollary 2.40], for a pointed space $i_X: \{*\} \rightarrow X$ and an object C of \mathcal{C} , $X \otimes_{\mathcal{C}} C$ can be computed as the cofiber of the morphism $i_X \odot C: \{*\} \odot C \rightarrow X \odot C$.*

For instance, when \mathcal{C} is the ∞ -category Sp of spectra, the functors $(-)\odot_{\text{Sp}} \mathbf{1}_{\text{Sp}}$ and $(-)\otimes_{\text{Sp}} \mathbf{1}_{\text{Sp}}$ are given by the suspension spectrum functors $\Sigma_+^\infty: \mathcal{S} \rightarrow \text{Sp}$ and $\Sigma^\infty: \mathcal{S}_* \rightarrow \text{Sp}$, respectively.

Viewing monoidal categories as algebras in Cat_∞ allows us to relate comodules over algebras in different ∞ -categories as follows.

⁴Most of our statements regarding presentable ∞ -categories can be generalized to ∞ -categories admitting all colimits, but we restrict our attention to the former class for the sake of convenience.

Construction 1.1.20. Let $F^\otimes: \mathcal{D}^\otimes \rightarrow \mathcal{E}^\otimes$ be a strongly monoidal functor between monoidal ∞ -categories and \mathcal{M} an ∞ -category left-tensored over \mathcal{E} encoded by an \mathcal{LM} -monoidal ∞ -category $\mathcal{M}^\otimes \rightarrow \mathcal{LM}^\otimes$.

Viewing F as a map in $\text{Alg}(\text{Cat}_\infty)$, we see that the left-tensoring of \mathcal{M} over \mathcal{D} given by the restriction of its left-tensoring over \mathcal{E} along F is encoded by a cartesian lift of F to $\text{LMod}(\text{Cat}_\infty)$ with target $\mathcal{M}^\otimes \rightarrow \mathcal{LM}^\otimes$. In other words, we have an \mathcal{LM} -monoidal ∞ -category $\mathcal{M}_F^\otimes \rightarrow \mathcal{LM}^\otimes$ equipped with an \mathcal{LM} -monoidal functor $\widehat{F}^\otimes: \mathcal{M}_F^\otimes \rightarrow \mathcal{M}^\otimes$ such that

- $\mathcal{M}_F^\otimes \times_{\mathcal{LM}^\otimes} \mathbb{E}_1^\otimes \simeq \mathcal{D}^\otimes$,
- the induced functor $\widehat{F}^\otimes \times_{\mathcal{LM}^\otimes} \mathbb{E}_1^\otimes$ coincides with $F^\otimes: \mathcal{D}^\otimes \rightarrow \mathcal{E}^\otimes$ under the above identification,
- \widehat{F}_m^\otimes is an equivalence, in particular $(\mathcal{M}_F^\otimes)_m \simeq \mathcal{M}$.

Hence \widehat{F}^\otimes induces a commutative diagram

$$\begin{array}{ccc} \text{LComod}^{\mathcal{D}}(\mathcal{M}) & \xrightarrow{\text{Coalg}/\mathcal{LM}(\widehat{F})} & \text{LComod}^{\mathcal{E}}(\mathcal{M}) \\ \theta_{\mathcal{D}} \downarrow & & \downarrow \theta_{\mathcal{E}} \\ \text{Coalg}(\mathcal{D}) & \xrightarrow{\text{Coalg}(F)} & \text{Coalg}(\mathcal{E}) \end{array}, \quad (1.1.21)$$

and similarly for modules.

The diagram (1.1.21) is in fact a pullback square, but we found it more appropriate to prove this statement in Corollary 1.3.21 after having discussed comonads.

1.2. Bialgebras

We now move on to bialgebras, which combine algebraic and coalgebraic structures and can be defined as algebras in an appropriate monoidal category of coalgebras (or as coalgebras in an appropriate monoidal category of algebras). In the context of 1-categories, the tensor product $D \otimes E$ of two coalgebras in a braided monoidal category \mathcal{C} admits a coalgebra structure with comultiplication

$$D \otimes E \xrightarrow{\Delta_D \otimes \Delta_E} D \otimes D \otimes E \otimes E \simeq D \otimes E \otimes D \otimes E$$

and counit

$$D \otimes E \xrightarrow{\epsilon_D \otimes \epsilon_E} \mathbf{1}_{\mathcal{C}} \otimes \mathbf{1}_{\mathcal{C}} \simeq \mathbf{1}_{\mathcal{C}},$$

and this operation defines a monoidal structure on the category of coalgebras.

Viewing \mathcal{C} as an \mathbb{E}_2 -monoidal category, another explanation of this phenomenon is as follows. Considering horizontal and vertical stacking of squares, the \mathbb{E}_2 -monoidal structure yields a horizontal and a vertical tensor product, which we denote by $|\cdot|$ and $\frac{?}{?}$, respectively. While these tensor products are equivalent, they play different roles

here. Given vertical comultiplications $\Delta_D: D \rightarrow \frac{D}{D}$ and $\Delta_E: E \rightarrow \frac{E}{E}$, we can define a horizontal comultiplication on the horizontal tensor product $D|E$ by

$$D|E \xrightarrow{\Delta_D|\Delta_E} \frac{D}{D}|\frac{E}{E}.$$

Hence the category of “vertical coalgebras” admit a monoidal structure given by the *horizontal* tensor product.

The distinction of the horizontal and the vertical binary operations of \mathbb{E}_2 can be encoded in a “product map” $\mathbb{E}_1^\otimes \times \mathbb{E}_1^\otimes \rightarrow \mathbb{E}_2^\otimes$ (cf. Example 1.2.4). One can generalize this perspective to obtain monoidal structures on ∞ -categories of (co)algebras as follows. Following [Lur17, Definition 2.2.5.3], we call a functor $f: \mathcal{P}^\otimes \times \mathcal{Q}^\otimes \rightarrow \mathcal{O}^\otimes$ a *bifunctor of ∞ -operads* if the diagram

$$\begin{array}{ccc} \mathcal{P}^\otimes \times \mathcal{Q}^\otimes & \xrightarrow{f} & \mathcal{O}^\otimes \\ \downarrow & & \downarrow \\ \text{Fin}_* \times \text{Fin}_* & \xrightarrow{\wedge} & \text{Fin}_* \end{array},$$

where \wedge denotes the smash product of pointed finite sets, commutes, and f maps pairs of inert morphisms to inert morphisms.

Construction 1.2.1 ([Lur17, Construction 3.2.4.1], see also [Lei22, Proposition E.4.1.4]). Let $f: \mathcal{P}^\otimes \times \mathcal{Q}^\otimes \rightarrow \mathcal{O}^\otimes$ be a bifunctor of ∞ -operads and $p: \mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes$ an \mathcal{O} -monoidal ∞ -category. By $\text{Fun}^{\text{in}}(\mathcal{Q}^\otimes, \mathcal{C}^\otimes) \subseteq \text{Fun}(\mathcal{Q}^\otimes, \mathcal{C}^\otimes)$ we denote the full subcategory spanned by functors that preserve inert morphisms. Let $\hat{f}: \mathcal{P}^\otimes \rightarrow \text{Fun}(\mathcal{Q}^\otimes, \mathcal{O}^\otimes)$ be the adjoint of f . We define $\text{Alg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})^\otimes$ as the pullback

$$\begin{array}{ccc} \text{Alg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})^\otimes & \longrightarrow & \text{Fun}^{\text{in}}(\mathcal{Q}^\otimes, \mathcal{C}^\otimes) \\ p_f^{\mathcal{C}} \downarrow & \lrcorner & \downarrow p_* \\ \mathcal{P}^\otimes & \xrightarrow{\hat{f}} & \text{Fun}(\mathcal{Q}^\otimes, \mathcal{O}^\otimes) \end{array},$$

and let $\text{Alg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})$ denote its fiber over $\mathcal{P} = \mathcal{P}_{(1)}^\otimes$.

Note that for all $\mathfrak{p} \in \mathcal{P}$, the fiber $\text{Alg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})_{\mathfrak{p}}^\otimes$ is the ∞ -category of ∞ -operad maps $\phi: \mathcal{Q}^\otimes \rightarrow \mathcal{C}^\otimes$ such that $p \circ \phi = \hat{f}(\mathfrak{p})$, i.e., $\text{Alg}_{\mathcal{Q}/\mathcal{O}}(\mathcal{C})$, where we consider the algebra category with respect to $\hat{f}(\mathfrak{p}): \mathcal{Q}^\otimes \rightarrow \mathcal{O}^\otimes$. We will therefore omit \mathcal{P} from the notation when $\mathcal{P} = \mathcal{P}_{(1)}^\otimes$ is contractible (in particular when $\mathcal{P} = \mathbb{E}_k$ for some $k \in \mathbb{N} \cup \{\infty\}$).

Fact 1.2.2 ([Lur17, Proposition 3.2.4.3]). *In the situation of Construction 1.2.1, $p_f^{\mathcal{C}}$ exhibits $\text{Alg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})$ as a \mathcal{P} -monoidal ∞ -category and an arrow in $\text{Alg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})^\otimes$, which is represented by a natural transformation τ between functors $\mathcal{Q}^\otimes \rightarrow \mathcal{C}^\otimes$, is $p_f^{\mathcal{C}}$ -cocartesian if and only if for all $\mathfrak{q} \in \mathcal{Q}$, its component $\tau_{\mathfrak{q}}$ is p -cocartesian in \mathcal{C}^\otimes .*

Remark 1.2.3. Construction 1.2.1 is functorial in \mathcal{P} , \mathcal{Q} and \mathcal{C} as follows.

Every map $g: \overline{\mathcal{P}}^\otimes \rightarrow \mathcal{P}^\otimes$ of ∞ -operads induces a bifunctor

$$f': \overline{\mathcal{P}}^\otimes \times \mathcal{Q}^\otimes \xrightarrow{g \times \mathcal{Q}} \mathcal{P}^\otimes \times \mathcal{Q}^\otimes \xrightarrow{f} \mathcal{O}^\otimes$$

whose adjoint \widehat{f}' can be factored as $\overline{\mathcal{P}}^\otimes \xrightarrow{g} \mathcal{P}^\otimes \xrightarrow{\widehat{f}} \text{Fun}(\mathcal{Q}^\otimes, \mathcal{O}^\otimes)$. Hence we have an equivalence $\text{Alg}_{\mathcal{Q}/\mathcal{O}}^{\overline{\mathcal{P}}}(\mathcal{C})^\otimes \simeq \text{Alg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})^\otimes \times_{\mathcal{P}^\otimes} \overline{\mathcal{P}}^\otimes$.

Similarly, every map $h: \overline{\mathcal{Q}}^\otimes \rightarrow \mathcal{Q}^\otimes$ of ∞ -operads induces a bifunctor

$$\mathcal{P}^\otimes \times \overline{\mathcal{Q}}^\otimes \xrightarrow{\mathcal{P}^\otimes \times h} \mathcal{P}^\otimes \times \mathcal{Q}^\otimes \xrightarrow{f} \mathcal{O}^\otimes$$

and precomposition with h induces a restriction functor $h^*: \text{Alg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C}) \rightarrow \text{Alg}_{\overline{\mathcal{Q}}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})$. Moreover, the componentwise description of $p_f^{\mathcal{C}}$ -cocartesian arrows from Fact 1.2.2 implies that h^* is strongly \mathcal{P} -monoidal. In particular, considering the case where $\overline{\mathcal{Q}}$ is the trivial ∞ -operad \mathcal{T} , we see that for every $\mathbf{q} \in \mathcal{Q}$, the “forgetful functor” $\text{Alg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C}) \rightarrow \text{Alg}_{\mathcal{T}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C}) \simeq \mathcal{C} \times_{\mathcal{O}} \mathcal{P}$ given by evaluation at \mathbf{q} is strongly \mathcal{P} -monoidal (cf. [Lur17, Remark 2.1.3.6]).

Analogously, postcomposition with an \mathcal{O} -monoidal functor $\mathcal{C} \rightarrow \overline{\mathcal{C}}$ induces a strongly \mathcal{P} -monoidal functor $\text{Alg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C}) \rightarrow \text{Alg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\overline{\mathcal{C}})$.

Example 1.2.4. Let $k, l \in \mathbb{N}$. Then, by [Lur17, Construction 5.1.2.1], there is a bifunctor $\mathbb{E}_k^\otimes \times \mathbb{E}_l^\otimes \rightarrow \mathbb{E}_{k+l}^\otimes$ of operads informally given by “taking products of configurations”, which induces an \mathbb{E}_k -monoidal structure on the ∞ -category of \mathbb{E}_l -algebras in an \mathbb{E}_{k+l} -monoidal ∞ -category \mathcal{C} . This bifunctor is in fact *universal* in the sense that $\text{Alg}_{\mathbb{E}_{k+l}}(\mathcal{C}) \simeq \text{Alg}_{\mathbb{E}_k}(\text{Alg}_{\mathbb{E}_l/\mathbb{E}_{k+l}}(\mathcal{C}))$ (cf. [Lur17, Theorem 5.1.2.2]).

Example 1.2.5. Let $p: \mathcal{Q}^\otimes \rightarrow \text{Fin}_*$ be an ∞ -operad. Then the composites

$$\mathbb{E}_\infty^\otimes \times \mathcal{Q}^\otimes = \text{Fin}_* \times \mathcal{Q}^\otimes \xrightarrow{\text{Fin}_* \times p} \text{Fin}_* \times \text{Fin}_* \xrightarrow{\wedge} \text{Fin}_* = \mathbb{E}_\infty^\otimes$$

and

$$\mathcal{Q}^\otimes \times \mathbb{E}_\infty^\otimes = \mathcal{Q}^\otimes \times \text{Fin}_* \xrightarrow{p \times \text{Fin}_*} \text{Fin}_* \times \text{Fin}_* \xrightarrow{\wedge} \text{Fin}_* = \mathbb{E}_\infty^\otimes$$

are bifunctors of operads.

In particular, the former bifunctor yields a symmetric monoidal structure on the ∞ -category of \mathcal{Q} -algebras in a symmetric monoidal ∞ -category (cf. [Lur17, Example 3.2.4.4]). Moreover, when $\mathcal{Q} = \mathbb{E}_\infty$, this symmetric monoidal structure is cocartesian by [Lur17, Proposition 3.2.4.7], i.e., the tensor product of commutative algebras is a coproduct in the ∞ -category of commutative algebras.

Convention 1.2.6. For $k, l \in \mathbb{N} \cup \{\infty\}$, whenever we use a bifunctor $\mathbb{E}_k^\otimes \times \mathbb{E}_l^\otimes \rightarrow \mathbb{E}_{k+l}^\otimes$ of ∞ -operads (where $k+l = \infty$ whenever k or l is ∞), we will mean the bifunctor discussed in Example 1.2.4 (if $k, l \in \mathbb{N}$) or Example 1.2.5 (otherwise).

We now have all the ingredients to define bialgebras.

Definition 1.2.7. Let $f: \mathcal{P}^\otimes \times \mathcal{Q}^\otimes \rightarrow \mathcal{O}^\otimes$ be a bifunctor of ∞ -operads and \mathcal{C} an \mathcal{O} -monoidal ∞ -category. Note that the fiberwise opposite of the cocartesian fibration $\text{Alg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C}^{\text{fop}})^\otimes \rightarrow \mathcal{P}^\otimes$ of Fact 1.2.2 defines a \mathcal{P} -monoidal structure on $\text{Coalg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})^\otimes := \text{Alg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C}^{\text{fop}})^{\text{fop}}$.

Moreover, note that taking pullbacks commutes with fiberwise opposites because under the straightening-unstraightening equivalence, the pullback of a cocartesian fibration along a functor corresponds to the restriction of its straightening along that functor. Hence, for all $\mathbf{p} \in \mathcal{P}$, $\text{Coalg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})_{\mathbf{p}}^\otimes \simeq \text{Coalg}_{\mathcal{Q}/\mathcal{O}}(\mathcal{C})$, where we consider coalgebras with respect to $\hat{f}(\mathbf{p}): \mathcal{Q}^\otimes \rightarrow \mathcal{O}^\otimes$.

We define the ∞ -category of $(\mathcal{P}, \mathcal{Q})$ -bialgebras in \mathcal{C} as

$$\text{Bialg}_{\mathcal{P}, \mathcal{Q}}(\mathcal{C}) := \text{Alg}_{/\mathcal{P}}(\text{Coalg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})).$$

For now, we view bialgebras as “algebras in coalgebras”, but Appendix A is dedicated to showing that $\text{Alg}_{/\mathcal{P}}(\text{Coalg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C}))$ is in fact equivalent to $\text{Coalg}_{/\mathcal{Q}}(\text{Alg}_{\mathcal{P}/\mathcal{O}}^{\mathcal{Q}}(\mathcal{C}))$, where the latter is constructed with respect to the bifunctor $\mathcal{Q}^\otimes \times \mathcal{P}^\otimes \simeq \mathcal{P}^\otimes \times \mathcal{Q}^\otimes \rightarrow \mathcal{O}^\otimes$ (cf. Corollary A.0.17). We will use this equivalent description of bialgebras in several places, most notably throughout Section 5.

Convention 1.2.8. In line with Convention 1.1.8, we will refer to $(\mathbb{E}_1, \mathbb{E}_1)$ -bialgebras simply as *bialgebras* and write Bialg instead of $\text{Bialg}_{\mathbb{E}_1, \mathbb{E}_1}$. Similarly, in line with Convention 1.1.9, we will refer to $(\mathbb{E}_\infty, \mathbb{E}_1)$ -bialgebras as *commutative bialgebras* and write CBialg instead of $\text{Bialg}_{\mathbb{E}_\infty, \mathbb{E}_1}$.

One of the main features of bialgebras is the fact that their (co)module categories admit a monoidal structure given by the tensor product of underlying objects. Indeed, given comodules M and N over a bialgebra H in a braided monoidal category, the composite

$$M \otimes N \xrightarrow{\rho_M \otimes \rho_N} H \otimes M \otimes H \otimes N \simeq H \otimes H \otimes M \otimes N \xrightarrow{\mu_H \otimes M \otimes N} H \otimes M \otimes N$$

defines a coaction of H on $M \otimes N$.

In order carry out an analogous construction for an \mathbb{E}_2 -monoidal ∞ -category \mathcal{C} , one considers the bifunctor $\mathbb{E}_1^\otimes \times \mathcal{LM}^\otimes \rightarrow \mathbb{E}_2^\otimes$ given by precomposing the standard bifunctor $\mathbb{E}_1^\otimes \times \mathbb{E}_1^\otimes \rightarrow \mathbb{E}_2^\otimes$ with the ∞ -operad map $\mathcal{LM}^\otimes \rightarrow \mathbb{E}_1^\otimes$, and the associated monoidal structure on $\text{LComod}(\mathcal{C})$ (more precisely, $\text{Coalg}_{\mathcal{LM}/\mathbb{E}_2}(\mathcal{C})$) obtained via Fact 1.2.2. An object of $\text{LComod}(\mathcal{C})$ corresponds to a pair (D, M) where D is a coalgebra coacting on an object M . Given another such pair (E, N) , the tensor product “ $(D, M) \otimes_{\text{LComod}(\mathcal{C})} (E, N)$ ” is given by $(D \otimes E, M \otimes N)$ with coaction map

$$M \otimes N \xrightarrow{\rho_M \otimes \rho_N} D \otimes M \otimes E \otimes N \simeq D \otimes E \otimes M \otimes N.$$

Hence, if $D = E = H$ is a bialgebra, the desired tensor product of M and N in $\text{LComod}_H(\mathcal{C})$ can be constructed as $(\mu_H)_*((H, M) \otimes_{\text{LComod}(\mathcal{C})} (H, N))$. We review a construction along these lines below.

While a significant portion of what we do with comodules over bialgebras can be generalized to a bifunctor $\mathcal{P}^\otimes \times \mathbb{E}_1^\otimes \rightarrow \mathcal{O}^\otimes$, we restrict our attention to $(\mathbb{E}_k, \mathbb{E}_1)$ -bialgebras

in \mathbb{E}_{k+1} -monoidal ∞ -categories for $k \in \mathbb{N} \cup \{\infty\}$ in order to avoid tracking bifunctors, colors etc. In fact, we will be mostly interested in the cases $k = 1$ and $k = \infty$.

Fact 1.2.9 ([Bea21, Proposition 3.16 and Theorem 3.18], [Lei22, Definition 3.4.2.1 and Remark 3.4.2.3]). *For $k \in \mathbb{N} \cup \{\infty\}$, consider the standard bifunctor $\mathbb{E}_k^\otimes \times \mathbb{E}_1^\otimes \rightarrow \mathbb{E}_{k+1}^\otimes$ and the ∞ -operad map $\mathcal{LM}^\otimes \rightarrow \mathbb{E}_1^\otimes$. Let \mathcal{C} be an \mathbb{E}_{k+1} -monoidal ∞ -category. Consider the \mathbb{E}_k -monoidal functor*

$$\theta^\otimes: \mathrm{LComod}(\mathcal{C})^\otimes = \mathrm{Coalg}_{\mathcal{LM}/\mathbb{E}_{k+1}}(\mathcal{C})^\otimes \rightarrow \mathrm{Coalg}_{\mathbb{E}_1/\mathbb{E}_{k+1}}(\mathcal{C})^\otimes = \mathrm{Coalg}(\mathcal{C})^\otimes$$

of Remark 1.2.3.

In this situation, θ^\otimes is a cocartesian fibration. Moreover, given an $(\mathbb{E}_k, \mathbb{E}_1)$ -bialgebra H in \mathcal{C} , pulling back θ^\otimes along the corresponding map $\mathbb{E}_k^\otimes \rightarrow \mathrm{Coalg}(\mathcal{C})^\otimes$ yields an \mathbb{E}_k -monoidal structure on $\mathrm{LComod}_H(\mathcal{C})$ such that $V_H: \mathrm{LComod}_H(\mathcal{C}) \rightarrow \mathcal{C}$ can be extended to a strongly \mathbb{E}_k -monoidal functor.

More generally, given a map $\xi: H \rightarrow H'$ of $(\mathbb{E}_k, \mathbb{E}_1)$ -bialgebras in \mathcal{C} , the functor induced on pullbacks of θ^\otimes extends the corestriction of scalars functor $\xi_*: \mathrm{LComod}_H(\mathcal{C}) \rightarrow \mathrm{LComod}_{H'}(\mathcal{C})$ to a strongly \mathbb{E}_k -monoidal functor.

Remark 1.2.10. In the situation of Fact 1.2.9, note that the strongly \mathbb{E}_k -monoidal structure on $V_H: \mathrm{LComod}_H(\mathcal{C}) \rightarrow \mathcal{C}$ induces a lax \mathbb{E}_k -monoidal structure on its right adjoint C_H by virtue of Fact 1.1.6. Unpacking the constructions, we see that its comparison transformation for $X, Y \in \mathcal{C}$ is given by the composite

$$\begin{aligned} C_H(X) \otimes C_H(Y) &\simeq (H \otimes X) \otimes (H \otimes Y) \xrightarrow{\Delta_H \otimes X \otimes \Delta_H \otimes Y} (H \otimes H \otimes X) \otimes (H \otimes H \otimes Y) \\ &\simeq (H \otimes H) \otimes ((H \otimes X) \otimes (H \otimes Y)) \\ &\xrightarrow{\mu_H \otimes H \otimes X \otimes H \otimes Y} H \otimes ((H \otimes X) \otimes (H \otimes Y)) \\ &\xrightarrow{H \otimes \epsilon_H \otimes X \otimes \epsilon_H \otimes Y} H \otimes (X \otimes Y) \simeq C_H(X \otimes Y), \end{aligned}$$

which can be simplified to

$$(H \otimes X) \otimes (H \otimes Y) \simeq (H \otimes H) \otimes (X \otimes Y) \xrightarrow{\mu_H \otimes X \otimes Y} H \otimes (X \otimes Y)$$

using the counitality of the comultiplication of H .

Definition 1.2.11. Let $k \in \mathbb{N} \cup \{\infty\}$, \mathcal{C} an \mathbb{E}_{k+1} -monoidal ∞ -category, and H an $(\mathbb{E}_k, \mathbb{E}_1)$ -bialgebra in \mathcal{C} . We call $\mathrm{Alg}_{/\mathbb{E}_k}(\mathrm{LComod}_H(\mathcal{C}))$ ∞ -category of H -comodule \mathbb{E}_k -algebras in \mathcal{C} .

Remark 1.2.12. In the situation of Definition 1.2.11, assume that \mathcal{C} admits geometric realizations and that the tensor product preserves them. Note that, by [Lur17, Corollary 4.2.3.3], the forgetful functor $V_H: \mathrm{LComod}_H(\mathcal{C}) \rightarrow \mathcal{C}$ reflects colimits. Therefore, $\mathrm{LComod}_H(\mathcal{C})$ also admits geometric realizations and its tensor product preserves them in each variable, which implies that extensions of scalars along maps of H -comodule algebras exist (cf. Fact 1.1.10).

Moreover, note that V_H is a strongly (\mathbb{E}_k) -monoidal functor which preserves geometric realizations, so as discussed in Remark 1.1.11, it is compatible with restrictions and extensions of scalars.

We conclude this subsection with an alternative description of comodule algebras.

Lemma 1.2.13. *Let $k \in \mathbb{N} \cup \{\infty\}$, \mathcal{C} an \mathbb{E}_{k+1} -monoidal ∞ -category, and H an $(\mathbb{E}_k, \mathbb{E}_1)$ -bialgebra in \mathcal{C} . Then $\text{Alg}_{/\mathbb{E}_k}(\text{LComod}_H(\mathcal{C}))$ is equivalent to the fiber of*

$$\text{Alg}_{/\mathbb{E}_k}(\theta): \text{Alg}_{/\mathbb{E}_k}(\text{LComod}(\mathcal{C})) \rightarrow \text{Alg}_{/\mathbb{E}_k}(\text{Coalg}(\mathcal{C}))$$

over H .

Proof. Since $\text{Alg}_{/\mathbb{E}_k}(\text{Coalg}(\mathcal{C}))$ is a full subcategory of $\text{Fun}_{\mathbb{E}_k^\otimes}(\mathbb{E}_k^\otimes, \text{Coalg}(\mathcal{C})^\otimes)$, H corresponds to a functor $h: \mathbb{E}_k^\otimes \rightarrow \text{Coalg}(\mathcal{C})^\otimes$. Moreover, the fiber $\text{Alg}_{/\mathbb{E}_k}(\text{LComod}(\mathcal{C}))_H$ can be identified with the pullback of the cospan

$$\{\text{Id}_{\mathbb{E}_k^\otimes}\} \xrightarrow{h \circ (-)} \text{Alg}_{/\mathbb{E}_k}(\text{Coalg}(\mathcal{C})) \xleftarrow{\text{Alg}_{/\mathbb{E}_k}(\theta)} \text{Alg}_{/\mathbb{E}_k}(\text{LComod}(\mathcal{C})^\otimes).$$

We would like to show that $\text{Alg}_{/\mathbb{E}_k}(\text{LComod}_H(\mathcal{C}))$ is also a pullback of this cospan.

Now consider the commutative diagram

$$\begin{array}{ccc} \text{LComod}(\mathcal{C})_H^\otimes & \xrightarrow{\iota^\otimes} & \text{LComod}(\mathcal{C})^\otimes \\ q \downarrow & \lrcorner & \downarrow \theta^\otimes \\ \mathbb{E}_k^\otimes & \xrightarrow{h} & \text{Coalg}(\mathcal{C})^\otimes \\ & \searrow & \downarrow p \\ & & \mathbb{E}_k^\otimes \end{array} ,$$

where the upper square is a pullback square. This pullback square yields another pullback square

$$\begin{array}{ccc} \text{Fun}_{\mathbb{E}_k^\otimes}(\mathbb{E}_k^\otimes, \text{LComod}_H(\mathcal{C})^\otimes) & \xrightarrow{\iota^\otimes \circ (-)} & \text{Fun}_{\mathbb{E}_k^\otimes}(\mathbb{E}_k^\otimes, \text{LComod}(\mathcal{C})^\otimes) \\ q \circ (-) \downarrow & & \downarrow \theta^\otimes \circ (-) \\ \{\text{Id}_{\mathbb{E}_k^\otimes}\} & \xrightarrow{h \circ (-)} & \text{Fun}_{\mathbb{E}_k^\otimes}(\mathbb{E}_k^\otimes, \text{Coalg}(\mathcal{C})^\otimes) \end{array} .$$

Hence, restricting the functor on the right to the full subcategories of functors that preserve inert arrows, it will suffice to show that a functor $a: \mathbb{E}_k^\otimes \rightarrow \text{LComod}_H(\mathcal{C})^\otimes$ over \mathbb{E}_k^\otimes is in $\text{Alg}_{/\mathbb{E}_k}(\text{LComod}_H(\mathcal{C}))$ if and only if $\iota^\otimes \circ a$ lies in $\text{Alg}_{/\mathbb{E}_k}(\text{LComod}(\mathcal{C}))$.

In order to do so, it will suffice to show that inert morphisms in $\text{LComod}_H(\mathcal{C})^\otimes$ are precisely those whose image under ι^\otimes is inert. Note that, by [Lur09, Proposition 2.4.1.3], an arrow in $\text{LComod}(\mathcal{C})^\otimes$ is inert if and only if it is a θ^\otimes -cocartesian lift of an inert arrow in $\text{Coalg}(\mathcal{C})^\otimes$. Therefore, as q -cartesian arrows are exactly those whose image under ι^\otimes is θ^\otimes -cocartesian, it will suffice to show that an arrow f in \mathbb{E}_k^\otimes is inert if and only if $h(f)$ is inert. If f is inert, then $h(f)$ is inert because h preserves inert morphisms. Conversely, if $h(f)$ is inert, then $p(h(f)) = f$ is also inert. \square

1.3. (Co)monads

Next, we review the theory of coendomorphism objects and comonads, which will be crucial for certain constructions on categories on comodules. We will also need the dual theory of endomorphism objects, but we expect the reader to dualize the statements as usual. Instead of the classical terminology of *(co)algebras* over a (co)monad, we will speak of *(co)modules* over a (co)monad, which is in line with our treatment of these objects as (co)modules in an ∞ -category left-tensored over a monoidal ∞ -category (cf. Example 1.3.4).

The citations in this section refer to different incarnations of (co)monads (the “operadic” theory of monads of [Lur17, Section 4.7], the “cosmological” approach to monads of [RV16] and [RV20, Chapter 10], and the “combinatorial” model for comonads of [Kra18, Chapter 2]), but they are all equivalent, for instance by an application of the Barr–Beck–Lurie (co)monadicity theorem (cf. [Lur17, Theorem 4.7.3.5] and [RV16, Theorem 7.2.7]).

Definition 1.3.1 ([Lur17, Section 4.7.1]). Let \mathcal{M} be an ∞ -category left-tensored over a monoidal ∞ -category \mathcal{C} , $M \in \mathcal{M}$, and $D \in \mathcal{C}$.

We say that a map $\rho: M \rightarrow D \otimes M$ exhibits D as a coendomorphism object for M if for all $X \in \mathcal{C}$, the composite

$$\mathrm{Map}_{\mathcal{C}}(D, X) \xrightarrow{(-) \otimes M} \mathrm{Map}_{\mathcal{M}}(D \otimes M, X \otimes M) \xrightarrow{(-) \circ \rho} \mathrm{Map}_{\mathcal{M}}(M, X \otimes M) \quad (1.3.2)$$

is an equivalence.

The key feature of a coendomorphism object is that it admits a coalgebra structure with a certain universal property.

Fact 1.3.3 ([Lur17, Corollary 4.7.1.40]). Let \mathcal{M} be an ∞ -category left-tensored over a monoidal ∞ -category \mathcal{C} and $\rho: M \rightarrow D \otimes M$ a map exhibiting $D \in \mathcal{C}$ as a coendomorphism object for $M \in \mathcal{M}$.

Then D admits a coalgebra structure, where, under the equivalence of (1.3.2), the comultiplication map $\Delta: D \rightarrow D \otimes D$ is given by the preimage of $(D \otimes \rho)\rho \in \mathrm{Map}_{\mathcal{M}}(M, D \otimes D \otimes M)$ and the counit map $\epsilon: D \rightarrow \mathbf{1}_{\mathcal{C}}$ is given by the preimage of $\mathrm{Id}_M \in \mathrm{Map}_{\mathcal{M}}(M, M)$.

Moreover, M admits a D -comodule structure where the coaction map is given by ρ , which we call the tautological coaction of D on M . This coaction makes D the initial \mathbb{E}_1 -coalgebra coacting on M in the sense that for every $E \in \mathrm{Coalg}(\mathcal{C})$, there is a natural equivalence

$$\mathrm{Map}_{\mathrm{Coalg}(\mathcal{C})}(D, E) \simeq \mathrm{LComod}_E(\mathcal{M}) \times_{\mathcal{M}} \{M\}$$

such that an E -comodule structure with coaction map $\rho': M \rightarrow E \otimes M$, corresponds to an upgrade of the preimage of ρ' under (1.3.2) to a coalgebra map $D \rightarrow E$.

Our first example concerns in fact endomorphism objects and lays the foundation for the study of (co)monads.

Example 1.3.4. Consider the action of the ∞ -category Cat_∞ of ∞ -categories equipped with the monoidal structure given by the cartesian product on itself. Let \mathcal{M} be an ∞ -category.

We can construct a monoidal structure on $\text{Fun}(\mathcal{M}, \mathcal{M})$ whose tensor product is given by the composition of functors by viewing it as a monoid object in marked simplicial sets (where the marked edges are equivalences) and applying the rectification result of [Lur17, Example 4.1.8.7]. (Co)algebras with respect to this monoidal structure are called *(co)monads*. Combining the point-set level description of this monoidal structure with a similar rectification result for modules ([Lur17, Theorem 4.3.3.17]), we see that ∞ -categories of the form $\text{Fun}(\mathcal{N}, \mathcal{M})$ are left-tensored over $\text{Fun}(\mathcal{M}, \mathcal{M})$ via composition.

Similarly, \mathcal{M} admits a left-tensoring over $\text{Fun}(\mathcal{M}, \mathcal{M})$ whose action map is the evaluation functor $e: \text{Fun}(\mathcal{M}, \mathcal{M}) \times \mathcal{M} \rightarrow \mathcal{M}$. In fact, e exhibits $\text{Fun}(\mathcal{M}, \mathcal{M})$ as an endomorphism object for \mathcal{M} . Indeed, for every ∞ -category \mathcal{D} , the composite

$$\begin{aligned} \text{Map}_{\text{Cat}_\infty}(\mathcal{D}, \text{Fun}(\mathcal{M}, \mathcal{M})) &\xrightarrow{(-) \times \mathcal{M}} \text{Map}_{\text{Cat}_\infty}(\mathcal{D} \times \mathcal{M}, \text{Fun}(\mathcal{M}, \mathcal{M}) \times \mathcal{M}) \\ &\xrightarrow{e_*} \text{Map}_{\text{Cat}_\infty}(\mathcal{D} \times \mathcal{M}, \mathcal{M}) \end{aligned}$$

is the restriction of the usual “uncurrying” equivalence to the maximal ∞ -groupoids of functor categories.

The universal property of $\text{Fun}(\mathcal{M}, \mathcal{M})$ as an endomorphism object implies that the adjoint $\mathcal{C} \rightarrow \text{Fun}(\mathcal{M}, \mathcal{M})$ of $\otimes: \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$ can be upgraded to a map of \mathbb{E}_1 -algebras in Cat_∞ , i.e., to a strongly monoidal functor (cf. Fact 1.1.16). In particular, a left action of $\text{Fun}(\mathcal{M}, \mathcal{M})$ on an ∞ -category can be restricted to a left action of \mathcal{C} along this functor.

As in the 1-categorical case, adjunctions give rise to (co)monads.

Example 1.3.5 ([Lur17, Lemma 4.7.3.1]). Let $F: \mathcal{D} \rightleftarrows \mathcal{E} : G$ be an adjunction with unit $u: \text{Id}_{\mathcal{D}} \rightarrow GF$ and counit $c: FG \rightarrow \text{Id}_{\mathcal{E}}$. Consider $\text{Fun}(\mathcal{D}, \mathcal{E})$ as left-tensored over $\text{Fun}(\mathcal{E}, \mathcal{E})$ via composition of functors.

Then the natural transformation $Fu: F \rightarrow FGF$ exhibits $FG \in \text{Fun}(\mathcal{E}, \mathcal{E})$ as a coendomorphism object for $F \in \text{Fun}(\mathcal{D}, \mathcal{E})$. In particular, FG admits the structure of a coalgebra in $\text{Fun}(\mathcal{E}, \mathcal{E})$, i.e., the structure of a comonad on \mathcal{E} .

The descriptions of Fact 1.3.3 imply that the comultiplication of this coalgebra structure is given by $FuG: FG \rightarrow FGFG$ and its counit by $c: FG \rightarrow \text{Id}_{\mathcal{E}}$. Moreover, given another comonad Θ on \mathcal{E} coacting on F with coaction map $r: F \rightarrow \Theta F$, the induced comonad map $FG \rightarrow \Theta$ is given by

$$FG \xrightarrow{rG} \Theta FG \xrightarrow{\Theta c} \Theta.$$

Next, we record some facts about how functors into the coalgebra category of a comonad encode “coalgebraic properties” of the comonad.

Fact 1.3.6 ([Kra18, Remark 2.4], see also [Lur17, Remark 4.7.3.8]). *Let $F: \mathcal{D} \rightarrow \mathcal{E}$ be a functor and Θ a comonad on \mathcal{E} . Then Θ -comodule structures on F correspond to lifts*

$$\begin{array}{ccc} & & \text{LComod}_\Theta(\mathcal{E}) \\ & \nearrow \widehat{F} & \downarrow v_\Theta \\ \mathcal{D} & \xrightarrow{F} & \mathcal{E} \end{array},$$

where a lift \hat{F} yields a coaction map

$$F \simeq V_\Theta \hat{F} \xrightarrow{V_\Theta u \hat{F}} V_\Theta C_\Theta V_\Theta \hat{F} \simeq \Theta F$$

using the unit $u: \text{Id}_\mathcal{E} \rightarrow V_\Theta C_\Theta$ of the adjunction $V_\Theta \dashv C_\Theta$.

Remark 1.3.7. Let $F: \mathcal{D} \rightleftarrows \mathcal{E} : G$ be an adjunction. Fact 1.3.6 applies in particular to the tautological coaction of the comonad FG on F discussed in Example 1.3.5, yielding the standard lift $\bar{F}: \mathcal{D} \rightarrow \text{LComod}_{FG}(\mathcal{E})$. The adjunction is called *comonadic* if \bar{F} is an equivalence. By the Barr–Beck–Lurie comonadicity theorem ([Lur17, Theorem 4.7.3.5]), this is the case if and only if F is conservative, and \mathcal{E} admits and F preserves limits of F -split cosimplicial objects.

Now let Θ be another comonad on \mathcal{E} such that F admits a lift $\hat{F}: \mathcal{D} \rightarrow \text{LComod}_\Theta(\mathcal{E})$. Applying the equivalence

$$\text{Map}_{\text{Coalg}(\text{Fun}(\mathcal{E}, \mathcal{E}))}(FG, \Theta) \simeq \text{LComod}_\Theta(\text{Fun}(\mathcal{D}, \mathcal{E})) \times_{\text{Fun}(\mathcal{D}, \mathcal{E})} \{F\}$$

of Fact 1.3.3 to the corresponding Θ -comodule structure on F , we obtain a comonad map $\zeta: FG \rightarrow \Theta$.

By the naturality of the aforementioned equivalence, the image of the tautological action of FG on F under ζ_* is the coaction of Θ on F induced by the lift \hat{F} . On the level of lifts, this yields a commutative diagram

$$\begin{array}{ccc} & & \downarrow \\ & \text{LComod}_{FG}(\mathcal{E}) & \xrightarrow{\zeta_*} \text{LComod}_\Theta(\mathcal{E}) \\ \hat{F} \uparrow & \downarrow V_{FG} & \uparrow V_\Theta \\ \mathcal{D} & \xrightarrow{F} \mathcal{E} & \end{array} \quad .$$

As for functors *out of* the coalgebra category of a comonad, they are often constructed by defining the functor on the full subcategory of cofree comodules and “resolving” more general comodules by cofree ones.

Definition 1.3.8. Let Θ be a comonad on an ∞ -category \mathcal{E} . A *resolution system* for Θ is a pair of functors $\tilde{R}_\Theta^\bullet(-)_+: \text{LComod}_\Theta(\mathcal{E}) \rightarrow \text{Fun}(\Delta_+, \text{LComod}_\Theta(\mathcal{E}))$ and $R_\Theta^\bullet(-)_\perp: \text{LComod}_\Theta(\mathcal{E}) \rightarrow \text{Fun}(\Delta_\perp, \mathcal{E})$ such that the following hold.

1. They fit into a commutative diagram

$$\begin{array}{ccccc} & & \tilde{R}_\Theta^\bullet(-) & & \\ & \searrow & \downarrow & \searrow & \\ \text{LComod}_\Theta(\mathcal{E}) & \xrightarrow{\tilde{R}_\Theta^\bullet(-)_+} \text{Fun}(\Delta_+, \text{LComod}_\Theta(\mathcal{E})) & \xrightarrow{(-)|_\Delta} \text{Fun}(\Delta, \text{LComod}_\Theta(\mathcal{E})) & & \\ \downarrow R_\Theta^\bullet(-)_\perp & \searrow R_\Theta^\bullet(-)_+ & \downarrow V_\Theta \circ (-) & \searrow R_\Theta^\bullet(-) & \downarrow V_\Theta \circ (-) \\ \text{Fun}(\Delta_\perp, \mathcal{E}) & \xrightarrow{(-)|_{\Delta_+}} \text{Fun}(\Delta_+, \mathcal{E}) & \xrightarrow{(-)|_\Delta} \text{Fun}(\Delta, \mathcal{E}) & & \end{array} \quad ,$$

where $\tilde{R}_\Theta^\bullet(-)$, $R_\Theta^\bullet(-)_+$ and $R_\Theta^\bullet(-)$ are defined as appropriate restrictions.

2. $\tilde{R}_\Theta^{-1}(-)_+ \simeq \text{Id}_{\text{LComod}_\Theta(\mathcal{E})}$.
3. For $k \geq 0$, $\tilde{R}_\Theta^k(-)_+$ factors through the full subcategory of cofree Θ -comodules.
4. For every $\tilde{M} \in \text{LComod}_\Theta(\mathcal{E})$, $\tilde{R}_\Theta^\bullet(\tilde{M})_+$ is a limiting cone.

We speak of *cobar* resolutions if for all $\tilde{M} \in \text{LComod}_\Theta(\mathcal{E})$, the following conditions hold as well.

- For $k \in \mathbb{Z}_{\geq -1}$, $\tilde{R}_\Theta^k(\tilde{M})_+ \simeq (C_\Theta V_\Theta)^{k+1}(\tilde{M})$.
- Setting $M := V_\Theta(\tilde{M})$, the resolutions can be depicted as

$$\begin{array}{ccccccc}
 M & \xrightarrow{\rho_M} & \Theta(M) & \xleftarrow{\epsilon_\Theta M} & \Theta^2(M) & \xleftarrow{\epsilon_\Theta \Theta^2 M} & \Theta^3(M) \quad \dots \\
 & & \xleftarrow{\Delta_\Theta M} & \xleftarrow{\Delta_\Theta M} & \xleftarrow{\Delta_\Theta M} & \xleftarrow{\Delta_\Theta M} & \\
 & & \xleftarrow{\Theta \epsilon_\Theta M} & \xleftarrow{\Theta \epsilon_\Theta M} & \xleftarrow{\Theta \epsilon_\Theta M} & \xleftarrow{\Theta \epsilon_\Theta M} & \\
 & & \xleftarrow{\Theta \rho_M} & \xleftarrow{\Theta \rho_M} & \xleftarrow{\Theta \rho_M} & \xleftarrow{\Theta \rho_M} &
 \end{array}$$

where the solid arrows depict the coaugmented cosimplicial object $\tilde{R}_\Theta^\bullet(\tilde{M})_+$ and the dashed arrows arise from its lift $\tilde{R}_\Theta^\bullet(\tilde{M})_\perp$ to a split coaugmented cosimplicial object. In other words, the coface maps of the resolutions are given by applying the unit of the adjunction $V_H \dashv C_H$, and the codegeneracy maps are given by applying its counit.

In this case, for $k \in \mathbb{Z}_{\geq -1}$, we denote the coaugmentation map $\tilde{M} \rightarrow (C_\Theta V_\Theta)^{k+1} \tilde{M}$ by $\rho_M^{(k+1)}$.

Fact 1.3.9 ([Lur17, Example 4.7.2.7 and Proposition 4.7.3.14], [RV20, Lemma 10.3.4 and Theorem 10.3.7]). *Let Θ be a comonad on an ∞ -category \mathcal{E} . Then there exists a cobar resolution system $(\tilde{R}_\Theta^\bullet(-)_+, \tilde{R}_\Theta^\bullet(-)_\perp)$ for Θ .*

Note that while any two cobar resolutions have homotopic coface and codegeneracy maps, we cannot necessarily construct cosimplicial comparison maps between them because a cosimplicial object in an ∞ -category also involves possibly non-trivial homotopies between composites of the generating maps. However, the arguments in this section will depend only on the resolution properties and the maps that occur in the resolution, so which “model” we use will not be relevant. The reason why we do not commit to a specific one is that later, in Section 5, we will make use of a concrete cobar resolution system that cannot be generalized to comodules over a general comonad (cf. Construction 5.1.17).

The idea of defining functors via resolutions is implemented as follows.

Proposition 1.3.10 (cf. [Kra18, Lemma 2.13]). *Let Λ and Θ be comonads on ∞ -categories \mathcal{D} and \mathcal{E} , respectively. Assume that $\text{LComod}_\Lambda(\mathcal{D})$ admits limits of cosimplicial*

objects and that we are given a commutative diagram

$$\begin{array}{ccc} \mathrm{LComod}_\Lambda(\mathcal{D}) & \xrightarrow{\tilde{F}} & \mathrm{LComod}_\Theta(\mathcal{E}) \\ \downarrow \mathrm{V}_\Lambda & & \downarrow \mathrm{V}_\Theta \\ \mathcal{D} & \xrightarrow{F} & \mathcal{E} \end{array}$$

such that F admits a right adjoint $G: \mathcal{E} \rightarrow \mathcal{D}$.

Then \tilde{F} admits a right adjoint $\tilde{G}: \mathrm{LComod}_\Theta(\mathcal{E}) \rightarrow \mathrm{LComod}_\Lambda(\mathcal{D})$ such that $\tilde{G} \circ C_\Theta \simeq C_\Lambda \circ G$.

Proof. Arguing as in the proof of [Lur17, Lemma 4.7.3.13], it will suffice to show that for every $M \in \mathrm{LComod}_\Theta(\mathcal{E})$, the functor $\mathrm{Map}_{\mathrm{LComod}_\Theta(\mathcal{E})}(\tilde{F}(-), \tilde{M}): \mathrm{LComod}_\Lambda(\mathcal{D}) \rightarrow \mathcal{S}$ is representable.

If $\tilde{M} \simeq C_\Theta(X)$ is cofree, then we have natural equivalences

$$\begin{aligned} \mathrm{Map}_{\mathrm{LComod}_\Theta(\mathcal{E})}(\tilde{F}(-), C_\Theta(X)) &\simeq \mathrm{Map}_\mathcal{E}(\mathrm{V}_\Theta(\tilde{F}(-)), X) \\ &\simeq \mathrm{Map}_\mathcal{E}(F(\mathrm{V}_\Lambda(-)), X) \\ &\simeq \mathrm{Map}_\mathcal{D}(\mathrm{V}_\Lambda(-), G(X)) \\ &\simeq \mathrm{Map}_{\mathrm{LComod}_\Lambda(\mathcal{D})}(-, C_\Lambda(G(X))), \end{aligned} \tag{1.3.11}$$

which defines the restriction \tilde{G}_C of \tilde{G} to the full subcategory $\mathrm{LComod}_\Theta(\mathcal{E})_C$ of cofree Θ -comodules with the desired property.

For general \tilde{M} , we use a resolution $\tilde{R}_\Theta^\bullet(\tilde{M})$, which exists by Fact 1.3.9⁵. We have $\tilde{M} \simeq \lim_{\Delta} \tilde{R}_\Theta^\bullet(\tilde{M})$ and $\tilde{R}_\Theta^\bullet(\tilde{M})$ is a diagram in $\mathrm{LComod}_\Theta(\mathcal{E})_C$. Hence we have natural equivalences

$$\begin{aligned} \mathrm{Map}_{\mathrm{LComod}_\Theta(\mathcal{E})}(\tilde{F}(-), \tilde{M}) &\simeq \mathrm{Map}_{\mathrm{LComod}_\Theta(\mathcal{E})}(\tilde{F}(-), \lim_{\Delta} \tilde{R}_\Theta^\bullet(\tilde{M})) \\ &\simeq \lim_{\Delta} \mathrm{Map}_{\mathrm{LComod}_\Theta(\mathcal{E})}(\tilde{F}(-), \tilde{R}_\Theta^\bullet(\tilde{M})) \\ &\simeq \lim_{\Delta} \mathrm{Map}_{\mathrm{LComod}_\Lambda(\mathcal{D})}(-, \tilde{G}_C \circ \tilde{R}_\Theta^\bullet(\tilde{M})) \\ &\simeq \mathrm{Map}_{\mathrm{LComod}_\Lambda(\mathcal{D})}(-, \lim_{\Delta} (\tilde{G}_C \circ \tilde{R}_\Theta^\bullet(\tilde{M}))). \end{aligned}$$

□

Remark 1.3.12. In the situation of Proposition 1.3.10, let $\tilde{M} \in \mathrm{LComod}_\Theta(\mathcal{E})$ with underlying object $M \in \mathcal{E}$. Given a cobar resolution $\tilde{R}_\Theta^\bullet(\tilde{M})$, we can describe the cosimplicial object $\tilde{G}_C \circ \tilde{R}_\Theta^\bullet(\tilde{M}): \Delta \rightarrow \mathrm{LComod}_\Lambda(\mathcal{D})$ used to define $\tilde{G}(\tilde{M})$ a bit more explicitly as follows.

⁵Note that while Fact 1.3.9 guarantees the existence of *cobar* resolutions, here we only need a resolution.

Note that for a map $f: C_\Theta(X) \rightarrow C_\Theta(Y)$ between cofree Θ -comodules, $\tilde{G}_C(f)$ is given by the image of $\text{Id}_{C_\Lambda(G(X))}$ under the composite

$$\begin{aligned} \text{Map}_{\text{LComod}_\Lambda(\mathcal{D})}(C_\Lambda(G(X)), C_\Lambda(G(X))) &\stackrel{(1.3.11)}{\simeq} \text{Map}_{\text{LComod}_\Theta(\mathcal{E})}(\tilde{F}(C_\Lambda(G(X))), C_\Theta(X)) \\ &\xrightarrow{f \circ (-)} \text{Map}_{\text{LComod}_\Theta(\mathcal{E})}(\tilde{F}(C_\Lambda(G(X))), C_\Theta(Y)) \\ &\stackrel{(1.3.11)}{\simeq} \text{Map}_{\text{LComod}_\Lambda(\mathcal{D})}(C_\Lambda(G(X)), C_\Lambda(G(Y))). \end{aligned}$$

While this yields a general formula for $\tilde{G}_C(f)$ in terms of units and counits of the adjunctions involved, it is rather long, so we will unpack it only in some specific cases.

If $f = C_\Theta(f')$ for some $f': X \rightarrow Y$, then the diagram

$$\begin{array}{ccc} \text{Map}_{\text{LComod}_\Lambda(\mathcal{D})}(C_\Lambda(G(X)), C_\Lambda(G(X))) & \simeq & \text{Map}_{\text{LComod}_\Theta(\mathcal{E})}(\tilde{F}(C_\Lambda(G(X))), C_\Theta(X)) \\ \downarrow C_\Lambda(G(f')) \circ (-) & & \downarrow C_\Theta(f') \circ (-) \\ \text{Map}_{\text{LComod}_\Lambda(\mathcal{D})}(C_\Lambda(G(X)), C_\Lambda(G(Y))) & \simeq & \text{Map}_{\text{LComod}_\Theta(\mathcal{E})}(\tilde{F}(C_\Lambda(G(X))), C_\Theta(Y)) \end{array}$$

commutes by the naturality of the equivalences (1.3.11), which implies that $\tilde{G}_C(f) \simeq C_\Lambda(G(f'))$.

If $Y = V_\Theta(C_\Theta(X))$ and f is the unit map $u: C_\Theta(X) \rightarrow C_\Theta(V_\Theta(C_\Theta(X)))$ of the adjunction $V_\Theta \dashv C_\Theta$, then $\tilde{G}_C(u)$ is a map of the form

$$\kappa_X: C_\Lambda(G(X)) \rightarrow C_\Lambda(G(V_\Theta(C_\Theta(X)))) \simeq C_\Lambda(G(\Theta(X))).$$

While “what it does” depends on the specific situation, it can in any case be thought of as encoding a “right coaction” $C_\Lambda \circ G \rightarrow (C_\Lambda \circ G) \circ \Theta$ of Θ on $C_\Lambda \circ G$.

Combining these, $\tilde{G}_C \circ \tilde{R}_\Theta^\bullet(\tilde{M})$ can hence be depicted as

$$\begin{array}{ccccc} & & \xrightarrow{\kappa_{\Theta(M)}} & & \\ & \xrightarrow{\kappa_M} & & \xleftarrow{C_\Lambda G \epsilon_{\Theta M}} & \\ C_\Lambda(G(M)) & \xleftarrow{C_\Lambda G \epsilon_{\Theta M}} & C_\Lambda(G(\Theta(M))) & \xleftarrow{C_\Lambda G \Delta_{\Theta M}} & C_\Lambda(G(\Theta^2(M))) \quad \dots, \\ & \xrightarrow{C_\Lambda G \rho_M^\sim} & & \xleftarrow{C_\Lambda G \Theta \epsilon_{\Theta M}} & \\ & & & \xrightarrow{C_\Lambda G \Theta \rho_M^\sim} & \end{array}$$

which can be thought of as a “two sided cobar construction”.

Corollary 1.3.13 (cf. [RV16, Theorem 7.2.4]). *Let $F: \mathcal{D} \rightleftarrows \mathcal{E} : G$ be an adjunction. Assume that \mathcal{D} admits limits of cosimplicial objects. Then the standard lift $\bar{F}: \mathcal{D} \rightarrow \text{LComod}_{FG}(\mathcal{E})$ admits a right adjoint \bar{G} .*

Proof. We apply Proposition 1.3.10 to the diagram

$$\begin{array}{ccc} \text{LComod}_{\text{Id}_\mathcal{D}}(\mathcal{D}) & \xrightarrow{\bar{F} \circ V_{\text{Id}_\mathcal{D}}} & \text{LComod}_{FG}(\mathcal{E}) \\ \downarrow V_{\text{Id}_\mathcal{D}} \simeq & & \downarrow V_{FG} \\ \mathcal{D} & \xrightleftharpoons[F]{F} & \mathcal{E} \\ & \xleftarrow[\text{dashed } G]{\perp} & \end{array} \quad .$$

□

We will often be interested in the comonad associated to an extension-restriction of scalars adjunction, for which we fix some terminology now.

Definition 1.3.14. Let \mathcal{C} be a monoidal ∞ -category. Assume that \mathcal{C} admits limits of cosimplicial objects and geometric realizations of simplicial objects, and that the tensor product preserves geometric realizations in each variable. Let $\psi: R \rightarrow S$ be a map of algebras in \mathcal{C} and consider the extension-restriction of scalars adjunction $\psi_! \dashv \psi^*$.

We call the comonad $\psi_! \psi^*$ on $\mathrm{LMod}_S(\mathcal{C})$ induced by $\psi_! \dashv \psi^*$ the *descent comonad* of ψ . Moreover, we say that ψ *admits descent*⁶ if the adjunction $\psi_! \dashv \psi^*$ is comonadic, i.e., if the standard lift $\overline{\psi}_!: \mathrm{LMod}_R(\mathcal{C}) \rightarrow \mathrm{LComod}_{\psi_! \psi^*}(\mathrm{LMod}_S(\mathcal{C}))$ of $\psi_!$ is an equivalence.

Now let M be an R -module. We call M *S -complete* if the unit map $M \rightarrow \overline{\psi}^*(\overline{\psi}_!(M))$ of the adjunction of Corollary 1.3.13 is an equivalence. Moreover, we set

$$\mathbf{C}^\bullet(\psi)(M) := \overline{\psi}^* \circ \tilde{\mathbf{R}}_{\psi_! \psi^*}^\bullet(\overline{\psi}_!(M)): \Delta \rightarrow \mathrm{LMod}_R(\mathcal{C}).$$

Note that as $\overline{\psi}^*(\overline{\psi}_!(M)) \simeq \lim_{\Delta} \mathbf{C}^\bullet(\psi)(M)$, the aforementioned unit map yields a lift of $\mathbf{C}^\bullet(\psi)(M)$ to an M -coaugmented cosimplicial object, which we denote by $\mathbf{C}_+^\bullet(\psi)(M)$. When $M = R$, we drop M from the notation and refer to $\mathbf{C}^\bullet(\psi)$ as the *Amitsur complex* of ψ .

Unpacking the maps discussed in Remark 1.3.12, $\mathbf{C}_+^\bullet(\psi)(M)$ can be depicted as

$$M \xrightarrow{\psi \otimes_R M} S \otimes_R M \xleftarrow[\begin{smallmatrix} -\psi \otimes_R S \otimes_R M \rightarrow \\ -S \otimes_R \psi \otimes_R M \rightarrow \end{smallmatrix}]{\begin{smallmatrix} -\psi \otimes_R S \otimes_R M \rightarrow \\ -\mu_{S \otimes_R M} \rightarrow \end{smallmatrix}} S \otimes_R S \otimes_R M \quad \dots$$

Moreover, note that $\overline{\psi}_!$ is fully faithful if every R -module is S -complete. Hence, S -completeness of R -modules can be viewed as a partial comonadicity condition for the adjunction $\psi_! \dashv \psi^*$.

As another application of Proposition 1.3.10, we can construct a right adjoint of corestriction of scalars functors in certain cases.

Corollary 1.3.15. Let \mathcal{M} be an ∞ -category left-tensored over a monoidal ∞ -category \mathcal{C} and $\zeta: D \rightarrow E$ a map of coalgebras in \mathcal{C} . Assume further that $\mathrm{LComod}_D(\mathcal{M})$ admits limits of cosimplicial objects.

Then the corestriction of scalars functor $\zeta_*: \mathrm{LComod}_D(\mathcal{M}) \rightarrow \mathrm{LComod}_E(\mathcal{M})$ admits a right adjoint $\zeta^\#: \mathrm{LComod}_E(\mathcal{M}) \rightarrow \mathrm{LComod}_D(\mathcal{M})$.

Proof. Using the Barr–Beck–Lurie comonadicity theorem [Lur17, Theorem 4.7.3.5], we identify the comodule categories of D and E with the comodule categories of the associated comonads $D \otimes (-)$ and $E \otimes (-)$, and ζ with a map $D \otimes (-) \rightarrow E \otimes (-)$ of comonads. The result then follows by applying Proposition 1.3.10 to the diagram

$$\begin{array}{ccc} \mathrm{LComod}_D(\mathcal{M}) & \xrightarrow{\zeta_*} & \mathrm{LComod}_E(\mathcal{M}) \\ \downarrow \mathbf{v}_D & & \downarrow \mathbf{v}_E \\ \mathcal{M} & \xlongequal{\quad} & \mathcal{M} \end{array} \quad .$$

⁶Various other notions of (admitting or satisfying) “descent” exist in the literature. In particular, our definition differs from [Mat16, Definition 3.17] (which requires pro-constantness of $\mathbf{C}^\bullet(\psi)$) and [RV20, Definition 10.5.12] (which only requires fully faithfulness of $\overline{\psi}_!$).

□

Notation 1.3.16. Let \mathcal{M} be an ∞ -category left-tensored over a monoidal ∞ -category \mathcal{C} and D a coalgebra in \mathcal{C} . When dealing with the various resolutions with respect to the associated comonad $D \otimes (-)$, we will simplify the notation by writing R_D instead of $R_{D \otimes (-)}$. Moreover, for a map $\zeta: D \rightarrow E$ of coalgebras in \mathcal{C} , we will denote the two sided cobar construction $(\zeta^\#)_C \circ \tilde{R}_E^\bullet(-)$ of Remark 1.3.12 by $\Omega_E^\bullet(D, -)$.

Unpacking the descriptions of Remark 1.3.12, the two-sided cobar construction $\Omega_E^\bullet(D, M)$ for a E -comodule can be depicted as

$$\begin{array}{ccccccc}
 & & & & & & -((D \otimes \zeta) \circ \Delta_D) \otimes E \otimes M \rightarrow \\
 & & & & & & \leftarrow D \otimes \epsilon_E \otimes E \otimes M \leftarrow \\
 D \otimes M & \xleftarrow{D \otimes \epsilon_E \otimes M} & D \otimes E \otimes M & \xleftarrow{D \otimes \Delta_E \otimes M} & D \otimes E \otimes E \otimes M & \dots, \\
 & \xrightarrow{D \otimes \rho_M} & & \xleftarrow{D \otimes E \otimes \epsilon_E \otimes M} & & \\
 & & & \xrightarrow{D \otimes E \otimes \rho_M} & &
 \end{array}$$

which can be thought of as being formally dual to the two-sided bar construction for an algebra. Hence $\zeta^\#$ can be thought of as “relative cotensor product” $D \square_E (-)$.

We now fix some terminology for coalgebras equipped with a coalgebra map from the tensor unit, which is the case, for instance, for bialgebras.

Definition 1.3.17. Let \mathcal{M} be an ∞ -category left-tensored over a monoidal ∞ -category \mathcal{C} and D a coaugmented coalgebra in \mathcal{C} , i.e., a coalgebra equipped with a coalgebra map $\eta_D: 1_{\mathcal{C}} \rightarrow D$.

Corestriction along η_D yields a functor

$$\text{Triv}_D: \mathcal{M} \simeq \text{LComod}_{1_{\mathcal{C}}}(\mathcal{M}) \xrightarrow{(\eta_D)_\star} \text{LComod}_D(\mathcal{M})$$

which we call the *trivial comodule functor*. Moreover, if \mathcal{M} admits limits of cosimplicial objects, then, by Corollary 1.3.15, Triv_D admits a right adjoint

$$\text{Prim}_D: \text{LComod}_D(\mathcal{M}) \xrightarrow{(\eta_D)^\#} \text{LComod}_{1_{\mathcal{C}}}(\mathcal{M}) \simeq \mathcal{M}.$$

which we call the *primitives functor*.

Remark 1.3.18. Note that in the situation of Definition 1.3.17, we have $\epsilon_D \circ \eta_D \simeq \text{Id}_{1_{\mathcal{C}}}$, which implies that $V_D \circ \text{Triv}_D \simeq \text{Id}_{\mathcal{M}}$. This equivalence of left adjoints induces an equivalence $\text{Prim}_D \circ C_D \simeq \text{Id}_{\mathcal{M}}$ of right adjoints.

Remark 1.3.19. Let \mathcal{C} be an \mathbb{E}_{k+1} -monoidal ∞ -category for $k \in \mathbb{N}$, and H an $(\mathbb{E}_k, \mathbb{E}_1)$ -bialgebra in \mathcal{C} , which is in particular a coaugmented coalgebra via the unit map of its algebra structure. Then, as a corestriction of scalars functor, $\text{Triv}_H: \mathcal{C} \rightarrow \text{LComod}_H(\mathcal{C})$ can be extended to a strongly \mathbb{E}_k -monoidal functor with respect to the \mathbb{E}_k -monoidal structure of Fact 1.2.9. Hence, by Fact 1.1.6, its right adjoint Prim_H admits a lax \mathbb{E}_k -monoidal structure.

We conclude this subsection with some properties of restricted left-tensorings of Construction 1.1.20.

Lemma 1.3.20. *Let $F: \mathcal{D} \rightarrow \mathcal{E}$ be a strongly monoidal functor between monoidal ∞ -categories, \mathcal{M} an ∞ -category left-tensored over \mathcal{E} (which we also view as left-tensored over \mathcal{D} via restriction along F), and D a coalgebra in \mathcal{D} . Then the functor*

$$\Xi_D: \mathrm{LComod}_D(\mathcal{M}) \rightarrow \mathrm{LComod}_{F(D)}(\mathcal{M})$$

induced by the functor $\mathrm{LComod}^{\mathcal{D}}(\mathcal{M}) \rightarrow \mathrm{LComod}^{\mathcal{E}}(\mathcal{M})$ of Construction 1.1.20 is an equivalence.

Proof. Note that Ξ_D and the forgetful functors fit into a commutative diagram

$$\begin{array}{ccc} \mathrm{LComod}_D(\mathcal{M}) & \xrightarrow{\Xi_D} & \mathrm{LComod}_{F(D)}(\mathcal{M}) \\ & \searrow V_D & \swarrow V_{F(D)} \\ & \mathcal{M} & \end{array},$$

to which we can apply the criterion of [Lur17, Corollary 4.7.3.16 and Remark 4.7.3.17]. As both V_D and $V_{F(D)}$ are left adjoints of comonadic adjunctions, it will suffice to show that for all $X \in \mathcal{M}$, the underlying map of the composite

$$\Xi_D C_D X \xrightarrow{u} C_{F(D)} V_{F(D)} \Xi_D C_D X \simeq C_{F(D)} V_D C_D X \xrightarrow{c} C_{F(D)} X,$$

where u is induced by the unit of adjunction $V_{F(D)} \dashv C_{F(D)}$, and c is induced by the counit of the adjunction $V_D \dashv C_D$, is an equivalence. This underlying map is given by the composite

$$F(D) \otimes X \xrightarrow{\Delta_{F(D)} \otimes X} F(D) \otimes F(D) \otimes X \xrightarrow{F(D) \otimes F(\epsilon_D) \otimes X \simeq F(D) \otimes \epsilon_{F(D)} \otimes X} F(D) \otimes X,$$

which is indeed an equivalence. \square

While the proof of Lemma 1.3.20 does not mention comonads explicitly, it uses a statement equivalent to the comonadicity theorem, which is arguably the “universal” case. Indeed, the left-tensorings of \mathcal{M} over \mathcal{C} and \mathcal{D} are given by restrictions of its left-tensoring over $\mathrm{Fun}(\mathcal{M}, \mathcal{M})$ along strongly monoidal functors $F_{\mathcal{C}}: \mathcal{C} \rightarrow \mathrm{Fun}(\mathcal{M}, \mathcal{M})$ and $F_{\mathcal{D}}: \mathcal{D} \rightarrow \mathrm{Fun}(\mathcal{M}, \mathcal{M})$. The comonadicity of the categories of D - and $F(D)$ -comodules corresponds to the result of the lemma for $F_{\mathcal{C}}$ and $F_{\mathcal{D}}$, which implies the result for F by a 2-out-of-3 argument.

Corollary 1.3.21. *Let $F: \mathcal{D} \rightarrow \mathcal{E}$ be a strongly monoidal functor between monoidal ∞ -categories, \mathcal{M} an ∞ -category left-tensored over \mathcal{E} , which we also view as left-tensored over \mathcal{D} via restriction along F . Then the diagram*

$$\begin{array}{ccc} \mathrm{LComod}^{\mathcal{D}}(\mathcal{M}) & \xrightarrow{\mathrm{Coalg}/\mathcal{L}\mathcal{M}(\widehat{F})} & \mathrm{LComod}^{\mathcal{E}}(\mathcal{M}) \\ \theta_{\mathcal{D}} \downarrow & & \downarrow \theta_{\mathcal{E}} \\ \mathrm{Coalg}(\mathcal{D}) & \xrightarrow{\mathrm{Coalg}(F)} & \mathrm{Coalg}(\mathcal{E}) \end{array}$$

of Construction 1.1.20 is a pullback square.

Proof. Recall that $\theta_{\mathcal{D}}$ and $\theta_{\mathcal{E}}$ are both cocartesian fibrations and in both cases, cocartesian edges are those whose underlying map in \mathcal{E} is an equivalence. Hence, the projection $\pi: \mathcal{P} \rightarrow \mathrm{Coalg}(\mathcal{D})$ of the pullback is a cocartesian fibration and the induced functor $\Xi: \mathrm{LComod}^{\mathcal{D}}(\mathcal{M}) \rightarrow \mathcal{P}$ preserves cocartesian edges. Therefore, by [Lur09, Corollary 2.4.4.4], it is enough to show that Ξ induces equivalences on the fibers, which is exactly the statement of Lemma 1.3.20. \square

Remark 1.3.22. Let $k \in \mathbb{N} \cup \{\infty\}$ and $F: \mathcal{D} \rightarrow \mathcal{E}$ a strongly \mathbb{E}_{k+1} -monoidal functor between \mathbb{E}_{k+1} -monoidal ∞ -categories. Then $\mathrm{Coalg}(F): \mathrm{Coalg}(\mathcal{D}) \rightarrow \mathrm{Coalg}(\mathcal{E})$ is strongly \mathbb{E}_k -monoidal with respect to the \mathbb{E}_k -monoidal structures discussed in Fact 1.2.2. Moreover, $\theta_{\mathcal{E}}: \mathrm{LComod}^{\mathcal{E}}(\mathcal{E}) \rightarrow \mathrm{Coalg}(\mathcal{E})$ is strongly \mathbb{E}_k -monoidal when we endow its source with the \mathbb{E}_k -monoidal structure discussed in Fact 1.2.9.

Therefore, since limits in $\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{\infty}}(\mathrm{Cat}_{\infty})$ can be computed in Cat_{∞} (cf. [Lur17, Corollary 3.2.2.4]), the diagram

$$\begin{array}{ccc} \mathrm{LComod}^{\mathcal{D}}(\mathcal{E}) & \xrightarrow{\mathrm{Coalg}/\mathcal{LM}(\widehat{F})} & \mathrm{LComod}^{\mathcal{E}}(\mathcal{E}) \\ \theta_{\mathcal{D}} \downarrow & & \downarrow \theta_{\mathcal{E}} \\ \mathrm{Coalg}(\mathcal{D}) & \xrightarrow{\mathrm{Coalg}(F)} & \mathrm{Coalg}(\mathcal{E}) \end{array}$$

of Corollary 1.3.21 can be upgraded to a pullback square of \mathbb{E}_k -monoidal ∞ -categories. In particular, $\mathrm{LComod}^{\mathcal{D}}(\mathcal{E})$ admits an \mathbb{E}_k -monoidal structure such that the functors $\mathrm{Coalg}/\mathcal{LM}(\widehat{F})$ and $\theta_{\mathcal{D}}$ are strongly \mathbb{E}_k -monoidal.

1.4. Linear functors

We will also need versions of Example 1.3.4 through Remark 1.3.7 that are in an appropriate sense compatible with the action of a monoidal ∞ -category, so we review the relevant theory now. Besides the usual algebra-coalgebra duality, some references to [Lur17, Chapter 4] in this subsection also implicitly switch the roles of left and right actions.

As discussed in [Lur17, Remark 4.3.3.7], for an algebra R in monoidal ∞ -category \mathcal{C} , $\mathrm{LMod}_R(\mathcal{C})$ admits a right-tensoring over \mathcal{C} , which is encoded by an \mathcal{RM} -monoidal ∞ -category. Informally speaking, if M is a left R -module with action map $\alpha_M: R \otimes M \rightarrow M$ and $X \in \mathcal{C}$, $M \otimes X$ admits a left action of R given by $\alpha_M \otimes X: R \otimes (M \otimes X) \rightarrow M \otimes X$. Dually, for every coalgebra D in \mathcal{C} , we obtain a right-tensoring of $\mathrm{LComod}_D(\mathcal{C})$ over \mathcal{C} by taking the fiberwise opposite of the right-tensoring of $\mathrm{LComod}_D(\mathcal{C})^{\mathrm{op}} \simeq \mathrm{LMod}_D(\mathcal{C}^{\mathrm{op}})$ over $\mathcal{C}^{\mathrm{op}}$. Moreover, note that if \mathcal{C} admits geometric realizations and its tensor product preserves them in each variable, also these right-tensorings preserve geometric realizations in each variable.

Given ∞ -categories \mathcal{M} and \mathcal{N} that are right-tensored over \mathcal{C} , we will be interested in functors $F: \mathcal{M} \rightarrow \mathcal{N}$ that are compatible with the tensoring in the sense that for all $M \in \mathcal{M}$ and $C \in \mathcal{C}$, $F(M \otimes C) \simeq F(M) \otimes C$, which can be formalized as follows.

Definition 1.4.1 ([Lur17, Definition 4.6.2.7, Remark 4.6.2.8]). Let \mathcal{C} be a monoidal ∞ -category, $\bar{\mathcal{M}}^\otimes \rightarrow \mathcal{R}\mathcal{M}^\otimes$ and $\bar{\mathcal{N}}^\otimes \rightarrow \mathcal{R}\mathcal{M}^\otimes$ $\mathcal{R}\mathcal{M}$ -monoidal ∞ -categories exhibiting $\mathcal{M} := \bar{\mathcal{M}}^\otimes_{\mathfrak{m}}$ and $\mathcal{N} := \bar{\mathcal{N}}^\otimes_{\mathfrak{m}}$ as right-tensored over \mathcal{C} , with specified identifications

$$\bar{\mathcal{M}} \times_{\mathcal{R}\mathcal{M}} \mathbb{E}_1^\otimes \simeq \mathcal{C}^\otimes \simeq \bar{\mathcal{N}} \times_{\mathcal{R}\mathcal{M}} \mathbb{E}_1^\otimes.$$

of pullbacks along the ∞ -operad map $\mathbb{E}_1^\otimes \rightarrow \mathcal{R}\mathcal{M}^\otimes$ that “classifies the algebra in $\mathcal{R}\mathcal{M}$ ”. Assume that \mathcal{M} and \mathcal{N} admit geometric realizations.

We call a strongly $\mathcal{R}\mathcal{M}$ -monoidal functor $F^\otimes: \bar{\mathcal{M}}^\otimes \rightarrow \bar{\mathcal{N}}^\otimes$ *right \mathcal{C} -linear* if its restriction along $\mathbb{E}_1^\otimes \rightarrow \mathcal{R}\mathcal{M}^\otimes$ induces $\text{Id}_{\mathcal{C}^\otimes}$ under the aforementioned identifications. The full subcategory of

$$\text{Fun}_{\mathcal{R}\mathcal{M}^\otimes}(\bar{\mathcal{M}}^\otimes, \bar{\mathcal{N}}^\otimes) \times_{\text{Fun}_{\mathbb{E}_1^\otimes}(\mathcal{C}^\otimes, \mathcal{C}^\otimes)} \{\text{Id}_{\mathcal{C}^\otimes}\}$$

spanned by right \mathcal{C} -linear functors $F^\otimes: \bar{\mathcal{M}}^\otimes \rightarrow \bar{\mathcal{N}}^\otimes$ such that $F_{\mathfrak{m}}^\otimes: \mathcal{M} \rightarrow \mathcal{N}$ preserves geometric realizations will be denoted by $\text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\mathcal{M}, \mathcal{N})$.

Note that under the correspondence of Fact 1.1.16, a right \mathcal{C} -linear functor can be viewed as map of right \mathcal{C} -modules in Cat_∞ . As we discuss below, there is a similar description if we add the condition of preserving geometric realizations.

Remark 1.4.2. Let \mathcal{C} be a monoidal ∞ -category that admits geometric realizations and whose tensor product preserves geometric realizations in each variable, and $\mathcal{M}, \mathcal{N}, \mathcal{L}$ ∞ -categories right-tensored over \mathcal{C} that admit geometric realizations and such that the right-tensoring preserves geometric realizations in each variable.

Note that composition of functors over $\mathcal{R}\mathcal{M}^\otimes$ preserves right \mathcal{C} -linearity and preservation of geometric realizations in the fiber over $\mathfrak{m} \in \mathcal{R}\mathcal{M} = \mathcal{R}\mathcal{M}_{\langle 1 \rangle}^\otimes$. Hence we can restrict the composition operation to a functor

$$\text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\mathcal{N}, \mathcal{L}) \times \text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\mathcal{M}, \mathcal{N}) \rightarrow \text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\mathcal{M}, \mathcal{L}).$$

These composition maps allow us to extend constructions of Example 1.3.4 from functor categories to categories of right \mathcal{C} -linear geometric-realization-preserving functors. For instance, considering $\text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\mathcal{M}, \mathcal{M})$ as a marked simplicial set by marking equivalences, the composition map yields a monoid object in marked simplicial sets and hence a monoidal structure on $\text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\mathcal{M}, \mathcal{M})$. Similarly, ∞ -categories of the form $\text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\mathcal{N}, \mathcal{M})$ are left-tensored over $\text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\mathcal{M}, \mathcal{M})$ via composition.

In fact, $\text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\mathcal{M}, \mathcal{N})$ can be seen as a “function object” in an appropriate ∞ -category as follows. By [Lur17, Remark 4.8.1.5] and [Lur09, Lemma 5.5.8.4], the cartesian monoidal structure of Cat_∞ restricts to a cartesian monoidal structure on the ∞ -category $\text{Cat}_\infty(\{\Delta^{\text{op}}\})$ of ∞ -categories admitting geometric realizations and functors that preserve geometric realizations. As discussed in [Lur17, Remark 4.8.1.9], the compatibility of tensor products with geometric realizations means that \mathcal{C} can be viewed as an algebra and $\mathcal{M}, \mathcal{N}, \mathcal{L}$ as right modules over it in this ∞ -category.

By [Lur17, Remark 4.8.4.14], $\text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(-, ?)$, can be seen as a functor

$$\text{RMod}_{\mathcal{C}}(\text{Cat}_\infty(\{\Delta^{\text{op}}\})) \times \text{RMod}_{\mathcal{C}}(\text{Cat}_\infty(\{\Delta^{\text{op}}\})) \rightarrow \text{Cat}_\infty(\{\Delta^{\text{op}}\}).$$

On the other hand, we can lift the functor $(-) \times \mathcal{M}: \text{Cat}_\infty(\{\Delta^{\text{op}}\}) \rightarrow \text{Cat}_\infty(\{\Delta^{\text{op}}\})$ to $\text{RMod}_{\mathcal{C}}(\text{Cat}_\infty(\{\Delta^{\text{op}}\}))$, for instance by considering \mathcal{M} as a $\mathbf{1}_{\text{Cat}_\infty(\{\Delta^{\text{op}}\})}$ - \mathcal{C} -bimodule. This functor is left adjoint to $\text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\mathcal{M}, -)$, where the unit and the counit transformations are induced by those of the adjunction $(-) \times \mathcal{M} \dashv \text{Fun}(\mathcal{M}, -)$. In particular, the evaluation functor $\text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\mathcal{M}, \mathcal{M}) \times \mathcal{M} \rightarrow \mathcal{M}$ induces an equivalence

$$\text{Fun}_{\text{Cat}_\infty(\{\Delta^{\text{op}}\})}((-), \text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\mathcal{M}, \mathcal{M})) \xrightarrow{\cong} \text{Fun}_{\text{RMod}_{\mathcal{C}}(\text{Cat}_\infty(\{\Delta^{\text{op}}\}))}((-) \times \mathcal{M}, \mathcal{M})$$

that exhibits $\text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\mathcal{M}, \mathcal{M})$ as an endomorphism object for \mathcal{M} .

We can therefore speak of right \mathcal{C} -linear geometric-realization-preserving adjunctions and (co)monads. In particular, one can construct linear (co)monads from linear adjunctions as in Example 1.3.5, identify (co)module structures over linear (co)monads on functors with certain linear lifts as in Fact 1.3.6 etc.

Lemma 1.4.3. *Let \mathcal{C} be a monoidal ∞ -category that admits geometric realizations and such that the tensor product preserves geometric realizations in each variable. Then, for every algebra R in \mathcal{C} , the adjunction $F_R: \mathcal{C} \rightleftarrows \text{LMod}_R(\mathcal{C}) : U_R$ is an adjunction of right \mathcal{C} -linear geometric-realization-preserving functors. Dually, for every coalgebra D in \mathcal{C} , the adjunction $V_D: \text{LComod}_D(\mathcal{C}) \rightleftarrows \mathcal{C} : C_D$ is also an adjunction of right \mathcal{C} -linear geometric-realization-preserving functors.*

Proof. Right \mathcal{C} -linearity of these adjunctions follows from [Lur17, Lemmas 4.8.4.10 and 4.8.4.12], so it is enough to show that the functors preserve geometric realizations. The left adjoints preserve colimits, in particular geometric realizations. Since the tensor product preserves geometric realizations in each variable, [Lur17, Corollary 4.2.3.5] implies that U_R preserves geometric realizations. As for C_D , first note that the composite $V_D \circ C_D$ preserves geometric realizations as it is given by tensoring with D . Now V_D reflects colimits, implying that C_D preserves geometric realizations as well. \square

Linear functors between module categories admit an “algebraic” description in terms of bimodule categories.

Fact 1.4.4. *Let \mathcal{C} be a monoidal ∞ -category and R, S algebras in \mathcal{C} . Assume that \mathcal{C} admits geometric realizations, and that the tensor product preserves them in each variable. Then, by [Lur17, Theorems 4.8.4.1 and 4.3.2.7], the ∞ -category ${}_R \text{Bimod}_S$ of R - S -bimodules is equivalent to the ∞ -category $\text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\text{LMod}_S(\mathcal{C}), \text{LMod}_R(\mathcal{C}))$ via functors informally given by*

$$\begin{aligned} \text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\text{LMod}_S(\mathcal{C}), \text{LMod}_R(\mathcal{C})) &\rightarrow {}_R \text{Bimod}_S \\ F &\mapsto F(S) \end{aligned}$$

and

$$\begin{aligned} {}_R \text{Bimod}_S &\rightarrow \text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\text{LMod}_S(\mathcal{C}), \text{LMod}_R(\mathcal{C})) \\ X &\mapsto X \otimes_S (-). \end{aligned}$$

Remark 1.4.5. Intuitively, the equivalences of Fact 1.4.4 can be thought of as the “morphism part” of an equivalence between two incarnations of the “Morita $(\infty, 2)$ -category” of \mathcal{C} (cf. [Lur17, discussion on page 738]). In particular, in the case where $R = S$, the equivalence ${}_R \text{Bimod}_R \simeq \text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\text{LMod}_R(\mathcal{C}), \text{LMod}_R(\mathcal{C}))$ is strongly monoidal when we equip ${}_R \text{Bimod}_R$ with the relative tensor product and $\text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\text{LMod}_R(\mathcal{C}), \text{LMod}_R(\mathcal{C}))$ with the composition monoidal structure. For the case where \mathcal{C} is the module category of an \mathbb{E}_2 -ring spectrum, such a strongly monoidal equivalence appears in [Bea21, Theorem 4.2].

What we discuss here only suffices to construct a particular monoidal structure on ${}_R \text{Bimod}_R$ whose the tensor product is given by \otimes_R by transferring the monoidal structure on $\text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\text{LMod}_R(\mathcal{C}), \text{LMod}_R(\mathcal{C}))$ along the aforementioned equivalence. In particular, it is not evident that this monoidal structure agrees with the one of [Lur17, Proposition 4.4.3.12].

However, the only case where we will need more coherence than the pointwise equivalences of the form $(X \otimes_R Y) \otimes_R (-) \simeq (X \otimes_R (-)) \circ (Y \otimes_R (-))$ is the case where $R = \mathbf{1}_{\mathcal{C}}$, and in that case one can check that the “standard” monoidal structure on $\mathcal{C} \simeq {}_{\mathbf{1}_{\mathcal{C}}} \text{Bimod}_{\mathbf{1}_{\mathcal{C}}}$ agrees with the transferred one by observing that their action maps $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ coincide and using the universal property of ${}_{\mathbf{1}_{\mathcal{C}}} \text{Bimod}_{\mathbf{1}_{\mathcal{C}}} \simeq \text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\mathcal{C}, \mathcal{C})$ as an endomorphism object for \mathcal{C} .

Remark 1.4.6. Let \mathcal{C} be a monoidal ∞ -category and D a coalgebra in \mathcal{C} . Assume that \mathcal{C} admits geometric realizations, and that the tensor product preserves geometric realizations in each variable.

Then, the forgetful-cofree adjunction $V_D : \text{LComod}_D(\mathcal{C}) \rightleftarrows \mathcal{C} : C_D$ is an adjunction of right \mathcal{C} -linear functors that preserve geometric realizations by Lemma 1.4.3. Hence, by the analogue of Example 1.3.5 for such functors, $V_D \circ C_D \simeq D \otimes (-)$ is a coendomorphism object for $V_D \in \text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\text{LComod}_D(\mathcal{C}), \mathcal{C})$ in $\text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\mathcal{C}, \mathcal{C})$. Restricting this along the strongly monoidal equivalence $\mathcal{C} \simeq \text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\mathcal{C}, \mathcal{C})$ of Remark 1.4.5 for $R = S = \mathbf{1}_{\mathcal{C}}$, we see that the natural transformation $V_D(-) \rightarrow D \otimes V_D(-)$ given by the coaction map exhibits D as a coendomorphism object for V_D .

2. Comodules over spaces

In this section, we review the theory of coalgebraic structures in the ∞ -category \mathcal{S} of spaces and give a simple description of comodules over spaces in the slice category \mathcal{S}_T of a space T (Proposition 2.0.6), which will be useful later when we consider Thom objects (cf. Example 3.1.3).

We start by recording a “purely categorical” description of coalgebras and comodules in a cartesian symmetric monoidal ∞ -category.

Fact 2.0.1. *Let \mathcal{D} be an ∞ -category equipped with a cartesian symmetric monoidal structure. Every object X of \mathcal{D} admits a unique coalgebra structure whose comultiplication is the diagonal map $(\mathrm{Id}_X, \mathrm{Id}_X): X \rightarrow X \times X$. More precisely, for all $l \in \mathbb{N} \cup \{\infty\}$, the (strongly symmetric monoidal) forgetful functor $\mathrm{Coalg}_{\mathbb{E}_l/\mathbb{E}_\infty}(\mathcal{D}) \rightarrow \mathcal{D}$ is an equivalence by [Lur17, Proposition 2.4.3.9]. In particular, for all $k \in \mathbb{N} \cup \{\infty\}$, we obtain an equivalence $\mathrm{Bialg}_{\mathbb{E}_k, \mathbb{E}_l}(\mathcal{D}) \simeq \mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_\infty}(\mathcal{D})$.*

Moreover, a coaction of an object X on another object Y is always of the form $(f, \mathrm{Id}_Y): Y \rightarrow X \times Y$ for some map $f: Y \rightarrow X$ (cf. [BP19, Corollary 2.2]). More precisely, by [HMS22, Corollary 2.6.6]⁷, we have an equivalence $\mathrm{LComod}(\mathcal{D}) \simeq \mathrm{Fun}([1], \mathcal{D})$ under which evaluation at 1 corresponds to the forgetful functor $\theta_{\mathcal{D}}: \mathrm{LComod}(\mathcal{D}) \rightarrow \mathrm{Coalg}(\mathcal{D}) \simeq \mathcal{D}$ and evaluation at 0 corresponds to the functor $\varsigma_{\mathcal{D}}: \mathrm{LComod}(\mathcal{D}) \rightarrow \mathcal{D}$ that picks the object being coacted on.

In this situation, we will often implicitly identify \mathcal{D} with $\mathrm{Coalg}(\mathcal{D})$. For instance, given an ∞ -category \mathcal{M} left-tensored over \mathcal{D} and an object $X \in \mathcal{D}$, we will write $\mathrm{LComod}_X(\mathcal{M})$ for the ∞ -category of comodules over the image of X under the equivalence $\mathcal{D} \simeq \mathrm{Coalg}(\mathcal{D})$.

Applying this to the cartesian symmetric monoidal ∞ -category \mathcal{S} , we obtain coalgebraic structures that can be transferred to every presentably monoidal ∞ -category as follows.

Remark 2.0.2. Let $k, l \in \mathbb{N} \cup \{\infty\}$ and \mathcal{C} a presentably \mathbb{E}_{k+l} -monoidal ∞ -category. Then, using the equivalences of Fact 2.0.1, we obtain lifts of the strongly \mathbb{E}_{k+l} -monoidal functor $(-) \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}: \mathcal{S} \rightarrow \mathcal{C}$ of Fact 1.1.18 to functors $\mathcal{S} \rightarrow \mathrm{Coalg}_{\mathbb{E}_l/\mathbb{E}_{k+l}}(\mathcal{C})$ and $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{S}) \rightarrow \mathrm{Bialg}_{\mathbb{E}_k, \mathbb{E}_l}(\mathcal{C})$. Moreover, $\mathrm{Coalg}_{\mathbb{E}_l/\mathbb{E}_{k+l}}(\mathcal{C})$ is presentable by [Pér20, Proposition 2.2.6] and colimits in $\mathrm{Coalg}_{\mathbb{E}_l/\mathbb{E}_{k+l}}(\mathcal{C})$ can be computed in \mathcal{C} by [Lur17, Corollary 3.2.2.4]. Hence the lift $\mathcal{S} \rightarrow \mathrm{Coalg}_{\mathbb{E}_l/\mathbb{E}_{k+l}}(\mathcal{C})$ of $(-) \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}$ can in fact be identified with $(-) \odot_{\mathrm{Coalg}_{\mathbb{E}_l/\mathbb{E}_{k+l}}(\mathcal{C})} \mathbf{1}_{\mathcal{C}}$.

Now let $l > 0$. As every presentable ∞ -category is left-tensored over \mathcal{S} in an *essentially unique* way, pulling back the left-tensoring of \mathcal{C} over itself along $(-) \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}: \mathcal{S} \rightarrow \mathcal{C}$ recovers the original left-tensoring of \mathcal{C} over \mathcal{S} . In particular we obtain equivalent ∞ -categories of comodules over spaces (which we all denote by $\mathrm{LComod}^{\mathcal{S}}(\mathcal{C})$) when we consider different \mathbb{E}_l -monoidal structures on \mathcal{C} .

However, the \mathbb{E}_k -monoidal structure on $\mathrm{LComod}^{\mathcal{S}}(\mathcal{C})$ discussed in Remark 1.3.22, and therefore the \mathbb{E}_k -monoidal structure on $\mathrm{LComod}_X(\mathcal{C}) \simeq \mathrm{LComod}_{X \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}}(\mathcal{C})$ for an \mathbb{E}_k -space

⁷The cited corollary yields in fact an equivalence between the ∞ -category of coalgebras over the ∞ -operad \mathcal{BM} , which classifies two (co)algebras and a bi(co)module over them, and $\mathrm{Fun}([1] \amalg_0 [1], \mathcal{D})$. The desired equivalence for $\mathrm{LComod}(\mathcal{D})$ can be obtained by restricting that equivalence to those \mathcal{BM} -comodules for which the coalgebra coacting from the right is the terminal object.

X , do depend on the \mathbb{E}_k -monoidal structure of $\mathrm{LComod}^{\mathcal{C}}(\mathcal{C})$, hence that of \mathcal{C} . We will therefore prefer to speak of $X \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}$ -comodules instead of X -comodules in \mathcal{C} when the \mathbb{E}_k -monoidal structure of the comodule category is relevant.

Moreover, note that every colimit-preserving functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between presentable ∞ -categories (i.e., every morphism in $\mathrm{Pr}^L \simeq \mathrm{LMod}_{\mathcal{S}}(\mathrm{Pr}^L)$) induces a functor

$$\mathrm{LComod}^{\mathcal{S}}(F): \mathrm{LComod}^{\mathcal{S}}(\mathcal{C}) \rightarrow \mathrm{LComod}^{\mathcal{S}}(\mathcal{D})$$

and similarly functors between comodule categories over a fixed space.

We now move on to our analysis of comodules over spaces in slice categories of the form $\mathcal{S}_{/T}$ for a space T . We start by recording some properties of this ∞ -category, which are essentially consequences of it being a slice ∞ -topos.

Fact 2.0.3 (cf. [Lur09, Proposition 6.3.5.1]). *For a space T , $\mathcal{S}_{/T}$ admits a cartesian presentably symmetric monoidal structure given by fiber products over T .*

Moreover, the forgetful functor $V_T: \mathcal{S}_{/T} \rightarrow \mathcal{S}$ admits a colimit-preserving right adjoint $C_T: \mathcal{S} \rightarrow \mathcal{S}_{/T}$ sending $X \in \mathcal{S}$ to $\mathrm{pr}_2: X \times T \rightarrow T$.⁸ In particular, V_T preserves colimits and thus induces functors on categories of comodules over spaces.

Viewing C_T as a colimit-preserving functor that is strongly symmetric monoidal with respect to the cartesian symmetric monoidal structures in its source and target, we see that it agrees with $(-) \odot_{\mathcal{S}_{/T}} \mathrm{Id}_T: \mathcal{S} \rightarrow \mathcal{S}_{/T}$ by the initiality of \mathcal{S} in $\mathrm{CAlg}(\mathrm{Pr}^L)$ (cf. Fact 1.1.18). Hence, for a space X and an object $f: Y \rightarrow T$ of $\mathcal{S}_{/T}$, $X \odot_{\mathcal{S}_{/T}} f \simeq (X \odot_{\mathcal{S}_{/T}} \mathrm{Id}_T) \times_T f$ is given by $C_T(X) \times_T f$, i.e., the composite $X \times Y \xrightarrow{\mathrm{pr}_2} Y \xrightarrow{f} T$.

Even though $\mathrm{LComod}^{\mathcal{S}}(\mathcal{S}_{/T})$ depends only on the underlying ∞ -category $\mathcal{S}_{/T}$, we will use the cartesian monoidal structure on $\mathcal{S}_{/T}$ to give an explicit description of comodules over spaces. On the other hand, in our application to Thom objects, we will consider the slice category of an \mathbb{E}_k -space and equip it with a non-cartesian monoidal structure.

Fact 2.0.4 ([Lur17, Theorem 2.2.2.4], [ABG18, Corollary 6.13]). *Let $k \in \mathbb{N} \cup \{\infty\}$ and P an \mathbb{E}_k -space. Then the slice category $\mathcal{S}_{/P}$ admits a presentably \mathbb{E}_k -monoidal structure, which we informally call the convolution monoidal structure, such that the forgetful functor $V_P: \mathcal{S}_{/P} \rightarrow \mathcal{S}$ is strongly \mathbb{E}_k -monoidal.*

Informally speaking, given two objects $f: Y \rightarrow P$ and $g: Z \rightarrow P$, their tensor product with respect to the convolution monoidal structure is given by the composite

$$Y \times Z \xrightarrow{f \times g} P \times P \xrightarrow{\mu_P} P,$$

and the unit of the convolution monoidal structure is the unit map $\eta_P: \{*\} \rightarrow P$ of P . Note that for a space X , $X \odot_{\mathcal{S}_{/P}} f$ can be alternatively described as the convolution product of $X \odot_{\mathcal{S}_{/P}} \eta_P$ and f , so it can be identified with

$$X \times Y \xrightarrow{\mathrm{const}_{\eta_P(*)} \times f} P \times P \xrightarrow{\mu_P} P.$$

⁸As the notation might suggest, the adjunction $V_T \dashv C_T$ corresponds to the forgetful-cofree adjunction under the equivalence $\mathcal{S}_{/T} \simeq \mathrm{LComod}_T(\mathcal{S})$.

Moreover, \mathbb{E}_k -algebras $f: X \rightarrow P$ with respect to this \mathbb{E}_k -monoidal structure correspond to maps of \mathbb{E}_k spaces. For instance, the multiplication map of such an f in $\mathcal{S}_{/P}$ is given by a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu_X} & X \\ f \times f \searrow & & \swarrow f \\ P \times P & \xrightarrow{\mu_P} & P \end{array} .$$

which witnesses the compatibility of the multiplications of X and P .

Notation 2.0.5. For a space T , we will abbreviate $\ominus_{\mathcal{S}_{/T}}$ as \ominus_T . For $k \in \mathbb{N}_+ \cup \{\infty\}$ and an \mathbb{E}_k -space P , we will denote the tensor product of the convolution monoidal structure of Fact 2.0.4 by \otimes_P and its unit $\eta_P: \{*\} \rightarrow P$ by $\mathbf{1}_P$. Moreover, when we need to disambiguate the monoidal structure we use, we will decorate the objects in question with \times_T (or \times_P) or \otimes_P .

The key observation about comodules over spaces in $\mathcal{S}_{/T}$ is that for a space X and an object $f: Y \rightarrow T$ of $\mathcal{S}_{/T}$, an X -coaction $f \rightarrow X \ominus_T f$ corresponds to a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\rho} & X \times Y \\ f \searrow & & \swarrow \text{pr}_2 \\ & Y & \\ f \swarrow & & \searrow f \\ & T & \end{array} .$$

For every map $g: Y \rightarrow X$ of spaces, we obtain such a diagram by setting $\rho := (g, \text{Id}_Y)$ and every such diagram is of this form. Moreover, by Fact 2.0.1, a map $Y \rightarrow X$ corresponds to an X -comodule structure on $Y \in \mathcal{S}$. This informal description can be made precise as follows.

Proposition 2.0.6. *Let T be a space. Then the diagram*

$$\begin{array}{ccc} \text{LComod}^{\mathcal{S}}(\mathcal{S}_{/T}) & \xrightarrow{\text{LComod}^{\mathcal{S}}(\mathbf{V}_T)} & \text{LComod}^{\mathcal{S}}(\mathcal{S}) \\ \varsigma_{\mathcal{S}_{/T}} \downarrow & & \downarrow \varsigma_{\mathcal{S}} \\ \mathcal{S}_{/T} & \xrightarrow{\mathbf{V}_T} & \mathcal{S} \end{array} , \quad (2.0.7)$$

where the vertical functors pick the objects being coacted on, is a pullback square.

Proof. Let us first describe the pullback. Consider the span category $K = (\bullet \leftarrow \bullet \rightarrow \bullet)$, which we identify with the pushout $[1] \amalg_0 [1]$ equipped with inclusions $i_1: [1] \hookrightarrow K$ and $i_2: [1] \hookrightarrow K$. Using the identification $\text{LComod}^{\mathcal{S}}(\mathcal{S}) \simeq \text{Fun}([1], \mathcal{S})$ of Fact 2.0.1, the pullback in question can be computed as the iterated pullback

$$\begin{array}{ccccc} ? & \longrightarrow & \text{Fun}(K, \mathcal{S}) & \xrightarrow{i_1^*} & \text{Fun}([1], \mathcal{S}) \\ \downarrow & \lrcorner & \downarrow i_2^* & \lrcorner & \downarrow \text{ev}_0 \\ \mathcal{S}_{/T} & \longrightarrow & \text{Fun}([1], \mathcal{S}) & \xrightarrow{\text{ev}_0} & \mathcal{S} \end{array} .$$

Hence it will be enough to show that the functor $\Gamma: \text{LComod}^{\mathcal{S}}(\mathcal{S}_{/T}) \rightarrow \text{Fun}(K, \mathcal{S})$ that sends an object $f: Y \rightarrow T$ of $\mathcal{S}_{/T}$ equipped with a coaction $(g, \text{Id}_Y): Y \rightarrow X \times Y$ to $T \xleftarrow{f} Y \xrightarrow{g} X$ exhibits its source as the fiber of i_2^* over $\mathcal{S}_{/T}$.

We can describe $\text{LComod}^{\mathcal{S}}(\mathcal{S}_{/T})$ more explicitly by considering the cartesian monoidal structure on $\mathcal{S}_{/T}$. Indeed, by Corollary 1.3.21, it is the pullback of the cospan

$$\text{Coalg}(\mathcal{S}) \xrightarrow{\text{Coalg}((-) \odot_T \text{Id}_T)} \text{Coalg}(\mathcal{S}_{/T}^{\times T}) \xleftarrow{\theta_{\mathcal{S}_{/T}^{\times T}}} \text{LComod}^{\mathcal{S}_{/T}^{\times T}}(\mathcal{S}_{/T}),$$

which, by Fact 2.0.1, is equivalent to the cospan

$$\mathcal{S} \xrightarrow{(-) \odot_T \text{Id}_T} \mathcal{S}_{/T} \xleftarrow{\text{ev}_1} \text{Fun}([1], \mathcal{S}_{/T}).$$

Now note that there is a pullback diagram

$$\begin{array}{ccc} \text{Fun}([1], \mathcal{S}_{/T}) & \xrightarrow{V_T \circ (-)} & \text{Fun}([1], \mathcal{S}) \\ \text{ev}_1 \downarrow & \lrcorner & \downarrow \text{ev}_1 \\ \mathcal{S}_{/T} & \xrightarrow{V_T} & \mathcal{S} \end{array} .$$

Hence, pasting pullback squares, we can identify $\text{LComod}^{\mathcal{S}}(\mathcal{S}_{/T})$ with the pullback of the cospan

$$\mathcal{S} \xrightarrow{(-) \odot_T \text{Id}_T} \mathcal{S}_{/T} \xrightarrow{V_T} \mathcal{S} \xleftarrow{\text{ev}_1} \text{Fun}([1], \mathcal{S}) . \quad (2.0.8)$$

$\underbrace{\hspace{10em}}_{(-) \times T}$

Now we “vary T ”. Note that the span category K is also isomorphic to the cone $([0] \amalg [0])^\triangleleft$. In this interpretation, the universal property of the product corresponds to a pullback square

$$\begin{array}{ccc} \text{Fun}(K, \mathcal{S}) & \xrightarrow{P} & \text{Fun}([1], \mathcal{S}) \\ (\text{ev}_{i_1(1)}, \text{ev}_{i_2(1)}) \downarrow & \lrcorner & \downarrow \text{ev}_1 \\ \mathcal{S} \times \mathcal{S} & \xrightarrow{\times} & \mathcal{S} \end{array} ,$$

where P maps a span $Z \xleftarrow{f} Y \xrightarrow{g} X$ to $(g, f): Y \rightarrow X \times Z$. Moreover, we can recover the pullback of the cospan (2.0.8) by taking the pullback of $(\text{ev}_{i_1(1)}, \text{ev}_{i_2(1)})$ along $\mathcal{S} \times \{T\} \hookrightarrow \mathcal{S} \times \mathcal{S}$.

Tracing the functors in question, we see that the “projection” $\text{LComod}^{\mathcal{S}}(\mathcal{S}_{/T}) \rightarrow \text{Fun}(K, \mathcal{S})$ induced by this pullback description indeed agrees with the functor Γ discussed above. Moreover, $(\text{ev}_{i_1(1)}, \text{ev}_{i_2(1)})$ can be factored as

$$\text{Fun}(K, \mathcal{S}) \xrightarrow{(\text{ev}_{i_1(1)}, i_2^*)} \mathcal{S} \times \text{Fun}([0], \mathcal{S}) \xrightarrow{\text{Id}_{\mathcal{S}} \times \text{ev}_1} \mathcal{S} \times \mathcal{S},$$

which exhibits its fiber over $\mathcal{S} \times \{T\}$ as the fiber of i_2^* over $\mathcal{S}_{/T}$. \square

We could have stated and proven Proposition 2.0.6 purely in terms of slice categories without mentioning monoidal structures or comodules, but the monoidal point of view will be important when we consider Thom objects. In particular, we will use that the pullback description of Proposition 2.0.6 is compatible with convolution monoidal structures in the following sense.

Lemma 2.0.9. *Let P be an \mathbb{E}_{k+1} -space for some $k \in \mathbb{N} \cup \{\infty\}$ and consider Proposition 2.0.6 for $T = P$. Then all the functors in the square (2.0.7) are strongly \mathbb{E}_k -monoidal with respect to the convolution monoidal structure on $\mathcal{S}_{/P}$. In particular, as limits of \mathbb{E}_k -monoidal ∞ -categories can be computed in Cat_∞ , (2.0.7) can be viewed as a pullback square of \mathbb{E}_k -monoidal ∞ -categories.*

Proof. The \mathbb{E}_k -monoidality of the bottom arrow and the right arrow is clear. For the top arrow and the left arrow, we consider the strongly \mathbb{E}_{k+1} -monoidal functors $F := (-) \otimes_P \mathbf{1}_P: \mathcal{S} \rightarrow \mathcal{S}_{/P}$ and $V_P: \mathcal{S}_{/P} \rightarrow \mathcal{S}$. Applying $\text{Coalg}_{/\mathcal{LM}}(-)$ to various restrictions of left-tensorings of \mathcal{S} and $\mathcal{S}_{/P}$ over themselves along these functors (cf. Construction 1.1.20), we obtain a commutative diagram

$$\begin{array}{ccccc}
 \text{LComod}^{\mathcal{S}}(\mathcal{S}_{/P}) & \xrightarrow{\text{LComod}^F(\mathcal{S}_{/P})} & \text{LComod}^{\mathcal{S}_{/P}^{\otimes P}}(\mathcal{S}_{/P}) & & \\
 \text{LComod}^{\mathcal{S}}(V_P) \downarrow & & \text{LComod}^{\mathcal{S}_{/P}^{\otimes P}}(V_P) \downarrow & \searrow \text{LComod}(V_P) & \\
 \text{LComod}^{\mathcal{S}}(\mathcal{S}) & \xrightarrow{\text{LComod}^F(\mathcal{S})} & \text{LComod}^{\mathcal{S}_{/P}^{\otimes P}}(\mathcal{S}) & \xrightarrow{\text{LComod}^{V_P}(\mathcal{S})} & \text{LComod}^{\mathcal{S}}(\mathcal{S})
 \end{array} \quad (2.0.10)$$

Now recall that the monoidal structure on $\text{LComod}^{\mathcal{S}}(\mathcal{S}_{/P})$ was defined in a way which makes $\text{LComod}^F(\mathcal{S}_{/P})$ tautologically strongly \mathbb{E}_k -monoidal (cf. Remark 1.3.22). As $\text{LComod}(V_P)$ is also strongly \mathbb{E}_k -monoidal, the commutativity of (2.0.10) implies that $\text{LComod}^{\mathcal{S}}(V_P)$, i.e., the top arrow in (2.0.7), is strongly \mathbb{E}_k -monoidal.

Now the left arrow in (2.0.7) is the composite of $\text{LComod}^F(\mathcal{S}_{/P})$ with the strongly \mathbb{E}_k -monoidal functor $\varsigma_{\mathcal{S}_{/P}}: \text{LComod}^{\mathcal{S}_{/P}^{\otimes P}}(\mathcal{S}_{/P}) \rightarrow \mathcal{S}_{/P}$ that picks the object being coacted on, so it is also \mathbb{E}_k -monoidal. \square

Given an object $f: X \rightarrow T$ of $\mathcal{S}_{/T}$, our description of comodules over spaces in $\mathcal{S}_{/T}$ allows us to view it as a comodule over X as follows.

Construction 2.0.11. We consider the regular coaction functor $R: \text{Coalg}(\mathcal{S}) \rightarrow \text{LComod}^{\mathcal{S}}(\mathcal{S})$ induced by the map of operads $\mathcal{LM}^{\otimes} \rightarrow \mathbb{E}_1^{\otimes}$, which equips the underlying object of a coalgebra D with a D -coaction given by its comultiplication. Note that under the equivalences $\text{Coalg}(\mathcal{S}) \simeq \mathcal{S}$ and $\text{LComod}^{\mathcal{S}}(\mathcal{S}) \simeq \text{Fun}([1], \mathcal{S})$, R sends a space X to Id_X .

Now let T be a space. Then R fits into a commutative diagram

$$\begin{array}{ccccc}
\mathcal{S}/T & \xrightarrow{V_T} & \mathcal{S} \simeq \text{Coalg}(\mathcal{S}) & \xrightarrow{R} & \text{Coalg}(\mathcal{S}) \\
\downarrow \mathfrak{S}_T & \searrow & \downarrow \mathfrak{S} & \nearrow & \downarrow \mathfrak{S} \\
\text{LComod}^{\mathcal{S}}(\mathcal{S}/T) & \xrightarrow{\text{LComod}^{\mathcal{S}}(V_T)} & \text{LComod}^{\mathcal{S}}(\mathcal{S}) & \xrightarrow{\theta_{\mathcal{S}}} & \text{Coalg}(\mathcal{S}) \\
\downarrow \mathfrak{S}_{\mathcal{S}/T} & \lrcorner & \downarrow \mathfrak{S} & & \\
\mathcal{S}/T & \xrightarrow{V_T} & \mathcal{S} & &
\end{array} \tag{2.0.12}$$

in particular inducing the dashed functor $\mathfrak{S}_T: \mathcal{S}/T \rightarrow \text{LComod}^{\mathcal{S}}(\mathcal{S}/T)$ by virtue of Proposition 2.0.6. Informally speaking, \mathfrak{S}_T sends an object $f: X \rightarrow T$ to an X -comodule with coaction map

$$\begin{array}{ccc}
X & \xrightarrow{(\text{Id}_X, \text{Id}_X)} & X \times X \\
f \searrow & & \swarrow f \circ \text{pr}_2 \\
& T &
\end{array} .$$

Note that if $T = P$ is an \mathbb{E}_{k+1} -space for some $k \in \mathbb{N} \cup \{\infty\}$, then all the functors in (2.0.12) are, hence \mathfrak{S}_P is strongly \mathbb{E}_k -monoidal when we equip \mathcal{S}/P with the convolution monoidal structure (cf. Lemma 2.0.9). On the level of \mathbb{E}_k -algebras, this yields a functor $\text{Alg}_{/\mathbb{E}_k}(\mathfrak{S}_P): \text{Alg}_{/\mathbb{E}_k}(\mathcal{S}_{/P}^{\otimes P}) \rightarrow \text{Alg}_{/\mathbb{E}_k}(\text{LComod}^{\mathcal{S}}(\mathcal{S}/P))$.

Now let X be a \mathbb{E}_k -space, which we view as an $(\mathbb{E}_k, \mathbb{E}_1)$ -bialgebra in \mathcal{S} . When we equip \mathcal{S}/T and $\text{LComod}^{\mathcal{S}}(\mathcal{S}/T)$ with the functors into $\text{Coalg}(\mathcal{S})$ depicted in (2.0.12), \mathfrak{S}_P can be viewed as a functor over $\text{Coalg}(\mathcal{S})$ and hence $\text{Alg}_{/\mathbb{E}_k}(\mathfrak{S}_P)$ as a functor over $\text{Alg}_{/\mathbb{E}_k}(\text{Coalg}(\mathcal{S}))$. Looking at fibers over $X \in \text{Alg}_{/\mathbb{E}_k}(\text{Coalg}(\mathcal{S}))$, we obtain a functor $\text{Alg}_{/\mathbb{E}_k}(\mathfrak{S}_P)_X: \text{Alg}_{/\mathbb{E}_k}(\mathcal{S}_{/P}^{\otimes P})_X \rightarrow \text{Alg}_{/\mathbb{E}_k}(\text{LComod}^{\mathcal{S}}(\mathcal{S}/P))_X$. Combining Remark 1.3.22 and Lemma 1.2.13, we see that its target $\text{Alg}_{/\mathbb{E}_k}(\text{LComod}^{\mathcal{S}}(\mathcal{S}/P))_X$ is equivalent to $\text{Alg}_{/\mathbb{E}_k}(\text{LComod}_{X \odot_P \mathbf{1}_P}(\mathcal{S}/P))$. As for the source, an algebra f in \mathcal{S}/P (i.e., a \mathbb{E}_k -map into P) is in $\text{Alg}_{/\mathbb{E}_k}(\mathcal{S}_{/P}^{\otimes P})_X$ if and only if its source is X . Hence, \mathfrak{S}_P maps a map $f: X \rightarrow P$ of \mathbb{E}_k -spaces not only to an X -comodule, but in fact to an $X \odot_P \mathbf{1}_P$ -comodule algebra.

From now on, when there is no room for confusion, we will abbreviate $\text{Alg}_{/\mathbb{E}_k}(\mathfrak{S}_P)$ and $\text{Alg}_{/\mathbb{E}_k}(\mathfrak{S}_P)_X$ as \mathfrak{S}_P .

Remark 2.0.13. We expect the results of this section to generalize from the slice category over a space (i.e., an ∞ -groupoid) to a slice category over a (presentable) ∞ -category \mathcal{C} as follows.

Consider the ∞ -category $\mathcal{S}_{/\mathcal{C}}$ of ∞ -groupoids over \mathcal{C} (i.e., the comma category of the cospan $\{\mathcal{C}\} \hookrightarrow \text{Cat}_{\infty} \leftarrow \mathcal{S}$). As in Proposition 2.0.6, we expect a comodule structure over a space X on an object $f: Y \rightarrow \mathcal{C}$ of $\mathcal{S}_{/\mathcal{C}}$ to be completely determined by a map $g: Y \rightarrow X$, and thus every such f to have a Y -comodule structure encoded by Id_Y as in Construction 2.0.11.

Moreover, for $k \in \mathbb{N} \cup \{\infty\}$, we expect an \mathbb{E}_{k+1} -monoidal structure on \mathcal{C} to give rise to a “convolution” \mathbb{E}_{k+1} -monoidal structure on $\mathcal{S}_{/\mathcal{C}}$ such that an \mathbb{E}_{k+1} -algebra $f: Y \rightarrow \mathcal{C}$ with respect to this monoidal structure corresponds to a map of \mathbb{E}_{k+1} -algebras in Cat_{∞} ,

i.e., a strongly \mathbb{E}_{k+1} -monoidal functor. As in Construction 2.0.11, a strongly \mathbb{E}_k -monoidal functor $f: Y \rightarrow \mathcal{C}$ would thus give rise to an $(Y \ominus_{\mathcal{S}/\mathcal{C}} \mathbf{1}_{\mathcal{S}/\mathcal{C}})$ -comodule \mathbb{E}_k -algebra using the aforementioned comodule structure.

3. Hopf–Galois extensions

In this section, we introduce one of the central concepts of this thesis.

Convention 3.0.1. We fix an \mathbb{E}_2 -monoidal ∞ -category \mathcal{C} and a bialgebra $H \in \text{Bialg}(\mathcal{C})$. We further assume that \mathcal{C} admits geometric realizations of simplicial objects and limits of cosimplicial objects, and that the tensor product preserves geometric realizations in each variable separately.

The examples we have in mind are the ∞ -category \mathcal{S} of spaces (equipped with the cartesian product), the ∞ -category Sp of spectra (equipped with the smash product), the ∞ -category $\text{LMod}_R(\text{Sp})$ of modules over an \mathbb{E}_n -ring spectrum for $n \geq 3$ (equipped with the relative smash product over R), and various localizations of those.

3.1. Hopf–Galois contexts

Definition 3.1.1. An H -Hopf–Galois-context is a map

$$\tilde{\varphi}: \text{Triv}_H(A) \rightarrow \tilde{B}$$

of comodule algebras, where $A \in \text{Alg}(\mathcal{C})$ is an algebra and $\tilde{B} \in \text{Alg}(\text{LComod}_H(\mathcal{C}))$ is an H -comodule algebra.

In this situation, we will denote the underlying algebra $V_H(\tilde{B}) \in \text{Alg}(\mathcal{C})$ by B and the underlying map $V_H(\tilde{\varphi}): A \simeq V_H(\text{Triv}_H(A)) \rightarrow B$ by φ .

Example 3.1.2. Let $A \in \text{Alg}(\mathcal{C})$. As the cofree comodule functor $C_H: \mathcal{C} \rightarrow \text{LComod}_H(\mathcal{C})$ is lax monoidal (cf. Remark 1.2.10), $C_H(A)$ lifts to an H -comodule algebra with multiplication map

$$C_H(A) \otimes C_H(A) \simeq H \otimes A \otimes H \otimes A \simeq H \otimes H \otimes A \otimes A \xrightarrow{\mu_H \otimes \mu_A} H \otimes A \simeq C_H(A).$$

In particular, the underlying algebra of H lifts to $\tilde{H} := C_H(\mathbf{1}_{\mathcal{C}}) \in \text{Alg}(\text{LComod}_H(\mathcal{C}))$.

Moreover, since both adjoints of the adjunction $V_H \dashv C_H$ are lax monoidal, it lifts to an adjunction between ∞ -categories of algebras (cf. Remark B.0.5). The adjoint of the equivalence $V_H(\text{Triv}_H(A)) \simeq A$ of Remark 1.3.18 can hence be upgraded to a map $\text{nb}_A: \text{Triv}_H(A) \rightarrow C_H(A)$ of H -comodule algebras, i.e., an H -Hopf–Galois context, which we call the *normal basis context* of A . Unpacking the definitions, we see that the underlying map of nb_A is given by $\eta_H \otimes A: A \rightarrow H \otimes A$. Accordingly, we will denote $\text{nb}_{\mathbf{1}_{\mathcal{C}}}$ also by $\tilde{\eta}_H: \text{Triv}_H(\mathbf{1}_{\mathcal{C}}) \rightarrow \tilde{H}$.

Example 3.1.3. Assume that \mathcal{C} is presentably \mathbb{E}_2 -monoidal and consider the Picard space $\text{Pic}(\mathcal{C})$ of \otimes -invertible objects of \mathcal{C} . By [ABG18, Theorem 1.5], there is a “Thom object” functor $M: \mathcal{S}_{/\text{Pic}(\mathcal{C})} \rightarrow \mathcal{C}$ that sends a map $X \rightarrow \text{Pic}(\mathcal{C})$ to the colimit of the composite $X \rightarrow \text{Pic}(\mathcal{C}) \hookrightarrow \mathcal{C}$ and is \mathbb{E}_2 -monoidal with respect to the convolution monoidal structure in the source. In particular, for every \mathbb{E}_1 -space X , M induces a functor $\tilde{M}_X: \text{Alg}(\text{LComod}_{X \odot \text{Pic}(\mathcal{C})}(\mathcal{S}_{/\text{Pic}(\mathcal{C})})) \rightarrow \text{Alg}(\text{LComod}_{X \odot \mathbf{1}_{\mathcal{C}}}(\mathcal{C}))$.

Now let $f: X \rightarrow \text{Pic}(\mathcal{C})$ be a map of \mathbb{E}_1 -spaces, which we view as an algebra in $\mathcal{S}_{/\text{Pic}(\mathcal{C})}$. In Construction 2.0.11, we saw that f can be lifted to an $(X \ominus_{\text{Pic}(\mathcal{C})} \mathbf{1}_{\text{Pic}(\mathcal{C})})$ -comodule algebra $\mathfrak{S}_{\text{Pic}(\mathcal{C})}(f)$ with coaction map

$$\begin{array}{ccc} X & \xrightarrow{(\text{Id}_X, \text{Id}_X)} & X \times X \\ & \searrow f & \swarrow \text{const}_{\mathbf{1}_{\mathcal{C}}} \times f \\ & \text{Pic}(\mathcal{C}) & \leftarrow \mu_{\text{Pic}(\mathcal{C})} \end{array} \quad \text{Pic}(\mathcal{C}) \times \text{Pic}(\mathcal{C})$$

Let $\eta_f: \text{Triv}_{X \ominus_{\text{Pic}(\mathcal{C})} \mathbf{1}_{\text{Pic}(\mathcal{C})}}(\mathbf{1}_{\text{Pic}(\mathcal{C})}) \rightarrow \mathfrak{S}_{\text{Pic}(\mathcal{C})}(f)$ denote its unit map, which we can view as an $(X \ominus_{\text{Pic}(\mathcal{C})} \mathbf{1}_{\text{Pic}(\mathcal{C})})$ -Hopf–Galois context in $\mathcal{S}_{/\text{Pic}(\mathcal{C})}$.

Applying M to f , we obtain a $(X \ominus_{\mathcal{C}} \mathbf{1}_{\mathcal{C}})$ -comodule algebra $\widetilde{M}(f) := \widetilde{M}_X(\mathfrak{S}_{\text{Pic}(\mathcal{C})}(f))$, whose coaction map $M(f) \rightarrow (X \ominus_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}) \otimes M(f) \simeq X \ominus_{\mathcal{C}} M(f)$ is an incarnation of the “Thom diagonal”. The unit map $\widetilde{M}_X(\eta_f): \text{Triv}_{X \ominus_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}}(\mathbf{1}_{\mathcal{C}}) \rightarrow \widetilde{M}(f)$ of $\widetilde{M}(f)$ yields an $(X \ominus_{\mathcal{C}} \mathbf{1}_{\mathcal{C}})$ -Hopf–Galois context, which we call the *Thom context of f* .

Remark 3.1.4. In the context of Example 3.1.3, one might ask whether colimits of certain functors $f: X \rightarrow \mathcal{C}$ that do not necessarily factor through $\text{Pic}(\mathcal{C})$ admit the structure of a $(X \ominus_{\mathcal{C}} \mathbf{1}_{\mathcal{C}})$ -comodule algebra. One possibility for this would be using the generalization sketched in Remark 2.0.13 and working with $\mathcal{S}_{/\mathcal{C}}$ instead of $\mathcal{S}_{/\text{Pic}(\mathcal{C})}$, which would yield such a comodule algebra structure on the colimit of a strongly monoidal functor $f: X \rightarrow \mathcal{C}$.⁹ Another approach that could work also for *lax* monoidal functors is as follows.

On the one hand, by [ABG18, Theorem 6.4], $\text{Fun}(X, \mathcal{C})$ admits a *pointwise* monoidal structure with respect to which $\text{const}: \mathcal{C} \rightarrow \text{Fun}(X, \mathcal{C})$ is strongly monoidal. This induces an oplax monoidal structure on its left adjoint $\text{colim}: \text{Fun}(X, \mathcal{C}) \rightarrow \mathcal{C}$. Hence, as every functor $f: X \rightarrow \mathcal{C}$ is a comodule over the unit $\text{const}_{\mathbf{1}_{\mathcal{C}}}$ of the pointwise monoidal structure, we obtain a coaction of $\text{colim}(\text{const}_{\mathbf{1}_{\mathcal{C}}}) \simeq X \ominus_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}$ on $\text{colim } f$ (cf. [Bea21, Corollary 4.14]).

On the other hand, when X is an \mathbb{E}_1 -space, $\text{Fun}(X, \mathcal{C})$ admits a *Day convolution* monoidal structure by [ABG18, Theorem 6.15], with respect to which $\text{colim}: \text{Fun}(X, \mathcal{C}) \rightarrow \mathcal{C}$ is strongly monoidal. Therefore, as algebras with respect to the Day convolution monoidal structure correspond to lax monoidal functors, a lax monoidal structure on a functor $f: X \rightarrow \mathcal{C}$ induces an algebra structure on $\text{colim } f$.

Hence the question is whether the algebra structure on the colimit of a lax monoidal functor $f: X \rightarrow \mathcal{C}$ is compatible with its X -comodule structure. We expect this to be the case because we expect the lift $\text{colim}_{\widetilde{X}}: \text{Fun}(X, \mathcal{C}) \rightarrow \text{LComod}_{X \ominus_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}}(\mathcal{C})$ of the colimit functor to be strongly monoidal with respect to the Day convolution monoidal structure in the source and the monoidal structure of Fact 1.2.9 in the target.

Indeed, the Day convolution monoidal structure can be constructed using a method similar to that of Fact 1.2.9 as follows. There is a cocartesian fibration $p: \mathcal{S}_{/\mathcal{C}}^{\otimes} \rightarrow \mathcal{S}^{\otimes}$ whose

⁹Note, however, that if X is a *grouplike* \mathbb{E}_1 -space, then every strongly monoidal functor $f: X \rightarrow \mathcal{C}$ factors through $\text{Pic}(\mathcal{C})$, so that case is already covered by Example 3.1.3.

straightening is a functor that extends $\text{Fun}(-, \mathcal{C}): \mathcal{S}_{(1)}^\otimes \simeq \mathcal{S} \rightarrow \text{Cat}_\infty$, where the functoriality is given by left Kan extension functors. The Day convolution monoidal structure on $\text{Fun}(X, \mathcal{C})$ can be obtained by pulling back p along the map $\bar{X}: \mathbb{E}_1^\otimes \rightarrow \mathcal{S}^\otimes$ representing the \mathbb{E}_1 -structure of X . Hence, if we can define a functor $\mathbf{colim}: \mathcal{S}_{\mathcal{C}}^\otimes \rightarrow \text{LComod}^{\mathcal{S}}(\mathcal{C})^\otimes$ over \mathcal{S}^\otimes whose fiber at $Y \in \mathcal{S} = \mathcal{S}_{(1)}^\otimes$ is $\text{colim}_{\tilde{Y}}: \text{Fun}(Y, \mathcal{C}) \rightarrow \text{LComod}_{Y \otimes_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}}(\mathcal{C})$ and that preserves cocartesian arrows, we can obtain an extension of $\text{colim}_{\tilde{X}}$ to a strongly monoidal functor by considering the pullback of \mathbf{colim} along the map $\bar{X}: \mathbb{E}_1^\otimes \rightarrow \mathcal{S}^\otimes$.

The main challenge here is constructing a functor \mathbf{colim} with the desired fibers; once such a functor is constructed we expect it to be straightforward to see that it preserves cocartesian arrows. For instance, for a map $u: Y \rightarrow Y'$ of spaces (i.e., a morphism in $\mathcal{S}_{(1)}^\otimes$), the underlying map of the comparison map $u_* \circ \text{colim}_{\tilde{Y}} \rightarrow \text{colim}_{\tilde{Y}'} \circ u_!$ would simply be the equivalence $\text{colim}_Y \simeq \text{colim}_{Y'} \circ u_!$.

Definition 3.1.5. We say that an H -Hopf-Galois-context $\tilde{\varphi}: \text{Triv}_H(A) \rightarrow \tilde{B}$ satisfies the *primitives condition* if the adjoint map $\hat{\varphi}: A \rightarrow \text{Prim}_H(\tilde{B})$ under the adjunction $\text{Triv}_H \dashv \text{Prim}_H$ of Definition 1.3.17 is an equivalence.

Example 3.1.6. Let A be an algebra in \mathcal{C} and consider the normal basis context $\text{nb}_A: \text{Triv}_H(A) \rightarrow C_H(A)$ of Example 3.1.2, which was defined as the adjoint of the equivalence $V_H(\text{Triv}_H(A)) \simeq A$ with respect to the adjunction $V_H \dashv C_H$. The adjoint of this under the adjunction $\text{Triv}_H \dashv \text{Prim}_H$ recovers the equivalence $A \simeq \text{Prim}_H(C_H(A))$ discussed in Remark 1.3.18, meaning that every normal basis context satisfies the primitives condition.

For Thom contexts, the primitives condition is usually verified via a completeness criterion that we will establish in Example 3.2.14.

Definition 3.1.7. Let $\tilde{\varphi}: \text{Triv}_H(A) \rightarrow \tilde{B}$ be an H -Hopf-Galois-context and $M \in \text{LMod}_B(\mathcal{C})$. The associated *shear map* is the composite¹⁰

$$\text{sh}_{\tilde{\varphi}} M: B \otimes_A M \xrightarrow{\rho_{\tilde{B}}^{\otimes_A M}} H \otimes B \otimes_A M \xrightarrow{H \otimes \alpha_M} H \otimes M.$$

When $M = B$, we will also simply write $\text{sh}_{\tilde{\varphi}}$ instead $\text{sh}_{\tilde{\varphi}} B$.

According to this definition, the shear map is only a map of objects in \mathcal{C} , but we will see in Corollary 3.2.8 that it underlies a more structured map.

Definition 3.1.8. We say that an H -Hopf-Galois-context $\tilde{\varphi}: \text{Triv}_H(A) \rightarrow \tilde{B}$ satisfies the *normal basis condition* if $\text{sh}_{\tilde{\varphi}}: B \otimes_A B \rightarrow H \otimes B$ is an equivalence.

Example 3.1.9. Let A be an algebra in \mathcal{C} and consider the normal basis context $\text{nb}_A: \text{Triv}_H(A) \rightarrow C_H(A)$ of Example 3.1.2. Note that we have an equivalence

$$\begin{array}{ccc} \text{sh}_{\text{nb}_A}: (H \otimes A) \otimes_A (H \otimes A) & \xrightarrow{(\Delta_H \otimes A) \otimes_A (H \otimes A)} & H \otimes (H \otimes A) \otimes_A (H \otimes A) \xrightarrow{H \otimes \mu_{H \otimes A}} H \otimes (H \otimes A) \\ \wr & & \wr \\ \text{sh}_{\eta_H} \otimes A: H \otimes H \otimes A & \xrightarrow{\Delta_H \otimes H \otimes A} & H \otimes H \otimes H \otimes A \xrightarrow{H \otimes \mu_{H \otimes A}} H \otimes H \otimes A \end{array}.$$

¹⁰ Note that the first map is well-defined because $\rho_{\tilde{B}}^{\otimes_A M} \simeq (H \otimes \varphi) \rho_{\text{Triv}_H(A)} \simeq (H \otimes \varphi)(\eta_H \otimes A) \simeq \eta_H \otimes \varphi$.

Hence sh_{nb_A} satisfies the normal basis condition if $\widetilde{\eta}_H: \text{Triv}_H(\mathbf{1}_C) \rightarrow \widetilde{H}$ does. Bialgebras H such that $\widetilde{\eta}_H$ satisfies the normal basis condition will be studied more closely in Section 4, where we call them *Hopf algebras*.

Example 3.1.10. Let X be an \mathbb{E}_1 -space, which we view as a bialgebra in \mathcal{S} . Then the shear map of the associated normal basis context $\widetilde{\eta}_X$ is given by

$$X \times X \xrightarrow{(\text{Id}_X, \text{Id}_X) \times X} X \times X \times X \xrightarrow{X \times \mu_X} X \times X.$$

\mathbb{E}_1 -spaces for which this map is an equivalence are called *grouplike* (cf. [Lur17, Definition 5.2.6.2]).

Example 3.1.11. Assume that \mathcal{C} is presentably \mathbb{E}_2 -monoidal, and let $f: X \rightarrow \text{Pic}(\mathcal{C})$ be a map of \mathbb{E}_1 -spaces. Consider the Thom context $\widetilde{M}_X(\eta_f): \text{Triv}_{X \odot_C \mathbf{1}_C}(\mathbf{1}_C) \rightarrow \widetilde{M}(f)$ of Example 3.1.3.

Note that as $M: \mathcal{S}_{/\text{Pic}(\mathcal{C})} \rightarrow \mathcal{C}$ is strongly \mathbb{E}_2 -monoidal, the shear map

$$M(f \otimes_{\text{Pic}(\mathcal{C})} f) \simeq M(f) \otimes M(f) \xrightarrow{\text{sh}_{\widetilde{M}_X(\eta_f)}} (X \odot_C \mathbf{1}_C) \otimes M(f) \simeq M((X \odot_{\text{Pic}(\mathcal{C})} \mathbf{1}_{\text{Pic}(\mathcal{C})}) \otimes_{\text{Pic}(\mathcal{C})} f)$$

agrees with $M(\text{sh}_{\eta_f})$. Viewing $\text{sh}_{\eta_f}: f \otimes_{\text{Pic}(\mathcal{C})} f \rightarrow (X \odot_{\text{Pic}(\mathcal{C})} \mathbf{1}_{\text{Pic}(\mathcal{C})}) \otimes_{\text{Pic}(\mathcal{C})} f$ as a map of spaces over $\text{Pic}(\mathcal{C})$, we can depict it as a commutative diagram

$$\begin{array}{ccccc} X \times X & \xrightarrow{(\text{Id}_X, \text{Id}_X) \times X} & X \times X \times X & \xrightarrow{X \times \mu_X} & X \times X \\ f \times f \downarrow & & \downarrow \text{const}_{\mathbf{1}_C} \times f \times f & & \downarrow \text{const}_{\mathbf{1}_C} \times f \\ \text{Pic}(\mathcal{C}) \times \text{Pic}(\mathcal{C}) & & \text{Pic}(\mathcal{C}) \times \text{Pic}(\mathcal{C}) \times \text{Pic}(\mathcal{C}) & & \text{Pic}(\mathcal{C}) \times \text{Pic}(\mathcal{C}) \\ & \searrow \mu_{\text{Pic}(\mathcal{C})} & \downarrow \mu_{\text{Pic}(\mathcal{C})} \circ (\text{Pic}(\mathcal{C}) \times \mu_{\text{Pic}(\mathcal{C})}) & \swarrow \mu_{\text{Pic}(\mathcal{C})} & \\ & & \text{Pic}(\mathcal{C}) & & \end{array}$$

Now sh_{η_f} is an equivalence in $\mathcal{S}_{/\text{Pic}(\mathcal{C})}$ if and only if the upper composite $(X \times \mu_X) \circ ((\text{Id}_X, \text{Id}_X) \times X)$, i.e., the shear map of the \mathbb{E}_1 -space X , is an equivalence of spaces. Hence $\text{sh}_{\widetilde{M}_X(\eta_f)}$ is an equivalence if X is a grouplike \mathbb{E}_1 -space, in which case $\text{sh}_{\widetilde{M}_X(\eta_f)}$ is an incarnation of the *Thom isomorphism* of [Mah79, Theorem 1.1].

Definition 3.1.12. An H -Hopf-Galois context $\widetilde{\varphi}: \text{Triv}_H(A) \rightarrow \widetilde{B}$ which satisfies both the primitives condition and the normal basis condition (i.e., one for which $\widehat{\varphi}: A \rightarrow \text{Prim}_H(\widetilde{B})$ and $\text{sh}_{\widetilde{\varphi}}: B \otimes_A B \rightarrow H \otimes B$ are equivalences) is called an *H-Hopf-Galois extension*.

This definition is a straightforward translation of [Rog08, Part I, Definition 12.1.5] for point-set models, variants of which appeared as [Rot09, Definition 5.7], [Hes09, Definition 3.2] and [Kar14, Definition 2.3.1]. For coactions of spaces on ring spectra (in particular for Thom spectra), it is essentially equivalent to [Bea16, Definition 4.2.2.5].

3.2. The structured shear map

Convention 3.2.1. For the rest of this section, we fix an H -Hopf-Galois context $\tilde{\varphi}: \text{Triv}_H(A) \rightarrow \tilde{B}$.

In applications such as Corollary 3.2.11 and Corollary 3.2.13, we will need a more structured version of the shear map $\text{sh}_{\tilde{\varphi}} M: B \otimes_A M \rightarrow H \otimes M$ introduced in Definition 3.1.7. First we note that the source $B \otimes_A M$ of the shear map can be identified with the underlying object of the image of M under the endofunctor

$$\varphi_! \varphi^* \simeq B \otimes_A (-): \text{LMod}_B(\mathcal{C}) \rightarrow \text{LMod}_B(\mathcal{C}).$$

Moreover, by Example 1.3.5, $\varphi_! \varphi^*$ admits a comonad structure which makes it the initial comonad coacting on $\varphi_!$.

Given this universal property of $\varphi_! \varphi^*$, we would like to lift $H \otimes (-)$ to a comonad $\Theta_{\tilde{B}}$ on $\text{LMod}_B(\mathcal{C})$ and construct an appropriate coaction of it on $\varphi_!$ such that for $M \in \text{LMod}_B(\mathcal{C})$, the component $\varphi_! \varphi^* M \simeq B \otimes_A M \rightarrow H \otimes M \simeq \Theta_{\tilde{B}} M$ of the comonad map $\varphi_! \varphi^* \rightarrow \Theta_{\tilde{B}}$ given by the initiality of $\varphi_! \varphi^*$ is the shear map.

We will realize the comonad $\Theta_{\tilde{B}}$ as the associated comonad of a “forgetful-cofree” adjunction $\text{LMod}_{\tilde{B}}(\text{LComod}_H(\mathcal{C})) \rightleftarrows \text{LMod}_B(\mathcal{C})$ constructed using the machinery of Appendix B. This method of constructing a comonad on $\text{LMod}_B(\mathcal{C})$ with the desired properties is due to Hadrian Heine.

Proposition 3.2.2. *There is a comonadic adjunction*

$$V_{\tilde{B}}: \text{LMod}_{\tilde{B}}(\text{LComod}_H(\mathcal{C})) \rightleftarrows \text{LMod}_B(\mathcal{C}) : C_{\tilde{B}} \quad (3.2.3)$$

lifting the forgetful-cofree adjunction $V_H: \text{LComod}_H(\mathcal{C}) \rightleftarrows \mathcal{C} : C_H$.

Proof. As $V_H: \text{LComod}_H(\mathcal{C}) \rightarrow \mathcal{C}$ admits a strongly monoidal structure and a right adjoint C_H , Example B.0.4 and Corollary B.0.10 yield the desired lift. \square

Definition 3.2.4. Let $\Theta_{\tilde{B}} := V_{\tilde{B}} C_{\tilde{B}}$, the comonad associated to the adjunction (3.2.3).

Remark 3.2.5. Note that $\Theta_{\tilde{B}}$ is a lift of $H \otimes (-) \simeq V_H C_H: \mathcal{C} \rightarrow \mathcal{C}$ to a functor $\text{LMod}_B(\mathcal{C}) \rightarrow \text{LMod}_B(\mathcal{C})$, so the effect of $\Theta_{\tilde{B}}$ on the underlying objects is indeed given by tensoring with H .

Remark 3.2.6. The identification $\text{LMod}_{\tilde{B}}(\text{LComod}_H(\mathcal{C})) \simeq \text{LComod}_{\Theta_{\tilde{B}}}(\text{LMod}_B(\mathcal{C}))$ allows us, in some sense, to swap the order in which we consider “ B -modules” and “ H -comodules”. One should, however, be careful about such statements because the coaction of H on B introduces certain “twists” (which is one of the reasons why we insist on making the comodule structure explicit by writing \tilde{B} instead of just B).

For example, unwinding the definitions of the functors constructed in the proof of Proposition B.0.6, we see that $C_{\tilde{B}}$ is in fact given by the composite

$$\text{LMod}_B(\mathcal{C}) \xrightarrow{C_H^B} \text{LMod}_{C_H(B)}(\text{LComod}_H(\mathcal{C})) \xrightarrow{(\rho_{\tilde{B}}: \tilde{B} \rightarrow C_H(B))^*} \text{LMod}_{\tilde{B}}(\text{LComod}_H(\mathcal{C})).$$

Therefore, given a B -module M , the B -action on $\Theta_{\tilde{B}}M \simeq V_{\tilde{B}}C_{\tilde{B}}M \simeq H \otimes M$ is informally given by $b \cdot (h \otimes m) = \rho(b) \cdot (h \otimes m)$, *not* by $b \cdot (h \otimes m) = h \otimes (b \cdot m)$ as one might naively expect. One could say that objects of $\text{LComod}_{\Theta_{\tilde{B}}}(\text{LMod}_B(\mathcal{C}))$ are B -modules equipped with a $(\rho_{\tilde{B}})$ -semilinear coaction of H .

Next, we construct an appropriate $\Theta_{\tilde{B}}$ -comodule structure on $\varphi_!$.

Lemma 3.2.7. $\varphi_! : \text{LMod}_A(\mathcal{C}) \rightarrow \text{LMod}_B(\mathcal{C})$ lifts to a functor

$$\widehat{\varphi}_! : \text{LMod}_A(\mathcal{C}) \rightarrow \text{LMod}_{\tilde{B}}(\text{LComod}_H(\mathcal{C})) \simeq \text{LComod}_{\Theta_{\tilde{B}}}(\text{LMod}_B(\mathcal{C}))$$

such that for all $N \in \text{LMod}_A(\mathcal{C})$, the induced coaction map

$$r_{\Theta_{\tilde{B}}} : \varphi_!N \simeq B \otimes_A N \rightarrow H \otimes B \otimes_A N \simeq \Theta_{\tilde{B}}\varphi_!N$$

of Fact 1.3.6 is given by $\rho_{\tilde{B}} \otimes_A N$.

Proof. We have a commutative diagram

$$\begin{array}{ccccc} & & \text{LMod}_{\text{Triv}_H(A)}(\text{LMod}_H(\mathcal{C})) & \xrightarrow{\tilde{\varphi}_!} & \text{LMod}_{\tilde{B}}(\text{LMod}_H(\mathcal{C})) \\ & \nearrow \text{Triv}_H^A & \downarrow \text{V}_H^{\text{Triv}_H(A)} & & \downarrow \text{V}_H^{\tilde{B}} \\ \text{LMod}_A(\mathcal{C}) & \xlongequal{\quad} & \text{LMod}_A(\mathcal{C}) & \xrightarrow{\varphi_!} & \text{LMod}_B(\mathcal{C}) \end{array},$$

yielding a lift $\widehat{\varphi}_! := \tilde{\varphi}_! \circ \text{Triv}_H^A$. Now, for $N \in \text{LMod}_A(\mathcal{C})$, the H -coaction on $\text{Triv}_H^A(N)$ is given by $\eta_H \otimes N : N \rightarrow H \otimes N$. $\tilde{\varphi}_!$ maps this to the coaction map

$$B \otimes_A N \xrightarrow{\rho_{\tilde{B}} \otimes_A (\eta_H \otimes N)} (H \otimes B) \otimes_A (H \otimes N) \simeq H \otimes H \otimes B \otimes_A N \xrightarrow{\mu_H \otimes B \otimes_A N} H \otimes B \otimes_A N,$$

which is homotopic to $\rho_{\tilde{B}}$ as $\mu_H \circ (H \otimes \eta_H) \simeq \text{Id}_H$. \square

Corollary 3.2.8. *There exists a comonad map $\varphi_!\varphi^* \rightarrow \Theta_{\tilde{B}}$ whose component at $M \in \text{LMod}_B(\mathcal{C})$ is given by the shear map $B \otimes_A M \simeq \varphi_!\varphi^*M \rightarrow \Theta_{\tilde{B}}M \simeq H \otimes M$.*

Proof. Applying Example 1.3.5 to the coaction of Lemma 3.2.7, we obtain a comonad map $\varphi_!\varphi^* \rightarrow \Theta_{\tilde{B}}$ with components

$$\varphi_!\varphi^*M \simeq B \otimes_A M \xrightarrow{r_{\Theta_{\tilde{B}}}\varphi^* \simeq \rho_{\tilde{B}} \otimes_A M} H \otimes B \otimes_A M \xrightarrow{\Theta_{\tilde{B}}c_\varphi \simeq H \otimes \alpha_M} H \otimes M \simeq \Theta_{\tilde{B}}M \quad (3.2.9)$$

as desired. \square

Remark 3.2.10. Note that both $\varphi_!\varphi^*$ and $\Theta_{\tilde{B}} \simeq V_H^{\tilde{B}} \circ (\rho_{\tilde{B}})^* \circ C_H^B$ are right \mathcal{C} -linear geometric-realization-preserving functors. Moreover, the factors that occur in (3.2.9), hence $\text{sh}_{\tilde{\varphi}}(-)$ can be viewed as morphisms of right \mathcal{C} -linear geometric-realization-preserving functors.

Therefore, using the equivalence $\text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\text{LMod}_B(\mathcal{C}), \text{LMod}_B(\mathcal{C})) \simeq {}_B\text{Bimod}_B$ of Fact 1.4.4, we see that the natural transformation $\text{sh}_{\tilde{\varphi}}(-)$ is an equivalence if and only if its component at B , i.e., $\text{sh}_{\tilde{\varphi}} : B \otimes_A B \rightarrow H \otimes B$ is.

In fact, one could equip $B \otimes_A B$ and $H \otimes B$ with appropriate B - B -bimodule coalgebra structures and define the shear map as a morphism of such objects, which is the perspective taken in [BH18, Definition 3.12].

An immediate consequence of this structured shear map is the following “Hopf–Galois descent” result.

Corollary 3.2.11. *Assume that the shear map $\text{sh}_{\tilde{\varphi}}: B \otimes_A B \rightarrow H \otimes B$ is an equivalence. Then $\varphi: A \rightarrow B$ admits descent if and only if the lift*

$$\widehat{\varphi}_! : \text{LMod}_A(\mathcal{C}) \rightarrow \text{LComod}_{\Theta_{\tilde{B}}}(\text{LMod}_B(\mathcal{C}))$$

of Lemma 3.2.7 is an equivalence.

Proof. As discussed in Remark 1.3.7, $\widehat{\varphi}_!$ can be factored as

$$\text{LMod}_A(\mathcal{C}) \xrightarrow{\overline{\varphi}_!} \text{LComod}_{\varphi_! \varphi^*}(\text{LMod}_B(\mathcal{C})) \xrightarrow{(\text{sh}_{\tilde{\varphi}})_*} \text{LComod}_{\Theta_{\tilde{B}}}(\text{LMod}_B(\mathcal{C})).$$

Now $(\text{sh}_{\tilde{\varphi}})_*$ is an equivalence as it is the corestriction of scalars functor along an equivalence of comonads, so $\widehat{\varphi}_!$ is an equivalence if and only if $\overline{\varphi}_!$ is, i.e., if and only if φ admits descent. \square

For Hopf–Galois contexts satisfying the normal basis condition, the structured shear map also allows us to replace the primitives condition by a completeness condition, which is an ∞ -categorical version of [Rog08, Part I, Proposition 12.1.8].

Lemma 3.2.12. *The functor*

$$\widehat{\varphi}_! : \text{LMod}_A(\mathcal{C}) \xrightarrow{\text{Triv}_H^A} \text{LMod}_{\text{Triv}_H(A)}(\text{LComod}_H(\mathcal{C})) \xrightarrow{\tilde{\varphi}_!} \text{LMod}_{\tilde{B}}(\text{LComod}_H(\mathcal{C})).$$

of Lemma 3.2.7 admits a right adjoint $P_{\tilde{\varphi}}$ whose effect on underlying H -comodules is given by taking primitives.

Proof. As discussed in Remark 1.2.12, $\tilde{\varphi}_!$ admits a right adjoint $\tilde{\varphi}^*$ which does not change the underlying H -comodule. Moreover, since $\text{Triv}_H: \mathcal{C} \rightarrow \text{LComod}_H(\mathcal{C})$ is strongly monoidal, Example B.0.4 and Corollary B.0.10 imply that the adjunction $\text{Triv}_H \dashv \text{Prim}_H$ can be lifted to an adjunction $\text{Triv}_H^A: \text{LMod}_A(\mathcal{C}) \rightleftarrows \text{LMod}_{\text{Triv}_H(A)}(\text{LComod}_H(\mathcal{C})) : P_A$. Composing these two right adjoints, we obtain a right adjoint of $\widehat{\varphi}_!$ with the desired property. \square

Corollary 3.2.13. *Assume that the shear map $\text{sh}_{\tilde{\varphi}}: B \otimes_A B \rightarrow H \otimes B$ is an equivalence. Then $A \rightarrow \text{Prim}_H(\tilde{B})$ is an equivalence if and only if $A \rightarrow \lim_{\Delta} \mathbf{C}^\bullet(\varphi)$ is an equivalence, i.e., if and only if A is B -complete.*

Proof. We have $\tilde{B} \simeq \widehat{\varphi}_!(A)$, and the first map can be identified with the unit map of the adjunction $\widehat{\varphi}_! : \text{LMod}_A(\mathcal{C}) \rightleftarrows \text{LComod}_{\Theta_{\tilde{B}}}(\text{LMod}_B(\mathcal{C})) : P_{\tilde{\varphi}}$ of Lemma 3.2.12. Moreover, the second map is the unit of the adjunction $\overline{\varphi}_! : \text{LMod}_A(\mathcal{C}) \rightleftarrows \text{LComod}_{\varphi_! \varphi^*}(\text{LMod}_B(\mathcal{C})) : \overline{\varphi}^*$ of Corollary 1.3.13. Since we have an equivalence $(\text{sh}_{\tilde{\varphi}})_* : \text{LComod}_{\varphi_! \varphi^*}(\text{LMod}_B(\mathcal{C})) \rightarrow \text{LComod}_{\Theta_{\tilde{B}}}(\text{LMod}_B(\mathcal{C}))$ such that $(\text{sh}_{\tilde{\varphi}})_* \circ \overline{\varphi}_! \simeq \widehat{\varphi}_!$, one is an equivalence if and only if the other is. \square

Example 3.2.14. Assume that \mathcal{C} is presentably \mathbb{E}_2 -monoidal. As discussed in Example 3.1.11, the Thom context $\widetilde{M}_X(\eta_f): \mathrm{Triv}_{X \otimes_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}}(\mathbf{1}_{\mathcal{C}}) \rightarrow \widetilde{M}(f)$ associated to a map $f: X \rightarrow \mathrm{Pic}(\mathcal{C})$ of grouplike \mathbb{E}_1 -spaces satisfies the normal basis condition, so it is a Hopf–Galois extension if and only if $\mathbf{1}_{\mathcal{C}}$ is $M(f)$ -complete.

For Thom *spectra*, i.e., in the case where $\mathcal{C} = \mathrm{Sp}$, this completeness condition is closely related to orientability (cf. [Rot09, Proposition 6.9]). Indeed, if the composite $X \xrightarrow{f} \mathrm{Pic}(\mathrm{Sp}) \xrightarrow{\mathrm{H}\mathbb{Z} \otimes (-)} \mathrm{Pic}(\mathrm{LMod}_{\mathrm{H}\mathbb{Z}}(\mathrm{Sp}))$ is null-homotopic, then $M(f)$ is connective, and using the Hurewicz theorem, its 0-th homotopy group can be computed as

$$\pi_0(M(f)) \simeq \pi_0(\mathrm{H}\mathbb{Z} \otimes M(f)) \simeq \pi_0(M(\mathrm{H}\mathbb{Z} \otimes f)) \simeq \pi_0(\Sigma_+^{\infty} X \otimes \mathrm{H}\mathbb{Z}) \simeq \mathrm{H}_0(X; \mathbb{Z}) \simeq \mathbb{Z}[\pi_0 X].$$

Hence $M(f)$ is a connective ring spectrum whose 0-th homotopy group is a group algebra. The monoidal unit $\mathbf{1}_{\mathrm{Sp}}$, i.e., the sphere spectrum is complete with respect to all such ring spectra (cf. [Bou79, Theorem 6.5]).

4. Hopf algebras

In this section, we introduce a notion of Hopf algebras in the ∞ -categorical setting and lift some results about Hopf algebras in the 1-categorical setting to ∞ -categories. We continue to work with Convention 3.0.1, i.e., we consider a bialgebra H in a suitable \mathbb{E}_2 -monoidal ∞ -category \mathcal{C} . Recall that H can be lifted to an H -comodule algebra $\tilde{H} := C_H(\mathbf{1}_{\mathcal{C}}) \in \text{Alg}(\text{LComod}_H(\mathcal{C}))$ (cf. Example 3.1.2).

Definition 4.0.1. A bialgebra H is called a *Hopf algebra* if the associated shear map

$$\text{sh}_{\eta_H}^\sim: H \otimes H \xrightarrow{\Delta_H \otimes H} H \otimes H \otimes H \xrightarrow{H \otimes \mu_H} H \otimes H.$$

of Definition 3.1.7 is an equivalence.

As discussed in Example 3.1.10, every grouplike \mathbb{E}_1 -space X is a Hopf algebra in \mathcal{S} . Consequently, if \mathcal{C} is presentably \mathbb{E}_2 -monoidal, $X \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}$ is a Hopf algebra in \mathcal{C} .

4.1. The antipode

In the 1-categorical setting, Hopf algebras are usually defined in terms of an “antipode” $H \rightarrow H$ (cf. [Mon93, Definition 1.5.1] and [Por15, Definition 32]). In this subsection, we relate our notion of a Hopf algebra to the classical definition.

Fact 4.1.1 ([Por15, Proposition 22]). *Given $f, g: H \rightarrow H$, we call*

$$f * g: H \xrightarrow{\Delta_H} H \otimes H \xrightarrow{f \otimes g} H \otimes H \xrightarrow{\mu_H} H$$

their convolution product. This operation endows $\pi_0 \text{Map}_{\mathcal{C}}(H, H) \simeq \text{Hom}_{\text{ho}(\mathcal{C})}(H, H)$ with the structure of a monoid with unit $\eta_H \circ \epsilon_H$.

Definition 4.1.2. A *homotopy antipode* for H is an inverse of Id_H with respect to the convolution product, i.e., a map $\chi_H: H \rightarrow H$ in \mathcal{C} such that

$$\mu_H(H \otimes \chi_H) \Delta_H = \eta_H \epsilon_H = \mu_H(\chi_H \otimes H) \Delta_H.$$

At first sight, a homotopy antipode is an additional structure map with which we can equip a bialgebra. However, since the inverse of an element in a monoid is unique if it exists, we can speak of the *property* of “admitting a homotopy antipode”.

In the 1-categorical setting, an antipode $\chi_H: H \rightarrow H$ always underlies a map of bialgebras when one considers the opposite multiplication and the opposite comultiplication in either the source or the target (cf. [Por15, Proposition 36]). In the ∞ -categorical setting, lifting a homotopy antipode to a map of bialgebras would involve an infinite hierarchy of coherence data and it is not clear whether it can always be done. This is the reason why we prefer the term *homotopy antipode*, which is in line with the terminology of [Rot09, Definition 5.9].

In order to relate the existence of a homotopy antipode with the invertibility of the shear map, we generalize the construction of the shear map as follows.

Proposition 4.1.3. *The assignment that sends a map $f: H \rightarrow H$ to the composite*

$$\Xi_H(f): H \otimes H \xrightarrow{\Delta_H \otimes H} H \otimes H \otimes H \xrightarrow{H \otimes f \otimes H} H \otimes H \otimes H \xrightarrow{H \otimes \mu_H} H \otimes H$$

defines a monoid homomorphism

$$\Xi_H: (\text{Hom}_{\text{ho}(C)}(H, H), *, \epsilon_H \eta_H) \rightarrow (\text{Hom}_{\text{ho}(C)}(H \otimes H, H \otimes H), \circ, \text{Id}_{H \otimes H}).$$

Proof. Let $f, g: H \rightarrow H$. Then we have a commutative diagram

$$\begin{array}{ccccccc} H \otimes H & \xrightarrow{\Delta_H \otimes H} & H \otimes H \otimes H & \xrightarrow{H \otimes \Delta_H \otimes H} & H \otimes H \otimes H \otimes H & \xrightarrow{H \otimes H \otimes g \otimes H} & H \otimes H \otimes H \otimes H \\ \Delta_H \otimes H \downarrow & & \nearrow \Delta_H \otimes H \otimes H & & & & \downarrow H \otimes f \otimes H \otimes H \\ H \otimes H \otimes H & & & & & & H \otimes H \otimes H \otimes H \\ H \otimes g \otimes H \downarrow & & & & & & \downarrow H \otimes \mu_H \otimes H \\ H \otimes H \otimes H & \xrightarrow{\Delta_H \otimes H \otimes H} & H \otimes H \otimes H \otimes H & & & & H \otimes H \otimes H \\ H \otimes \mu_H \downarrow & & \downarrow H \otimes H \otimes \mu_H & & \nwarrow H \otimes H \otimes \mu_H & & \downarrow H \otimes \mu_H \\ H \otimes H & \xrightarrow{\Delta_H \otimes H} & H \otimes H \otimes H & \xrightarrow{H \otimes f \otimes H} & H \otimes H \otimes H & \xrightarrow{H \otimes \mu_H} & H \otimes H \end{array} .$$

Now the left vertical composite is $\Xi_H(g)$, the bottom horizontal composite is $\Xi_H(f)$, and the composite of the top horizontal and the right vertical arrows is $\Xi_H(f * g)$. Hence $\Xi_H(f * g) = \Xi_H(f) \circ \Xi_H(g)$.

As for units, we have a commutative diagram

$$\begin{array}{ccccccc} H \otimes H & \xrightarrow{\Delta_H \otimes H} & H \otimes H \otimes H & \xrightarrow{H \otimes \epsilon_H \otimes H} & H \otimes H & \xrightarrow{H \otimes \eta_H \otimes H} & H \otimes H \otimes H \xrightarrow{H \otimes \mu_H} H \otimes H \\ & \searrow \text{Id}_H \otimes H & & & & & \nearrow H \otimes \text{Id}_H \end{array}$$

implying that $\Xi_H(\eta_H \epsilon_H) = \text{Id}_{H \otimes H}$. □

Corollary 4.1.4. *If H admits a homotopy antipode, then H is a Hopf algebra, i.e., its shear map $\text{sh}_{\eta_H}^\sim: H \otimes H \rightarrow H \otimes H$ is an equivalence.*

Proof. Note that $\Xi_H(\text{Id}_H) = \text{sh}_{\eta_H}^\sim$. Hence, since Ξ_H is a monoid homomorphism, $\text{sh}_{\eta_H}^\sim$ is \circ -invertible if Id_H is $*$ -invertible. □

We can also go back from endomorphisms of $H \otimes H$ to endomorphisms of H .

Proposition 4.1.5. *The assignment that sends a map $u: H \otimes H \rightarrow H \otimes H$ to the composite*

$$\Xi'_H(u): H \xrightarrow{H \otimes \eta_H} H \otimes H \xrightarrow{u} H \otimes H \xrightarrow{\epsilon_H \otimes H} H$$

defines a left inverse of the map $\Xi_H: \text{Hom}_{\text{ho}(C)}(H, H) \rightarrow \text{Hom}_{\text{ho}(C)}(H \otimes H, H \otimes H)$ of Proposition 4.1.3.

Proof. Let $f: H \rightarrow H$. Then we have a commutative diagram

$$\begin{array}{ccccccc}
H & \xrightarrow{H \otimes \eta_H} & H \otimes H & \xrightarrow{\Delta_H \otimes H} & H \otimes H \otimes H & \xrightarrow{H \otimes f \otimes H} & H \otimes H \otimes H & \xrightarrow{H \otimes \mu_H} & H \otimes H & \xrightarrow{\epsilon_H \otimes H} & H \\
& \searrow \Delta_H & & \nearrow H \otimes H \otimes \eta_H & & & \searrow \epsilon_H \otimes H \otimes H & & \nearrow \mu_H & & \\
& & H \otimes H & \xrightarrow{\epsilon_H \otimes H} & H & \xrightarrow{f} & H & \xrightarrow{H \otimes \eta_H} & H \otimes H & &
\end{array},$$

where the upper composite is $\Xi'_H(\Xi_H(f))$ and the lower composite is f . \square

Example 4.1.6. Consider the case where \mathcal{C} is a preadditive (1-)category equipped with the symmetric monoidal structure given by its biproduct \oplus . Then every object X admits a bialgebra structure whose multiplication is the fold map $X \oplus X \rightarrow X$ and whose comultiplication is the diagonal $X \rightarrow X \oplus X$. Moreover, the convolution product of endomorphisms of X coincides with the addition of endomorphisms obtained from the preadditive structure.

Identifying $\text{Hom}_{\mathcal{C}}(X \oplus X, X \oplus X)$ with 2×2 matrices with entries in $\text{Hom}_{\mathcal{C}}(X, X)$, Ξ_X sends an endomorphism $f: X \rightarrow X$ to the matrix

$$\begin{pmatrix} \text{Id}_X & f \\ 0_X & \text{Id}_X \end{pmatrix},$$

which is where the terminology of “shear” maps comes from. Moreover, under this identification, the retraction Ξ'_X sends a matrix A to the entry $A_{1,2}$. In particular, we see that Ξ'_X is not a monoid homomorphism in general because a matrix does not necessarily have identities on the diagonal.

However, the restriction of Ξ'_H to the image of Ξ_H is a homomorphism because the inverse of a bijective monoid homomorphism is a monoid homomorphism. Therefore, as monoid homomorphisms preserve invertibility, we have the following partial converse of Corollary 4.1.4.

Corollary 4.1.7. *If the shear map $\text{sh}_{\eta_H}: H \otimes H \rightarrow H \otimes H$ admits an inverse that is in the image of Ξ_H , i.e., is of the form*

$$H \otimes H \xrightarrow{\Delta_H \otimes H} H \otimes H \otimes H \xrightarrow{H \otimes \chi \otimes H} H \otimes H \otimes H \xrightarrow{H \otimes \mu_H} H \otimes H,$$

then χ is a homotopy antipode for H .

As we will see in Proposition 5.4.1, another setting where we can extract a homotopy antipode from an inverse of the shear map is that of (co)commutative Hopf algebras. This applies in particular to a grouplike \mathbb{E}_1 -space X , whose comultiplication $(\text{Id}_X, \text{Id}_X): X \rightarrow X \times X$ is cocommutative. An antipode $X \rightarrow X$ sends a point $x \in X$ to an inverse of x with respect to the multiplication of X .

4.2. Hopf modules and descent

We now consider objects equipped with an action and a coaction of a bialgebra that are compatible in an appropriate sense.

Definition 4.2.1. The category of (left) H -Hopf modules in \mathcal{C} is

$$\mathrm{HopfMod}_H(\mathcal{C}) := \mathrm{LMod}_{\tilde{H}}(\mathrm{LComod}_H(\mathcal{C})) \stackrel{3.2.2}{\simeq} \mathrm{LComod}_{\Theta_{\tilde{H}}}(\mathrm{LMod}_H(\mathcal{C})),$$

where $\Theta_{\tilde{H}}$ is the comonad of Definition 3.2.4.

In the 1-categorical setting, the *fundamental theorem of Hopf modules* states that category of Hopf modules is equivalent to the base category (cf. [Mon93, Theorem 1.9.4]). We prove an ∞ -categorical version of this statement in Corollary 4.2.9, which requires some preparation.

Let us first sketch the proof of the 1-categorical statement, on which our approach is based. Note that every cofree H -comodule $H \otimes X$ has a compatible H -module structure given by $\mu_H \otimes X: H \otimes (H \otimes X) \rightarrow X$, yielding a lift $\widehat{C}_H: \mathcal{C} \rightarrow \mathrm{HopfMod}_H(\mathcal{C})$ of the cofree H -comodule functor. Now \widehat{C}_H admits a left quasi-inverse because $\mathrm{Prim}_H \circ U_{\tilde{H}} \circ \widehat{C}_H \simeq \mathrm{Prim}_H \circ C_H \simeq \mathrm{Id}_{\mathcal{C}}$. As for the right quasi-inverse, we can check that for every Hopf module M , the composite

$$M \xrightarrow{\rho_M} H \otimes M \xrightarrow{\Delta_{H \otimes M}} H \otimes H \otimes M \xrightarrow{H \otimes \chi_H \otimes M} H \otimes H \otimes M \xrightarrow{H \otimes \alpha_M} H \otimes M \quad (4.2.2)$$

lands in $H \otimes \mathrm{Prim}_H(M)$ and defines an inverse to the map $H \otimes \mathrm{Prim}_H(M) \rightarrow M$ given by the restriction of α_M , hence yielding a natural isomorphism $\mathrm{Id}_{\mathrm{HopfMod}_H(\mathcal{C})} \simeq \widehat{C}_H \circ \mathrm{Prim}_H \circ U_{\tilde{H}}$.

In the 1-categorical setting $\mathrm{Prim}_H(M)$ is an equalizer, whereas in the ∞ -categorical setting it is the limit of a cosimplicial diagram (cf. the proof of Proposition 1.3.10), which makes it harder to show that the analogue of (4.2.2) factors through $H \otimes \mathrm{Prim}_H(M)$. We circumvent this issue by working with a “dual shear map” instead of the antipode (cf. Construction 4.2.6) and making the equivalence $M \simeq H \otimes \mathrm{Prim}_H(M)$ implicit in equivalences of module categories induced by this variant of this shear map.

We start with a general “codescent” statement for coaugmented coalgebras, which yields an ∞ -category equivalent to \mathcal{C} that is easier to compare with $\mathrm{HopfMod}_H(\mathcal{C})$.

Remark 4.2.3. Let $\eta_D: \mathbf{1}_{\mathcal{C}} \rightarrow D$ be a coaugmented coalgebra. Then the cofree comodule functor $C_D: \mathcal{C} \rightarrow \mathrm{LComod}_D$ is conservative because it admits a left inverse $\mathrm{Prim}_D: \mathrm{LComod}_D \rightarrow \mathcal{C}$.

Proposition 4.2.4. Let $\eta_D: \mathbf{1}_{\mathcal{C}} \rightarrow D$ be a coaugmented coalgebra. Then the adjunction $V_D \dashv C_D$ is monadic, i.e., the standard lift $\overline{C}_D: \mathcal{C} \rightarrow \mathrm{LMod}_{C_D V_D}(\mathrm{LComod}_D(\mathcal{C}))$ of C_D to the module category of the monad $C_D V_D$ is an equivalence.

Proof. We apply the Barr–Beck–Lurie monadicity theorem ([Lur17, Theorem 4.7.3.5]). First, note that C_D is conservative by Remark 4.2.3.

Moreover, as the forgetful functor of a comodule category, $V_D: \text{LComod}_D(\mathcal{C}) \rightarrow \mathcal{C}$ reflects colimits, so C_D -split simplicial objects in $\text{LComod}_D(\mathcal{C})$ admit geometric realizations, as their images in \mathcal{C} do.

Now let X^\triangleright be a colimiting cocone of a C_D -split simplicial object. Then $D \otimes X^\triangleright \simeq V_D C_D X^\triangleright$ is a colimiting cocone as the tensor product of \mathcal{C} preserves geometric realizations, so $C_D X^\triangleright$ is a colimiting cocone because V_D reflects colimits. \square

Remark 4.2.5. Remark 4.2.3 can be formally dualized to augmented algebras, i.e., algebras R equipped with an algebra map $\epsilon_R: R \rightarrow \mathbf{1}_{\mathcal{C}}$. In that case, the functor $F_R \simeq (\eta_R)_!: \mathcal{C} \rightarrow \text{LMod}_R(\mathcal{C})$ is conservative as it admits a left inverse $(\epsilon_R)_!: \text{LMod}_R(\mathcal{C}) \rightarrow \mathcal{C}$. On the other hand, the proof of Proposition 4.2.4 cannot be directly dualized to augmented algebras as $R \otimes -$ does not necessarily preserve limits of cosimplicial objects.

However, when R is dualizable¹¹, $R \otimes -$ is right adjoint to $R^\vee \otimes -$, so it does preserve limits. Hence, dualizing the argument of Proposition 4.2.4, we see that augmented *dualizable* algebras admit descent.

We now construct a lift of C_H to $\text{HopfMod}_H(\mathcal{C})$ and a dual version of the structured shear map¹².

Construction 4.2.6. As $C_H: \mathcal{C} \rightarrow \text{LComod}_H(\mathcal{C})$ is lax monoidal, it sends $\mathbf{1}_{\mathcal{C}}$ -modules to $C_H(\mathbf{1}_{\mathcal{C}}) = \widetilde{H}$ -modules. More precisely, we have lift

$$\begin{array}{ccc} \text{LMod}_{\mathbf{1}_{\mathcal{C}}}(\mathcal{C}) & \longrightarrow & \text{LMod}_{\widetilde{H}}(\text{LComod}_H(\mathcal{C})) = \text{HopfMod}_H(\mathcal{C}) \\ \wr & \nearrow \widehat{C_H} & \downarrow U_{\widetilde{H}} \\ \mathcal{C} & \xrightarrow{C_H} & \text{LComod}_H(\mathcal{C}) \end{array}.$$

This lift exhibits the functor $C_H: \mathcal{C} \rightarrow \text{LComod}_H(\mathcal{C})$ as a module over the monad $U_{\widetilde{H}} F_{\widetilde{H}}$ on $\text{LComod}_H(\mathcal{C})$. Unpacking the definitions, we see that the associated action map at $X \in \mathcal{C}$ is given by

$$U_{\widetilde{H}} F_{\widetilde{H}} C_H X \simeq H \otimes H \otimes X \xrightarrow{\mu_H \otimes X} H \otimes X \simeq C_H X.$$

Now consider the monad $C_H V_H$ on $\text{LComod}_H(\mathcal{C})$, which is an endomorphism object for C_H , i.e., a terminal monad acting on it. The aforementioned action of $U_{\widetilde{H}} F_{\widetilde{H}}$ yields a map $\widetilde{\text{sh}}_H: U_{\widetilde{H}} F_{\widetilde{H}} \rightarrow C_H V_H$ of monads, which we call the *dual shear map*. Unpacking the constructions, we see that its component at $M \in \text{LComod}_H(\mathcal{C})$ is given by

$$U_{\widetilde{H}} F_{\widetilde{H}} M \simeq H \otimes M \xrightarrow{H \otimes \rho_M} H \otimes H \otimes M \xrightarrow{\mu_H \otimes M} H \otimes M \simeq C_H V_H M.$$

¹¹The notion of dualizability will be briefly reviewed in the beginning of Subsection 4.3

¹²We could formally dualize the construction in Subsection 3.2 where we consider maps of comodule algebras to maps of module coalgebras, and apply it to the map $\epsilon_H: H \rightarrow \mathbf{1}_{\mathcal{C}}$ in order obtain a map from a certain monad to $C_H V_H$. However, Construction 4.2.6 has the advantage of explicitly describing the source as the monad associated to the comodule algebra \widetilde{H} .

Moreover, dualizing the discussion in Remark 1.3.7, we obtain a commutative diagram

$$\begin{array}{ccc}
& & \text{LMod}_{C_H \vee_H}(\text{LComod}_H(\mathcal{C})) \xrightarrow{(\widetilde{\text{sh}}_H)^*} \text{LMod}_{\widetilde{H}}(\text{LComod}_H(\mathcal{C})) \\
\widehat{C}_H \uparrow & \nearrow \overline{C}_H & \downarrow U_{C_H \vee_H} \\
\mathcal{C} & \xrightarrow{C_H} & \mathcal{C} \xleftarrow{V_{\widetilde{H}}}
\end{array}$$

Lemma 4.2.7. *Let H be a Hopf algebra with a homotopy antipode and $M \in \text{LComod}_H(\mathcal{C})$. Then the dual shear map $\widetilde{\text{sh}}_H(M): H \otimes M \rightarrow H \otimes M$ of Construction 4.2.6 is an equivalence.*

Proof. Diagram chases similar to the one in the proof of Proposition 4.1.3 show that

$$H \otimes M \xrightarrow{H \otimes \rho_M} H \otimes H \otimes M \xrightarrow{H \otimes \chi_H \otimes M} H \otimes H \otimes M \xrightarrow{\mu \otimes M} H \otimes M.$$

is a two sided inverse for $\widetilde{\text{sh}}_H(M): H \otimes M \rightarrow H \otimes M$. \square

Corollary 4.2.8. *If H is a Hopf algebra with a homotopy antipode, then the dual shear map $\widetilde{\text{sh}}_H: U_{\widetilde{H}} F_{\widetilde{H}} \rightarrow C_H V_H$ induces an equivalence $\widetilde{\text{sh}}_H^*: \text{LMod}_{C_H \vee_H}(\text{LComod}_H(\mathcal{C})) \xrightarrow{\cong} \text{HopfMod}_H(\mathcal{C})$.*

Now we can prove our version of the fundamental theorem of Hopf modules.

Corollary 4.2.9. *If H is a Hopf algebra with a homotopy antipode, then the composite $\text{Prim}_H \circ U_{\widetilde{H}}: \text{HopfMod}_H(\mathcal{C}) \rightarrow \mathcal{C}$ is an inverse of the functor $\widehat{C}_H: \mathcal{C} \rightarrow \text{HopfMod}_H(\mathcal{C})$ of Construction 4.2.6.*

Proof. Combining Proposition 4.2.4 and Corollary 4.2.8, we see that $\widehat{C}_H \simeq \widetilde{\text{sh}}_H^* \circ \overline{C}_H$ is an equivalence. Moreover, we have $\text{Prim}_H \circ U_{\widetilde{H}} \circ \widehat{C}_H \simeq \text{Prim}_H \circ C_H \simeq \text{Id}_{\mathcal{C}}$, meaning that $\text{Prim}_H \circ U_{\widetilde{H}}$ is a left inverse for \widehat{C}_H . As \widehat{C}_H is an equivalence, $\text{Prim}_H \circ U_{\widetilde{H}}$ is also a right inverse. \square

We end this subsection with another application of the dual shear map.

Lemma 4.2.10. *If H is a Hopf algebra with a homotopy antipode, then the functor $\widehat{C}_H: \mathcal{C} \rightarrow \text{HopfMod}_H(\mathcal{C})$ is equivalent to the functor $(\widehat{\eta_H})_!: \mathcal{C} \rightarrow \text{HopfMod}_H(\mathcal{C})$ of Lemma 3.2.7.*

Proof. Note that the adjoint $F_{C_H \vee_H} \rightarrow \overline{C}_H V_H$ of the unit transformation $\text{Id}_{\text{LComod}_H(\mathcal{C})} \rightarrow C_H V_H \simeq U_{C_H \vee_H} \overline{C}_H V_H$ is an equivalence because its image under the conservative functor $U_{C_H \vee_H}$ is an equivalence $U_{C_H \vee_H} F_{C_H \vee_H} \simeq C_H V_H \simeq U_{C_H \vee_H} \overline{C}_H V_H$. Hence we have a commutative diagram

$$\begin{array}{ccccc}
& & \text{LMod}_{C_H \vee_H}(\text{LComod}_H(\mathcal{C})) & \xrightarrow{\widetilde{\text{sh}}_H^*} & \text{HopfMod}_H(\mathcal{C}) \\
& \nearrow F_{C_H \vee_H} & \downarrow U_{C_H \vee_H} & \nwarrow U_{\widetilde{H}} & \\
\mathcal{C} & \xrightarrow{\text{Triv}_H} \text{LComod}_H(\mathcal{C}) \xrightarrow{V_H} \mathcal{C} & \xrightarrow{\overline{C}_H} \text{LComod}_H(\mathcal{C}) & \xrightarrow{C_H} & \text{LComod}_H(\mathcal{C})
\end{array}$$

which implies that

$$\widetilde{\text{sh}}_H^* \circ F_{C_H \vee H} \circ \text{Triv}_H \simeq \widetilde{\text{sh}}_H^* \circ \overline{C}_H \circ V_H \circ \text{Triv}_H \simeq \widetilde{\text{sh}}_H^* \circ \overline{C}_H \simeq \widehat{C}_H. \quad (4.2.11)$$

Now recall that $(\widehat{\eta_H})_!$ is given by the composite

$$\mathcal{C} \xrightarrow{\text{Triv}_H} \text{LComod}_H(\mathcal{C}) \xrightarrow{F_{\widetilde{H}} \simeq (\widehat{\eta_H})_!} \text{LMod}_{\widetilde{H}}(\text{LComod}_H(\mathcal{C})) = \text{HopfMod}_H(\mathcal{C}).$$

Moreover, $\widetilde{\text{sh}}_H$ is an equivalence by Corollary 4.2.8, so $F_{\widetilde{H}} \simeq \widetilde{\text{sh}}_H^* \circ F_{C_H \vee H}$ as both functors are left adjoint to $U_{\widetilde{H}} \simeq U_{C_H \vee H} \circ (\widetilde{\text{sh}}_H^*)^{-1}$. Hence the left hand side of (4.2.11) is in fact equivalent to $(\widehat{\eta_H})_!$. \square

Combining Lemma 4.2.10, Corollary 4.2.9 and Corollary 3.2.11, we obtain the following.

Corollary 4.2.12. *If H is a Hopf algebra with a homotopy antipode, then we have equivalences*

$$\mathcal{C} \xrightarrow[\simeq]{(\widehat{\eta_H})_!} \text{LComod}_{(\eta_H)_!(\eta_H)^*}(\mathcal{C}) \xrightarrow[\simeq]{(\widetilde{\text{sh}}_{\eta_H})_*} \text{HopfMod}_H(\mathcal{C}).$$

$\underbrace{\hspace{15em}}_{(\widehat{\eta_H})_!}$

In particular, $\eta_H: \mathbf{1}_{\mathcal{C}} \rightarrow H$ admits descent.

4.3. Comodules over dualizable coalgebras

Recall that an object $X \in \mathcal{C}$ is called *right dualizable* if there exists an object X^\vee (a *right dual* of X) together with morphisms $c_X: \mathbf{1}_{\mathcal{C}} \rightarrow X \otimes X^\vee$ and $e_X: X^\vee \otimes X \rightarrow \mathbf{1}_{\mathcal{C}}$ such that the composites

$$X \xrightarrow{c_X \otimes X} X \otimes X^\vee \otimes X \xrightarrow{X \otimes e_X} X$$

and

$$X^\vee \xrightarrow{X^\vee \otimes c_X} X^\vee \otimes X \otimes X^\vee \xrightarrow{e_X \otimes X^\vee} X^\vee$$

are equivalent to identities, in which case we also say that X^\vee is *left dualizable* with *left dual* X (cf. [Lur17, Definition 4.6.1.1]). Note that this structure exhibits the functor $X \otimes (-)$ as a right adjoint of the functor $X^\vee \otimes (-)$ and determines X^\vee up to equivalence.

An object X is called *dualizable* if it is both left and right dualizable. In general, left dualizability is not equivalent to right dualizability. However, if the braiding $\tau: (-) \otimes (?) \simeq (?) \otimes (-)$ squares to the identity, $\tau_{X, X^\vee} \circ c_X$ and $e_X \circ \tau_{X^\vee, X}$ do exhibit X^\vee as a left dual of X , so that we can speak of *the* dual of X . We strengthen our assumption on \mathcal{C} in order to make use of this phenomenon.

Convention 4.3.1. In the rest of this section, we assume that the base category \mathcal{C} is \mathbb{E}_3 -monoidal. This implies in particular that the braiding of \mathcal{C} squares to the identity and thus the notions of left and right dualizability coincide. We denote by $\mathcal{C}_{\text{fd}} \subseteq \mathcal{C}$ the full subcategory spanned by dualizable objects.

Lemma 4.3.2. *The \mathbb{E}_3 -monoidal structure of \mathcal{C} restricts to a \mathbb{E}_3 -monoidal structure on \mathcal{C}_{fd} , and the assignment $X \mapsto X^\vee$ can be extended to a strongly \mathbb{E}_2 -monoidal equivalence $(-)^\vee: \mathcal{C}_{\text{fd}} \rightarrow \mathcal{C}_{\text{fd}}^{\text{op}}$.*

*Proof*¹³. Note that $\mathbf{1}_{\mathcal{C}}$ is dualizable, and if X and Y are dualizable, then so is $X \otimes Y$. Now all the operations of \mathbb{E}_3 are generated by the unit and the tensor product, meaning that \mathcal{C}_{fd} is closed under *all* operations of \mathbb{E}_3 . Hence, by [Lur17, Proposition 2.2.1.1], the \mathbb{E}_3 -monoidal structure of \mathcal{C} restricts to a \mathbb{E}_3 -monoidal structure on \mathcal{C}_{fd} .

For the functoriality of $(-)^\vee$, we can adapt the argument of [Lur18, Proposition 3.2.4] (which deals with the symmetric monoidal case) as follows. We consider the cospan $\mathcal{C}_{\text{fd}} \times \mathcal{C}_{\text{fd}} \xrightarrow{\otimes} \mathcal{C}_{\text{fd}} \xleftarrow{\text{pr}} (\mathcal{C}_{\text{fd}})_{/\mathbf{1}_{\mathcal{C}}}$. The fact that every object of \mathcal{C}_{fd} is dualizable implies the projection $\lambda: (\mathcal{C}_{\text{fd}} \times \mathcal{C}_{\text{fd}}) \times_{\mathcal{C}_{\text{fd}}} (\mathcal{C}_{\text{fd}})_{/\mathbf{1}_{\mathcal{C}}} \rightarrow \mathcal{C}_{\text{fd}} \times \mathcal{C}_{\text{fd}}$ is a perfect pairing in the sense of [Lur17, Definition 5.2.1.8 and Corollary 5.2.1.22], yielding an equivalence $(-)^\vee: \mathcal{C}_{\text{fd}} \rightarrow \mathcal{C}_{\text{fd}}^{\text{op}}$. Now, by the Dunn additivity theorem ([Lur17, Theorem 5.1.2.2]), \mathcal{C}_{fd} can be seen as an algebra in the ∞ -category of \mathbb{E}_2 -monoidal ∞ -categories, implying that the functor $\otimes: \mathcal{C}_{\text{fd}} \times \mathcal{C}_{\text{fd}} \rightarrow \mathcal{C}_{\text{fd}}$ is strongly \mathbb{E}_2 -monoidal. Moreover, $(\mathcal{C}_{\text{fd}})_{/\mathbf{1}_{\mathcal{C}}}$ admits an \mathbb{E}_3 -monoidal structure such that the projection $(\mathcal{C}_{\text{fd}})_{/\mathbf{1}_{\mathcal{C}}} \rightarrow \mathcal{C}_{\text{fd}}$ is \mathbb{E}_3 -monoidal (and hence \mathbb{E}_2 -monoidal). This means that λ can be upgraded to a pairing of \mathbb{E}_2 -monoidal ∞ -categories in the sense of [Lur17, Definition 5.2.2.20], lifting the associated equivalence $(-)^\vee: \mathcal{C}_{\text{fd}} \rightarrow \mathcal{C}_{\text{fd}}^{\text{op}}$ to a strongly \mathbb{E}_2 -monoidal equivalence. \square

The monoidal equivalence $(-)^\vee: \mathcal{C}_{\text{fd}} \rightarrow \mathcal{C}_{\text{fd}}^{\text{op}}$ gives us more possibilities of dualizing algebraic structures to coalgebraic structures and vice versa. For instance, we will show that for every dualizable coalgebra D , there is an equivalence $\text{LComod}_D(\mathcal{C}) \simeq \text{LMod}_{D^\vee}(\mathcal{C})$ that does not change the underlying objects (cf. Proposition 4.3.10), which generalizes an analogous statement in the 1-categorical setting (cf. [Mon93, Lemma 1.6.4]¹⁴).

Proposition 4.3.3. *Let \mathcal{M} be an ∞ -category left-tensored over \mathcal{C}_{fd} , $D \in \mathcal{C}_{\text{fd}}$ and $M \in \mathcal{M}$. A map $\rho: M \rightarrow D \otimes M$ exhibits D as a coendomorphism object for M in \mathcal{C}_{fd} if and only if the adjoint map $\hat{\rho}: D^\vee \otimes M \rightarrow M$ exhibits D^\vee as an endomorphism object for M in \mathcal{C}_{fd} .*

Proof. For all $X \in \mathcal{C}_{\text{fd}}$, there is a commutative diagram

$$\begin{array}{ccccc}
\text{Map}_{\mathcal{C}_{\text{fd}}}(X^\vee, D^\vee) & \xrightarrow{(-) \otimes M} & \text{Map}_{\mathcal{M}}(X^\vee \otimes M, D^\vee \otimes M) & \xrightarrow{\hat{\rho} \circ (-)} & \text{Map}_{\mathcal{M}}(X^\vee \otimes M, M) \\
\wr & & & & \parallel \\
\text{Map}_{\mathcal{C}_{\text{fd}}}(X^\vee \otimes D, \mathbf{1}_{\mathcal{C}}) & \xrightarrow{(-) \otimes M} & \text{Map}_{\mathcal{M}}(X^\vee \otimes D \otimes M, M) & \xrightarrow{(-) \circ (X^\vee \otimes \rho)} & \text{Map}_{\mathcal{M}}(X^\vee \otimes M, M) \\
\wr & & \wr & & \wr \\
\text{Map}_{\mathcal{C}_{\text{fd}}}(D, X) & \xrightarrow{(-) \otimes M} & \text{Map}_{\mathcal{M}}(D \otimes M, X \otimes M) & \xrightarrow{(-) \circ \rho} & \text{Map}_{\mathcal{M}}(M, X \otimes M)
\end{array}$$

¹³Note that for $k \geq 3$, this proof applies verbatim to an \mathbb{E}_k -monoidal ∞ -category \mathcal{D} to yield a strongly \mathbb{E}_{k-1} -monoidal equivalence $(-)^\vee: \mathcal{D}_{\text{fd}} \rightarrow \mathcal{D}_{\text{fd}}^{\text{op}}$.

¹⁴Note that the cited lemma compares *right* comodules with *left* modules. We will comment on this discrepancy in Remark 4.3.5.

Since $(-)^{\vee}$ is an equivalence, the upper composite is an equivalence for all X if and only if $\hat{\rho}: D^{\vee} \otimes M \rightarrow M$ exhibits D^{\vee} as an endomorphism object for M , and the lower composite is an equivalence for all X if and only if $\rho: M \rightarrow D \otimes M$ exhibits D as a coendomorphism object for M . \square

Corollary 4.3.4. *If D is a dualizable coalgebra in \mathcal{C} , then there exists an algebra structure on D^{\vee} and a right \mathcal{C} -linear geometric-realization-preserving lift*

$$\begin{array}{ccc} & & \text{LMod}_{D^{\vee}}(\mathcal{C}) \\ & \nearrow \Phi_D & \downarrow U_{D^{\vee}} \\ \text{LComod}_D(\mathcal{C}) & \xrightarrow{V_D} & \mathcal{C} \end{array}$$

sending a D -comodule M with coaction map $\rho: M \rightarrow D \otimes M$ to a D^{\vee} -module with the same underlying object and action map given by the adjoint $\hat{\rho}: D^{\vee} \otimes M \rightarrow M$.

Proof. As discussed in Remark 1.4.6, the natural transformation $\rho_{(-)}: V_D(-) \rightarrow D \otimes V_D(-)$ exhibits D as a coendomorphism object for $V_D \in \text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\text{LComod}_D(\mathcal{C}), \mathcal{C})$. Now, by Proposition 4.3.3, the adjoint transformation $\hat{\rho}_{(-)}: D^{\vee} \otimes V_D(-) \rightarrow V_D(-)$ exhibits D^{\vee} as an endomorphism object for V_D . In particular, we obtain an algebra structure on D^{\vee} and an action of it on V_D extending $\hat{\rho}_{(-)}$. This action induces a lift $\Phi_D: \text{LComod}_D(\mathcal{C}) \rightarrow \text{LMod}_{D^{\vee}}(\mathcal{C})$ of V_D against $U_{D^{\vee}}$ with the desired properties. \square

Remark 4.3.5. Note that we have two different ways of dualizing the coalgebra structure of a dualizable coalgebra D to an algebra structure on its dual D^{\vee} . Namely, we could view D as an algebra in $\mathcal{C}_{\text{fd}}^{\text{op}}$ and transfer this algebra structure along the monoidal equivalence $(-)^{\vee}: \mathcal{C}_{\text{fd}}^{\text{op}} \rightarrow \mathcal{C}_{\text{fd}}$ of Lemma 4.3.2, or use Corollary 4.3.4. In this subsection we work with the latter structure, but we expect the two to be opposites of each other. We need to consider opposite algebras because Lemma 4.3.2 identifies $(X \otimes Y)^{\vee}$ with $X^{\vee} \otimes Y^{\vee}$, whereas composing the adjunctions $X^{\vee} \otimes (-) \dashv X \otimes (-)$ and $Y^{\vee} \otimes (-) \dashv Y \otimes (-)$ as implicitly done in Proposition 4.3.3 yields an identification $(X \otimes Y)^{\vee} \simeq Y^{\vee} \otimes X^{\vee}$.

The following lemma provides some evidence that the algebra structure of Corollary 4.3.4 is the “correct” one.

Lemma 4.3.6. *If D is a dualizable coalgebra in \mathcal{C} , then the coalgebra structure on $D \simeq (D^{\vee})^{\vee}$ constructed by iterating Corollary 4.3.4 agrees with its original coalgebra structure.*

Proof. Let D_o denote the original coalgebra structure and D_n the new one. By definition, D_n is a coendomorphism object for $U_{D^{\vee}} \in \text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\text{LMod}_{D^{\vee}}(\mathcal{C}), \mathcal{C})$. Precomposing the coaction of D_n on $U_{D^{\vee}}$ with the functor $\Phi_D: \text{LComod}_D(\mathcal{C}) \rightarrow \text{LMod}_{D^{\vee}}(\mathcal{C})$ of Corollary 4.3.4, we obtain a coaction of D_n on $U_{D^{\vee}} \Phi_D \simeq V_D \in \text{LinFun}_{\mathcal{C}}^{\{\Delta^{\text{op}}\}}(\text{LComod}_D(\mathcal{C}), \mathcal{C})$, whose coaction map $V_D(-) \rightarrow D \otimes V_D(-)$ agrees with the coaction map $\rho_{(-)}$ of D_o as the adjoint of $\hat{\rho}_{(-)}$ is $\rho_{(-)}$. Now, $\rho_{(-)}$ exhibits D_o as a coendomorphism object for V_D , so by its universal property Id_D extends to an equivalence $D_o \simeq D_n$ of coalgebras. \square

Remark 4.3.7. Let D be a dualizable coalgebra in \mathcal{C} . Applying (the dual of) Proposition 4.3.3 to D^\vee , we obtain a lift

$$\begin{array}{ccc} & & \text{LComod}_D(\mathcal{C}) \\ & \nearrow \Psi_{D^\vee} & \downarrow V_D \\ \text{LMod}_{D^\vee}(\mathcal{C}) & \xrightarrow{U_{D^\vee}} & \mathcal{C} \end{array},$$

where the coalgebra structure on D agrees with the original one by Lemma 4.3.6.

Remark 4.3.8. The functoriality of our construction of the dual algebra structure is witnessed on the level of (co)module categories. Indeed, a map $\zeta: D \rightarrow E$ of dualizable coalgebras in \mathcal{C} yields a functor

$$\text{LMod}_{D^\vee}(\mathcal{C}) \xrightarrow{\Psi_{D^\vee}} \text{LComod}_D(\mathcal{C}) \xrightarrow{\zeta_*} \text{LComod}_E(\mathcal{C}) \xrightarrow{\Phi_E} \text{LMod}_{E^\vee}(\mathcal{C}) \quad (4.3.9)$$

lifting $U_{D^\vee}: \text{LMod}_{D^\vee} \rightarrow \mathcal{C}$ against $U_{E^\vee}: \text{LMod}_{E^\vee} \rightarrow \mathcal{C}$ and thus inducing an action of E^\vee on U_{D^\vee} .

As the action map of a D^\vee -module M is mapped as

$$\begin{aligned} (D^\vee \otimes M \xrightarrow{\alpha_M} M) &\xrightarrow{\Psi_{D^\vee}} (M \xrightarrow{\widehat{\alpha_M}} D \otimes M) \\ &\xrightarrow{\zeta_*} (M \xrightarrow{\widehat{\alpha_M}} D \otimes M \xrightarrow{\zeta \otimes M} E \otimes M) \\ &\xrightarrow{\Phi_E} (E^\vee \otimes M \xrightarrow{\zeta^\vee \otimes M} D^\vee \otimes M \xrightarrow{\alpha_M} M), \end{aligned}$$

the aforementioned action of E^\vee on U_{D^\vee} is given by

$$E^\vee \otimes U_{D^\vee}(-) \xrightarrow{\zeta^\vee \otimes U_{D^\vee}(-)} D^\vee \otimes U_{D^\vee}(-) \xrightarrow{\alpha(-)} U_{D^\vee}(-).$$

Now, by the universal property of D^\vee as an endomorphism object for U_{D^\vee} , this yields an extension of $\zeta^\vee: E^\vee \rightarrow D^\vee$ to an algebra map such that $(\zeta^\vee)^*$ agrees with the composite (4.3.9). Compatibility with identities and composition can be computed similarly by considering actions on forgetful functors of module categories.

Proposition 4.3.10. *If D is a dualizable coalgebra in \mathcal{C} , then the functors*

$$\begin{aligned} \text{LComod}_D(\mathcal{C}) &\xrightarrow{\Phi_D} \text{LMod}_{D^\vee}(\mathcal{C}) & \text{LMod}_{D^\vee}(\mathcal{C}) &\xrightarrow{\Psi_{D^\vee}} \text{LComod}_D(\mathcal{C}) \\ (\rho: M \rightarrow D \otimes M) &\longmapsto (\widehat{\rho}: D^\vee \otimes M \rightarrow M) & (\alpha: D^\vee \otimes M \rightarrow M) &\longmapsto (\widehat{\alpha}: M \rightarrow D \otimes M) \end{aligned}$$

of Corollary 4.3.4 and Remark 4.3.7 are equivalences.

Proof. We start by showing that $\Psi_{D^\vee} \circ \Phi_D: \text{LComod}_D(\mathcal{C}) \rightarrow \text{LComod}_D(\mathcal{C})$ is an equivalence. We have a commutative diagram

$$\begin{array}{ccc} \text{LComod}_D(\mathcal{C}) & \xrightarrow{\Psi_{D^\vee} \circ \Phi_D} & \text{LComod}_D(\mathcal{C}) \\ & \searrow V_D \quad \swarrow V_D & \\ & \mathcal{C} & \end{array}$$

where V_D is part of a comonadic adjunction $V_D \dashv C_D$ with unit $u: \text{Id}_{\text{LComod}_D(\mathcal{C})} \rightarrow C_D V_D$ and counit $c: V_D C_D \rightarrow \text{Id}_{\mathcal{C}}$. Applying [Lur17, Corollary 4.7.3.16 and Remark 4.7.3.17], it will suffice to show that for all $X \in \mathcal{C}$, the underlying map of

$$\Psi_{D^\vee} \Phi_D C_D X \xrightarrow{u_{\Psi_{D^\vee} \Phi_D C_D X}} C_D V_D \Psi_{D^\vee} \Phi_D C_D X \simeq C_D V_D C_D X \xrightarrow{C_D c_X} C_D X \quad (4.3.11)$$

is an equivalence. Now, the underlying map of $u_{\Psi_{D^\vee} \Phi_D C_D X}$ is simply the coaction map of $\Psi_{D^\vee} \Phi_D C_D X$, which agrees with the coaction map of $C_D X$ as it is given by the “adjoint of its adjoint”. Therefore, the underlying map of (4.3.11) is equivalent to the composite

$$D \otimes X \xrightarrow{\Delta_{D \otimes X}} D \otimes D \otimes X \xrightarrow{D \otimes \epsilon_{D \otimes X}} D \otimes X$$

which is an indeed equivalence.

By a dual argument, $\Phi_D \circ \Psi_{D^\vee}: \text{LMod}_{D^\vee}(\mathcal{C}) \rightarrow \text{LComod}_D(\mathcal{C})$ is also an equivalence. Hence Φ_D and Ψ_{D^\vee} are both equivalences. \square

Convention 4.3.12. For a dualizable coalgebra D in \mathcal{C} , we will from now on implicitly identify $\text{LComod}_D(\mathcal{C})$ with $\text{LMod}_{D^\vee}(\mathcal{C})$ via Φ_D and Ψ_{D^\vee} .

Remark 4.3.13. Let $\zeta: D \rightarrow E$ be a map of dualizable coalgebras in \mathcal{C} . Then we have an adjunction

$$\begin{array}{ccc} \text{LMod}_{E^\vee}(\mathcal{C}) & \begin{array}{c} \xrightarrow{(\zeta^\vee)_!} \\ \perp \\ \xleftarrow{(\zeta^\vee)^*} \end{array} & \text{LMod}_{D^\vee}(\mathcal{C}) \\ \wr & & \wr \\ \text{LComod}_E(\mathcal{C}) & \xleftarrow{\zeta_*} & \text{LComod}_D(\mathcal{C}) \end{array}$$

where the lower square commutes (cf. Remark 4.3.8). This yields a left adjoint of ζ_* , which we, in line with Convention 4.3.12, also denote by $(\zeta^\vee)_!$. Note that left adjoints obtained this way are compatible with identities and composition of coalgebra maps as the corresponding right adjoints are.

We have seen that, up to homotopy, the algebra structure on D^\vee is functorial and Φ_D is natural with respect to coalgebra maps. We expect that this functoriality and naturality can be made coherent, which we make precise below.

The statement will make use of a duality notion for (co)cartesian fibrations. Recall that a cartesian fibration $p: X \rightarrow S$ classifies a functor $S^{\text{op}} \rightarrow \text{Cat}_\infty$, which is also classified by a *cocartesian* fibration $p^\vee: X^\vee \rightarrow S^{\text{op}}$. An explicit model for $(-)^\vee$ can be found in [BGN18, Definition 3.4]. Alternatively, p^\vee can be thought of as $(p^{\text{op}})^{\text{fop}}$. Intuitively, this construction allows us to “take the opposite in the transversal direction without changing the fibers”.

Conjecture 4.3.14. *For every dualizable coalgebra D , the algebra structure on D^\vee constructed in Corollary 4.3.4 agrees with the opposite of the algebra structure obtained by transferring the coalgebra structure of D along $(-)^{\vee}: \mathcal{C}_{\text{fd}}^{\text{op}} \rightarrow \mathcal{C}_{\text{fd}}$.*

Under this identification, the functor $\Phi_D: \text{LComod}_D(\mathcal{C}) \rightarrow \text{LMod}_{D^\vee}(\mathcal{C})$ is the fiber of an equivalence Φ that fits into a commutative diagram

$$\begin{array}{ccc} \text{LComod}^{\mathcal{C}_{\text{fd}}}(\mathcal{C}) & \xrightarrow[\simeq]{\Phi} & \text{LMod}^{\mathcal{C}_{\text{fd}}}(\mathcal{C})^\vee \\ \theta_{\mathcal{C}_{\text{fd}}}^{\text{coalg}} \downarrow & & \downarrow (\theta_{\mathcal{C}_{\text{fd}}}^{\text{alg}})^\vee \\ \text{Coalg}(\mathcal{C}_{\text{fd}}) = \text{Alg}(\mathcal{C}_{\text{fd}}^{\text{op}})^{\text{op}} & \xrightarrow[\text{Alg}((-)^\vee)^{\text{op}}]{\simeq} \text{Alg}(\mathcal{C}_{\text{fd}})^{\text{op}} & \xrightarrow[(-)^{\text{rev}}]{\simeq} \text{Alg}(\mathcal{C}_{\text{fd}})^{\text{op}} \end{array}, \quad (4.3.15)$$

where $(-)^{\text{rev}}$ denotes the opposite algebra functor (cf. [Lur17, Remark 4.1.1.7]). Moreover, the equivalence Φ is strongly monoidal with respect to the monoidal structures induced by the bifunctor $\mathbb{E}_1^\otimes \times \mathcal{LM}^\otimes \rightarrow \mathbb{E}_2^\otimes$ (cf. Fact 1.2.2).

Remark 4.3.16. Compatibilities encoded in Conjecture 4.3.14 include the following.

For a dualizable bialgebra H , it implies that the equivalence $\Phi_H: \text{LComod}_H(\mathcal{C}) \simeq \text{LMod}_{H^\vee}(\mathcal{C})$ is strongly monoidal with respect to the monoidal structure of Fact 1.2.9 in the source and the analogous monoidal structure in the target. Indeed, the former monoidal structure can be constructed as the pullback of $(\theta_{\mathcal{C}_{\text{fd}}}^{\text{coalg}})^\otimes: \text{LComod}^{\mathcal{C}_{\text{fd}}}(\mathcal{C})^\otimes \rightarrow \text{Coalg}(\mathcal{C}_{\text{fd}})^\otimes$ along the map $h: \mathbb{E}_1^\otimes \rightarrow \text{Coalg}(\mathcal{C}_{\text{fd}})^\otimes$ that corresponds to H and the latter as the pullback of $((\theta_{\mathcal{C}_{\text{fd}}}^{\text{alg}})^\otimes)^\vee$ along the composite of h with the lower composite $\text{Coalg}(\mathcal{C}_{\text{fd}})^\otimes \rightarrow (\text{Alg}(\mathcal{C}_{\text{fd}})^{\text{op}})^\otimes$ of (4.3.15), which, under the equivalence $\text{Bialg}(\mathcal{C}_{\text{fd}}) \simeq \text{Coalg}(\text{Alg}(\mathcal{C}_{\text{fd}}))$ of Corollary A.0.17, represents a dual bialgebra structure on H^\vee .

Moreover, as the lower composite in (4.3.15) maps a coalgebra of the form $D \otimes E^{\text{rev}}$ to the algebra $D^\vee \otimes (E^\vee)^{\text{rev}}$, the conjectural equivalence Φ maps $(D \otimes E^{\text{rev}})$ -comodules, i.e., D - E -bicomodules to $D^\vee \otimes (E^\vee)^{\text{rev}}$ -modules, i.e., D^\vee - E^\vee -bimodules.

We conclude this section with an informal discussion of how Conjecture 4.3.14 could be used to develop an analogue of equivariant homotopy theory for dualizable Hopf algebras. Consider a topological group (or more generally, a grouplike \mathbb{E}_k -space) G and for the sake of this illustration, assume that \mathcal{C} is *presentably* \mathbb{E}_3 -monoidal. Then $G \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}$, which is an algebra (in fact, a Hopf algebra with a homotopy antipode) in \mathcal{C} , can be thought of as a *group algebra*, and the module category $\text{LMod}_{G \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}}(\mathcal{C})$ can be thought of as the ∞ -category of objects of \mathcal{C} with a G -action.

Note that a large portion of (naive) G -equivariant homotopy theory can be formulated in terms of the bialgebra structure of $G \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}$. For instance, equipping the tensor product $X \otimes Y$ of two G -objects with the diagonal G -action corresponds to their tensor product with respect to the lift of the monoidal structure of \mathcal{C} to $\text{LMod}_{G \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}}(\mathcal{C})$ obtained via the dual of Fact 1.2.9. Moreover, restriction of scalars along the augmentation $\epsilon_{G \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}}: G \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}} \rightarrow \{*\} \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}} \simeq \mathbf{1}_{\mathcal{C}}$ corresponds to equipping an object with a trivial G -action and extension of scalars along $\epsilon_{G \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}}$ to the homotopy orbits functors.

Now assume that $G \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}$ is dualizable. This is for example the case when G is a compact Lie group, and \mathcal{C} is the ∞ -category Sp of spectra (or more generally, a stable ∞ -category). There are also groups that are not dualizable in Sp but become such after a localization, which have been studied under the name of *stably dualizable groups* (cf. [Rog08, Part II]). Then, using the equivalence $\text{LMod}_{G \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}}(\mathcal{C}) \simeq \text{LComod}_{(G \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}})^\vee}(\mathcal{C})$ and

the identification $(\epsilon_{G \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}})^{\vee} \simeq \eta_{(G \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}})^{\vee}}$, we can interpret G -homotopy fixed points as primitives with respect to $(G \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}})^{\vee}$.

This interplay between modules over the group algebra and comodules over its dual is also crucial for norm maps, whose analogue for general dualizable Hopf algebras with a homotopy antipode we sketch now. We start with an analogue of homotopy orbits for a dualizable coalgebra.

Definition 4.3.17. Let $\eta_D: \mathbf{1}_{\mathcal{C}} \rightarrow D$ be a coaugmented dualizable coalgebra in \mathcal{C} . Remark 4.3.13 implies in particular that $\mathrm{Triv}_D \simeq (\eta_D)_* \simeq (\eta_D^{\vee})^*$ admits a left adjoint given by $(\eta_D^{\vee})_!$. We call this left adjoint the *coorbits functor* and denote it by $\mathrm{Coorb}_D: \mathrm{LComod}_D(\mathcal{C}) \rightarrow \mathcal{C}$.

Now consider the *alternate norm map* $(-)^{hG} \rightarrow ((\Sigma_+^{\infty} G)^{\vee})^{hG} \otimes (-)^{hG}$ of [Rog08, Part II, Definition 5.2.7]. The analogue of the *inverse dualizing object* $((\Sigma_+^{\infty} G)^{\vee})^{hG}$ in our setting is $(D \otimes_{D^{\vee}} \mathbf{1}_{\mathcal{C}})$. The observation that $(D \otimes_{D^{\vee}} \mathbf{1}_{\mathcal{C}}) \otimes (-): \mathcal{C} \rightarrow \mathcal{C}$ can be lifted to $\mathrm{LComod}_D(\mathcal{C}) \rightarrow \mathrm{LComod}_D(\mathcal{C})$ as $D \otimes_{D^{\vee}} (-)$ leads to the following definition.

Construction 4.3.18 (assuming Conjecture 4.3.14). Let $\eta_D: \mathbf{1}_{\mathcal{C}} \rightarrow D$ be a coaugmented dualizable coalgebra in \mathcal{C} . Note that the composite

$$\mathrm{LComod}_D(\mathcal{C}) \xrightarrow{\mathrm{Coorb}_D \simeq (\eta_D^{\vee})_!} \mathcal{C} \xrightarrow{\mathrm{Triv}_D \simeq (\eta_D^{\vee})^*} \mathrm{LComod}_D(\mathcal{C})$$

is given by tensoring over D^{\vee} with the D^{\vee} - D^{\vee} bimodule $\mathbf{1}_{\mathcal{C}}$. Now $\eta_D: \mathbf{1}_{\mathcal{C}} \rightarrow D$, considered as a map of D - D -bicomodules and thus of D^{\vee} - D^{\vee} bimodules, induces a natural transformation

$$\mathrm{Triv}_D \mathrm{Coorb}_D \simeq \mathbf{1}_{\mathcal{C}} \otimes_{D^{\vee}} (-) \xrightarrow{\eta_D \otimes_{D^{\vee}} (-)} D \otimes_{D^{\vee}} (-).$$

Using the adjunction $\mathrm{Triv}_D \dashv \mathrm{Prim}_D$, this yields a natural transformation

$$N'_D: \mathrm{Coorb}_D \rightarrow \mathrm{Prim}_D(D \otimes_{D^{\vee}} (-)),$$

which we call the *alternate conorm transformation*.

If we had a suitable theory of a relative cotensor product \square_D , we could define an analogue of the *norm map* $((\Sigma_+^{\infty} G)^{hG} \otimes (-))^{hG} \rightarrow (-)^{hG}$ of [Rog08, Part II, Definition 5.2.2] dually as the adjoint $\mathrm{Coorb}_D(D^{\vee} \square_D (-)) \rightarrow \mathrm{Prim}_D$ of the transformation $D^{\vee} \square_D (-) \rightarrow \mathbf{1}_{\mathcal{C}} \square_D (-) \simeq \mathrm{Triv}_D \circ \mathrm{Prim}_D$ induced by the D - D -bicomodule map $\eta_D^{\vee}: D^{\vee} \rightarrow \mathbf{1}_{\mathcal{C}}$. However, as discussed in Remark 1.1.14, we are lacking such a theory for many of the examples we are interested in.

For a dualizable Hopf algebra H with a homotopy antipode, we expect that one can circumvent this issue by showing that the twist $\mathfrak{C}_H := H \otimes_{H^{\vee}} (-)$ is invertible and defining the norm map as $N'_H(\mathfrak{C}_H^{-1}(-))$.

Indeed, in the case of G -equivariant homotopy theory, the inverse dualizing object $((\Sigma_+^{\infty} G)^{\vee})^{hG}$ is, as the name suggests, \otimes -invertible, and its inverse is $(\Sigma_+^{\infty} G)^{hG}$ (cf. [Rog08, Part II, Theorem 3.3.4]). Hence, at least on the level of underlying twists, one might

expect that $H \otimes_{H^\vee} \mathbf{1}_\mathcal{C}$ is \otimes -invertible with inverse $\text{Prim}_H^r(H^\vee)$, where the superscript r indicates primitives with respect to the right coaction.

Now consider the variant of the equivalence $\mathcal{C} \simeq \text{HopfMod}_H(\mathcal{C})$ of Corollary 4.2.9 for *right* Hopf modules. We expect that the right action map $\mu_H^\vee: H^\vee \otimes H \rightarrow H^\vee$ can be lifted to a right Hopf module structure on H^\vee , which would imply that

$$H^\vee \simeq C_H^r(\text{Prim}_H^r(H^\vee)) \simeq \text{Prim}_H^r(H^\vee) \otimes H,$$

which is an analogue of [Rog08, Part II, Theorem 3.1.4]. Applying $(-) \otimes_{H^\vee} \mathbf{1}_\mathcal{C}$, this would yield the desired \otimes -invertibility result as

$$\mathbf{1}_\mathcal{C} \simeq H^\vee \otimes_{H^\vee} \mathbf{1}_\mathcal{C} \simeq \text{Prim}_H^r(H^\vee) \otimes (H \otimes_{H^\vee} \mathbf{1}_\mathcal{C}).$$

A construction of a norm map for cocommutative bialgebras assuming the existence of such invertible objects was carried out in [Rak20, Subsection 2.4].

A sufficiently structured correspondence between G -objects and $(G \odot_\mathcal{C} \mathbf{1}_\mathcal{C})^\vee$ -comodules would also allow us to make the relationship between Galois and Hopf–Galois extensions sketched in the introduction precise.

Remark 4.3.19 (cf. [Rog08, Part I, Example 12.1.6]¹⁵). Assume that \mathcal{C} is presentably \mathbb{E}_3 -monoidal, and let G be a topological group such that $G \odot_\mathcal{C} \mathbf{1}_\mathcal{C}$ is dualizable. Let $\varphi: A \rightarrow B$ be a map of algebras with G -action (i.e., algebras in $\text{LMod}_{G \odot_\mathcal{C} \mathbf{1}_\mathcal{C}}(\mathcal{C})$), where the action on the source is trivial. Let $F_\mathcal{C}(G, -)$ be a right adjoint of $G \odot_\mathcal{C} (-)$, for instance $(G \odot_\mathcal{C} \mathbf{1}_\mathcal{C})^\vee \otimes (-)$. Then φ is a G -Galois extension in the sense of [Rog08, Part I, Definition 4.1.3] if the induced map $A \rightarrow B^{hG}$ and the adjoint $B \otimes_A B \rightarrow F_\mathcal{C}(G, B)$ of $G \odot_\mathcal{C} B \otimes_A B \xrightarrow{\alpha_B \otimes_A B} B \otimes_A B \xrightarrow{\mu_B} B$ are equivalences.

Assuming that Conjecture 4.3.14 holds, we can transfer φ to a $(G \odot_\mathcal{C} \mathbf{1}_\mathcal{C})^\vee$ -Hopf–Galois context via the strongly monoidal equivalence $\text{LMod}_{G \odot_\mathcal{C} \mathbf{1}_\mathcal{C}}(\mathcal{C}) \simeq \text{LComod}_{(G \odot_\mathcal{C} \mathbf{1}_\mathcal{C})^\vee}(\mathcal{C})$. Under this identification, the first map corresponds to the map $A \rightarrow \text{Prim}_{(G \odot_\mathcal{C} \mathbf{1}_\mathcal{C})^\vee} B$, so the first condition can be viewed as a primitives condition. Moreover, the second map can be identified with the shear map

$$B \otimes_A B \xrightarrow{\widehat{\alpha_B \otimes_A B}} (G \odot_\mathcal{C} \mathbf{1}_\mathcal{C})^\vee \otimes B \otimes_A B \xrightarrow{(G \odot_\mathcal{C} \mathbf{1}_\mathcal{C})^\vee \otimes \mu_B} B,$$

so the second condition can be viewed as a normal basis condition. This means that φ is a Galois extension if and only if its image in $\text{LComod}_{(G \odot_\mathcal{C} \mathbf{1}_\mathcal{C})^\vee}(\mathcal{C})$ is a Hopf–Galois extension.

¹⁵Note however, the cited example assumes that dual group algebras in EKMM spectra can be rigidified, which is unlikely for non-abelian groups in light of [PS18, Theorem 1.1], which says that all comodules in EKMM spectra are cocommutative.

5. Hopf–Galois extensions of commutative algebras

Recall that by [Lur17, Proposition 3.2.4.7], the tensor product of commutative algebras in a symmetric monoidal ∞ -category is a coproduct in the ∞ -category of commutative algebras. In this section, we employ this fact to study Hopf–Galois extensions of commutative algebras more closely.

Convention 5.0.1. Throughout this section, we work with a symmetric monoidal ∞ -category \mathcal{C} .

Due to the cocartesianness of the symmetric monoidal structure of $\mathrm{CAlg}(\mathcal{C})$, it will be more convenient to consider coalgebraic structures in $\mathrm{CAlg}(\mathcal{C})$ instead of algebraic structures in coalgebra or comodule categories.

Convention 5.0.2. In this section, our preferred model for the ∞ -category of commutative bialgebras in \mathcal{C} will be $\mathrm{Coalg}(\mathrm{CAlg}(\mathcal{C}))$, which is equivalent to $\mathrm{CAlg}(\mathrm{Coalg}(\mathcal{C}))$ used in Definition 1.2.7 by Corollary A.0.17.

Similarly, we will work with $\mathrm{LComod}(\mathrm{CAlg}(\mathcal{C}))$ instead of $\mathrm{CAlg}(\mathrm{LComod}(\mathcal{C}))$ and given $H \in \mathrm{Coalg}(\mathrm{CAlg}(\mathcal{C})) \simeq \mathrm{Alg}(\mathrm{CAlg}(\mathcal{C}))$, with $\mathrm{LComod}_H(\mathrm{CAlg}(\mathcal{C}))$ instead of $\mathrm{CAlg}(\mathrm{LComod}_H(\mathcal{C}))$ by employing the equivalences of Example A.0.18.

5.1. Overview of cosimplicial models

Coalgebras and comodules in cocartesian symmetric monoidal categories (such as $\mathrm{CAlg}(\mathcal{C})$) can be identified with cosimplicial objects that satisfy certain “dual Segal conditions”.

Before we describe these “cosimplicial models”, let us first informally review the general machinery that is used to construct such models.. We discuss the dual case of algebraic structures, i.e., algebras over an ∞ -operad \mathcal{O} , in a cartesian symmetric monoidal ∞ -category \mathcal{D} to avoid cluttering the notation with fiberwise opposites.

- [Lur17, Definition 2.4.2.1] An \mathcal{O} -monoid in \mathcal{D} is a functor $M: \mathcal{O}^\otimes \rightarrow \mathcal{D}$ such that for every $\mathfrak{o}_1, \dots, \mathfrak{o}_n \in \mathcal{O}_{\langle 1 \rangle}^\otimes$, a certain comparison map $M(\mathfrak{o}_1 \oplus \dots \oplus \mathfrak{o}_n) \rightarrow \prod_{i=1}^n M(\mathfrak{o}_i)$ is an equivalence. Let $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{O}^\otimes, \mathcal{C})$ spanned by \mathcal{O} -monoids.
- [Lur17, Proposition 2.4.1.7] Let $\mathcal{D}^\times \rightarrow \mathrm{Fin}_*$ be the cocartesian fibration encoding the cartesian symmetric monoidal structure of \mathcal{D} and $\pi: \mathcal{D}^\times \rightarrow \mathcal{D}$ be the functor that sends a tuple $(D_1, \dots, D_n) \in \mathcal{D}_{\langle n \rangle}^\times$ to $\prod_{i=1}^n D_i$. Then the functor $\pi \circ (-): \mathrm{Fun}(\mathcal{O}^\otimes, \mathcal{D}^\times) \rightarrow \mathrm{Fun}(\mathcal{O}^\otimes, \mathcal{D})$ restricts to an equivalence $\mathrm{Alg}_{\mathcal{O}/\mathbb{E}_\infty}(\mathcal{C}) \simeq \mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$.
- [Lur17, Definition 2.4.2.8 and Proposition 2.4.2.11] This “monoid model” can be simplified further if we have a suitable *weak approximation* $f: \mathcal{E} \rightarrow \mathcal{O}^\otimes$ to \mathcal{O} in the sense of [Lur17, Definition 2.3.3.6]. In that case, $(-) \circ f: \mathrm{Mon}_{\mathcal{O}}(\mathcal{D}) \rightarrow \mathrm{Fun}(\mathcal{E}, \mathcal{D})$ is fully faithful and its image can be described in terms of certain “Segal-type conditions”.

In [Lur17, Construction 4.1.2.9], a weak approximation $\Delta^{\text{op}} \rightarrow \mathbb{E}_1^\otimes$ is constructed, which yields the following.

Fact 5.1.1 ([Lur17, Definition 4.1.2.5 and Proposition 4.1.2.10]). *The ∞ -category $\text{CBialg}(\mathcal{C}) \simeq \text{Coalg}(\text{CAlg}(\mathcal{C}))$ is equivalent to the full subcategory of $\text{Fun}(\Delta, \text{CAlg}(\mathcal{C}))$ spanned by cosimplicial objects \mathfrak{H} such that for every $n \in \mathbb{N}$, the coface maps*

$$\{\mathfrak{H}(\{i-1, i\})\} \xrightarrow{\mathfrak{H}(\{i-1, i\} \hookrightarrow [n])} \mathfrak{H}([n])\}_{1 \leq i \leq n}$$

exhibit $\mathfrak{H}([n])$ as a coproduct of $\{\mathfrak{H}(\{i-1, i\})\}_{1 \leq i \leq n}$. We will refer to this condition as the dual Segal condition and objects satisfying it as comonoid objects.

Under this identification, the forgetful functor $\text{CBialg}(\mathcal{C}) \rightarrow \text{CAlg}(\mathcal{C})$ corresponds to evaluation at $[1] \in \Delta$.

Notation 5.1.2. When a commutative algebra H in \mathcal{C} underlies a commutative bialgebra, we will denote the corresponding cosimplicial commutative algebra by $\Omega_H^\bullet: \Delta \rightarrow \text{CAlg}(\mathcal{C})$.

Remark 5.1.3. Let H be a commutative bialgebra in \mathcal{C} . Unpacking the weak approximation $\Delta^{\text{op}} \rightarrow \mathbb{E}_1^\otimes$ of [Lur17, Construction 4.1.2.9], we see that the coalgebra structure of H is encoded in Ω_H^\bullet as follows. First we note that for all $n \in \mathbb{N}$, we have $\Omega_H^n \simeq H^{\otimes n}$ by the dual Segal condition. Under this identification, the codegeneracy map s^i is given by

$$\Omega_H^{n+1} \simeq H^{\otimes i} \otimes H \otimes H^{\otimes(n-i)} \xrightarrow{H^{\otimes i} \otimes \epsilon_H \otimes H^{\otimes(n-i)}} H^{\otimes n} \simeq \Omega_H^n.$$

As for coface maps, d^0 is given by

$$\Omega_H^{n-1} \simeq \mathbf{1}_{\mathcal{C}} \otimes H^{\otimes(n-1)} \xrightarrow{\eta_H \otimes H^{\otimes(n-1)}} H^{\otimes n} \simeq \Omega_H^n,$$

d^n by

$$\Omega_H^{n-1} \simeq H^{\otimes(n-1)} \otimes \mathbf{1}_{\mathcal{C}} \xrightarrow{H^{\otimes(n-1)} \otimes \eta_H} H^{\otimes n} \simeq \Omega_H^n,$$

and d^i for $0 < i < n$ by

$$\Omega_H^{n-1} \simeq H^{\otimes(i-1)} \otimes H \otimes H^{\otimes(n-i-1)} \xrightarrow{H^{\otimes(i-1)} \otimes \Delta_H \otimes H^{\otimes(n-i-1)}} H^{\otimes n} \simeq \Omega_H^n.$$

Moreover, the cosimplicial identities encode the coassociativity and the counitality of the coalgebra structure.

Lemma 5.1.4. *Let K be a simplicial set such that $\text{CAlg}(\mathcal{C})$ admits colimits of diagrams of shape K . Using the equivalence of Fact 5.1.1, we view $\text{Coalg}(\text{CAlg}(\mathcal{C}))$ as a full subcategory of $\text{Fun}(\Delta, \text{CAlg}(\mathcal{C}))$. Then $\text{Coalg}(\text{CAlg}(\mathcal{C}))$ is closed under colimits of diagrams of shape K in $\text{Fun}(\Delta, \text{CAlg}(\mathcal{C}))$.*

Proof. Let $f: K \rightarrow \text{Coalg}(\text{CAlg}(\mathcal{C}))$ be a diagram and $\bar{f}: K \rightarrow \text{Fun}(\Delta, \text{CAlg}(\mathcal{C}))$ its composite with the “inclusion” $\text{Coalg}(\text{CAlg}(\mathcal{C})) \rightarrow \text{Fun}(\Delta, \text{CAlg}(\mathcal{C}))$. Now $\text{colim}_K \bar{f} \in \text{Fun}(\Delta, \text{CAlg}(\mathcal{C}))$ satisfies the dual Segal condition because each $\bar{f}(k)$ for $k \in K$ satisfies the dual Segal condition and colimits commute with colimits, so we indeed have $\text{colim}_K \bar{f} \in \text{Coalg}(\text{CAlg}(\mathcal{C}))$. \square

In [Lur17, Remark 4.2.2.8], a weak approximation $\Delta^{\text{op}} \times [1] \rightarrow \mathcal{LM}^{\otimes}$ is constructed, which yields a cosimplicial model for comodules. For the sake of notational convenience, we consider $[1] \times \Delta^{\text{op}}$ instead of $\Delta^{\text{op}} \times [1]$ that appears there and identify $([1] \times \Delta^{\text{op}})^{\text{op}} \simeq [1]^{\text{op}} \times \Delta$ with $[1] \times \Delta$.

Fact 5.1.5 ([Lur17, Definition 4.2.2.2 and Proposition 4.2.2.9]). $\text{LComod}(\text{CAlg}(\mathcal{C}))$ is equivalent to the full subcategory of $\text{Fun}([1] \times \Delta, \text{CAlg}(\mathcal{C}))$ spanned by \mathfrak{B} such that

- $\mathfrak{B}|_{\{0\} \times \Delta}$ is a comonoid object (i.e., encodes a commutative bialgebra in the sense of the cosimplicial description of Fact 5.1.1),
- for all $n \in \mathbb{N}$, the maps $\mathfrak{B}(0, [n]) \rightarrow \mathfrak{B}(1, [n])$ and $\mathfrak{B}(1, 0 \mapsto n): \mathfrak{B}(1, [0]) \rightarrow \mathfrak{B}(1, [n])$ exhibit $\mathfrak{B}(1, [n])$ as a coproduct of $\mathfrak{B}(0, [n])$ and $\mathfrak{B}(1, [0])$.

We will refer to this condition as the left coaction condition.

This identification yields a commutative diagram

$$\begin{array}{ccc}
 \text{Coalg}(\text{CAlg}(\mathcal{C})) & \xleftarrow{\theta} & \text{LComod}(\text{CAlg}(\mathcal{C})) \\
 \downarrow & & \downarrow \\
 \text{Fun}(\Delta, \text{CAlg}(\mathcal{C})) & \xleftarrow[\substack{(\Delta \cong \{0\} \times \Delta \hookrightarrow [1] \times \Delta)^*}]{} & \text{Fun}([1] \times \Delta, \text{CAlg}(\mathcal{C}))
 \end{array}
 \begin{array}{c}
 \nearrow \varsigma \\
 \searrow \text{ev}_{(1, [0])}
 \end{array}
 \text{CAlg}(\mathcal{C}), \quad (5.1.6)$$

where θ forgets and ς picks the object being coacted on, and the vertical arrows are equivalences onto the full subcategories of comonoid and left coaction objects, respectively.

Notation 5.1.7. Let B be a commutative algebra in \mathcal{C} equipped with a coaction of a commutative bialgebra H in \mathcal{C} . We will depict the functor $[1] \times \Delta \rightarrow \text{CAlg}(\mathcal{C})$ witnessing the coaction as a map $\iota_{H,B}^{\bullet}: \Omega_H^{\bullet} \rightarrow \Omega_H^{\bullet}(B)$ of cosimplicial commutative algebras.

Remark 5.1.8. Let B be a commutative algebra in \mathcal{C} equipped with a coaction of a commutative bialgebra H in \mathcal{C} . Unpacking the weak approximation $\Delta^{\text{op}} \times [1] \rightarrow \mathcal{LM}^{\otimes}$ of [Lur17, Remark 4.2.2.8], we see that $\Omega_H^{\bullet}(B)$ encodes the coaction of H on B as follows. For all $n \in \mathbb{N}$, we have $\Omega_H^n(B) \simeq \Omega_H^n \otimes B \simeq H^{\otimes n} \otimes B$ by the left coaction condition. Under this identification, the coface map d^n is given by

$$\Omega_H^{n-1}(B) \simeq H^{\otimes(n-1)} \otimes B \xrightarrow{H^{\otimes(n-1)} \otimes \rho_B} H^{\otimes n} \otimes B \simeq \Omega_H^n(B).$$

The other coface maps and the codegeneracy maps are given by tensoring the respective maps of Ω_H^{\bullet} with B .

It is not a coincidence that the coface and codegeneracy maps of $\Omega_H^{\bullet}(B)$ are reminiscent of those of the cobar resolutions of Definition 1.3.8. Indeed, unpacking the definitions, Lurie’s version of cobar construction $R_H^{\bullet}(B)_{\perp}: \Delta_{\perp} \rightarrow \text{CAlg}(\mathcal{C})$ from [Lur17, Example 4.7.2.7] can be identified as the restriction of $\Omega_H^{\bullet}(B)$ along the functor $\Delta_{\perp} \rightarrow \Delta$ that forgets the distinction of the bottom element. In Construction 5.1.17, we will construct a version of cobar resolutions that is “internal to the cosimplicial models” and also has this property, and see in Remark 5.1.23 that $\Omega_H^{\bullet}(B)$ can be identified with the two sided cobar construction $\Omega_H^{\bullet}(\mathbf{1}_{\mathcal{C}}, B)$ when working with these cobar resolutions.

Notation 5.1.9. Note that in the cosimplicial context of Fact 5.1.5, a morphism in $\text{LComod}(\text{CAlg}(\mathcal{C}))$ corresponds to a commutative square

$$\begin{array}{ccc} \Omega_H^\bullet & \xrightarrow{\xi^\bullet} & \Omega_{H'}^\bullet \\ \iota_{H,B}^\bullet \downarrow & & \downarrow \iota_{H',B'}^\bullet \\ \Omega_H^\bullet(B) & \xrightarrow{\phi^\bullet} & \Omega_{H'}^\bullet(B') \end{array} \quad (5.1.10)$$

of cosimplicial commutative algebras, where ξ^\bullet corresponds to a map of commutative bialgebras with underlying commutative algebra map $\xi := \xi^1: H \rightarrow H'$ by virtue of Fact 5.1.1 and $\phi := \phi^0: B \rightarrow B'$ is a map of commutative algebras.

We will informally denote such a morphism as $(\xi^\bullet, \phi^\bullet): (H, B) \rightarrow (H', B')$.

Remark 5.1.11. Let H be a commutative bialgebra in \mathcal{C} . Considering the fiber of $(\Delta \cong \{0\} \times \Delta \hookrightarrow [1] \times \Delta)^*: \text{Fun}([1] \times \Delta, \text{CAlg}(\mathcal{C})) \rightarrow \text{Fun}(\Delta, \text{CAlg}(\mathcal{C}))$ over Ω_H^\bullet , we see that in the cosimplicial model of Fact 5.1.5, morphisms in $\text{LComod}_H(\text{CAlg}(\mathcal{C}))$ correspond to commutative squares of the form (5.1.10) where ξ^\bullet is $\text{Id}_{\Omega_H^\bullet}$. Alternatively, as the aforementioned functor corresponds to $\text{ev}_0: \text{Fun}([1], \text{Fun}(\Delta, \text{CAlg}(\mathcal{C}))) \rightarrow \text{Fun}(\Delta, \text{CAlg}(\mathcal{C}))$, $\text{LComod}_H(\text{CAlg}(\mathcal{C}))$ can be viewed as the full subcategory of $\text{Fun}(\Delta, \text{CAlg}(\mathcal{C}))_{\Omega_H^\bullet/}$ spanned by the objects of the form $\iota_{H,B}^\bullet: \Omega_H^\bullet \rightarrow \Omega_H^\bullet(B)$.

Next, we give explicit descriptions of various constructions for comodule algebras in the cosimplicial model.

Lemma 5.1.12. *Assume that \mathcal{C} admits geometric realizations and that the tensor product preserves geometric realizations in each variable. Let $\xi: H \rightarrow H'$ be a map of commutative bialgebras in \mathcal{C} and B a commutative H -comodule algebra.*

Then, in the cosimplicial description of Fact 5.1.5, the pushout square

$$\begin{array}{ccc} \Omega_H^\bullet & \xrightarrow{\xi^\bullet} & \Omega_{H'}^\bullet \\ \iota_{H,B}^\bullet \downarrow & & \downarrow \\ \Omega_H^\bullet(B) & \xrightarrow{\quad} & ? \end{array} \quad (5.1.13)$$

of simplicial commutative algebras corresponds to a cocartesian lift of ξ along the forgetful functor $\theta: \text{LComod}(\text{CAlg}(\mathcal{C})) \rightarrow \text{Coalg}(\text{CAlg}(\mathcal{C}))$. In particular, the right vertical arrow, which we denote by $\iota_{H',B}^\bullet: \Omega_{H'}^\bullet \rightarrow \Omega_{H'}^\bullet(B)$, can be identified with $\xi_\star B$.

Proof. Note that by [Lur09, Lemma 6.1.1.1], cartesian arrows with respect to the functor $\text{ev}_0: \text{Fun}([1], \text{Fun}(\Delta, \text{CAlg}(\mathcal{C}))) \rightarrow \text{Fun}(\Delta, \text{CAlg}(\mathcal{C}))$ correspond to such pushout squares. Hence, as ev_0 “restricts to θ ” (cf. the diagram (5.1.6)), it is enough to show that the right arrow in (5.1.13) lies in the full subcategory that corresponds to $\text{LComod}(\text{CAlg}(\mathcal{C}))$, i.e., that it satisfies the left coaction condition. This is the case because $\iota_{H,B}^\bullet: \Omega_H^\bullet \rightarrow \Omega_H^\bullet(B)$ satisfies the left coaction condition and colimits commute with colimits. \square

The following combinatorial constructions will be needed while dealing with cofree comodule algebras and cobar resolutions.

Construction 5.1.14. Let $X^\bullet: \Delta \rightarrow \mathcal{D}$ be a cosimplicial object and $Y_+^\bullet: \Delta_+ \rightarrow \mathcal{D}$ a coaugmented cosimplicial object in an ∞ -category. Consider the ordinal sum functor $\star: \Delta_+ \times \Delta_+ \rightarrow \Delta_+$, which sends $([k], [l]) \in \Delta_+ \times \Delta_+$ to $[k + l + 1]$. We denote its various restrictions to Δ , Δ_\top and Δ_\perp also by \star .

For $q \in \mathbb{N}$, we will denote the cosimplicial object given by precomposing X^\bullet with $(-)\star[q-1]: \Delta \rightarrow \Delta$ informally also by $X^{\bullet+q}$, and use the notation $Y_+^{\bullet+q}$ for the analogous construction for Y_+^\bullet .

Now let $q \geq 1$. Then $(-)\star[q-1]$ can be viewed as a functor $\Delta_+ \rightarrow \Delta$, meaning that $X^{\bullet+q}$ can be extended to a coaugmented cosimplicial object $\Delta_+ \rightarrow \mathcal{D}$. Unpacking the constructions, we see that its coaugmentation map $X^{q-1} \rightarrow X^{m+q}$ at the m -th level is induced by the map $[q-1] \rightarrow [m] \star [q-1] \cong [m+q]$ that maps the q elements of $[q-1]$ to the last q elements of $[m+q]$ in order.

Moreover, for a map $\beta: [k] \rightarrow [l]$ in Δ_+ , the map $\beta \star \text{Id}_{[q-1]}: [k] \star [q-1] \rightarrow [l] \star [q-1]$ maps the top q elements to the top q elements in order. This means that $(-)\star[q-1]: \Delta_+ \rightarrow \Delta$ can be factored as

$$\begin{array}{ccc} \Delta_+ & \xrightarrow{(-)\star[q-1]} & \Delta \\ \text{add disjoint } \top \downarrow & & \uparrow \text{forget distinction of } \top \\ \Delta_\top & \xrightarrow{(-)\star[q-2]} & \Delta_\top \end{array}$$

Therefore, $X^{\bullet+q}: \Delta_+ \rightarrow \mathcal{D}$ can be extended to a (right) split coaugmented cosimplicial object $\Delta_\top \rightarrow \mathcal{D}$.

Similarly, for $q \geq 1$, $[q-1] \star (-): \Delta_+ \rightarrow \Delta$ factors through $[q-2] \star (-): \Delta_\perp \rightarrow \Delta_\perp$ and $X^{[q-1]\star\bullet}$ admits a (left) split coaugmentation by X^{q-1} whose m -th level $X^{q-1} \rightarrow X^{q+m}$ is induced by the map $[q] \rightarrow [q-1] \star [m] \cong [q+m]$ that maps the elements of $[q-1]$ to the first elements of $[q+m]$ in order.

Moreover, note that for every map $\beta: [k] \rightarrow [l]$ in Δ_+ , we have a commutative diagram

$$\begin{array}{ccc} [k] & \xrightarrow{\delta^{k+1}} & [k+1] \\ \beta \downarrow & & \downarrow \beta \star [0] \\ [l] & \xrightarrow{\delta^{l+1}} & [l+1] \end{array}$$

which yields a natural transformation $\delta^{\bullet+1}: \text{Id}_{\Delta_+} \rightarrow (-)\star[0]$, hence natural transformations $d^{\bullet+1}: X^\bullet \rightarrow X^{\bullet+1}$ and $d^{\bullet+1}: Y_+^\bullet \rightarrow Y_+^{\bullet+1}$.

We now generalize [Lur17, Example 4.2.2.4] to a description of cofree comodule algebras.

Lemma 5.1.15. *Let H be a commutative bialgebra in \mathcal{C} . Then, in the cosimplicial description of Fact 5.1.5, the cofree functor $C_H: \text{CAlg}(\mathcal{C}) \rightarrow \text{LComod}_H(\text{CAlg}(\mathcal{C}))$ corresponds to the functor that sends $A \in \text{CAlg}(\mathcal{C})$ to $d_{\Omega_H}^{\bullet+1} \otimes \eta_A: \Omega_H^\bullet \rightarrow \Omega_H^{\bullet+1} \otimes A$.*

Proof. First we show that $d_{\Omega_H}^{\bullet+1} \otimes \eta_A$ satisfies the left coaction condition, so that we obtain a well-defined functor $C': \text{CAlg}(\mathcal{C}) \rightarrow \text{LComod}_H(\text{CAlg}(\mathcal{C}))$. When considered as a functor $[1] \times \Delta \rightarrow \text{CAlg}(\mathcal{C})$, $d_{\Omega_H}^{\bullet+1} \otimes \eta_A$ maps $(0, [n]) \rightarrow (1, [n])$ to the map

$$d^{n+1} \otimes \eta_A \simeq H^{\otimes n} \otimes \eta_H \otimes \eta_A: H^{\otimes n} \rightarrow H^{\otimes n} \otimes H \otimes A$$

and $(1, \beta) := (1, 0 \mapsto n)$ to

$$\Omega_H^{\beta \star [0]} \otimes A \simeq (\eta_H)^{\otimes n} \otimes H \otimes A: H \otimes A \rightarrow H^{\otimes n} \otimes H \otimes A,$$

as β is the composite of “ n δ^0 ’s”. These two maps do exhibit their target as their coproduct.

In order to identify C' with C_H , we show that the former is also a right adjoint of the forgetful functor $V_H: \text{LComod}_H(\text{CAlg}(\mathcal{C})) \rightarrow \text{CAlg}(\mathcal{C})$, which maps $\Omega_H^\bullet \rightarrow \Omega_H^\bullet(B)$ to $\Omega_H^0(B) \simeq B$. We define the counit c to be the transformation whose value at $A \in \text{CAlg}(\mathcal{C})$ is given by $\epsilon_H \otimes A: V_H(C'(A)) \simeq H \otimes A \rightarrow A$.

In order to define the unit, consider a commutative H -comodule algebra $\iota_{H,B}^\bullet: \Omega_H^\bullet \rightarrow \Omega_H^\bullet(B)$. Note that the cosimplicial map

$$\iota_{H,B}^{\bullet+1}: \Omega_H^{\bullet+1} \simeq H^{\otimes(\bullet+1)} \xrightarrow{H^{\otimes(\bullet+1)} \otimes \eta_B} H^{\otimes(\bullet+1)} \otimes B \simeq \Omega_H^{\bullet+1}(B)$$

and the coaugmentation

$$\Omega_H^{0 \mapsto (\bullet+1)}(B): \Omega_H^0(B) \simeq B \xrightarrow{(\eta_H)^{\otimes(\bullet+1)} \otimes B} H^{\otimes(\bullet+1)} \otimes B \simeq \Omega_H^{\bullet+1}(B)$$

exhibit the target as the coproduct $\Omega_H^{\bullet+1} \otimes B$. Hence we have a commutative diagram

$$\begin{array}{ccc} & \iota_{H,B}^\bullet & \rightarrow \Omega_H^\bullet(B) \\ & \searrow & \downarrow d_{\Omega_H(B)}^{\bullet+1} \\ \Omega_H^\bullet & \xrightarrow{d_{\Omega_H}^{\bullet+1}} \Omega_H^{\bullet+1} & \xrightarrow{\iota_{H,B}^{\bullet+1}} \Omega_H^{\bullet+1}(B) \\ & \searrow \Omega_H^{\bullet+1} \otimes \eta_B & \uparrow \simeq \mu_{\Omega_H^{\bullet+1}(B)} \circ \left(\iota_{H,B}^{\bullet+1} \otimes \Omega_H^{0 \mapsto (\bullet+1)}(B) \right) \\ & \searrow d_{\Omega_H}^{\bullet+1} \otimes \eta_B & \rightarrow \Omega_H^{\bullet+1} \otimes B \end{array} \quad (5.1.16)$$

Composing in the vertical direction, this yields a natural map $\iota_{H,B}^\bullet \rightarrow d_{\Omega_H}^{\bullet+1} \otimes \eta_B \simeq C'(V_H(\iota_{H,B}^\bullet))$, which we take to be the value of the unit transformation u at $\iota_{H,B}^\bullet: \Omega_H^\bullet \rightarrow \Omega_H^\bullet(B)$.

Now the triangle identity for V_H holds because the composite

$$V_H(\iota_{H,B}^\bullet) \simeq B \xrightarrow{V_H(u_{\iota_{H,B}^\bullet}) \simeq V_H(d_{\Omega_H(B)}^{\bullet+1}) \simeq d^1 \simeq \rho_B} H \otimes B \xrightarrow{c_{V_H(\iota_{H,B}^\bullet)} \simeq \epsilon_H \otimes B} B \simeq V_H(\iota_{H,B}^\bullet)$$

is the identity and the triangle identity for C' holds because the composite

$$\begin{aligned} C'(A) &\simeq H^{\otimes(\bullet+1)} \otimes A \xrightarrow{u_{C'(A)} \simeq d_{C'(A)}^{\bullet+1} \simeq d_{\Omega_H}^{\bullet+2} \otimes A \simeq H^{\otimes \bullet} \otimes \Delta_H \otimes A} H^{\otimes(\bullet+2)} \otimes A \\ &\xrightarrow{C'(c_A) \simeq H^{\otimes(\bullet+1)} \otimes \epsilon_H \otimes A} H^{\otimes(\bullet+1)} \otimes A \simeq C'(A) \end{aligned}$$

is the identity. □

Construction 5.1.17. Let H be a commutative bialgebra. We now construct a cobar resolution system for H in the sense of Definition 1.3.8 using the description of cofree H -comodule algebras discussed in Lemma 5.1.15 (and its proof), which will be our preferred model of cobar resolutions for commutative comodule algebras.

Informally, this new model of the cobar resolution of a commutative H -comodule algebra $\iota_{H,B}^\bullet: \Omega_H^\bullet \rightarrow \Omega_H^\bullet(B)$ can be depicted as

$$\begin{array}{ccccccc} \Omega_H^\bullet & \xlongequal{\quad} & \Omega_H^\bullet & \xlongequal{\quad} & \Omega_H^\bullet & \xlongequal{\quad} & \Omega_H^\bullet \quad \dots \\ \iota_{H,B}^\bullet \downarrow & & d^{\bullet+1} \iota_{H,B}^\bullet \downarrow & & d^{\bullet+2} d^{\bullet+1} \iota_{H,B}^\bullet \downarrow & & \downarrow d^{\bullet+3} d^{\bullet+2} d^{\bullet+1} \iota_{H,B}^\bullet \\ \Omega_H^\bullet(B) & \xrightarrow{-d^{\bullet+1}} & \Omega_H^{\bullet+1}(B) & \xleftarrow{s^{\bullet+1}} & \Omega_H^{\bullet+2}(B) & \xrightarrow{-d^{\bullet+2}} & \Omega_H^{\bullet+3}(B) \quad \dots \\ & & & \xleftarrow{-d^{\bullet+2}} & & \xleftarrow{s^{\bullet+2}} & \\ & & & & & \xleftarrow{-d^{\bullet+3}} & \end{array} \quad (5.1.18)$$

where we think of the horizontal maps as defining a coaugmented cosimplicial object \mathfrak{R}_+ in $\text{Fun}([1] \times \Delta, \text{CAlg}(\mathcal{C}))$.

Note that we can identify each column $\Omega_H^\bullet \rightarrow \Omega_H^{\bullet+k}(B)$ with $(C_H V_H)^k(B)$ by iterating the equivalence $\Omega_H^{\bullet+1}(B) \simeq \Omega_H^{\bullet+1} \otimes B$ depicted in (5.1.16), so \mathfrak{R}_+ can be viewed as a coaugmented cosimplicial commutative H -comodule algebra. Moreover, unpacking the construction of the unit and counit maps of the adjunction $V_H \dashv C_H$ from the proof of Lemma 5.1.15, one sees that the coface and codegeneracy maps of this coaugmented cosimplicial object are indeed the ones expected from a cobar resolution.

In order to implement this idea, we first consider the functors $F_0, F_1: \Delta_+ \times \Delta \rightarrow [1] \times \Delta$ given by $F_0([k], [n]) = (0, [n])$ and $F_1([k], [n]) = (1, [n] \star [k])$ for all $[k] \in \Delta_+$ and $[n] \in \Delta$. There is a natural transformation $F_0 \rightarrow F_1$ whose component at $([k], [n]) \in \Delta_+ \times \Delta$ is given by $(0 \rightarrow 1, [n] \hookrightarrow [n] \star [k])$, which gives rise to a functor $F: \Delta_+ \times [1] \times \Delta \rightarrow [1] \times \Delta$.

Precomposition with F induces a functor

$$F^*: \text{Fun}([1] \times \Delta, \text{CAlg}(\mathcal{C})) \rightarrow \text{Fun}(\Delta_+ \times [1] \times \Delta, \text{CAlg}(\mathcal{C})),$$

which sends $\iota_{H,B}^\bullet: \Omega_H^\bullet \rightarrow \Omega_H^\bullet(B)$ to a coaugmented cosimplicial object of the form (5.1.18). By the description of the levels of this coaugmented cosimplicial object (i.e., columns of (5.1.18)) discussed above, $F^*(\iota_{H,B}^\bullet)$ factors through the full subcategory of objects satisfying the left coaction condition, so F^* can be restricted to a functor

$$\tilde{R}_H^\bullet(-)_+: \text{LComod}_H(\text{CAlg}(\mathcal{C})) \rightarrow \text{Fun}(\Delta_+, \text{LComod}_H(\text{CAlg}(\mathcal{C}))),$$

whose levels, face and degeneracy maps are those expected from a cobar resolution.

We move on to proving the remaining resolution properties. Consider the lower row of (5.1.18) as a functor $\mathfrak{X}: \Delta_+ \times \Delta \rightarrow \text{CAlg}(\mathcal{C})$, which is given by $([k], [n]) \mapsto \Omega_H^{[n] \star [k]}(B)$. Note that the n -th row of \mathfrak{X} is $\Omega_H^{[n] \star \bullet}(B)$, which can be extended to a (left) split coaugmented object because, as discussed at the end of Construction 5.1.14, the functor $[n] \star (-): \Delta_+ \rightarrow \Delta$ can be extended to a functor $\Delta_\perp \rightarrow \Delta$.

Note that these row-wise splittings are not compatible with each other for varying n because $\star: \Delta_+ \times \Delta \rightarrow \Delta$ cannot be extended “globally” to a functor $\Delta_\perp \times \Delta \rightarrow \Delta$. However, the existence of levelwise splittings does mean that $\tilde{R}_H^\bullet(B)_+$ defined above is

a limiting cone because limits in functor categories are computed pointwise (and the “ Ω_H^\bullet -component”, which is constant, is evidently also split).

Now consider $V_H \circ \tilde{R}_H^\bullet(B)_+$, which is simply the 0-th row of \mathfrak{X} , i.e., $\Omega_H^{[0]\star\bullet}(B)$. As discussed above, it admits a splitting (that is natural in B). Hence we obtain a functor $R_H^\bullet(-)_\perp: \text{LComod}_H(\text{CAlg}(\mathcal{C})) \rightarrow \text{Fun}(\Delta_\perp, \text{CAlg}(\mathcal{C}))$ such that $R_H^\bullet(-)_\perp|_{\Delta_+} \simeq V_H \circ \tilde{R}_H^\bullet(-)_+$.

We conclude this subsection with cosimplicial descriptions of trivial coactions and primitives.

Lemma 5.1.19. *In the cosimplicial description of Fact 5.1.5, an object of the ∞ -category $\text{Fun}([1] \times \Delta, \text{CAlg}(\mathcal{C}))$ is equivalent to one in $\text{LComod}_{\mathbf{1}_\mathcal{C}}(\text{CAlg}(\mathcal{C}))$ if and only if it is equivalent to one of the form $\text{const}_{\eta_A}: \text{const}_{\mathbf{1}_\mathcal{C}} \rightarrow \text{const}_A$ for some $A \in \text{CAlg}(\mathcal{C})$.*

Proof. Note that $\text{const}_{\mathbf{1}_\mathcal{C}}$ is an initial object of $\text{Fun}(\Delta, \text{CAlg}(\mathcal{C}))$ and satisfies the dual Segal condition, so by Lemma 5.1.4, it is an initial object of $\text{Coalg}(\text{CAlg}(\mathcal{C}))$, which is indeed the commutative bialgebra structure on $\mathbf{1}_\mathcal{C}$ we consider.

Now let $A \in \text{CAlg}(\mathcal{C})$. Then, as $\eta_A: \mathbf{1}_\mathcal{C} \rightarrow A$ and $\text{Id}_A: A \rightarrow A$ exhibit A as a coproduct of $\mathbf{1}_\mathcal{C}$ and A , $\text{const}_{\eta_A}: \text{const}_{\mathbf{1}_\mathcal{C}} \rightarrow \text{const}_A$ satisfies the left coaction condition, i.e., lies in $\text{LComod}_{\mathbf{1}_\mathcal{C}}(\text{CAlg}(\mathcal{C}))$. Moreover, the underlying commutative algebra of const_{η_A} , i.e., its evaluation at $(1, [0]) \in [1] \times \Delta$ is A .

Hence $A \mapsto \text{const}_{\eta_A}$ yields a section of the forgetful functor $\text{LComod}_{\mathbf{1}_\mathcal{C}}(\text{CAlg}(\mathcal{C})) \rightarrow \text{CAlg}(\mathcal{C})$ and is thus an equivalence as the forgetful functor is. \square

Corollary 5.1.20. *Let H be a commutative bialgebra in \mathcal{C} . Then, in the cosimplicial description of Fact 5.1.5, the functor $\text{Triv}_H: \text{CAlg}(\mathcal{C}) \rightarrow \text{LComod}_H(\text{CAlg}(\mathcal{C}))$ corresponds to the functor that sends $A \in \text{CAlg}(\mathcal{C})$ to $\Omega_H^\bullet \otimes \eta_A: \Omega_H^\bullet \rightarrow \Omega_H^\bullet \otimes A$.*

Proof. Consider the equivalence $\text{CAlg}(\mathcal{C}) \simeq \text{LComod}_{\mathbf{1}_\mathcal{C}}(\text{CAlg}(\mathcal{C}))$, under which A corresponds to $\text{const}_{\eta_A}: \text{const}_{\mathbf{1}_\mathcal{C}} \rightarrow \text{const}_A$ by Lemma 5.1.19 and thus $\text{Triv}_H A$ corresponds to $(\eta_H)_\star(\text{const}_{\eta_A})$. Now, by Lemma 5.1.12, $(\eta_H)_\star(\text{const}_{\eta_A})$ is given by the pushout of $\text{const}_{\eta_A}: \text{const}_{\mathbf{1}_\mathcal{C}} \rightarrow \text{const}_A$ along $\eta_H^\bullet: \text{const}_{\mathbf{1}_\mathcal{C}} \rightarrow \Omega_H^\bullet$, which is equivalent to $\Omega_H^\bullet \otimes \eta_A: \Omega_H^\bullet \rightarrow \Omega_H^\bullet \otimes A$. \square

Lemma 5.1.21. *Let H be a commutative bialgebra in \mathcal{C} . Assume that $\text{CAlg}(\mathcal{C})$ admits limits of cosimplicial objects. Then, in the cosimplicial description of Fact 5.1.5, the primitives functor $\text{Prim}_H: \text{LComod}_H(\text{CAlg}(\mathcal{C})) \rightarrow \text{CAlg}(\mathcal{C})$ corresponds to the functor sending $\Omega_H^\bullet \rightarrow \Omega_H^\bullet(B)$ to $\lim_\Delta \Omega_H^\bullet(B)$.*

Proof. As Prim_H is a right adjoint of Triv_H , it will suffice to show that the functor sending $\Omega_H^\bullet \rightarrow \Omega_H^\bullet(B)$ to $\lim_\Delta \Omega_H^\bullet(B)$ is also a right adjoint of Triv_H .

Let $A \in \text{CAlg}(\mathcal{C})$ and $B \in \text{LComod}_H(\text{CAlg}(\mathcal{C}))$. Then, by Corollary 5.1.20, $\text{Triv}_H A$ is given by $\Omega_H^\bullet \otimes \eta_A: \Omega_H^\bullet \rightarrow \Omega_H^\bullet \otimes A$. Moreover, identifying $\text{LComod}_H(\text{CAlg}(\mathcal{C}))$ with a full subcategory of $\text{Fun}(\Delta, \text{Coalg}(\mathcal{C}))_{\Omega_H^\bullet/}$ as discussed in Remark 5.1.11, we see that a map

$\varphi: \text{Triv}_H A \rightarrow B$ corresponds to a map $\varphi^\bullet: \Omega_H^\bullet \otimes A \rightarrow \Omega_H^\bullet(B)$ such that $\varphi^\bullet \circ (\Omega_H^\bullet \otimes \eta_A) \simeq \iota_{H,B}^\bullet$. Depicting $\Omega_H^\bullet \otimes A$ as a pushout, every such φ^\bullet yields commutative diagram

$$\begin{array}{ccccc} \text{const}_{1_C} & \xrightarrow{\eta_H^\bullet} & \Omega_H^\bullet & \xlongequal{\quad} & \Omega_H^\bullet \\ \text{const}_{\eta_A} \downarrow & & \downarrow \Omega_H^\bullet \otimes \eta_A & & \downarrow \iota_{H,B}^\bullet \\ \text{const}_A & \xrightarrow{\eta_H^\bullet \otimes A} & \Omega_H^\bullet \otimes A & \xrightarrow{\varphi^\bullet} & \Omega_H^\bullet(B) \end{array}$$

Translating this description to mapping spaces, we obtain a commutative diagram¹⁶

$$\begin{array}{ccccc} \text{Map}_{\text{LComod}_H(\text{CAlg}(\mathcal{C}))}(\text{Triv}_H A, B) & \longrightarrow & [\Omega_H^\bullet \otimes A, \Omega_H^\bullet(B)] & \xrightarrow{(\eta_H^\bullet \otimes A)^*} & [\text{const}_A, \Omega_H^\bullet(B)] \\ \downarrow & \lrcorner & (\Omega_H^\bullet \otimes \eta_A)^* \downarrow & \lrcorner & \downarrow (\text{const}_{\eta_A})^* \\ \{\iota_{H,B}^\bullet\} & \longleftarrow & [\Omega_H^\bullet, \Omega_H^\bullet(B)] & \xrightarrow{(\eta_H^\bullet)^*} & [\text{const}_{1_C}, \Omega_H^\bullet(B)] \end{array}$$

Note that the left square is a pullback square by the aforementioned description of $\text{LComod}_H(\text{CAlg}(\mathcal{C}))$ as the full subcategory of a coslice category. Moreover, the right square is a pullback square by the universal property of $\Omega_H^\bullet \otimes A$ as a pushout. Hence the composite square is also a pullback square. Now, $\text{Map}_{\text{Fun}(\Delta, \text{CAlg}(\mathcal{C}))}(\text{const}_{1_C}, \Omega_H^\bullet(B)) \simeq \{*\}$ by the initiality of const_{1_C} , implying that the composite of the upper horizontal arrows is in fact an equivalence as the lower composite is.

Hence we obtain natural equivalences

$$\begin{aligned} \text{Map}_{\text{LComod}_H(\text{CAlg}(\mathcal{C}))}(\text{Triv}_H A, B) &\simeq \text{Map}_{\text{Fun}(\Delta, \text{CAlg}(\mathcal{C}))}(\text{const}_A, \Omega_H^\bullet(B)) \\ &\simeq \text{Map}_{\text{CAlg}(\mathcal{C})}(A, \lim_{\Delta} \Omega_H^\bullet(B)), \end{aligned}$$

which yields the desired adjunction. \square

Example 5.1.22. Let H be a commutative bialgebra and A a commutative algebra in \mathcal{C} . Then, by Lemma 5.1.15, $\Omega_H^\bullet(C_H(A)) \simeq \Omega_H^{\bullet+1} \otimes A$. As discussed in Construction 5.1.14, this cosimplicial object admits a split coaugmentation by $\Omega_H^0 \otimes A \simeq 1_C \otimes A \simeq A$, implying that $\text{Prim}_H(C_H(A)) \simeq \lim_{\Delta} (\Omega_H^{\bullet+1} \otimes A) \simeq A$ as expected.

Remark 5.1.23. Let B be a commutative algebra in \mathcal{C} on which a commutative bialgebra H coacts. Note that, with respect to the version of the cobar resolution from Construction 5.1.17, the two sided cobar construction $\Omega_H^\bullet(1_C, B)$ can be computed as

$$\Omega_H^\bullet(1_C, B) = \text{Prim}_H \circ \tilde{R}_H^\bullet(B) \stackrel{5.1.21}{\simeq} \lim_{[n] \in \Delta} \tilde{R}_H^\bullet(B)^n \stackrel{5.1.17}{\simeq} \lim_{[n] \in \Delta} \Omega_H^{[n]\star\bullet}(B).$$

Now consider the bicosimplicial object $([k], [n]) \mapsto \Omega_H^{[n]\star[k]}(B)$. Analogous to the coaugmentation in the horizontal direction discussed in Construction 5.1.17, its k -th column $\Omega_H^{?\star[k]}(B)$ is equipped with a (right) split coaugmentation by $\Omega_H^k(B)$ obtained using the extension of $(-)\star[k]: \Delta_+ \rightarrow \Delta$ to a functor $\Delta_\top \rightarrow \Delta$ discussed in Construction 5.1.14. This exhibits each $\Omega_H^k(B)$ as the limit $\lim_{[n] \in \Delta} \Omega_H^{[n]\star[k]}(B)$ and hence yields an equivalence $\Omega_H^\bullet(1_C, B) \simeq \Omega_H^\bullet(B)$.

¹⁶Here we use $[-, -]$ instead of $\text{Map}_{\text{Fun}(\Delta, \text{CAlg}(\mathcal{C}))}(-, -)$ to save space.

5.2. Relative Hopf–Galois contexts

Convention 5.2.1. For the rest of this section, we assume that \mathcal{C} admits geometric realizations of simplicial objects and limits of cosimplicial objects, and that the tensor product preserves geometric realizations in each variable.

Note that, by [Lur17, Corollaries 3.2.2.5 and 3.2.3.2], these assumptions imply that $\mathrm{CAlg}(\mathcal{C})$ also admits limits of cosimplicial objects and geometric realizations of simplicial objects, and that these limits and colimits can be computed in \mathcal{C} .

In this subsection, we introduce a relative variant of Hopf–Galois contexts for commutative algebras. Even though it will not be important for the results of this thesis, the relative perspective is crucial when one compares Hopf–Galois-extensions over different bialgebras (cf. [Kar14, Chapter 4]).

Definition 5.2.2. A *relative Hopf–Galois context of commutative algebras* is a morphism in $\mathrm{LComod}(\mathrm{CAlg}(\mathcal{C}))$.

Note that the datum of a relative Hopf–Galois context $(\gamma^\bullet, \varphi^\bullet): (K, A) \rightarrow (H, B)$ of commutative algebras is equivalent to that of a map $\gamma: K \rightarrow H$ of commutative bialgebras and a map $\varphi: \gamma_\star(A) \rightarrow B$ of commutative H -comodule algebras. In [BH18, Section 3], the point-set analogue of this kind of datum is used to define relative Hopf–Galois extensions, which transfer to our setting below.

We could have defined relative Hopf–Galois contexts also in the non-commutative setting as morphisms in $\mathrm{LComod}(\mathrm{Alg}(\mathcal{C})) \simeq \mathrm{Alg}(\mathrm{LComod}(\mathcal{C}))$. However, we refrained from working in this more general setup because even our most basic constructions for relative Hopf–Galois contexts such as Definition 5.2.9 rely on the cocartesian symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{C})$. We nevertheless expect that most of the constructions of [BH18, Section 3] can be transferred to the non-commutative setting as well.

Remark 5.2.3. Since the unit map $\eta_H: \mathbf{1}_{\mathcal{C}} \rightarrow H$ of a commutative bialgebra H is the unique map of commutative bialgebras it receives from $\mathbf{1}_{\mathcal{C}}$, a relative Hopf–Galois context of commutative algebras of the form $(\mathbf{1}_{\mathcal{C}}, A) \rightarrow (H, B)$ is uniquely determined by a morphism $(\eta_H)_\star(A) \simeq \mathrm{Triv}_H(A) \rightarrow B$ of commutative H -comodule algebras. The underlying H -comodule algebra map of such a map yields an H -Hopf–Galois context in the sense of Definition 3.1.1, but in general this notion is stronger as it requires an “ H -coaction via \mathbb{E}_∞ -maps”.

For example, given a map $f: X \rightarrow \mathrm{Pic}(\mathcal{C})$ of \mathbb{E}_∞ -spaces, $X \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}$ is a *commutative* bialgebra and the Thom context $\mathrm{Triv}_{X \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}}}(\mathbf{1}_{\mathcal{C}}) \rightarrow \mathrm{M}(f)$ of Example 3.1.3 can be upgraded to such a map of *commutative* $(X \odot_{\mathcal{C}} \mathbf{1}_{\mathcal{C}})$ -comodule algebras.

Example 5.2.4. Let $\gamma: K \rightarrow H$ be a map of commutative bialgebras and A a commutative algebra. Then we have a commutative diagram

$$\begin{array}{ccc} \Omega_K^\bullet & \xrightarrow{\gamma^\bullet} & \Omega_H^\bullet \\ d_{\Omega_K^\bullet}^{+1} \otimes \eta_A \downarrow & & \downarrow d_{\Omega_H^\bullet}^{+1} \otimes \eta_A \\ \Omega_K^{\bullet+1} \otimes A & \xrightarrow{\gamma^{\bullet+1} \otimes A} & \Omega_H^{\bullet+1} \otimes A \end{array}$$

Note that, by Lemma 5.1.15, the vertical arrows can be identified with $C_K(A)$ and $C_H(A)$, respectively. Hence this diagram depicts a relative Hopf–Galois context $(K, K \otimes A) \rightarrow (H, H \otimes A)$ of commutative algebras which can be thought of as a relative and commutative version of *normal basis contexts* of Example 3.1.2.

The primitives condition in the relative setting involves primitives of both the source and the target.

Definition 5.2.5. We say that a relative Hopf–Galois context $(\gamma^\bullet, \varphi^\bullet): (K, A) \rightarrow (H, B)$ of commutative algebras satisfies the *primitives condition* if the induced map

$$\lim_{\Delta} \varphi^\bullet: \lim_{\Delta} \Omega_K^\bullet(A) \xrightarrow{5.1.21} \text{Prim}_K(A) \rightarrow \text{Prim}_H(B) \xrightarrow{5.1.21} \lim_{\Delta} \Omega_H^\bullet(B) \quad (5.2.6)$$

is an equivalence.

Remark 5.2.7. Let $(\eta_H^\bullet, \varphi^\bullet): (\mathbf{1}_C, A) \rightarrow (H, B)$ be a relative Hopf–Galois context of commutative algebras. Then the map (5.2.6) agrees with the map $A \rightarrow \text{Prim}_H(B)$ adjoint to the map $\text{Triv}_H(A) \rightarrow B$ induced by $(\eta_H^\bullet, \varphi^\bullet)$. Therefore, $(\eta_H^\bullet, \varphi^\bullet)$ satisfies the primitives condition in the sense of Definition 5.2.5 if and only if the associated H -Hopf–Galois context satisfies the primitives condition in the sense of Definition 3.1.5.

Example 5.2.8. Consider a relative normal basis context $(\gamma^\bullet, \gamma^{\bullet+1} \otimes A): (K, K \otimes A) \rightarrow (H, H \otimes A)$ as discussed in Example 5.2.4. As in the absolute case (cf. Example 3.1.6), such a context always satisfies the primitives condition. In fact, by virtue of the split coaugmentations $\Omega_K^{0 \rightarrow (\bullet+1)}: \mathbf{1}_C \rightarrow \Omega_K^{\bullet+1}$ and $\Omega_H^{0 \rightarrow (\bullet+1)}: \mathbf{1}_C \rightarrow \Omega_H^{\bullet+1}$ of Construction 5.1.14, $\lim_{\Delta}(\gamma^{\bullet+1} \otimes A)$ can be identified with $\gamma^0 \otimes A \simeq \text{Id}_{\mathbf{1}_C} \otimes A: \mathbf{1}_C \otimes A \rightarrow \mathbf{1}_C \otimes A$.

Next, we would like to define a variant of the shear map in the relative setting. Recall that for an H -Hopf–Galois context $\text{Triv}_H(A) \rightarrow \tilde{B}$, we needed the triviality of the H -coaction on A in order to have a well-defined map $\rho_{\tilde{B}} \otimes_A B: B \otimes_A B \rightarrow H \otimes B \otimes_A B$, which is the first factor of the shear map (cf. Footnote 10 on page 51). Therefore, in order to construct a shear map for a relative Hopf–Galois context, we first “trivialize” the coaction on the source.

Definition 5.2.9. For a map $\gamma: K \rightarrow H$ of commutative bialgebras, we define $\text{Cof}(\gamma) := H \otimes_K \mathbf{1}_C$, which is a pushout $H \amalg_K \mathbf{1}_C$ in $\text{CAlg}(\mathcal{C})$. Using Lemma 5.1.4, we equip it with the commutative bialgebra structure witnessed by the pushout $\Omega_H^\bullet \amalg_{\Omega_K^\bullet} \Omega_{\mathbf{1}_C}^\bullet$ (computed in $\text{Fun}(\Delta, \text{CAlg}(\mathcal{C}))$). Let $\pi_\gamma^\bullet: \Omega_H^\bullet \rightarrow \Omega_{\text{Cof}(\gamma)}^\bullet$ denote the quotient map.

Construction 5.2.10. Let $(\gamma^\bullet, \varphi^\bullet): (K, A) \rightarrow (H, B)$ be a relative Hopf–Galois context of commutative algebras. Consider the commutative cube

$$\begin{array}{ccccc} \Omega_K^\bullet & \xrightarrow{\gamma^\bullet} & \Omega_H^\bullet & & \\ \downarrow \epsilon_K^\bullet & \searrow & \downarrow & \searrow \pi_\gamma^\bullet & \\ \Omega_K^\bullet(A) & \xrightarrow{\varphi^\bullet} & \Omega_H^\bullet(B) & & \\ \downarrow \Gamma & \searrow & \downarrow \Gamma & \searrow \Gamma & \\ \Omega_{\mathbf{1}_C}^\bullet(A) & \xrightarrow{\quad} & \Omega_{\text{Cof}(\gamma)}^\bullet(B) & & \end{array}, \quad (5.2.11)$$

where all the faces except for the front and the back are pushout squares, and the left and right squares witness corestrictions of scalars by Lemma 5.1.12. We denote the coaction map $\Omega_{\text{Cof}(\gamma)}^{\delta^1}(B) : \Omega_{\text{Cof}(\gamma)}^0(B) \simeq B \rightarrow \text{Cof}(\gamma) \otimes B \simeq \Omega_{\text{Cof}(\gamma)}^1(B)$ by $\bar{\rho}_B$.

Now $\Omega_{1_c}^\bullet(A)$ is a constant cosimplicial object by Lemma 5.1.19, so the lower front arrow $\text{const}_A \simeq \Omega_{1_c}^\bullet(A) \rightarrow \Omega_{\text{Cof}(\gamma)}^\bullet(B)$ yields a coaugmentation of the cosimplicial object $\Omega_{\text{Cof}(\gamma)}^\bullet$ by A , i.e., a functor $\Omega_{\text{Cof}(\gamma)}^\bullet(B)_+ : \Delta_+ \rightarrow \text{CAlg}(\mathcal{C})$ such that $\Omega_{\text{Cof}(\gamma)}^{-1}(B)_+ \simeq A$ and $\Omega_{\text{Cof}(\gamma)}^\bullet(B)_+|_{\Delta} \simeq \Omega_{\text{Cof}(\gamma)}^\bullet(B)$. Moreover, under these identifications, the map $\Omega_{\text{Cof}(\gamma)}^{\delta^0}(B)_+ : \Omega_{\text{Cof}(\gamma)}^{-1}(B)_+ \rightarrow \Omega_{\text{Cof}(\gamma)}^0(B)_+$ corresponds to the map $A \rightarrow \Omega_{\text{Cof}(\gamma)}^0(B) \simeq B$ induced by the lower front arrow, i.e., to φ .

Definition 5.2.12. Let $(\gamma^\bullet, \varphi^\bullet) : (K, A) \rightarrow (H, B)$ be a relative Hopf–Galois context of commutative algebras. We define the associated *shear map* to be the composite

$$\text{sh}_{(\gamma^\bullet, \varphi^\bullet)} : B \otimes_A B \xrightarrow{\bar{\rho}_B \otimes_A B} \text{Cof}(\gamma) \otimes B \otimes_A B \xrightarrow{\text{Cof}(\gamma) \otimes \mu_B} \text{Cof}(\gamma) \otimes B,$$

which is a morphism of commutative algebras as the coaction map $\bar{\rho}_B : B \rightarrow \text{Cof}(\gamma) \otimes B$ and the multiplication map $\mu_B : B \otimes_A B \rightarrow B$ are.

Remark 5.2.13. Note that for a relative Hopf–Galois context $(\gamma^\bullet, \varphi^\bullet) : (K, A) \rightarrow (H, B)$, the underlying map of the shear map $\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}$ of Definition 5.2.12 agrees with the shear map of Definition 3.1.7 associated to the $\text{Cof}(\gamma)$ -Hopf–Galois context $\text{Triv}_{\text{Cof}(\gamma)}(A) \rightarrow B$ witnessed by the front face in the diagram (5.2.11) of Construction 5.2.10.

Lemma 5.2.14. Let $(\gamma^\bullet, \varphi^\bullet) : (K, A) \rightarrow (H, B)$ be a relative Hopf–Galois context of commutative algebras. Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \varphi \downarrow & & \downarrow \bar{\rho}_B \\ B & \xrightarrow{\eta_{\text{Cof}(\gamma)} \otimes B} & \text{Cof}(\gamma) \otimes B \end{array} \quad (5.2.15)$$

obtained by applying $\Omega_{\text{Cof}(\gamma)}^\bullet(B)_+$ to the identity $\delta^0 \delta^0 = \delta^1 \delta^0 : [-1] \rightarrow [0]$. Then the associated map $B \otimes_A B \rightarrow \text{Cof}(\gamma) \otimes B$ induced by the universal property of $B \otimes_A B$ as a pushout of commutative algebras agrees with the shear map $\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}$ of Definition 5.2.12.

Proof. By the universal property of the pushout, it is enough to check that the restriction of $\text{sh}_{(\gamma^\bullet, \varphi^\bullet)} : B \otimes_A B \rightarrow \text{Cof}(\gamma) \otimes B$ along the first and the second factor coincides with $\bar{\rho}_B$ and $\eta_{\text{Cof}(\gamma)} \otimes B$, respectively. This is indeed the case, as witnessed by the commutativity of the diagram

$$\begin{array}{ccccc} B & & & & \\ & \searrow \bar{\rho}_B \otimes_A \varphi & & \searrow \bar{\rho}_B & \\ B \otimes_A B & \xrightarrow{\bar{\rho}_B \otimes_A B} & \text{Cof}(\gamma) \otimes B \otimes_A B & \xrightarrow{\text{Cof}(\gamma) \otimes \mu_B} & \text{Cof}(\gamma) \otimes B \\ & \nearrow \varphi \otimes_A B & \nearrow \eta_{\text{Cof}(\gamma)} \otimes \varphi \otimes_A B & \nearrow \eta_{\text{Cof}(\gamma)} \otimes B & \\ B & & & & \end{array}$$

□

Definition 5.2.16. We say that a relative Hopf–Galois context $(\gamma^\bullet, \varphi^\bullet): (K, A) \rightarrow (H, B)$ of commutative algebras satisfies the *normal basis condition* if $\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}: B \otimes_A B \rightarrow \text{Cof}(\gamma) \otimes B$ is an equivalence.

Remark 5.2.17. Let $(\eta_H^\bullet, \varphi^\bullet): (\mathbf{1}_C, A) \rightarrow (H, B)$ be a relative Hopf–Galois context of commutative algebras. Then $\text{Cof}(\eta_H) \simeq H \otimes_{\mathbf{1}_C} \mathbf{1}_C \simeq H$ and $\text{sh}_{(\eta_H^\bullet, \varphi^\bullet)}: B \otimes_A B \rightarrow H \otimes B$ agrees with the shear map of the associated H -Hopf–Galois context. Therefore, $(\eta_H^\bullet, \varphi^\bullet)$ satisfies the normal basis condition in the sense of Definition 5.2.16 if and only if the associated H -Hopf–Galois context satisfies the normal basis condition in the sense of Definition 3.1.8.

Example 5.2.18. Consider a relative normal basis context $(\gamma^\bullet, \gamma^{\bullet+1} \otimes A): (K, K \otimes A) \rightarrow (H, H \otimes A)$ as discussed in Example 5.2.4. Then the square (5.2.15) of Lemma 5.2.14 is of the form

$$\begin{array}{ccc} K \otimes A & \xrightarrow{\gamma \otimes A} & H \otimes A \\ \gamma \otimes A \downarrow & & \downarrow \Delta_{H \otimes A} \\ & H \otimes H \otimes A & \\ & \downarrow \pi_\gamma \otimes H \otimes A & \\ H \otimes A & \xrightarrow[\eta_{\text{Cof}(\gamma) \otimes H \otimes A}]{} & \text{Cof}(\gamma) \otimes H \otimes A \end{array},$$

i.e., given by tensoring the corresponding square for the relative Hopf–Galois context $(\gamma^\bullet, \gamma^{\bullet+1}): (K, K) \rightarrow (H, H)$ by A . As $(-)\otimes A$ preserves colimits of commutative algebras, Lemma 5.2.14 implies that $\text{sh}_{(\gamma^\bullet, \gamma^{\bullet+1} \otimes A)}$ can be identified with $\text{sh}_{(\gamma^\bullet, \gamma^{\bullet+1})} \otimes A$.

Therefore, as in Example 3.1.9, $(\gamma^\bullet, \gamma^{\bullet+1} \otimes A)$ satisfies the normal basis condition if $(\gamma^\bullet, \gamma^{\bullet+1})$ does. A bialgebra map $\gamma: K \rightarrow H$ satisfying this condition is called a *relative Hopf algebra*.

Definition 5.2.19. A relative Hopf–Galois context $(\gamma^\bullet, \varphi^\bullet): (K, A) \rightarrow (H, B)$ of commutative algebras which satisfies both the primitives and the normal basis condition (i.e., one for which $\lim_\Delta \varphi^\bullet: \text{Prim}_K(A) \rightarrow \text{Prim}_H(B)$ and $\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}: B \otimes_A B \rightarrow \text{Cof}(\gamma) \otimes B$ are equivalences) is called a *relative Hopf–Galois extension of commutative algebras*.

5.3. Another structured shear map

In this subsection, we discuss an extension of the shear map $B \otimes_A B \rightarrow \text{Cof}(\gamma) \otimes B$ associated to a relative Hopf–Galois context $(\gamma^\bullet, \varphi^\bullet): (K, A) \rightarrow (H, B)$ to a cosimplicial map $\mathbf{C}^\bullet(\varphi) \rightarrow \Omega_{\text{Cof}(\gamma)}^\bullet(B)$ and its consequences. This extension is essentially equivalent to the structured shear map of Subsection 3.2, but its construction is arguably simpler.

Remark 5.3.1. By [Lur17, Corollary 2.4.3.10], the forgetful functor $\text{CAlg}(\text{CAlg}(\mathcal{C})) \rightarrow \text{CAlg}(\mathcal{C})$ is a (strongly symmetric monoidal) equivalence. Hence we can view a morphism ψ in $\text{CAlg}(\mathcal{C})$ as a morphism in $\text{CAlg}(\text{CAlg}(\mathcal{C}))$ and thus its coaugmented Amitsur complex as a functor $\mathbf{C}_+^\bullet(\psi): \Delta_+ \rightarrow \text{CAlg}(\mathcal{C})$. We will use this variant of $\mathbf{C}_+^\bullet(\psi)$ throughout the remainder of this section.

In order to construct such a cosimplicial map, we will interpret the Amitsur complex $\mathbf{C}^\bullet(\psi)$ of a map $\psi: R \rightarrow S$ of commutative algebras as a “dual Čech nerve”. Indeed, writing $\mathrm{Spec} T$ for the object of $\mathrm{CAlg}(\mathcal{C})^{\mathrm{op}}$ corresponding to $T \in \mathrm{CAlg}(\mathcal{C})$, $\mathbf{C}^1(\psi) \simeq S \otimes_R S$ corresponds to $\mathrm{Spec} S \times_{\mathrm{Spec} R} \mathrm{Spec} S$, $\mathbf{C}^2(\psi) \simeq S \otimes_R S \otimes_R S$ to a “triple self-intersection” etc. In fact, when studying descent for commutative algebras, one usually *defines* the Amitsur complex as such a dual Čech nerve (cf. [Lur11, Proposition 5.7] and [GL21, text between Propositions 6.4 and 6.5]), but we proceed by relating our existing notions to the ones used in the aforementioned references.

The idea of a dual Čech nerve can be formalized as follows (cf. [Lur09, discussion after Proposition 6.1.2.11]).

Lemma 5.3.2. *Let $\psi: R \rightarrow S$ be a map of commutative algebras in \mathcal{C} . Then the coaugmented Amitsur complex $\mathbf{C}_+(\psi): \Delta_+ \rightarrow \mathrm{CAlg}(\mathcal{C})$ is the left Kan extension of its restriction to $\Delta_+^{\leq 0} \cong [1]$ (which is given by $\psi: R \rightarrow S$).*

Proof. We verify the conditions of [Lur09, Proposition 6.1.2.11]. We need to show that $d^0, d^1: \mathbf{C}^0(\psi) \rightarrow \mathbf{C}^1(\psi)$ exhibit $\mathbf{C}^1(\psi)$ as the pushout of $d^0: \mathbf{C}^{-1}(\psi) \rightarrow \mathbf{C}^0(\psi)$ along itself, which is indeed the case because

$$\begin{array}{ccc} R & \xrightarrow{\psi} & S \\ \psi \downarrow & & \downarrow S \otimes_R \psi \\ S & \xrightarrow{\psi \otimes_R \psi} & S \otimes_R S \end{array}$$

is a pushout square in $\mathrm{CAlg}(\mathcal{C})$. Moreover, we need to show that for all $m, n \in \mathbb{N}$, $\mathbf{C}^{[m] \hookrightarrow [m] \star [n]}(\psi)$ and $\mathbf{C}^{[n] \hookrightarrow [m] \star [n]}(\psi)$ exhibit $\mathbf{C}^{[n] \star [m]}(\psi)$ as a pushout of $\mathbf{C}^{0 \rightarrow n+1}(\psi)$ along $\mathbf{C}^{0 \rightarrow m+1}(\psi)$, which follows from the fact that

$$\begin{array}{ccc} S & \xrightarrow{\psi \otimes_R^n \otimes_R S} & S^{\otimes_R(n+1)} \\ S \otimes_R \psi^{\otimes_R m} \downarrow & & \downarrow S^{\otimes_R(n+1)} \otimes_R \psi^{\otimes_R m} \\ S^{\otimes_R(m+1)} & \xrightarrow[\psi \otimes_R^n \otimes_R S^{\otimes_R(m+1)}]{} & S^{\otimes_R(n+m+1)} \end{array}$$

is a pushout square. □

Next, we would like to describe the comodule category $\mathrm{LComod}_{\psi_! \psi^*}(\mathrm{LMod}_S(\mathcal{C}))$ of the descent comonad in terms of the cosimplicial commutative algebra $\mathbf{C}^\bullet(\psi)$, for which some preparation will be needed.

Definition 5.3.3 ([Lur09, Definition 7.3.1.2]). Consider a commutative square

$$\begin{array}{ccc} \mathcal{D}' & \xrightarrow{F'} & \mathcal{E}' \\ U_1 \downarrow & & \downarrow U_2 \\ \mathcal{D} & \xrightarrow{F} & \mathcal{E} \end{array} \tag{5.3.4}$$

of ∞ -categories, i.e., an equivalence $FU_1 \simeq U_2F'$ of functors.

If F admits a right adjoint R (with unit transformation u) and F' admits a right adjoint R' (with counit transformation c'), we obtain a *mate transformation*

$$U_1R' \xrightarrow{uU_1R'} RFU_1R' \simeq RU_2F'R' \xrightarrow{RU_2c'} RU_2.$$

We call the square (5.3.4) *right adjointable* if F and F' admits right adjoints and the associated mate transformation is an equivalence.

Dually, if F admits a left adjoint L and F' admits a left adjoint L' , we obtain a mate transformation $L'U_2 \rightarrow U_1L$, and call the square *left adjointable* if it is an equivalence.

Lemma 5.3.5.¹⁷ Let $\psi: R \rightarrow S$ and $\nu: R \rightarrow T$ be maps of commutative algebras in \mathcal{C} . Then the diagram

$$\begin{array}{ccc} \mathrm{LMod}_R(\mathcal{C}) & \xrightarrow{\psi_!} & \mathrm{LMod}_S(\mathcal{C}) \\ \nu_! \downarrow & & \downarrow (S \otimes_R \nu)_! \\ \mathrm{LMod}_T(\mathcal{C}) & \xrightarrow{(\psi \otimes_R T)_!} & \mathrm{LMod}_{S \otimes_R T}(\mathcal{C}) \end{array}$$

is right adjointable.

Hence every diagram of ∞ -categories that is obtained by applying $\mathrm{LMod}_{(-)}(\mathcal{C})$ (with functoriality given by extension of scalars) to a pushout diagram of commutative algebras is right adjointable.

Proof. The functors $\psi_!$ and $(\psi \otimes T)_!$ admit right adjoints ψ^* and $(\psi \otimes T)^*$, respectively. The component of the associated mate transformation at $M \in \mathrm{LMod}_S(\mathcal{C})$ is given by

$$T \otimes_R M \xrightarrow{\psi \otimes_R T \otimes_R M} S \otimes_R T \otimes_R M \simeq T \otimes_R S \otimes_R M \xrightarrow{T \otimes_R \alpha_M} T \otimes_R M,$$

which is indeed an equivalence. \square

Proposition 5.3.6. Let $\psi: R \rightarrow S$ be a map of commutative algebras in \mathcal{C} . Then there exists an equivalence $\mathrm{LComod}_{\psi_! \psi^*}(\mathrm{LMod}_S(\mathcal{C})) \simeq \lim_{\Delta} \mathrm{LMod}_{\mathbf{C}^\bullet(\psi)}(\mathcal{C})$ that is compatible with the forgetful functor $V_{\psi_! \psi^*}: \mathrm{LComod}_{\psi_! \psi^*}(\mathrm{LMod}_S(\mathcal{C})) \rightarrow \mathrm{LMod}_S(\mathcal{C})$ and the projection $\lim_{\Delta} \mathrm{LMod}_{\mathbf{C}^\bullet(\psi)}(\mathcal{C}) \rightarrow \mathrm{LMod}_{\mathbf{C}^0(\psi)}(\mathcal{C}) \simeq \mathrm{LMod}_S(\mathcal{C})$.

Proof. First we note that for every $\beta: [m] \rightarrow [n]$ in Δ_+ , the diagram

$$\begin{array}{ccc} \mathrm{LMod}_{S \otimes_R^{(m+1)}}(\mathcal{C}) & \xrightarrow{(S \otimes_R^{(m+1)} \otimes_R \psi)_!} & \mathrm{LMod}_{S \otimes_R^{(m+2)}}(\mathcal{C}) \\ \mathbf{C}^\beta(\psi)_! \downarrow & & \downarrow \mathbf{C}^{\beta \star [0]}(\psi)_! \simeq (\mathbf{C}^\beta(\psi) \otimes_R S)_! \\ \mathrm{LMod}_{S \otimes_R^{(n+1)}}(\mathcal{C}) & \xrightarrow{(S \otimes_R^{(n+1)} \otimes_R \psi)_!} & \mathrm{LMod}_{S \otimes_R^{(n+2)}}(\mathcal{C}) \end{array} \quad (5.3.7)$$

¹⁷It is likely that this lemma is a special case of the intended form of [Lur11, Lemma 6.15], but in the cited lemma some arrows are reversed, and the proposition cited in its proof cannot be found anymore, so we sketch a proof here.

is right adjointable, which follows from Lemma 5.3.5 as the corresponding diagram before applying $\text{LMod}_{(-)}(\mathcal{C})$ is a pushout diagram of commutative algebras.

This means in particular that [Lur17, Theorem 4.7.5.2] is applicable to the composite¹⁸

$$\Delta \xrightarrow{(-)^{\text{op}}} \Delta \xrightarrow{\mathbf{C}^\bullet(\psi)} \text{CAlg}(\mathcal{C}) \xrightarrow{\text{LMod}_{(-)}(\mathcal{C})} \text{Cat}_\infty,$$

which yields a right adjoint $C: \text{LMod}_S(\mathcal{C}) \rightarrow \lim_{\Delta} \text{LMod}_{\mathbf{C}^\bullet(\psi)}(\mathcal{C})$ to the projection $V: \lim_{\Delta} \text{LMod}_{\mathbf{C}^\bullet(\psi)}(\mathcal{C}) \rightarrow \text{LMod}_{\mathbf{C}^0(\psi)}(\mathcal{C}) \simeq \text{LMod}_S(\mathcal{C})$ such that the adjunction $V \dashv C$ is comonadic and $V \circ C \simeq (S \otimes_R \psi)^*(\psi \otimes_R S)_!$. Note that $(S \otimes_R \psi)^*(\psi \otimes_R S)_! \simeq \psi_! \psi^*$ by Lemma 5.3.5 applied to the pushout of ψ along itself. This defines a comonad structure on $\psi_! \psi^*$, which will turn out to be equivalent to that of Definition 1.3.14. When we want to notationally distinguish these two comonad structures in this proof, we will refer to the “new” comonad as Θ .

Now the coaugmentation of $\mathbf{C}^\bullet(\psi)$ by R induces a coaugmentation of $\text{LMod}_{\mathbf{C}^\bullet(\psi)}(\mathcal{C})$ by $\text{LMod}_R(\mathcal{C})$, hence a lift $\widehat{\psi}_!: \text{LMod}_R(\mathcal{C}) \rightarrow \lim_{\Delta} \text{LMod}_{\mathbf{C}^\bullet(\psi)}(\mathcal{C}) \simeq \text{LComod}_{\Theta}(\text{LMod}_S(\mathcal{C}))$ of $\psi_!$ against $V \simeq V_{\Theta}$. By Fact 1.3.6, this lift yields a coaction of Θ on $\psi_!$, which, by the initiality of the coaction of $\psi_! \psi^*$ on $\psi_!$ discussed in Example 1.3.5, induces a comonad map $\psi_! \psi^* \rightarrow \Theta$. Writing u for the unit of the adjunction $V \dashv C$ and c for the counit of the adjunction $\psi_! \dashv \psi^*$, the underlying map of this comonad map can be unpacked as

$$\psi_! \psi^* \simeq V \widehat{\psi}_! \psi^* \xrightarrow{V u \widehat{\psi}_! \psi^*} V C V \widehat{\psi}_! \psi^* \simeq V C \psi_! \psi^* \xrightarrow{V C c} V C.$$

Note that this map can be obtained by applying V to the right adjointability mate transformation of the commutative square

$$\begin{array}{ccc} \text{LMod}_R(\mathcal{C}) & \xrightarrow{\psi_!} & \text{LMod}_S(\mathcal{C}) \\ \widehat{\psi}_! \downarrow & & \parallel \\ \text{LComod}_{\Theta}(\text{LMod}_S(\mathcal{C})) & \xrightarrow{V} & \text{LMod}_S(\mathcal{C}) \end{array}, \quad (5.3.8)$$

so it will suffice show that this square is right adjointable. In order to do so, we will make use of the ∞ -category $\text{Fun}^{\text{RAd}}([1], \text{Cat}_\infty)$ of [Lur17, Definition 4.7.4.16], whose objects are arrows in Cat_∞ that admit right adjoints (which we display horizontally) and whose morphisms correspond to right adjointable squares.

Consider the functor $\Delta_+ \rightarrow \text{Fun}([1], \text{Cat}_\infty)$ corresponding to the natural transformation $d^{\bullet+1}: \text{LMod}_{\mathbf{C}_+^\bullet(\psi)}(\mathcal{C}) \rightarrow \text{LMod}_{\mathbf{C}_+^{\bullet+1}(\psi)}(\mathcal{C})$ (cf. Construction 5.1.14). We would like to lift it to a functor $\mathcal{A}^\bullet: \Delta_+ \rightarrow \text{Fun}^{\text{RAd}}([1], \text{Cat}_\infty)$, which amounts to showing that the naturality squares for $d^{\bullet+1}$ are right adjointable. These squares are all of the form (5.3.7), so they are indeed right adjointable.

¹⁸Precomposing a cosimplicial object X^\bullet with $(-)^{\text{op}}$ in particular swaps $d^0: X^n \rightarrow X^{n+1}$, which is used in the aforementioned theorem, and $d^{n+1}: X^n \rightarrow X^{n+1}$, which we consider in our application. This proposition could be proven without this “twist”, but the twist will be crucial for cosimplicial objects of the form $\Omega_H^\bullet(B)$ (cf. Proposition 5.4.24).

Now, by [Lur17, Corollary 4.7.4.18], limits in $\text{Fun}^{\text{RAd}}([1], \text{Cat}_\infty)$ can be computed in $\text{Fun}([1], \text{Cat}_\infty)$. Moreover, limits in $\text{Fun}([1], \text{Cat}_\infty)$ can be computed pointwise. Hence, since the coaugmentation of $\text{LMod}_{\mathbf{C}^{\bullet+1}(\psi)}(\mathcal{C})$ by $\text{LMod}_{\mathbf{C}^0(\psi)}(\mathcal{C}) \simeq \text{LMod}_S(\mathcal{C})$ is split, the limit of $\mathcal{A}^\bullet|_\Delta$ is given by $V: \text{LComod}_\Theta(\text{LMod}_S(\mathcal{C})) \simeq \lim_\Delta \text{LMod}_{\mathbf{C}^\bullet(\psi)}(\mathcal{C}) \rightarrow \text{LMod}_S(\mathcal{C})$. Moreover, the underlying map of the induced map $\mathcal{A}^{-1} \rightarrow \lim_\Delta (\mathcal{A}^\bullet|_\Delta)$ in $\text{Fun}([1], \text{Cat}_\infty)$ corresponds to the square (5.3.8), which shows that it is right adjointable as desired. \square

The dual Čech nerve interpretation of the coaugmented Amitsur complex allows us to extend the shear map to a cosimplicial map as follows.

Construction 5.3.9. Let $(\gamma^\bullet, \varphi^\bullet): (K, A) \rightarrow (H, B)$ be a relative Hopf–Galois context of commutative algebras. Recall that by Lemma 5.3.2, $\mathbf{C}_+^\bullet(\varphi)$ is the left Kan extension of $\varphi: [1] \rightarrow \text{CAlg}(\mathcal{C})$ along the inclusion $[1] \cong \Delta_+^{\leq 0} \hookrightarrow \Delta_+$, meaning that we have equivalences

$$\begin{aligned} \text{Map}_{\text{Fun}(\Delta_+, \text{CAlg}(\mathcal{C}))}(\mathbf{C}_+^\bullet(\varphi), \Omega_{\text{Cof}(\gamma)}^\bullet(B)_+) &\simeq \text{Map}_{\text{Fun}([1], \text{CAlg}(\mathcal{C}))}(\varphi, \Omega_{\text{Cof}(\gamma)}^{\delta^0: [-1] \rightarrow [0]}(B)_+) \\ &\simeq \text{Map}_{\text{Fun}([1], \text{CAlg}(\mathcal{C}))}(\varphi, \varphi). \end{aligned}$$

We let $\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}^\bullet: \mathbf{C}_+^\bullet(\varphi) \rightarrow \Omega_{\text{Cof}(\gamma)}^\bullet(B)_+$ be the map that corresponds to Id_φ under this identification.

Note that we have equivalences

$$\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}^1 \circ (B \otimes_A \varphi) \simeq \text{sh}_{(\gamma^\bullet, \varphi^\bullet)}^1 \circ \mathbf{C}^{\delta^1}(\varphi) \simeq \Omega_{\text{Cof}(\gamma)}^{\delta^1}(B) \circ \text{sh}_{(\gamma^\bullet, \varphi^\bullet)}^0 \simeq \bar{\rho}_B \circ \text{Id}_B \simeq \bar{\rho}_B$$

and similarly

$$\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}^1 \circ (\varphi \otimes_A B) \simeq \text{sh}_{(\gamma^\bullet, \varphi^\bullet)}^1 \circ \mathbf{C}^{\delta^0}(\varphi) \simeq \Omega_{\text{Cof}(\gamma)}^{\delta^0}(B) \circ \text{sh}_{(\gamma^\bullet, \varphi^\bullet)}^0 \simeq \eta_{\text{Cof}(\gamma)} \otimes B.$$

Hence, by Lemma 5.2.14, $\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}$ can be recovered as $\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}^1$.

Lemma 5.3.10. *Let $(\gamma^\bullet, \varphi^\bullet): (K, A) \rightarrow (H, B)$ be a relative Hopf–Galois context of commutative algebras. Then $\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}^\bullet: \mathbf{C}_+^\bullet(\varphi) \rightarrow \Omega_{\text{Cof}(\gamma)}^\bullet(B)_+$ is an equivalence if and only if its first level $\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}: B \otimes_A B \rightarrow \text{Cof}(\gamma) \otimes B$ is an equivalence.*

Proof. The “only if” direction is clear, so we show that $\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}^\bullet$ is an equivalence if $\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}$ is an equivalence. Since the restriction of $\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}^\bullet$ to $\Delta_+^{\leq 0} \cong [1]$ is Id_φ , it will suffice to show that $\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}^n$ is an equivalence for all $n > 0$, which we do by induction.

The case $n = 1$ is precisely the assumption that $\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}$ is an equivalence. Now let $n > 1$. By the universal property of $\mathbf{C}_+^\bullet(\varphi)$, $\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}^n$ can be described as the dashed arrow in the commutative diagram

$$\begin{array}{ccccc} B & \xrightarrow{B \otimes_A \varphi^{\otimes A(n-1)}} & B^{\otimes A n} & \xrightarrow{\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}^{n-1}} & \text{Cof}(\gamma)^{\otimes(n-1)} \otimes B \\ \downarrow \varphi \otimes_A B & \searrow & \downarrow \bar{\rho}_B^{(n-1)} & & \downarrow \eta_{\text{Cof}(\gamma)} \otimes \text{Cof}(\gamma)^{\otimes(n-1)} \otimes B \\ B & \xrightarrow{\bar{\rho}_B^{(n-1)}} & B & \xrightarrow{\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}^n} & \text{Cof}(\gamma) \otimes B \\ \downarrow \eta_{\text{Cof}(\gamma)} \otimes B & \searrow & \downarrow \Gamma & & \downarrow \eta_{\text{Cof}(\gamma)} \otimes \text{Cof}(\gamma)^{\otimes(n-1)} \otimes B \\ B \otimes_A B & \xrightarrow{\eta_{\text{Cof}(\gamma)} \otimes B} & B^{\otimes A(n+1)} & \xrightarrow{\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}^n} & \text{Cof}(\gamma)^{\otimes n} \otimes B \\ \downarrow \text{sh}_{(\gamma^\bullet, \varphi^\bullet)} & \searrow & \downarrow & & \downarrow \\ \text{Cof}(\gamma) \otimes B & \xrightarrow{\text{Cof}(\gamma) \otimes \bar{\rho}_B^{(n-1)}} & & & \end{array}$$

induced by the universal property of the pushout. Note that the front face is a pushout square too. Now $\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}$ and $\text{sh}_{(\gamma^{\bullet-1}, \varphi^\bullet)}$ are equivalences by the induction hypothesis, so the map $\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}$ they induce on the pushouts is also an equivalence. \square

Corollary 5.3.11. *Let $(\gamma^\bullet, \varphi^\bullet): (K, A) \rightarrow (H, B)$ be a relative Hopf–Galois context of commutative algebras whose shear map $\text{sh}_{(\gamma^\bullet, \varphi^\bullet)}: B \otimes_A B \rightarrow \text{Cof}(\gamma) \otimes B$ is an equivalence. Then $\text{LComod}_{\varphi! \varphi^*}(\text{LMod}_S(\mathcal{C})) \simeq \lim_{\Delta} \text{LMod}_{\Omega_{\text{Cof}(\gamma)}^\bullet(B)}(\mathcal{C})$.*

Proof. By Lemma 5.3.10, we have a cosimplicial equivalence $\mathbf{C}^\bullet(\varphi) \simeq \Omega_{\text{Cof}(\gamma)}^\bullet(B)$. Hence

$$\text{LComod}_{\varphi! \varphi^*}(\text{LMod}_B(\mathcal{C})) \stackrel{5.3.6}{\simeq} \lim_{\Delta} \text{LMod}_{\mathbf{C}^\bullet(\varphi)}(\mathcal{C}) \simeq \lim_{\Delta} \text{LMod}_{\Omega_{\text{Cof}(\gamma)}^\bullet(B)}(\mathcal{C}).$$

\square

Example 5.3.12. Let H be a commutative bialgebra and A a commutative algebra. Consider the normal basis context $(\eta_H^\bullet, \eta_H^{\bullet+1} \otimes A): (\mathbf{1}_{\mathcal{C}}, A) \rightarrow (H, H \otimes A)$. Then the associated coaugmentation $\eta_H^{\otimes(\bullet+1)} \otimes A: A \rightarrow \Omega_H^{\bullet+1} \otimes A \simeq \Omega_H^\bullet(H \otimes A)$ of Construction 5.2.10 is split. Hence the induced coaugmentation $\text{LMod}_A(\mathcal{C}) \rightarrow \text{LMod}_{\Omega_H^\bullet(H \otimes A)}(\mathcal{C})$ is also split, which means that it induces an equivalence $F: \text{LMod}_A(\mathcal{C}) \simeq \lim_{\Delta} \text{LMod}_{\Omega_H^\bullet(H \otimes A)}(\mathcal{C})$.

This equivalence can be used to obtain an incarnation of descent for Hopf algebras (Corollary 4.2.12) as follows. If H is a Hopf algebra, then $\text{sh}_{(\eta_H^\bullet, \eta_H^{\bullet+1} \otimes A)} \stackrel{5.2.18}{\simeq} \text{sh}_{(\eta_H^\bullet, \eta_H^{\bullet+1})} \otimes A$ is an equivalence. Moreover, the functor F can be factored as

$$\text{LMod}_A(\mathcal{C}) \xrightarrow{(\eta_H \otimes A)_!} \text{LComod}_{(\eta_H \otimes A)_!, (\eta_H \otimes A)^*}(\text{LMod}_B(\mathcal{C})) \stackrel{5.3.11}{\simeq} \lim_{\Delta} \text{LMod}_{\Omega_H^\bullet(H \otimes A)}(\mathcal{C}),$$

implying that $(\eta_H \otimes A)_!$ is also an equivalence, i.e., that $\eta_H \otimes A: A \rightarrow H \otimes A$ admits descent.

5.4. Semilinear coactions of a commutative Hopf algebra

Let $\tilde{\varphi}: \text{Triv}_H(A) \rightarrow \tilde{B}$ be an H -Hopf–Galois context of commutative algebras. We have realized the associated shear map as a morphism $\varphi! \varphi^* \simeq B \otimes_A (-) \rightarrow H \otimes (-) \simeq \Theta_{\tilde{B}}$ of comonads in Corollary 3.2.8 and seen in Proposition 5.3.6 that the comodule category of the source is equivalent to $\lim_{\Delta} \text{LMod}_{\mathbf{C}^\bullet(\varphi)}(\mathcal{C})$.

In light of the extension $\mathbf{C}^\bullet(\varphi) \rightarrow \Omega_H^\bullet(B)$ of the shear map to a cosimplicial map constructed in Construction 5.3.9, it is natural to ask whether the comodule category of $\Theta_{\tilde{B}}$, which depends only on the H -comodule algebra \tilde{B} and not on the Hopf–Galois context, is equivalent to $\lim_{\Delta} \text{LMod}_{\Omega_H^\bullet(B)}(\mathcal{C})$ in general. We will see in Proposition 5.4.24 that this is indeed the case when H is a Hopf algebra.

We start with some generalities about commutative Hopf algebras.

Proposition 5.4.1. *If H is a commutative Hopf algebra, then H admits a homotopy antipode. Moreover, this homotopy antipode can be lifted to a map of commutative algebras.*

Proof. Note that for commutative algebra maps $f, g: H \rightarrow H$, their convolution product $f * g = \mu_H(f \otimes g)\Delta_H$ is also a map of commutative algebras as μ_H , $f \otimes g$ and Δ_H are. Moreover, $\eta_H \epsilon_H$ and Id_H are also maps of commutative algebras. Hence the convolution product monoid structure of Fact 4.1.1 can be restricted to a monoid structure on $\text{Hom}_{\text{ho}(\text{CAlg}(\mathcal{C}))}(H, H)$, and it is enough to show that Id_H has an inverse in this monoid. We show that this monoid is in fact a group.

Since $H \otimes H$ is a coproduct in $\text{CAlg}(\mathcal{C})$, we have

$$\text{Hom}_{\text{ho}(\text{CAlg}(\mathcal{C}))}(H \otimes H, H) \simeq \text{Hom}_{\text{ho}(\text{CAlg}(\mathcal{C}))}(H, H) \times \text{Hom}_{\text{ho}(\text{CAlg}(\mathcal{C}))}(H, H).$$

Under this identification, the map induced by precomposition with $\text{sh}_{(\eta_H^\bullet, \eta_H^{\bullet+1})}$ acts as

$$\begin{aligned} (f, g) &\mapsto \mu_H \circ (f \otimes g) \\ &\mapsto \mu_H \circ (f \otimes g) \circ \text{sh}_{(\eta_H^\bullet, \eta_H^{\bullet+1})} \\ &\mapsto \left(\mu_H \circ (f \otimes g) \circ \text{sh}_{(\eta_H^\bullet, \eta_H^{\bullet+1})} \circ (H \otimes \eta_H), \mu_H \circ (f \otimes g) \circ \text{sh}_{(\eta_H^\bullet, \eta_H^{\bullet+1})} \circ (\eta_H \otimes H) \right). \end{aligned}$$

By Lemma 5.2.14, the first component can be simplified to $\mu_H \circ (f \otimes g) \circ \Delta_H = f * g$ and the second component to $\mu_H \circ (f \otimes g) \circ (\eta_H \otimes H) = \mu_H \circ (\eta_H \otimes g) = g$.

Hence, since $\text{sh}_{(\eta_H^\bullet, \eta_H^{\bullet+1})}$ is an equivalence, the map $(f, g) \mapsto (f * g, g)$ is a bijection. Considering a preimage of $(\eta_H \epsilon_H, g)$ under this map, we see that every commutative algebra map $g: H \rightarrow H$ has a left inverse with respect to convolution, so $\text{Hom}_{\text{ho}(\text{CAlg}(\mathcal{C}))}(H, H)$ is a group as desired. \square

Remark 5.4.2. Dualizing the proof of Proposition 5.4.1, we see that also every *cocommutative* Hopf algebra admits a homotopy antipode.

We have seen in Lemma 4.2.7 that the dual shear map associated to a Hopf algebra with a homotopy antipode is an equivalence, so we obtain the following.

Corollary 5.4.3. *Let H be a commutative Hopf algebra and M an H -comodule. Then the dual shear map*

$$\widetilde{\text{sh}}_H(M): H \otimes M \xrightarrow{H \otimes \rho_M} H \otimes H \otimes M \xrightarrow{\mu_H \otimes M} H \otimes M$$

of Construction 4.2.6 is an equivalence.

Next, we give an alternative description of the dual shear map that will be useful later.

Lemma 5.4.4. *Let H be a commutative bialgebra coacting on a commutative algebra B . Then the map $H \otimes B \rightarrow H \otimes B$ of commutative algebras induced by the cospan*

$$B \xrightarrow{\rho_B} H \otimes B \xleftarrow{H \otimes \eta_B} H \quad (5.4.5)$$

by considering the source as the coproduct of H and B in $\text{CAlg}(\mathcal{C})$ agrees with the dual shear map $\widetilde{\text{sh}}_H(B): H \otimes B \rightarrow H \otimes B$.

In particular, if H is a Hopf algebra, (5.4.5) is a coproduct cospan because in that case, the dual shear map is an equivalence by Corollary 5.4.3.

Proof. As in the proof of Lemma 5.2.14, it is enough to check that restrictions of $\widetilde{\text{sh}}_H(B)$ along the structure maps of the coproduct recover the maps in (5.4.5), which is witnessed by the commutativity of the diagram

$$\begin{array}{ccccc}
H & & & & \\
\downarrow H \otimes \eta_B & \searrow H \otimes \eta_H \otimes \eta_B & & \searrow H \otimes \eta_B & \\
H \otimes B & \xrightarrow{H \otimes \rho_B} & H \otimes H \otimes B & \xrightarrow{\mu_H \otimes B} & H \otimes B \\
\uparrow \eta_H \otimes B & & \uparrow \eta_H \otimes H \otimes B & & \\
B & \xrightarrow{\rho_B} & H \otimes B & &
\end{array}$$

□

So far, we have been rather imprecise about the distinction between commutative comodule algebras and their underlying commutative algebras, but we fix the following convention to match the notation of Section 3.

Notation 5.4.6. Given a commutative algebra B on which a commutative bialgebra H acts, we use the notation \tilde{B} to refer to the corresponding object in $\text{CAlg}(\text{LComod}_H(\mathcal{C}))$.

Before delving into the details, let us describe our overall strategy for proving the equivalence $\text{LComod}_{\Theta_{\tilde{B}}}(\text{LMod}_B(\mathcal{C})) \simeq \lim_{\Delta} \text{LMod}_{\Omega_H^\bullet(B)}(\mathcal{C})$. Instead of $\text{LComod}_{\Theta_{\tilde{B}}}(\text{LMod}_B(\mathcal{C}))$, we work with the equivalent ∞ -category $\text{LMod}_{\tilde{B}}(\text{LComod}_H(\mathcal{C}))$ (cf. Definition 3.2.4). The idea is to construct a coaugmentation of the cosimplicial ∞ -category $\text{LMod}_{\Omega_H^\bullet(B)}(\mathcal{C})$ by $\text{LMod}_{\tilde{B}}(\text{LComod}_H(\mathcal{C}))$ whose n -th level is given by the composite

$$\begin{aligned}
\text{LMod}_{\tilde{B}}(\text{LComod}_H(\mathcal{C})) & \xrightarrow{((C_H \vee_H)^n (\rho_{\tilde{B}} \circ \dots \circ \rho_{\tilde{B}}))_! \simeq (\rho_{\tilde{B}}^{(n+1)})_!} \text{LMod}_{(C_H \vee_H)^{n+1} \tilde{B}}(\text{LComod}_H(\mathcal{C})) \\
& \xrightarrow{\text{Prim}_H^{(C_H \vee_H)^{n+1} \tilde{B}}} \text{LMod}_{H^{\otimes n} \otimes B}(\mathcal{C})
\end{aligned} \tag{5.4.7}$$

and show that the induced map to the limit is an equivalence.

While this resembles composing the coaugmentation of $\text{LMod}_{\tilde{R}_H^\bullet(B)}(\text{LComod}_H(\mathcal{C}))$ with a cosimplicial map $\text{LMod}_{\tilde{R}_H^\bullet(B)}(\text{LComod}_H(\mathcal{C})) \rightarrow \text{LMod}_{\Omega_H^\bullet(B)}(\mathcal{C})$, the functors induced by Prim_H cannot necessarily be assembled into such a cosimplicial map because extensions of scalars are not necessarily compatible with taking primitives. However, the extensions of scalars in question will turn out to be compatible with taking primitives after pre-composing with the coaugmentation of $\text{LMod}_{\tilde{R}_H^\bullet(B)}(\text{LComod}_H(\mathcal{C}))$. In order to express this phenomenon, we work with the associated cocartesian fibrations over Δ instead of cosimplicial ∞ -categories.

Construction 5.4.8. Let H be a commutative bialgebra coacting on a commutative algebra B . Consider the following functors.

- The projection $\text{LMod}_{\tilde{B}}(\text{LComod}_H(\mathcal{C})) \times \Delta \rightarrow \Delta$. It is a bicartesian fibration and the straightening of the underlying cocartesian fibration corresponds to the constant cosimplicial category on $\text{LMod}_{\tilde{B}}(\text{LComod}_H(\mathcal{C}))$.

- $\tilde{\mathbf{L}}_{\tilde{B}} \rightarrow \Delta$ given by the pullback of the forgetful functor $\tilde{\theta}: \mathbf{LMod}(\mathbf{LComod}_H(\mathcal{C})) \rightarrow \mathbf{Alg}(\mathbf{LComod}_H(\mathcal{C}))$ along the composite

$$\tilde{A}: \Delta \xrightarrow{\tilde{R}_H^\bullet(B)} \mathbf{LComod}_H(\mathbf{CAlg}(\mathcal{C})) \simeq \mathbf{CAlg}(\mathbf{LComod}_H(\mathcal{C})) \rightarrow \mathbf{Alg}(\mathbf{LComod}_H(\mathcal{C})).$$

As the pullback of a bicartesian fibration (cf. Remark 1.2.12), it is a bicartesian fibration. The straightening of its underlying cocartesian fibration is the cosimplicial ∞ -category $\mathbf{LMod}_{\tilde{R}_H^\bullet(B)}(\mathbf{LComod}_H(\mathcal{C}))$.

- $\mathbf{L}_{\tilde{B}} \rightarrow \Delta$ given by the pullback of the forgetful functor $\theta: \mathbf{LMod}(\mathcal{C}) \rightarrow \mathbf{Alg}(\mathcal{C})$ along the composite

$$A: \Delta \xrightarrow{\Omega_H^\bullet(B)} \mathbf{CAlg}(\mathcal{C}) \rightarrow \mathbf{Alg}(\mathcal{C})$$

As the pullback of a bicartesian fibration (cf. Fact 1.1.10), it is a bicartesian fibration. The straightening of its underlying cocartesian fibration is the cosimplicial ∞ -category $\mathbf{LMod}_{\Omega_H^\bullet(B)}(\mathcal{C})$.

Note that the lax monoidal functor $\mathbf{Prim}_H: \mathbf{LComod}_H(\mathcal{C}) \rightarrow \mathcal{C}$ gives rise to a commutative diagram

$$\begin{array}{ccc} & \mathbf{Alg}(\mathbf{LComod}_H(\mathcal{C})) & \xleftarrow{\tilde{\theta}} \mathbf{LMod}(\mathbf{LComod}_H(\mathcal{C})) \\ \tilde{A} \nearrow & \downarrow \mathbf{Alg}(\mathbf{Prim}_H) & \downarrow \mathbf{LMod}(\mathbf{Prim}_H) \\ \Delta & \xrightarrow{5.1.23} & \\ A \searrow & \mathbf{Alg}(\mathcal{C}) & \xleftarrow{\theta} \mathbf{LMod}(\mathcal{C}) \end{array}$$

Taking pullbacks, we obtain a functor $P_{\tilde{B}}: \tilde{\mathbf{L}}_{\tilde{B}} \rightarrow \mathbf{L}_{\tilde{B}}$ over Δ . As discussed in Remark 1.1.11, $\mathbf{LMod}(\mathbf{Prim}_H)$ maps $\tilde{\theta}$ -cartesian arrows to θ -cartesian arrows, so the pullback $P_{\tilde{B}}$ also preserves cartesian arrows. In other words, taking primitives is compatible with restrictions of scalars.

Moreover, unstraightening the natural transformation

$$\mathrm{const}_{\mathbf{LMod}_{\tilde{B}}(\mathbf{LComod}_H(\mathcal{C}))} \rightarrow \mathbf{LMod}_{\tilde{R}_H^\bullet(B)}(\mathbf{LComod}_H(\mathcal{C}))$$

induced by the coaugmentation of the cobar resolution $\tilde{R}_H^\bullet(B)$ by \tilde{B} , we obtain a functor $E_{\tilde{B}}: \mathbf{LMod}_{\tilde{B}}(\mathbf{LComod}_H(\mathcal{C})) \times \Delta \rightarrow \tilde{\mathbf{L}}_{\tilde{B}}$ over Δ that preserves cocartesian arrows.

Remark 5.4.9. In the context of Construction 5.4.8, the fiber of the composite $P_{\tilde{B}} \circ E_{\tilde{B}}: \mathbf{LMod}_{\tilde{B}}(\mathbf{LComod}_H(\mathcal{C})) \times \Delta \rightarrow \mathbf{L}_{\tilde{B}}$ at $[n] \in \Delta$ coincides with the desired coaugmentation map $\mathbf{LMod}_{\tilde{B}}(\mathbf{LComod}_H(\mathcal{C})) \rightarrow \mathbf{LMod}_{H^{\otimes n} \otimes B}(\mathcal{C})$ of (5.4.7). Therefore, we can obtain the desired coaugmentation of $\mathbf{LMod}_{\Omega_H^\bullet(B)}(\mathcal{C})$ by showing that $P_{\tilde{B}} \circ E_{\tilde{B}}$ preserves cocartesian arrows and considering the natural transformation $P_{\tilde{B}} \circ E_{\tilde{B}}$ induces between straightenings. As $E_{\tilde{B}}$ preserves cocartesian arrows, it will be enough to show that $P_{\tilde{B}}$ preserves cocartesian arrows that are in the image of $E_{\tilde{B}}$.

Now let a be a cocartesian lift of a map $\beta: [m] \rightarrow [n]$ in $\tilde{\mathbf{L}}_{\tilde{B}}$, exhibiting an object of $\mathrm{LMod}_{\tilde{R}_H^n(B)}(\mathrm{LComod}_H(\mathcal{C}))$ as the image of an object $M' \in \mathrm{LMod}_{\tilde{R}_H^m(B)}(\mathrm{LComod}_H(\mathcal{C}))$ under $\tilde{R}_H^\beta(B)_!$. As discussed in [Lur09, Remark 7.3.1.3], since $P_{\tilde{B}}$ preserves cartesian arrows, we have a simple criterion for determining whether $P_{\tilde{B}}(a)$ is a cocartesian lift of β . Namely, this is the case if and only if the component

$$\Omega_H^\beta(B)_! \mathrm{Prim}_H^{\tilde{R}_H^m(B)} M' \rightarrow \mathrm{Prim}_H^{\tilde{R}_H^n(B)} \tilde{R}_H^\beta(B)_! M' \quad (5.4.10)$$

of the mate transformation associated to the commutative square

$$\begin{array}{ccc} \mathrm{LMod}_{\tilde{R}_H^n(B)}(\mathrm{LComod}_H(\mathcal{C})) & \xrightarrow{\tilde{R}_H^\beta(B)^*} & \mathrm{LMod}_{\tilde{R}_H^m(B)}(\mathrm{LComod}_H(\mathcal{C})) \\ \mathrm{Prim}_H^{\tilde{R}_H^n(B)} \downarrow & & \downarrow \mathrm{Prim}_H^{\tilde{R}_H^m(B)} \\ \mathrm{LMod}_{\Omega_H^n(B)}(\mathcal{C}) & \xrightarrow{\Omega_H^\beta(B)^*} & \mathrm{LMod}_{\Omega_H^m(B)}(\mathcal{C}) \end{array}$$

is an equivalence.

Specializing to the case where a is in the image of $E_{\tilde{B}}$, it will be enough to show that (5.4.10) is an equivalence when M' is of the form $(\rho_{\tilde{B}}^{(m+1)})_! M$ for some $M \in \mathrm{LMod}_{\tilde{B}}(\mathrm{LComod}_H(\mathcal{C}))$.

In other words, we would like to show that for every $\beta: [m] \rightarrow [n]$, the natural transformation

$$\Omega_H^\beta(B)_! \mathrm{Prim}_H^{\tilde{R}_H^m(B)} (\rho_{\tilde{B}}^{(m+1)})_! \rightarrow \mathrm{Prim}_H^{\tilde{R}_H^n(B)} \tilde{R}_H^\beta(B)_! (\rho_{\tilde{B}}^{(m+1)})_!$$

obtained by precomposing (5.4.10) with $(\rho_{\tilde{B}}^{(m+1)})_!$, which we call the *critical mate* for $\tilde{R}_H^\beta(B)$, is an equivalence. Moreover, by the horizontal pasting law for mate transformations, it will be enough to consider the critical mates for the generating maps of Δ , i.e., coface and codegeneracy maps of $\tilde{R}_H^\bullet(B)$.

We start with a lemma that will allow us to identify objects of the form $(\rho_{\tilde{B}}^{(m+1)})_! M$. For $\tilde{B} = \mathrm{Triv}_H(\mathbf{1}_{\mathcal{C}})$, it specializes to the equivalence $\tilde{H} \otimes (-) \simeq C_H V_H$ given by the dual shear map of Construction 4.2.6.

Lemma 5.4.11. *Let B be a commutative algebra on which a commutative Hopf algebra H coacts. Then the functor $(\rho_{\tilde{B}})_!: \mathrm{LMod}_{\tilde{B}}(\mathrm{LComod}_H(\mathcal{C})) \rightarrow \mathrm{LMod}_{C_H(B)}(\mathrm{LComod}_H(\mathcal{C}))$ is equivalent to the composite*

$$\mathrm{LMod}_{\tilde{B}}(\mathrm{LComod}_H(\mathcal{C})) \xrightarrow{V_H^{\tilde{B}}} \mathrm{LMod}_B(\mathcal{C}) \xrightarrow{C_H^B} \mathrm{LMod}_{C_H(B)}(\mathrm{LComod}_H(\mathcal{C})).$$

Proof. Recall that by Proposition 3.2.2 (or rather Proposition B.0.6 from which it follows), $V_H^{\tilde{B}}$ admits a right adjoint given by $\rho_{\tilde{B}}^* \circ C_H^B$, which is a lift of the right adjoint C_H of V_H . Let $u: \mathrm{Id}_{\mathrm{LMod}_{\tilde{B}}(\mathrm{LComod}_H(\mathcal{C}))} \rightarrow (\rho_{\tilde{B}}^* \circ C_H^B) \circ V_H^{\tilde{B}}$ denote the associated unit transformation,

whose underlying map is simply the coaction map of the underlying H -comodule. We claim that for all $M \in \text{LMod}_{\tilde{B}}(\text{LComod}_H(\mathcal{C}))$, the adjoint

$$(\rho_{\tilde{B}})_! M \xrightarrow{(\rho_{\tilde{B}})_! u_M} (\rho_{\tilde{B}})_! (\rho_{\tilde{B}})^* C_H^B V_H^{\tilde{B}} M \xrightarrow{\alpha_{C_H^B(M)}} C_H^B V_H^{\tilde{B}} M \quad (5.4.12)$$

of u_M is an equivalence.

It is enough to show that the underlying map in \mathcal{C} is an equivalence. For a map $\psi: R \rightarrow S$ of algebras, we will write $S_\psi \otimes_R (-)$ instead of just $S \otimes_R (-)$ in order to distinguish tensor products with respect to different morphisms. Now, using the description of the “summands” of the dual shear map $\widetilde{\text{sh}}_H(B): H \otimes B \rightarrow H \otimes B$ from Lemma 5.4.4, we see that it induces equivalences that fit into a commutative diagram

$$\begin{array}{ccc} H \otimes M & \xrightarrow{H \otimes \rho_M} & H \otimes H \otimes M \\ \wr & & \wr \\ (H \otimes B)_{\eta_H \otimes B} \otimes_B M & \xrightarrow{(H \otimes B)_{\eta_H \otimes B} \otimes_B \rho_M} & (H \otimes B)_{\eta_H \otimes B} \otimes_B \rho_B^*(H \otimes M) \\ \widetilde{\text{sh}}_H(B) \otimes_B M \downarrow \simeq & & \widetilde{\text{sh}}_H(B) \otimes_B \rho_B^*(H \otimes M) \downarrow \simeq \\ (H \otimes B)_{\rho_B} \otimes_B M & \xrightarrow{(H \otimes B)_{\rho_B} \otimes_B \rho_M} & (H \otimes B)_{\rho_B} \otimes_B \rho_B^*(H \otimes M) \xrightarrow{\alpha_{H \otimes M}} H \otimes M \end{array},$$

where the right triangle commutes because

$$\begin{aligned} & \alpha_{H \otimes M}(\widetilde{\text{sh}}_H(B) \otimes H \otimes M)(H \otimes \eta_B \otimes H \otimes M) \\ & \simeq \alpha_{H \otimes M}(\mu_H \otimes B \otimes H \otimes M)(H \otimes \rho_B \otimes H \otimes M)(H \otimes \eta_B \otimes H \otimes M) \\ & \simeq \alpha_{H \otimes M}(\mu_H \otimes B \otimes H \otimes M)(H \otimes \eta_H \otimes \eta_B \otimes H \otimes M) \\ & \simeq \alpha_{H \otimes M}(H \otimes \eta_B \otimes H \otimes M) \simeq \mu_H \otimes M. \end{aligned}$$

Note that the composite of the upper and the right curved arrow is the dual shear map $\widetilde{\text{sh}}_H(M)$ of M , which is an equivalence by Corollary 5.4.3. Therefore, the composite of the lower arrows, hence (5.4.12) is also an equivalence. \square

Remark 5.4.13. Let B be a commutative algebra on which a commutative Hopf algebra H coacts and $n \in \mathbb{N}$. Recall that the coaugmentation map $\rho_{\tilde{B}}^{(n+1)}: \tilde{B} \rightarrow (C_H V_H)^{n+1} \tilde{B}$ is the composite $\rho_{(C_H V_H)^n \tilde{B}} \circ \dots \circ \rho_{C_H V_H \tilde{B}} \circ \rho_{\tilde{B}}$. Hence, Lemma 5.4.11 implies that $(\rho_{\tilde{B}}^{(n+1)})_!$ factors through the functor

$$C_H^{V_H(C_H V_H)^n \tilde{B}}: \text{LMod}_{V_H(C_H V_H)^n \tilde{B}}(\mathcal{C}) \rightarrow \text{LMod}_{(C_H V_H)^{n+1} \tilde{B}}(\text{LComod}_H(\mathcal{C})).$$

Recall that all coface and codegeneracy maps that occur in $\tilde{R}_H^\bullet(B)$ except for “ d^0 ’s” are of the form $C_H(\psi)$ for a map ψ of algebras in \mathcal{C} (cf. Definition 1.3.8), so the following lemma about such maps will imply that their critical mates are equivalences.

Lemma 5.4.14. *Let H be a bialgebra, $\psi: R \rightarrow S$ a map of commutative algebras and $M \in \text{LMod}_R(\mathcal{C})$. We identify $\text{Prim}_H(C_H(\psi))$ with ψ . Then the component*

$$\psi_! \text{Prim}_H^{C_H(R)} C_H^R M \rightarrow \text{Prim}_H^{C_H(S)} C_H(\psi)_! C_H^R M$$

of the mate transformation induced by the equivalence $\psi^* \text{Prim}_H^{C_H(S)} \simeq \text{Prim}_H^{C_H(R)} C_H(\psi)^*$ is an equivalence.

Proof. Recall that like Prim_H , C_H is also lax monoidal and thus compatible with restrictions of scalars. Hence we have a commutative diagram

$$\begin{array}{ccc}
 \text{LMod}_S(\mathcal{C}) & \xrightarrow{\psi^*} & \text{LMod}_R(\mathcal{C}) \\
 \downarrow C_H^S & & \downarrow C_H^R \\
 \text{LMod}_{C_H(S)}(\text{LComod}_H(\mathcal{C})) & \xrightarrow{C_H(\psi)^*} & \text{LMod}_{C_H(R)}(\text{LComod}_H(\mathcal{C})) \\
 \downarrow \text{Prim}_H^{C_H(S)} & & \downarrow \text{Prim}_H^{C_H(R)} \\
 \text{LMod}_S(\mathcal{C}) & \xrightarrow{\psi^*} & \text{LMod}_R(\mathcal{C})
 \end{array}$$

By the vertical pasting law for mates, the mate transformation for the composite square is given by the composite

$$\psi_! \text{Prim}_H^{C_H(R)} C_H^R M \xrightarrow{\nu_{C_H^R(M)}} \text{Prim}_H^{C_H(S)} C_H(\psi)_! C_H^R M \xrightarrow{\text{Prim}_H^{C_H(S)} \xi_M} \text{Prim}_H^{C_H(S)} C_H^S \psi_! M, \quad (5.4.15)$$

where ν is the mate transformation associated to the lower square and ξ is the mate transformation associated to the upper square.

In order to show that $\nu_{C_H^R(M)}$ is an equivalence, it is enough to show that ξ_M and the composite (5.4.15) are equivalences, i.e., that the upper square and the composite square are left adjointable. The composite square is left adjointable because it can be identified with a “degenerate square” whose mate transformation is an equivalence by the triangle identity for the left adjoint $\psi_!$ of adjunction $\psi_! \dashv \psi^*$.

As for the upper square, it is enough to show that $V_H^{C_H(S)} \xi_M$ is an equivalence because V_H is conservative. Now the adjunction $C_H(\psi)_! \dashv C_H(\psi)^*$ is a lift of the adjunction $(H \otimes \psi)_! \dashv (H \otimes \psi)^*$ along $V_H^{C_H(R)}$ and $V_H^{C_H(S)}$, so under the identification $V_H^{C_H(-)} C_H^{(-)} \simeq (\eta_H \otimes (-))_!$, $V_H^{C_H(S)} \xi_M$ corresponds to the mate transformation associated to the commutative square

$$\begin{array}{ccc}
 \text{LMod}_S(\mathcal{C}) & \xrightarrow{\psi^*} & \text{LMod}_R(\mathcal{C}) \\
 (\eta_H \otimes S)_! \downarrow & & \downarrow (\eta_H \otimes R)_! \\
 \text{LMod}_{H \otimes S}(\mathcal{C}) & \xrightarrow{(H \otimes \psi)^*} & \text{LMod}_{H \otimes R}(\mathcal{C})
 \end{array}$$

whose commutativity is witnessed by the mate transformation associated to the equivalence $(H \otimes \psi)_!(\eta_H \otimes R)_! \simeq (\eta_H \otimes S)_! \psi_!$ (cf. Lemma 5.3.5). Hence, when we consider the mate associated to this square, the unit and counit transformations “cancel each other” by triangle identities and we recover the original equivalence $(H \otimes \psi)_!(\eta_H \otimes R)_! \simeq (\eta_H \otimes S)_! \psi_!$, implying that $V_H^{C_H(S)} \xi_M$ is an equivalence. \square

As alluded to before, Remark 5.4.13 and Lemma 5.4.14 together yield the following.

Corollary 5.4.16. *Let B be a commutative algebra on which a commutative Hopf algebra H coacts and $\beta: [n] \rightarrow [m]$ a map in Δ of the form $\sigma^i: [m+1] \rightarrow [m]$ for $0 \leq i \leq m$ or $\delta^i: [n] \rightarrow [n+1]$ for $0 < i \leq n+1$. Then the critical mate for $\tilde{R}_H^\beta(B)$ is an equivalence.*

The remaining coface maps in $\tilde{R}_H^\bullet(B)$ are of the form $\rho_{(C_H \vee_H)^k \tilde{B}}$, which we take care of now.

Lemma 5.4.17. *Let B be a commutative algebra on which a commutative Hopf algebra H coacts and $M \in \text{LMod}_B(\mathcal{C})$. Consider the map $\rho_{C_H \vee_H \tilde{B}}: C_H \vee_H \tilde{B} \rightarrow (C_H \vee_H)^2 \tilde{B}$ of commutative H -comodule algebras, whose image under Prim_H we identify with the map $\eta_H \otimes B: B \rightarrow H \otimes B$. Then the component*

$$(\eta_H \otimes B)_! \text{Prim}_H^{C_H \vee_H \tilde{B}} C_H^B M \rightarrow \text{Prim}_H^{(C_H \vee_H)^2 \tilde{B}} (\rho_{C_H \vee_H \tilde{B}})_! C_H^B M \quad (5.4.18)$$

of the mate transformation associated to the equivalence $(\eta_H \otimes B)^* \text{Prim}_H^{(C_H \vee_H)^2 \tilde{B}} \simeq \text{Prim}_H^{C_H \vee_H \tilde{B}} (\rho_{C_H \vee_H \tilde{B}})^*$ is an equivalence.

Proof. The map (5.4.18) is the composite

$$\begin{aligned} (\eta_H \otimes B)_! \text{Prim}_H^{C_H \vee_H \tilde{B}} C_H^B M &\xrightarrow{v} (\eta_H \otimes B)_! \text{Prim}_H^{C_H \vee_H \tilde{B}} (\rho_{C_H \vee_H \tilde{B}})^* (\rho_{C_H \vee_H \tilde{B}})_! C_H^B M \\ &\simeq (\eta_H \otimes B)_! (\eta_H \otimes B)^* \text{Prim}_H^{(C_H \vee_H)^2 \tilde{B}} (\rho_{C_H \vee_H \tilde{B}})_! C_H^B M \\ &\xrightarrow{\omega} \text{Prim}_H^{(C_H \vee_H)^2 \tilde{B}} (\rho_{C_H \vee_H \tilde{B}})_! C_H^B M, \end{aligned}$$

where v induced by the unit of the adjunction $(\rho_{C_H \vee_H \tilde{B}})_! \dashv (\rho_{C_H \vee_H \tilde{B}})^*$ and ω is induced by the counit of the adjunction $(\eta_H \otimes B)_! \dashv (\eta_H \otimes B)^*$. We identify the underlying maps of v and ω as follows.

First, consider the equivalence $\hat{u}_{C_H^B M}: (\rho_{C_H \vee_H \tilde{B}})_! C_H^B M \simeq C_H^{V_H C_H B} V_H^{C_H B} C_H^B M$ of Lemma 5.4.11. By its construction, precomposing $(\rho_{C_H \vee_H \tilde{B}})^* \hat{u}_{C_H^B M}$ with the unit map $C_H^B M \rightarrow (\rho_{C_H \vee_H \tilde{B}})^* (\rho_{C_H \vee_H \tilde{B}})_! C_H^B M$ of the adjunction $(\rho_{C_H \vee_H \tilde{B}})_! \dashv (\rho_{C_H \vee_H \tilde{B}})^*$ recovers the unit map $u_{C_H^B M}: C_H^B M \rightarrow (\rho_{C_H \vee_H \tilde{B}})^* C_H^{V_H C_H B} V_H^{C_H B} C_H^B M$ of the adjunction $V_H^{C_H B} \dashv (\rho_{C_H \vee_H \tilde{B}})^* C_H^{V_H C_H B}$, whose underlying comodule map is the “coaction map” $\rho_{C_H^B M}$. This means that the composite

$$\begin{aligned} v': (\eta_H \otimes B)_! \text{Prim}_H^{C_H \vee_H \tilde{B}} C_H^B M &\xrightarrow{v} (\eta_H \otimes B)_! \text{Prim}_H^{C_H \vee_H \tilde{B}} (\rho_{C_H \vee_H \tilde{B}})^* (\rho_{C_H \vee_H \tilde{B}})_! C_H^B M \\ &\simeq (\eta_H \otimes B)_! \text{Prim}_H^{C_H \vee_H \tilde{B}} (\rho_{C_H \vee_H \tilde{B}})^* C_H^{V_H C_H B} V_H^{C_H B} C_H^B M \end{aligned}$$

can be identified with $(\eta_H \otimes B)_! \text{Prim}_H^{C_H \vee_H \tilde{B}} u_{C_H^B M}$. Hence, since $\text{Prim}_H(\rho_{C_H^B M}) \simeq \eta_H \otimes M$, the underlying map of v' is given by

$$(H \otimes B) \otimes_B M \xrightarrow{(H \otimes B) \otimes_B (\eta_H \otimes M)} (H \otimes B) \otimes_B (H \otimes M). \quad (5.4.19)$$

Now, under the identification

$$\mathrm{Prim}_H^{(C_H \vee_H)^{2\tilde{B}}} (\rho_{C_H \vee_H \tilde{B}})_! C_H^B M \simeq \mathrm{Prim}_H^{(C_H \vee_H)^{2\tilde{B}}} C_H^{V_H C_H B} V_H^{C_H B} C_H^B M \simeq H \otimes M,$$

ω corresponds to the action map

$$(H \otimes B) \otimes_B (H \otimes M) \rightarrow H \otimes M,$$

whose composite with (5.4.19) is an equivalence. Hence the mate transformation (5.4.18) is an equivalence because its underlying map can be identified with this composite. \square

Remark 5.4.13 and Lemma 5.4.17 together yield the following.

Corollary 5.4.20. *Let B be a commutative algebra on which a commutative Hopf algebra H coacts and consider the coface map $\delta^0: [n] \rightarrow [n+1]$ for some $n \in \mathbb{N}$. Then the critical mate for $\tilde{R}_H^{\delta^0}(B)$ is an equivalence.*

Notation 5.4.21. Let B be a commutative algebra on which a commutative Hopf algebra H coacts. Corollaries 5.4.16 and 5.4.20 imply that the cosimplicial ∞ -category $\mathrm{LMod}_{\Omega_H^\bullet(B)}(\mathcal{C})$ admits a coaugmentation by $\mathrm{LMod}_{\tilde{B}}(\mathrm{LComod}_H(\mathcal{C}))$ as discussed in Remark 5.4.9. We denote the associated functor $\Delta_+ \rightarrow \mathrm{Cat}_\infty$ by $\mathcal{L}_{\tilde{B}}$.

Remark 5.4.22. Let B be a commutative algebra on which a commutative Hopf algebra H coacts and $n \in \mathbb{N}$. Then the coaugmentation functor $\mathrm{LMod}_{\tilde{B}}(\mathrm{LComod}_H(\mathcal{C})) \rightarrow \mathrm{LMod}_{\Omega_H^n(B)}(\mathcal{C})$ encoded in $\mathcal{L}_{\tilde{B}}$ can be simplified as

$$\begin{aligned} & \mathrm{Prim}_H^{(C_H \vee_H)^{n+1}\tilde{B}} (\rho_{\tilde{B}}^{(n+1)})_! \\ & \simeq \mathrm{Prim}_H^{(C_H \vee_H)^{(n+1)\tilde{B}}} \circ (\rho_{(C_H \vee_H)^n \tilde{B}})_! \circ \dots \circ (\rho_{\tilde{B}})_! \\ & \stackrel{5.4.11}{\simeq} \mathrm{Prim}_H^{(C_H \vee_H)^{n+1}\tilde{B}} \circ (C_H^{V_H(C_H \vee_H)^n \tilde{B}} \circ V_H^{(C_H \vee_H)^n \tilde{B}}) \circ \dots \circ (C_H^{V_H \tilde{B}} \circ V_H^{\tilde{B}}) \\ & \simeq (V_H^{(C_H \vee_H)^n \tilde{B}} \circ C_H^{V_H(C_H \vee_H)^{n-1} \tilde{B}}) \circ \dots \circ (V_H^{(C_H \vee_H) \tilde{B}} \circ C_H^{V_H \tilde{B}}) \circ V_H^{\tilde{B}} \\ & \simeq (\eta_H \otimes H^{\otimes(n-1)} \otimes B)_! \circ \dots \circ (\eta_H \otimes B)_! \circ V_H^{\tilde{B}} \simeq ((\eta_H)^{\otimes n} \otimes B)_! \circ V_H^{\tilde{B}}. \end{aligned}$$

We would like to show that $\mathcal{L}_{\tilde{B}}$ is a limiting cone, for which we will need one last lemma.

Lemma 5.4.23. *Let H be a commutative Hopf algebra coacting on a commutative algebra B . Then the diagram*

$$\begin{array}{ccc} B & \xrightarrow{\rho_B} & H \otimes B \\ \rho_B \downarrow & & \downarrow \Delta_{H \otimes B} \\ H \otimes B & \xrightarrow{H \otimes \rho_B} & H \otimes H \otimes B \end{array}$$

is a pushout square.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
& & \eta_{H \otimes B} & & \\
& \nearrow & & \searrow & \\
1_{\mathcal{C}} & \xrightarrow{\eta_B} & B & \xrightarrow{\rho_B} & H \otimes B \\
\eta_H \downarrow & & \downarrow \rho_B & & \downarrow \Delta_{H \otimes B} \\
H & \xrightarrow{H \otimes \eta_B} & H \otimes B & \xrightarrow{H \otimes \rho_B} & H \otimes H \otimes B \\
& \searrow & & \nearrow & \\
& & H \otimes \eta_{H \otimes B} & &
\end{array}$$

Note that the left square is a pushout square by Lemma 5.4.4. Moreover, the cospan formed by the lower composite and the right map is of the form (5.4.5) for the coaction of H on the cofree comodule algebra $H \otimes B$. Hence, again by Lemma 5.4.4, the composite square is also a pushout square. Therefore the right square is a pushout square. \square

Proposition 5.4.24. *Let H be a commutative Hopf algebra coacting on a commutative algebra B . Then the functor $\mathrm{LMod}_{\tilde{B}}(\mathrm{LComod}_H(\mathcal{C})) \rightarrow \lim_{\Delta} \mathrm{LMod}_{\Omega_{\bullet}(B)}(\mathcal{C})$ induced by $\mathcal{L}_{\tilde{B}}$ is an equivalence.*

Proof. We apply [Lur17, Corollary 4.7.5.3] to the composite $\Delta_+ \xrightarrow{(-)^{\mathrm{op}}} \Delta_+ \xrightarrow{\mathcal{L}_{\tilde{B}}} \mathrm{Cat}_{\infty}$.¹⁹

First, we need to show that $\mathcal{L}_{\tilde{B}}^{\delta^0}: \mathcal{L}_{\tilde{B}}^{-1} \rightarrow \mathcal{L}_{\tilde{B}}^0$ is conservative, $\mathcal{L}_{\tilde{B}}^{-1}$ admits limits of $\mathcal{L}_{\tilde{B}}^{\delta^0}$ -split cosimplicial objects and that $\mathcal{L}_{\tilde{B}}^{\delta^0}$ preserves such limits. Now $\mathcal{L}_{\tilde{B}}^{\delta^0}$ is given by the composite

$$\mathrm{LMod}_{\tilde{B}}(\mathrm{LComod}_H(\mathcal{C})) \xrightarrow{(\rho_{\tilde{B}})_!} \mathrm{LMod}_{(C_H \vee_H) \tilde{B}}(\mathrm{LComod}_H(\mathcal{C})) \xrightarrow{\mathrm{Prim}_H^{(C_H \vee_H) \tilde{B}}} \mathrm{LMod}_B(\mathcal{C}),$$

which is, by Lemma 5.4.11, equivalent to $\mathrm{Prim}_H^{(C_H \vee_H) \tilde{B}} C_H^B \vee_H^{\tilde{B}} \simeq \vee_H^{\tilde{B}}$. The functor $\vee_H^{\tilde{B}}$ is the left adjoint in a comonadic adjunction (cf. Proposition 3.2.2), so it has all the desired properties.

The other condition we need to check is the right adjointability of squares of the form

$$\begin{array}{ccc}
\mathcal{L}_{\tilde{B}}^m & \xrightarrow{\mathcal{L}_{\tilde{B}}^{\delta^{m+1}}} & \mathcal{L}_{\tilde{B}}^{m+1} \\
\mathcal{L}_{\tilde{B}}^{\beta} \downarrow & & \downarrow \mathcal{L}_{\tilde{B}}^{\beta \star [0]} \\
\mathcal{L}_{\tilde{B}}^n & \xrightarrow{\mathcal{L}_{\tilde{B}}^{\delta^{n+1}}} & \mathcal{L}_{\tilde{B}}^{n+1}
\end{array} \tag{5.4.25}$$

for all $\beta: [m] \rightarrow [n]$ in Δ_+ . By the vertical pasting law for mate transformations, it is enough consider the case where β is one of the generating morphisms of Δ_+ .

¹⁹As discussed in Footnote 18 on page 86, $(-)^{\mathrm{op}}$ swaps the roles of $d^0, d^{n+1}: \Omega_H^n(B) \rightarrow \Omega_H^{n+1}(B)$, which makes a substantial difference here because the former “inserts a unit” whereas the latter “applies ρ_B ”.

For $\beta = \delta^0: [-1] \rightarrow [0]$, we decompose the mate transformation associated to the square (5.4.25) as²⁰

$$\begin{array}{ccccc}
\text{LMod}_B & & & & \\
\downarrow C_H^B & & & & \\
\text{LMod}_{C_H \vee_H \tilde{B}} & \xrightarrow{c_1} & & & \text{LMod}_B \\
\downarrow (\rho_{\tilde{B}})^* & & \xrightarrow{V_H^{\tilde{B}}} & & \downarrow C_H^B \\
\text{LMod}_{\tilde{B}} & \xrightarrow{u_1} & \text{LMod}_{C_H \vee_H \tilde{B}} & \xrightarrow{\nu} & \text{LMod}_B \\
\parallel & & \parallel & & \parallel \\
\text{LMod}_{\tilde{B}} & \xrightarrow{(\rho_{\tilde{B}})^*} & \text{LMod}_{C_H \vee_H B} & \xrightarrow{\text{Prim}_H^{C_H \vee_H B}} & \text{LMod}_B \\
\downarrow (\rho_{\tilde{B}})_! & \searrow (\rho_{\tilde{B}})_! & \downarrow (\rho_{C_H \vee_H \tilde{B}})_! & & \downarrow (\eta_H \otimes B)_! \\
\text{LMod}_{C_H \vee_H \tilde{B}} & \xrightarrow{C_H(\rho_{\tilde{B}})_!} & \text{LMod}_{(C_H \vee_H)^2 \tilde{B}} & \xrightarrow{\xi} & \\
\parallel & \searrow C_H(\rho_{\tilde{B}})^* & \downarrow \text{Prim}_H^{(C_H \vee_H)^2 \tilde{B}} & & \downarrow (\eta_H \otimes B)_! \\
\text{LMod}_{C_H \vee_H \tilde{B}} & \xrightarrow{\wr} & \text{LMod}_{H \otimes B} & & \downarrow (\rho_B)^* \\
\downarrow \text{Prim}_H^{C_H \vee_H \tilde{B}} & \searrow (\rho_B)^* & \xrightarrow{c_2} & & \downarrow (\rho_B)^* \\
\text{LMod}_B & \xrightarrow{(\rho_B)_!} & \text{LMod}_{H \otimes B} & \xrightarrow{u_2} & \text{LMod}_B
\end{array}$$

By the triangle identities for adjunctions, pasting c_1 and u_1 yields an equivalence, and similarly for c_2 and u_2 . Moreover, pasting ν and ξ yields an equivalence by Lemma 5.4.17. Hence it is enough to show that the square \diamond is right adjointable. By Lemma 5.3.5 (applied to the symmetric monoidal ∞ -category $\text{LComod}_H(\mathcal{C})$), it will suffice to show that the corresponding diagram of commutative H -comodule algebras before applying $\text{LMod}_{(-)}(\text{LComod}_H(\mathcal{C}))$ is a pushout square. As V_H is strongly symmetric monoidal and reflects colimits, it is enough to show that the underlying diagram of commutative algebras in \mathcal{C} is a pushout square, which follows from Lemma 5.4.23.

The other generating maps $\beta: [m] \rightarrow [n]$ of Δ_+ lie in Δ , in which case the square (5.4.25) is of the form

$$\begin{array}{ccc}
\text{LMod}_{H^{\otimes m} \otimes B}(\mathcal{C}) & \xrightarrow{(H^{\otimes m} \otimes \rho_B)_!} & \text{LMod}_{H^{\otimes(m+1)} \otimes B}(\mathcal{C}) \\
\Omega_H^\beta(B)_! \downarrow & & \downarrow \Omega_H^{\beta \star [0]}(B)_! \\
\text{LMod}_{H^{\otimes n} \otimes B}(\mathcal{C}) & \xrightarrow{(H^{\otimes n} \otimes \rho_B)_!} & \text{LMod}_{H^{\otimes(n+1)} \otimes B}(\mathcal{C})
\end{array}$$

²⁰We omit the “base categories” of module categories from the notation in order to save space.

Hence, by Lemma 5.3.5, it is enough to show that the corresponding diagram of commutative algebras before applying $\mathrm{LMod}_{(-)}(\mathcal{C})$ is a pushout square.

If $\beta = \delta^n: [n-1] \rightarrow [n]$ for some $n \geq 1$, then the diagram in question is of the form

$$\begin{array}{ccc} H^{\otimes(n-1)} \otimes B & \xrightarrow{H^{\otimes(n-1)} \otimes \rho_B} & H^{\otimes(n-1)} \otimes H \otimes B \\ \downarrow H^{\otimes(n-1)} \otimes \rho_B & & \downarrow H^{\otimes(n-1)} \otimes \Delta_H \otimes B \\ H^{\otimes(n-1)} \otimes H \otimes B & \xrightarrow{H^{\otimes(n-1)} \otimes H \otimes \rho_B} & H^{\otimes n} \otimes H \otimes H \otimes B \end{array}$$

This square can be obtained by tensoring the pushout square of Lemma 5.4.23 with $H^{\otimes(n-1)}$, so it is indeed a pushout square.

Otherwise, the square in question is of the form

$$\begin{array}{ccc} H^{\otimes m} \otimes B & \xrightarrow{H^{\otimes m} \otimes \rho_B} & H^{\otimes m} \otimes H \otimes B \\ \downarrow \Omega_H^\beta \otimes B & & \downarrow \Omega_H^\beta \otimes H \otimes B \\ H^{\otimes n} \otimes B & \xrightarrow{H^{\otimes n} \otimes \rho_B} & H^{\otimes n} \otimes H \otimes B \end{array}$$

which is also a pushout square. □

As many “derivates” of Cat_∞ admit limits that can be computed in Cat_∞ , Proposition 5.4.24 can be refined to yield equivalences in these categories. For instance, since $\mathrm{LMod}_{\Omega_H^\bullet(B)}(\mathcal{C})$ can be viewed as a diagram in Pr^L if \mathcal{C} is presentably symmetric monoidal and limits in Pr^L can be computed in Cat_∞ by [Lur09, Proposition 5.5.3.13], we have the following.

Corollary 5.4.26. *Assume that \mathcal{C} is presentably symmetric monoidal. Let H be a commutative Hopf algebra in \mathcal{C} and \tilde{B} a commutative H -comodule algebra. Then $\mathrm{LMod}_{\tilde{B}}(\mathrm{LComod}_H(\mathcal{C}))$ is presentable. In particular, $\mathrm{LMod}_{\mathrm{Triv}_H(\mathbf{1}_\mathcal{C})}(\mathrm{LComod}_H(\mathcal{C})) \simeq \mathrm{LComod}_H(\mathcal{C})$ is presentable.*

Similarly, as limits in $\mathrm{CAlg}(\mathrm{Cat}_\infty)$, i.e., the ∞ -category of symmetric monoidal ∞ -categories, can be computed in $\mathrm{CAlg}(\mathrm{Cat}_\infty)$, we expect to have a symmetric monoidal version of Proposition 5.4.24 as follows. The module category of a commutative algebra R admits a symmetric monoidal structure given by \otimes_R , and extensions of scalars along morphisms of commutative algebras are strongly symmetric monoidal (cf. [Lur17, Theorem 4.5.3.1]). In particular, $\mathrm{LMod}_{\Omega_H^\bullet(B)}(\mathcal{C})$ can be viewed as a cosimplicial object in $\mathrm{CAlg}(\mathrm{Cat}_\infty)$. Moreover, in Remark 5.4.22 we have seen that the n -th coaugmentation functor $\mathrm{LMod}_{\tilde{B}}(\mathcal{C}) \rightarrow \mathrm{LMod}_{\Omega_H^n(B)}(\mathcal{C})$ is equivalent to $((\eta_H)^{\otimes n} \otimes B)_! \circ V_H^{\tilde{B}}$, which can be lifted to a strongly symmetric monoidal functor. It is therefore plausible that the cosimplicial object $\mathcal{L}_{\tilde{B}}: \Delta_+ \rightarrow \mathrm{Cat}_\infty$ can be lifted to $\mathrm{CAlg}(\mathrm{Cat}_\infty)$, which would lift the equivalence $\mathrm{LMod}_{\tilde{B}}(\mathrm{LComod}_H(\mathcal{C})) \simeq \lim_{\Delta} \mathrm{LMod}_{\Omega_H^\bullet(B)}(\mathcal{C})$ to a symmetric monoidal one.

5.5. Tensors with spaces

We conclude this section with an interesting consequence of the normal basis condition for certain tensors of commutative algebras with spaces.

Convention 5.5.1. For the rest of this section, we will assume that \mathcal{C} is *presentably* symmetric monoidal.

Remark 5.5.2. By [Lur17, Corollary 3.2.3.5], the assumption that \mathcal{C} is presentably symmetric monoidal implies that $\mathrm{CAlg}(\mathcal{C})$ is presentable. Now let $R \in \mathrm{CAlg}(\mathcal{C})$. Then the coslice category $\mathrm{CAlg}(\mathcal{C})_{R/}$ is presentable by [Lur09, Proposition 5.5.3.11]. Moreover, by [Lur17, Corollary 3.4.1.7 and Theorem 5.1.4.10], we have an equivalence $\mathrm{CAlg}(\mathcal{C})_{R/} \simeq \mathrm{CAlg}(\mathrm{LMod}_R(\mathcal{C}))$ when we equip $\mathrm{LMod}_R(\mathcal{C})$ with the relative tensor product over R . Similarly, the slice category $\mathrm{CAlg}(\mathcal{C})_{R//R}$ is presentable by [Lur09, Proposition 5.5.3.10] and the aforementioned equivalence yields an equivalence $\mathrm{CAlg}(\mathcal{C})_{R//R} \simeq \mathrm{CAlg}(\mathrm{LMod}_R(\mathcal{C}))_{/R}$. Moreover, $\mathrm{CAlg}(\mathcal{C})_{R//R}$ is pointed with zero object $R = R = R$ (cf. [Lur09, Lemma 7.2.2.9]).

Notation 5.5.3. Given a commutative algebra R in a presentably symmetric monoidal ∞ -category \mathcal{D} , we abbreviate $\ominus_{\mathrm{CAlg}(\mathcal{D})_{R/}}: \mathcal{S} \times \mathrm{CAlg}(\mathcal{D})_{R/} \rightarrow \mathrm{CAlg}(\mathcal{D})_{R/}$ as \ominus_R (cf. Fact 1.1.18) and $\otimes_{\mathrm{CAlg}(\mathcal{D})_{R//R}}: \mathcal{S}_* \times \mathrm{CAlg}(\mathcal{D})_{R//R} \rightarrow \mathrm{CAlg}(\mathcal{D})_{R//R}$ as \otimes_R (cf. Fact 1.1.19). Moreover, we will implicitly identify left and right R -modules (cf. [Lur17, Section 4.5.1]) and denote the free *left* R -module functor also by $(-) \otimes R$.

The following lemma will allow us to relate tensors in different slice categories.

Lemma 5.5.4. *Let $F: \mathcal{D} \rightarrow \mathcal{E}$ be a colimit-preserving strongly symmetric monoidal functor between presentably symmetric monoidal ∞ -categories. Then the induced functor $\mathrm{CAlg}(\mathcal{D})_{/1_{\mathcal{D}}} \rightarrow \mathrm{CAlg}(\mathcal{E})_{/1_{\mathcal{E}}}$ preserves colimits. In particular, it is compatible with the respective tensorings over \mathcal{S}_* , i.e., for a pointed space Y and an augmented commutative algebra A in \mathcal{D} , we have a natural equivalence. $Y \otimes_{1_{\mathcal{E}}} F(A) \simeq F(Y \otimes_{1_{\mathcal{D}}} A)$.*

Proof. As colimits in slice categories can be computed in the underlying category, it will suffice to show that the induced functor $\mathrm{CAlg}(F): \mathrm{CAlg}(\mathcal{D}) \rightarrow \mathrm{CAlg}(\mathcal{E})$ preserves colimits. Combining [Lur09, Corollary 4.2.3.12] and [Lur17, Lemma 1.3.3.10], it will suffice to show that $\mathrm{CAlg}(F)$ preserves finite coproducts and sifted colimits. Now finite coproducts in $\mathrm{CAlg}(\mathcal{D})$ are given by the tensor product, which is preserved as F strongly symmetric monoidal. Moreover, by [Lur17, Corollary 3.2.3.2], sifted colimits in $\mathrm{CAlg}(\mathcal{D})$ and $\mathrm{CAlg}(\mathcal{E})$ can be computed in \mathcal{D} and \mathcal{E} , respectively, so $\mathrm{CAlg}(F)$ preserves sifted colimits as F does. \square

Note that for a commutative algebra R in \mathcal{C} , the free module functor $(-) \otimes R: \mathcal{C} \rightarrow \mathrm{LMod}_R(\mathcal{C})$ preserves colimits as it admits a right adjoint and is strongly symmetric monoidal by [Lur17, Theorem 4.5.2.1], which implies the following.

Corollary 5.5.5. *For a commutative algebra R , the functor*

$$\mathrm{CAlg}(\mathcal{C})_{1_{\mathcal{C}}//1_{\mathcal{C}}} \simeq \mathrm{CAlg}(\mathcal{C})_{/1_{\mathcal{C}}} \xrightarrow{(-) \otimes R} \mathrm{CAlg}(\mathrm{LMod}_R(\mathcal{C}))_{/R} \simeq \mathrm{CAlg}(\mathcal{C})_{R//R}$$

preserves colimits. Moreover, for a pointed space Y and an augmented commutative algebra A , we have a natural equivalence $Y \otimes_R (A \otimes R) \simeq (Y \otimes_{\mathbf{1}_C} A) \otimes R$.

Now we can carry out the key construction of this subsection.

Construction 5.5.6. Let $(\eta_H^\bullet, \varphi^\bullet): (\mathbf{1}_C, A) \rightarrow (H, B)$ be a relative Hopf–Galois context of commutative algebras. By [RS20, Proposition 3.20]²¹, there is a colimit-preserving functor $T_\varphi: \mathcal{S}_* \rightarrow \text{CAlg}(\mathcal{C})_{B//B}$ sending $Y \in \mathcal{S}_*$ to

$$B \simeq \{*\} \ominus_A B \xrightarrow{i_Y \ominus_A B} Y \ominus_A B \xrightarrow{p_Y \ominus B} \{*\} \ominus_A B \simeq B,$$

where $i_Y: \{*\} \rightarrow Y$ denotes the inclusion of the base point and $p_Y: Y \rightarrow \{*\}$ the projection. Moreover, by Corollary 5.5.5, the composite

$$T'_{(H,B)}: \mathcal{S}_* \xrightarrow{(-) \otimes H} \text{CAlg}(\mathcal{C})_{\mathbf{1}_C//\mathbf{1}_C} \xrightarrow{(-) \otimes B} \text{CAlg}(\mathcal{C})_{B//B},$$

which is equivalent to $(-) \otimes_B (H \otimes B)$, also preserves colimits.

Now, by [RSV19, Theorem 2.29], evaluation at $S^0 \in \mathcal{S}_*$ induces an equivalence $\text{Fun}^L(\mathcal{S}_*, \text{CAlg}(\mathcal{C})_{B//B}) \simeq \text{CAlg}(\mathcal{C})_{B//B}$. In particular, every map $S^0 \ominus_A B \simeq B \otimes_A B \rightarrow H \otimes B \simeq (S^0 \otimes_{\mathbf{1}_C} H) \otimes B$ of augmented commutative B -algebras extends uniquely to a natural transformation $T_\varphi \rightarrow T'_{(H,B)}$.

We claim that the shear map associated to $(\eta_H^\bullet, \varphi^\bullet)$ is such a map. Indeed, we have a commutative diagram

$$\begin{array}{ccccc} B & \xrightarrow{\quad \quad \quad} & B & & \\ \downarrow \varphi \otimes_A B & \searrow \eta_H \otimes \varphi \otimes_A B & \searrow \eta_H \otimes B & & \downarrow \eta_H \otimes B \\ B \otimes_A B & \xrightarrow{\rho_B \otimes B} & H \otimes B \otimes_A B & \xrightarrow{H \otimes \mu_B} & H \otimes B \\ & \searrow \epsilon_H \otimes B \otimes B & \downarrow \epsilon_H \otimes B & & \downarrow \epsilon_H \otimes B \\ & & B \otimes_A B & \xrightarrow{\mu_B} & B \\ \downarrow \mu_B & & & & \downarrow \mu_B \\ B & \xrightarrow{\quad \quad \quad} & B & & \end{array}, \quad (5.5.7)$$

where the left column depicts $T_\varphi(S^0)$, the right column $T'_{(H,B)}(S^0)$ and the middle row $\text{sh}_{(\eta_H^\bullet, \varphi^\bullet)}$.

The value of the natural transformation $T_\varphi \rightarrow T'_{(H,B)}$ induced by the shear map at $Y \in \mathcal{S}_*$ is a map

$$\text{sh}_{\varphi^\bullet}^Y: Y \ominus_A B \rightarrow (Y \otimes_{\mathbf{1}_C} H) \otimes B$$

of augmented commutative B -algebras. Employing [RSV19, Theorem 2.29] again, we see that $\text{sh}_{\varphi^\bullet}^Y$ is an equivalence for all Y if and only if it is an equivalence for $Y = S^0$, i.e., if and only if the shear map $B \otimes_A B \rightarrow H \otimes B$ is an equivalence.

²¹In [RS20], the authors work with commutative algebras in the ∞ -category of spectra, but the argument for the cited proposition applies verbatim to any presentably symmetric monoidal ∞ -category.

Remark 5.5.8. Using the notation of Construction 5.5.6, [RS20, Proposition 3.20] also yields an identification $T_\varphi(Y) \simeq Y \otimes_B (B \otimes_A B)$ under which sh_φ^Y is equivalent to the map given by applying $Y \otimes_B (-)$ to the shear map $B \otimes_A B \rightarrow H \otimes B$.

Example 5.5.9. In the situation of Construction 5.5.6, we in particular have a comparison map $\text{sh}_\varphi^{S^1}: S^1 \ominus_A B \rightarrow (S^1 \otimes_{\mathbf{1}_C} H) \otimes B$. Note that the source $S^1 \ominus_A B$ can be identified with the topological Hochschild homology $\text{THH}^A(B)$ of B relative to A .

We can describe this map more explicitly by considering S^1 as the pushout of the span $\{*\} \leftarrow S^0 \rightarrow \{*\}$. Applying $(-) \ominus_A B$ and $((-) \otimes_{\mathbf{1}_C} H) \otimes B$ to this span and using the commutativity of the lower half of (5.5.7), we obtain a map of spans

$$\begin{array}{ccccc} B & \xleftarrow{\mu_B} & B \otimes_A B & \xrightarrow{\mu_B} & B \\ \parallel & & \downarrow \text{sh}_{(\eta_H^\bullet, \varphi^\bullet)} & & \parallel \\ B & \xleftarrow{\epsilon_H \otimes B} & H \otimes B & \xrightarrow{\epsilon_H \otimes B} & B \end{array} .$$

Now the pushout of the upper span is equivalent to $\text{THH}^A(B)$, the pushout of the lower span is equivalent to $(\mathbf{1}_C \otimes_H \mathbf{1}_C) \otimes B$, and the map induced on the pushouts corresponds to $\text{sh}_\varphi^{S^1}$. This is an analogue of the description of the relative Hochschild homology of Hopf–Galois extensions in EKMM spectra from [Rot09, Theorem 10.7].

Example 5.5.10. Consider a map $f: X \rightarrow \text{Pic}(\mathcal{C})$ of grouplike \mathbb{E}_∞ -spaces. Then, as discussed in Example 3.1.11, the shear map $M(f) \otimes M(f) \rightarrow (X \ominus_{\mathcal{C}} \mathbf{1}_C) \otimes M(f) \simeq X \ominus_{\mathcal{C}} M(f)$ associated to the unit map $\mathbf{1}_C \rightarrow M(f)$ is an equivalence.

Hence, for $Y \in \mathcal{S}_*$, Construction 5.5.6 yields an equivalence

$$Y \ominus_{\mathbf{1}_C} M(f) \simeq (Y \otimes_{\mathbf{1}_C} (X \ominus_{\mathcal{C}} \mathbf{1}_C)) \otimes M(f).$$

Now $(-) \ominus_{\mathcal{C}} \mathbf{1}_C: \mathcal{S} \rightarrow \mathcal{C}$ is a colimit-preserving symmetric monoidal functor, so Lemma 5.5.4 implies that $Y \otimes_{\mathbf{1}_C} (X \ominus_{\mathcal{C}} \mathbf{1}_C) \simeq (Y \otimes_{\{*\}} X) \ominus_{\mathcal{C}} \mathbf{1}_C$. We therefore have an equivalence

$$Y \ominus_{\mathbf{1}_C} M(f) \simeq ((Y \otimes_{\{*\}} X) \ominus_{\mathcal{C}} \mathbf{1}_C) \otimes M(f) \simeq (Y \otimes_{\{*\}} X) \ominus_{\mathcal{C}} M(f).$$

When \mathcal{C} is the module category of a commutative ring spectrum, this recovers [RSV19, Theorem 4.13].

A. Algebras in coalgebras versus coalgebras in algebras

Convention A.0.1. For the rest of this appendix, we fix a bifunctor $f: \mathcal{P}^\otimes \times \mathcal{Q}^\otimes \rightarrow \mathcal{O}^\otimes$ of ∞ -operads and a \mathcal{O} -monoidal ∞ -category $p: \mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes$.

In Definition 1.2.7, we defined the ∞ -category of $(\mathcal{P}, \mathcal{Q})$ -bialgebras as the ∞ -category of \mathcal{P} -algebras in a \mathcal{P} -monoidal category of \mathcal{Q} -coalgebras and would now like to show that this is equivalent to the ∞ -category of \mathcal{Q} -coalgebras in a \mathcal{Q} -monoidal ∞ -category of \mathcal{P} -algebras.

Equivalences along these lines were constructed in [Rak20, Propositions 2.1.2 and 2.2.10] in the case where \mathcal{P} or \mathcal{Q} is \mathbb{E}_∞ , but the proofs rely on the fact that the symmetric monoidal structure of $\mathrm{CAlg}(\mathcal{C})$ is cocartesian (and the dual property of cocommutative coalgebras). In the case where f is the standard bifunctor $\mathbb{E}_1^\otimes \times \mathbb{E}_1^\otimes \rightarrow \mathbb{E}_2^\otimes$, the desired equivalence could be recovered from [Tor21, Theorem 1.1] by realizing the \mathbb{E}_2 -monoidal ∞ -category \mathcal{C} as a *duoidal ∞ -category* in the sense of [Tor21, Definition 4.14].

Our proof strategy, which is due to Hadrian Heine, is as follows. Note that unpacking Construction 1.2.1 and Fact 1.2.2, we can identify $\mathrm{Alg}_{/\mathcal{P}}(\mathrm{Alg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C}))$ with bifunctors $\mathcal{P}^\otimes \times \mathcal{Q}^\otimes \rightarrow \mathcal{C}^\otimes$ of ∞ -operads over \mathcal{O}^\otimes . Inspired by this and using the machinery of *monoidal envelopes* of [Lur17, Section 2.2.4], we will show that both algebras in coalgebras and coalgebras in algebras admit a description in terms of a notion of bifunctor, which we define now.

Definition A.0.2. Let $q: \mathcal{D}^\otimes \rightarrow \mathcal{Q}^\otimes$ be a \mathcal{Q} -monoidal ∞ -category and $s: \mathcal{E}^\otimes \rightarrow \mathcal{P}^\otimes$ a \mathcal{P} -monoidal ∞ -category. A *bifunctor of monoidal ∞ -categories* is a functor $\phi: \mathcal{E}^\otimes \times \mathcal{D}^\otimes \rightarrow \mathcal{C}^\otimes$ making the diagram

$$\begin{array}{ccc} \mathcal{E}^\otimes \times \mathcal{D}^\otimes & \xrightarrow{\phi} & \mathcal{C}^\otimes \\ s \times q \downarrow & & \downarrow p \\ \mathcal{P}^\otimes \times \mathcal{Q}^\otimes & \xrightarrow{f} & \mathcal{O}^\otimes \end{array}$$

commute and such that for every s -cocartesian morphism t in \mathcal{E}^\otimes and every q -cocartesian morphism u in \mathcal{D}^\otimes , the morphism $\phi(t, u)$ is p -cocartesian.

We denote the full subcategory of $\mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{E}^\otimes \times \mathcal{D}^\otimes, \mathcal{C}^\otimes)$ spanned by bifunctors of monoidal ∞ -categories by $\mathrm{BiFun}^\otimes(\mathcal{E}, \mathcal{D}; \mathcal{C})$.

Remark A.0.3. Note that Definition A.0.2 is “symmetric in \mathcal{E} and \mathcal{D} ” in the sense that the natural equivalence $(-) \times (?) \simeq (?) \times (-)$ induces an equivalence

$$\mathrm{BiFun}^\otimes(\mathcal{E}, \mathcal{D}; \mathcal{C}) \simeq \mathrm{BiFun}^\otimes(\mathcal{D}, \mathcal{E}; \mathcal{C}).$$

Next, we would like to define an \mathcal{P} -monoidal ∞ -category whose fiber at $\mathbf{p} \in \mathcal{P}$ correspond to \mathcal{Q} -monoidal functors $\mathcal{D} \rightarrow \mathcal{C} \times_{\mathcal{O}} \mathcal{Q}$, where the pullback is taken with respect to $\hat{f}(\mathbf{p}): \mathcal{Q}^\otimes \rightarrow \mathcal{O}^\otimes$.

Definition A.0.4. Let $q: \mathcal{D}^\otimes \rightarrow \mathcal{Q}^\otimes$ be a \mathcal{Q} -monoidal ∞ -category. As q is in particular a map of ∞ -operads by [Lur17, Remark 2.1.2.14], we can consider $\mathrm{Alg}_{\mathcal{D}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})^\otimes$ with respect to the bifunctor $f \circ (\mathcal{P} \times q): \mathcal{P}^\otimes \times \mathcal{D}^\otimes \rightarrow \mathcal{O}^\otimes$.

We define $\text{Fun}_f^\otimes(\mathcal{D}, \mathcal{C})^\otimes \subseteq \text{Alg}_{\mathcal{D}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})^\otimes$ as the full subcategory spanned by the objects whose associated functor $g: \mathcal{D}^\otimes \rightarrow \mathcal{C}^\otimes$ maps q -cocartesian edges to p -cocartesian edges and denote by $p_f^{\mathcal{D}, \mathcal{C}}: \text{Fun}_f^\otimes(\mathcal{D}, \mathcal{C})^\otimes \rightarrow \mathcal{P}^\otimes$ the restriction of $p_{f \circ (\mathcal{P}^\otimes \times q)}^{\mathcal{C}}: \text{Alg}_{\mathcal{D}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})^\otimes \rightarrow \mathcal{P}^\otimes$.

Proving that this indeed yields a \mathcal{P} -monoidal structure on an appropriate ∞ -category of \mathcal{Q} -monoidal functors (Proposition A.0.10) will require some preparation. We start by an alternative description of $p_f^{\mathcal{D}, \mathcal{C}}$ that makes it evident that it is a cocartesian fibration.

Definition A.0.5. Given cocartesian fibrations $g: \mathcal{X} \rightarrow \mathcal{A}$ and $h: \mathcal{Y} \rightarrow \mathcal{B}$ between ∞ -categories, let $\text{Fun}(g, h)$ denote the pullback

$$\begin{array}{ccc} \text{Fun}(g, h) & \longrightarrow & \text{Fun}(\mathcal{X}, \mathcal{Y}) \\ \text{pr}_{g, h} \downarrow & \lrcorner & \downarrow h_* \\ \text{Fun}(\mathcal{A}, \mathcal{B}) & \xrightarrow{g^*} & \text{Fun}(\mathcal{X}, \mathcal{B}) \end{array} .$$

We define $\text{Fun}^{\text{cocart}}(\mathcal{X}, \mathcal{Y}) \subseteq \text{Fun}(\mathcal{X}, \mathcal{Y})$ as the full subcategory spanned by the functors which map g -cocartesian arrows to h -cocartesian arrows, and let h_*^{cocart} be the restriction of h_* to $\text{Fun}^{\text{cocart}}(\mathcal{X}, \mathcal{Y})$. Moreover, we define $\text{pr}_{g, h}^{\text{cocart}}: \text{Fun}^{\text{cocart}}(g, h) \rightarrow \text{Fun}(\mathcal{A}, \mathcal{B})$ as the pullback of h_*^{cocart} along g^* .

For cartesian fibrations, $\text{Fun}^{\text{cart}}(-, ?)$ is defined analogously.

Lemma A.0.6. *In the situation of Definition A.0.5, h_* , h_*^{cocart} , $\text{pr}_{g, h}$ and $\text{pr}_{g, h}^{\text{cocart}}$ are all cocartesian fibrations.*

Proof. As cocartesian fibrations are stable under pullback, it will suffice to show that h_* and h_*^{cocart} are cocartesian fibrations. By [Lur17, Proposition 3.1.2.1], h_* is a cocartesian fibration, and a natural transformation is h_* -cocartesian if and only if its components in \mathcal{Y} are h -cocartesian.

As for h_*^{cocart} , it will suffice to show that for every h_* -cocartesian morphism $\tau: \beta \rightarrow \gamma$ such that β maps g -cocartesian arrows to h -cocartesian arrows, γ maps g -cocartesian arrows to h -cocartesian as well. Now let $a: v \rightarrow w$ be a g -cocartesian arrow. Then we have a naturality square

$$\begin{array}{ccc} \beta(v) & \xrightarrow{\beta(a)} & \beta(w) \\ \tau_v \downarrow & & \downarrow \tau_w \\ \gamma(v) & \xrightarrow{\gamma(a)} & \gamma(w) \end{array} .$$

Note that τ_v and τ_w are h -cocartesian by the componentwise description of h_* -cocartesian arrows. Therefore, as the composite $\gamma(a) \circ \tau_v \simeq \tau_w \circ \beta(a)$ and τ_v are h -cocartesian, $\gamma(a)$ is h -cocartesian by [Lur09, Proposition 2.4.1.7]. \square

Remark A.0.7. Let $q: \mathcal{D}^\otimes \rightarrow \mathcal{Q}^\otimes$ be a \mathcal{Q} -monoidal ∞ -category. Note that the adjoint $\mathcal{P}^\otimes \rightarrow \text{Fun}(\mathcal{D}^\otimes, \mathcal{O}^\otimes)$ of $f \circ (\mathcal{P}^\otimes \times q)$ can be factored as

$$\mathcal{P}^\otimes \xrightarrow{\hat{f}} \text{Fun}(\mathcal{Q}^\otimes, \mathcal{O}^\otimes) \xrightarrow{q^*} \text{Fun}(\mathcal{D}^\otimes, \mathcal{O}^\otimes).$$

Therefore, we can describe $\text{Fun}_f^\otimes(\mathcal{D}, \mathcal{C})^\otimes$ as the iterated pullback

$$\begin{array}{ccccc} \text{Fun}_f^\otimes(\mathcal{D}, \mathcal{C})^\otimes & \longrightarrow & \text{Fun}^{\text{cocart}}(q, p) & \longrightarrow & \text{Fun}^{\text{cocart}}(\mathcal{D}^\otimes, \mathcal{C}^\otimes) \\ p_f^{\mathcal{D}, \mathcal{C}} \downarrow & \lrcorner & \downarrow \text{pr}_{q, p} & \lrcorner & \downarrow p_* \\ \mathcal{P}^\otimes & \xrightarrow{\hat{f}} & \text{Fun}(\mathcal{Q}^\otimes, \mathcal{O}^\otimes) & \xrightarrow{q^*} & \text{Fun}(\mathcal{D}^\otimes, \mathcal{O}^\otimes) \end{array}.$$

In particular, $p_f^{\mathcal{D}, \mathcal{C}}$ is a cocartesian fibration as it is the pullback of the cocartesian fibration $\text{pr}_{q, p}$.

Next, we discuss some properties of inert and more general cocartesian morphisms in the ∞ -operads in question. The following lemma will allow us to do induction along decompositions of the form $z \simeq v \oplus w$ in an \mathcal{O} -monoidal ∞ -category. It is essentially a repacking of the argument of [Lur17, Remark 2.2.4.8], but we include it here for the sake of completion.

Lemma A.0.8. *Let \mathcal{R} be an ∞ -operad, $p: \mathcal{F}^\otimes \rightarrow \mathcal{R}^\otimes$ an \mathcal{R} -monoidal ∞ -category. Assume that we have a commutative diagram*

$$\begin{array}{ccccc} v & \xleftarrow{g} & z & \xrightarrow{h} & w \\ t_v \downarrow & & \downarrow t_z & & \downarrow t_w \\ v' & \xleftarrow{g'} & z' & \xrightarrow{h'} & w' \end{array}$$

in \mathcal{F}^\otimes , where g, h, g' and h' are inert, and t_v and t_w are p -cocartesian. Then t_z is also p -cocartesian.

Proof. Let $\bar{t}: z \rightarrow \bar{z}$ be a p -cocartesian lift of $p(t_z)$. Then we have a commutative diagram

$$\begin{array}{ccccc} v & \xleftarrow{g} & z & \xrightarrow{h} & w \\ t_v \downarrow & & \searrow t_z & & \downarrow t_w \\ v' & \xleftarrow{g'} & z' & \xrightarrow{h'} & w' \\ & \nwarrow \bar{g} & \downarrow \bar{t} & \nearrow \bar{h} & \\ & \bar{z} & & & \end{array},$$

where the dashed arrows are obtained using the universal property of \bar{t} as a cocartesian arrow. As the composite $\bar{g} \circ \bar{t} \simeq t_v \circ g$ and \bar{t} are p -cocartesian, \bar{g} is p -cocartesian by [Lur09, Proposition 2.4.1.7], and similarly for \bar{h} .

Now let $n, m \in \mathbb{N}$ such that $v' \in \mathcal{F}_{\langle n \rangle}^\otimes$ and $w' \in \mathcal{F}_{\langle m \rangle}^\otimes$. As \bar{g} and g' are both p -cocartesian, the equivalence $\text{Id}_{v'} \circ \bar{g} \simeq g' \circ s$ implies that the image of s under the projection $\mathcal{F}_{\langle m+n \rangle}^\otimes \rightarrow \mathcal{F}_m^\otimes$ is an equivalence, and similarly for its image under $\mathcal{F}_{\langle m+n \rangle}^\otimes \rightarrow \mathcal{F}_n^\otimes$. As these projections exhibit $\mathcal{F}_{\langle m+n \rangle}^\otimes$ as a product, this means that s is an equivalence. Hence t_z is p -cocartesian as it is equivalent to a p -cocartesian arrow. \square

As a first application, we show that the cocartesian morphisms discussed in Fact 1.2.2 satisfy a stronger condition.²²

Lemma A.0.9. *Consider a cocartesian arrow with respect to $p_f^{\mathcal{C}}: \text{Alg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})^{\otimes} \rightarrow \mathcal{P}^{\otimes}$, which is represented by a natural transformation $\tau: \phi \rightarrow \phi'$ between functors $\mathcal{Q}^{\otimes} \rightarrow \mathcal{C}^{\otimes}$. Then τ_v is p -cocartesian for all vertices v of \mathcal{Q}^{\otimes} .*

Proof. Let $n \in \mathbb{N}$ such that $v \in \mathcal{Q}_{\langle n \rangle}^{\otimes}$. We argue by induction on n . For $n \leq 1$, τ_v is p -cocartesian by the description of Fact 1.2.2, so let $n > 1$. Then there exist $v_1 \in \mathcal{Q}_{\langle 1 \rangle}^{\otimes}$ and $v_2 \in \mathcal{Q}_{\langle n-1 \rangle}^{\otimes}$ such that $v \simeq v_1 \oplus v_2$, and τ_{v_1} and τ_{v_2} are p -cocartesian by the induction hypothesis. Under this identification, the naturality of τ yields a commutative diagram

$$\begin{array}{ccccc} \phi(v_1) & \xleftarrow{\phi(\pi_{v_1})} & \phi(v_1 \oplus v_2) & \xrightarrow{\phi(\pi_{v_2})} & \phi(v_2) \\ \tau_{v_1} \downarrow & & \downarrow \tau_v & & \downarrow \tau_{v_2} \\ \phi'(v_1) & \xleftarrow{\phi'(\pi_{v_1})} & \phi'(v_1 \oplus v_2) & \xrightarrow{\phi'(\pi_{v_2})} & \phi'(v_2) \end{array} .$$

Now the horizontal arrows are inert as ϕ and ϕ' preserve inert morphisms. Hence τ_v is p -cocartesian by Lemma A.0.8. \square

We can now show that $p_f^{\mathcal{D}, \mathcal{C}}$ defines a \mathcal{P} -monoidal ∞ -category.

Proposition A.0.10. *Let $q: \mathcal{D}^{\otimes} \rightarrow \mathcal{Q}^{\otimes}$ be a \mathcal{Q} -monoidal ∞ -category. Then the functor $p_f^{\mathcal{D}, \mathcal{C}}: \text{Fun}_f^{\otimes}(\mathcal{D}, \mathcal{C})^{\otimes} \rightarrow \mathcal{P}^{\otimes}$ of Definition A.0.4 exhibits $\text{Fun}_f^{\otimes}(\mathcal{D}, \mathcal{C}) := \text{Fun}_f^{\otimes}(\mathcal{D}, \mathcal{C})_{\langle 1 \rangle}^{\otimes}$ as a \mathcal{P} -monoidal ∞ -category.*

Proof. As discussed in Remark A.0.7, $p_f^{\mathcal{D}, \mathcal{C}}$ is a cocartesian fibration. By [Lur17, Proposition 2.1.2.12], it will suffice to show that for all $n \in \mathbb{N}$ and all $\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathcal{P}$, the comparison functor $\text{Fun}_f^{\otimes}(\mathcal{D}, \mathcal{C})_{\mathbf{p}_1 \oplus \dots \oplus \mathbf{p}_n}^{\otimes} \rightarrow \prod_{i=1}^n \text{Fun}_f^{\otimes}(\mathcal{D}, \mathcal{C})_{\mathbf{p}_i}^{\otimes}$ induced by $\pi_{\mathbf{p}_i}$'s is an equivalence. As this functor is the restriction of the analogous functor for the \mathcal{P} -monoidal ∞ -category $\text{Alg}_{\mathcal{D}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})$, it will suffice to show that an object of $\text{Alg}_{\mathcal{D}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})_{\mathbf{p}_1 \oplus \dots \oplus \mathbf{p}_n}^{\otimes}$ lies in $\text{Fun}_f^{\otimes}(\mathcal{D}, \mathcal{C})_{\mathbf{p}_1 \oplus \dots \oplus \mathbf{p}_n}^{\otimes}$ if its image in $\prod_{i=1}^n \text{Alg}_{\mathcal{D}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})_{\mathbf{p}_i}^{\otimes}$ lies in $\prod_{i=1}^n \text{Fun}_f^{\otimes}(\mathcal{D}, \mathcal{C})_{\mathbf{p}_i}^{\otimes}$, which we do by induction on n .

For $n \leq 1$, the statement holds tautologically, so let $n > 1$. Set $v := \mathbf{p}_1 \oplus \dots \oplus \mathbf{p}_{n-1}$. As the comparison functor $\text{Fun}_f^{\otimes}(\mathcal{D}, \mathcal{C})_v^{\otimes} \rightarrow \prod_{i=1}^{n-1} \text{Fun}_f^{\otimes}(\mathcal{D}, \mathcal{C})_{\mathbf{p}_i}^{\otimes}$ is an equivalence by the inductive hypothesis, it will suffice to show that every object of $\text{Alg}_{\mathcal{D}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})_{v \oplus \mathbf{p}_n}^{\otimes}$ whose image in $\text{Alg}_{\mathcal{D}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})_v^{\otimes} \times \text{Alg}_{\mathcal{D}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})_{\mathbf{p}_n}^{\otimes}$ lies in $\text{Fun}_f^{\otimes}(\mathcal{D}, \mathcal{C})_v^{\otimes} \times \text{Fun}_f^{\otimes}(\mathcal{D}, \mathcal{C})_{\mathbf{p}_n}^{\otimes}$ is in $\text{Fun}_f^{\otimes}(\mathcal{D}, \mathcal{C})_{v \oplus \mathbf{p}_n}^{\otimes}$. Let such an object be given, which is represented by a functor $\phi: \mathcal{D}^{\otimes} \rightarrow \mathcal{C}^{\otimes}$ preserving inert morphisms such that $p \circ \phi \simeq \hat{f}(v \oplus \mathbf{p}_n) \circ q$. We would like to show that for all q -cocartesian morphisms $t: w \rightarrow w'$ in \mathcal{D}^{\otimes} , $\phi(t)$ is p -cocartesian.

²²In fact, the machinery of *categorical patterns* of [Lur17, Appendix B] used in the proof of [Lur17, Proposition 3.2.4.3] seems to yield this apparently stronger notion, which has probably been simplified in the statement of the cited proposition. Categorical patterns could possibly also be used to prove Proposition A.0.10, but we include a more explicit proof here.

Consider a $p_{f \circ (\mathcal{P}^\otimes \times q)}^{\mathcal{C}}$ -cocartesian lift of $\pi_v: v \oplus \mathfrak{p}_n \rightarrow v$, which, by Lemma A.0.9, corresponds to a natural transformation $\bar{\pi}_v: \phi \rightarrow \phi_1$ lying over $\hat{f}(\pi_v)$ whose components are p -cocartesian. Now for all $z \in \mathcal{Q}^\otimes$, $\hat{f}(\pi_v)(z) = f(\pi_v, z)$ is an inert morphism in \mathcal{O}^\otimes as f is a bifunctor of ∞ -operads. Hence the components of $\bar{\pi}_v$ are in fact inert morphisms in \mathcal{C}^\otimes , as they are p -cocartesian lifts of inert morphisms. We similarly have a $p_{f \circ (\mathcal{P}^\otimes \times q)}^{\mathcal{C}}$ -cocartesian lift of $\pi_{\mathfrak{p}_n}$ given by a natural transformation $\bar{\pi}_{\mathfrak{p}_n}: \phi \rightarrow \phi_2$ with inert components.

Using the naturality of $\bar{\pi}_v$ and $\bar{\pi}_{\mathfrak{p}_n}$, we obtain a commutative diagram

$$\begin{array}{ccccc} \phi_1(w) & \xleftarrow{(\bar{\pi}_v)_w} & \phi(w) & \xrightarrow{(\bar{\pi}_{\mathfrak{p}_n})_w} & \phi_2(w) \\ \phi_1(t) \downarrow & & \downarrow \phi(t) & & \downarrow \phi_2(t) \\ \phi_1(w') & \xleftarrow{(\bar{\pi}_v)_{w'}} & \phi(w') & \xrightarrow{(\bar{\pi}_{\mathfrak{p}_n})_{w'}} & \phi_2(w') \end{array},$$

where the horizontal morphisms are inert. Hence, using Lemma A.0.8, it will suffice to show that $\phi_1(t)$ and $\phi_2(t)$ are cocartesian.

Now ϕ_1 represents the first component of the image of ϕ in $\text{Alg}_{\mathcal{D}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})_v^\otimes \times \text{Alg}_{\mathcal{D}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})_{\mathfrak{p}_n}^\otimes$, and ϕ_2 its second component, which, by assumption, lie in $\text{Fun}_f^\otimes(\mathcal{D}, \mathcal{C})_v^\otimes$ and $\text{Fun}_f^\otimes(\mathcal{D}, \mathcal{C})_{\mathfrak{p}_n}^\otimes$, respectively. Therefore ϕ_1 and ϕ_2 map the q -cocartesian arrow t to a p -cocartesian arrow. \square

Remark A.0.11. Let $q: \mathcal{D}^\otimes \rightarrow \mathcal{Q}^\otimes$ be a \mathcal{Q} -monoidal ∞ -category and $s: \mathcal{E}^\otimes \rightarrow \mathcal{P}^\otimes$ a \mathcal{P} -monoidal ∞ -category. In light of Proposition A.0.10, we can describe bifunctors $\mathcal{E}^\otimes \times \mathcal{D}^\otimes \rightarrow \mathcal{C}^\otimes$ of monoidal ∞ -categories alternatively as follows.

Viewing $\text{Fun}_f^\otimes(\mathcal{D}, \mathcal{C})^\otimes$ as the pullback $\text{Fun}^{\text{cocart}}(q, p) \times_{\text{Fun}(\mathcal{Q}^\otimes, \mathcal{O}^\otimes)} \mathcal{P}^\otimes$ and using the equivalence $\text{Fun}(\mathcal{E}^\otimes, \text{Fun}(\mathcal{D}^\otimes, \mathcal{C}^\otimes)) \simeq \text{Fun}(\mathcal{E}^\otimes \times \mathcal{D}^\otimes, \mathcal{C}^\otimes)$, $\text{Fun}_{\mathcal{P}^\otimes}(\mathcal{E}^\otimes, \text{Fun}_f^\otimes(\mathcal{D}, \mathcal{C})^\otimes)$ can be identified with the full subcategory of $\text{Fun}_{\mathcal{O}^\otimes}(\mathcal{E}^\otimes \times \mathcal{D}^\otimes, \mathcal{C}^\otimes)$ spanned by functors $\phi: \mathcal{E}^\otimes \times \mathcal{D}^\otimes \rightarrow \mathcal{C}^\otimes$ over \mathcal{O}^\otimes such that for every object v in \mathcal{E}^\otimes and every q -cocartesian morphism u in \mathcal{D}^\otimes , the morphism $\phi(\text{Id}_v, u)$ is p -cocartesian.

Moreover, by the componentwise description of cocartesian morphisms in $\text{Fun}_f^\otimes(\mathcal{D}, \mathcal{C})^\otimes$, the adjoint $\hat{\phi}: \mathcal{E}^\otimes \rightarrow \text{Fun}_f^\otimes(\mathcal{D}, \mathcal{C})^\otimes$ of such a ϕ maps s -cocartesian morphisms to $p_f^{\mathcal{D}, \mathcal{C}}$ -cocartesian morphisms if and only if for every s -cocartesian morphism t in \mathcal{E}^\otimes and every object w in \mathcal{D}^\otimes , $\phi(t, \text{Id}_w)$ is p -cocartesian. As every morphism (t, u) in $\mathcal{E}^\otimes \times \mathcal{D}^\otimes$ admits a decomposition of the form $(t, \text{Id}_w) \circ (\text{Id}_v, u)$, this is the case if and only if ϕ is a bifunctor of monoidal ∞ -categories. This means that the equivalence $\text{Fun}(\mathcal{E}^\otimes, \text{Fun}(\mathcal{D}^\otimes, \mathcal{C}^\otimes)) \simeq \text{Fun}(\mathcal{E}^\otimes \times \mathcal{D}^\otimes, \mathcal{C}^\otimes)$ restricts to an equivalence

$$\text{Fun}_{\mathcal{P}^\otimes}^\otimes(\mathcal{E}, \text{Fun}_f^\otimes(\mathcal{D}, \mathcal{C})) \simeq \text{BiFun}^\otimes(\mathcal{E}, \mathcal{D}; \mathcal{C}).$$

We now employ the monoidal envelopes of [Lur17, Section 2.2.4]. By [Lur17, Proposition 2.2.4.9]), for every map $h: \overline{\mathcal{Q}}^\otimes \rightarrow \mathcal{Q}^\otimes$ of ∞ -operads that is also a categorical fibration, there exists a \mathcal{Q} -monoidal ∞ -category $e_{\overline{\mathcal{Q}}}: \text{Env}_{\mathcal{Q}}(\overline{\mathcal{Q}})^\otimes \rightarrow \mathcal{Q}^\otimes$ and an ∞ -operad map

$i_{\overline{\mathcal{Q}}}: \overline{\mathcal{Q}}^{\otimes} \rightarrow \text{Env}_{\mathcal{Q}}(\overline{\mathcal{Q}})^{\otimes}$ that fit into a commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{Q}}^{\otimes} & \xrightarrow{i_{\overline{\mathcal{Q}}}} & \text{Env}_{\mathcal{Q}}(\overline{\mathcal{Q}})^{\otimes} \\ & \searrow h \quad \swarrow e_{\overline{\mathcal{Q}}} & \\ & \mathcal{Q}^{\otimes} & \end{array}$$

and such that for every \mathcal{Q} -monoidal ∞ -category \mathcal{D} , restriction along $i_{\overline{\mathcal{Q}}}$ induces an equivalence $\text{Fun}_{\mathcal{Q}}^{\otimes}(\text{Env}_{\mathcal{Q}}(\overline{\mathcal{Q}}), \mathcal{D}) \simeq \text{Alg}_{\overline{\mathcal{Q}}/\mathcal{Q}}(\mathcal{D})$. We now give an analogous description of $\text{Alg}_{\overline{\mathcal{Q}}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})$.

Proposition A.0.12. *Let $g: \overline{\mathcal{Q}}^{\otimes} \rightarrow \mathcal{Q}^{\otimes}$ be a fibration of ∞ -operads (i.e., a map of ∞ -operads that is also a categorical fibration). Then the functor*

$$\text{Fun}_f^{\otimes}(\text{Env}_{\mathcal{Q}}(\overline{\mathcal{Q}}), \mathcal{C})^{\otimes} \rightarrow \text{Alg}_{\overline{\mathcal{Q}}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})^{\otimes}$$

induced by $i_{\overline{\mathcal{Q}}}: \mathcal{Q}^{\otimes} \rightarrow \text{Env}_{\mathcal{Q}}(\overline{\mathcal{Q}})^{\otimes}$ is a strongly \mathcal{P} -monoidal equivalence.

Proof. This comparison functor is the restriction of the strongly \mathcal{P} -monoidal functor $\text{Alg}_{\text{Env}_{\mathcal{Q}}(\overline{\mathcal{Q}})/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})^{\otimes} \rightarrow \text{Alg}_{\overline{\mathcal{Q}}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})^{\otimes}$ (cf. Remark 1.2.3) to a full subcategory closed under cocartesian morphisms, so it maps $p_f^{\text{Env}_{\mathcal{Q}}(\overline{\mathcal{Q}}), \mathcal{C}}$ -cocartesian morphisms to $p_{f \circ (\mathcal{P}^{\otimes} \times g)}^{\mathcal{C}}$ -cocartesian morphisms, i.e., is indeed strongly \mathcal{P} -monoidal. Hence, as discussed in [Lur17, Remark 2.1.3.8], it will suffice to show that for all $\mathfrak{p} \in \mathcal{P}$, the induced functor $\text{Fun}_f^{\otimes}(\text{Env}_{\mathcal{Q}}(\overline{\mathcal{Q}}), \mathcal{C})_{\mathfrak{p}}^{\otimes} \rightarrow \text{Alg}_{\overline{\mathcal{Q}}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})_{\mathfrak{p}}^{\otimes}$ is an equivalence. Considering algebras with respect to the ∞ -operad map $\hat{f}(\mathfrak{p}): \mathcal{Q}^{\otimes} \rightarrow \mathcal{O}^{\otimes}$, this functor can be identified with the composite

$$\text{Fun}_{\mathcal{Q}}^{\otimes}(\text{Env}_{\mathcal{Q}}(\overline{\mathcal{Q}}), \mathcal{C} \times_{\mathcal{O}} \mathcal{Q}) \rightarrow \text{Alg}_{\overline{\mathcal{Q}}/\mathcal{Q}}(\mathcal{C} \times_{\mathcal{O}} \mathcal{Q}) \rightarrow \text{Alg}_{\overline{\mathcal{Q}}/\mathcal{O}}(\mathcal{C}),$$

where the first arrow is an equivalence by [Lur17, Proposition 2.2.4.9] and the second by the universal property of the pullback. \square

Next, we would like to identify $\text{Coalg}_{\mathcal{Q}/\mathcal{O}}(\mathcal{C}) \simeq \text{Fun}_{\mathcal{Q}}^{\otimes}(\text{Env}_{\mathcal{Q}}(\mathcal{Q}), \mathcal{C}^{\text{fop}} \times_{\mathcal{O}} \mathcal{Q})$ with $\text{Fun}_{\mathcal{Q}}^{\otimes}(\text{Env}_{\mathcal{Q}}(\mathcal{Q})^{\text{fop}}, \mathcal{C} \times_{\mathcal{O}} \mathcal{Q})^{\text{op}}$, so that we can relate it to other constructions such as $\text{Alg}_{\mathcal{P}/\mathcal{O}}(\mathcal{C})$ that involve functors into \mathcal{C}^{\otimes} . For this, we need to show that taking fiberwise opposites is compatible with $\text{Fun}^{\text{cocart}}(-, ?)$ in an appropriate sense.

Lemma A.0.13. *Let $g: \mathcal{X} \rightarrow \mathcal{A}$ and $h: \mathcal{Y} \rightarrow \mathcal{B}$ be cocartesian fibrations between ∞ -categories. Then the cocartesian fibrations*

$$(\text{pr}_{g,h}^{\text{cocart}})^{\text{fop}}: \text{Fun}^{\text{cocart}}(g, h)^{\text{fop}} \rightarrow \text{Fun}(\mathcal{A}, \mathcal{B})$$

and

$$\text{pr}_{g^{\text{fop}}, h^{\text{fop}}}^{\text{cocart}}: \text{Fun}^{\text{cocart}}(g^{\text{fop}}, h^{\text{fop}}) \rightarrow \text{Fun}(\mathcal{A}, \mathcal{B})$$

are naturally equivalent.

Proof. Intuitively speaking, $\text{Fun}^{\text{cocart}}(g, h)^{\text{fop}}$ is the ∞ -category of commutative squares

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\bar{\gamma}} & \mathcal{Y} \\ g \downarrow & & \downarrow h \\ \mathcal{A} & \xrightarrow{\gamma} & \mathcal{B} \end{array}$$

such that $\bar{\gamma}$ preserves cocartesian morphisms, but where a morphism $\bar{\gamma} \rightarrow \bar{\gamma}'$ in the fiber over $\gamma \in \text{Fun}(\mathcal{A}, \mathcal{B})$ is given by a natural transformation $\bar{\gamma}' \rightarrow \bar{\gamma}$ in the opposite direction. On the other hand, $\text{Fun}^{\text{cocart}}(g^{\text{fop}}, h^{\text{fop}})$ is the ∞ -category of commutative squares

$$\begin{array}{ccc} \mathcal{X}^{\text{fop}} & \xrightarrow{\hat{\gamma}} & \mathcal{Y}^{\text{fop}} \\ g^{\text{fop}} \downarrow & & \downarrow h^{\text{fop}} \\ \mathcal{A} & \xrightarrow{\gamma} & \mathcal{B} \end{array}$$

such that $\hat{\gamma}$ preserves cocartesian morphisms. These two types of squares correspond to each other by taking fiberwise opposites of the vertical arrows.

Making this intuition precise requires a “sufficiently functorial” construction of $(-)^{\text{fop}}$ that induces a comparison functor between $\text{Fun}^{\text{cocart}}(g, h)^{\text{fop}}$ and $\text{Fun}^{\text{cocart}}(g^{\text{fop}}, h^{\text{fop}})$. As we discuss below, a construction suitable for this purpose is the point-set level description of $(-)^{\text{fop}}$ given in [BGN18, Theorem 1.4]. Namely, there is a functorial construction $(-)^{\vee}$ sending a cocartesian fibration $q: \mathcal{Z} \rightarrow \mathcal{D}$ over an ∞ -category to a cartesian fibration $q^{\vee}: \mathcal{Z}^{\vee} \rightarrow \mathcal{D}^{\text{op}}$ such that $(q^{\vee})^{\text{op}}$ is a model for q^{fop} . Similarly, there is a duality construction mapping cartesian fibrations to cocartesian fibrations that is also denoted by $(-)^{\vee}$.

Therefore, taking opposites, it will suffice to construct an equivalence $\text{Fun}^{\text{cocart}}(g, h)^{\vee} \simeq \text{Fun}^{\text{cart}}(g^{\vee}, h^{\vee})$ over $\text{Fun}(\mathcal{A}, \mathcal{B})^{\text{op}} \simeq \text{Fun}(\mathcal{A}^{\text{op}}, \mathcal{B}^{\text{op}})$. Moreover, in order to avoid cluttering the notation with even more $(-)^{\text{op}}$ ’s, we will implicitly dualize our cocartesian fibrations g and h to *cartesian* fibrations by taking opposites and construct an equivalence $\text{Fun}^{\text{cart}}(g, h)^{\vee} \simeq \text{Fun}^{\text{cocart}}(g^{\vee}, h^{\vee})$ over $\text{Fun}(\mathcal{A}, \mathcal{B})^{\text{op}} \simeq \text{Fun}(\mathcal{A}^{\text{op}}, \mathcal{B}^{\text{op}})$.

In order to construct the comparison functor, we need to unpack the definition of $(-)^{\vee}$ a bit. In [BGN18, Definition 3.2], for an ∞ -category \mathcal{D} equipped with suitable subcategories $\mathcal{D}_{\dagger}, \mathcal{D}^{\dagger} \subseteq \mathcal{D}$, the *effective Burnside ∞ -category* $A^{\text{eff}}(\mathcal{D}, \mathcal{D}_{\dagger}, \mathcal{D}^{\dagger})$ is constructed, whose objects are those of \mathcal{D} and whose morphisms are spans $w \leftarrow u \rightarrow v$ in \mathcal{D} such that $u \rightarrow v$ lies in \mathcal{D}_{\dagger} and $u \rightarrow w$ lies in \mathcal{D}^{\dagger} . Given a cartesian fibration $q: \mathcal{Z} \rightarrow \mathcal{D}$, let $\iota\mathcal{D}$ be the subcategory of \mathcal{D} that contains all objects and whose morphisms are equivalences, and let $\iota^{\mathcal{D}}\mathcal{Z}$ be the subcategory of \mathcal{Z} that contains all objects and whose morphisms are q -cartesian morphisms. By [BGN18, Proposition 3.3], q induces a cocartesian fibration

$$q': A^{\text{eff}}(\mathcal{Z}, \mathcal{Z} \times_{\mathcal{D}} \iota\mathcal{D}, \iota^{\mathcal{D}}\mathcal{Z}) \rightarrow A^{\text{eff}}(\mathcal{D}, \iota\mathcal{D}, \mathcal{D})$$

such that a morphism $w \leftarrow u \rightarrow v$ in $A^{\text{eff}}(\mathcal{Z}, \mathcal{Z} \times_{\mathcal{D}} \iota\mathcal{D}, \iota^{\mathcal{D}}\mathcal{Z})$ is cocartesian if and only if $u \rightarrow v$ is an equivalence. Then, in [BGN18, Definition 3.4], q^{\vee} is defined as the pullback of q' along the equivalence

$$i_{\mathcal{D}}: \mathcal{D}^{\text{op}} \rightarrow A^{\text{eff}}(\mathcal{D}, \iota\mathcal{D}, \mathcal{D})$$

that sends a morphism $v \rightarrow w$ to the span $w \leftarrow v = v$.

In order to save space, we will abbreviate $A^{\text{eff}}(-, (-)_{\dagger}, (-)^{\dagger})$ as $A^{\text{eff}}(-)$ when $(-)_{\dagger}$ and $(-)^{\dagger}$ are clear from the context.

The next relevant property of $A^{\text{eff}}(\mathcal{D}, \mathcal{D}_{\dagger}, \mathcal{D}^{\dagger})$ is the fact that it is a certain subsimplicial set of a nerve construction $N_{\tilde{\mathcal{O}}([\bullet])^{\text{op}}}(\mathcal{D})$ with respect to the cosimplicial object $\tilde{\mathcal{O}}([\bullet])^{\text{op}}$ given by opposites of twisted arrow categories (cf. [BGN18, Definition 2.1]). As $N_{\tilde{\mathcal{O}}([\bullet])^{\text{op}}}$ commutes with products, we have a natural transformation

$$\Phi: N_{\tilde{\mathcal{O}}([\bullet])^{\text{op}}}(\text{Fun}(-, ?)) \rightarrow \text{Fun}(N_{\tilde{\mathcal{O}}([\bullet])^{\text{op}}}(-), N_{\tilde{\mathcal{O}}([\bullet])^{\text{op}}}(?)).$$

This natural transformation yields a commutative diagram

$$\begin{array}{ccc} A^{\text{eff}}(\text{Fun}^{\text{cart}}(g, h)) & \xrightarrow{\quad \bar{\Phi} \quad} & \text{Fun}^{\text{cocart}}(g', h') \\ (\text{pr}_{g,h}^{\text{cart}})' \downarrow & & \downarrow \text{pr}_{g',h'}^{\text{cocart}} \\ A^{\text{eff}}(\text{Fun}(\mathcal{A}, \mathcal{B})) & \xrightarrow{\quad \underline{\Phi} \quad} & \text{Fun}(A^{\text{eff}}(\mathcal{A}), A^{\text{eff}}(\mathcal{B})) \\ & \nwarrow i_{\text{Fun}(\mathcal{A}, \mathcal{B})} \quad \nearrow F & \\ & \text{Fun}(\mathcal{A}, \mathcal{B})^{\text{op}} & \end{array} \quad , \quad (\text{A.0.14})$$

where

- $\bar{\Phi}$ and $\underline{\Phi}$ are obtained by restricting Φ to appropriate subsimplicial sets,
- $\bar{\Phi}$ maps $(\text{pr}_{g,h}^{\text{cart}})'$ -cocartesian arrows to $\text{pr}_{g',h'}^{\text{cocart}}$ -cocartesian arrows,
- F is the functor induced by the functoriality of $A^{\text{eff}}(-, \iota(-), -)$.

Now, by the naturality of $i_{(-)}$, F can be identified with $i_{\mathcal{B}} \circ (-) \circ i_{\mathcal{A}}^{-1}: \text{Fun}(\mathcal{A}^{\text{op}}, \mathcal{B}^{\text{op}}) \rightarrow \text{Fun}(A^{\text{eff}}(\mathcal{A}), A^{\text{eff}}(\mathcal{B}))$. Moreover, note that the equivalence $(-)^{\vee} \simeq (-)'$ of cocartesian fibrations induces a commutative diagram

$$\begin{array}{ccc} \text{Fun}^{\text{cocart}}(g^{\vee}, h^{\vee}) & \xrightarrow{\quad \simeq \quad} & \text{Fun}^{\text{cocart}}(g', h') \\ \text{pr}_{g^{\vee}, h^{\vee}}^{\text{cocart}} \downarrow & & \downarrow \text{pr}_{g', h'}^{\text{cocart}} \\ \text{Fun}(\mathcal{A}^{\text{op}}, \mathcal{B}^{\text{op}}) & \xrightarrow[\quad i_{\mathcal{B}} \circ (-) \circ i_{\mathcal{A}}^{-1} \quad]{\quad \simeq \quad} & \text{Fun}(A^{\text{eff}}(\mathcal{A}), A^{\text{eff}}(\mathcal{B})) \end{array} \quad ,$$

which we can in particular view as a pullback square.

Hence, considering the functor $\bar{\Phi}$ induces between the pullbacks along the lower diagonal arrows in (A.0.14), we obtain a commutative diagram

$$\begin{array}{ccc} \text{Fun}^{\text{cart}}(g, h)^{\vee} & \xrightarrow{\quad \hat{\Phi} \quad} & \text{Fun}^{\text{cocart}}(g^{\vee}, h^{\vee}) \\ (\text{pr}_{g,h}^{\text{cart}})^{\vee} \searrow & & \swarrow \text{pr}_{g^{\vee}, h^{\vee}}^{\text{cocart}} \\ & \text{Fun}(\mathcal{A}, \mathcal{B})^{\text{op}} \simeq \text{Fun}(\mathcal{A}^{\text{op}}, \mathcal{B}^{\text{op}}) & \end{array} \quad ,$$

where $\widehat{\Phi}$ maps $(\mathrm{pr}_{g,h}^{\mathrm{cart}})^\vee$ -cocartesian arrows to $\mathrm{pr}_{g^\vee,h^\vee}^{\mathrm{cocart}}$ -cocartesian arrows.

We now use the fiberwise criterion of [Lur09, Corollary 2.4.4.4] to show that $\widehat{\Phi}$ is an equivalence. Let $\gamma: \mathcal{A} \rightarrow \mathcal{B}$ be a functor, which we can equivalently view as a functor $\gamma^{\mathrm{op}}: \mathcal{A}^{\mathrm{op}} \rightarrow \mathcal{B}^{\mathrm{op}}$. Then $\widehat{\Phi}_\gamma$ coincides with the functor $\mathrm{Fun}^{\mathrm{cart}}(g, h)_\gamma^{\mathrm{op}} \rightarrow \mathrm{Fun}^{\mathrm{cocart}}(g^\vee, h^\vee)_\gamma$ induced by the functoriality of $(-)^\vee$. Now $(-)^\vee$ is an equivalence as the “double dual” construction is equivalent to the identity by [BGN18, Proposition 4.1], so $\widehat{\Phi}_\gamma$ is indeed an equivalence. \square

Corollary A.0.15. *Let $q: \mathcal{D}^\otimes \rightarrow \mathcal{Q}^\otimes$ be a \mathcal{Q} -monoidal ∞ -category. Then the cocartesian fibrations $(p_f^{\mathcal{D}, \mathcal{C}})^{\mathrm{fop}}$ and $p_f^{\mathcal{D}^{\mathrm{fop}}, \mathcal{C}^{\mathrm{fop}}}$ are equivalent, i.e., we have a strongly \mathcal{P} -monoidal equivalence $\mathrm{Fun}_f^\otimes(\mathcal{D}, \mathcal{C})^{\mathrm{fop}} \simeq \mathrm{Fun}_f^\otimes(\mathcal{D}^{\mathrm{fop}}, \mathcal{C}^{\mathrm{fop}})$.*

Proof. By Remark A.0.7, $p_f^{\mathcal{D}, \mathcal{C}}: \mathrm{Fun}_f^\otimes(\mathcal{D}, \mathcal{C}) \rightarrow \mathcal{P}$ is the pullback of $\mathrm{pr}_{q,p}$ along \widehat{f} and $p_f^{\mathcal{D}^{\mathrm{fop}}, \mathcal{C}^{\mathrm{fop}}}$ is the pullback of $\mathrm{pr}_{q^{\mathrm{fop}}, p^{\mathrm{fop}}}$ along \widehat{f} . Now $(\mathrm{pr}_{q,p})^{\mathrm{fop}} \simeq \mathrm{pr}_{q^{\mathrm{fop}}, p^{\mathrm{fop}}}$ by Lemma A.0.13, so the result follows by the compatibility of fiberwise opposites with pullbacks. \square

Remark A.0.16. Applying Corollary A.0.15 to the case where \mathcal{P} is the trivial operad, we obtain an equivalence

$$\mathrm{Fun}_{\mathcal{Q}}^\otimes(\mathcal{D}, \mathcal{C} \times_{\mathcal{O}} \mathcal{Q})^{\mathrm{op}} \simeq \mathrm{Fun}_{\mathcal{Q}}^\otimes(\mathcal{D}^{\mathrm{fop}}, \mathcal{C}^{\mathrm{fop}} \times_{\mathcal{O}} \mathcal{Q})$$

for every map $\mathcal{Q} \rightarrow \mathcal{O}$ of ∞ -operads.

We can now show that ∞ -categories of algebras in coalgebras and algebras in coalgebras are equivalent.

Corollary A.0.17. *There is an equivalence²³*

$$\mathrm{Alg}_{/\mathcal{P}}(\mathrm{Alg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C}^{\mathrm{fop}})^{\mathrm{fop}}) \simeq \mathrm{Alg}_{/\mathcal{Q}}(\mathrm{Alg}_{\mathcal{P}/\mathcal{O}}^{\mathcal{Q}}(\mathcal{C})^{\mathrm{fop}})^{\mathrm{op}}$$

that is natural in \mathcal{P}^\otimes , \mathcal{Q}^\otimes and \mathcal{C}^\otimes , where the right hand side is constructed with respect to the bifunctor $f': \mathcal{Q}^\otimes \times \mathcal{P}^\otimes \simeq \mathcal{P}^\otimes \times \mathcal{Q}^\otimes \rightarrow \mathcal{O}^\otimes$.

²³Or using the notation of Definition 1.2.7, $\mathrm{Alg}_{/\mathcal{P}}(\mathrm{Coalg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C})) \simeq \mathrm{Coalg}_{/\mathcal{Q}}(\mathrm{Alg}_{\mathcal{P}/\mathcal{O}}^{\mathcal{Q}}(\mathcal{C}))$.

Proof. We have a chain

$$\begin{aligned}
\mathrm{Alg}_{/\mathcal{P}}(\mathrm{Alg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C}^{\mathrm{fop}})^{\mathrm{fop}}) &\stackrel{[\mathrm{Lur17}, \text{Proposition 2.2.4.9}]}{\simeq} \mathrm{Fun}_{\mathcal{P}}^{\otimes}(\mathrm{Env}_{\mathcal{P}}(\mathcal{P}), \mathrm{Alg}_{\mathcal{Q}/\mathcal{O}}^{\mathcal{P}}(\mathcal{C}^{\mathrm{fop}})^{\mathrm{fop}}) \\
&\stackrel{\mathrm{A.0.12}}{\simeq} \mathrm{Fun}_{\mathcal{P}}^{\otimes}(\mathrm{Env}_{\mathcal{P}}(\mathcal{P}), \mathrm{Fun}_f^{\otimes}(\mathrm{Env}_{\mathcal{Q}}(\mathcal{Q}), \mathcal{C}^{\mathrm{fop}})^{\mathrm{fop}}) \\
&\stackrel{\mathrm{A.0.15}}{\simeq} \mathrm{Fun}_{\mathcal{P}}^{\otimes}(\mathrm{Env}_{\mathcal{P}}(\mathcal{P}), \mathrm{Fun}_f^{\otimes}(\mathrm{Env}_{\mathcal{Q}}(\mathcal{Q})^{\mathrm{fop}}, \mathcal{C})) \\
&\stackrel{\mathrm{A.0.11}}{\simeq} \mathrm{BiFun}^{\otimes}(\mathrm{Env}_{\mathcal{P}}(\mathcal{P}), \mathrm{Env}_{\mathcal{Q}}(\mathcal{Q})^{\mathrm{fop}}; \mathcal{C}) \\
&\stackrel{\mathrm{A.0.3}}{\simeq} \mathrm{BiFun}^{\otimes}(\mathrm{Env}_{\mathcal{Q}}(\mathcal{Q})^{\mathrm{fop}}, \mathrm{Env}_{\mathcal{P}}(\mathcal{P}); \mathcal{C}) \\
&\stackrel{\mathrm{A.0.11}}{\simeq} \mathrm{Fun}_{\mathcal{Q}}^{\otimes}(\mathrm{Env}_{\mathcal{Q}}(\mathcal{Q})^{\mathrm{fop}}, \mathrm{Fun}_{f'}^{\otimes}(\mathrm{Env}_{\mathcal{P}}(\mathcal{P}), \mathcal{C})) \\
&\stackrel{\mathrm{A.0.12}}{\simeq} \mathrm{Fun}_{\mathcal{Q}}^{\otimes}(\mathrm{Env}_{\mathcal{Q}}(\mathcal{Q})^{\mathrm{fop}}, \mathrm{Alg}_{\mathcal{P}/\mathcal{O}}^{\mathcal{Q}}(\mathcal{C})) \\
&\stackrel{\mathrm{A.0.16}}{\simeq} \mathrm{Fun}_{\mathcal{Q}}^{\otimes}(\mathrm{Env}_{\mathcal{Q}}(\mathcal{Q}), \mathrm{Alg}_{\mathcal{P}/\mathcal{O}}^{\mathcal{Q}}(\mathcal{C})^{\mathrm{fop}})^{\mathrm{op}} \\
&\stackrel{[\mathrm{Lur17}, \text{Proposition 2.2.4.9}]}{\simeq} \mathrm{Alg}_{/\mathcal{Q}}(\mathrm{Alg}_{\mathcal{P}/\mathcal{O}}^{\mathcal{Q}}(\mathcal{C})^{\mathrm{fop}})^{\mathrm{op}}
\end{aligned}$$

of natural equivalences. \square

We end this appendix with a generalization of [Rak20, Proposition and 2.2.10].

Example A.0.18. Let $k \in \mathbb{N} \cup \{\infty\}$ and \mathcal{C} an \mathbb{E}_{k+1} -monoidal ∞ -category. Consider the standard bifunctor $\mathbb{E}_k^{\otimes} \times \mathbb{E}_1^{\otimes} \rightarrow \mathbb{E}_{k+1}^{\otimes}$ and the bifunctor $\mathbb{E}_k^{\otimes} \times \mathcal{LM}^{\otimes} \rightarrow \mathbb{E}_{k+1}^{\otimes}$ induced by the ∞ -operad map $\mathcal{LM}^{\otimes} \rightarrow \mathbb{E}_1^{\otimes}$. As discussed in Remark 1.2.3, we identify $\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}^{\mathcal{LM}}(\mathcal{C})^{\otimes}$ with $\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})^{\otimes} \times_{\mathbb{E}_1^{\otimes}} \mathcal{LM}^{\otimes}$.

In this situation, Corollary A.0.17 yields a commutative diagram

$$\begin{array}{ccc}
\mathrm{Alg}_{/\mathbb{E}_k}(\mathrm{LComod}(\mathcal{C})) & \simeq & \mathrm{LComod}(\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})) \\
\downarrow & & \downarrow \\
\mathrm{Alg}_{/\mathbb{E}_k}(\mathrm{Coalg}(\mathcal{C})) & \simeq & \mathrm{Coalg}(\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C}))
\end{array} .$$

Now let $H \in \mathrm{Coalg}(\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})) \simeq \mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathrm{Coalg}(\mathcal{C}))$ be a bialgebra. Then, considering fibers over H , we obtain an equivalence

$$\mathrm{Alg}_{/\mathbb{E}_k}(\mathrm{LComod}_H(\mathcal{C})) \stackrel{1.2.13}{\simeq} \mathrm{Alg}_{/\mathbb{E}_k}(\mathrm{LComod}(\mathcal{C}))_H \simeq \mathrm{LComod}_H(\mathrm{Alg}_{\mathbb{E}_k/\mathbb{E}_{k+1}}(\mathcal{C})).$$

B. Lifting adjunctions

In this appendix, we study certain adjunctions induced by an adjunction $F: \mathcal{D} \rightleftarrows \mathcal{E} : G$ of lax monoidal functors. In particular, we show that such an adjunction gives rise to an adjunction between $\mathrm{LMod}_R(\mathcal{D})$ and $\mathrm{LMod}_{F(R)}(\mathcal{E})$ for every algebra R in \mathcal{D} , which is used in Subsection 3.2 to construct a comonad on the ∞ -category of modules over the underlying algebra of a comodule algebra. We start by introducing some terminology.

Definition B.0.1. Let $U_1: \mathcal{D}' \rightarrow \mathcal{D}$, $U_2: \mathcal{E}' \rightarrow \mathcal{E}$ be functors and $F: \mathcal{D} \rightleftarrows \mathcal{E} : G$ an adjunction. We call an adjunction $F': \mathcal{D}' \rightleftarrows \mathcal{E}' : G'$ a *lift of $F \dashv G$ (along U_1 and U_2)* if the front and back faces of the diagram

$$\begin{array}{ccc} \mathcal{D}' & \begin{array}{c} \xrightarrow{F'} \\ \perp \\ \xleftarrow{G'} \end{array} & \mathcal{E}' \\ U_1 \downarrow & & \downarrow U_2 \\ \mathcal{D} & \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} & \mathcal{E} \end{array}$$

commute, the image of the unit of $F' \dashv G'$ under U_1 is equivalent to the unit of $F \dashv G$, and the image of the counit of $F' \dashv G'$ under U_2 is equivalent to the counit of $F \dashv G$.

When $\mathcal{C} := \mathcal{D} = \mathcal{E}$ and $F \dashv G$ is the identity adjunction, we call $F' \dashv G'$ an *adjunction relative to \mathcal{C}* (cf. [Lur17, Remark 7.3.2.3]).

Remark B.0.2. Note that by [RV16, Theorem 4.4.11], there is (up to homotopy) at most one adjunction extending the datum of two functors and *either* a candidate unit *or* a candidate counit transformation. Therefore, in the context of Definition B.0.1, the compatibility with the unit implies the compatibility with the counit and vice versa.

As alluded to before, we will be interested in lifts of adjunctions of lax monoidal functors.

Definition B.0.3. Let \mathcal{O} be an ∞ -operad, \mathcal{D} and \mathcal{E} \mathcal{O} -monoidal ∞ -categories. An *adjunction of lax \mathcal{O} -monoidal functors* is an adjunction $F^\otimes: \mathcal{D}^\otimes \rightleftarrows \mathcal{E}^\otimes : G^\otimes$ relative to \mathcal{O}^\otimes such that F^\otimes and G^\otimes are lax \mathcal{O} -monoidal.

Example B.0.4. Let \mathcal{O} be an ∞ -operad, \mathcal{D} and \mathcal{E} \mathcal{O} -monoidal ∞ -categories. Our main source of adjunctions of lax \mathcal{O} -monoidal functors are strongly \mathcal{O} -monoidal functors $F^\otimes: \mathcal{D}^\otimes \rightarrow \mathcal{E}^\otimes$ such that for every $\mathfrak{o} \in \mathcal{O} = \mathcal{O}_{(1)}^\otimes$, $F_\mathfrak{o}^\otimes$ admits a right adjoint $G_\mathfrak{o}$.

Indeed, viewing the underlying functor of F^\otimes as the underlying functor of an oplax \mathcal{O} -monoidal functor, Fact 1.1.6 yields a lax monoidal functor $G^\otimes: \mathcal{E}^\otimes \rightarrow \mathcal{D}^\otimes$ whose fiber over $\mathfrak{o} \in \mathcal{O}$ is given by $G_\mathfrak{o}$. Moreover, arguing as in the proof of [Lur17, Proposition 7.3.2.6], the adjunctions $F_\mathfrak{o}^\otimes \dashv G_\mathfrak{o}$ can be extended to an adjunction $F^\otimes \dashv G^\otimes$ relative to \mathcal{O}^\otimes .

Alternatively, one can apply [Lur17, Corollary 7.3.2.7]²⁴ to F^\otimes in order to obtain a right adjoint with the desired properties directly. In fact, the “only if” part of [Lur17,

²⁴Note that while it is not made explicit in its statement, this corollary applies only to *strongly* \mathcal{O} -monoidal functors because its proof uses [Lur17, Proposition 7.3.2.6], which requires the functor in question to preserve locally (co)cartesian morphisms.

Proposition 7.3.2.6] implies that every adjunction $\mathcal{D}^\otimes \rightleftarrows \mathcal{E}^\otimes$ relative to \mathcal{O}^\otimes is induced by such an F^\otimes through this mechanism.

Remark B.0.5. Let $f: \mathcal{P}^\otimes \rightarrow \mathcal{O}^\otimes$ be a map of operads, \mathcal{D} and \mathcal{E} \mathcal{O} -monoidal ∞ -categories, and $F^\otimes: \mathcal{D}^\otimes \rightleftarrows \mathcal{E}^\otimes : G^\otimes$ an adjunction of lax \mathcal{O} -monoidal functors.

Using [Lur17, Proposition 7.3.2.5], we can pull back this adjunction along f to obtain an adjunction $F_{\mathcal{P}}^\otimes: \mathcal{D}^\otimes \times_{\mathcal{O}^\otimes} \mathcal{P}^\otimes \rightleftarrows \mathcal{E}^\otimes \times_{\mathcal{O}^\otimes} \mathcal{P}^\otimes : G_{\mathcal{P}}^\otimes$ relative to \mathcal{P}^\otimes . Note that $F_{\mathcal{P}}^\otimes \dashv G_{\mathcal{P}}^\otimes$ is an adjunction of lax \mathcal{P} -monoidal functors because inert morphisms in $\mathcal{D}^\otimes \times_{\mathcal{O}^\otimes} \mathcal{P}^\otimes$ and $\mathcal{E}^\otimes \times_{\mathcal{O}^\otimes} \mathcal{P}^\otimes$ are detected in \mathcal{D}^\otimes and \mathcal{E}^\otimes , respectively.

Similarly, we can pull back $F^\otimes: \mathcal{D}^\otimes \rightleftarrows \mathcal{E}^\otimes : G^\otimes$ to an adjunction $F_{\langle 1 \rangle}^\otimes: \mathcal{D} \rightleftarrows \mathcal{E} : G_{\langle 1 \rangle}^\otimes$. Moreover, we can lift $F_{\langle 1 \rangle}^\otimes \dashv G_{\langle 1 \rangle}^\otimes$ to an adjunction

$$\mathrm{Alg}_{\mathcal{P}/\mathcal{O}}(F^\otimes): \mathrm{Alg}_{\mathcal{P}/\mathcal{O}}(\mathcal{D}) \rightleftarrows \mathrm{Alg}_{\mathcal{P}/\mathcal{O}}(\mathcal{E}) : \mathrm{Alg}_{\mathcal{P}/\mathcal{O}}(G^\otimes)$$

by considering $\mathrm{Alg}_{\mathcal{P}/\mathcal{O}}(-)$ as a full subcategory of $\mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{P}^\otimes, -)$, and applying F^\otimes , G^\otimes , the unit transformation and the counit transformation “objectwise” (cf. [Lei22, Proposition E.3.3.1]).

We now move on to the main result of this appendix.

Proposition B.0.6. *Assume that we are given the following.*

- \mathcal{LM} -monoidal categories \mathcal{D} and \mathcal{E} (exhibiting $\mathcal{D}_\mathfrak{m}$ as left-tensored over $\mathcal{D}_\mathfrak{a}$ and $\mathcal{E}_\mathfrak{m}$ as left-tensored over $\mathcal{E}_\mathfrak{a}$, respectively).
- An adjunction $F^\otimes: \mathcal{D}^\otimes \rightleftarrows \mathcal{E}^\otimes : G^\otimes$ of lax \mathcal{LM}^\otimes -monoidal functors.
- An algebra $R \in \mathrm{Alg}(\mathcal{D}_\mathfrak{a})$.

Then there exists a lifted adjunction

$$\begin{array}{ccc} \mathrm{LMod}_R(\mathcal{D}_\mathfrak{m}) & \begin{array}{c} \xrightarrow{F_R} \\ \perp \\ \xleftarrow{G_R} \end{array} & \mathrm{LMod}_{F(R)}(\mathcal{E}_\mathfrak{m}) \\ \mathrm{U}_R \downarrow & & \downarrow \mathrm{U}_{F(R)} \\ \mathcal{D}_\mathfrak{m} & \begin{array}{c} \xrightarrow{F_\mathfrak{m}} \\ \perp \\ \xleftarrow{G_\mathfrak{m}} \end{array} & \mathcal{E}_\mathfrak{m} \end{array} \quad .$$

Proof. As discussed in Remark B.0.5, we can pull back the adjunction $F^\otimes \dashv G^\otimes$ along the ∞ -operad map $\mathbb{E}_1^\otimes \rightarrow \mathcal{LM}^\otimes$ to obtain an adjunction $F_\mathbb{E}: \mathcal{D}_\mathfrak{a}^\otimes \rightleftarrows \mathcal{E}_\mathfrak{a}^\otimes : G_\mathbb{E}$ of lax monoidal functors. Moreover, $F^\otimes \dashv G^\otimes$ gives rise another adjunction

$$F_{\mathrm{LMod}}: \mathrm{LMod}(\mathcal{D}_\mathfrak{m}) = \mathrm{Alg}_{/\mathcal{LM}}(\mathcal{D}) \rightleftarrows \mathrm{Alg}_{/\mathcal{LM}}(\mathcal{E}) = \mathrm{LMod}(\mathcal{E}_\mathfrak{m}) : G_{\mathrm{LMod}},$$

which is a lift of $F_{\langle 1 \rangle}^\otimes \dashv G_{\langle 1 \rangle}^\otimes$ (and hence in particular of $F_\mathfrak{m}^\otimes \dashv G_\mathfrak{m}^\otimes$). Similarly, we obtain an adjunction $F_{\mathrm{Alg}}: \mathrm{Alg}(\mathcal{D}_\mathfrak{a}) \rightleftarrows \mathrm{Alg}(\mathcal{E}_\mathfrak{a}) : G_{\mathrm{Alg}}$ lifting $F_\mathfrak{a} \dashv G_\mathfrak{a}$. By their “objectwise”

description, these adjunctions fit into a tower

$$\begin{array}{ccc}
\mathrm{LMod}(\mathcal{D}_{\mathfrak{m}}) & \begin{array}{c} \xrightarrow{F_{\mathrm{LMod}}} \\ \perp \\ \xleftarrow{G_{\mathrm{LMod}}} \end{array} & \mathrm{LMod}(\mathcal{E}_{\mathfrak{m}}) \\
\theta_{\mathcal{D}} \downarrow & \begin{array}{c} \xrightarrow{F_{\mathrm{Alg}}} \\ \perp \\ \xleftarrow{G_{\mathrm{Alg}}} \end{array} & \downarrow \theta_{\mathcal{E}} \\
\mathrm{Alg}(\mathcal{D}_{\mathfrak{a}}) & & \mathrm{Alg}(\mathcal{E}_{\mathfrak{a}}) \\
\downarrow & \begin{array}{c} \xrightarrow{F_{\mathfrak{a}}} \\ \perp \\ \xleftarrow{G_{\mathfrak{a}}} \end{array} & \downarrow \\
\mathcal{D}_{\mathfrak{a}} & & \mathcal{E}_{\mathfrak{a}}
\end{array}$$

of lifts.

Considering fibers over $R \in \mathrm{Alg}(\mathcal{D}_{\mathfrak{a}})$, $F(R) \in \mathrm{Alg}(\mathcal{E}_{\mathfrak{a}})$ and $G(F(R)) \in \mathrm{Alg}(\mathcal{D}_{\mathfrak{a}})$, we obtain functors $F_R: \mathrm{LMod}_R(\mathcal{D}_{\mathfrak{m}}) \rightarrow \mathrm{LMod}_{F(R)}(\mathcal{E}_{\mathfrak{m}})$ and $G'_R: \mathrm{LMod}_{F(R)}(\mathcal{E}_{\mathfrak{m}}) \rightarrow \mathrm{LMod}_{G(F(R))}(\mathcal{D}_{\mathfrak{m}})$ lifting $F_{\mathfrak{m}}$ and $G_{\mathfrak{m}}$, respectively. Moreover, note that the restriction of scalars $u_R^*: \mathrm{LMod}_{G(F(R))}(\mathcal{D}_{\mathfrak{m}}) \rightarrow \mathrm{LMod}_R(\mathcal{D}_{\mathfrak{m}})$ along the unit $u_R: R \rightarrow G(F(R))$ associated to the adjunction $F_{\mathrm{Alg}} \dashv G_{\mathrm{Alg}}$ is a lift of $\mathrm{Id}_{\mathcal{D}_{\mathfrak{m}}}$. Therefore, setting $G_R := u_R^* \circ G'_R$, we obtain a functor $\mathrm{LMod}_{F(R)}(\mathcal{E}_{\mathfrak{m}}) \rightarrow \mathrm{LMod}_R(\mathcal{D}_{\mathfrak{m}})$ that is also a lift of $G_{\mathfrak{m}}$.

For $S \in \mathrm{Alg}(\mathcal{D}_{\mathfrak{a}})$, let $\iota_S: \mathrm{LMod}_S(\mathcal{D}_{\mathfrak{m}}) \rightarrow \mathrm{LMod}(\mathcal{D}_{\mathfrak{m}})$ denote the inclusion (and similarly for algebras in $\mathcal{E}_{\mathfrak{a}}$). Let $v: \iota_R u_R^* \rightarrow \iota_{G(F(R))}$ be a natural transformation whose component at $K \in \mathrm{LMod}_{G(F(R))}(\mathcal{D}_{\mathfrak{m}})$ is given by a cartesian lift of $u_R: \theta_{\mathcal{D}}(u_R^*(K)) \simeq R \rightarrow G(F(R)) \simeq \theta_{\mathcal{D}}(K)$. Let $c_{\mathrm{LMod}}: F_{\mathrm{LMod}} G_{\mathrm{LMod}} \rightarrow \mathrm{Id}_{\mathrm{LMod}(\mathcal{E})}$ be the counit of the adjunction $F_{\mathrm{LMod}} \dashv G_{\mathrm{LMod}}$. We define the counit $c_R: F_R G_R \rightarrow \mathrm{Id}_{\mathrm{LMod}_{F(R)}(\mathcal{E})}$ as the restriction of the composite

$$\begin{aligned}
\iota_{F(R)} \circ F_R \circ G_R &= \iota_{F(R)} \circ F_R \circ u_R^* \circ G'_R \simeq F_{\mathrm{LMod}} \circ \iota_R \circ u_R^* \circ G'_R \\
&\xrightarrow{F_{\mathrm{LMod}} \circ v \circ G'_R} F_{\mathrm{LMod}} \circ \iota_{G(F(R))} \circ G'_R \simeq F_{\mathrm{LMod}} \circ G_{\mathrm{LMod}} \circ \iota_{F(R)} \\
&\xrightarrow{c_{\mathrm{LMod}} \circ \iota_{F(R)}} \iota_{F(R)}
\end{aligned}$$

to $\mathrm{LMod}_{F(R)}(\mathcal{D}_{\mathfrak{m}})$.

Now, as v is a lift of the identity transformation of $\mathrm{Id}_{\mathcal{D}_{\mathfrak{m}}}$ and c_{LComod} is a lift the counit of $F_{\mathfrak{m}} \dashv G_{\mathfrak{m}}$, c_R is a lift of the counit of $F_{\mathfrak{m}} \dashv G_{\mathfrak{m}}$ as well. Hence, by Remark B.0.2, it is enough to show that c_R is the counit of an adjunction $F_R \dashv G_R$. It will therefore suffice to show that for all $M \in \mathrm{LMod}_R(\mathcal{D}_{\mathfrak{m}})$ and $N \in \mathrm{LMod}_{F(R)}(\mathcal{E}_{\mathfrak{m}})$, the composite

$$\mathrm{Map}_R(M, G_R N) \xrightarrow{F_R} \mathrm{Map}_{F(R)}(F_R M, F_R G_R N) \xrightarrow{(c_R)^*} \mathrm{Map}_{F(R)}(F_R M, N) \quad (\text{B.0.7})$$

is an equivalence.

On the level of LMod and Alg , the relevant mapping spaces fit into a commutative

diagram²⁵

$$\begin{array}{ccccc}
[M, u_R^* G_{\text{LMod}} N] & \xrightarrow{\theta_{\mathcal{D}}} & [R, R] & \xrightarrow{F_{\text{Alg}}} & [F_{\text{Alg}} R, F_{\text{Alg}} R] \\
\downarrow v_* & \searrow F_{\text{LMod}} & \downarrow (u_R)_* & \searrow F_{\text{Alg}} & \downarrow F_{\text{Alg}}(u_R)_* \\
& [F_{\text{LMod}} M, F_{\text{LMod}} u_R^* G_{\text{LMod}} N] & \xrightarrow{\theta_{\mathcal{E}}} & [F_{\text{Alg}} R, F_{\text{Alg}} R] & \\
& \downarrow F_{\text{LMod}} v_* & \downarrow & \downarrow & \\
[M, G_{\text{LMod}} N] & \xrightarrow{\theta_{\mathcal{D}}} & [R, G_{\text{Alg}} F_{\text{Alg}} R] & \xrightarrow{F_{\text{Alg}}} & [F_{\text{Alg}} R, F_{\text{Alg}} G_{\text{Alg}} F_{\text{Alg}} R] \\
\downarrow F_{\text{LMod}} & \searrow F_{\text{LMod}} v_* & \downarrow & \searrow F_{\text{Alg}} & \downarrow F_{\text{Alg}}(u_R)_* \\
& [F_{\text{LMod}} M, F_{\text{LMod}} G_{\text{LMod}} N] & \xrightarrow{\theta_{\mathcal{E}}} & [F_{\text{Alg}} R, F_{\text{Alg}} G_{\text{Alg}} F_{\text{Alg}} R] & \\
& \downarrow (c_{\text{LMod}})_* & \downarrow & \downarrow (c_{\text{Alg}})_* & \\
& [F_{\text{LMod}} M, N] & \xrightarrow{\theta_{\mathcal{E}}} & [F_{\text{Alg}} R, F_{\text{Alg}} R] & \\
& \nearrow \simeq & & \nearrow \simeq &
\end{array} \quad . \tag{B.0.8}$$

Since $v_{G_{\text{LMod}} N}$ was chosen to be a cartesian lift of u_R and therefore $F_{\text{LMod}} v_{G_{\text{LMod}} N}$ is a cartesian lift of $F_{\text{Alg}} u_R$ (as its underlying morphism in \mathcal{E}_{m} is an equivalence), the upper front and back squares in this diagram are pullback squares by [Lur09, Proposition 2.4.4.3]. In particular, the maps induced by v_* and $F_{\text{LMod}} v_*$ on the fibers are equivalences.

Tracing $\text{Id}_R \in [R, R]$ on the right face of (B.0.8), we obtain a commutative diagram

$$\begin{array}{ccc}
\{\text{Id}_R\} & \xrightarrow{F_{\text{Alg}}} & \{\text{Id}_{F_{\text{Alg}} R}\} \\
(u_R)_* \downarrow & & \downarrow F_{\text{Alg}}(u_R)_* \\
\{u_R\} & \xrightarrow{F_{\text{Alg}}} & \{F_{\text{Alg}} u_R\} \\
& \searrow \simeq & \downarrow (c_{\text{Alg}})_* \\
& & \{\text{Id}_{F_{\text{Alg}} R}\}
\end{array} \quad ,$$

which, taking fibers in the horizontal direction in (B.0.8), yields a commutative diagram

$$\begin{array}{ccc}
\text{Map}_R(M, G_R N) & \xrightarrow{F_R} & \text{Map}_R(F_R M, F_R G_R N) \\
(v_{G_{\text{LMod}} N})_* \downarrow \simeq & & \simeq \downarrow F_{\text{LMod}}(v_{G_{\text{LMod}} N})_* \\
\text{fib}_{u_R}(\theta_{\mathcal{D}}) & \xrightarrow{F_{\text{LMod}}} & \text{fib}_{F_{\text{Alg}} u_R}(\theta_{\mathcal{E}}) \\
& \searrow \simeq & \downarrow (c_{\text{LMod}})_* \\
& & \text{Map}_{F(R)}(F_R M, N)
\end{array} \quad .$$

Now the composite of the top horizontal map and the right vertical maps is precisely the composite (B.0.7). As it is homotopic to the equivalence given by the composite of the left vertical map and the lower diagonal map, it is itself an equivalence. \square

²⁵Here we omit ι 's and use $[-, -]$ instead of $\text{Map}_?(-, -)$ to save space.

Next, we observe that the lifting process of Proposition B.0.6 “preserves comonadicity”.

Lemma B.0.9. *In the situation of Proposition B.0.6, if the adjunction $F_{\mathfrak{m}}: \mathcal{D}_{\mathfrak{m}} \rightleftarrows \mathcal{E}_{\mathfrak{m}} : G_{\mathfrak{m}}$ is comonadic, then so is the lifted adjunction*

$$F_R: \mathrm{LMod}_R(\mathcal{D}_{\mathfrak{m}}) \rightleftarrows \mathrm{LMod}_{F(R)}(\mathcal{E}_{\mathfrak{m}}) : G_R.$$

Proof. We would like to apply the Barr–Beck–Lurie comonadicity theorem ([Lur17, Theorem 4.7.3.5]), so we need to show that F_R is conservative, $\mathrm{LMod}_R(\mathcal{D}_{\mathfrak{m}})$ admits limits of F_R -split cosimplicial objects, and F_R preserves such limits. For this, we consider the commutative diagram

$$\begin{array}{ccc} \mathrm{LMod}_R(\mathcal{D}_{\mathfrak{m}}) & \xrightarrow{F_R} & \mathrm{LMod}_{F(R)}(\mathcal{E}_{\mathfrak{m}}) \\ \downarrow U_R & & \downarrow U_{F(R)} \\ \mathcal{D}_{\mathfrak{m}} & \xrightarrow{F_{\mathfrak{m}}} & \mathcal{E}_{\mathfrak{m}} \end{array} \quad .$$

Let f be a morphism in $\mathrm{LMod}_R(\mathcal{D}_{\mathfrak{m}})$ such that $F_R f$ is an equivalence. Then the morphism $U_{F(R)} F_R f \simeq F_{\mathfrak{m}} U_R f$ in $\mathcal{E}_{\mathfrak{m}}$ is also equivalence. Now, $F_{\mathfrak{m}}$ is conservative as the adjunction $F_{\mathfrak{m}} \dashv G_{\mathfrak{m}}$ is comonadic, so $U_R f$ is an equivalence. Moreover, U_R is conservative as the forgetful functor of a module category, so f is an equivalence. Hence F_R is conservative.

Now let $X: \Delta \rightarrow \mathrm{LMod}_R(\mathcal{D}_{\mathfrak{m}})$ be an F_R -split cosimplicial object. Then $U_{F(R)} F_R X \simeq F_{\mathfrak{m}} U_R X$ is also split, implying that $U_R X$ is $F_{\mathfrak{m}}$ -split. Hence, as $F_{\mathfrak{m}} \dashv G_{\mathfrak{m}}$ is comonadic, $U_R X$ admits a limit in $\mathcal{D}_{\mathfrak{m}}$. Now, by [Lur17, Corollary 4.2.3.3], U_R reflects limits, so X admits a limit as well.

The argument for preservation is similar. Let $X^{\triangleleft}: \Delta^{\triangleleft} \rightarrow \mathrm{LMod}_R(\mathcal{D}_{\mathfrak{m}})$ be a limiting cone for X . Then $U_R X^{\triangleleft}$ is a limiting cone in $\mathcal{D}_{\mathfrak{m}}$ as U_R also preserves limits by the aforementioned corollary. Since $F_{\mathfrak{m}} \dashv G_{\mathfrak{m}}$ is comonadic, $F_{\mathfrak{m}}$ preserves limits of $F_{\mathfrak{m}}$ -split cosimplicial objects, so $F_{\mathfrak{m}} U_R X^{\triangleleft}$ is a limiting cone in $\mathcal{E}_{\mathfrak{m}}$. Now the limit-reflecting functor $U_{F(R)}$ maps $F_R X^{\triangleleft}$ to the limiting cone $F_{\mathfrak{m}} U_R X^{\triangleleft}$, so $F_R X^{\triangleleft}$ is also a limiting cone. \square

We conclude our discussion by specializing Proposition B.0.6 to \mathcal{LM} -monoidal ∞ -categories induced by monoidal ∞ -categories.

Corollary B.0.10. *Let \mathcal{D} and \mathcal{E} be monoidal ∞ -categories and $F^{\otimes}: \mathcal{D}^{\otimes} \rightleftarrows \mathcal{E}^{\otimes} : G^{\otimes}$ an adjunction of lax monoidal functors. Then the adjunction $F: \mathcal{D} \rightleftarrows \mathcal{E} : G$ lifts to an adjunction*

$$F_R: \mathrm{LMod}_R(\mathcal{D}) \rightleftarrows \mathrm{LMod}_{F(R)}(\mathcal{E}) : G_R.$$

Moreover, if $F \dashv G$ is comonadic, then so is $F_R \dashv G_R$.

Proof. We consider the left-tensorings of \mathcal{D} and \mathcal{E} over themselves. As discussed in Remark B.0.5, we can pull back the adjunction $F^{\otimes} \dashv G^{\otimes}$ along the map $\mathcal{LM}^{\otimes} \rightarrow \mathbb{E}_1^{\otimes}$ of ∞ -operads to obtain an adjunction

$$F_{\mathcal{LM}}^{\otimes}: \mathcal{D}^{\otimes} \times_{\mathbb{E}_1^{\otimes}} \mathcal{LM}^{\otimes} \rightleftarrows \mathcal{E}^{\otimes} \times_{\mathbb{E}_1^{\otimes}} \mathcal{LM}^{\otimes} : G_{\mathcal{LM}}^{\otimes}$$

of lax \mathcal{LM} -monoidal functors. Now we can obtain the desired results by applying Proposition B.0.6 and Lemma B.0.9 to the adjunction $F_{\mathcal{LM}}^{\otimes} \dashv G_{\mathcal{LM}}^{\otimes}$. \square

References

- [ABG18] Matthew Ando, Andrew J. Blumberg, and David Gepner. “Parametrized spectra, multiplicative Thom spectra and the twisted Umkehr map”. English. In: *Geom. Topol.* 22.7 (2018), pp. 3761–3825. ISSN: 1465-3060. DOI: [10.2140/gt.2018.22.3761](https://doi.org/10.2140/gt.2018.22.3761) (cit. on pp. 42, 49, 50).
- [Bea16] Jonathan Beardsley. “Coalgebraic structure and Intermediate Hopf-Galois Extensions of Thom Spectra in Quasicategories”. PhD thesis. Johns Hopkins University, March 2016. URL: <https://jscholarship.library.jhu.edu/handle/1774.2/39729> (cit. on pp. 10, 52).
- [Bea21] Jonathan Beardsley. *On Bialgebras, Comodules, Descent Data and Thom Spectra in ∞ -categories*. 2021. arXiv: [1810.00734](https://arxiv.org/abs/1810.00734) [[math.AT](#)] (cit. on pp. 10, 13, 26, 40, 50).
- [BGN18] Clark Barwick, Saul Glasman, and Denis Nardin. “Dualizing cartesian and cocartesian fibrations”. In: *Theory Appl. Categ.* 33 (2018), pp. 67–94. ISSN: 1201-561X. URL: www.tac.mta.ca/tac/volumes/33/4/33-04abs.html (cit. on pp. 14, 67, 109–111).
- [BH18] Alexander Berglund and Kathryn Hess. “Homotopic Hopf-Galois extensions revisited.” In: *J. Noncommut. Geom.* 12.1 (2018), pp. 107–155. ISSN: 1661-6952; 1661-6960/e. DOI: [10.4171/JNCG/272](https://doi.org/10.4171/JNCG/272) (cit. on pp. 10, 54, 80).
- [Bou79] A. K. Bousfield. “The localization of spectra with respect to homology”. In: *Topology* 18 (1979), pp. 257–281. ISSN: 0040-9383. DOI: [10.1016/0040-9383\(79\)90018-1](https://doi.org/10.1016/0040-9383(79)90018-1) (cit. on p. 56).
- [BP19] Jonathan Beardsley and Maximilien Péroux. *Koszul Duality in Higher Topoi*. 2019. arXiv: [1909.11724](https://arxiv.org/abs/1909.11724) [[math.AT](#)] (cit. on p. 41).
- [GL21] David Gepner and Tyler Lawson. “Brauer groups and Galois cohomology of commutative ring spectra”. English. In: *Compos. Math.* 157.6 (2021), pp. 1211–1264. ISSN: 0010-437X. DOI: [10.1112/S0010437X21007065](https://doi.org/10.1112/S0010437X21007065) (cit. on p. 84).
- [Gre92] Cornelius Greither. *Cyclic Galois extensions of commutative rings*. Vol. 1534. Lect. Notes Math. Berlin: Springer-Verlag, 1992. ISBN: 3-540-56350-4. DOI: [10.1007/BFb0089165](https://doi.org/10.1007/BFb0089165) (cit. on p. 10).
- [Hes09] Kathryn Hess. “Homotopic Hopf-Galois extensions: foundations and examples.” In: *New topological contexts for Galois theory and algebraic geometry (BIRS 2008). Proceedings of the workshop, Banff, Canada, March 9–14, 2008*. Coventry: Geometry & Topology Publications, 2009, pp. 79–132. DOI: [10.2140/gtm.2009.16.79](https://doi.org/10.2140/gtm.2009.16.79) (cit. on pp. 10, 52).
- [HHLN21] Rune Haugseng, Fabian Hebestreit, Sil Linskens, and Joost Nuiten. *Lax monoidal adjunctions, two-variable fibrations and the calculus of mates*. 2021. arXiv: [2011.08808](https://arxiv.org/abs/2011.08808) [[math.CT](#)] (cit. on p. 16).

- [HMS22] Rune Haugseng, Valerio Melani, and Pavel Safronov. “Shifted coisotropic correspondences”. English. In: *J. Inst. Math. Jussieu* 21.3 (2022), pp. 785–849. ISSN: 1474-7480. DOI: [10.1017/S1474748020000274](https://doi.org/10.1017/S1474748020000274) (cit. on p. 41).
- [Hop41] Heinz Hopf. “Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen”. German. In: *Ann. Math. (2)* 42 (1941), pp. 22–52. ISSN: 0003-486X. DOI: [10.2307/1968985](https://doi.org/10.2307/1968985) (cit. on p. 9).
- [Kar14] Varvara Karpova. “Homotopic Hopf-Galois extensions of commutative differential graded algebras”. PhD thesis. École Polytechnique Fédérale de Lausanne, June 2014. URL: <https://infoscience.epfl.ch/record/199513> (cit. on pp. 10, 52, 80).
- [Kra18] Achim Krause. “Periodicity in motivic homotopy theory and over BP^*BP ”. PhD thesis. Rheinische Friedrich-Wilhelms-Universität Bonn, July 2018. URL: <http://hdl.handle.net/20.500.11811/7592> (cit. on pp. 28, 29, 31).
- [KT81] H. F. Kreimer and M. Takeuchi. “Hopf algebras and Galois extensions of an algebra”. In: *Indiana Univ. Math. J.* 30 (1981), pp. 675–692. ISSN: 0022-2518 (cit. on p. 10).
- [Lei22] Malte Leip. “On the Hochschild homology of hypersurfaces as a mixed complex”. PhD thesis. University of Copenhagen, July 2022. ISBN: 978-87-7125-055-8. URL: <http://malte-leip.net/phd/> (cit. on pp. 13, 23, 26, 114).
- [Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Princeton, NJ: Princeton University Press, 2009, pp. xv + 925. ISBN: 978-0-691-14049-0/pbk; 978-0-691-14048-3/hbk (cit. on pp. 10, 11, 14, 19, 21, 27, 37, 38, 42, 74, 84, 92, 99, 100, 104, 105, 111, 116).
- [Lur11] Jacob Lurie. *Derived Algebraic Geometry VII: Spectral Schemes*. November 2011. URL: <https://www.math.ias.edu/~lurie/papers/DAG-VII.pdf> (cit. on pp. 84, 85).
- [Lur17] Jacob Lurie. *Higher Algebra*. September 2017. URL: <https://www.math.ias.edu/~lurie/papers/HA.pdf> (cit. on pp. 10, 11, 13–19, 21, 23, 24, 26, 28–32, 34, 36–42, 52, 60, 63, 64, 67, 68, 71–73, 75, 80, 83, 86, 87, 97, 99, 100, 103–108, 112–114, 117).
- [Lur18] Jacob Lurie. *Elliptic Cohomology I: Spectral Abelian Varieties*. February 2018. URL: <https://www.math.ias.edu/~lurie/papers/Elliptic-I.pdf> (cit. on p. 64).
- [Mah79] Mark Mahowald. “Ring spectra which are Thom complexes”. In: *Duke Math. J.* 46 (1979), pp. 549–559. ISSN: 0012-7094. DOI: [10.1215/S0012-7094-79-04628-3](https://doi.org/10.1215/S0012-7094-79-04628-3) (cit. on p. 52).

- [Mat16] Akhil Mathew. “The Galois group of a stable homotopy theory”. English. In: *Adv. Math.* 291 (2016), pp. 403–541. ISSN: 0001-8708. DOI: [10.1016/j.aim.2015.12.017](https://doi.org/10.1016/j.aim.2015.12.017) (cit. on p. 34).
- [Mon93] Susan Montgomery. *Hopf algebras and their actions on rings*. Vol. 82. CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1993, pp. xiv+238. ISBN: 0-8218-0738-2. DOI: [10.1090/cbms/082](https://doi.org/10.1090/cbms/082) (cit. on pp. 57, 60, 64).
- [Pér20] Maximilien Péroux. “Highly Structured Coalgebras and Comodules”. PhD thesis. University of Illinois at Chicago, August 2020. DOI: [10.25417/uic.13475667.v1](https://doi.org/10.25417/uic.13475667.v1). URL: https://indigo.uic.edu/articles/thesis/Highly_Structured_Coalgebras_and_Comodules/13475667/1 (cit. on pp. 13, 41).
- [Por15] Hans-E. Porst. “The formal theory of Hopf algebras. Part I: Hopf monoids in a monoidal category”. In: *Quaest. Math.* 38.5 (2015), pp. 631–682. ISSN: 1607-3606. DOI: [10.2989/16073606.2014.981736](https://doi.org/10.2989/16073606.2014.981736) (cit. on p. 57).
- [PS18] Maximilien Péroux and Brooke Shipley. “Coalgebras in symmetric monoidal categories of spectra.” In: *Homology Homotopy Appl.* 21.1 (2018), pp. 1–18. ISSN: 1532-0073; 1532-0081/e (cit. on pp. 10, 70).
- [Rak20] Arpon Raksit. *Hochschild homology and the derived de Rham complex revisited*. 2020. arXiv: [2007.02576](https://arxiv.org/abs/2007.02576) [math.AG] (cit. on pp. 70, 103, 112).
- [Rog08] John Rognes. “Galois extensions of structured ring spectra. Stably dualizable groups.” In: *Mem. Am. Math. Soc.* 898 (2008), p. 137. ISSN: 0065-9266; 1947-6221/e (cit. on pp. 10, 52, 55, 68–70).
- [Rot09] Fridolin Roth. “Galois and Hopf-Galois Theory of Associative S -Algebras”. PhD thesis. Universität Hamburg, August 2009. URL: <https://ediss.sub.uni-hamburg.de/volltexte/2009/4292/index.html> (cit. on pp. 10, 52, 56, 57, 102).
- [RS20] Nima Rasekh and Bruno Stonek. “The cotangent complex and Thom spectra”. English. In: *Abh. Math. Semin. Univ. Hamb.* 90.2 (2020), pp. 229–252. ISSN: 0025-5858. DOI: [10.1007/s12188-020-00226-8](https://doi.org/10.1007/s12188-020-00226-8) (cit. on pp. 101, 102).
- [RSV19] Nima Rasekh, Bruno Stonek, and Gabriel Valenzuela. *Thom spectra, higher THH and tensors in ∞ -categories*. 2019. arXiv: [1911.04345](https://arxiv.org/abs/1911.04345) [math.AT] (cit. on pp. 21, 101, 102).
- [RV16] Emily Riehl and Dominic Verity. “Homotopy coherent adjunctions and the formal theory of monads”. In: *Adv. Math.* 286 (2016), pp. 802–888. ISSN: 0001-8708. DOI: [10.1016/j.aim.2015.09.011](https://doi.org/10.1016/j.aim.2015.09.011) (cit. on pp. 28, 33, 113).
- [RV20] Emily Riehl and Dominic Verity. “Elements of ∞ -Category Theory”. Original draft. August 2020. URL: <https://emilyriehl.github.io/files/more-elements.pdf> (cit. on pp. 28, 31, 34).

- [Sta63] James D. Stasheff. “Homotopy associativity of H -spaces. I, II”. In: *Trans. Am. Math. Soc.* 108 (1963), pp. 275–292, 293–312. ISSN: 0002-9947. DOI: [10.2307/1993608](https://doi.org/10.2307/1993608) (cit. on p. 9).
- [Tho54] René Thom. “Quelques propriétés globales des variétés différentiables”. French. In: *Comment. Math. Helv.* 28 (1954), pp. 17–86. ISSN: 0010-2571. DOI: [10.1007/BF02566923](https://doi.org/10.1007/BF02566923) (cit. on p. 9).
- [Tor21] Takeshi Torii. *On duoidal ∞ -categories*. 2021. arXiv: [2106.14121](https://arxiv.org/abs/2106.14121) [[math.CT](#)] (cit. on p. 103).

Curriculum vitae

Personal information

name **Aras Ergus**
date of birth **May 19, 1993**
place of birth **Osmangazi, Bursa, Turkey**

Employment

09/2017 – **Doctoral assistant**
08/2022 *École Polytechnique Fédérale de Lausanne* (Lausanne, Switzerland)
10/2014 – **Teaching assistant**
03/2017 *Rheinische Friedrich-Wilhelms-Universität Bonn* (Bonn, Germany)

Education

2017 – 2022 **PhD (mathematics)**
 École Polytechnique Fédérale de Lausanne (Lausanne, Switzerland)
 Thesis title: Hopf algebras and Hopf–Galois extensions in ∞ -categories
 Advisor: Kathryn Hess
2015 – 2017 **MSc (mathematics)**
 Rheinische Friedrich-Wilhelms-Universität Bonn (Bonn, Germany)
 Thesis title: Operads, Duality and the Gravity Operad
 Advisor: Yuri Manin
2012 – 2015 **BSc (mathematics)**
 Rheinische Friedrich-Wilhelms-Universität Bonn (Bonn, Germany)
 Thesis title: Loop Objects in Pointed Derivators
 Advisor: Moritz Rahn (formerly Moritz Groth)
2007 – 2012 **High school**
 Istanbul Lisesi (Istanbul, Turkey)

Seminar talks

05/2022 **Homotopy coherent Hopf algebras**
 EPFL Topology seminar
10/2020 **Localization and Dévissage for Algebraic K-Theory**
 electronic Computational Homotopy Theory K-Theory Reading Seminar
12/2019 **Computation of π_*TMF via the Descent and Adams–Novikov Spectral Sequences**
 Bayerische Arbeitsgemeinschaft: Topological Modular Forms
12/2019 **The localization of spectra with respect to homology**
 electronic Computational Homotopy Theory Kan Seminar

- 03/2019 **The moduli stack \mathcal{M}_{FG} of formal groups & the MU -construction**
 Bayerische kleine Arbeitsgemeinschaft: Towards Chromatic Homotopy Theory – The Landweber Exact Functor Theorem
- 02/2018 **The Dennis trace map**
 EPFL Topology seminar

Teaching experience

- 2017 – 2022 **Teaching assistantship**
École Polytechnique Fédérale de Lausanne
 Courses assisted: Group theory, Algebraic topology, Algebraic K-theory, Metric and topological spaces, Homotopical algebra.
- 2019 – 2021 **Semester project supervision**
École Polytechnique Fédérale de Lausanne
 Projects: “Homology and Cohomology of Groups” by Ilan Perez, “Steenrod Algebra” by Sacha Bors, “Bordism: an extraordinary homology theory” by Max Duparc.
- 2013 – 2021 **Talks and courses for high school and early bachelor students**
Quod Erat Demonstrandum e.V.
 Course topics: Geometric group theory (with Robin Stoll), Knot theory (with Robin Stoll), Scissors congruence, Combinatorial topology, Functional programming with Haskell, Type theory.
- 2014 – 2017 **Teaching assistantship**
Rheinische Friedrich-Wilhelms-Universität Bonn
 Courses assisted: Representation theory II, Representation theory I, Foundations of representation theory, Models of set theory I, Set theory.