Elastodynamics
notes for earthquake source physics

version 0.4

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Notations

We use a Cartesian (otherwise stated) reference frame with vector basis $\boldsymbol{e}_i$ - such that the position vector $\boldsymbol{x} = x_i \boldsymbol{e_i}$. We use the convention of summation on repeated indices. and will use a dot $\dot{}$ to denote the time-derivative, use and write spatial derivatives as $\partial_{x_i}\cdot$, or either as $\cdot_{,i}$. Vector notation with lower case bold for vector may also be use, similarly for the gradient ($\boldsymbol{\nabla}$) and divergence operator ($\nabla\cdot$).

1 Preliminaries

The linear balance of momentum for a solid of density ρ reads in local form

$$\rho \ddot{u}_i = \partial_{x_j}\sigma_{ij} + f_i$$

where u_i is the displacement vector, σ_{ij} the stress tensor and f_i a body forces. This field equation is valid everywhere in the solid domain Ω of boundaries Γ. This is nothing else that Newton's law applied to a deformable body where forces acting on a surface (internal or external to the body) derived from a second order tensor describing the internal stress in the solid. The angular balance of momentum imply the symmetry of the stress tensor $\sigma_{ij} = \sigma_{ji}$.

The following boundaries conditions can be applied (displacement or velocities, and tractions)

$$\begin{aligned} &u_i = u_i^g \qquad \text{on } \Gamma_{u_i} \qquad \text{(or similarly on velocity } \dot{u}_i) \\ &T_i(n_i) = \sigma_{ij} n_j \qquad \text{on } \Gamma_{t_i} \text{with normal } n_i \end{aligned}$$

with $\Gamma = \Gamma_{u_i} \bigcup \Gamma_{t_i}$, $i = 1, 3$. In addition, we usually deal with initial "quiet" conditions, i.e.

$$u_i(\boldsymbol{x}, t = 0) = \dot{u}_i(\boldsymbol{x}, t = 0) = 0$$

Linear small strain elastic behavior provide the following strain-stress relation

$$\sigma_{ij} - \sigma^o_{ij} = c_{ijkl}\epsilon_{kl}$$

where σ^o_{ij} is an initial stress field (valid at $t = 0$, solution of a static problem i.e. thus satisfying equilibrium), and ϵ_{ij} is the small-strain tensor defined as

$$\epsilon_{ij} = \frac{1}{2}\left(\partial_{x_j} u_i + \partial_{x_i} u_j\right)$$

. The elastic stiffness tensor c_{ijkl} has the following symmetries:

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{jilk}.$$

Due to symmetry, we have $c_{ijkl}\epsilon_{kl} = c_{ijkl}u_{k,l}$. For an isotropic material, we have

$$c_{ijkl} = (k - 2/3g)\delta_{ij}\delta_{kl} + g(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

where g is the shear modulus and k the bulk (volumetric) modulus. Note that in rate form, we have

$$\dot{\sigma}_{ij} = c_{ijkl}\dot{\epsilon}_{kl} = c_{ijkl}\dot{u}_{k,l}$$

1.1 Maxwell-Betti theorem

Multiplying the balance of momentum by a displacement field v_i (solution of another elastodynamics problem with body forces g_i instead of f_i), integrating over the domain and using the divergence theorem, we obtain

$$\int_\Omega v_i\left(\rho\ddot{u}_i - f_i\right)\mathrm{d}V - \int_\Gamma v_i T_i(n_i, u_i)\mathrm{d}S + \int_\Omega v_{i,j}\sigma_{ij}(u_i)\mathrm{d}V = 0$$

Similarly for the problem with v_i as the displacement field solution, we multiply by u_i and obtain

$$\int_\Omega u_i\left(\rho\ddot{v}_i - g_i\right)\mathrm{d}V - \int_\Gamma u_i T_i(n_i, v_i)\mathrm{d}S + \int_\Omega u_{i,j}\sigma_{ij}(v_i)\mathrm{d}V = 0$$

In the absence of initial stress or when they are absorbed in the body forces vector / or traction (which will always be the case), using the elastic constitutive law, we obtain

$$\int_\Omega v_i\left(\rho\ddot{u}_i - f_i\right) - u_i\left(\rho\ddot{v}_i - g_i\right)\mathrm{d}V = \int_\Gamma v_i T_i(n_i, u_i) - u_i T_i(n_i, v_i)\mathrm{d}S$$

which states the reciprocity of two elastodynamic states.

Note that the previous relation holds for any times for either u_i and v_i. Writing $t_1 = t$ for u_i and $t_2 = \tau - t$ for v_i , we can first show that

$$\int_0^\tau \rho(\ddot{u}_i(t)v_i(\tau - t) - \ddot{v}_i(\tau - t)u_i(t))\mathrm{d}t = 0$$

when both states are “quiet” initially.

We can then by integrating in time obtain the following Maxwell-Betti theorem for elastodynamics:

$$\int_0^\tau\int_\Omega\left[\rho u_i(\boldsymbol{x}, t)g_i(\boldsymbol{x}, \tau - t) - \rho v_i(\boldsymbol{x}, \tau - t)f_i(\boldsymbol{x}, t)\right]\mathrm{d}\Omega\mathrm{d}t =$$
$$\int_0^\tau\int_\Omega\left[v_i(\boldsymbol{x}, \tau - t)T_i(u_i, \boldsymbol{x}, \boldsymbol{n}, t) - u_i(\boldsymbol{x}, t)T_i(v_i, \boldsymbol{x}, \boldsymbol{n}, \tau - t)\right]\mathrm{d}S\mathrm{d}t$$

1.2 Green’s Point source function & representation

We denote $G_i^k(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau)$ the displacement at $\boldsymbol{x}$ for time t due to an unit impulse point force applied at position $\boldsymbol{\xi}$ and time τ in the k-direction, i.e solution of the elastodynamics problem

$$\rho\ddot{G}_i^k(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) = \delta_{ik}\delta(\boldsymbol{x} - \boldsymbol{\xi})\delta(t - \tau) + \partial_{x_j}\left(c_{ijkl}\partial_{x_l}G_i^k\right)$$

with quiet initial condition, (and some given boundary conditions for the surface of the domain). Maxwell-betti theorem do provide some relation, notably

$$G_i^k(\boldsymbol{x}, \tau; \boldsymbol{y}, 0) = G_k^i(\boldsymbol{y}, \tau; \boldsymbol{x}, 0)$$

Note that G_i^k is for an unit impulse point force and as such has dimension of 1 / (meter stress) (we will see that more clearly when deriving such a fundamental solution).

1.3 Displacement representation

Using the unit impulse Green's function for the domain as the solution v_i in the time-integrated Maxwell betti theorem, we obtain (for $g_i = \delta_{ik}\delta(\boldsymbol{x}-\boldsymbol{\xi})\delta(t-\tau)$):

$$
\begin{aligned}
u_k(\boldsymbol{\xi},\tau) &= \int_0^\tau \mathrm{d}t \int_\Omega f_i(\boldsymbol{x},t) G_i^k(\boldsymbol{x},\tau-t;\boldsymbol{\xi},0)\mathrm{d}\Omega_x \\
&\quad + \int_0^\tau \mathrm{d}t \int_\Gamma \left[G_i^k(\boldsymbol{x},\tau-t;\boldsymbol{\xi},0)T_i(u_i(\boldsymbol{x},t),\boldsymbol{n}) - u_i(\boldsymbol{x},t)c_{ijmn}n_j(\boldsymbol{x})G_{m,n}^k(\boldsymbol{x},\tau-t;\boldsymbol{\xi},0)\right]\mathrm{d}\Gamma_x
\end{aligned}
$$

In other words, if we know the solution for the displacement u_i and traction T_i on the boundaries of the solid domain Γ, knowing the Green's function, we can compute the displacement everywhere! Note that we can differentiate in time to obtain the velocity (acceleration), or differentiate in space to obtain the strain and then the stress. Note that such a representation formula holds strictly for $\boldsymbol{\xi}$ inside the domain (and the Green's function are singular as $\boldsymbol{\xi} \to \boldsymbol{x}$).

In some simple cases, for example for geophysics application, the surface of the earth is free of traction (such that the term with T_i disappears for the free surface). In addition, if we use an appropriate Green's function incorporating the free-surface the last term also vanishes (on the part of Γ associated with the free surface).

Notation-wise, we can switch ξ and $\boldsymbol{x}$, τ and t (this is often done).

2 Representation of earthquakes sources

2.1 Displacement discontinuity across a fault surface

A fault can be considered as an internal surface which can host relative motions. Let denote the fault surface as Γ_f with normal $n_i = -n_i^+ = n_i^-$ (we have chosen the normal to be n_i^- directed from Γ_f^- toward Γ_f^+as in Aki K. & Richards P. G. (2002)), the displacement jump

$$\Delta u_i = u_i^+ - u_i^-$$

represent the relative motion - it can be a function of space (localized on the fault) and time. It is important to note that to ensure the balance of momentum in the medium, the traction vector must be continuous across this internal surface, i.e.

$$\sigma_{ij}^+ n_j^+ = \sigma_{ij}^- n_j^-$$

in other words

$$T_i(u_i, n_i^+) = T_i(u_i, n_i^-).$$

Assuming for simplicity that the body forces f_i are absent, and assuming the case of a "full" space for which they are no boundaries beside the fault, we obtain after summing the integrals for the positive and negative sides of the fault, the displacement representation formula as:

$$u_k(\boldsymbol{\xi},\tau) = \int_0^\tau \mathrm{d}t \int_{\Gamma_f} \left[\Delta u_i(\boldsymbol{x},t)c_{ijmn}n_j(\boldsymbol{x})G_{m,n}^k(\boldsymbol{x},\tau-t;\boldsymbol{\xi},0)\right]\mathrm{d}\Gamma_x \tag{1}$$

[Note a minus sign would have been obtained if we chose the normal to be n_i^+ directed from Γ_f^+ toward Γ_f^-]. This expression relates fault movement with observable displacement (or velocity / acceleration if we take its derivatives).

Note that a-priori, we do not know the fault orientation, such that what we can "invert" from observables is the quantity $\Delta u_i(\boldsymbol{x},t)n_j$, more generally in seismology, we invert for

$$m_{mn} = c_{ijmn}\Delta u_i(\boldsymbol{x},t)n_j$$

which is referred to as the moment tensor (dimension of stress).

2.2 Transformation strain and moment tensor

We can take an alternative point of view, and think that the source of the earthquake can be represent by a transformation strain ϵ^*_{kl} occuring in the solid domain - We could think of a volumetric source due to an underground explosion, or an earthquake (we will see the connection with the above sub-section).

Let ϵ^*_{kl} be an irrerversible transformation strain located in the domain, the elastic stress-strain relation becomes (neglecting any initial static stress - as in elasticity we can always solve for the mju7y8tu09098'perturbed state only)

$$\sigma_{ij} = c_{ijkl}(\epsilon_{kl} - \epsilon^*_{kl})$$

such that the balance of momentum can be re-written in the absence of any body forces

$$\rho \ddot{u}_i = \partial_{x_j} c_{ijkl}\epsilon_{kl} - \partial_{x_j} c_{ijkl}\epsilon^*_{kl}$$

we thus see that the effect of the transformation strain is similar to a body force

$$f_i = -\partial_{x_j} c_{ijkl}\epsilon^*_{kl}$$

Let introduce the moment tensor as the transformation stress

$$m_{ij} = c_{ijkl}\epsilon^*_{kl}$$

The problem can be recasted in an equivalent elastic problem with body forces, i.e. solution of

$$\rho \ddot{u}_i = \left(c_{ijkl}u_{k,l}\right)_{,j} - m_{ij,j}$$

For a problem where some of the surfaces are free of traction $T_i = \sigma_{ij}n_j$, the traction boundary condition for the transformed problem becomes

$$T_i = c_{ijkl}u_{k,l}n_j = m_{ij}n_j$$

Taking as example, such a case of a free surface, we can express the representation formula for that problem as:

$$u_k(\boldsymbol{\xi},\tau) = -\int_0^\tau \mathrm{d}t \int_\Omega m_{ij,j}(\boldsymbol{x},t)G^k_i(\boldsymbol{x},\tau-t;\boldsymbol{\xi},0)\mathrm{d}\Omega_x + \int_0^\tau \mathrm{d}t \int_\Gamma \left[G^k_i(\boldsymbol{x},\tau-t;\boldsymbol{\xi},0)m_{ij}(\boldsymbol{x},t)n_j(\boldsymbol{x}) - u_i(\boldsymbol{x},t)c_{ijmn}n_j(\boldsymbol{x})G^k_{m,n}(\boldsymbol{x},\tau-t;\boldsymbol{\xi},0)\right]\mathrm{d}\Gamma_x$$

Using the divergence theorem and the chain rules $((a_ib_{ij})_{,j} = a_{i,j}b_{ij} + a_ib_{ij,j}$, we obtain

$$u_k(\boldsymbol{\xi},\tau) = \int_0^\tau \mathrm{d}t \int_\Omega m_{ij}(\boldsymbol{x},t)G^k_{i,j}(\boldsymbol{x},\tau-t;\boldsymbol{\xi},0)\mathrm{d}\Omega_x - \int_0^\tau \mathrm{d}t \int_\Gamma \left[u_i(\boldsymbol{x},t)c_{ijmn}n_j(\boldsymbol{x})G^k_{m,n}(\boldsymbol{x},\tau-t;\boldsymbol{\xi},0)\right]\mathrm{d}\Gamma_x$$

For the case of an infinite medium or when using a Green's function satisfying traction free boundary conditions on Γ - the second integral disappears. Focusing on that case particularly relevant for earthquake, we observe that the representation formula has a similar expression that Eq.(1) with the difference that it is now a volume integral.

Localized transformation strain

The transformation strain ϵ^*_{kl} (and its associated transformation stress m_{ij}) acts in an usually small subdomain Ω^* of Ω. Restricting to Green's function with zero traction on outer boundaries, we obtain

$$u_k(\boldsymbol{\xi},\tau) = \int_0^\tau \mathrm{d}t \int_{\Omega^*} m_{ij}(\boldsymbol{x},t)G^k_{i,j}(\boldsymbol{x},\tau-t;\boldsymbol{\xi},0)\mathrm{d}\Omega_x$$

Let's assume that the transformation strain occurs on a fault of finite but negliglible thickness h. The model of a crack or fault as a surface hosting a displacement discontinuity Δu_i can be viewed as a limit of a distribution of transformation strain ϵ^*_{ij} over a narrow zone. One lets the thickness of the fault goes to zero with appropriate component of the transformation strain then going to infinity. Such a transformation eigenstrain in a surface Γ_f of normal n_i with a "finite thickness" h (such that $\mathrm{Meas}(\Omega^*) = \mathrm{Meas}(\Gamma_f) \times h$) can be written as

$$\epsilon_{ij}^* = \frac{1}{2h} \left(\Delta u_i n_j + n_i \Delta u_j \right) \delta_{dirac}(\Gamma_f \pm h/2)$$

where $\delta_{dirac}(\Gamma_f \pm h/2)$ denotes the volume dirac delta for such a finite thickness fault. Assuming uniform transformation across the fault thickness, the previous representation formula can be re-written as

$$u_k(\boldsymbol{\xi}, \tau) = \int_0^\tau \mathrm{d}t \int_{\Gamma_f} c_{ijkl} \left(\frac{1}{2} \left(\Delta u_k(\boldsymbol{x}, t) n_l(\boldsymbol{x}) + n_l(\boldsymbol{x}) \Delta u_k(\boldsymbol{x}, t) \right) \right) G_{i,j}^k(\boldsymbol{x}, \tau - t; \boldsymbol{\xi}, 0) \mathrm{d}\Gamma_x$$

Note that due to the symmetry of the elastic stiffness tensor,

$$c_{ijkl} \left(\frac{1}{2} \left(\Delta u_k(\boldsymbol{x}, t) n_l(\boldsymbol{x}) + n_l(\boldsymbol{x}) \Delta u_k(\boldsymbol{x}, t) \right) \right) = c_{ijkl} \Delta u_k(\boldsymbol{x}, t) n_l(\boldsymbol{x})$$

and we recover exactly the expression of the displacement representation obtained from the approach based directly on displacement discontinuity across the fault surface.

Note that it is usual to write the transformation strain for a zero thickness fault as

$$\epsilon_{ij}^* = \frac{1}{2} \left(\Delta u_i n_j + n_i \Delta u_j \right) \delta_{dirac}(\Gamma_f)$$

(which as dimension of displacement).

2.3 Examples of infinetesimal source mechanims

Let us look at source mechanisms, i.e. $m_{ij} = c_{ijkl} \epsilon_{kl}^*$. First, we will restrict to an isotropic medium for clarity

$$m_{ij} = 2g\epsilon_{ij}^* + (k - 2/3g)\epsilon_{kk}^* \delta_{ij}$$

2.3.1 Explosive type

The transformation eigenstrain in 3D is akin to a center of dilation (explosion type)

$$\epsilon_{ij}^* = \frac{1}{3} \frac{\Delta V}{V} \delta_{ij}$$

$$m_{ij} = k \frac{\Delta V}{V} \delta_{ij}$$

2.3.2 Fault

In the reference frame of the fault with normal $\boldsymbol{n}$ aligned along $\boldsymbol{e}_3$.

Pure shear displacement discontinuity (called double-couple in seismology) in direction l_i (such that $l_k n_k = 0$), we can write for a slip $\Delta u_i = \Delta u l_i$

$$\epsilon_{ij}^* = \frac{\Delta u}{2} \left(l_i n_j + n_i l_j \right)$$

thus

$$m_{ij} = g \Delta u \left(l_i n_j + n_i l_j \right)$$

in matrix form with the fault normal $\boldsymbol{n}$ aligned along $\boldsymbol{e}_3$, and slip direction $\boldsymbol{l}$ equals to e_1

$$\boldsymbol{m} = \left[\begin{array}{ccc} 0 & 0 & g\Delta u \\ 0 & 0 & 0 \\ g\Delta u & 0 & 0 \end{array} \right]$$

Pure Opening mode crack

$$\epsilon^*_{ij} = \Delta u \times (n_i n_j)$$
$$m_{ij} = \Delta u \times (2g n_i n_j + (k - 2/3g)\delta_{ij})$$

in matrix form with the fault normal $\boldsymbol{n}$ aligned along $\boldsymbol{e}_3$,

$$\boldsymbol{m} = \Delta u \begin{bmatrix} k-2/3g & 0 & 0 \\ 0 & k-2/3g & 0 \\ 0 & 0 & k+4/3g \end{bmatrix}$$

2.4 Finite source seen as a point in the far-field

The displacement due to a moment tensor source (either in a fault surface as displacement discontinuity or within a finite volumetric domain as transformation strain can be obtained as (with the proper Green's function satisfying the boundary condition)

$$u_k(\boldsymbol{\xi}, \tau) = \int_0^\tau \mathrm{d}t \int_{\Gamma_f} m_{ij}(\boldsymbol{x}, t) G^k_{i,j}(\boldsymbol{x}, \tau - t; \boldsymbol{\xi}, 0)\mathrm{d}\Gamma_x$$

Let's assume that Γ_f the part of faul that ruptures, has a characteristic lengthscale ℓ_*, and denotes $\boldsymbol{\xi}_o$ the "center" of this rupture area. For a point $\boldsymbol{\xi}$ such that $\|\boldsymbol{\xi} - \boldsymbol{\xi}_o\| \gg \ell_*$, we can perform a far-field spatial moment expansion (Note that it is strictly speaking better to take first the Fourier transform in time, to do the reasoning in frequencies), and show that in the far-field

$$u_k(\boldsymbol{\xi}, \tau) \approx \int_0^\tau \mathrm{d}t \underbrace{\int_{\Gamma_f} m_{ij}(\boldsymbol{x}, t)\mathrm{d}\Gamma_x}_{M_{ij}(t)} \times G^k_{i,j}(\boldsymbol{0}, \tau - t; \boldsymbol{\xi} - \boldsymbol{\xi}_o, 0) + \mathrm{h.o.t}(\ell_*/\|\boldsymbol{\xi} - \boldsymbol{\xi}_o\|)$$

Far away, one does not see the details of the shape of the rupture but just its first order moment M_{ij} (the terminology moment tensor actually comes from there).

3 Fundamental solutions for the full-space

3.1 The scalar wave equation

Let's first focus on the scalar wave equation in $\mathbb{R}^3$, with an unit impulse located in $\boldsymbol{x} = 0$ (i.e $r = 0$) at $t = 0$:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = \delta(x)\delta(t)$$

where ∇^2 is the Laplacian operator, c the wave velocity. and δ denotes the Dirac Delta. Note that the operator

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = \Box \phi$$

is sometimes referred to as the D'Alembertian (In reference to Jean Le Rond D'Alembert 1717-1783).

In $\mathbb{R}^3$, we can anticipate that the impulse will generate a purely spherical wave, which will expand as it travels from the origin to infinity at a velocity c.

3.1.1 Solution via Fourier transform

Taking the Fourier transform in time of the previous equation, with the definition of the FT as

$$\hat{\phi}(\boldsymbol{x}, \omega) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \phi(\boldsymbol{x}, t) \mathrm{e}^{-\imath\omega t}\mathrm{d}t$$

we have

$$-\left(\nabla^2 \hat{\phi} + \frac{\omega^2}{c^2}\hat{\phi}\right) = \frac{1}{\sqrt{2\pi}}\delta(\boldsymbol{x})$$

writing as $k = \omega/c$ wave number. Recognizing that the wave is spherical, we switch to a spherical coordinate system and recognize that ϕ will be solely function of the cartesian radius and time, i.e.

$$\frac{1}{r}\frac{\partial^2 r\hat{\phi}}{\partial r^2} + k^2\hat{\phi} = -\frac{1}{\sqrt{2\pi}}\delta(r)$$

for $r \neq 0$, the LHS is zero, we thus obtain the homogeneous solution

$$r\hat{\phi} = A\mathrm{e}^{\imath kr} + B\mathrm{e}^{-\imath kr}$$

Near $r \to 0$, the term in $k^2\hat{\phi}$ becomes negligeable compared to $\nabla^2\hat{\phi}$, such that $\hat{\phi}$ must satisfy near $r = 0$

$$\nabla^2\hat{\phi} = -\frac{1}{\sqrt{2\pi}}\delta(r)$$

we thus see that we need the fundamental solution for the Laplacian !

Fundamental solution for the Laplacian in $\mathbb{R}^3$,
Let's do a detour and derive the fundamental solution for the Laplacian in a full-space

$$\nabla^2\phi + \delta(r) = 0$$

Integrating over a domain V, and using Green's theorem

$$\int_V \nabla\cdot\boldsymbol{\nabla}\phi \mathrm{dV} + 1 = 0$$

$$\int_S \boldsymbol{\nabla}\phi\boldsymbol{n}\mathrm{d}S + 1 = 0$$

Switching to spherical coordinates, postulating that ϕ depends only on r, integrating over the sphere S center at $r = 0$, we obtain

$$\int_0^\pi\int_0^{2\pi}\frac{\mathrm{d}\phi}{\mathrm{d}r}r^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi + 1 = 0$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = -\frac{1}{4\pi r^2}$$

$$\phi = \frac{1}{4\pi r}$$

Note for later that we can re-write this as

$$\nabla^2\left(-\frac{1}{4\pi r}\right) = \delta(r).$$

Back to the scalar wave equation
Now we see that we must have (to satisfy the dirac delta)

$$A + B = \frac{1}{4\pi}\times\frac{1}{\sqrt{2\pi}}$$

Restricting to outgoing wave (positive times), i.e. $B = 0$, we thus obtain via the inverse Fourier transform:

$$\phi(r,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{4\pi}\times\frac{1}{\sqrt{2\pi}}\frac{\mathrm{e}^{\imath\omega r/c}}{r}\mathrm{e}^{\imath\omega t}\mathrm{d}t = \frac{1}{4\pi r}\delta(t - r/c)$$

We thus see that the wave front is spherical, is growing outward at the velocity c and exhibit a geometric attenuation in $1/r$ associated with its expansion.

[Note that you can recover this results via different techniques: fourier transform in space, directly in space and time etc.]

3.2 Elastodynamics - point force impulse solution

Let's now move to the elastodynamics system. Things are more "complex" as the equation is of vectorial nature: the displacement vector u_i is the solution, and we have 3 components of the linear balance of momentum (in 3D). Note we restrict in the following to isotropic material.

$$c_{ijkl} = \underbrace{(k - 2/3g)}_{\lambda} \delta_{ij}\delta_{kl} + \underbrace{g}_{\mu} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

[The following derivation is taken from Rice (2004)].

3.2.1 Navier Equation

Introducing the elastic stress-strain relation in the balance of momentum, we obtain:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = (g + \lambda) \frac{\partial^2 u_j}{\partial x_i \partial x_j} + g \frac{\partial^2 u_i}{\partial x_j \partial x_j} + f_i$$

i.e. in bold face for vector notation

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (g + \lambda) \boldsymbol{\nabla} (\nabla \cdot \boldsymbol{u}) + g \nabla^2 \boldsymbol{u} + \boldsymbol{f}$$

We look at solution for the case of a time-dependent point force located at $\boldsymbol{x} = \boldsymbol{0}$, and write

$$\boldsymbol{f} = \delta(\boldsymbol{x}) \boldsymbol{F}(t).$$

Orthogonal operators

For any vector field $\boldsymbol{v}$, let's define the following two differential operators:

$$\boldsymbol{M}^p \boldsymbol{v} = \boldsymbol{\nabla} (\nabla \cdot \boldsymbol{v}) \qquad \boldsymbol{M}^s \boldsymbol{v} = \nabla^2 \boldsymbol{v} - \boldsymbol{\nabla} (\nabla \cdot \boldsymbol{u}) = -\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{v}$$

such that

$$\boldsymbol{M}^p \boldsymbol{v} + \boldsymbol{M}^s \boldsymbol{v} = \nabla^2 \boldsymbol{v}$$

which verify the following identities

$$\boldsymbol{M}^p (\boldsymbol{M}^s \boldsymbol{v}) = \boldsymbol{M}^s (\boldsymbol{M}^p \boldsymbol{v}) = \boldsymbol{0}$$

and

$$\boldsymbol{M}^p (\boldsymbol{M}^p \boldsymbol{v}) = \boldsymbol{M}^p (\nabla^2 \boldsymbol{v})$$
$$\boldsymbol{M}^s (\boldsymbol{M}^s \boldsymbol{v}) = \boldsymbol{M}^s (\nabla^2 \boldsymbol{v})$$

Navier equation can be further expressed as

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \rho c_p^2 \boldsymbol{M}^p \boldsymbol{u} + \rho c_s^2 \boldsymbol{M}^s \boldsymbol{u} + \boldsymbol{f}$$

with c_p and c_s the P (compressional) and S (shear) waves velocities

$$c_p^2 = (\lambda + g)/\rho \qquad c_s^2 = g/\rho$$

Now, if we write the displacement as the following combination

$$\boldsymbol{u} = \boldsymbol{M}^p \boldsymbol{A}^p(\boldsymbol{x}, t) + \boldsymbol{M}^s \boldsymbol{A}^s(\boldsymbol{x}, t)$$

then thanks to the previous identities for M^p and M^s, the Navier equation becomes

$$\rho \boldsymbol{M}^p \left(c_p^2 \boldsymbol{\nabla}^2 \boldsymbol{A}^p - \frac{\partial^2 \boldsymbol{A}^p}{\partial t^2} \right) + \rho \boldsymbol{M}^s \left(c_s^2 \boldsymbol{\nabla}^2 \boldsymbol{A}^s - \frac{\partial^2 \boldsymbol{A}^s}{\partial t^2} \right) + \boldsymbol{f} = 0$$

Time-dependent point force solution

We recall that the fundamental solution for the Laplacian is such that

$$\nabla^2 \left(-1/(4\pi r)\right) = \delta(\boldsymbol{x})$$

we can thus re-write the time dependent point force $\boldsymbol{f}$ as

$$\boldsymbol{f} = \delta(\boldsymbol{x})\boldsymbol{F}(t) = -\nabla^2 \left(\frac{\boldsymbol{F}(t)}{4\pi r}\right) = -\left(\boldsymbol{M}^p + \boldsymbol{M}^s\right)\left(\frac{\boldsymbol{F}(t)}{4\pi r}\right)$$

the Navier equations will thus be satisfied if both $\boldsymbol{A}^p$ and $\boldsymbol{A}^s$ satisfy

$$c_p^2 \boldsymbol{\nabla}^2 \boldsymbol{A}^p - \frac{\partial^2 \boldsymbol{A}^p}{\partial t^2} = \frac{1}{\rho}\frac{\boldsymbol{F}(t)}{4\pi r}$$

$$c_s^2 \boldsymbol{\nabla}^2 \boldsymbol{A}^s - \frac{\partial^2 \boldsymbol{A}^s}{\partial t^2} = \frac{1}{\rho}\frac{\boldsymbol{F}(t)}{4\pi r}$$

where we see that our detour toward the scalar wave equation will finally be useful, as the form is similar.... For generality, let's assume that there is a vector function $\boldsymbol{P}(t)$ such that

$$\frac{\partial^2 \boldsymbol{P}}{\partial t^2} = \boldsymbol{F}$$

The equation

$$c^2 \boldsymbol{\nabla}^2 \boldsymbol{A} - \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = \frac{1}{\rho}\frac{1}{4\pi r}\frac{\partial^2 \boldsymbol{P}}{\partial t^2}$$

has spherical symmetry, i.e in spherical coordinates after multiplying by r:

$$c^2 \frac{\partial^2 r\boldsymbol{A}}{\partial r^2} - \frac{\partial^2 r\boldsymbol{A}}{\partial t^2} = \frac{1}{\rho}\frac{1}{4\pi}\frac{\partial^2 \boldsymbol{P}}{\partial t^2}$$

such that the general solution is

$$r\boldsymbol{A} = -\frac{\boldsymbol{P}(t)}{\rho 4\pi} + \boldsymbol{Q}_1(t - r/c) + \boldsymbol{Q}_2(t + r/c)$$

restricting to outgowing waves ($\boldsymbol{Q}_2 = 0$), in order to avoid singularity as $r \to 0$, we must set

$$\boldsymbol{Q}_1(t) = \frac{\boldsymbol{P}(t)}{\rho 4\pi}$$

The solutions for $\boldsymbol{A}^p$ and $\boldsymbol{A}^s$ are therefore

$$\boldsymbol{A}^p = \frac{\boldsymbol{P}(t - r/c_p) - \boldsymbol{P}(t)}{4\pi\rho r} \qquad \boldsymbol{A}^s = \frac{\boldsymbol{P}(t - r/c_s) - \boldsymbol{P}(t)}{4\pi\rho r}$$

with $r = \|\boldsymbol{x}\|$, such that the displacement solution for such a point force is finally obatined as:

$$\boldsymbol{u} = -\boldsymbol{\nabla}^2\left(\frac{\boldsymbol{P}(t)}{4\pi\rho r}\right) + \boldsymbol{\nabla}\left(\nabla \cdot \frac{\boldsymbol{P}(t - r/c_p)}{4\pi\rho r}\right) + \boldsymbol{\nabla}^2 \frac{\boldsymbol{P}(t - r/c_s)}{4\pi\rho r} - \boldsymbol{\nabla}\left(\nabla \cdot \frac{\boldsymbol{P}(t - r/c_s)}{4\pi\rho r}\right)$$

where the first term vanishes for $r > 0$

$$u_i = -\delta_{ik}\frac{\partial^2}{\partial x_j \partial x_j}\left(\frac{P_k(t)}{4\pi\rho r}\right) + \frac{\partial^2}{\partial x_i \partial x_k}\frac{P_k(t - r/c_p)}{4\pi\rho r} + \delta_{ik}\frac{\partial^2}{\partial x_j \partial x_j}\frac{P_k(t - r/c_s)}{4\pi\rho r} - \frac{\partial^2}{\partial x_i \partial x_k}\frac{P_k(t - r/c_s)}{4\pi\rho r}$$

Noting that

$$\frac{\partial^2 P_k(t - r/c)}{\partial x_i \partial x_k} = \frac{1}{c^2}P_k^{''}(t - r/c)\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_k} - \frac{1}{c}P_k^{'}(t - r/c)\frac{\partial^2 r}{\partial x_i \partial x_k}$$

Moreover

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} = \gamma_i \qquad \frac{\partial^2 r}{\partial x_i \partial x_k} = -\frac{\gamma_i \gamma_k}{r}$$

which can help simplifying further to [re-derive as exercice;)], recalling that $P''_k = F_k$

$$u_i = \underbrace{\frac{3\gamma_i\gamma_j - \delta_{ij}}{4\pi\rho}\frac{1}{r^3}\int_{r/c_p}^{r/c_s}\tau F_j(t-\tau)\mathrm{d}\tau}_{\text{near-field term}} + \underbrace{\frac{1}{4\pi\rho c_p^2}\frac{\gamma_i\gamma_j}{r}F_j(t-r/c_p)}_{u_i^P} + \underbrace{\frac{1}{4\pi\rho c_s^2}\frac{(\delta_{ij}-\gamma_i\gamma_j)}{r}F_j(t-r/c_s)}_{u_i^S} \tag{2}$$

Note the first term is dominant in the near-field (for F_j non zero for time shorter than r/c_p & r/c_s, it behaves as r^{-2}), while the later two terms are dominant in the far-field (and corresponds the far-field P and S waves). It is also important to note that $\gamma_i = x_i/r = \partial_{x_i} r$ is the unit vector directed from the source (located here at the origin) and the receiver (located at x_i). We thus see that

- both waves (P and S) attenuates as r^{-1}, travels respectively at c_p and c_s velocities and have a waveform proportional to the applied force at retarded time.
- the P-wave is such that its displacement at a receiver point is parallel to the direction γ_i from the source
- the S-wave is such that its displacement at a receiver point is perpendicular to the γ_i from the source

Green's function

For an unit impulse

$$F_i = \delta_{ik}\delta(t)$$

such that P_k is the unit ramp function

$$P_k(t) = \begin{cases} t & t > 0 \\ 0 & t < 0 \end{cases}$$

and in that case we write

$$u_i(\boldsymbol{x}, t; F_k) \equiv G_i^k$$

Static limit

Let s assume that F remains constant for $t > t_f$, i.e.

$$F_i = \delta_{ik}H(t - t_f)$$

where H is the heavyside function. For time larger than t_f, we have $P_i = \left(C_1 + C_2 t + F_i t^2/2\right)$, such that now whenener $t - r/c > t_f$

$$\frac{P_k(t-r/c) - P_k(t)}{4\pi\rho r} = \frac{-C_2 r/c + F_k r^2/2c^2}{4\pi\rho r} = -\frac{C_2}{4\pi\rho c} + \frac{F_k r}{8\pi\rho c^2}$$

only the last term matters when applying the spatial differential operator. such that in the static limit which is established just behind the shear wave front, we have

$$u_i = \frac{\partial^2}{\partial x_i \partial x_k}\left(\frac{F_k r}{8\pi\rho c_p^2}\right) + \delta_{ik}\frac{\partial^2}{\partial x_j \partial x_j}\left(\frac{F_k r}{8\pi\rho c_s^2}\right) - \frac{\partial^2}{\partial x_i \partial x_k}\left(\frac{F_k r}{8\pi\rho c_s^2}\right)$$

[You can show that you finally recover the static Green's function].

3.3 Elastodynamics - fundamental moment tensor / dislocation solution in a full-space

In order to obtain the displacement due to a point dislocation source, we have seen when deriving the representation formula that we need to derivate spatially the point force Green's function, i.e. recall that for a finite fault

$$\begin{aligned} u_k(\boldsymbol{\xi}, \tau) &= \int_0^\tau \mathrm{d}t \int_{\Omega^*} m_{ij}(\boldsymbol{x}, t) G_{i,j}^k(\boldsymbol{x}, \tau - t; \boldsymbol{\xi}, 0)\mathrm{d}\Omega_x \\ m_{ij} &= \frac{c_{ijkl}}{2}\left(\Delta u_k n_l + \Delta u_l n_k\right) \end{aligned}$$

For a point moment tensor M_{ij}, we need to have the third order tensor $G^k_{i,j}$. Derivations are lengthy and obtained starting from eq.(2), and finally we can express the displacement due to a point moment tensor M_{ij} as

$$\begin{aligned} u_k(\boldsymbol{x},t) = M_{ij} * G^k_{i,j} = & \frac{15\gamma_k\gamma_i\gamma_j - 3\gamma_k\delta_{ij} - 3\gamma_i\delta_{jk} - 3\gamma_j\delta_{ik}}{4\pi\rho}\frac{1}{r^4}\int_{r/c_p}^{r/c_s}\tau M_{ij}(t-\tau)\mathrm{d}\tau \\ & + \frac{6\gamma_k\gamma_i\gamma_j - \gamma_k\delta_{ij} - \gamma_i\delta_{jk} - \gamma_j\delta_{ik}}{4\pi\rho c_p^2}\frac{1}{r^2}M_{ij}(t-r/c_p) \\ & - \frac{6\gamma_k\gamma_i\gamma_j - \gamma_k\delta_{ij} - \gamma_i\delta_{jk} - 2\gamma_j\delta_{ik}}{4\pi\rho c_s^2}\frac{1}{r^2}M_{ij}(t-r/c_s) \\ & + \frac{\gamma_k\gamma_i\gamma_j}{4\pi\rho c_p^3}\frac{1}{r}\dot{M}_{ij}(t-r/c_p) - \frac{\gamma_k\gamma_i - \delta_{ik}}{4\pi\rho c_s^2}\frac{\gamma_j}{r}\dot{M}_{ij}(t-r/c_s) \end{aligned}$$

where

$$\gamma_i = \frac{x_i}{r}$$

3.3.1 Point shear dislocation case (also referred to as the double-couple)

The solution can be further particularized for the of shear slip on a fault of normal n_i, in that case the moment tensor can be written (in the fault coordiantes system with the normal defining e_3):

$$M_{ij}(t) = M_o(t)\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Taking a spherical coordinates system (see Fig. 1) such that the cartesian point $\boldsymbol{x}$ transforms into r, θ, ϕ, for a shear relative slip in the direction $\boldsymbol{e}_1$ of the cartesian system, we have (see Aki K. & Richards P. G. (2002) pg 79)

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}=(r,\theta,\phi),t) = & \frac{1}{4\pi\rho}\boldsymbol{A}^N(\theta,\phi)\frac{1}{r^4}\int_{r/c_p}^{r/c_s}\tau M_o(t-\tau)\mathrm{d}\tau \\ & + \frac{1}{4\pi\rho c_p^2}\boldsymbol{A}^{IP}(\theta,\phi)\frac{1}{r^2}M_o(t-r/c_p) + \frac{1}{4\pi\rho c_s^2}\boldsymbol{A}^{IS}(\theta,\phi)\frac{1}{r^2}M_o(t-r/c_p) \\ & + \frac{1}{4\pi\rho c_p^3}\boldsymbol{A}^{FP}(\theta,\phi)\frac{1}{r}\dot{M}_o(t-r/c_p) + \frac{1}{4\pi\rho c_s^3}\boldsymbol{A}^{FS}(\theta,\phi)\frac{1}{r}\dot{M}_o(t-r/c_s) \end{aligned}$$

where a dot denotes the time derivative. The first term is the near-field term, while the two last terms are the far-field term and the two middle ones are near-field / intermediate terms.

The $\boldsymbol{A}$s are the radiation pattern and are only function of the Euler angles (θ,ϕ), not distance to the source. They are given as

$$\begin{aligned} \boldsymbol{A}^N(\theta,\phi) &= 9\sin 2\theta\cos\phi\hat{\boldsymbol{r}} - 6(\cos 2\theta\cos\phi\hat{\boldsymbol{\theta}} - \cos\theta\sin\phi\hat{\boldsymbol{\phi}}) \\ \boldsymbol{A}^{IP}(\theta,\phi) &= 4\sin 2\theta\cos\phi\hat{\boldsymbol{r}} - 2(\cos 2\theta\cos\phi\hat{\boldsymbol{\theta}} - \cos\theta\sin\phi\hat{\boldsymbol{\phi}}) \\ \boldsymbol{A}^{IS}(\theta,\phi) &= -3\sin 2\theta\cos\phi\hat{\boldsymbol{r}} + 3(\cos 2\theta\cos\phi\hat{\boldsymbol{\theta}} - \cos\theta\sin\phi\hat{\boldsymbol{\phi}}) \\ \boldsymbol{A}^{FP}(\theta,\phi) &= \sin 2\theta\cos\phi\hat{\boldsymbol{r}} \\ \boldsymbol{A}^{FS}(\theta,\phi) &= \cos 2\theta\cos\phi\hat{\boldsymbol{\theta}} - \cos\theta\sin\phi\hat{\boldsymbol{\phi}} \end{aligned}$$

with

$$\begin{aligned} \hat{\boldsymbol{r}} &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\ \hat{\boldsymbol{\theta}} &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \\ \hat{\boldsymbol{\phi}} &= (-\sin\phi, \cos\phi, 0) \end{aligned}$$

The far-field ratation pattern are particularly important, as they describe how the waves are radiated away from the source. In particular the most important for observations are the far-field ones, where the dependence on $\sin 2\theta$ and $\cos 2\theta$ defines "lobes" and the direction where the energy radiated is the largest.

Note that in Page100 of Udías et al. (2014) the far-field pattern of a general point moment tensor is given for the full-space.

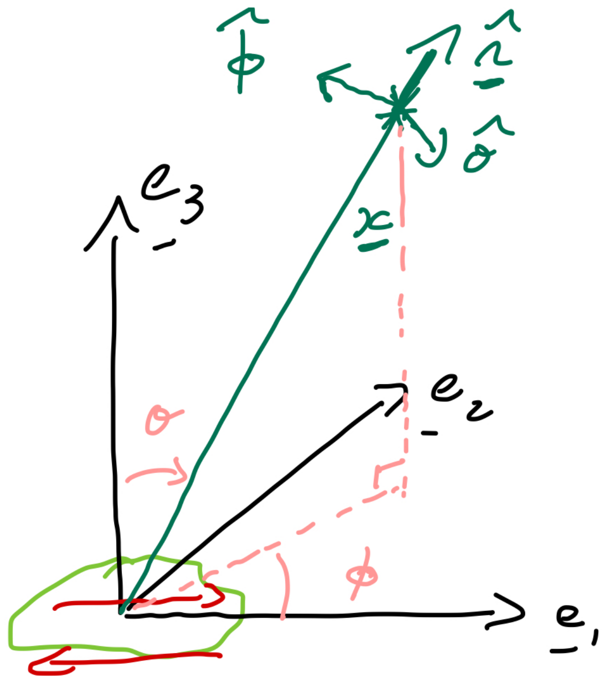

Figure 1: Cartesian and spherical coordinates system for the analysis of the diplacement radiated by a shear dislocation (area A viewed as a point). Figure adapted from Aki K. & Richards P. G. (2002).

3.3.2 The centroid moment tensor

It corresponds to a point moment tensor and is defined by its centroid, origin time and $M_{ij}(t)$. For a simple shear dislocation (double-couple mechanism), it reduces to M_o. Its location is different than the hypocenter (the location of the start of the rupture) - determined from first arrivals.

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