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Summary

A fundamental question in mathematical analysis is that of recovering a function f on $\mathbb{R}^d$, taken from some given function space V, only from its restriction and the restriction of its Fourier transform $\hat{f}$ to two given subsets $A, B \subseteq \mathbb{R}^d$. In particular, one is interested in sufficient and necessary conditions on A, B, V under which the restriction map $f \mapsto (f|_A, \hat{f}|_B)$ is injective. This is the Fourier uniqueness problem contained in the title of this thesis, which has received considerable attention in recent years because of a remarkable new Fourier interpolation formula proved by Radchenko and Viazovska. Aiming to generalize that formula to higher dimensions in different ways we present several new results related to the Fourier uniqueness problem. These all involve the sequences of origin-centered spheres whose radii are square roots of integers and all of our proofs feature modular forms, in one way or another.

To be more specific, we prove that for all dimensions $d \geq 2$, any Schwartz function f on $\mathbb{R}^d$ can be recovered only from the restrictions of f and the restrictions of its Fourier transform $\hat{f}$ to the sequence of origin-centered spheres $\sqrt{n}S^{d-1}$, where n traverses the non-negative integers. We use harmonic analysis on spheres and on $\mathbb{R}^d$ to relate this problem to the same problem for *radial* functions in dimension $d, d+2, d+4, \ldots$ and solve those radial interpolation problems via the construction of modular integrals with preassigned period functions on Hecke-type groups (of finite or infinite covolume). For these constructions, we use a modular Green-type kernel function and an associated integral transform and also a series construction similar to the one for classical Poincaré series (but with a real instead of an integral indexing parameter). We complement these so-called Fourier uniqueness results with a Fourier uniqueness result for *surfaces* close to the spheres $\sqrt{n}S^{d-1}$ and *sets of discrete points* contained in them, which we obtain via functional-analytic perturbation methods. In a complementary, opposite direction, we construct infinite dimensional spaces of Schwartz Fourier eigenfunctions on $\mathbb{R}^d$, which vanish on certain discrete sets coming from totally real number fields, contained in the spheres $\sqrt{n}S^{d-1}$.

Keywords

Fourier transform, modular forms.

Zusammenfassung

Eine zentrale Frage der mathematischen Analysis ist die der Rekonstruktion einer Funktion f auf $\mathbb{R}^d$, aus einem gegebenen Funktionenraum V, nur durch Kenntnis ihrer Einschränkungen und der Einschränkungen ihrer Fourier Transformierten $\hat{f}$ auf zwei gegebene Teilmengen $A, B \subseteq \mathbb{R}^d$. Insbesondere interessiert man sich für notwendige und hinreichende Bedingungen an A, B, V unter denen die Einschränkungsabbildung $f \mapsto (f|_A, \hat{f}|_B)$ injektiv ist. Dies ist das Fourier-Eindeutigkeitsproblem im Titel dieser Arbeit, welchem in den vergangenen Jahren beachtliches Interesse geschenkt wurde dank einer erstaunlichen Interpolationsformel von Radchenko und Viazovska. Mit dem Ziel diese Formel auf höhre Dimensionen zu verallgemeinern, legen wir in dieser Arbeit neue Ergebnisse zum Fourier-Eindeutigkeitsproblem vor, welche alle die Folge der kozentrischen Sphären mit Radius $\sqrt{n}$ und Mittelpunk dem Ursprung beinhalten. In all unseren Beweisen tauchen in der ein oder anderen Form Modulformen auf.

Genauer gesagt beweisen wir, dass in jeder Dimension $d \geq 2$ jede beliebige Schwartz funktion f auf $\mathbb{R}^d$ nur durch ihre Einschränkungen und die Einschränkungen ihrer Fourier-Transformierten auf die Sphären $\sqrt{n}S^{d-1}$ rekonstruiert werden kann. Wir verwenden harmonische Analysis auf $\mathbb{R}^d$ und auf Sphären um dieses Problem auf dasselbe Problem für rotationsinvariante Funktionen zu beziehen und lösen letzteres durch die Konstruktion modularer Integrale mit vorgeschriebenen Perioden auf Gruppen vom Hecke Typus (von endlichem oder unendlichem Kovolumen). Für dies benutzen wir einerseits eine Integraltransformation mit einem Integralkern vom Green'schen Typ und andererseits eine Konstruktion ähnlich zu jener der Poincaréschen Reihen (mit einem reellen anstatt eines ganzen Paramteres). Wir ergänzen dieses (sogenannte) Fourier-Eindeutigkeitsergebnis durch eines, das Flächen. die nah den oben genannten Sphären sind, beinhaltet und durch ein anderes mit diskreten Punktmengen auf diesen Sphären. Zum Beweis dafür verwenden wir elementare Methoden der Funktionalanalysis. In einer entgegengesetzten, ergänzenden Richtung konstruieren wir einen unendlich dimensionalen Raum von Schwartz-Fourier- Eigenfunktionen auf $\mathbb{R}^d$, die auf gewissen diskreten Punktmengen in den Sphären $\sqrt{n}S^{d-1}$ verschwinden. Diese Punktmengen stammen von total reellen Zahlkörpern ab.

Schlüsselwörter

Fourier transformation, Modulformen.

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Contents

1 Introduction

We are interested in two kinds of problems concerning the Fourier transform in Euclidean space, called Fourier uniqueness and Fourier interpolation. Among those, we are particularly interested in the ones that have connections to the theory of modular forms and to number theory more generally. We start by giving a definition of a Fourier uniqueness pair, give some motivation and context for its study and then outline the contents of this thesis. As indicated in the abstract, this thesis is centered around a very special Fourier uniqueness pair, given by the origin-centered spheres $\sqrt{n}S^{d-1}$, where $n \geq 0$ is an integer.

Let V be a space of complex-valued, continuous and integrable functions on $\mathbb{R}^d$. Let $A, B \subseteq \mathbb{R}^d$ be subsets. Let $R = R_{A,B} : V \to C(A) \times C(B)$ denote the map given by $R(f) = (f|_A, \widehat{f}|_B)$, where $\widehat{f}$ denotes the Fourier transform of f (normalized as in §1.1 below) and where, for any topological space X, we write $C(X)$ for the space of continuous $\mathbb{C}$-valued functions on X.

Definition. We say that the pair (A, B) is a *(Fourier) uniqueness pair* for the space V, if the map $R = R_{A,B}$, defined as above, is injective. If R is not injective, then (A, B) is a *(Fourier) non-uniqueness pair.* If $A = B$, we simply speak of (Fourier) uniqueness *sets* or (Fourier) non-uniqueness sets.

To get used to the definition, we start by noting the following trivial examples of Fourier uniqueness pairs: $(\mathbb{R}^d, \emptyset)$, $(\emptyset, \mathbb{R}^d)$, $(\mathbb{Q}^d, \emptyset)$ and $(\emptyset, \mathbb{Q}^d)$. Another, slightly less trivial example is $(\mathbb{R}^d \smallsetminus S, \mathbb{R}^d \smallsetminus T)$ for any two bounded subsets $S, T \subseteq \mathbb{R}^d$. If (A, B) is a Fourier uniqueness pair for V, then so is $(gA, g^{-t}B)$, for any $g \in \mathrm{GL}_d(\mathbb{R})$. An easy general example of a *non*-uniqueness pair is $(D, \emptyset)$ or $(\emptyset, D)$ for any *discrete* subset D, assuming the space V contains compactly supported functions. For a Fourier uniqueness pair to be interesting, we want it to be *minimal* (or *tight* or *sharp*) in the sense that A and B are as "small as possible" and that V is as "large as possible". What exactly "minimal" means needs to be made precise depending on context. Do we mean set-theoretic minimality (inclusion), measure-theoretic minimality or minimality with respect to a certain notion of density of A and B (if these sets are discrete)?

In addition, we are interested in uniqueness pairs which are implied by the existence of a *Fourier interpolation formula.* By that we mean, roughly speaking, a formula that reconstructs the values of any given $f \in V$ using only the information $R(f)$ and that takes the form

$$f(x) = \int_A f(a)K(x,a)da + \int_B \widehat{f}(b)\tilde{K}(x,b)db, \tag{1.1}$$

for some kernel functions $K, \tilde{K}$ and suitable measures da on A, db on B. The precise meaning of those as well as the meaning of the equality in (1.1) needs of course to be specified for any given instance, but the existence of such a formula clearly *implies* that (A, B) is a Fourier uniqueness pair. Note also that any formula like (1.1) can be seen as a generalization of Fourier inversion, which itself corresponds to the case $A = \emptyset$ (and $\tilde{K}(x, b) = e^{2\pi i x \cdot b}$ when using a suitable normalization).

In this thesis, the term *(Fourier-) interpolation formula* will be used in the above general and somewhat loosely defined sense. In particular, when we use the word *interpolation* in the term Fourier interpolation (formula), we do not necessarily have in mind the problem of finding necessary and sufficient conditions for when a pair of functions $(g, h) \in C(A) \times C(B)$ belongs to the image of $R_{A,B}$, although this problem will sometimes also be discussed. Thus, the term "Fourier reconstruction formula" may perhaps be more accurate at certain places.

Let us now discuss a classical example, the Whittaker–Shannon interpolation formula. Fix some positive real number $w > 0$, a band*w*idth. Consider the Payley–Wiener space $\mathrm{PW}_w(\mathbb{R})$ consisting of all $f \in L^2(\mathbb{R})$ such that $\hat{f}(\xi) = 0$ for almost every $|\xi| \geq w/2$. Here, we use the extension of the Fourier transform from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ (using the normalization in item (7) in section 1.1). As is well-known such functions identify with (restrictions of) entire functions of exponential type $\leq 2\pi/w$, so that their evaluation at points of $\mathbb{R}$ is canonically defined. The Whittaker–Shannon interpolation formula says that for every $f \in \mathrm{PW}_w(\mathbb{R})$ and $x \in \mathbb{R}$, we have

$$f(x) = \sum_{n \in \mathbb{Z}} f\left(\frac{n}{w}\right) \frac{\sin\left(\pi w(n - x)\right)}{\pi w(x - n)}. \tag{1.2}$$

It can be proved by combining L^2-Fourier inversion on $\mathbb{R}$ and L^2-Fourier inversion on $\mathbb{R}/(w\mathbb{Z})$. Since the value $f(n/w)$ is the nth Fourier coefficient of $\widehat{f}$ (viewed as an element of $L^2(\mathbb{R}/(w\mathbb{Z}))$), we have $\{f(n/w)\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ and from this it follows that convergence in (1.2) holds not only in the L^2-sense but also point-wise and uniformly on compact subsets of $\mathbb{C}$. The Whittaker–Shannon interpolation formula is an important tool in signal processing and connects to many different areas in time-frequency analysis. A trivial corollary of (1.2) is that the pair $((1/w)\mathbb{Z}, \mathbb{R} \setminus (-w/2, w/2))$ is a Fourier uniqueness pair for the space $L^2(\mathbb{R})$. The latter space is neither a space of continuous nor L^1-integrable functions on $\mathbb{R}$, but we will sometimes use the notion of a uniqueness pair for other function spaces, with obvious modifications.

More generally, in an L^2- (or L^p-)space setting, Fourier uniqueness pairs have been extensively studied, not only on $\mathbb{R}^d$ but also on other locally compact abelian groups. They are then sometimes called "annihilating pairs". It is beyond the scope of this thesis to give a sensible overview of these results. We only mention the works of Amrein–Berthier [AB77], Nazarov [Naz93], Jaming [Jam07] and the references therein to give a sample of some of these results.

Perhaps surprisingly, the case where A and B are both *discrete* infinite subsets of $\mathbb{R}^d$ has not been explored up until recently. In 2017, Danylo Radchenko and Maryna Viazovska [RV19] proved a remarkable, novel Fourier interpolation formula for Schwartz functions on the real line. Using integral transforms of weakly holomorphic modular forms, they constructed even Schwartz functions $a_n : \mathbb{R} \to \mathbb{R}$ such that for all even Schwartz functions $f : \mathbb{R} \to \mathbb{C}$ and all $x \in \mathbb{R}$, one has

$$f(x) = \sum_{n=0}^{\infty} a_n(x) f(\sqrt{n}) + \sum_{n=0}^{\infty} \widehat{a_n}(x) \widehat{f}(\sqrt{n}), \tag{1.3}$$

where both series converge absolutely. (In fact, the sums on the right both converge absolutely with respect to any continuous semi-norm on $\mathcal{S}(\mathbb{R})$.) They also proved a similar formula for odd Schwartz functions on $\mathbb{R}$, which uses the derivatives of f and $\hat{f}$ at zero instead of the values at zero. By subsequent work of Bondarenko, Radchenko and Seip [BRS20], it is known that (1.3) holds not only for even *Schwartz* functions, but in a much larger function space, described by decay and regularity conditions on f and $\widehat{f}$ (a weaker result of this type is already contained in [RV19]). To the author's knowledge, (1.3) is the first Fourier interpolation formula of its kind, in the sense that it uses point-wise values of f *and* $\widehat{f}$ from *discrete* subsets of their respective domains. The implied Fourier uniqueness result is moreover sharp in the sense that for all $n_0 \in \mathbb{Z}_{\geq 0}$, the set

$\{\sqrt{n} : n \in \mathbb{Z}_{\geq 0}, n \neq n_0\}$ is *not* a Fourier uniqueness set for $\mathcal{S}_{\text{even}}(\mathbb{R})$ (the space of even Schwartz functions on $\mathbb{R}$).

In a subsequent, related work, done jointly [CKM+21] with Cohn, Kumar and Miller, Radchenko and Viazovska proved an interpolation theorem for radial Schwartz functions on $\mathbb{R}^8$ and $\mathbb{R}^{24}$ that recover any $f \in \mathcal{S}_{\text{rad}}(\mathbb{R}^d)$, $d = 8, 24$ only from the values of f and $\hat{f}$ and the radial derivatives $f', \hat{f}'$ at all points $\sqrt{2n}$, for integers $n \geq n_0(d)$, where $n_0(8) = 1$, $n_0(24) = 2$. Using these formulas, the 5 authors were able to prove the universal optimality of the E_8 and the Leech lattice as energy minimizing configurations among all configurations of a given density, with respect to a certain class of potential functions (see [CKM+21] for a precise formulation). Through this interpolation theorem their work put in a more conceptual framework the previous work by Viazovska [Via17] and the same five authors [CKM+17] on sphere packing and the construction of so-called magic functions used to solve them. In view of these results, the work by Radchenko–Viazovska [RV19] is a prequel to the one just mentioned and partially motivates its study.

As we will explain in more detail in the body of the thesis, the square root structure in the nodes of these formulas connects them inextricably to the theory of modular forms. But besides this connection, the particular sequence of nodes $\{\sqrt{n}\}_{n\in\mathbb{N}}$ is also special from another perspective. To explain it on an informal level, we briefly mention that, by yet unpublished work of Nazarov, Kulikov and Sodin, it is now known that "basically" any sequence of points $\lambda_n \in \mathbb{R}_{\geq 0}$ which concentrates more densely at infinity than $\sqrt{n}$ (in the sense of how the differences $\lambda_{n+1} - \lambda_n$ decay as $n \to \infty$) is a Fourier uniqueness set for a (large) space of functions containing the Schwartz space, while any sequence that concentrates less densely (i.e. is sparser) than $\sqrt{n}$, is a Fourier non-uniqueness set.

The main goal of this thesis is to study different generalizations of the formula (1.3) to functions on $\mathbb{R}^d$ when $d \geq 2$. As already mentioned, the generalizations are centered around replacing the sequence $\sqrt{n}$ in $\mathbb{R}$ by the sequence of spheres $\sqrt{n}S^{d-1}$ in $\mathbb{R}^d$ and we will discover some new phenomena while studying this problem.

Let us turn to an overview of the chapters and the main results contained in them. Given that this is an overview, we will not give the most complete and precise formulations of the main results, postponing such formulations to the individual chapters, after the necessary objects and notations have been introduced.

Chapter 2: Fourier interpolation for radial functions

There is one way of generalizing (1.3) which is almost implicit in [RV19], namely to *radial* functions in higher dimensions. (Note that, by slight abuse of notation, (1.3) makes sense if f is an element $\mathcal{S}_{\text{rad}}(\mathbb{R}^d)$, the space of radial Schwartz functions on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$. This abuse of notation will be committed throughout the thesis whenever convenient.)

After gathering some background material in §2.1 we will explain in §2.2 how the problem of proving such a generalization to radial functions is equivalent to the problem of finding a family of modular integrals for the group $\Gamma(2)$ with pre-assigned period functions. We will explain what the last few terms mean in their classical context. We then present two solutions of that problem and thereby obtain two generalizations of (1.3) to radial Schwartz functions on $\mathbb{R}^d$.

The first solution and result, Theorem 1 in §2.3, says that for all $d \geq 1$ and all integers $n_0, \hat{n}_0 \geq 0$ such that $n_0 + \hat{n}_0 = 1 + \lfloor d/4 \rfloor$, there exist $a_{d,n}, \tilde{a}_{d,n} \in \mathcal{S}_{\text{rad}}(\mathbb{R}^d)$ such that for all $f \in \mathcal{S}_{\text{rad}}(\mathbb{R}^d)$ we have

$$f(x) = \sum_{n=n_0}^{\infty} a_{d,n}(x) f(\sqrt{n}) + \sum_{n=\hat{n}_0}^{\infty} \tilde{a}_{d,n}(x) \hat{f}(\sqrt{n}) \tag{1.4}$$

with natural convergence properties. Moreover, (1.4) holds naturally for the basis functions $a_{d,n}$, $\tilde{a}_{d,n}$ themselves, in the sense that we have $a_n(\sqrt{m}) = \delta_{nm}$ and $\widehat{a_n}(\sqrt{m}) = 0$ for all integers n, m

in the appropriate ranges; see Theorem 1 for a precise formulation. The number $1 + \lfloor d/4 \rfloor$ equals the dimension of a space of modular forms of weight $d/2$ for the group $\Gamma(2) \leq \mathrm{SL}_2(\mathbb{Z})$ and we show that this space precisely describes the image of the restriction map $R : \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{N}_0)^2$, $f \mapsto ((f(\sqrt{n}))_{n\in\mathbb{N}_0}, ((\hat{f}(\sqrt{n}))_{n\in\mathbb{N}_0})$, where $\mathcal{S}(\mathbb{N}_0)$ denotes the space of all rapidly decaying sequences of complex numbers. We will also write down the inverse of this restriction map R. As an immediate corollary of Theorem 1, we obtain that the pair of subsets

$$\left(\bigcup_{n\geq n_0} \sqrt{n}S^{d-1}, \bigcup_{n\geq \hat{n}_0} \sqrt{n}S^{d-1}\right) \tag{1.5}$$

is a Fourier uniqueness pair for the space $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$. It is minimal in the sense that we cannot remove a single sphere from either set without losing the uniqueness property, as the basis functions in (1.4) give counterexamples. (This will be clear from part (ii) of Theorem 1).

The proof of Theorem 1 uses a singular integral transform of a separately meromorphic and modular kernel function $\mathcal{K}(\tau, z) = \mathcal{K}_{k,n_0,\hat{n}_0}(\tau, z)$ of $(\tau, z) \in \mathbb{H}\times\mathbb{H}$ which is being integrated against the Gaussian $e^{\pi i z|x|^2}$ over a suitable path (here and henceforth, $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ is the upper-half plane). After a certain process of analytic continuation, this integral transform yields a 2-periodic holomorphic function $F_{k,n_0,\hat{n}_0}(\tau, x)$ of $\tau \in \mathbb{H}$, whose Fourier coefficients are the functions $a_{d,n}(x)$ in the above formula (1.4). The method is very similar to the one in [RV19], but it is not the most direct generalization of that method to radial functions in higher dimensions (i.e. the one we called "implicit" further above). The latter is contained in [BRS20]. As opposed to that work, the main difference in our approach (which was worked out independently) is that we work on a smaller congruence subgroup and we do not divide the problem into Fourier eigenspaces (to be explained in §2.2.2.1).

The second main result of chapter 2, Theorem 2, asserts the following. For every real number $h \geq 2$ and for every integer $d \geq 5$, there exist even entire functions $b_{d/2,h,n}, \tilde{b}_{d/2,h,n} : \mathbb{C} \to \mathbb{C}$, indexed by natural numbers $n \geq 1$, such that for all radial $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ and all $x \in \mathbb{R}^d$ we have

$$f(x) = \sum_{n=1}^{\infty} b_{k,h,n}(|x|) f(\sqrt{2n/h}) + \sum_{n=1}^{\infty} \tilde{b}_{k,h,n}(|x|) \hat{f}(\sqrt{2n/h}) \tag{1.6}$$

where $k = d/2$ and the series converge absolutely and uniformly on $\mathbb{R}^d$. Moreover, if $h > 2$, then, for every integer $n_0 \geq 1$, there exists an interpolation formula like (1.6), with both series starting at $n = n_0$ and different functions $b_{k,h,n}$, $\tilde{b}_{k,h,n}$. We also prove bounds for these functions that are explicit in all involved parameters (see parts (ii) and (iii) of Theorem 1), whose relevance will be explained further below.

While Theorem 2 is in a certain sense weaker in its formulation, its proof will be completely different from the one of Theorem 1 and is comparatively simple. Based on an idea by my advisor, M. Viazovska, we will write down a series

$$F_{k,h}(\tau, r) = -\sum_{\gamma\in\mathcal{V}_h} j_k(\gamma, \tau)^{-1} e^{\pi i (\gamma\tau) r^2}, \qquad \tau \in \mathbb{H}$$

which solves a modular integral equation corresponding to (1.6) (to be explained in §2.2.2), is h-periodic and holomorphic in the variable $\tau \in \mathbb{H}$ and whose Fourier coefficients $b_{k,h,n}(r)$ are the ones in (1.6). The above construction is closely related to the construction of a Poincaré series for the group $\Gamma(h) \leq \mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{Aut}_{\mathrm{hol}}(\mathbb{H})$, generated by the translation $\tau \mapsto \tau + h$ and the inversion $\tau \mapsto -1/\tau$. We will define the meaning of all the terms in the above series in §2.4, but just to say a few words about the connection to Poincaré series now, let us mention that, if $h = 2$, $k = d/2 \geq 3$ is an even integer and if $r = \sqrt{m}$ is the square root of an integer $m \geq 0$, then

$$-F_{k,2}(2\tau, m) = -e^{2\pi i m\tau} + \sum_{\gamma \in \Gamma_0(4)_\infty \backslash \Gamma_0(4)} (c_\gamma \tau + d_\gamma)^{-k} e^{2\pi i (\gamma\tau) m}$$

becomes the "actual", classical Poincaré series $P_m^k(\tau)$ on $\Gamma_0(4)$ of weight k, modified by removing the identity coset in its defining series. The set $\mathcal{V}_2$ is a set representing these cosets (the corresponding ones in $\Gamma(2)$). Here, the group $\Gamma_0(4)$ arises because that group is conjugate to $\Gamma(2)$.[1] In that sense the series $F_{k,h}(\tau, r)$ "interpolate" the Poincaré series $P_m^k(\tau)$ from integers m to complex numbers r. By exploiting this connection to Poincaré series we can prove that, while the functions $b_{k,h,n}(r)$ are smooth and have some decay with respect to real $r \in \mathbb{R}_{\geq 0}$, they are not of *rapid* decay and in particular not in the Schwartz class (when restricted to $\mathbb{R}$). More precisely, Proposition 2.12 will show that, if $d \geq 6$ is even and if the nth Poncaré series on $\Gamma_0(4)$ (with respect to a character if $d/2$ is odd) does not vanish identically, then $m \mapsto b_{d/2,2,n}(\sqrt{m})$ does not decay rapidly in $m \in \mathbb{N}$ and hence $b_{d/2,2,n}(r)$ is not rapidly decaying in $r \in \mathbb{R}_{\geq 0}$ either.

Chapter 3: Fourier interpolation from spheres

As stated above, one of the main goals of this work was to generalize the Radchenko–Viazovska formula not only to radial (Schwartz-) functions in all dimensions $d \geq 2$, but to *all* (Schwartz-) functions in all dimensions. Building upon Theorems 1 and 2 mentioned above and upon various harmonic analysis results that we prove in the beginning of chapter 3, we could eventually achieve this goal in the following way. As formulated more precisely in Theorem 3, we constructed smooth kernel functions $A_n^d, \tilde{A}_n^d : \mathbb{R}^d \times S^{d-1} \to \mathbb{C}$ and $c_d, \tilde{c}_d \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R})$ such that for all $f \in \mathcal{S}(\mathbb{R}^d)$ and all $x \in \mathbb{R}^d$, we have

$$f(x) = c_d(|x|)f(0) + \sum_{n=1}^{\infty} \int_S A_n^d(x,\zeta) f(\sqrt{n}\zeta) d\zeta + \tilde{c}_d(|x|)\widehat{f}(0) + \sum_{n=1}^{\infty} \int_S \tilde{A}_n^d(x,\zeta) \widehat{f}(\sqrt{n}\zeta) d\zeta, \quad (1.7)$$

where $S = S^{d-1}$ is the unit sphere in $\mathbb{R}^d$ and where $d\zeta$ denotes integration with respect to the unique rotation-invariant Radon probability measure on S.[2] The convergence in (1.7) is absolute for every fixed $x \in \mathbb{R}^d$ and uniform and rapid with respect to any partial derivative, on any fixed compact subset of $\mathbb{R}^d \setminus \{0\}$. If $d \geq 4$, we can take $c_d = 0 = \tilde{c}_d = 0$ and for $d \leq 4$ the convergence is uniform on any compact subset of $\mathbb{R}^d$.

Let us explain the harmonic analysis input for the proof of (1.7) in the case $d = 2$, where we can write things in terms of familiar Fourier series. For general $d \geq 2$, we will use harmonic polynomials. For a slightly oversimplified explanation of the strategy in the case $d = 1$, involving even and odd functions and the relation to radial functions on $\mathbb{R}^3$, which is already implicit in Radchenko and Viazovska's paper [RV19], we refer to the introduction of our paper [Sto21].

We identify, as usual, the set $\mathbb{C}$ with $\mathbb{R}^2$ using $1_{\mathbb{C}} = (1, 0)$ and $i = (0, 1)$. For $x \in \mathbb{C}$, we write $e(x) = e^{2\pi i x}$. Fix $x \in \mathbb{R}^2 \setminus \{0\}$ and write

$$x = |x|(x/|x|) = |x|e(\theta_x), \qquad \theta_x \in \mathbb{R}/\mathbb{Z}.$$

[1] It doesn't arise naturally, but we find the translation/transition to $\Gamma_0(4)$ sometimes helpful, if only for ease of reference to the literature.

[2] Throughout the thesis, we will often use the abbreviation $S = S^{d-1}$, which sometimes helps to make formulas fit on one line together with their label, for example (1.7).

Let $f \in \mathcal{S}(\mathbb{R}^2)$ be arbitrary. We can interpret the value of f at the point x as the value of the smooth function $\theta \mapsto f(|x|e(\theta))$ at $\theta = \theta_x$. We expand that function into a Fourier series:

$$f(x) = \sum_{n\in\mathbb{Z}} e(n\theta) \int_0^1 f(|x|e(\varphi))e(-n\varphi)d\varphi = R_0 f(|x|) + \sum_{m=1}^{\infty} R_m f(|x|, \theta),$$

where, for any $r \in \mathbb{R}^\times = \mathbb{R} \smallsetminus \{0\}$, we define $R_0 f(r) := \int_0^1 f(|r|e(\varphi))d\varphi$ and for any $m \geq 1$, $r > 0$ and $\theta \in \mathbb{R}/\mathbb{Z}$, we define

$$R_m f(r, \theta) := \sum_{\pm} e(\pm m\theta) \int_0^1 f(|r|e(\varphi))e(\mp m\varphi)d\varphi = \int_0^1 f(|r|e(\varphi)) 2\cos(2\pi m(\theta - \varphi))d\varphi,$$

For uniform notation, we also define $R_0(f)(r, \theta) := R_0 f(r)$ for any $\theta \in \mathbb{R}/\mathbb{Z}$. Fix temporarily $m \in \mathbb{N}_0$ and $\theta \in \mathbb{R}/\mathbb{Z}$. We will verify (Proposition 3.4) that the function $r \mapsto R_m f(r, \theta)$ extends to an even Schwartz function of $r \in \mathbb{R}$ and that, for every integer $p \geq 1$, it can be viewed as a radial Schwartz function on $\mathbb{R}^p$, when we replace r by the Euclidean norm $|x| = r$ of $x \in \mathbb{R}^p$. Consequently, we may use the formula (1.4) (or (1.6) with $h = 2$) for radial Schwartz functions on $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^p)$, to express $R_m f(|x|, \theta)$ in terms of its values at $\sqrt{n}$, and the values of its Fourier transform (computed on $\mathbb{R}^p$!) at $\sqrt{n}$. Here, we have the freedom of choosing p as a function of m and, if we take $p = 2 + 2m$, then we can relate the Fourier transform on $\mathbb{R}^{2+2m}$ to the Fourier transform of the original function f on $\mathbb{R}^2$ in a simple way. This is (a somewhat indirect) consequence of the well-known Bochner formula for the Fourier transform of a radial function times a harmonic polynomial.

In this way, we obtain a formula for $f(x)$ of the form

$$f(x) = \sum_{m=0}^{\infty} R_m f(|x|, \theta_x) = \sum_{m=0}^{\infty} \left[\sum_{n=0}^{\infty} (**) \int_0^1 (**) f(\sqrt{n}e(\varphi))d\varphi + \sum_{n=0}^{\infty} (**) \int_0^1 (**) \widehat{f}(\sqrt{n}e(\varphi))d\varphi \right],$$

where the unspecified terms $(**)$ are all explicitly expressible only in terms of[3] x and $a_{d+2m,n}(|x|)$, $\tilde{a}_{d+2m,n}(|x|)$. By formally manipulating these series and integrals we arrive at a formula like (1.7). To make this into a rigorous proof, we need to

- precisely formulate and prove the statements about the spherical averages $R_m f$, introduced above. In chapter 3, these averages will (up to a factor) be denoted as $L_u f$, and indexed by harmonic polyomials u on $\mathbb{R}^m$. We will establish their main properties (smoothness, decay, Fourier transform) in Propositions 3.3 and 3.4.
- take care of the origin, which was excluded at the very beginning. Once we have established the basic properties of the $L_u f$, this will not be very difficult.
- take care of convergence. This is the most serious problem. In order to rearrange the above double sums and the integrals to obtain a formula (1.7), we need to have estimates for $a_{2+2m,n}(|x|)$ and $\tilde{a}_{2+2m,n}(|x|)$ which are explicit in m, n *and* $|x|$. In fact, we were not able to prove good enough bounds for the radial functions from (1.4). This problem was the main reason to use a different approach in the radial case which led to Theorem 2 in chapter 2. The functions in those theorems can be estimated explicitly in all parameters and one can prove bounds that are sufficient to justify the above maneuvers. On the other hand, these functions only exist in dimension ≥ 5 and as a consequence, for $2 \leq d \leq 4$, we will need *both* Theorems 1 and 2 to prove (1.7).

[3] in fact, polynomials in the real and imaginary part of x, obtained by using relations like $e(m\theta_x) = e(\theta_x)^m = x^m/|x|^m$, where $|x|^m$ cancels another term $|x|^m$ that shows up. This becomes somewhat cleaner if one consistently works only with harmonic polynomials.

Despite the convergence problem last described, it is possible to almost directly deduce the corollary concerning only the Fourier *uniqueness* aspect of (1.7). For example, we get directly from Theorem 1 only (and the harmonic analytic results in chapter 3) that the pair (1.5) is a Fourier uniqueness pair for $\mathcal{S}(\mathbb{R}^d)$ (not only for $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$). In fact, we don't need the full force of Theorem 1 for that, but only the uniqueness corollary it entails; see Proposition 3.7 for precise statements.

Another section in chapter 3 is concerned with *perturbations* of (1.7). These were obtained in joint work with João Ramos in [RS21] using methods from functional analysis. What is being perturbed is the uniqueness pair (1.5) given by the union of spheres $\sqrt{n}S^{d-1}$ in two different ways:

- we can replace $\sqrt{n}S^{d-1}$ by $\{(\sqrt{n}+\varepsilon_n(\zeta))\zeta \,:\, \zeta \in S^{d-1}\}$, where $\varepsilon_n : S^{d-1} \to \mathbb{R}$ is any sufficiently small, continuous function.

- we can replace $\sqrt{n}S^{d-1}$ by $\sqrt{n}D_n^d$, where $D_n^d \subseteq S^{d-1}$ is any sufficiently uniformly and densely distributed *finite* set of points.

We formulate these results more precisely in Theorem 4. Let us mention that the word "sufficiently" comes with a slight caveat. We can prove (i) only under the assumption that the functions ε_n obey $\sup_{\zeta\in S^{d-1}} |\varepsilon_n(\zeta)| = O(n^{-10n-(5/2)d-c})$ for some $c > 0$ (not depending on anything) and we can prove (ii) under the assumption the points belong to partitions of S^{d-1} into subsets of diameter bounded in terms of n in a similar way. We will not fight for numerical improvements of these numbers here but we believe that these results remain true with more slowly decreasing perturbations (as functions of n). More precisely, at least for the discrete uniqueness results.

The final result of chapter 3 is a conditional result, Theorem 6. Under the assumption that the interpolation theorem [CKM+21] by Cohn, Kumar, Miller, Radchenko and Viazovska generalizes (in a suitable way) to all even dimension $d \geq 8$, we show that the so-called magic functions for the spheres packing problem in 8 and 24 dimensions constructed by Viazovska [Via17] and the 5 authors just mentioned [CKM+17], are unique among *all* Schwartz functions which are admissible and optimal for the so-called Cohn–Elkies linear program. We refer the reader to §3.4 for the formulation of the precise result we prove.

Chapter 4: Fourier non-uniqueness and totally real number fields

Chapter 4 is concerned with a different line of generalization of the Radchenko–Viazovska formula and based on joint work with Danylo Radchenko [RS]. We started to wonder whether the objects in this formula and in its proof can be replaced in the following way:

- the set of nodes $\sqrt{\mathbb{Z}} := \{x \in \mathbb{R} \,:\, x^2 \in \mathbb{Z}\}$ by the set

  $$\sqrt{\Lambda} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, (x_1^2, \ldots, x_n^2) \in \Lambda\}$$

  where $\Lambda \subseteq \mathbb{R}^n$ is a lattice coming from a totally real number field K over $\mathbb{Q}$ of degree n. To be more specific, Λ is given as the image of the inverse different $\mathfrak{d}_K^{-1}$ of K under the Minkowski embedding $K \hookrightarrow \mathbb{R}^n$,

- even Schwartz functions by Schwartz functions on $\mathbb{R}^n$ that are even in each variable,

- the one-variable Gaussian $e^{\pi i z x^2}$, $z \in \mathbb{H}$, by an n-variable Guassian $e^{\pi i z_1 x_1^2} \cdots e^{\pi i z_n x_n^2}$, where $x_j \in \mathbb{R}$, $z_j \in \mathbb{H}$.

- modular forms by certain Hilbert modular forms.

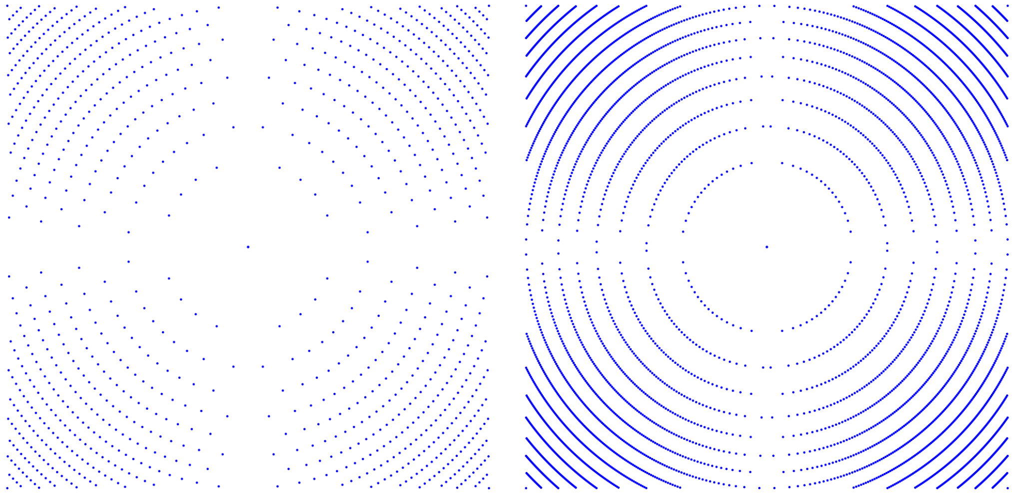

Figure 1: Non-uniqueness set (1.8) for $K = \mathbb{Q}(\sqrt{17})$ and $\mathbb{Q}(\sqrt{257})$

While initially, this seemed natural and while the Fourier reconstruction problem with nodes $\sqrt{\Lambda}$ also admits an equivalent formulation in terms of modular integral equations for a subgroup Γ_Λ of the Hilbert modular group of K, we soon found out that there is a group theoretic obstruction to the existence of any such formula. In its more general form, it can be nicely explained via Margulis' normal subgroup theorem. We will explain this in Chapter 4, in the discussion leading up to Proposition 4.2. On the other hand, there is a way to exploit this group theoretic obstruction to obtain the opposite of the Fourier uniqueness result that we were initially hoping for. More precisely, we proved the following result (Theorem 8). Let K be a totally real number field of degree $n \geq 2$ over $\mathbb{Q}$, and let $\sigma_1, \dots, \sigma_n : K \to \mathbb{R}$ denote the real embeddings. Define the set

$$\sqrt{\mathfrak{d}_K^{-1}} := \left\{(x_1, \dots, x_n) \in \mathbb{R}^n \,:\, (x_1^2, \dots, x_n^2) = (\sigma_1(\alpha), \dots, \sigma_n(\alpha)) \text{ for some } \alpha \in \mathfrak{d}_K^{-1}\right\}. \tag{1.8}$$

Then, for each $\epsilon \in \{\pm 1\}$, the space of $f \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\hat{f} = \epsilon f$ and $f(x) = 0$ for all $x \in \sqrt{\mathfrak{d}_K^{-1}}$, is infinite dimensional. We also proved that another discrete subset of $\mathbb{R}^n$ derived from the inverse different of K is a Fourier non-uniqueness set, see Theorem 7.

The main reason why the above Fourier non-uniqueness set is interesting is that its points are (again) contained in the spheres $\sqrt{m}S^{n-1}$, for the non-negative integers $m \geq 0$ which are traces of totally non-negative elements in the inverse different $\mathfrak{d}_K^{-1}$. Furthermore, we have the following asymptotic formula for the number of these points

$$\left|\sqrt{\mathfrak{d}^{-1}} \cap \sqrt{m}S^{n-1}\right| = 2^n \frac{\sqrt{|\operatorname{disc}(K)|}}{(n-1)!} m^{n-1} + O(m^{n-2}), \quad m \to \infty,$$

with implied constant depending only on K (hence on n). On the one hand, this is quite a large number (roughly the square of the surface area of $\sqrt{m}S^{n-1}$), but on the other hand, because of the nature of the square root map, the points are *not* uniformly distributed on $\sqrt{m}S^{d-1}$ in the sense that they "avoid" the regions close to the coordinate axis (see Figure 1).

Before proving Theorems 7 and 8, we will start Chapter 4 by studying the Fourier interpolation formulas with nodes $\sqrt{\Lambda}$ and formulate certain two natural conditions for their existence in terms of the group $\Gamma_\Lambda \leq \mathrm{PSL}_2(\mathbb{R})^n$ mentioned above. One condition is that Γ_Λ is discrete (condition D), the other one, condition (F), is a purely group theoretic one, ruling out the existence of certain relations between two subgroups of Γ_Λ. We conjecture the two conditions together are necessary and sufficient for the existence for these kinds of interpolation formulas. We rigorously proof that condition (F) is necessary and we also show that, if $n \geq 2$, then there is no lattice Λ such that Γ_Λ satisfies both conditions simultaneously. The latter result reproves that there cannot exist an interpolation formula when Λ comes from a totally real number field and the necessity of condition (F) proves, for instance, that there is no such formula in the case $\Lambda = \mathbb{Z}^n$, $n \geq 2$.

Let us mention here that another necessary condition for the existence of Fourier interpolation formulas has recently been obtained by Kulikov [Kul21] in the case $n = 1$, by considering certain counting functions of the set of interpolation nodes.

To mention another related result, Sardari [Sar21] recently studied the Fourier interpolation problem for radial Schwartz functions on $\mathbb{R}^2$ with nodes given by the lengths of vectors of the A_2 lattice to answer a question raised in [CKM$^+$21]. His work is also based on a basic relation between the generators of Hecke triangle groups.

Reading suggestions and comments on style of presentation

The thesis can be read linearly and its contents are ordered roughly from "old" to "new" with respect to the point in time at which they were obtained. The chapters are essentially self-contained. Chapter 3 depends upon chapter 2 only insofar as the two main *results* of that chapter will be used, so that, in order to gain a quick overview, the reader may only read the statements of Theorems 1 and Theorem 2 (but not their proofs) and then see how they are applied in Chapter 3. Chapter 4 is essentially self-contained, but it may be helpful to have some understanding of the strategy in §2.2.2 and to know the statements of Theorem 3 and Theorem 4 for motivation.

We intended to give all proofs in complete detail, including details in calculations. We have attempted to include many reminders and paragraphs of orientation for the reader's (and author's) convenience, thereby allowing certain repetitions and redundancies. We hope that the greater amount of detail and background material, compared to our papers [Sto21], [RS], [RS21], turns this text into a friendly introduction to "modular" Fourier interpolation and -uniqueness.

1.1 Some notation and conventions

We collect here some notations and conventions that will be used in this thesis. Occasionally, we will refer back to this section in the body of the text.

(1) We clarify that, in this thesis, $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of natural numbers and that $0 \notin \mathbb{N}$. We write $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The adjective "positive" implies nonzero, "negative" implies nonzero and "smooth" means C^∞.

(2) *Asymptotic notation.* We write $A \lesssim B$ to express that there is a constant $C > 0$ so that $|A| \leq C|B|$ for a range of parameters implicitly understood (or indicated explicitly, for emphasis). If we wish to emphasize that C may depend on a parameter p, or on several parameters $p_1, \dots p_n$, we write $A \lesssim_p B$ or $A \lesssim_{p_1,\dots,p_n} B$. The notation $A = O(B)$ bears the same meaning as $A \lesssim B$, but $A = O(B)$ will more often be used when the implied estimate holds for "large enough" parameters. The notation $A \asymp B$ means $A \lesssim B$ *and* $B \lesssim A$. If A and B are functions of some parameter n (a positive integer, say), then $A(n) \sim B(n)$ as $n \to \infty$, means that $A(n)/B(n) \to 1$ as $n \to \infty$ and $A(n) = o(B(n))$ means $A(n)/B(n) \to 0$.

(3) We denote by $\mathrm{Hol}(U, V)$ the set of all holomorphic maps $f : U \to V$ between open subsets $U, V \subseteq \mathbb{C}$.

(4) *Complex powers.* For $z \in \mathbb{H}$, the number z/i belongs to the simply connected, open right half plane $\mathcal{H} := \{w \in \mathbb{C} : \operatorname{Re}(w) > 0\}$. On $\mathcal{H}$, there is a unique holomorphic function $L : \mathcal{H} \to \mathbb{C}$ such that $L(1) = 0$ and $\exp(L(w)) = w$ for all $w \in \mathcal{H}$. It is given by $L(w) = \int_1^w d\zeta/\zeta$. It may be proved that for all $r > 0$ and $\theta \in (-\pi/2, \pi/2)$ we have $L(re^{i\theta}) = \log(r) + \theta$, where $\log : (0, +\infty) \to \mathbb{R}$ denotes the inverse of $t \mapsto e^t$, $\mathbb{R} \to (0, +\infty)$. For $k \in \mathbb{C}$ and $z \in \mathbb{H}$, we define
$$(z/i)^k := \exp(kL(z/i)).$$
In the body of the thesis, the letter L will not bear the same meaning as the one above. (The author knows that the standard terminology of L is "principal branch", but prefers not to use it.)

(5) *Assignments between functions.* As we will be dealing many times with linear maps between vector spaces of functions, we occasionally allow ourselves the standard abuse notation that consists in confusing a function with its values (e.g. writing $f(x) \mapsto f(x^2)$ to define a map between functions on $\mathbb{R}$). Similarly, when appropriate, we might confuse an element of $L^p(\mathbb{R}^d)$ with one of its representatives.

(6) *Cocycles*: Let M be an abelian group written additively. Let G be a group acting on the right on M via group automorphisms $m \mapsto m|g$, $m \in M$, $g \in G$. A 1-cocycle of G with values in M is a map $c : G \to M$ satisfying $c(g_1 g_2) = c(g_1)|g_2 + c(g_2)$ for all $g_1, g_2 \in G$ and (hence) $c(1_G) = 0_M$. A 1-coboundary of G with values of M is a 1-cocycle $c : G \to M$ of the form $c(g) = m|g$ for some fixed $m \in M$. As we will be dealing only with 1-cocycles or 1-coboundaries (and not n-cocycles with $n \geq 2$) we will just speak of cocycles. The terminology will be used for G a subgroup of $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{R})$ and M equal to the vector space of all V-valued functions $f : \mathbb{H} \to V$, for some fixed complex vector space V, in which case the action of G will be given by some "slash action" that we will specify. We will also apply the terminology for the multiplicatively written G-module $M = \mathrm{Hol}(\mathbb{H}, \mathbb{C}^\times)$.

(7) *Radial functions.* A function $f : \mathbb{R}^d \to \mathbb{C}$ is *radial* if any of the following equivalent conditions hold:

   (i) For all $x, y \in \mathbb{R}^d$ with the same Euclidean norm $|x| = |y|$, one has $f(x) = f(y)$.
   (ii) For all $r > 0$, the function f is constant on rS^{d-1}.
   (iii) f is fixed by the action of the orthogonal group $\mathrm{O}(d)$.
   (iv) There exists a function $f_0 : [0, \infty) \to \mathbb{C}$ so that $f(x) = f_0(|x|)$ for all $x \in \mathbb{R}^d$.

If $d = 1$, a radial function is the same as an even function. If $d \geq 2$, we could also replace $\mathrm{O}(d)$ with $\mathrm{SO}(d)$ in (iii). More often than not, we will confuse a radial function f with the function f_0 given in (iv), but the notation f_0 will occasionally be used as well, for emphasis.

(8) *Fourier transform.* For $f \in L^1(\mathbb{R}^d)$, we normalize its Fourier transform by
$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle} dx, \quad \xi \in \mathbb{R}^d,$$
where $\langle x, \xi \rangle = \sum_{i=1}^d \xi_i x_i$ for $x = (x_1, \ldots, x_d), \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$. Occasionally, we will also use the notations
$$\widehat{f} = \mathcal{F}(f) = \mathcal{F}_{\mathbb{R}^d}(f).$$
In particular, we might use the last version with the subscript $\mathbb{R}^d$, if f can be viewed as a radial function in different dimensions and we wish to indicate in which dimension we take the Fourier transform.

(9) *Elements of* $\mathrm{SL}_2(R)$ *and* $\mathrm{PSL}_2(R)$. Let R be a commutative unital ring. Given $a,b,c,d \in R$ satisfying $ad - bc = 1$ we denote by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(R)$$

a *matrix* and we use

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(R) = \mathrm{SL}_2(R)/\{\pm I\} = \{\{g, -g\} \,:\, g \in \mathrm{SL}_2(R)\}$$

to denote the corresponding element of $\mathrm{PSL}_2(R)$, which is not a matrix. For an element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(R)$ we sometimes use the notation $a = a_\gamma$, $b = b_\gamma$, $c = c_\gamma$, $d = d_\gamma$ for its entries. We use the same notation if $\gamma \in \mathrm{PSL}_2(R)$, provided the expression or condition in terms of $a_\gamma, b_\gamma, c_\gamma, d_\gamma$ we write is well-defined for $\gamma \in \mathrm{PSL}_2(R)$. For example, it makes sense to describe a subset of $\mathrm{PSL}_2(R)$ by imposing the condition $c_\gamma \neq 0$.

(10) *Fractional linear transformations.* We denote them as follows. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathbb{H}$, then

$$gz := \frac{az+b}{cz+d} \in \mathbb{H}.$$

(11) *Multi index notation.* Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$. We write

$$|\alpha| = \sum_{i=1}^{d} \alpha_i \qquad \text{and} \qquad \alpha! = \alpha_1! \cdots \alpha_d!.$$

For $f \in C^\infty(\mathbb{R}^d)$, we use $\partial^\alpha f$ as a short hand for $\frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we write x^α as a shorthand for the monomial $x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. The generalized Leibniz rule for partial derivative of products of functions is

$$\partial^\alpha (f_1 f_2) = \sum_{\gamma_1 + \gamma_2 = \alpha} \frac{\alpha!}{\gamma_1! \gamma_2!} (\partial^{\gamma_1} f_1)(\partial^{\gamma_2} f_2).$$

(12) *Schwartz space.* We denote by $\mathcal{S}(\mathbb{R}^d)$ the space of Schwartz functions on $\mathbb{R}^d$, topologized via the semi-norms $\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta f(x)|$ for $\alpha, \beta \in \mathbb{N}_0^d$ in the usual way. A *continuous semi-norm* on $\mathcal{S}(\mathbb{R}^d)$ is, by definition, a semi-norm $\mathcal{S}(\mathbb{R}^d) \to \mathbb{R}_{\geq 0}$ which is continuous for this topology. Any continuous semi-norm is bounded by a constant positive multiple of a finite sum of semi-norms $\|\cdot\|_{\alpha,\beta}$. We denote by $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d)^{\mathrm{O}(d)}$ the subpsace of radial Schwartz functions. It is closed and thus complete.

(13) *Sequences.* We denote by $\mathcal{S}(\mathbb{N}_0)$ the space of *rapidly decaying sequences* of complex numbers x_n, i.e. the ones that satisfy $\sup_{n \in \mathbb{N}_0} |x_n|(1+n)^A < \infty$ for all $A > 0$. It is a Fréchet space for the collection of semi-norms given by these suprema.

We denote by $\mathcal{P}(\mathbb{N}_0)$ the space of all polynomially bounded sequences of complex numbers x_n, i.e., the ones that satisfy $\sup_{n \in \mathbb{N}_0} |x_n|(1+n)^{-A} < \infty$ for some $A > 0$.

(14) *Lattices.* A *lattice* in a finite dimensional real vector space is a discrete cocompact subgroup. If $\Lambda \subseteq \mathbb{R}^d$ is a lattice we denote its dual lattice by

$$\Lambda^\vee = \{x \in \mathbb{R}^d \,:\, \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in \Lambda\}$$

A *lattice* Γ in a real Lie group G is a discrete subgroup for which the quotient space $\Gamma\backslash G$ admits a finite right G-invariant Radon measure. The latter notion will only appear once in this thesis.

(15) *Periodic holomorphic functions.* Let $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$ denote the open unit disc in $\mathbb{C}$ and let $\mathbb{D}^\times := \mathbb{D} \smallsetminus \{0\}$ be the punctured open unit disc. Fix some real number $h > 0$. Suppose $F : \mathbb{H} \to \mathbb{C}$ is a holomorphic function satisfying $F(z+h) = F(z)$ for all $z \in \mathbb{H}$. There is then a unique holomorphic function $F_{\mathrm{disc}} : \mathbb{D}^\times \to \mathbb{C}$ with the property that $F(z) = F_{\mathrm{disc}}(e^{2\pi i z/h})$ for all $z \in \mathbb{H}$. We define the *Fourier coefficients* $\widehat{F}(n) \in \mathbb{C}$, $n \in \mathbb{Z}$, of F by

$$\widehat{F}(n) = \frac{1}{h}\int_{iy-h/2}^{iy+h/2} F(z)e^{-2\pi i n z/h} dz = \frac{1}{2\pi i}\int_{|w|=\delta} F_{\mathrm{disc}}(w)\frac{dw}{w^{n+1}}, \tag{1.9}$$

where the (contour-)integrals and identities are independent of $y > 0$ and $\delta \in (0,1)$ (by Cauchy's theorem). We have the Fourier- and Laurent expansions

$$F(z) = \sum_{n\in\mathbb{Z}} \widehat{F}(n)e^{2\pi i n z/h}, \qquad F_{\mathrm{disc}}(w) = \sum_{n\in\mathbb{Z}} \widehat{F}(n)w^n,$$

which converge absolutely and uniformly on horizontal strips in $\mathbb{H}$ and annuli in $\mathbb{D}^\times$ respectively. We say that F is

- *meromorphic at infinity*, if there is $n_0 \in \mathbb{Z}$ so that $\widehat{F}(n) = 0$ for all $n < n_0$,
- *holomorphic at infinity*, if $\widehat{F}(n) = 0$ for all $n < 0$,
- *vanishes at infinity*, if $\widehat{F}(n) = 0$ for all $n \leq 0$.

If F is meromorphic at infinity we define its *valuation at infinity* to be the valuation of F_{disc} at 0 and denote it by $\nu_\infty(F) = \nu_0(F_{\mathrm{disc}}) \in \mathbb{Z}$ (it is the order of zero in case F_{disc} vanishes at zero, or minus the order of pole in case F_{disc} has a pole at zero, and zero otherwise). This discussion extends in a natural way to functions F that are meromorphic in $\mathbb{H}$.

2 Fourier interpolation for radial functions

The main results of this chapter are two interpolation theorems, Theorem 1 and Theorem 2, valid for radial Schwartz functions $f : \mathbb{R}^d \to \mathbb{C}$, of the form

$$f(x) = \sum_{n=n_0}^{\infty} a_n(x) f(\sqrt{n/\alpha}) + \sum_{n=\hat{n}_0}^{\infty} \tilde{a}_n(x) \hat{f}(\sqrt{n/\beta}) \tag{2.1}$$

for certain parameters $n_0, \hat{n}_0 \in \mathbb{N}_0$ and $\alpha, \beta > 0$. After giving some general background on modular forms and modular integrals in §2.1, we explain in §2.2.2 how the problem of finding $a_n(x)$ and $\tilde{a}_n(x)$ satisfying (2.1) is equivalent to the problem of finding certain modular integrals $F : \mathbb{H} \to \mathbb{C}$ (one for each real number $r = |x|$ with $x \in \mathbb{R}^d$, with pre-assigned period functions). We explain what these terms mean in their classical context in §2.1.3. An important technical result that makes this equivalence work is the fact that the span of all Gaussians $e^{\pi i z |x|^2}$, $z \in \mathbb{H}$, is dense in the space $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$. We discuss this and related preliminary results in §2.2.1. Theorem 1 will be proved using a method closely related to the one employed by Radchenko–Viazovska in [RV19], who construct the modular integral F as a contour integral of the Gaussian against a meromorphic modular kernel function on $\mathbb{H} \times \mathbb{H}$. Theorem 2 will be proved differently, by constructing F more directly using a construction closely related to the construction of Poincaré series. The outcomes of the two methods are quite different and we end the chapter with a comparison in §2.5. Both of these Theorems will be applied in Chapter 3. The reader who is mainly interested in that chapter may only read the statements of Theorems 1 and Theorem 2.

2.1 Background on modular forms and modular integrals

In this section, we review some basic definitions and examples of modular forms and we provide some background on so-called modular integrals and Eichler cohomology of modular forms.

2.1.1 Groups, fundamental domains, slash action

The results we recall here are mostly standard and may be found in the union of [BK08], [Iwa97], [Mum94], [Ser73], [Hec36], [Ran77].

2.1.1.1 Groups

Recall that the group $\mathrm{PSL}_2(\mathbb{R})$ acts faithfully on the upper half plane $\mathbb{H}$ via fractional linear transformations. Since the action is faithful, we will often identify $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ with the associated automorphism it defines on $\mathbb{H}$, in particular when writing compositions of maps. For $x, y \in \mathbb{R}$, we define the following elements of $\mathrm{PSL}_2(\mathbb{R})$:

$$T^x := \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \qquad V^y := \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix}, \qquad S := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We also use the abbreviations $T := T^1$, $V := V^1 = ST^{-1}S$. For every real $h > 0$, we define the subgroup

$$\Gamma_\theta(h) := \langle S, T^h \rangle \leq \mathrm{PSL}_2(\mathbb{R}). \tag{2.2}$$

The study of these subgroups $\Gamma_\theta(h)$ goes back at least to Hecke [Hec36].[4] It is known (see [BK08]) that $\Gamma_\theta(h)$ is discrete precisely, when $h \geq 2$ or when $h = 2\cos(\pi/q)$ for some integer $q \geq 3$, in which case $\Gamma_\theta(h)$ is called a Hecke triangle group. We also define

$$\Gamma(h) := \langle T^h, V^h \rangle \leq \Gamma_\theta(h),$$

consistent with the standard notation $\Gamma(N)$ for the principal congruence subgroup of level N. We use the abbreviation $\Gamma_\theta := \Gamma_\theta(2)$, which is known as the theta (sub)group. Just as $\Gamma(2)$ is normal in Γ_θ, the group $\Gamma(h)$ is always normal in $\Gamma_\theta(h)$. It will sometimes be useful to allow h to be a complex number in these definitions, in which case $\Gamma_\theta(h) \leq \Gamma(h) \leq \mathrm{PSL}_2(\mathbb{C})$.

Lemma 2.1. *For all $h \in \mathbb{C}$ such that $|h| \geq 2$, the group $\Gamma(h) \leq \mathrm{PSL}_2(\mathbb{C})$ is* freely *generated by T^h, V^h.*

Proof. Consider the following subsets of $\mathbb{P}^1(\mathbb{C})$:

$$X_h := \{z \in \mathbb{C} \,:\, |z| \leq 1/(|h|-1)\}, \quad Y_h := \{\infty\} \cup \{z \in \mathbb{C} \,:\, |z| \geq (|h|-1)\}.$$

Let $m \in \mathbb{Z} \smallsetminus \{0\}$. By the Ping Pong lemma, it suffices to show that

$$T^{mh} X_h \subseteq Y_h, \qquad V^{mh} Y_h \subseteq X_h.$$

Since $SX_h = Y_h$, $S^2 = 1$ and $ST^{mh}S = V^{-mh}$, it suffices to prove the first of these containments. And indeed, for $z \in X_h$, we have

$$|T^{mh}z| = |z + mh| \geq |m||h| - |z| \geq |h| - |z| \geq |h| - \frac{1}{|h|-1} \geq |h| - 1,$$

since the last inequality is equivalent to $|h| \geq 2$. □

Remark 2.1. Later, in Corollary 2.2, we will also show that, when $|h| \geq 2$, then $S \notin \Gamma(h)$ and that the only relation in $\Gamma_\theta(h)$ is $S^2 = 1$.

2.1.1.2 Fundamental domains

In this thesis, fundamental domains $\mathcal{D} \subseteq \mathbb{H}$ for quotients $\Gamma\backslash\mathbb{H}$ of the upper half plane $\mathbb{H}$ by discrete subgroups $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ are required to be open and connected, to intersect every Γ-orbit in at most one point while the closure $\overline{\mathcal{D}}$ must intersect every orbit in at least one and at most two points. A fundamental domain in the strict sense may be obtained by adding to $\mathcal{D}$ certain subsets of the boundary to the open fundamental domain.

For $h > 0$ such that $\Gamma_\theta(h)$ is discrete in $\mathrm{PSL}_2(\mathbb{R})$, it is known ([BK08, Thm 3.1]) that the set

$$\mathcal{D}_h := \{z \in \mathbb{H} \,:\, |\operatorname{Re}(z)| < h/2, \quad |z| > 1\} \tag{2.3}$$

is a fundamental domain for $\Gamma_\theta(h)\backslash\mathbb{H}$. For $h \geq 2$, the set

$$\mathcal{D}_h \cup S\mathcal{D}_h = \{z \in \mathbb{H} \,:\, |\operatorname{Re}(z)| < h/2,\ |z - 1/h| > 1/h,\ |z + 1/h| > 1/h\} \tag{2.4}$$

is a fundamental domain for $\Gamma(h)\backslash\mathbb{H}$.

[4] We remark that h is denoted as λ in [Hec36]. We will reserve λ for the modular λ-invariant, introduced below.

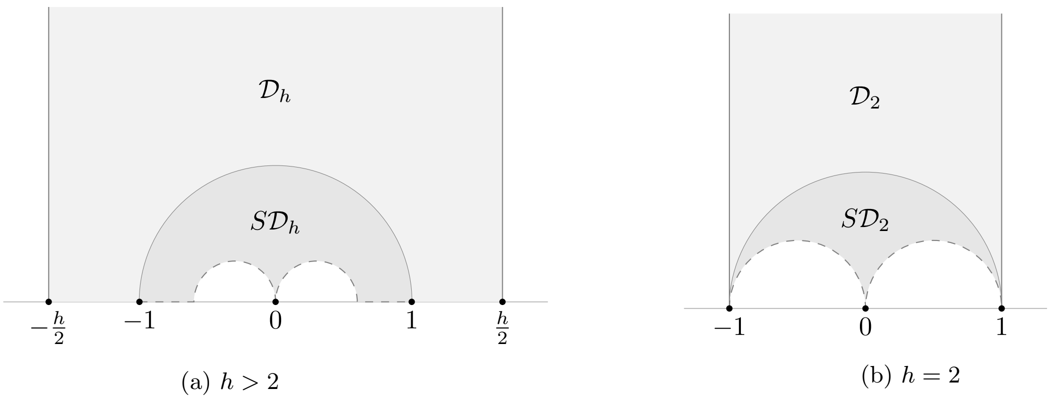

Figure 2: Fundamental domains for $\Gamma_\theta(h)$

2.1.1.3 Slash action and cocycles

Here, we will define a certain slash-action of subgroups $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ on functions on $\mathbb{H}$ with values in $\mathbb{C}$ or a complex vector space. This notation gives us an efficient tool to describe various identities satisfied by such functions.

We define a cocycle $\mu : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{Hol}\,(\mathbb{H}, \mathbb{C}^\times)$ by

$$\mu(\gamma)(z) := \mu(\gamma, z) := c_\gamma z + d_\gamma, \quad \gamma \in \mathrm{SL}_2(\mathbb{R}), z \in \mathbb{H}. \tag{2.5}$$

Recall that c_γ, d_γ denote the lower row entries of $\gamma \in \mathrm{SL}_2(\mathbb{R})$ (item (9) in §1.1). To say that μ is a cocycle means that we have

$$\mu(\gamma_1\gamma_2) = (\mu(\gamma_1) \circ \gamma_2) \cdot \mu(\gamma_2) \quad \text{for all } \gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{R}).$$

Since we have $\mu(-\gamma, z) = -\mu(\gamma, z)$, this cocyle is *not* well-defined on $\mathrm{PSL}_2(\mathbb{R})$ (but $|\mu|$ or μ^2 are, for example). Not also that $\mu(\gamma)^{-2}$ is the complex derivative of the fractional linear transformation $\gamma : \mathbb{H} \to \mathbb{H}$.

If $k \in \mathbb{Z}$ is an integer, we can define the "standard" slash-action in *weight* k of the group $\mathrm{SL}_2(\mathbb{R})$ on complex-valued (or complex-vector-space-valued) functions f on $\mathbb{H}$ by

$$(f|_k\gamma) = \mu(\gamma)^{-k}(f \circ \gamma). \tag{2.6}$$

We may extend it to formal $\mathbb{C}$-linear combinations of γ's, or, in other words, to the group algebra $\mathbb{C}[\mathrm{SL}_2(\mathbb{R})]$. If k is an even integer, we can also work with $\mathrm{PSL}_2(\mathbb{R})$ instead of $\mathrm{SL}_2(\mathbb{R})$.

In fact, we will rarely work with the slash action defined as in (2.6). Instead, we will work with the group $\Gamma_\theta(h)$ and define, for any *real* number k, the cocycle

$$j_k : \Gamma_\theta(h) \to \mathrm{Hol}(\mathbb{H}, \mathbb{C}^\times), \tag{2.7}$$

by prescribing its values on generators to be

$$j_k(S, z) = (z/i)^k, \quad j_k(T^h, z) = 1 \tag{2.8}$$

and in general by requiring that $j_k(\gamma_1\gamma_2) = (j_k(\gamma_1) \circ \gamma_2) j_k(\gamma_2)$ holds for all $\gamma_1, \gamma_2 \in \Gamma_\theta(h)$. Since the only relation $S^2 = 1$ in $\Gamma_\theta(h)$ is preserved, this is well-defined. Based upon this, we define a slash action of $\Gamma_\theta(h)$ on complex-(vector-space)-valued functions f on $\mathbb{H}$ by

$$f|_k\gamma = j_k(\gamma)^{-1}(f \circ \gamma). \tag{2.9}$$

Of course, both the cocycle j_k and the slash-action just introduced, depend upon the parameter h in $\Gamma_\theta(h)$, but we will suppress this in the notation. Note that, when, k is an integer divisible by four, then $\mu(\gamma)^k = j_k(\gamma)$ for all $\gamma \in \Gamma_\theta(h)$. Indeed, this equality can be checked on generators, in which case it is clear.

We will also use that for all $\gamma \in \Gamma_\theta(h)$, all $k \in \mathbb{R}$ and $z \in \mathbb{H}$ we have

$$|j_k(\gamma, z)| = |c_\gamma z + d_\gamma|^k. \tag{2.10}$$

To see that this is true, we note that both sides of the equality (2.10) define cocycles $\Gamma_\theta(h) \to C(\mathbb{H}, \mathbb{R}_{>0})$, so that the equality can again be checked on the generators $\gamma = T^h$ and $\gamma = S$ of $\Gamma_\theta(h)$, in which case it is clear.

2.1.2 Theta functions and the modular lambda invariant

Here, we recall various properties of three well-known theta functions and the modular lambda invariant, which will play an important role in our proofs, especially in the proof of Theorem 1.

For $\tau \in \mathbb{H}$, we define the theta functions

$$\begin{aligned}
\Theta_2(\tau) = \theta_{10}(\tau) &= \sum_{n\in\mathbb{Z}} e^{\pi i (n+1/2)^2 \tau},\\
\Theta_3(\tau) = \theta_{00}(\tau) &= \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau},\\
\Theta_4(\tau) = \theta_{01}(\tau) &= \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i n^2 \tau}.
\end{aligned}$$

By direct computation using the Poisson summation formula, we see that these functions have the following transformation properties

$$\Theta_2(\tau+1) = e^{\pi i/4}\Theta_2(\tau) \qquad \Theta_2(-1/\tau) = (\tau/i)^{1/2}\Theta_4(\tau) \tag{2.11}$$

$$\Theta_3(\tau+1) = \Theta_4(\tau) \qquad \Theta_3(-1/\tau) = (\tau/i)^{1/2}\Theta_3(\tau) \tag{2.12}$$

$$\Theta_4(\tau+1) = \Theta_3(\tau) \qquad \Theta_4(-1/\tau) = (\tau/i)^{1/2}\Theta_2(\tau) \tag{2.13}$$

For any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $c = c_\gamma > 0$, $d = d_\gamma$ and such that γ represents an element of Γ_θ, we have

$$\Theta_3(\gamma z) = \frac{g_c(d)}{\sqrt{c}}((cz+d)/i)^{1/2}\Theta_3(z) = g_c(d)((z+d/c)/i)^{1/2}\Theta_3(z), \tag{2.14}$$

where $g_c(d)$ is defined in terms of the Gauss sums $G(a,q) = \sum_{m=0}^{q-1} e^{2\pi i m^2/q}$ by

$$g_c(d) := \begin{cases} G(2d, c) & \text{if } c \equiv 1 \pmod 2,\\ \frac{1}{2}G(d, 2c) & \text{if } c \equiv 0 \pmod 2.\end{cases}$$

Remark 2.2. We give an extended remark concerning the notations $\theta_{00}, \theta_{10}, \theta_{01}$ for Θ_3, Θ_2 and Θ_4. This will be inessential to what follows, but it gives us the chance to provide a bit more context, introduce two-variable Jacobi theta function and write down the Jacobi triple product formula

$$\vartheta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau} e^{2\pi i n z} = \prod_{m=1}^{\infty} (1 - e^{2\pi i m\tau})(1 + e^{\pi i(2m-1)\tau}e^{2\pi i z})(1 + e^{\pi i(2m-1)\tau}e^{-2\pi i z}). \tag{2.15}$$

This holds for all $\tau \in \mathbb{H}$ and all $z \in \mathbb{C}$ and both sides are absolutely convergent.

We start by stating the following fact: For fixed $\tau \in \mathbb{H}$, the entire function $z \mapsto \vartheta(z, \tau)$ spans the *one-dimensional* space of entire functions $f : \mathbb{C} \to \mathbb{C}$ obeying the following transformation laws with respect to the group action of the lattice $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z} \subseteq \mathbb{C}$ on $\mathbb{C}$:

$$f(z + a\tau + b) = e^{-\pi i a^2 \tau - 2\pi i a z} f(z), \qquad a, b \in \mathbb{Z}.$$

For the proof, we may note that these transformation laws yield a simple recursion formula for the Fourier coefficients of $f(z) = \sum_{n\in\mathbb{Z}} a_f(\tau, n) e^{2\pi i n z}$.

We may reformulate and generalize this fact in more group theoretic terms as follows. Let $\mathcal{E}$ denote the space of all entire functions $f : \mathbb{C} \to \mathbb{C}$. Fix $\tau \in \mathbb{H}$. For $a, b \in \mathbb{R}$ and $f \in \mathcal{E}$, we define[5] $T_a f, S_b f \in \mathcal{E}$ by

$$T_a f(z) = e^{\pi i a^2 \tau + 2\pi i a z} f(z + a\tau), \qquad S_b f(z) = f(z + b), \quad z \in \mathbb{C}.$$

It is easy to verify that $a \mapsto T_a$ and $b \mapsto S_b$ define group homomorphisms $\mathbb{R} \to \mathrm{Aut}_{\mathbb{C}}(\mathcal{E})$. Let $\mathcal{H}_\tau \leq \mathrm{Aut}\,(\mathcal{E})$ denote the subgroup generated by all automorphisms T_a, S_b, $a, b \in \mathbb{R}$. The relation $[T_a, S_b] = e^{-2\pi i a b}\,\mathrm{id}_{\mathcal{E}}$ shows that any element of $\mathcal{H}_\tau$ is of the form $\lambda(T_a \circ S_b)$ for some $\lambda \in \mathbb{C}^{(1)}$, $a, b \in \mathbb{R}$. In fact, $\mathcal{H}$ is isomorphic to the central extension $\mathcal{H} := \mathbb{C}^{(1)} \times \mathbb{R} \times \mathbb{R}$ of $\mathbb{R} \times \mathbb{R}$, with group law

$$(\lambda_1, a_1, b_1)(\lambda_2, a_2, b_2) := (\lambda_1 \lambda_2 e^{2\pi i b_1 a_2}, a_1 + a_2, b_1 + b_2), \quad a_i, b_i \in \mathbb{R}, \lambda_i \in \mathbb{C}^{(1)}.$$

Moreover, $\Phi = \Phi_\tau : \mathcal{H} \to \mathcal{H}_\tau$, $\Phi(\lambda, a, b) := \lambda(T_a \circ S_b)$ is an isomorphism of groups. For every integer $\ell \geq 1$, consider the subgroup $\Gamma_\ell := (\{1\} \times \ell\mathbb{Z} \times \ell\mathbb{Z}) \leq \mathcal{H}$. One may show that $\dim(\mathcal{E}^{\Gamma_\ell}) = \ell^2$ for all $\ell \geq 1$, with the case $\ell = 1$ corresponding to the fact about $z \mapsto \vartheta(z, \tau)$ already discussed. In particular, in the the case $\ell = 2$, we obtain a 4-dimensional space.

For $u, v \in \{0, 1\}$, the functions $\theta_{uv}(\tau) = \theta_{u,v}(\tau)$ defined above are derived as follows from members of this 4-dimensional space. They are given as the following "Nullwerte" of translates of $\vartheta(z, \tau)$:

$$\theta_{u,v}(\tau) := (T_{\frac{u}{2}} S_{\frac{v}{2}} \vartheta(\cdot, \tau))(z)\Big|_{z=0} = e^{\pi i \frac{u^2}{4}\tau + \pi i z} \vartheta(z + \tfrac{u}{2}\tau + \tfrac{v}{2}, \tau)\Big|_{z=0} = e^{\pi i \frac{u^2}{4}} \vartheta(\tfrac{u}{2}\tau + \tfrac{v}{2}, \tau).$$

So, more explicitly,

$$\theta_{10}(\tau) = \vartheta(\tfrac{\tau}{2}, \tau) e^{\pi i \frac{\tau}{4}}, \qquad \theta_{00}(\tau) = \vartheta(0, \tau), \qquad \theta_{01}(\tau) = \vartheta(\tfrac{1}{2}, \tau) \tag{2.16}$$

and there is also the "missing" theta function

$$\theta_{11}(\tau) = \vartheta(\tfrac{\tau}{2} + \tfrac{1}{2}, \tau) e^{\pi i \frac{\tau}{4}} = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau} e^{2\pi i n(\frac{\tau}{2} + \frac{1}{2}) + \pi i \frac{\tau}{4}} = \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i (n+1/2)^2 \tau} = 0,$$

which can be seen either by pairing $n \in \mathbb{N}_0$ with $-n-1$ or by twice applying the identity $\theta_{11}(-1/\tau) = -i(\tau/i)^{1/2}\theta_{11}(\tau)$ which in turn follows from Poisson summation.

It is well-known that the fourth powers $\Theta_2^4, \Theta_3^4, \Theta_4^4$ define modular forms for the group $\Gamma(2)$ of weight 2 and that they are related by Jacobi's identity

$$\Theta_3(z)^4 = \Theta_2(z)^4 + \Theta_4(z)^4. \tag{2.17}$$

[5]There is no relationship between the matrices S, T defined above and the linear maps T_a, S_b defined here; we simply follow Mumford's notation [Mum94].

	∞	1	0
θ_{10}^4	1	0	0
θ_{00}^4	0	1	0
θ_{01}^4	0	0	1
λ	1	-1	0
λ'	1	-1	1

Table 1: Valuations at the cusps

Recall that the group $\Gamma(2)$ has three cusps, namely $0, 1, \infty$. More precisely, we have

$$\Gamma(2)\backslash\mathbb{P}^1(\mathbb{Q}) = \{\Gamma(2)0, \Gamma(2)1, \Gamma(2)\infty\}.$$

In general, for an even integer $k \in 2\mathbb{Z}$ and any nonzero meromorphic[6] modular form $f : \mathbb{H} \to \mathbb{P}^1(\mathbb{C})$ of weight k for $\Gamma(2)$, the following valency formula holds

$$\nu_0(f) + \nu_1(f) + \nu_\infty(f) + \sum_{[z]\in\Gamma(2)\backslash\mathbb{H}} \nu_z(f) = \frac{k}{2}, \tag{2.18}$$

where:

- we used $[z] = \Gamma(2)z$ to denote the orbit of $z \in \mathbb{H}$ under $\Gamma(2)$,
- for $z \in \mathbb{H}$, $\nu_z(f) \in \mathbb{Z}$ is the valuation of f at the point in the sense of complex analysis, i.e. the order of zero, or minus the order of pole. This depends only on $[z]$ and not on the representative of that orbit.
- for $s \in \{0, 1, \infty\}$, the valuation $\nu_s(f)$ is defined as

  $$\nu_s(f) = \nu_\infty(f|_k\gamma), \qquad \text{for any } \gamma \in \Gamma(1) \text{ such that } \gamma\infty = s,$$

  which is well-defined in the sense that, firstly, $f|_k\gamma$ is 2-periodic (in fact it is modular of weight k for the group $\gamma^{-1}\Gamma(2)\gamma = \Gamma(2)$) so that the definition given in item (15) of §1.1 applies and, secondly, the valuation is also independent of the choice of γ with $\gamma\infty = \gamma$, since any two choices differ by a power of T, which only multiplies the Fourier coefficients by a root of unity.

A proof of (2.18) can be given via integration of $f'(z)/f(z)$ over the boundary of a (suitably modified) fundamental domain for $\Gamma(2)$, given as in (2.4). Table 1 lists the valuations of the Θ_2^4, Θ_3^4 and Θ_4^4 and it follows from this table and the valency formula (2.18) that none of these functions has a zero in the upper half plane. Alternatively, this follows from (2.16) and Jacobi's triple product formula (2.15). We will also be working with the *elliptic modular lambda invariant* $\lambda : \mathbb{H} \to \mathbb{C}$, which is defined as

$$\lambda(z) := \frac{\Theta_2(z)^4}{\Theta_3(z)^4}. \tag{2.19}$$

Since Θ_2^4 and Θ_3^4 have no zeros in $\mathbb{H}$ and are both modular of weight 2 for $\Gamma(2)$, the function λ is well-defined, holomorphic, zero-free in $\mathbb{H}$ and it is $\Gamma(2)$-invariant. It transforms as follows under S, T:

$$\lambda(Sz) = 1 - \lambda(z), \qquad \lambda(Tz) = \frac{\lambda(z)}{\lambda(z) - 1}. \tag{2.20}$$

[6]including meromorphic at the cusps

This follows easily from (2.11), (2.12), (2.13) and Jacoboi's identity (2.17). From the first identity and the fact that $\lambda(Sz) \neq 0$ for all $z \in \mathbb{H}$, it also follows that $\lambda(z) \neq 1$ for all $z \in \mathbb{H}$. The behavior of λ at the cusps is recorded in Table 1. From these properties, we may deduce that for any $w \in \mathbb{C} \setminus \{0,1\}$, the function $z \mapsto \lambda(z) - w$ is holomorphic in $\mathbb{H}$, modular of weight 0 for $\Gamma(2)$, does not vanish at the cusps ∞, 0, but has a pole at the cusp 1, so that, by the valency formula (2.18), it must have a zero in $\mathbb{H}$. In other words, $\lambda : \mathbb{H} \to \mathbb{C} \setminus \{0,1\}$ is surjective. The following formula will also be used

$$\lambda'(z) = \pi i \Theta_3(z)^4 \lambda(z)(1-\lambda(z)) \tag{2.21}$$

We prove this formula as an exercise. Let temporarily $\varphi(z) = \lambda(z)(1-\lambda(z))\Theta_3(z)^4$. Then $\lambda'(z)/\varphi(z)$ is a $\Gamma(2)$-invariant holomorphic function on $\mathbb{H}$ being the ratio of two forms of weight 2. To show that it is constant, it suffices to show that it is holomorphic at all three cusps (because $X(2) = \Gamma(2)\backslash(\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}))$ can be made into a compact Riemann surface and φ would then define a holomorphic function on it). First, we note that with $q = e^{\pi i z}$, we have, using the notation from item (15) in §1.1,

$$\lambda'(z) = \pi i q \frac{d}{dq} \lambda_{\mathrm{disc}}(q) = \pi i \sum_{n=1}^{\infty} n \widehat{\lambda}(n) q^n,$$

so that $\nu_\infty(\lambda') = 1$, since it is easy to see that $\widehat{\lambda}(1) = 16$. To determine $\nu_0(\lambda')$ and $\nu_1(\lambda')$, we first differentiate the relations

$$\lambda(Sz) = 1 - \lambda(z), \qquad \lambda(TSz) = 1 - 1/\lambda(z)$$

to obtain

$$\lambda'(Sz)z^{-2} = (\lambda|_2 S)(z) = -\lambda'(z), \qquad \lambda'(TSz)z^{-2} = (\lambda'|_2(TS))(z) = \frac{\lambda'(z)}{\lambda(z)^2}$$

and then deduce $\nu_0(\lambda') = 1$, $\nu_1(\lambda') = 1 - 2 = -1$. On the other hand, from what we already know about λ, Θ_3^4, we can compute

$$\nu_\infty(\varphi) = 1 + 0 + 0 = 1, \qquad \nu_0(\varphi) = 0 + 1 + 0 = 1, \qquad \nu_1(\varphi) = -1 + -1 + 1 = -1,$$

where the order in the summation corresponds to the order of the factors $\lambda, (1-\lambda), \Theta_3^4$ in the definition of φ. Since the valuations of φ and λ' agree at all cusps, the function φ is indeed constant on $X(2)$ and hence on $\mathbb{H}$. That the constant equals πi, follows from $\lambda'(z) \sim (16)(\pi i)e^{\pi i z}$, $\Theta_3(z) \sim 1$, $\lambda(z) \sim 16e^{\pi i z}$,$1 - \lambda(z) \sim 1$ as $\mathrm{Im}(z) \to \infty$. This finishes the proof of (2.21).

2.1.2.1 More on cocycles and slash actions

Here, we add a few more comments on cocycles and slash actions, which are specific to the groups Γ_θ and $\Gamma(2)$. We will refer back to this in the proof of Theorem 1, but it is not essential for any other part of this thesis.

We abbreviate $\Theta_3(z)$ to $\Theta(z)$. We define the following holomorphic logarithm of Θ:

$$L_\Theta(z) := -\int_z^{i\infty} \Theta'(w)/\Theta(w)dw, \qquad z \in \mathbb{H}.$$

This makes sense, since $\Theta'(z)/\Theta(z)$ is a 2-periodic function on $\mathbb{H}$ that decays exponentially as $\mathrm{Im}(z) \to \infty$. It follows that $L_\Theta(z) \to 0$ as $\mathrm{Im}(z) \to \infty$ and $L'_\Theta(z) = \Theta'(z)/\Theta(z)$ so that we indeed have $\exp(L_\Theta(z)) = \Theta(z)$ for all $z \in \mathbb{H}$. (Differentiate the ratio to see that it's constant and note that both sides tend to 1 as $\mathrm{Im}(z) \to \infty$). Moreover, since $\Theta(iy) \in \mathbb{R}_{>0}$ and $\Theta'(iy) \in i\mathbb{R}$ for all

$y > 0$, we see that $L_\Theta(iy) = \log(\Theta(iy))$ for all $y > 0$, where on the right we mean, of course, the usual real logarithm of positive real numbers. By 2-periodicity of the integrand $\Theta'(z)/\Theta(z)$ (and by Cauchy's Theorem), it also follows that $L_\Theta(z+2) = L_\Theta(z)$ for all $z \in \mathbb{H}$. We define

$$\Theta(z)^k := \Theta^k(z) := \exp(kL_\Theta(z)) \in \mathbb{C}^\times, \quad k \in \mathbb{C}, z \in \mathbb{H}.$$

Then, with $j_k : \Gamma_\theta \to \mathbb{C}^\times$ defined as in (2.8) we have

$$j_k(\gamma)(z) = \Theta^{2k}(\gamma z)/\Theta^{2k}(z) \quad \text{for all } \gamma \in \Gamma_\theta, z \in \mathbb{H}. \tag{2.22}$$

Indeed, it suffices to verify this for $\gamma = T^2$, in which case both sides are equal to 1 and for $\gamma = S$, we can argue as follows. We need to show that $\Theta_3^{2k}(Sz)/\Theta_3^{2k}(z) = (z/i)^k$, for all $z \in \mathbb{H}$, so we can specialize to $z = iy$ for $y > 0$ and compute

$$\begin{aligned}\Theta^{2k}(S(iy)) &= \exp\left(2kL_\Theta(i/y)\right) = \exp\left(2k\log\left(\Theta(i/y)\right)\right) = \exp\left(2k(\log\left(y^{1/2}\Theta(iy)\right)\right)\\ &= \exp\left(2k(\log\left(y^{1/2}\right) + \log\left(\Theta(iy)\right))\right) = y^k\Theta(iy)^{2k} = (iy/i)^k\Theta(iy)^{2k},\end{aligned}$$

as desired. We might occasionally also use the notation

$$j_\Theta(\gamma)(z) := j_\Theta(\gamma, z) := \Theta(\gamma z)/\Theta(z),$$

so that (2.22) can be written as $j_\Theta(\gamma)^{2k} = j_k(\gamma)$ for all $\gamma \in \Gamma_\theta$ (where $j_\Theta(\gamma)^{2k}$ is defined as on the right hand side of that equation). Note that $j_\Theta(\gamma)$ is defined for all $\gamma \in \mathrm{PSL}_2(\mathbb{R})$, as opposed to $j_k(\gamma)$, which is a prioroi only defined on $\Gamma_\theta(h)$.

Definition 2.1 (in force when working with $\Gamma(2)$ and $k \in \mathbb{R}$)**.** Let $k \in \mathbb{R}$ and $\Gamma \in \{\Gamma(2), \Gamma_\theta\}$. A function $\varphi : \mathbb{H} \to \mathbb{C}$ is modular of weight k with respect to Γ if $\varphi|_k\gamma = \varphi$ for all $\gamma \in \Gamma$. Such a function is a *modular form of weight* k if it is in addition holomorphic and of moderate growth (see Definition 2.2 below).

Let us also extend the notation of valuations at cusps of $\Gamma(2)$ to the case of real weights $k \in \mathbb{R}$. Suppose that a holomorphic (or meromorphic) function $\varphi : \mathbb{H} \to \mathbb{C}$ is modular of weight k with respect to $\Gamma(2)$. For $s \in \{0, 1, \infty\}$ define $A_s \in \Gamma_\theta$ by

$$A_\infty := 1, \quad A_0 := S, \quad A_1 := TS,$$

Note that $z \mapsto (\varphi|_kA_s)(z)$ is 2-periodic (in fact, it is modular of weight k for $\Gamma(2)$, since $\Gamma(2)$ is normal in Γ_θ). We say that φ has *valuation* $m \in \mathbb{R}$ at the cusp s, if for some $c \in \mathbb{C}^\times$ we have

$$(\varphi|_kA_s)(z) \sim ce^{\pi imz}, \quad \text{as} \quad \mathrm{Im}(z) \to \infty.$$

In this case, we write $\nu_s(\varphi) = m$.

2.1.3 Period functions and modular integrals

The purpose of this section is to collect some background material on modular integrals. To motivate this subject, we begin by discussing the notion of a (rational) *period* function for the group $\mathrm{SL}_2(\mathbb{Z})$. To that end, we start with a familiar example, the Eisenstein series of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$. Recall that it is given by

$$E_2(z) = 1 + \frac{1}{2\zeta(2)}\sum_{0\neq m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^2} = 1 - 24\sum_{\ell=1}^{\infty}\sigma_1(\ell)e^{2\pi i\ell z} = \frac{1}{2\pi i}\frac{\Delta'(z)}{\Delta(z)}, \tag{2.23}$$

where:

- the first series over n, m converges in the indicated order of summation,
- $\sigma_s(\ell) = \sum_{d|\ell} d^s$ for any $s \in \mathbb{C}$, $\ell \in \mathbb{N}$,
- $\Delta(z) = e^{2\pi i z} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})^{24}$ is the modular discriminant.

As is very well-known, E_2 is holomorphic and 1-periodic, but not a modular form for $\mathrm{SL}_2(\mathbb{Z})$. (If we have forgotten it) we can easily derive its transformation behavior under $\mathrm{SL}_2(\mathbb{Z})$ by using its expression as a logarithmic derivative of a modular form. To explain how, consider more generally a meromorphic modular form f on $\mathbb{H}$ for the group $\mathrm{SL}_2(\mathbb{Z})$ of weight $k \in 2\mathbb{Z}$. By differentiating the relation

$$(f \circ \gamma) = \mu(\gamma)^k \cdot f, \qquad \gamma \in \mathrm{PSL}_2(\mathbb{Z})$$

we obtain

$$(f' \circ \gamma)\mu(\gamma)^{-2} = k\mu(\gamma)^{k-1} c_\gamma f + \mu(\gamma)^k f'$$

and then, by dividing by $f \circ \gamma = \mu(\gamma)^k f$ and multiplying by $\mu(\gamma)^2$, the identity

$$\frac{f' \circ \gamma}{f \circ \gamma} = k\mu(\gamma)c_\gamma + \mu(\gamma)^2 \frac{f'}{f}. \tag{2.24}$$

Specializing to $k = 12$, $f = \Delta$, we get

$$(E_2 \circ \gamma)\mu(\gamma)^{-2} - E_2 = \frac{12 c_\gamma}{(2\pi i)\mu(\gamma)} = \frac{6}{\pi i} \frac{c_\gamma}{\mu(\gamma)}. \tag{2.25}$$

As a side remark, if, conversely, we have proved (2.25) in some other way, one can deduce that Δ, when defined by the above product, is modular of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$.

Using the slash action in weight k introduced in (2.6), we can write (2.25) as

$$E_2|_2\gamma - E_2 = \frac{6}{\pi i} \frac{c_\gamma}{\mu(\gamma)}$$

which shows that $\gamma \mapsto \varphi_\gamma := \frac{6}{\pi i} \frac{c_\gamma}{\mu(\gamma)}$ is a cocycle of the group $\mathrm{PSL}_2(\mathbb{Z})$ with values in the space of (nowhere vanishing) rational functions of $z \in \mathbb{H}$, because it is in fact a coboundary (see item (6) in §1.1). In other words $\{\varphi_\gamma\}_{\gamma \in \Gamma}$ is a collection of (nowhere vanishing) rational functions on $\mathbb{H}$, satisfying

$$\varphi_{\gamma_1\gamma_2} = \varphi_{\gamma_1}|_{\gamma_2} + \varphi_{\gamma_2} \tag{2.26}$$

for all $\gamma_1, \gamma_2 \in \Gamma$. Note that any $\mathrm{PSL}_2(\mathbb{Z})$-cocycle is determined by its values on the generators S and T and that, in the above example, the values are the functions $\varphi_T = 0$ and $\varphi_S(z) = \frac{6}{\pi i} \frac{1}{z}$.

Let us now generalize the setting. Fix a weight $k \in 2\mathbb{Z}$ and let $\mathcal{V}$ be a space of (holomorphic) functions on $\mathbb{H}$ which is stable under the slash action in weight k of the group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. As already mentioned, a Γ-cocycle $\gamma \mapsto q_\gamma$ $\Gamma \to \mathcal{V}$ is then uniquely determined by its values $q_T \in \mathcal{V}$ and $q_S \in \mathcal{V}$. It is called *parabolic*, if $q_T = 0$. Note that if a parabolic cocycle is a coboundary, that is, if $q_\gamma = F|(\gamma - 1)$, for some $F \in \mathcal{V}$, then F is necessarily 1-periodic. Note also that a parabolic cocycle $q : \Gamma \to \mathcal{V}$ is uniquely determined by the function $q_S \in \mathcal{V}$. We call $f = q_S$ the *period function* of the parabolic cocycle q. If we write $U = TS$, we see that the period function f satisfies the following two identities:

$$\begin{aligned}
0 &= q_1 = q_{S^2} = f|S + f = f|(1 + S), \\
0 &= q_1 = q_{U^3} = q_U|U^2 + q_U|U + q_U = q_U|(1 + U + U^2) = f|(1 + U + U^2).
\end{aligned}$$

Here we used that, since $q_T = 0$, we have

$$q_U = q_T|S + q_S = 0 + f = f.$$

Conversely, for any given $f \in \mathcal{V}$, there exists a parabolic cocycle Γ-cocyle $\{p_\gamma\}_{\gamma\in\Gamma}$ with period function $p_S = f$, if and only if $f|(1+S) = 0$ and $f|(1+U+U^2) = 0$.

Motivated by the study of the Eisenstein series of weight 2, Marvin Knopp started to investigate the problem of characterizing Γ-cocycles of positive weight $2k \in \mathbb{Z}_{>0}$, with values in the space of *rational functions* on $\mathbb{H}$, with no poles in $\mathbb{H}$ and discovered that this question is connected to the real quadratic fields. In his papers [Kno78], [Kno81] he proved – among many other things – that the poles of any such rational period function are on the real line and given by real quadratic irrationals. We refer the reader also to the papers [CZ93], [Kno89] for related results.

A related question – which is more important for us – is that of determining when a parabolic cocycle $\{q_\gamma\}_{\gamma\in\Gamma}$ attached to some rational period function $f = q_S$, is equal to a coboundary. In other words, when can one solve the equation(s) $F|(\gamma-1) = q_\gamma$, $\gamma \in \Gamma$, or equivalently, the two equations $F|(T-1) = 0$, $F|(S-1) = f$ for a given period function f? In his paper [Kno74], Knopp answered this question in great generality and often in the affirmative. The main idea is to use a generalized Poincaré series, due to Eichler [Eic65]. Let us present Eichler's construction.

We take an auxiliary weight $m \in 2\mathbb{Z}_{\geq 2}$, which should be thought of as "sufficiently large" for convergence. Consider the series

$$\Phi := \sum_{[\gamma]\in\Gamma_\infty\backslash\Gamma} q_\gamma \cdot \mu(\gamma)^{-m}.$$

It makes sense to sum over cosets $[\gamma] \in \Gamma_\infty\backslash\Gamma$, because for $\gamma \in \Gamma$ and $\gamma_\infty = T^\ell \in \Gamma_\infty$, we have

$$q_{\gamma_\infty\gamma} \cdot \mu(\gamma_\infty\gamma)^{-m} = (q_{\gamma_\infty}|_k\gamma + q_\gamma)\left((\mu(\gamma_\infty)\circ\gamma)\cdot m(\gamma)\right)^{-m} = q_\gamma\mu(\gamma)^{-m}$$

since both the "additive" cocycle q and the "multiplicative" cocycle μ^2 are trivial on $\Gamma_\infty \leq \Gamma$. We assume in the following that the series Φ converges absolutely and we compute, for any $\rho \in \Gamma$, that

$$\begin{aligned}
\Phi\circ\rho &= \sum_{[\gamma]\in\Gamma_\infty\backslash\Gamma} (q_\gamma\circ\rho)\cdot(\mu(\gamma)\circ\rho)^{-m} \\
&= \sum_{[\gamma]\in\Gamma_\infty\backslash\Gamma} (q_\gamma\circ\rho)\mu(\rho)^{-k}\mu(\rho)^k\cdot((\mu(\gamma)\circ\rho)\mu(\rho))^{-m}\mu(\rho)^m \\
&= \mu(\rho)^{m+k} \sum_{[\gamma]\in\Gamma_\infty\backslash\Gamma} (q_\gamma|_k\rho)\mu(\gamma\rho)^{-m} \\
&= \mu(\rho)^{m+k} \sum_{[\gamma]\in\Gamma_\infty\backslash\Gamma} (q_{\gamma\rho} - q_\rho)\mu(\gamma\rho)^{-m} \\
&= \mu(\rho)^{m+k}\Phi - \mu(\rho)^{m+k}q_\rho E_m,
\end{aligned}$$

where $E_m = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\mu(\gamma)^{-m}$ is the Eisenstein series of weight m. Multiplying by $\mu(\rho)^{-k-m}$, we get the identity

$$(\Phi\circ\rho)\mu(\rho)^{-k-m} = \Phi - q_\rho E_m. \tag{2.27}$$

Now, the a priori only meromorphic function F on $\mathbb{H}$, given by the ratio $F := -\frac{\Phi}{E_m}$ satisfies

$$F|_k\rho - F = -\frac{(\Phi\circ\rho)\mu(\rho)^{-k}}{E_m\circ\rho} + \frac{\Phi}{E_m} = -\frac{(\Phi\circ\rho)\mu(\rho)^{-k-m}}{E_m} + \frac{\Phi}{E_m} = -\frac{\Phi - q_\rho E_m}{E_m} + \frac{\Phi}{E_m} = q_\rho$$

where we used (2.27) in the second last step. This solves the problem formally, but there are two obvious problems:

- We can only handle cocycles $\gamma \mapsto q_\gamma$ whose growth we can control to make Φ a nicely convergent series.

- Since E_m has zeros on the upper half plane, the function $F = -\Phi/E_m$ may have poles at the zeros of E_m. We need to remove them by subtracting from F a suitable (meromorphic) modular form of weight k. Note that for every modular form f of weight k, we have $(F - f)|_k(\gamma - 1) = F|_k(\gamma - 1)$, so the desired identity $F_k|(\gamma - 1) = q_\gamma$ is not changed, when we replace F by $F - f$.

These problems are both addressed (in great generality) in Knopp's paper [Kno74]. We will not make use of his results. But there is a certain similarity of this construction to the construction that we will be using in §2.4.

2.1.4 Functions of moderate growth

Here, we collect some definitions and facts about functions of moderate growth on $\mathbb{H}$ most of which are taken from [CKM+21, §4] with some minor modifications. For some of the proofs, we refer to the cited reference.

Recall that the Frobenius norm of a matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{R})$ is defined as

$$\|g\|_{\mathrm{Fr}} = (\mathrm{Tr}(gg^t))^{1/2} = (a^2 + b^2 + c^2 + d^2)^{1/2}.$$

It is a sub-multiplicative matrix norm on $M_2(\mathbb{R})$ which is invariant under right- or left multiplication by elements of $\mathrm{SO}_2(\mathbb{R})$. For $g \in \mathrm{SL}_2(\mathbb{R})$ one has $\|g^{-1}\|_{\mathrm{Fr}} = \|g\|_{\mathrm{Fr}}$ and $\|g\|_{\mathrm{Fr}} \geq \sqrt{2}$, as follows for instance from the Iwasawa decomposition for $\mathrm{SL}_2(\mathbb{R})$. We have

$$\|g\|_{\mathrm{Fr}}^{-1} \leq |\mu(g, i)| = (c_g^2 + d_g^2)^{1/2} \leq \|g\|_{\mathrm{Fr}}, \tag{2.28}$$

for all $g \in \mathrm{SL}_2(\mathbb{R})$. The upper bound is trivial and the lower bound is implied by $(a^2+b^2)(c^2+d^2) \geq 1 = ad - bc$, where the last inequality follows from the Cauchy–Schwarz inequality.

Definition 2.2. Let $F : \mathbb{H} \to \mathbb{C}$ be a continuous function and let $\Omega \subseteq \mathbb{H}$ be a subset. We say that F has *moderate growth on* Ω, if there exist constants $C, N \geq 0$ such that for all $g \in \mathrm{SL}_2(\mathbb{R})$ one has

$$gi \in \Omega \quad \Rightarrow \quad |F(gi)| \leq C\|g\|_{\mathrm{Fr}}^N. \tag{2.29}$$

We say that F is *of moderate growth*, if it has moderate growth on $\mathbb{H}$.

The following lemma gives an equivalent condition to moderate growth, which is also useful.

Lemma 2.2. *Let $F : \mathbb{H} \to \mathbb{C}$ be a continuous function and let $\Omega \subseteq \mathbb{H}$ be a subset. Then F has moderate growth on Ω if and only if there exist constants $C, \alpha, \beta \geq 0$ such that*

$$|F(\tau)| \leq C(|\tau|^\alpha + \mathrm{Im}(\tau)^{-\beta}) \tag{2.30}$$

for all $\tau \in \Omega$.

Proof. We closely follow [CKM+21][§3]. Suppose first hat (2.30) holds. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ be such that $gi \in \Omega$. We have, by (2.28),

$$|gi|^2 = \frac{|ai + b|^2}{|ci + d|^2} = \frac{(a^2 + b^2)}{(c^2 + d^2)} \leq \|g\|_{\mathrm{Fr}}^2 \frac{1}{(c^2 + d^2)} \leq \|g\|_{\mathrm{Fr}}^4 \quad \Longrightarrow \quad |gi|^\alpha \leq \|g\|_{\mathrm{Fr}}^{2\alpha}.$$

Similarly,

$$\operatorname{Im}(gi)^{-1} = \frac{1}{c^2+d^2} \le \|g\|_{\mathrm{Fr}}^2 \quad \Longrightarrow \quad \operatorname{Im}(gi)^{-\beta} \le \|g\|_{\mathrm{Fr}}^{2\beta}.$$

Therefore, applying (2.30) with $\tau = gi$ (and using that $\|g\|_{\mathrm{Fr}} \ge \sqrt{2} \ge 1$), we obtain

$$|F(gi)| \le C(\|g\|_{\mathrm{Fr}}^{2\alpha} + \|g\|_{\mathrm{Fr}}^{2\beta}) \le 2C\|g\|_{\mathrm{Fr}}^{\max(2\alpha,2\beta)}$$

Suppose now that (2.29) holds. Let $\tau \in \Omega$. Write $\tau = x+iy$. Consider the matrix $g = g_\tau = \left(\begin{smallmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. By assumption, there are constants $C, N \ge 0$ so that

$$|F(\tau)| = |F(g_\tau i)| \le C\|g_\tau\|_{\mathrm{Fr}}^N = C(y + x^2/y + 1/y)^{N/2} = C(1+|\tau|^2)^{N/2}\operatorname{Im}(\tau)^{-N/2}.$$

If $|\tau| \ge 1$, then we bound

$$(1+|\tau|^2)^{N/2}\operatorname{Im}(\tau)^{-N/2} \le 2^{N/2}|\tau|^N \operatorname{Im}(\tau)^{-N/2} \le 2^{N/2-1}\left(|\tau|^{2N} + \operatorname{Im}(\tau)^{-N}\right).$$

If $|\tau| \le 1$, then also $\operatorname{Im}(\tau) \le 1$ and so

$$(1+|\tau|^2)^{N/2}\operatorname{Im}(\tau)^{-N/2} \le 2^{N/2}\operatorname{Im}(\tau)^{-N/2} \le 2^{-N/2}\operatorname{Im}(\tau)^{-N} \le 2^{N/2}\left(|\tau|^{2N} + \operatorname{Im}(\tau)^{-N}\right).$$

In both cases, the bound $|F(\tau)| \le C2^{N/2}(|\tau|^{2N} + \operatorname{Im}(\tau)^{-N})$ holds, which is of the desired shape (2.30). □

Lemma 2.3. *Let $h > 0$ and let $F : \mathbb{H} \to \mathbb{C}$ be a holomorphic h-periodic function of moderate growth. Then F is holomorphic at infinity and there is $\beta \ge 0$ so that $\widehat{F}(n) = O(n^\beta)$.*

Proof. Let $C, \alpha, \beta \ge 0$ be such that (2.30) holds for all $\tau \in \mathbb{H}$. By applying the triangle inequality to the first formula in (1.9), we obtain

$$|\widehat{F}(n)| \le e^{-2\pi ny/h} \sup_{|x|\le h/2} |F(x+iy)| \le Ce^{-2\pi ny/h}\left(y^{-\beta} + (h/2+y)^\beta)\right),$$

for all $y > 0$ and $n \in \mathbb{Z}$. If $n < 0$, then the expression on the right tends to zero as $y \to \infty$, hence $\widehat{F}(n) = 0$ for those n, showing that F is holomorphic at infinity. For $n \ge 1$ we can choose $y = 1/n$ to deduce $\widehat{F}(n) = O(n^\beta)$ as $n \to \infty$. □

We remark (but will not use) that a holomorphic h-periodic function $F : \mathbb{H} \to \mathbb{C}$, which is holomorphic at infinity with Fourier coefficients of polynomial growth, is of moderate growth on $\mathbb{H}$, as a simple estimate using the geometric series shows. Indeed, if $\widehat{F}(n) = O(n^b)$ for some $b \ge 0$, then the characterization of Lemma 2.2 holds with $\alpha = 0$ and with $\beta = b+1$.

The next lemma implies that the space of functions of moderate growth on $\mathbb{H}$ is closed under the slash action in weight k with respect to the group $\Gamma_\theta(h)$. (In fact, the lemma holds with any kind of slash action defined with respect to a factor of automorphy whose absolute value equals $|\mu|$.)

Lemma 2.4. *Let $k \in \mathbb{R}$, $h \ge 2$ and let $\gamma \in \Gamma_\theta(h)$. Let $F : \mathbb{H} \to \mathbb{C}$ be a function which is of moderate growth on the subset $\Omega \subseteq \mathbb{H}$. Let $C, N \ge 0$ be constants such that (2.29) holds for all $g \in \mathrm{SL}_2(\mathbb{R})$. Let $\gamma \in \Gamma_\theta$ be arbitrary. Then, for all $g \in \mathrm{SL}_2(\mathbb{R})$ one has*

$$\gamma g i \in \Omega \quad \Rightarrow \quad |(F|_k\gamma)(g\cdot i)| \le C\|g\gamma\|_{Fr}^{N+|k|}\|g\|_{Fr}^{|k|} \le \|\gamma\|_{Fr}^{N+|k|}\|g\|_{Fr}^{N+2|k|}.$$

Thus, $F|_k\gamma$ is of moderate growth on $\gamma^{-1}\Omega$.

Proof. Indeed, for $\gamma gi \in \Omega$ we have, using (2.10) and the assumption on F,

$$|(F|_k\gamma)(gi)| = |j_k(\gamma, gi)^{-1}||F(\gamma gi)| \leq |\mu(\gamma, gi)|^{-k}C\|\gamma g\|_{\mathrm{Fr}}^N.$$

Since μ is a cocycle, we also have

$$|\mu(\gamma, gi)| = |\mu(\gamma g, i)||\mu(g, i)|^{-1},$$

so that the desired result follows from (2.28). □

2.2 Gaussians and generating series

In this section, we collect or establish some auxiliary results of general nature that will be used in the proofs of Theorem 1 and Theorem 2. In §2.2.1 we gather some technical results on smooth radial (Schwartz-) functions and in particular, give a proof of the density of Gaussians (see Proposition 2.3). In §2.2.2 we describe a translation of the problem of finding a Fourier interpolation formula for radial Schwartz functions, using the nodes $\sqrt{n}$, $n \in \mathbb{N}_0$ to the problem of finding certain modular integrals, making the link to what we discussed in the previous paragraph.

2.2.1 Preliminaries on radial functions and complex Gaussians

This preliminary subsection is about results on smooth radial functions on $\mathbb{R}^d$, which are all related to the issue that the Euclidean norm is not differentiable at the origin. The latter "problem" is going to be present also at other places, in particular in Chapter 3. We will also prove a density result for complex Gaussians, generalizing known ones.

We start by considering the linear map

$$Q : C^\infty(\mathbb{R}) \to C^\infty_{\text{even}}(\mathbb{R}) \qquad Qg(t) = g(t^2), \quad t \in \mathbb{R}.$$

It is natural to ask if Q is onto. In other words, given $f \in C^\infty_{\text{even}}(\mathbb{R})$, does the function $g(t) = f(\sqrt{t})$, which is defined and continuous for $t \geq 0$, extend to a smooth function on $\mathbb{R}$? Note that the extension of g to the negative axis can be arbitrary, as long as it is smooth on $\mathbb{R}$. The first who thoroughly investigated this question was H. Whitney [Whi43], who answered it in the affirmative. We state his result as a lemma here. Whitney prove it via a clever application of Taylor's theorem.

Lemma 2.5 ([Whi43])**.** *The map Q, defined as a above, is surjective.*

Note the following immediate consequence of this lemma. For every $d \geq 1$, the linear map

$$E_d : C^\infty_{\text{even}}(\mathbb{R}) \to C^\infty(\mathbb{R}^d) \qquad (E_d f)(x) = f(|x|), \quad x \in \mathbb{R}^d, f \in C^\infty_{\text{even}}(\mathbb{R})$$

is *well-defined*. For our purposes, we need to know that E_d induces a well-defined *continuous* linear map between Schwartz spaces. This was proved by Grafakos and Teschl in [GT11] and their proof also relied on Lemma 2.5.

Proposition 2.1 ([GT11])**.** *The map E_d induces a continuous linear map $\mathcal{S}_{even}(\mathbb{R}) \to \mathcal{S}_{rad}(\mathbb{R}^d)$.*

Before we turn to Gaussians, we need a further generalization of Whitney's Lemma 2.5. To explain it, note that we may view the space of radial Schwartz functions on $\mathbb{R}^d$ as the subspace of functions fixed under the action of the orthogonal group $\mathrm{O}(d)$. Adopting this point of view, we write $\mathcal{S}_{\text{rad}}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R})^{\mathrm{O}(d)}$ and in particular $\mathcal{S}_{\text{even}}(\mathbb{R}) = \mathcal{S}(\mathbb{R})^{\mathrm{O}(1)}$ with $\mathrm{O}(1) = \{\pm \mathrm{id}_{\mathbb{R}}\}$. In his paper [Sch75], Gerald Schwarz gave a beautiful characterization of the smooth functions on

$\mathbb{R}^d$ which are invariant under the action of any compact subgroup $K \leq \mathrm{O}(d)$. We only need the case of K finite, which was also considered by Bierstone [Bie75] (and apparently, by several authors roughly at the same time, as is written in cited references). So consider a finite group G acting linearly on $\mathbb{R}^d$. By a famous result of Hilbert, the algebra $\mathcal{P}(\mathbb{R}^d)^G$ of G-invariant polynomial functions is finitely generated. Here, let us temporarily work with real-valued functions only, which causes no loss of generality as far as smoothness and decay properties are concerned. Choose a finite generating set $P_1, \ldots, P_n \in \mathcal{P}(\mathbb{R}^d)^G$ of $\mathcal{P}(\mathbb{R}^d)^G$. We then have a well-defined map

$$C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^d)^G, \qquad h \mapsto (x \mapsto h(P_1(x), \ldots, P_n(x))), \quad x \in \mathbb{R}^d, h \in C^\infty(\mathbb{R}^n). \tag{2.31}$$

Proposition 2.2 ([Sch75] [Bie75][7])**.** *The map in* (2.31) *is surjective.*

Further below in this subsection, we only need the following corollary of Proposition 2.2, which, for $n = 1$, specializes again to Whitney's lemma 2.5.

Corollary 2.1. *Let $n \geq 1$ and let $f : \mathbb{R}^n \to \mathbb{C}$ be a smooth compactly supported function, which is even in each variable, i.e., satisfies $f(\epsilon_1 t_1, \ldots, \epsilon_n t_n) = f(t_1, \ldots, t_n)$ for all $\epsilon_j \in \{\pm 1\}$ and all $t_j \in \mathbb{R}$. Then there exists a smooth, compactly supported function $h : \mathbb{R}^n \to \mathbb{C}$ such that $f(t_1, \ldots, t_n) = h(t_1^2, \ldots, t_n^2)$.*

Proof. We may assume that f is real-valued. It is easy to see that the algebra of real polynomials $P(x_1, \ldots, x_n)$ which are even in each variable, is generated by the monomials $x_1^2, \ldots, x_n^2$. It follows from Proposition 2.2 that there is some $h \in C^\infty(\mathbb{R}^d)$ such that $f(t_1, \ldots, t_n) = h(t_1^2, \ldots, t_n^2)$. Since f is compactly supported, $h|_{[0,\infty)^n}$ is also compactly supported. To make h have compact support on all of $\mathbb{R}^d$, we can multiply it by a tensor product of one-dimensional functions that are identically equal to one on $[-1, \infty)$ and identically equal to zero on $(-\infty, -2]$, say. ☐

2.2.1.1 Gaussians

We now turn our attention to complex Gaussians $e^{\pi i z|x|^2}$, where $z \in \mathbb{H}$ is in the upper half plane and $x \in \mathbb{R}^d$. We abbreviate them by

$$g_d(z, x) := g_d(z)(x) := e^{\pi i z|x|^2}, \tag{2.32}$$

but we might often drop the subscript d from the notation, when it is clear from context. As any other function of two variables, we can view g_d as a function-valued function and as such, it is continuous, as the next lemma shows.

Lemma 2.6. *The map $g_d : \mathbb{H} \to \mathcal{S}_{rad}(\mathbb{R}^d)$, $z \mapsto g(z)$, is continuous.*

Proof. It suffices to verify continuity in the case $d = 1$, since the natural map $\mathcal{S}(\mathbb{R})^{\otimes d} \to \mathcal{S}(\mathbb{R}^d)$ is continuous. To do this, fix any $z \in \mathbb{H}$ and suppose given a sequence of points $z_k \in \mathbb{H}$ converging to $z \in \mathbb{H}$, as $k \to \infty$. By differentiating the Gaussian several times, we reduce the proof of $g_1(z_k) \to g_1(z)$ in $\mathcal{S}(\mathbb{R}^1)$, to the task of showing that for all fixed integers $j, \nu \geq 0$, we have

$$\sup_{x \in \mathbb{R}} |z_k^j x^\nu e^{\pi i z_k x^2} - z^j x^\nu e^{\pi i z x^2}| \to 0.$$

[7] Strictly speaking, Theorem 3 in [Bie75] only proves an assertion about germs of functions near the origin, but Bierstone remarks (Remark 2 in [Bie75]) that it is possible to extend it to all functions. In any event, Theorem 1 in [Sch75] gives exactly the statement we need for all compact G.

We write

$$\begin{aligned}
z_k^j x^\nu e^{\pi i z_k x^2} - z^j x^\nu e^{\pi i z x^2} &= z_k^j x^\nu (e^{\pi i z_k x^2} - e^{\pi i z x^2}) - (z^j - z_k^j) x^\nu e^{\pi i z x^2} \\
&= z_k^j x^\nu \int_z^{z_k} (2\pi i x) e^{\pi i w x^2}\, dw + (z^j - z_k^j) x^\nu e^{\pi i z x^2}
\end{aligned}$$

and apply the triangle inequality to get, for all k,

$$\sup_{x\in\mathbb{R}} |z_k^j x^\nu e^{\pi i z_k x^2} - z^j x^\nu e^{\pi i z x^2}| \lesssim |z - z_k| \sup_{x\in\mathbb{R}} |x^{\nu+1} e^{-\pi\delta x^2}| + |z^j - z_k^j| \sup_{x\in\mathbb{R}} |x^\nu e^{-\pi\delta x^2}|,$$

where the implied constant is independent of k and where $\delta = \min\{\mathrm{Im}(z), \inf_k \mathrm{Im}(z_k)\} > 0$. This proves what we want. □

Remark 2.3. Suppose for simplicity of discussion of this remark that d is even. It is known (see e.g. [Lan85, ch. XI, p. 211]) that there is a unique morphism of groups $\rho = \rho_d : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{Aut}(\mathcal{S}(\mathbb{R}^d))$ (the group of automorphism of the topological vector space $\mathcal{S}(\mathbb{R}^d)$), called the Weil representation, such that the elements

$$n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbb{R} \qquad t(y) := \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, y \in \mathbb{R}^\times \qquad w := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

act in the following way on $f \in \mathcal{S}(\mathbb{R}^d)$ via ρ:

$$\rho(n(b))f(x) = e^{\pi i b |x|^2} f(x), \qquad \rho(t(y))f(x) = |y|^{d/2} f(yx), \qquad \rho(w)f = \hat{f}.$$

It may be proved that ρ is strongly continuous, in the sense that for all $f \in \mathcal{S}(\mathbb{R}^d)$ the map $\mathrm{SL}_2(\mathbb{R}) \to \mathcal{S}(\mathbb{R}^d)$, $g \mapsto \rho(g)f$ is continuous. It is not hard to see that this is true on the subgroup of upper-triangular matrices. This is consistent with Lemma 2.6 since for all $z = b + iy \in \mathbb{H}$ and all $x \in \mathbb{R}^d$, we have

$$\rho(n(b)t(\sqrt{y}))g_d(i)(x) = \sqrt{y}^{-d/2} g_d(z).$$

Lemma 2.7. *For all $z \in \mathbb{H}$ and all $d \geq 1$, we have*

$$\mathcal{F}_{\mathbb{R}^d}(g_d(z)) = (z/i)^{-d/2} g_d(-1/z) \tag{2.33}$$

where $(z/i)^{-d/2}$ is defined as in item (4) in §1.1.

Proof. This is well-known. We will recall the proof of a more general formula (involving a harmonic polynomial) in Proposition 3.1 below. □

Remark 2.4. For $z = iy$, $y > 0$, the formula (2.33) is also consistent with Remark 2.3, from which it may be “deduced” by means of the the relation

$$wt(y^{1/2})w^{-1} = t(y^{-1/2})$$

together with the fact that $g_d(i)$ is fixed under $\rho(w) = \mathcal{F}_{\mathbb{R}^d}$.

Proposition 2.3. *For all $d \geq 1$, the linear span of all Gaussians $g_d(z)$, $z \in \mathbb{H}$ is dense in $\mathcal{S}_{rad}(\mathbb{R}^d)$.*

Informally speaking, Proposition 2.3 says that Gaussians are "test functions" in $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ in the sense that any continuous $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$-statement can be tested on them. More formally, a continuous linear map from $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ to another topological vector space is zero, if and only if its kernel contains all Gaussians $g_d(z)$. We can apply this for instance to prove an interpolation formula for radial Schwartz functions, if we know that both sides of that formula depend continuously upon the input function.

For $n = 1$, a proof of Proposition 2.3 was given by Johnson-McDaniel in [JM12, Lemma 2]. His proof relied on Hermite polynomials and -expansions and shows that the statement remains true if one restricts to a subset of $z \in \mathbb{H}$ with an accumulation point. A completely different proof idea is given in the paper by Radchenko and Viazovska [RV19, Sec. 6], which was then reformulated in terms of density of Gaussians in [CKM+21, Lemma 2.2].

We will give a more general density result below, involving tensor products of Gaussians, generalizing the idea [RV19, Sec. 6] further. We remark that [RV19, Sec. 6] and [CKM+21, Lemma 2.2] both elide what is equivalent to the content of the non-trivial Corollary 2.1 for $n = 1$, that is, Whitney's lemma 2.5.

Proposition 2.4. *Let $d, n, d_1, \ldots, d_n \geq 1$ be integers such that $d = d_1 + \cdots + d_n$. View the Euclidean space $\mathbb{R}^d$ as a product space $\mathbb{R}^d = \prod_{j=1}^n \mathbb{R}^{d_j}$ and correspondingly write $x \in \mathbb{R}^d$ as $x = (x_1, \ldots, x_n)$ with $x_j \in \mathbb{R}^{d_j}$. Consider the linear span V consisting of all tensor products of Gaussians $e^{\pi i z_1 |x_1|^2} \cdots e^{\pi i z_n |x_n|^2}$ where $z_j \in \mathbb{H}$. Then V is dense in the space of all $f \in \mathcal{S}(\mathbb{R}^d)$ which are invariant under the natural action of $\mathrm{O}(d_1) \times \cdots \times \mathrm{O}(d_n)$.*

Proof. Abbreviate $H := \mathrm{O}(d_1) \times \cdots \times \mathrm{O}(d_n) \hookrightarrow \mathrm{O}(d)$. We need to show that the linear span $W \subseteq \mathcal{S}(\mathbb{R}^d)^H$, of all Gaussians

$$g(z)(x) = e^{\pi i \sum_{j=1}^n z_j |x_j|^2}, \quad z \in \mathbb{H}^n, x_j \in \mathbb{R}^{d_j},$$

is dense in $\mathcal{S}(\mathbb{R}^d)^H$.

Step 1: By adapting the proof of the fact that $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^d)$, one may show that $C_c^\infty(\mathbb{R}^d)^{\mathrm{O}(d)}$ is dense in $\mathcal{S}(\mathbb{R}^d)^H$. In particular the larger space $C_c^\infty(\mathbb{R}^d)^H$ is dense in $\mathcal{S}(\mathbb{R}^d)^H$.

For completeness, we include also this part of the argument. Fix $w \in C_c^\infty(\mathbb{R})$ such that $0 \leq w \leq 1$, $w(t) = 0$, if $|t| > 4$, and $w(t) = 1$, if $|t| \leq 1$. Then define $\phi : \mathbb{R}^d \to \mathbb{R}$ by $\phi(x) := w(|x|^2)$. We have $\phi \in C_c^\infty(\mathbb{R}^d)^{\mathrm{O}(d)} \subseteq C_c^\infty(\mathbb{R}^d)^H$ and $0 \leq \phi \leq 1$ as well as $\phi(x) = 0$, if $|x| > 2$ and $\phi(x) = 1$, if $|x| \leq 1$. Now given $f \in \mathcal{S}(\mathbb{R}^d)^H$, consider $f_k(x) := f(x)\phi(x/k)$, $k \in \mathbb{N}$. Then each f_k belongs to $C_c^\infty(\mathbb{R}^d)^H$ and we have $f_k \to f$ as $k \to \infty$ in the Schwartz topology. The intuition here is of course that $\phi(x/k)$ will approximate the constant function 1 as k gets large, while f and all its derivatives are very small once the norm is large. To make this rigorous, compute that for all $\alpha, \beta \in \mathbb{N}_0^d$,

$$x^\alpha \partial^\beta f(x) - x^\alpha \partial^\beta f_k(x) = \sum_{\substack{\gamma_1 + \gamma_2 = \beta \\ \gamma_1 \neq 0}} \frac{\beta!}{\gamma_1! \gamma_2!} k^{-|\gamma_1|} (\partial^{\gamma_1}\phi)(x/k) x^\alpha \partial^{\gamma_2} f(x) + (\phi(x/k) - 1) x^\alpha \partial^\beta f(x).$$

This difference tends to zero as $k \to \infty$, uniformly in $x \in \mathbb{R}^d$.

Step 2: We now fix $f \in C_c^\infty(\mathbb{R}^d)^H$ (not the same as in step 1) and aim to show that $f \in \overline{W}$. Fix positive reals $b_1, \ldots, b_n > 0$ and consider the function

$$h(x) := f(x) e^{\pi \sum_{j=1}^n b_j |x_j|^2}, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^d, \quad x_j \in \mathbb{R}^{d_j}.$$

Then $h \in C_c^\infty(\mathbb{R}^d)^H$. We claim that there exists a function $\eta \in C_c^\infty(\mathbb{R}^n)$ such that

$$h(x) = \eta(|x_1|^2, \ldots, |x_n|^2) \quad \text{for all } x \in \mathbb{R}^d. \tag{2.34}$$

To prove this, let us fix the unit vectors $e_j \in S^{d_j-1} \subseteq \mathbb{R}^{d_j}$ and define $h_0 \in C_c^\infty(\mathbb{R}^n)^{\mathrm{O}(1)\times\cdots\times\mathrm{O}(1)}$ by $h_0(t_1,\ldots,t_n) := h(t_1e_1,\ldots,t_ne_n)$. By Corollary 2.1, there is $\eta \in C_c^\infty(\mathbb{R}^n)$ such that $h_0(t_1,\ldots,t_n) = \eta(t_1^2,\ldots,t_n^2)$ for all $t_j \in \mathbb{R}$. This function then satisfies (2.34).

Step 3: For a function $u \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{u}$ is compactly supported, but otherwise unspecified for the moment, we write

$$\begin{aligned} f(x) &= h(x)g_{(ib_1,\ldots,ib_n)}(x) \\ &= (\eta-u)(|x_1|^2,\ldots,|x_n|^2)g((ib_1,\ldots,ib_n),x) + u(|x_1|^2,\ldots,|x_n|^2)g((ib_1,\ldots,ib_n),x) \\ &= (\eta-u)(|x_1|^2,\ldots,|x_n|^2)g((ib_1,\ldots,ib_n),x) + \int_{\mathbb{R}^n}\widehat{u}(\xi)g((ib_1+2\xi_1,\ldots,ib_n+2\xi_n),x)d\xi, \end{aligned}$$

where we applied the Fourier inversion on $\mathbb{R}^n$ in the last step. The latter integral belongs to $\overline{W}$, regardless of the choice of u, as long as $\widehat{u}$ has compact support. This follows from integration theory[8] in Fréchet spaces and continuity of the map $\mathbb{H}^n \to \mathcal{S}(\mathbb{R}^d)$, $z \mapsto g_z$ (or alternatively by approximation via Riemann sums). It therefore suffices to show that the term involving $\eta - u$ can be made arbitrarily small in the Schwartz topology. To see this, consider the linear map $E : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^d)^H$, defined by $E\varphi(x) := \varphi(|x_1|^2,\ldots,|x_n|^2)$. It continuous for the Schwartz topology and multiplication by $g((ib_1,\ldots,ib_n))$ is continuous. Since the space of $u \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{u}$ has compact support is dense in $\mathcal{S}(\mathbb{R}^n)$ and E is continuous, we can choose u in such a way that $E(\eta-u)$ is in any prescribed open zero neighborhood of $\mathcal{S}(\mathbb{R}^d)$. This finishes the proof of Proposition 2.4. □

Remark 2.5. In the above proof, we used integration theory in locally convex vector spaces. Let us therefore recall elements of the theory of (weak) Gelfand–Pettis integrals, as we will also use it it in some other sections of this thesis. We refer to [Gar18, ch 14] for a more detailed treatment.

Let V be a Hausdorff locally convex vector space over the complex numbers. Let X be a *compact*, second countable Hausdorff space, equipped with a Radon measure μ. Let $f : X \to V$ be a continuous map. A vector $v \in V$ is called a *weak integral* for f, if for all continuous linear functionals $\ell : V \to \mathbb{C}$ we have $\ell(v) = \int_X \ell(f(x))d\mu(x)$ (note that the integral on the right always exists). As a consequence of the fact that the continuous linear functionals of V separate its points, we see that any weak integral (if it exists) is unique. Under the stated compactness assumptions on X, and under the assumptions that the closed convex hull of any compact subset of V is also compact[9], a weak integral f exists and belongs to the (compact) closed convex hull of $f(X) \subseteq V$. The existence proof is not difficult; it uses the finite intersection property and separation of convex sets by hyperplanes in finite dimensional Euclidean spaces. We denote *the* weak integral of f by $\int_X f d\mu$ or $\int_X f(x)d\mu(x)$. It is trivial to prove (from its characterization) that for any continuous linear $T : V \to W$ one has $\int_X f(x)d\mu(x) = \int_X T(f(x))d\mu(x)$.

Applying this to $V = \mathcal{S}(\mathbb{R}^d)$ (or any closed subspace thereof) and to $X \subseteq Y$ a compact set containing the support of some compactly supported continuous function $g : Y \to V = \mathcal{S}(\mathbb{R}^d)$ (on some locally compact space Y), we can define the Schwartz function $\int_Y f(x)d\mu(x)$. The fact that weak integrals commute with linear maps allows for trivial justification of operations such as taking the Fourier transform under the integral sign or differentiation under the integral sign.

[8] We say precisely what we need after this proof

[9] which is satisfied by all locally convex V that are quasi-complete: all bounded Cauchy-sequences in V converge. All Fréchet spaces are quasi-complete. We will always assume that V has (at least) the stated property for closed convex hulls of compact sets.

2.2.1.2 Non-integral dimensions

Here, we comment on the possibility of extending many of the results in this thesis concerning radial functions, to non-integral dimensions, but we will not carry out this extension.

Recall the definition of the J-Bessel function as a power series:

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu}, \qquad x > 0, \nu > -1.$$

and define $\tilde{J}_\nu(x) := x^{-\nu} J_\nu(x)$, which is (at at the least) continuous near $x = 0$. As is well-known we can use this function as an integral kernel to describe the Fourier transform on radial functions. To explain how, let us define for any real $k > 0$ and any $\varphi \in \mathcal{S}_{\mathrm{even}}(\mathbb{R})$ the new function $\mathcal{H}_k\varphi : [0, \infty) \to \mathbb{C}$ by

$$\mathcal{H}_k\varphi(\rho) := 2\pi \int_0^\infty \varphi(r)(r/\rho)^{k-1} J_{k-1}(2\pi r\rho) r dr = (2\pi)^k \int_0^\infty \varphi(r) r^{2k-1} \tilde{J}_{k-1}(2\pi r\rho) dr. \tag{2.35}$$

The letter $\mathcal{H}$ is chosen because the above formula is a (renormalized) Hankel transform. For any $k \in (1/2)\mathbb{N}$ and $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^{2k})$ and $\rho > 0$, we then have [Gra08, Appendix B.5, p. 429]

$$(\mathcal{F}_{\mathbb{R}^{2k}} f)_0(\rho) = (\mathcal{H}_k f_0)(\rho). \tag{2.36}$$

Via the Hankel transform $\mathcal{H}_k$, the formula for the Fourier transform of the Gaussian (2.33) extends to all real dimension $d = 2k > 0$ in a natural way. More precisely, we have

$$(\mathcal{H}_k g_1(z))(\rho) = (z/i)^{-k} g_1(-1/z) = (z/i)^{-k} e^{\pi i(-1/z)\rho^2} \tag{2.37}$$

for all real $k > 0$. For the proof of (2.37), we reduce via analyticity to the case $z = iy$ and then cite [ZMGR15, p. 706, 6.631], which asserts that

$$\int_0^\infty x^\nu e^{-\alpha x^2} J_\nu(\beta x) dx = \frac{\beta^\nu}{(2\alpha)^{\nu+1}} e^{-\frac{\beta^2}{4\alpha}}, \quad \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\nu) > -1, \mu > 0.$$

We then apply this formula with

$$\mu = k, \quad \alpha = \pi y, \quad \nu = k - 1, \quad \mu = k, \quad \beta = 2\pi\rho.$$

Via (2.36) and (2.37), one should be able to replace $(\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d), \mathcal{F}_{\mathbb{R}^d}, g_d)$ by $(\mathcal{S}_{\mathrm{even}}(\mathbb{R}), \mathcal{H}_k, g_1)$ for any real $k > 0$, in some of the results and proofs in this thesis.

2.2.2 Generating series and functional equations

In this section, we explain the equivalence of the problem of finding a Fourier interpolation formula for radial Schwartz functions that involves the pair of nodes $\{\sqrt{n/\alpha}\}_{n\in\mathbb{N}_0}, \{\sqrt{n/\beta}\}_{n\in\mathbb{N}_0}$ (for some fixed $\alpha, \beta > 0$) and the problem of finding a continuous family of modular integrals for a Hecke-type group inside $\mathrm{PSL}_2(\mathbb{R})$ depending on α, β. In Chapter 4 we will generalize this equivalence by working with not necessarily radial functions on $\mathbb{R}^n$, $n \geq 2$, by replacing the nodes mentioned above with the sets of points $(x_1, \dots, x_n) \in \mathbb{R}^n$ such that $(x_1^2, \dots, x_n^2) \in \Lambda$ for some fixed lattice $\Lambda \subseteq \mathbb{R}^n$. That generalization motivates the specific shape of interpolation nodes we consider here. The general discussion that follows is the starting point for the proof of Theorem 1 and Theorem 2. It is implicit in [RV19] or [CKM+17].

We start by asking when it is possible to find functions $a_n, \tilde{a}_n : \mathbb{R} \to \mathbb{C}$ so that for all $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ and all $x \in \mathbb{R}^d$, we have

$$f(x) = \sum_{n=0}^{\infty} a_n(|x|) f(\sqrt{n/\alpha}) + \sum_{n=0}^{\infty} \tilde{a}_n(|x|) \widehat{f}(\sqrt{n/\beta}) \tag{2.38}$$

with absolute convergence. Here, the integer $d \geq 1$ and the real numbers $\alpha, \beta > 0$ are given and considered as fixed. By a straightforward scaling argument, we see that it suffices to consider the case $\alpha = \beta$. We henceforth consider a fixed $h > 0$ and the problem of finding $a_n, \tilde{a}_n$ satisfying (2.38) in the case $h/2 = \alpha = \beta$, i.e. such that

$$f(x) = \sum_{n=0}^{\infty} a_n(|x|) f(\sqrt{2n/h}) + \sum_{n=0}^{\infty} \tilde{a}_n(|x|) \widehat{f}(\sqrt{2n/h}). \tag{2.39}$$

There are different ways of interpreting the equal sign in (2.39):

- at the point-wise level: we fix x and interpret the right hand side as a formula for the functional $\mathrm{ev}_x(f) = f(x)$. From this point of view, we need two families of sequence of *numbers* $a_n(r)$, $\tilde{a}_n(r)$. In principle, it could happen that such families exist, but that the resulting functions $r \mapsto a_n(r)$ are badly behaved.
- at the level of functions: we seek sequences of functions a_n, $\tilde{a}_n$ in some "nice" space V of functions on $\mathbb{R}^d$ so that for all radial functions $f \in V$, we have

$$f = \sum_{n=0}^{\infty} f(\sqrt{2n/h}) a_n + \sum_{n=0}^{\infty} \widehat{f}(\sqrt{2n/h}) \tilde{a}_n \tag{2.40}$$

  with convergence in topology of that space. In that case, we can also view the the right hand side of (2.40) as an artful way of expressing id_V:

$$\mathrm{id}_V = \sum_{n=0}^{\infty} (\mathrm{ev}_{r_n} \otimes a_n) + \sum_{n=0}^{\infty} (\mathcal{F}^*(\mathrm{ev}_{r_n}) \otimes \tilde{a}_n), \quad r_n := \sqrt{2n/h}.$$

  Here $\mathcal{F}^*$ denotes the distributional Fourier transform and $\otimes$ is used in reference to the natural bilinear map $V^* \times V \to \mathrm{End}(V)$. This viewpoint will be more relevant in §3.3.

Before we do anything fancier, we address the first of these viewpoints and restrict attention to the function space $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$. Thus, we *fix* temporarily some $x \in \mathbb{R}^d$ and ask about the existence of two sequences of complex numbers $a_n(r)$, $\tilde{a}_n(r)$, $r = |x|$, such that (2.39) holds for all $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$. It is then reasonable to impose the growth condition $a_n(r), \tilde{a}_n(r) = O(n^a)$ for some a, possibly depending on r. Indeed, in that case, the right hand side of (2.39) defines a continuous functional on $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ so that its equality with the evaluation functional ev_x can be tested on a dense subspace of $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$, e.g. on the span of all complex Gaussians $g_d(z) \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$, $z \in \mathbb{H}$, by Proposition 2.3. This technical reduction and the square root structure are what allows the connection to the theory of modular forms, as will become clear in the sequel.

Assume first that numbers $a_n(r)$, $\tilde{a}_n(r)$ exist with the previously mentioned properties. We can then consider the following generating functions of these numbers

$$F(z) = \sum_{n=0}^{\infty} a_n(r) e^{2\pi i n z/h}, \qquad \tilde{F}(z) := \sum_{n=0}^{\infty} \tilde{a}_n(r) e^{2\pi i n z/h}, \quad z \in \mathbb{H}. \tag{2.41}$$

By our polynomial growth assumptions, these series define h-periodic, holomorphic functions $F, \tilde{F} : \mathbb{H} \to \mathbb{C}$. Moreover, since (2.39) holds for $f = g(z)$, since $\mathcal{F}_{\mathbb{R}^d}(g(z)) = (z/i)^{-d/2} g(-1/z)$ and since $g(z)(\sqrt{2n/h}) = e^{2\pi i n z/h}$, the functions F and $\tilde{F}$ must satisfy

$$F(z) + (z/i)^{-d/2}\tilde{F}(1/z) = g(z)(r) = e^{\pi i z r^2}. \tag{2.42}$$

We constructed F and $\tilde{F}$ assuming the existence of $a_n(r)$, $\tilde{a}_n(r)$. Suppose conversely that we are given two h-periodic, holomorphic functions $F, \tilde{F} : \mathbb{H} \to \mathbb{C}$ related by (2.42). Then these functions admit Fourier expansions as in (2.41) with Fourier coefficients given by

$$a_n(r) = \frac{1}{h}\int_{iy-h/2}^{iy+h/2} F(z) e^{-2\pi i n z/h} dz, \qquad \tilde{a}_n(r) = \frac{1}{h}\int_{iy-h/2}^{iy+h/2} \tilde{F}(z) e^{-2\pi i n z/h} dz, \tag{2.43}$$

where $y > 0$ is arbitrary. If one can show that the coefficients $a_n(r)$ and $\tilde{a}_n(r)$ are zero for $n < 0$ and are polynomally bounded, then the interpolation formula (2.39) will hold with these numbers, for all Gaussians, hence for all Schwartz functions, as just explained. By Lemma 2.3, these conditions on the coefficients will hold if F and $\tilde{F}$ are both of moderate growth. (In fact, the conditions are equivalent).

This was just a reformulation of the problem using the density of Gaussians. To make progress, we will connect the problem to symmetries of the upper half plane and the groups $\Gamma_\theta(h)$, $\Gamma(h)$ that already implicitly appeared via the transformations $z \mapsto z + h$ and $z \mapsto -1/z$. To that end, recall from §2.1.1 the definition of the cocycle $j_k : \Gamma_\theta \to \mathrm{Hol}(\mathbb{H}, \mathbb{C}^\times)$ and the slash action (2.9) derived from it. If $d = 2k$, then what we need are two holomorphic functions F, $\tilde{F} : \mathbb{H} \to \mathbb{C}$ (depending on the fixed real number $r \geq 0$) of moderate growth, satisfying

(i) $F|_k(T^h - 1) = 0$. This says that F is h-periodic.

(ii) $\tilde{F}|_k(T^h - 1) = 0$. This says that $\tilde{F}$ is h-periodic.

(iii) $F + \tilde{F}|_k S = g$. This rewrites the equation (2.42). Here, we temporarily abuse notation and write g for the function $g(z) = e^{\pi i z r^2}$.

Again, we have not done much else other than rewriting things with the newly introduced slash-action. Our next, slightly less trivial step, consists in eliminating $\tilde{F}$ from these equations. More precisely, we claim that it suffices to find (only) F satisfying

(a) $F|_k(T^h - 1) = 0$,

(b) $F|_k(V^{-h} - 1) = g|(V^{-h} - 1)$.

We recall here that $V^x = ST^{-x}S$ for all $x \in \mathbb{R}$. To explain why (a) (b) imply (i), (ii) and (iii), note that, if we have solved (a), (b), we can define $\tilde{F}$ by $\tilde{F} = (g - F)|_k S$ and (i) and (iii) will hold trivially and (ii) holds because

$$\begin{aligned} ((g - F)|_k S)|_k(T^h - 1) &= (g - F)|_k(S(T^h - 1)) \\ &= (g - F)|_k(S(T^h - 1)SS) \\ &= ((g - F)|_k(V^{-h} - 1))|_k S = 0|_k S = 0 \end{aligned}$$

Moreover, if F is of moderate growth on $\mathbb{H}$, then so is $\tilde{F}$ by Lemma 2.4 and since g is obviously of moderate growth.

To make further progress in finding F satisfying (a) and (b), let us generalize the setting, "forget" that we actually care about the Gaussian g on the right hand side of (b) and let us instead consider the following problem: Given a real number $h \geq 2$ and given a holomorphic function $\varphi : \mathbb{H} \to \mathbb{C}$ of moderate growth, find a holomorphic function $F : \mathbb{H} \to \mathbb{C}$ of moderate growth, such that

(A) $F|_k(T^h - 1) = 0$

(B) $F|_k(V^{-h} - 1) = \varphi|_k(V^{-h} - 1)$.

Moreover, if F is of moderate growth, then so will be $\tilde{F}$, by Lemma 2.4. If we can solve this problem, we can solve (all) the previous one(s), since for all $r \in \mathbb{R}$, the function $\varphi(z) = g_r(z) = e^{\pi i r^2 z}$ is holomorphic and of moderate growth (simply because it is always bounded by 1). In a second step, we can attempt to analyze how the solutions vary with r, to say something towards the second bullet point containing (2.40).

In solving (A) and (B), the first elementary but useful observation to make is the following: When $\varphi = 0$, henceforth referred to as the *homogeneous case/system*, the solutions are exactly modular forms for the subgroup generated by T^h and V^h, that is, the group $\Gamma(h)$, introduced in §2.1.1.

In order to actually find F, we will use two different approaches, which will be presented in the remaining sections 2.3 and 2.4. We believe, but have not checked in complete detail, that if the weight k is bigger than 2, then Knopp's general results [Kno74], briefly discussed in §2.1.3, should prove the *existence* of F by other means. Since the construction is rather complicated and not explicit enough for our purposes, we will not comment further on this.

2.2.2.1 Decomposition into Fourier eigenspaces

For later purposes, we record here an alternative version of the strategy outlined in the previous section, which is also the approach taken by Radchenko–Viazovska in [RV19]. For $\epsilon \in \{\pm 1\}$, define

$$\mathcal{S}^{\epsilon}_{\mathrm{rad}}(\mathbb{R}^d) := \{f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d) \,:\, \widehat{f} = \epsilon f\}.$$

Then any given $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ can be written as

$$f = \frac{f + \hat{f}}{2} + \frac{f - \hat{f}}{2} = f_+ + f_-,$$

where $f_\epsilon \in \mathcal{S}^{\epsilon}_{\mathrm{rad}}(\mathbb{R}^d)$ (we use here that the Fourier transform is an involution on even functions, in particular on radial functions). Using this decomposition, we see that, in order to prove an interpolation formula of the shape (2.39), it suffices to find $b^{\epsilon}_n(r)$ so that

$$f(r) = \sum_{n=0}^{\infty} b^{\epsilon}_n(r) f(\sqrt{2n/h}), \qquad \text{for all } f \in \mathcal{S}^{\epsilon}_{\mathrm{rad}}(\mathbb{R}^d). \tag{2.44}$$

Indeed, the functions

$$a_n := \frac{b^+_n + b^-_n}{2}, \qquad \tilde{a}_n := \frac{b^+_n - b^-_n}{2}$$

will then satisfy (2.39). To find b^{ϵ}_n satisfying (2.44), it suffices to find h-periodic, holomorphic functions $F^+, F^- : \mathbb{H} \times \mathbb{R} \to \mathbb{C}$ of moderate growth such that

$$F^{\epsilon}(z, r) + \epsilon (z/i)^{-k} F^{\epsilon}(-1/z, r) = g(z, r) + \epsilon (z/i)^{-k} g(-1/z, r)$$

for all $z \in \mathbb{H}$, $r \in \mathbb{R}$. This is because the mappings $f \mapsto \frac{f + \epsilon \hat{f}}{2}$ are continuous surjective projections onto $\mathcal{S}^{\epsilon}_{\mathrm{rad}}(\mathbb{R}^d)$ so that, as a consequence of Proposition 2.3, the set of all functions $g(z) + \epsilon \widehat{g(z)} =$

$g(z) + \epsilon(z/i)^{-k}g(-1/z)$, $z \in \mathbb{H}$, is dense in $\mathcal{S}^{\epsilon}_{\mathrm{rad}}(\mathbb{R}^d)$. Just as before, we may also rewrite the transformation behavior of F^{ϵ} more algebraically using an extension of the slash action $|_k$ defined via j_k in (2.8). Namely, let $\chi_\epsilon : \Gamma_\theta(h) \to \{\pm 1\}$ denote the group homomorphism satisfying $\chi_\epsilon(T^h) = 1$, $\chi_\epsilon(S) = \epsilon$. Let us then define

$$\varphi|_k^{\epsilon}\gamma := \chi_\epsilon(\gamma) j_k(\gamma)^{-1}(\varphi \circ \gamma)$$

for any function $\varphi : \mathbb{H} \to \mathbb{C}$ and $\gamma \in \Gamma_\theta(h)$. Then the functions F^{ϵ} must satisfy

(1) $F^{\epsilon}|_k^{\epsilon}(1 - T^h) = 0$,

(2) $F^{-\epsilon}|_k^{\epsilon}(1 - S) = g|_k^{\epsilon}(1 - S)$.

Note the sign change in (ii). Here we have again extended the slash action to the group algebra $\mathbb{C}[\Gamma_\theta(h)]$. Moreover, by means of the relationships

$$\begin{pmatrix} F \\ \tilde{F} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} F^+ \\ F^- \end{pmatrix} \iff \begin{pmatrix} F^+ \\ F^- \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} F \\ \tilde{F} \end{pmatrix}, \tag{2.45}$$

$$\begin{pmatrix} a_n \\ \tilde{a}_n \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} b_n^+ \\ b_n^- \end{pmatrix} \iff \begin{pmatrix} b_n^+ \\ b_n^- \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a_n \\ \tilde{a}_n \end{pmatrix}, \tag{2.46}$$

the problem of finding (F^+, F^-) and (b_n^+, b_n^-) is equivalent to the problem of finding $(F, \tilde{F})$ and $(a_n, \tilde{a}_n)$ described in the previous section.

2.3 Modular integrals via Green-type kernels on $\Gamma(2)$

In this section, we will construct modular integral(s) F and $\tilde{F}$ having the properties stated in the previous section, in the case $h = 2$ and $\varphi_r(z) = e^{\pi i z r^2}$, as an integral transform of a meromorphic kernel function $\mathcal{K}(\tau, z)$ of $\tau, z \in \mathbb{H}$. The approach is similar to the approach taken by Radchenko and Viazovska [RV19], but is not the most direct generalization of their proof to radial functions in higher dimensions (see [BRS20] and [RS21] for that generalization). Specifically, instead of working on the congruence subgroup Γ_θ of $\mathrm{PSL}_2(\mathbb{Z})$ we work on the (smaller, normal) congruence subgroup $\Gamma(2) \le \Gamma_\theta$, which has three cusps instead of two.

The structure of §2.3 is as follows:

- In §2.3.1 we state the main result, Theorem 1.
- In §2.3.2 we prove one part of this theorem which follows from the other parts and shows that $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ is isomorphic (as topological vector spaces, in different ways) to a certain space of pairs of sequences, which we will describe precisely. This shows that Theorem 1 provides *free* interpolation formula.
- In §2.3.3 we will define the already mentioned kernel functions $\mathcal{K}(\tau, z)$ and list some of their important properties.
- In §2.3.4 we will compute the Fourier expansions of $\tau \mapsto \mathcal{K}(\tau, z)$ which are weakly holomorphic modular forms in z.
- In §2.3.5 we will construct the pair of generating functions $F, \tilde{F}$, introduced in the previous section as contour integrals involving the kernel $\mathcal{K}(\tau, z)$ and the Gaussian.
- In §2.3.6 we will show that the generating functions F and $\tilde{F}$ so defined are of moderate growth. This will be a big part of the work and quite technical.
- In §2.3.7 we will combine all of the previous parts to give the proof of Theorem 1.

2.3.1 Main result

The main result of §2.3 is the following theorem.

Theorem 1. *Let $d \geq 1$ and $n_0, \hat{n}_0 \geq 0$ be integers such that $n_0 + \hat{n}_0 = 1 + \lfloor d/4 \rfloor$. Set $k = d/2$. There exist two unique sequences*[10] *$(a_{k,n})_{n \geq n_0}, (\tilde{a}_{k,n})_{n \geq \hat{n}_0}$ of real-valued radial Schwartz functions on $\mathbb{R}^d$, having the following two properties.*

(i) For every $f \in \mathcal{S}_{rad}(\mathbb{R}^d)$ we have

$$f = \sum_{n=n_0}^{\infty} f(\sqrt{n}) a_{k,n} + \sum_{n=\hat{n}_0}^{\infty} \hat{f}(\sqrt{n}) \tilde{a}_{k,n}, \tag{2.47}$$

with convergence in the Schwartz space topology.

(ii) For all integers $n, m \geq n_0$ and all $p, q \geq \hat{n}_0$ we have

$$a_{k,n}(\sqrt{m}) = \delta_{m,n} \qquad \mathcal{F}(a_{k,p})(\sqrt{q}) = 0$$
$$\tilde{a}_{k,n}(\sqrt{m}) = 0 \qquad \mathcal{F}(\tilde{a}_{k,p})(\sqrt{q}) = \delta_{p,q}$$

Moreover, for every continuous semi-norm $\|\cdot\|$ on $\mathcal{S}(\mathbb{R}^d)$, the sequences $(\|a_{k,n}\|)_{n \in \mathbb{N}_0}, (\|\tilde{a}_{k,n}\|)_{n \in \mathbb{N}_0}$ are polynomially bounded. The map

$$\Phi_d : \mathcal{S}_{rad}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{N}_0) \times \mathcal{S}(\mathbb{N}_0), \quad \Phi_d(f) = \big((f(\sqrt{n}))_{n \in \mathbb{N}_0}, (\hat{f}(\sqrt{n}))_{n \in \mathbb{N}_0}\big) \tag{2.48}$$

is a continuous linear injection and defines an isomorphism of Fréchet spaces onto a closed subspace of co-dimension $1 + \lfloor d/4 \rfloor$, which is the space of vectors annihilated by the image of a linear injection $M_k(\Gamma(2)) \to (\mathcal{S}(\mathbb{N}_0) \times \mathcal{S}(\mathbb{N}_0))^*$. The inverse map is given by

$$\Psi_d((x_n), (y_n)) := \sum_{n=n_0}^{\infty} x_n a_{k,n} + \sum_{n=\hat{n}_0}^{\infty} y_n \tilde{a}_{k,n} \tag{2.49}$$

for $x = (x_n), y = (y_n)$ in that subspace; see Proposition 2.5 for the precise description of the image of Φ_d.

Here, $\mathcal{S}(\mathbb{N}_0)$ is the space of rapidly decreasing sequences and $M_k(\Gamma(2))$ is the space of modular forms of weight k for the group $\Gamma(2) \leq \mathrm{PSL}_2(\mathbb{Z})$.

Remark 2.6. Note that the properties (i) and (ii) force the uniqueness of the functions $a_{k,n}, \tilde{a}_{k,n}$ and that (ii) implies that, **if** $n_0 = \hat{n}_0$, then $\tilde{a}_{k,n} = \mathcal{F}(a_{k,n})$ for all $n \geq n_0$. The assertion concerning the isomorphism with the space of pairs of sequences will be formulated more precisely and proved in Proposition 2.5 in §2.3.2.

2.3.2 The image of Φ_d and its inverse

We let, as usual, $d \geq 1$ be an integer and write $k = d/2$ and view these parameters fixed. The goal of this section is to explain and prove the assertions concerning the mappings $\Phi = \Phi_d$ and $\Psi = \Psi_d$

[10] the functions $a_{k,n}$ and $\tilde{a}_{k,n}$ also depend on n_0 and $\hat{n}_0$, but we do not display this dependence in the notation. In the following, we also set $a_{k,n} = 0$ if $n < n_0$ and $\tilde{a}_{k,n} = 0$ for $n < \hat{n}_0$.

given in (2.48) and (2.49) in Theorem 1 assuming the other parts of that theorem. Recall from item (13) in §1.1 the definition of the spaces $\mathcal{S}(\mathbb{N}_0)$, $\mathcal{P}(\mathbb{N}_0)$. There is the natural bilinear pairing

$$\langle \cdot, \cdot \rangle : \mathcal{S}(\mathbb{N}_0) \times \mathcal{P}(\mathbb{N}_0) \to \mathbb{C} \quad \text{given by} \quad \langle (x_n), (c_n) \rangle := \sum_{n=0}^{\infty} x_n c_n. \tag{2.50}$$

We say that a sequence of Schwartz functions $(f_n)_{n \in \mathbb{N}_0}$ on $\mathbb{R}^d$ has polynomial semi-norm growth, if for all fixed $\alpha, \beta \in \mathbb{N}_0^d$, the sequence of semi-norms $\|f_n\|_{\alpha,\beta}$ belongs to $\mathcal{P}(\mathbb{N}_0)$. Next, recall (from item (15) in §1.1) that any 2-periodic holomorphic function $\vartheta : \mathbb{H} \to \mathbb{C}$ admits a Fourier expansion $\vartheta(z) = \sum_{n \in \mathbb{Z}} \hat{\vartheta}(n) e^{\pi i n z}$. Assuming that $(\hat{\vartheta}(n))_{n \in \mathbb{N}_0}$ has polynomial growth we can attach to it a continuous linear functional $\vartheta^* \in \mathcal{S}(\mathbb{N}_0)^*$ via the above paring; it is given by

$$\vartheta^*(x) := \langle x, \vartheta^* \rangle := \langle (x_n)_{n \in \mathbb{N}_0}, (\hat{\vartheta}(n))_{n \in \mathbb{N}_0} \rangle := \sum_{n=0}^{\infty} x_n \hat{\vartheta}(n).$$

Now consider some $\vartheta \in M_k(\Gamma(2))$. We compute the action of ϑ^* on a Gaussian $g(z) = g_d(z) \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$, $z \in \mathbb{H}$ and its Fourier transform $\widehat{g(z)}$ and see that $\langle g(z), \vartheta^* \rangle = \sum_{n=0}^{\infty} \hat{\vartheta}(n) e^{\pi i n z} = \vartheta(z)$ and

$$\langle \widehat{g(z)}, \vartheta^* \rangle = \sum_{n=0}^{\infty} \hat{\vartheta}(n) (z/i)^{-k} e^{\pi i n S z} = (\vartheta|_k S)(z) = \sum_{n=0}^{\infty} \widehat{(\vartheta|S)}(n) e^{\pi i n z}.$$

Recall that $\vartheta|S$ is 2-periodic with polynomially bounded Fourier coefficients. By replacing ϑ by $\vartheta|S$ is the last computation, we obtain

$$\langle g(z), \vartheta^* \rangle - \langle \widehat{g(z)}, (\vartheta|S)^* \rangle = \vartheta(z) - \vartheta(z) = 0,$$

for all $z \in \mathbb{H}$. Let us define the linear map

$$\mathcal{A} : M_k(\Gamma(2)) \to (\mathcal{S}(\mathbb{N}_0) \times \mathcal{S}(\mathbb{N}_0))^*, \quad \vartheta \mapsto \mathcal{A}(\vartheta),$$

by

$$\langle (x, y), \mathcal{A}(\vartheta) \rangle := \langle x, \vartheta^* \rangle - \langle y, (\vartheta|S)^* \rangle = \sum_{n=0}^{\infty} (\hat{\vartheta}(n) x_n - \widehat{(\vartheta|S)}(n) y_n).$$

By continuity of $f \mapsto \langle \Phi(f), \mathcal{A}(\vartheta) \rangle$ and the density of the span of all Gaussians $g(z)$ in $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ (Proposition 2.3) we then have have $\langle \Phi(f), \mathcal{A}(\vartheta) \rangle = 0$ for all $\vartheta \in M_k(\Gamma(2))$ and all $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$. Formulated differently, we have shown that the image of Φ_d is contained in ${}^{\perp}M_k(\Gamma(2))$, where[11]

$${}^{\perp}M_k(\Gamma(2)) := \{(x, y) \in \mathcal{S}(\mathbb{N}_0) \times \mathcal{S}(\mathbb{N}_0) \,:\, \langle (x, y), \mathcal{A}(\vartheta) \rangle = 0 \text{ for all } \vartheta \in M_k(\Gamma(2))\}.$$

The next proposition will show that image of Φ is in fact equal to that space and that Ψ defines the inverse of Φ on its image.

Proposition 2.5. *Let $n_0, \hat{n}_0 \in \mathbb{N}_0$ be such that $n_0 + \hat{n}_0 = 1 + \lfloor d/4 \rfloor$. Let $(a_{k,n})_{n \geq n_0}$ and $(\tilde{a}_{k,n})_{n \geq \hat{n}_0}$ denote the two (necessarily unique) sequences of radial Schwartz functions on $\mathbb{R}^d$ having properties (i) and (ii) stated in Theorem 1 and enjoying polynomial semi-norm growth. Let $\Phi = \Phi_d$ and $\Psi = \Psi_d$ be defined as in (2.48) and (2.49) respectively. Then Ψ_d is a well-defined continuous linear map, and the compositions $\Phi \circ \Psi$ and $\Psi \circ \Phi$ satisfy*

$$\Psi(\Phi(f)) = f \qquad \text{for all } f \in \mathcal{S}_{rad}(\mathbb{R}^d), \tag{2.51}$$

$$\Phi(\Psi(x, \tilde{x})) = (x, \tilde{x}) \quad \text{for all } (x, \tilde{x}) \in {}^{\perp}M_k(\Gamma(2)). \tag{2.52}$$

[11] The more accurate notation for this space is perhaps ${}^{\perp}\mathcal{A}(M_k(\Gamma(2)))$, but note that $\mathcal{A}$ is injective.

Proof. That Ψ is a well-defined follows form sequential completeness of space of radial Schwartz functions and the assumption that the $a_{k,n}$, $\tilde{a}_{k,n}$ have polynomial semi-norm growth. Continuity holds from the same reason (use the triangle inequality). Note that (2.51) is just a rewriting of (the assumed) part (i) of Theorem 1. It remains to establish (2.52). So let $(x,\tilde{x}) \in {}^{\perp}M_k(\Gamma(2))$. By the assumed part (ii) of Theorem 1, the pair $(y,\tilde{y}) := \Phi(\Psi(x,\tilde{x})) - (x,\tilde{x})$ is of the form

$$(y,\tilde{y}) = ((s_0,\dots,s_{n_0-1},0,0,\dots),(t_0,\dots,t_{\hat{n}_0-1},0,0,\dots)) \in {}^{\perp}M_k(\Gamma(2)),$$

for some $s_j, t_j \in \mathbb{C}$. For integers $a,b \geq 0$ such that $a+b \leq \lfloor k/2 \rfloor = n_0 + \hat{n}_0 - 1$, consider

$$\vartheta_{a,b} := \Theta_3^d \lambda^a (1-\lambda)^b \in M_k(\Gamma(2)).$$

Note that $\mathrm{ord}_\infty(\vartheta_{a,b}) = a$ and that $\vartheta_{a,b}|S = \vartheta_{b,a}$ and so

$$0 = \langle (y,\tilde{y}), \mathcal{A}(\vartheta_{a,b})\rangle = \langle y, \vartheta_{a,b}^*\rangle - \langle \tilde{y}, \vartheta_{b,a}^*\rangle = \sum_{n=0}^{n_0-1} y_n \widehat{\vartheta_{a,b}}(n) - \sum_{n=0}^{\hat{n}_0-1} \tilde{y}_n \widehat{\vartheta_{b,a}}(n).$$

Taking $a = n_0$ and letting b range in $\{0,1\dots,\hat{n}_0-1\}$ yields a triangular system of homogeneous linear equations in the $\tilde{y}_n$ and shows that they are all zero. Similarly, taking $b = \hat{n}_0$ and letting a range in $\{0,1,\dots,n_0-1\}$ yields a triangular system of homogeneous linear equations involving only the y_n and shows that they are all zero, as desired. $\square$

We point out that the map Φ_d and its image depend only upon d, whereas the map Ψ_d depends a priori also on the parameters n_0 and $\hat{n}_0$. However, the above proposition implies that all of the maps $\Psi_d = \Psi_{d,n_0,\hat{n}_0}$ agree on the finite co-dimensional subspace $\mathrm{image}(\Phi_d) = {}^{\perp}M_k(\Gamma(2)) \subseteq \mathcal{S}(\mathbb{N}_0) \times \mathcal{S}(\mathbb{N}_0)$.

2.3.3 Definition of the kernels

Given $k \in \mathbb{R}$, $\ell, \hat{\ell} \in \mathbb{Z}$, we define a function $\mathcal{K}_{k,\ell,\hat{\ell}}(\tau,z)$ of $(\tau,z) \in \mathbb{H}\times\mathbb{H}$ such that $z \notin \Gamma(2)\tau$, by

$$\mathcal{K}_{k,\ell,\hat{\ell}}(\tau,z) = \frac{1}{\pi i}\frac{\lambda'(z)}{\lambda(z)-\lambda(\tau)}\frac{\Theta(\tau)^{2k}}{\Theta(z)^{2k}}\frac{\lambda(\tau)^{\ell}\lambda(S\tau)^{\hat{\ell}}}{\lambda(z)^{\ell}\lambda(Sz)^{\hat{\ell}}}. \tag{2.53}$$

Let us remark that for all $z,\tau \in \mathbb{H}$, we have that $\lambda(z) = \lambda(\tau)$ if and only if $z \in \Gamma(2)\tau$ (as can be proved using (2.18)). Let us write down some properties of this kernel. First, for fixed $\tau \in \mathbb{H}$, the function $z \mapsto \mathcal{K}_{k,\ell,\hat{\ell}}(\tau,z)$ is meromorphic and modular of weight $2-k$, all of its poles are simple, contained in $\Gamma(2)\tau$ and for each $\gamma \in \Gamma(2)$, we have

$$\mathrm{Res}_{z=\gamma\tau}\,\mathcal{K}_{k,\ell,\hat{\ell}}(\tau,z) = \frac{j_k(\gamma)(\tau)^{-1}}{\pi i}. \tag{2.54}$$

To see this, we compute that for $z \notin \Gamma(2)\tau$ and $z \to \gamma\tau$ we have

$$\pi i(z-\gamma\tau)\mathcal{K}_{k,\ell,\hat{\ell}}(\tau,z) = \frac{\lambda'(z)}{\frac{\lambda(z)-\lambda(\gamma\tau)}{z-\gamma\tau}}\frac{\Theta(\tau)^{2k}}{\Theta(z)^{2k}}\frac{\lambda(\tau)^{\ell}\lambda(S\tau)^{\hat{\ell}}}{\lambda(z)^{\ell}\lambda(Sz)^{\hat{\ell}}} \longrightarrow \frac{\Theta(\tau)^{2k}}{\Theta(\gamma\tau)^{2k}} = j_k(\gamma,\tau)^{-1}.$$

Second, again for fixed $\tau \in \mathbb{H}$, the function

$$\kappa_\tau(z) = \mathcal{K}_{k,\ell,\hat{\ell}}(\tau,z) = \frac{1}{\lambda(z)-\lambda(\tau)}\frac{\Theta(\tau)^{2k}}{\Theta(z)^{2k-4}}\frac{\lambda(\tau)^{\ell}\lambda(S\tau)^{\hat{\ell}}}{\lambda(z)^{\ell-1}\lambda(Sz)^{\hat{\ell}-1}}$$

(where we used the formula (2.21) $\lambda' = \pi i\lambda(1-\lambda)\Theta^4$) has the following valuations at the cusps

$$\nu_\infty(\kappa_\tau) = 1-\ell, \quad \nu_0(\kappa_\tau) = 1-\hat{\ell}, \quad \nu_1(\kappa_\tau) = 1 + \frac{4-2k}{4} + (\ell-1) + (\hat{\ell}-1) = \ell + \hat{\ell} - \frac{k}{2}$$

2.3.4 Expansions into weakly holomorphic modular forms

The goal of this section is to show that the Fourier coefficients of the 2-periodic meromorphic function $\tau \mapsto \mathcal{K}_{k,\ell,\hat{\ell}}(\tau,z)$ are weakly holomorphic modular forms for $\Gamma(2)$, which themselves have an interesting Fourier expansion at infinity. We will ultimately use this to prove the interpolatory properties of the basis functions entering Theorem 1, that is, part (ii) of that theorem.

This discussion is similar to [RV19, Thm 3] and to what is done in the paper [DJ08] by Duke and Jenkins in the case of $\mathrm{PSL}_2(\mathbb{Z})$. Compared to the latter paper, we replace (roughly speaking) $(\mathrm{PSL}_2(\mathbb{Z}), \Delta, j)$ by $(\Gamma(2), \Theta_3, \lambda)$.

We fix $k \in \mathbb{R}$ and two non-negative integers $\ell, \hat{\ell}$. We also fix $z \in \mathbb{H}$ and abbreviate

$$a := a_z := \lambda(z), \qquad \kappa(\tau) := \kappa_z(\tau) := \mathcal{K}_{k,\ell,\hat{\ell}}(\tau, z).$$

Let us also define

$$\Omega_z := \left\{ \tau' \in \mathbb{H} \,:\, \mathrm{Im}(\tau') \geq 1 + \sup_{\gamma \in \Gamma(2)} \mathrm{Im}(\gamma \cdot z) \right\}. \tag{2.55}$$

For $\tau \in \Omega_z$, the function $\kappa(\tau)$ is holomorphic and if $\mathrm{Im}(\tau)$ is in addition large enough so that $|\lambda(\tau)/\lambda(z)| < 1$, we have

$$\begin{aligned}
\kappa(\tau) &= \Theta(z)^{4-2k} \lambda(Sz)^{1-\hat{\ell}} \frac{1}{1 - \lambda(\tau)/\lambda(z)} \frac{\lambda(\tau)^\ell}{\lambda(z)^\ell} \Theta(\tau)^{2k} \lambda(S\tau)^{\hat{\ell}} \\
&= \Theta(z)^{4-2k} \lambda(Sz)^{1-\hat{\ell}} \sum_{n=\ell}^{\infty} \frac{\lambda(\tau)^n}{\lambda(z)^n} \Theta(\tau)^{2k} \lambda(S\tau)^{\hat{\ell}}.
\end{aligned}$$

This suggests that there are polynomials $P_{k,n}(X) = P_{k,n}^{(\ell,\hat{\ell})}(X) \in X^\ell \mathbb{Q}[X]$ of degree $n \geq \ell$, so that

$$\kappa(\tau) = \sum_{n=\ell}^{\infty} \left(\Theta(z)^{4-2k} \lambda(Sz)^{1-\hat{\ell}} P_{k,n}^{(\ell,\hat{\ell})}(1/\lambda(z)) \right) e^{\pi i n \tau}. \tag{2.56}$$

We will now make this rigorous. We introduce $\varphi(z) = \Theta(z)^{2k-4} \lambda(z)^{\ell-1} \lambda(Sz)^{\hat{\ell}-1}$ and recall formula (2.21), which was $\lambda' = \pi i \Theta_3 \lambda(1-\lambda)$, to write the kernel as

$$\kappa(\tau) = \frac{1}{\pi i} \frac{\lambda'(\tau)}{a - \lambda(\tau)} \frac{\varphi(\tau)}{\varphi(z)} = \frac{w(\lambda_{\mathrm{disc}})'(w)}{a - \lambda_{\mathrm{disc}}(w)} \frac{\varphi_{\mathrm{disc}}(w)}{\varphi(z)}, \qquad w = e^{\pi i \tau}.$$

Let $n \geq \ell$ and let $\delta > 0$ be such that for $|w| < 2\delta$ we have $|\lambda_{\mathrm{disc}}(w)| < |a|$. Then

$$\widehat{\kappa}(n) = \frac{1}{\varphi(z)} \int_{|w|=\delta} \frac{\kappa_{\mathrm{disc}}(w)}{w^{n+1}} \frac{dw}{2\pi i} = \frac{1}{\varphi(z)} \int_{|w|=\delta} \frac{\varphi_{\mathrm{disc}}(w)}{a - \lambda_{\mathrm{disc}}(w)} \frac{(\lambda_{\mathrm{disc}})'(w)}{w^n} \frac{dw}{2\pi i}.$$

Since $\lambda_{\mathrm{disc}}(w)$ vanishes to order exactly 1 at $w = 0$, it defines a biholomorphic map between $|w| < \delta$ and some open neighborhood of 0. We would thus like to make the change of variables $\zeta = \lambda_{\mathrm{disc}}(w)$. In order to get a "nice" expression in terms of ζ after we do this, we want to be able rewrite the term $\frac{\varphi_{\mathrm{disc}}(w)}{w^n}$ in terms of a function of $\lambda_{\mathrm{disc}}(w)$. To do so, we look for a polynomial $Q_n = Q_{k,n}^{(\ell,\hat{\ell})}$ so that

$$\frac{\varphi_{\mathrm{disc}}(w)}{w^n} - Q_n(1/\lambda_{\mathrm{disc}}(w)) = O(1), \quad \text{when } w \to 0. \tag{2.57}$$

We prove the existence of such polynomials Q_n later. We add and subtract $Q(1/\lambda_{\text{disc}}(w))$ to the integral expressing $\widehat{\kappa}(n)$ above to get rid of $\frac{\varphi_{\text{disc}}(w)}{w^n}$. Then, we change variables $\zeta = \lambda_{\text{disc}}(w)$ as explained before, to obtain

$$\begin{aligned}\widehat{\kappa_z}(n) &= \frac{1}{\varphi(z)}\int_{|w|=\delta}\frac{1}{a-\lambda_{\text{disc}}(w)}Q_n(1/\lambda_{\text{disc}}(w))(\lambda_{\text{disc}})'(w)\frac{dw}{2\pi i}\\ &= \frac{1}{\varphi(z)}\int_{|\zeta|=\varepsilon}\frac{1}{a-\zeta}Q_n(1/\zeta)\frac{d\zeta}{2\pi i} = \frac{1}{\varphi(z)}\int_{|\xi|=1/\varepsilon}\frac{1}{a-\xi^{-1}}Q_n(\xi)\frac{1}{\xi^2}\frac{d\xi}{2\pi i}\\ &= \frac{1}{a\varphi(z)}\int_{|\xi|=1/\varepsilon}\frac{1}{\xi-a^{-1}}\frac{Q_n(\xi)}{\xi}\frac{d\xi}{2\pi i} = \frac{1}{a\varphi(z)}\frac{Q_n(a^{-1})}{a^{-1}} = \frac{1}{\varphi(z)}Q_n(1/\lambda(z)),\end{aligned}$$

where the second last step is an application of Cauchy's theorem whose justification also requires that $Q_n(0)=0$. Once we find such Q_n, the polynomials $P_{k,n}(X) = X^{\ell-1}Q_n(X)$ will then indeed express $\widehat{\kappa(n)}$ as in (2.56). We add the argument for this to the proof of the following summarizing result.

Proposition 2.6. *Let k be a real number and let $\ell, \hat{\ell} \geq 0$ be integers*[12]*. For every integer $n \geq \ell$, there exists a unique polynomial $P_{k,n}^{(\ell,\hat{\ell})} \in X^\ell\mathbb{C}[X]$ such that* $\deg P_{k,n}^{(\ell,\hat{\ell})} = n$ *and such that the weakly holomorphic modular form*

$$\vartheta_{k,n}^{(\ell,\hat{\ell})}(z) := \Theta(z)^{4-2k}\lambda(Sz)^{1-\hat{\ell}}P_{k,n}^{(\ell,\hat{\ell})}(1/\lambda(z))$$

satisfies

$$(\vartheta_{k,n}^{(\ell,\hat{\ell})})^\wedge(-m) = \delta_{n,m}$$

for all integers $n, m \geq \ell$. Moreover, for each $z \in \mathbb{H}$ and all $\tau \in \Omega_z$ (as defined in (2.55)*) we have*

$$\mathcal{K}_{k,\ell,\hat{\ell}}(\tau,z) = \sum_{n=\ell}^{\infty}\vartheta_{k,n}^{(\ell,\hat{\ell})}(z)e^{\pi i n\tau}. \tag{2.58}$$

Proof. Let $n \geq \ell$ be an integer. Consider a general polynomial $P(X) = \sum_{j=0}^{n-\ell} a_j X^{j+\ell} \in X^\ell\mathbb{C}[X]$ of degree n and the function

$$\vartheta_P(z) = \Theta(z)^{4-2k}\lambda(Sz)^{1-\hat{\ell}}P(1/\lambda(z)).$$

This is a weakly holomorphic modular form for $\Gamma(2)$ of weight $2-k$ with a pole at infinity of order at most the degree P, so we need to show that there is a unique P for which we have $\vartheta_P^\wedge(-m) = \delta_{n,m}$ for all $m \in \{\ell, \ell+1, \ldots, n\}$. Since $\Theta(z)^{4-2k}\lambda(Sz)^{1-\hat{\ell}}$ is regular and non-vanishing at ∞, the condition $\vartheta_P^\wedge(-n) = 1$ fixes the leading coefficient of P and the other coefficients are then determined successively from the condition $\vartheta_P^\wedge(-m) = 0$ for $\ell \leq m < n$. Thus there is indeed a unique $P = P_{k,n}^{(\ell,\hat{\ell})}$ with the stated properties.

To prove that (2.58) holds, it suffices to show that the polynomial $Q_n(X) = Q_n^{(\ell,\hat{\ell})}(X) = X^{-\ell+1}P_{k,n}^{(\ell,\hat{\ell})}(X)$ is such that (2.57) holds. To see that it does, we write, for $n \geq \ell$,

$$\begin{aligned}\frac{\varphi_{\text{disc}}(w)}{w^n} - Q_n(1/\lambda_{\text{disc}}(w)) &= \varphi_{\text{disc}}(w)\left(\frac{1}{w^n} - \frac{\lambda_{\text{disc}}(w)^{\ell-1}}{\varphi_{\text{disc}}(w)}P_n(1/\lambda_{\text{disc}}(w))\right)\\ &= \varphi_{\text{disc}}(w)O(w^{-\ell+1}) = O(1)\end{aligned}$$

as $w \to 0$, as desired. ☐

[12] not necessarily satisfying $\ell + \hat{\ell} = 1 + \lfloor k/2 \rfloor$, as in many other parts of this chapter.

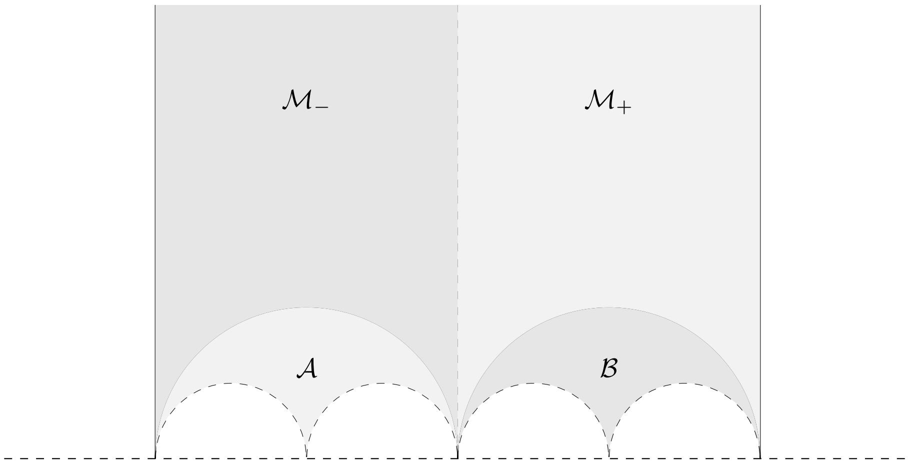

Figure 3: The sets $\mathcal{M}_\pm$, $\mathcal{A}$,$\mathcal{B}$

2.3.5 Singular integral transforms

In this section, we will construct the modular integral $F_{k,\ell,\hat{\ell}}(\tau, x)$ whose Fourier coefficients are the functions $a_{k,n}$ in Theorem 1 as an integral transform of the kernel function $\mathcal{K}_{k,\ell,\hat{\ell}}(\tau, z)$ defined earlier.

2.3.5.1 Definition of the integral

Fix a real number $k \geq 0$ and integers $\ell, \hat{\ell} \in \mathbb{N}_0$ such that $\ell + \hat{\ell} = 1 + \lfloor k/2 \rfloor$. We abbreviate by $\mathcal{K} := \mathcal{K}_{k,\ell,\hat{\ell}}$, the function defined in (2.53). Let

$$\mathcal{M} := \mathcal{D}_2 \cup S\mathcal{D}_2 = \{z \in \mathbb{H} \,:\, |\operatorname{Re}(z)| < 1, |z - 1/2| > 1/2, |z + 1/2| > 1/2\}$$

denote "the" fundamental domain for $\Gamma(2)\backslash\mathbb{H}$, as introduced in (2.3), (2.4). Let $\alpha_0, \beta_0 : [0, 1] \to \mathbb{H} \cup \mathbb{R}$, denote the following pieces of the boundary of $\mathcal{M}$:

$$\alpha_0(t) = -1/2 + (1/2)e^{\pi i(1-t)}, \qquad \beta_0(t) = 1/2 + (1/2)e^{\pi i(1-t)}.$$

We orient these paths clock-wise. We write α_0^- and β_0^- for the reversed paths, oriented counter-clockwise. Recall that $V = ST^{-1}S$ and note that

$$\alpha_0^- = V^{-2}\beta_0, \qquad \beta_0^- = V^2\alpha_0. \tag{2.59}$$

Given a point $\tau \in \mathcal{M}$, we call a piece-wise smooth path $\gamma = \gamma_\tau : [0, 1] \to \mathbb{C}$ *admissible* for τ, if:

- $\gamma((0, 1)) \subseteq \mathbb{H}$, $\gamma(0) = -1$, $\gamma(1) = 1$,
- there exists $t_0 \in (1/2, 1)$ such that γ_τ is the concatenation of $\alpha_0|_{[0,t_0]}$, the line segment joining $\alpha(t_0)$ to $\beta(1 - t_0)$ and $\beta|_{[1-t_0,1]}$,
- $\operatorname{Im}(\alpha_0(t_0)) < \operatorname{Im}(\tau)$, so that the line segment is below the point τ.

Thus, an admissible path is a slight "tweaking" near 0 of the path $\alpha_0 + \beta_0$. This is to avoid the pole of $z \mapsto \mathcal{K}(\tau, z)$ at the cusp 0 (and the pole at $z = \tau$ in $\mathcal{M}$ itself of course). Note that, if $\hat{\ell} = 0$, then $z \mapsto \mathcal{K}(\tau, z)$ has no pole at 0 and no such modification would be necessary, that is to say, we could set $\gamma_\tau = \alpha_0 + \beta_0$ for all $\tau \in \mathcal{M}$ during the entire following analysis.

Proposition 2.7. *Let $\varphi : \mathbb{H} \to \mathbb{C}$ be a holomorphic function of moderate growth. For $\tau \in \mathcal{M}$ the integral*

$$F(\tau) := F_{k,\ell,\hat{\ell}}(\tau,\varphi) := \frac{1}{2}\int_{\gamma_\tau} \mathcal{K}(\tau,z)\varphi(z)dz, \tag{2.60}$$

taken over any admissible path γ_τ of $\tau \in \mathcal{M}$ is well-defined and defines a holomorphic function of τ. It extends to a holomorphic function on all of $\mathbb{H}$ where it satisfies the functional equations

$$F|_k(T^2-1) = 0, \qquad F|_k(V^{-2}-1) = \varphi|_k(V^{-2}-1). \tag{2.61}$$

Proof. Fix a point $\tau \in \mathcal{M}$, and an open ball B, centered at τ such that $\overline{B} \subseteq \mathcal{M}$, and such that we can find a single admissible path $\gamma = \gamma_\tau$, which is also admissible for all $\tau' \in \overline{B}$. Fix a parametrization $\gamma : [0,1] \to \mathbb{C}$ of this path. Let $X = X_B$ denote the space of $\mathbb{C}$-valued continuous functions on $\overline{B}$ which are holomorphic in B. Equipped with the sup norm, X becomes a Banach algebra. Consider the map

$$f : (0,1) \to X, \qquad f(t) := \mathcal{K}(\cdot,\gamma(t))\varphi(\gamma(t))\gamma'(t).$$

We claim that f is continuous and that it extends continuously to $[0,1]$ by $f(0) = f(1) = 0 \in X$. To prove this, we may replace f by

$$\tilde{f}(t) := \frac{\lambda(\gamma(t))}{\lambda(\gamma(t)) - \lambda(\cdot)}\vartheta(\gamma(t))\varphi(\gamma(t)), \quad \vartheta(z) = \Theta(z)^{4-2k}\lambda(Sz)^{-\hat{\ell}+1}\lambda(z)^{-\ell}.$$

For this reduction we used that multiplication by a complex scalar or an element of X are continuous operations on X. Let $t_i \in (0,1)$ be a sequence such that $t_i \to t_0 \in [0,1]$. Write $z_i = \gamma(t_i)$. Suppose first that $t_0 \in (0,1)$. To show that $\tilde{f}(t_i) \to \tilde{f}(t_0)$, it suffices to show that

$$\sup_{\tau \in \overline{B}} \left| \frac{\lambda(z_i)}{\lambda(z_i) - \lambda(\tau)} - \frac{\lambda(z_0)}{\lambda(z_0) - \lambda(\tau)} \right| = \sup_{\tau \in \overline{B}} \frac{|\lambda(\tau)|}{|\lambda(z_0) - \lambda(\tau)|} \frac{|\lambda(z_0) - \lambda(z_i)|}{|\lambda(z_i) - \lambda(\tau)|}$$

tends to zero as $i \to \infty$. This is clear, since the denominators can be bounded uniformly from below by compactness and continuity. Suppose now that $t_0 = 0$. Then, to show that $\tilde{f}(t_i) \to 0 \in X$, it suffices to show that

$$\sup_{t \in [0,1], \tau \in \overline{B}} \left| \frac{\lambda(\gamma(t))}{\lambda(\gamma(t)) - \lambda(\tau)} \right| < \infty, \quad \text{and} \quad \lim_{t \to 0} \vartheta(\gamma(t))\varphi(\gamma(t)) = 0.$$

The second assertion holds since

$$\operatorname{ord}_1(\vartheta) = \frac{4-2k}{4} + (\hat{\ell}-1) + \ell = -\frac{k}{2} + 1 + \left\lfloor \frac{k}{2} \right\rfloor > 0,$$

so that, as $t \to 0$, we have $\vartheta(\gamma(t)) \lesssim e^{-\alpha \operatorname{Im}(\gamma(t))^{-1}}$ for some $\alpha > 0$. Since φ is assumed to be of moderate growth, the vanishing of the limit follows. The finiteness of the supremum follows by writing

$$\frac{\lambda(\gamma(t))}{\lambda(\gamma(t)) - \lambda(\tau)} = \frac{1}{1 - \lambda(\tau)/\lambda(\gamma(t))}$$

and using that $\lambda(\gamma(t)) \to 0$ as $t \to 0$ (or 1) and, away from $t = 0$, using compactness and continuity. These arguments prove that the original map $f(t)$ extends to a continuous X-valued function on $[0,1]$ and can thus be integrated; see §2.5. This shows that the integral (2.60) is well-defined and holomorphic on $\mathcal{M}$. It is clear that it does not depend upon the choice of the admissible path.

Now we turn to the analytic continuation. We introduce the following sets (see Figure 3).

$$\mathcal{M}_+ := \{z \in \mathcal{M} : \mathrm{Re}(z) > 0\}, \qquad \mathcal{M}_- := \{z \in \mathcal{M} : \mathrm{Re}(z) < 0\} = S\mathcal{M}_+.$$

Note that the open set

$$\mathcal{A} := ST^2S\mathcal{M}_+ = V^{-2}\mathcal{M}_+ = \Delta(-1, -1/2, 0)$$

is a hyperbolic triangle[13] below $\mathcal{M}_-$, with one arc given by α_0. Similarly, the open set

$$\mathcal{B} := ST^{-2}S\mathcal{M}_- = V^2\mathcal{M}_- = \Delta(0, 1/2, 1)$$

is a hyperbolic triangle below $\mathcal{M}_+$ with one arc given by β_0. We shall first analytically continue $F(\tau)$ from $\mathcal{M}_+$ to $\overline{\mathcal{M}_+} \cup \mathcal{B}$ and then say that one can proceed similarly to analytically continue $F(\tau)$ to $\overline{\mathcal{M}_-} \cup \mathcal{A}$.

For $\tau \in \mathcal{B}$ (or on β_0) we define

$$F_{\mathrm{cont}}(\tau) = \frac{1}{2}\int_{\tilde{\gamma}_\tau} \mathcal{K}(\tau, z)\varphi(z)dz$$

where $\tilde{\gamma}_\tau$ is a path described as follows:

- In $\mathbb{H}_- = \{z \in \mathbb{H} : \mathrm{Re}(z) < 0\}$, the path $\tilde{\gamma}_\tau$ is contained in $\overline{\mathcal{M}_-}$, joins -1 to (essentially) 0 and runs "above" γ_τ, so as to enclose with γ_τ (in $\mathbb{H}_-$) a domain $\Omega_-(ST^2S\tau)$ that contains $ST^2S\tau \in \mathcal{M}_-$ and no other $\Gamma(2)$ conjugates in that domain,
- in $\mathbb{H}_+$, the path $\tilde{\gamma}_\tau$ is contained in $\overline{\mathcal{B}}$ and joins 0 to 1 along the two geodesic arcs with endpoints $0, 1/2, 1$ (except that we have to modify the paths near the points 0 and $1/2$ to avoid possible poles of $z \mapsto \mathcal{K}(\tau, z)$ at 0 and at $\Gamma(2)\infty = \Gamma(2)(1/2)$) so as to enclose with γ_τ (in $\mathbb{H}_+$) a domain $\Omega_+(\tau)$ that contains τ and no other $\Gamma(2)$-conjugates in that domain

It then follows from the modularity of the kernels in the τ-variables and the residue formulas (2.54) that

$$\begin{aligned} F_{\mathrm{cont}}(\tau) - j_k(ST^2S, \tau)^{-1}F(ST^2S\tau) &= \frac{1}{2}\int_{\partial\Omega_+(\tau)} \mathcal{K}(\tau, z)\varphi(z)dz - \frac{1}{2}\int_{\partial\Omega_-(ST^2S\tau)} \mathcal{K}(\tau, z)\varphi(z)dz \\ &= \pi i \,\mathrm{Res}_{z=\tau}\left(\mathcal{K}(\tau, z)\varphi(z)\right) - \pi i \,\mathrm{Res}_{z=ST^2S\tau}\left(\mathcal{K}(\tau, z)\varphi(z)\right) \\ &= \varphi(\tau) - j_k(ST^2S, \tau)^{-1}\varphi(ST^2S\tau) \end{aligned}$$

as desired (here, both boundaries are oriented counter-clock wise and we also used that the poles of $\mathcal{K}$ are simple and that $z \mapsto \varphi(z)$ has no zeros and no poles). We should also note that $F_{\mathrm{cont}}(\tau) = F(\tau)$ for $\tau \in \mathcal{M}_+$, by changing the contour from γ_τ back to $\tilde{\gamma}_\tau$ (without crossing any poles).

Remark 2.7. We pause to point out a further technical point. A priori, we cannot apply the residue theorem to the domains $\Omega = \Omega_+(\tau)$ or $\Omega = \Omega_-(ST^2S\tau)$, as they are not open subsets contained in $\mathbb{H}$. Instead, we need to replace Ω by $\Omega_\delta = \Omega \smallsetminus \Sigma_\delta$, where Σ_δ is a sector of a disc of radius δ, centered at 1 or -1 respectively and apply the residue theorem to Ω_δ. Since $z \mapsto K(\tau, z)\varphi(z) \to 0$, as $z \to \pm 1$ within $\overline{\mathcal{M}}$, we can let $\delta \to 0$.

In a completely similar way, we can analytically continue $F(\tau)$ from $\mathcal{M}_-$ to $\overline{\mathcal{M}_-} \cup \mathcal{A}$. In this way, we have analytically continued $F(\tau)$ to an open subset $U \subseteq \mathbb{H}$ containing $\overline{\mathcal{M}}$. We obtain then an analytic continuation to any translate γU of $\gamma \in \Gamma(2)$ by writing γ as a product of the generators T^2, V^{-2} of $\Gamma(2)$ and requiring that the functional equations (2.61) hold. □

[13] For pair-wise distinct $p, q, r \in \mathbb{P}^1(\mathbb{R})$ we denote by $\Delta(p, q, r)$ the hyperbolic triangle with vertices p, q, r.

To make a next step towards Theorem (1), we will specialize the function φ to the functions $z \mapsto g_d(z,x) = e^{\pi i z|x|^2}$ with $k = d/2$ and $x \in \mathbb{R}^d$. In the following, we will reserve the letter r for $r = |x|$.

Lemma 2.8. *Fix $d \geq 1$. The Gaussian $g_d : \mathbb{H} \to \mathcal{S}_{rad}(\mathbb{R}^d)$, $g_d(z,x) = g_d(z)(x) = e^{\pi i z|x|^2}$, $x \in \mathbb{R}^d$ is of moderate growth, in the sense that for all $\alpha, \beta \in \mathbb{N}_0^d$, the function $z \mapsto \sup_{x\in\mathbb{R}^d} |x^\alpha \partial_x^\beta e^{\pi i z|x|^2}|$ is of moderate growth on $\mathbb{H}$.*

Proof. To verify the moderate growth condition, when composed with semi-norms, it suffices to bound $\sup_{x\in\mathbb{R}^d} |z^j x^\nu e^{\pi i z|x|^2}|$ for integers j such that $0 \leq j \leq |\beta|$ and multi-indices $\nu \in \mathbb{N}_0^n$ with $|\nu| \leq |\alpha| + |\beta|$. To do that, we use that for all $a, b > 0$, we have $\sup_{t>0} (t^a e^{-bt^2}) \leq (a/2eb)^{a/2}$ and then estimate

$$\left|z^j x^\nu e^{\pi i z|x|^2}\right| \leq |z|^j \prod_{i=1}^d x_i^{\nu_i} e^{-\pi \operatorname{Im}(z)|x_i|^2} \leq |z|^j \prod_{i=1}^d \left(\frac{\nu_i}{2\pi e \operatorname{Im}(z)}\right)^{\nu_i/2} \lesssim_{\nu,d} |z|^j \operatorname{Im}(z)^{-|\nu|/2}, \quad (2.62)$$

which (together with $2AB \leq A^2 + B^2$), clearly implies the moderate growth condition. ☐

Proposition 2.8. *Let $d \in \mathbb{N}$, $k = d/2$, and $\ell, \hat{\ell} \in \mathbb{N}_0$ such that $\ell + \hat{\ell} = 1 + \lfloor k/2 \rfloor$. The function*

$$F_{k,\ell,\hat{\ell}}(\tau, x) := F_{k,\ell,\hat{\ell}}(\tau, \varphi_r) = \frac{1}{2}\int_{\gamma_\tau} \mathcal{K}_{k,\ell,\hat{\ell}}(\tau, z) g_d(z,x) dz,$$

defined for each fixed $x \in \mathbb{R}^d$ via Proposition 2.7, is such that $\tau \mapsto F_{k,\ell,\hat{\ell}}(\tau, \cdot)$ defines a continuous function $\mathbb{H} \to \mathcal{S}_{rad}(\mathbb{R}^d)$. Moreover, we have

$$F_{k,\ell,\hat{\ell}}(\tau) + (\tau/i)^{-k} \left(\mathcal{F}_{\mathbb{R}^{2k}} \circ F_{k,\hat{\ell},\ell}\right)(-1/\tau) = \varphi(\tau) \quad (2.63)$$

for all $\tau \in \mathbb{H}$. Note the order of the indices ℓ and $\hat{\ell}$.

Proof. By the proof of the analytic continuation, for any fixed $\tau \in \overline{\mathcal{M}}$, there is an open ball B containing τ and a path γ, joining -1 and 1 with endings in $\overline{\mathcal{M}}$, such that for all $\tau \in B$ we have

$$F_{k,\ell,\hat{\ell}}(\tau, x) = \frac{1}{2}\int_{\gamma} \mathcal{K}_{k,\ell,\hat{\ell}}(\tau, z) g_d(z,x) dz.$$

As a function of $z \in \gamma$, the integrand defines a continuous $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$-valued map with value the zero function at the endpoints. It follows from abstract integration theory (§2.5) that this integral defines a Schwartz function. To prove continuity, it suffices to prove continuity on B by the functional equations.

To do that, let us write

$$\mathcal{K}_{k,\ell,\hat{\ell}}(\tau, z) = \frac{\phi(\tau)\vartheta(z)}{\lambda(\tau) - \lambda(z)}$$

and compute, for $\tau' \in B$,

$$\mathcal{K}_{k,\ell,\hat{\ell}}(\tau, z) - \mathcal{K}_{k,\ell,\hat{\ell}}(\tau', z) = \frac{\vartheta(z)}{\lambda(\tau) - \lambda(z)} \frac{\phi(\tau) - \phi(\tau')}{(\lambda(\tau)/\lambda(z)) - 1} + \vartheta(z) \frac{\phi(\tau')\lambda(\tau) - \lambda(\tau')\phi(\tau)}{(\lambda(\tau') - \lambda(z))(\lambda(\tau) - \lambda(z))}$$

Multiplying with the Gaussian $g_d(z)$ and integrating over $z \in \gamma$, we see that, as $\tau' \to \tau$ the resulting Schwartz function tends to zero (in the Schwartz topology), as desired.

Now let us prove (2.63). By analyticity, it suffices to consider $\tau \in \mathcal{M}$. Fix an admissible path γ_τ for τ. We have

$$\begin{aligned}(\tau/i)^{-k}\left(\mathcal{F}_{\mathbb{R}^{2k}} \circ F_{k,\hat{\ell},\ell}\right)(-1/\tau) &= \frac{1}{2}\int_{\gamma_\tau} (\tau/i)^{-k}\mathcal{K}_{k,\hat{\ell},\ell}(-1/\tau, z)(z/i)^{-k}\varphi_r(-1/z)dz \\ &= \frac{1}{2}\int_{\gamma_\tau} -\mathcal{K}_{k,\ell,\hat{\ell}}(\tau, Sz)\varphi_r(-1/z)(z/i)^{-2}dz \\ &= -\frac{1}{2}\int_{S\circ\gamma_\tau} \mathcal{K}_{k,\ell,\hat{\ell}}(\tau, z)\varphi_r(z)dz\end{aligned}$$

Note the change in the subscripts ℓ and $\hat{\ell}$, which comes from terms such as $\lambda(\tau)^{\ell}\lambda(S\tau)^{\hat{\ell}}$ in the definition of the kernel and the formula $\lambda(S\tau) = 1 - \lambda(\tau)$, which causes a sign change in the denominator $\lambda(z) - \lambda(\tau)$. Note moreover that $S \circ \gamma_\tau$ is the concatenation of three paths, namely $\pm 1 + i[0, Y]$ and a path joining $-1 + iY$ and $1 + iY$ (for some Y given as the imaginary part of $S\alpha_0(t_0)$, where $t_0 \in (1/2, 1)$ is as in the definition of "admissible path"). It follows that

$$F_{k,\ell,\hat{\ell}}(\tau) + (\tau/i)^{-k}\left(\mathcal{F}_{\mathbb{R}^{2k}} \circ F_{k,\hat{\ell},\ell}\right)(-1/\tau) = \frac{1}{2}\int_{\gamma_\tau + S\gamma_\tau} \mathcal{K}_{k,\ell,\hat{\ell}}(\tau, z)\varphi(z)dz = \varphi(\tau)$$

because of the residue property and since the concatenation $\gamma_\tau + S\gamma_\tau$ encloses counterclockwise a domain containing τ. □

2.3.6 Moderate growth bounds

As before, we fix an integer $d \geq 1$, set $k = d/2$ and fix two integers $\ell, \hat{\ell}$ such that $\ell + \hat{\ell} = 1 + \lfloor d/4 \rfloor$. The goal of this section is to prove, in a first step, that $\tau \mapsto F_{k,\ell,\hat{\ell}}(\tau, \cdot)$ is of moderate growth on the fundamental domain $\mathcal{M}$ of $\Gamma(2)$ and, in a second step, on all of $\mathbb{H}$.

To achieve the first step, we divide the latter into cuspidal regions (defined below) and the compact complement in $\mathcal{M}$, on which moderate growth holds by Proposition 2.8. We define the cuspidal regions, for $a > 1/2$ and $\varepsilon \in (0, 1/2)$ to be the following subsets of the fundamental domain $\mathcal{M}$:

$$\begin{aligned}\mathcal{R}_{\infty,a} &:= \{\tau \in \mathcal{M} : \operatorname{Im}(\tau) \geq a\} \\ \mathcal{R}_{0,\varepsilon} &:= \{\tau \in \mathcal{M} : \operatorname{Im}(\tau) < \varepsilon, \quad \operatorname{Re}(\tau) \in (-1/2, 1/2)\} \\ \mathcal{R}_{1,\varepsilon} &:= \{\tau \in \mathcal{M} : \operatorname{Im}(\tau) < \varepsilon, \quad \operatorname{Re}(\tau) \notin [-1/2, 1/2]\}\end{aligned}$$

Proposition 2.9. *For all $\varepsilon \in (0, 1/2)$ and $a > 1/2$, for all $\mathcal{R} \in \{\mathcal{R}_{\infty,a}, \mathcal{R}_{0,\varepsilon}, \mathcal{R}_{1,\varepsilon}\}$ and for all $\alpha, \beta \in \mathbb{N}_0^d$, the function*

$$\tau \mapsto \sup_{x\in\mathbb{R}^d} |x^\alpha \partial_x^\beta F_{k,\ell,\hat{\ell}}(\tau, x)|$$

is of moderate growth on $\mathcal{R}$ and hence (by Proposition 2.8) of moderate growth on all of $\overline{\mathcal{M}}$. In fact, we have

$$\sup_{x\in\mathbb{R}^d} |x^\alpha \partial_x^\beta F_{k,\ell,\hat{\ell}}(\tau, x)| \leq C(1 + \operatorname{Im}(\tau)^{-b}) \tag{2.64}$$

for all $\tau \in \overline{\mathcal{M}}$, for some constant $C > 0$ depending only on $k, \ell, \hat{\ell}, \alpha, \beta$ and some constant b depending only on k, α, β.

Remark 2.8. It is very likely that there is $c \geq 0$ so that $b = k + \frac{|\alpha|+|\beta|}{2} + c$ is admissible in (2.64) and that C as a function of k grows at most exponentially with k, but we have not carried out the estimates carefully enough to guarantee this.[14]

All our estimate only use contour-shifts combined with compactness and continuity arguments (and the properties of the fundamental domains). The contour-shifts are performed depending on τ in such a way that, for z on the contour, one can control the denominator $\lambda(z) - \lambda(\tau)$ appearing in $\mathcal{K}$ and creating the singularities. At some place, the following lemma will therefore be useful.

Lemma 2.9. *There exists an absolute constant $Y > 0$ such that for all $z, \tau \in \mathbb{H}$ we have*

$$\min\left(\mathrm{Im}(z), \mathrm{Im}(\tau)\right) \geq Y \quad \Rightarrow \quad |\lambda(z) - \lambda(\tau)| \geq 8|e^{\pi i z} - e^{\pi i \tau}|.$$

Proof. This is based on $\lambda_{\mathrm{disc}}(0) = 0$ and $(\lambda_{\mathrm{disc}})'(0) = 16 \neq 0$. More generally, consider any holomorphic function $f : \mathbb{D} \to \mathbb{C}$ satisfying $f(0) = 0$ and $f'(0) = a \neq 0$. Let $\delta \in (0, 1)$ and $w_1, w_2 \in \mathbb{D}$ with $|w_1|, |w_2| \leq \delta$. Then

$$f(w_1) = \int_0^{w_1} f'(w)dw = \int_0^{w_1} (a + (f'(w) - a))dw = aw_1 + \int_0^{w_1} (f'(w) - a)dw.$$

Hence

$$f(w_1) - f(w_2) = a(w_1 - w_2) + \int_{w_2}^{w_1} (f'(w) - a)dw. \tag{2.65}$$

Since $f'(0) - a = 0$ we have $\sup_{|w|\leq\delta} |f'(w) - a| \to 0$ as $\delta \to 0$ and hence, by applying the triangle inequality to (2.65), we get

$$|w_1 - w_2|(|a|/2) \leq |f(w_1) - f(w_1)| \leq |w_1 - w_2|(3|a|/2),$$

for all sufficiently small δ. This translates in particular to the claimed lower bound. □

2.3.6.1 Temporary notations and conventions

The following subsections 2.3.6.2, 2.3.6.3, 2.3.6.4 are devoted to the proof of Proposition 2.9 and the following notations will be in force.

We abbreviate $\mathcal{K} = \mathcal{K}_{k,\ell,\hat{\ell}}$, $g = g_d$ and $F(\tau, x) = F_{k,\ell,\hat{\ell}}$. We might also use r to denote $|x|$, where $x \in \mathbb{R}^d$ and $k = d/2$ as always. All contour shifts involving paths ending in the cusps -1 or 1 are performed as explained in Remark 2.7. All implied constants are allowed to depend on $k, \ell, \hat{\ell}$.

Let us also introduce short hands

$$\phi(\tau) := \Theta(\tau)^d \lambda(\tau)^\ell \lambda(S\tau)^{\hat{\ell}}, \qquad \psi(z) := \Theta_3(z)^{4-d} \lambda(z)^{1-\ell} \lambda(Sz)^{1-\hat{\ell}} \tag{2.66}$$

so that, by definition (and by (2.21)), we can write

$$\mathcal{K}(\tau, z) = \frac{\phi(\tau)\psi(z)}{\lambda(z) - \lambda(\tau)}.$$

For later reference, we compute (and recall) the following table of valuations at cusps.

[14] For a related painful exercise of this flavor resulting in such an estimate, we refer to §2 in our preprint [RS21]. We have not attempted to reproduce it here, because of the additional complications of our setting here, e.g. the fact that we have three cusps.

f	$\nu_\infty(f)$	$\nu_0(f)$	$\nu_1(f)$
ϕ	ℓ	$\hat{\ell}$	$k/2 - \lfloor k/2 \rfloor - 1$
ψ	$1-\ell$	$1-\hat{\ell}$	$\lfloor k/2 \rfloor - k/2$
λ	1	0	-1
Θ_3	0	0	$1/4$

2.3.6.2 Bounds in $\mathcal{R}_{\infty,a}$

This is the simplest region, where we don't have to do any contour shifts and only use compactness and continuity. Let $a > 1/2$. We fix a path γ, which is admissible for all $\tau \in \mathcal{R}_{a,\infty}$. We also fix $\alpha, \beta \in \mathbb{N}_0^d$. We have, for $\tau \in \mathcal{R}_{a,\infty}$,

$$x^\alpha \partial_x^\beta F(\tau, x) = \phi(\tau) \frac{1}{2} \int_\gamma \frac{1}{1 - \lambda(\tau)/\lambda(z)} \frac{\psi(z)}{\lambda(z)} x^\alpha \partial_x^\beta g_d(z, x) dz.$$

We apply the triangle inequality and use that for $\tau \in \mathcal{R}_{a,\infty}$ we have $|\phi(\tau)| \leq C$, for some constant C depending only on $d, \ell, \hat{\ell}$ and a. We use that

$$\inf_{(\tau,z) \in \mathcal{R}_{a,\infty} \times \gamma} |1 - \lambda(\tau)/\lambda(z)| > 0$$

which follows from the fact that $\lambda(\tau) \to 0$ as $\tau \to i\infty$, that $1/\lambda(z) \to 0$ as z tends to either 1 or -1 along γ and from the fact that $\lambda(z) \neq \lambda(\tau)$ for all $(\tau, z) \in \mathcal{R}_{a,\infty}$ (note that there are no problems when $\tau \in 1 + i\mathbb{R}_{>0}$, say as its only $\Gamma(2)$-conjugates are on $-1 + i\mathbb{R}$ but not on α_0 or β_0 (these two pieces are identified under the generator V^2). Finally, as $z \to 1$, or as $z \to -1$ along γ, we have

$$\sup_{x \in \mathbb{R}} |\psi(z)\lambda(z)^{-1} x^\alpha \partial_x^\beta g_d(z, x)| \longrightarrow 0.$$

(This was already used in the proof of Proposition 2.7, where we used the notation $\vartheta(z)$ for $\psi(z)\lambda(z)^{-1}$). We deduce that $\sup_{x \in \mathbb{R}^d} |x^\alpha \partial_x^\beta F(\tau, x)|$ is bounded on $\mathcal{R}_{a,\infty}$ and in particular of moderate growth.

2.3.6.3 Bounds in $\mathcal{R}_{0,\varepsilon}$

Let Y be the smallest constant having the property of Lemma 2.9. For $\tau \in \mathcal{R}_{0,\varepsilon}$ we write $\tilde{\tau} := S\tau$ and note that $\mathrm{Im}(\tilde{\tau}) \geq 1/(2\varepsilon) \geq 1$. Suppose that $1/(2\varepsilon) \geq \max(1, Y)$. Set $c = 4$ (other positive real numbers > 2 should also do the job). Recall that α_0, β_0 are the two bounding semi-circles of $\mathcal{M}$. Define $t_0 = t_0(\tau, c) \in [0,1]$ by requiring that $\alpha_0(t_0)$ is the point of intersection of α_0 and the semi-circle of radius $r(\tau) := c\,\mathrm{Im}(\tau)$ centered at 0. In the integral expressing $x^\alpha \partial_x^\beta F(\tau, x)$, we shift contours from an admissible path γ_τ, to the path which is the concatenation of

- the path $\tilde{\alpha}_\tau := \alpha_0|_{[0,t_0]}$.
- the path p_τ, defined as the segment of the circle with radius $r(\tau)$ and center 0, with endpoints $\alpha_0(t_0)$ and $\beta_0(1 - t_0)$,
- the path $\tilde{\beta}_\tau := \beta_0|_{[1-t_0,1]}$.

Also changing variables $z \leftrightarrow Sz$, we obtain

$$x^\alpha \partial_x^\beta F(\tau, x) = x^\alpha \partial_x^\beta e^{\pi i \tau |x|^2} + \frac{1}{2} \sum_{p \in \{\tilde{\alpha}_\tau, \tilde{\beta}_\tau, p_\tau\}} \int_{Sop} \mathcal{K}(S\tilde{\tau}, Sz) x^\alpha \partial_x^\beta e^{\pi i Sz|x|^2} \frac{dz}{z^2}.$$

Here, the supremum over $x \in \mathbb{R}^d$ of the residue $x^\alpha \partial_x^\beta e^{\pi i \tau |x|^2}$ may be bounded by a negative power of $\mathrm{Im}(\tau)$ as in (2.62). In the remaining integral, we may replace $x^\alpha \partial_x^\beta e^{\pi i S z|x|^2}$ by $x^\nu (Sz)^j e^{\pi i Sz|x|^2}$ for some integer $j \le |\beta|$ and multi-index $\nu \in \mathbb{N}_0^d$ with $|\nu| \le |\alpha| + |\beta|$. By definition, there exists $h(\tau) > 0$ such that

$$S\tilde{\alpha}_\tau = 1 + i[0, h(\tau)], \quad S\tilde{\beta}_\tau = -1 + i[0, h(\tau)],$$

the latter with opposite orientation. The path $S \circ p_\tau$ is the *segment* of the semi-circle with center 0, radius $\frac{1}{r(\tau)} = \frac{1}{c\,\mathrm{Im}(\tau)}$ connecting $S\beta_0(1 - t_0) = -1 + ih(\tau)$ to $S\alpha_0(t_0) = 1 + ih(\tau)$ (Note that this is similar to the contour shift performed in the proof of Proposition 2.8). The point $\tilde{\tau} = S\tau$ (which belongs to $\mathcal{M}$) lies *above* the integration variable z, since

$$\mathrm{Im}(\tilde{\tau}) - \frac{1}{r(\tau)} \ge \frac{1}{2\,\mathrm{Im}(\tau)} - \frac{1}{c\,\mathrm{Im}(\tau)} = \frac{1}{\mathrm{Im}(\tau)}\left(\frac{1}{2} - \frac{1}{c}\right) = \frac{1}{4\,\mathrm{Im}(\tau)}.$$

To simplify a bit more, we shift contours so that all paths of integration become straight line segments and it remains to estimate

$$\sum_{\sigma \in \{\pm\}} \sigma \int_\sigma^{\sigma + i/(4\,\mathrm{Im}(\tau))} K(S\tilde{\tau}, Sz)(Sz)^j x^\nu e^{\pi i Sz|x|^2} \frac{dz}{z^2} - \int_{-1+i/(4\,\mathrm{Im}(\tau))}^{1+i/(4\,\mathrm{Im}(\tau))} K(S\tilde{\tau}, Sz)(Sz)^j x^\nu e^{\pi i Sz|x|^2} \frac{dz}{z^2}.$$

Using $\lambda(Sz) - \lambda(S\tilde{\tau}) = \lambda(z) - \lambda(\tilde{\tau})$ and the definition of the slash action, we write the transformed kernel as

$$\mathcal{K}(S\tilde{\tau}, Sz) = \frac{(\tilde{\tau}/i)^k (\phi|_k S)(\tilde{\tau})(z/i)^{2-k}(\psi|_{2-k}S)(z)}{\lambda(\tilde{\tau}) - \lambda(z)}. \tag{2.67}$$

We split up the integrals over the vertical segments as $\int_\sigma^{\sigma+iY}(\dots) + \int_{\sigma+iY}^{\sigma+i/(4\,\mathrm{Im}(\tau))}(\dots)$. To estimate the contribution from the piece $\sigma + i[0, Y]$, we choose $\varepsilon' > 0$ such that for all $\varepsilon \in (0, \varepsilon']$ and all $z \in \sigma + i(0, Y]$,

$$\begin{aligned} |\lambda(z) - \lambda(\tilde{\tau})| &\ge \frac{|\lambda(z)|}{2} + \frac{|\lambda(z)|}{2} - |\lambda(\tilde{\tau})| \\ &\ge \frac{|\lambda(z)|}{2} + \frac{1}{2}\Big(\inf_{t \in (0,Y]} |\lambda(\sigma + it)|\Big) - \Big(\sup_{\mathrm{Im}(\tau') \ge 1/(2\varepsilon)} |\lambda(\tau')|\Big) \ge \frac{|\lambda(z)|}{2}. \end{aligned}$$

This is possible because the above infimum is strictly positive and the supremum tends to zero with $\varepsilon \to 0$. We henceforth assume that $\varepsilon \le \varepsilon'$. We can then use

$$\begin{aligned} &\int_\sigma^{\sigma+iY} \mathcal{K}(S\tilde{\tau}, Sz) x^\nu (Sz)^j e^{\pi i Sz|x|^2} \frac{dz}{z^2} \\ &\lesssim |\tilde{\tau}|^k |(\phi|S)(\tilde{\tau})| \int_\sigma^{\sigma+iY} \frac{2}{|\lambda(z)|} |z|^{-k} |(\psi|S)(z)| (Sz)^j |x^\nu e^{\pi i Sz|x|^2}| |dz| \\ &\lesssim |\tilde{\tau}|^k e^{-\pi\,\mathrm{Im}(\tilde{\tau})\nu_0(\phi)} \lesssim \mathrm{Im}(\tau)^{-k} \end{aligned}$$

where we bounded the Schwartz function by using that $(\psi|S)(z)/\lambda(z) g_d(z)$ tends to $0 \in \mathcal{S}(\mathbb{R}^d)$ as $s \to \sigma$. Thus, the implied constants depend at most on $k, \varepsilon, \sigma, Y, j$ and ν, but not on x. On all of the remaining integrals, the integration variable z satisfies

$$Y \le \mathrm{Im}(z) \le \mathrm{Im}(\tilde{\tau}) - (4\,\mathrm{Im}(\tau))^{-1} \le \mathrm{Im}(\tilde{\tau}) - (4\varepsilon)^{-1}.$$

Thus, by (2.67), Lemma 2.9 and by (2.62), we have on these remaining integrals

$$\begin{aligned}|\mathcal{K}(S\tilde{\tau}, Sz)(Sz)^j x^\nu e^{\pi i (Sz)|x|^2} z^{-2}| &\lesssim \frac{1}{|\lambda(z) - \lambda(\tilde{\tau})|} |(\psi|S)(z)||z|^{-k}|(\phi|S)(\tilde{\tau})||\tilde{\tau}|^k |Sz|^j \operatorname{Im}(Sz)^{-|\nu|/2} \\ &\lesssim \frac{1}{|1 - e^{\pi i(\tilde{\tau} - z)}|} e^{\pi \operatorname{Im}(z)} e^{-\pi \operatorname{Im}(z)(1-\hat{\ell})} e^{-\pi \operatorname{Im}(\tilde{\tau})\hat{\ell}} |z|^{k-j} |\tilde{\tau}|^k \operatorname{Im}(Sz)^{-|\nu|/2} \\ &\lesssim |\tilde{\tau}|^k |z|^{k+|\nu|-j} \operatorname{Im}(z)^{-|\nu|/2}\end{aligned}$$

When integrated over the remaining 3 line segments (with respect to $|dz|$), this is clearly bounded by an power of $\operatorname{Im}(\tau)^{-1} \asymp \operatorname{Im}(\tilde{\tau}) \asymp |\tilde{\tau}|$. In all estimates, the implied constant depends at most on ε, ν, j and k, but not on x or τ.

2.3.6.4 Bounds in $\mathcal{R}_{1,\varepsilon}$

This is the most complicated case. Let $\varepsilon \in (0, 1/2)$ and let $\tau \in \mathcal{R}_{1,\varepsilon}$. Define

$$\tilde{\tau} := \begin{cases} ST\tau & \text{if } \operatorname{Re}(\tau) < 0, \\ ST^{-1}\tau & \text{if } \operatorname{Re}(\tau) > 0. \end{cases}$$

Note that $\tilde{\tau} \in \mathcal{M}$ and $\operatorname{Im}(\tilde{\tau}) \geq (2\varepsilon)^{-1} \geq 1$.

At various places in the proof, we will assume that ε is sufficiently small. We first shift contours (with modifications explained in Remark 2.7) from an admissible path γ_τ to the path that is the concatenation of the paths $\pm 1 + i[0, 3\varepsilon]$ and the shifted piece $\alpha_0 + \beta_0 + i(3\varepsilon)$ of $\partial\mathcal{M}$. By the residue theorem, we thus obtain

$$F(\tau, x) = g(\tau, x) + \frac{1}{2} \int_{\alpha_0 + \beta_0 + i(3\varepsilon)} \mathcal{K}(\tau, z) g(z, x) dz + \sum_{\pm} \int_{\pm 1 + i(3\varepsilon)} \mathcal{K}(\tau, z) g(z, x) dz$$

and the same holds if we apply $x^\alpha \partial_x^\beta$. We may bound $g(\tau, x)$ and the integral over $\alpha_0 + \beta_0 + i(3\varepsilon)$ via compactness and continuity as in the analysis on $\mathcal{R}_{\infty,a}$. By differentiating the Gaussian (as in the proof of Lemma 2.8), we reduce our task to bounding the following integrals, for each fixed integer $j \geq 0$ and each fixed multi-index $\nu \in \mathbb{N}_0$:

$$I_\pm := \int_{\pm 1 + i(3\varepsilon)} \mathcal{K}(\tau, z) z^j x^\nu g(z, x) dz$$

We focus on I_+, the computations for I_- are almost identical. In order to "see" better how to avoid the singularities of the kernel, we proceed as follows. Let $A = TS$, so that $A\infty = 1$. We change variables to the image of the segment $1 + i(0, \varepsilon)$ under A^{-1} and obtain

$$\begin{aligned}I_+ &= \int_{i\infty}^{i/(3\varepsilon)} \mathcal{K}(A\tilde{\tau}, Az)(Az)^j x^\nu g(Az, x) A'(z) dz \\ &= \int_{i\infty}^{i/(3\varepsilon)} \frac{\lambda(z)\lambda(\tilde{\tau})}{\lambda(z) - \lambda(\tilde{\tau})} (\phi|_k A)(\tilde{\tau}) j_\Theta(A, \tilde{\tau})^k (\psi|_{2-k} A)(z) j_\Theta(A, z)^{2-k} (1 - 1/z)^j x^\nu e^{\pi i(1-1/z)|x|^2} \frac{dz}{z^2}\end{aligned}$$

where we used that $\lambda(Az) = \lambda(TSz) = 1 - 1/\lambda(z)$ which follows from the formulas recorded in (2.20). The virtue of this maneuver is that, on the new path of integration, the singularities to avoid are exactly $\Gamma(2)\tilde{\tau} \cap \{z \in \mathbb{H} : \operatorname{Im}(z) \geq 1\} = 2\mathbb{Z} + \tilde{\tau}$. We will avoid them by modifying the path by a little semi-circle centered at $i \operatorname{Im}(\tilde{\tau})$ (this technique is inspired from [CKM+21]). More precisely, we write $I_+ = I_{+,1} + I_{+,2} + I_{+,3}$, where:

- the integral $I_{+,1}$ is taken over the vertical line segment $i[1/(3\varepsilon), \operatorname{Im}(\tilde{\tau}) - \delta]$,
- the integral $I_{+,2}$ is taken over the semi-circle

$$p_\tau(t) := i\operatorname{Im}(\tilde{\tau}) + \delta e^{\frac{\pi i}{2}(3 - t\operatorname{sgn}(\operatorname{Re}(\tilde{\tau})))}, \quad 0 \le t \le 2,$$

  of radius $\delta := 1/4$. Thus, p_τ is in $\mathcal{M}_-$ if $\tilde{\tau} \in \mathcal{M}_+$ and vice versa)
- the integral over $I_{+,3}$ is taken over the vertical ray $i[\operatorname{Im}(\tilde{\tau}) + \delta, \infty)$.

Having performed this shift of countour, we estimate the integrals $I_{+,j}$ one after the other with the triangle inequality and in each case we will use that, uniformly in z on the path of integration, we have $|1 + 1/z|^j \lesssim_{\varepsilon,j} 1$ and (as in (2.62)),

$$\sup_{x \in \mathbb{R}^d} |x^\nu e^{\pi i(1-1/z)|x|^2}| \lesssim \operatorname{Im}(1 - 1/z)^{-|\nu|/2} = |z|^{|\nu|} \operatorname{Im}(z)^{-|\nu|/2}.$$

We will also assume that we have chosen ε so that $1/(3\varepsilon) \ge Y \ge 1$, where Y is the smallest admissible constant in Lemma 2.9. Note also that $j_\Theta(A, \tilde{\tau}) = |\tilde{\tau}|$ and $|j_\Theta(A, z)|^{2-k} = |z|^{2-k} = |z|^2|z|^{-k}$ and we may cancel $|z|^2$ with $1/z^2$ in dz/z^2. Thus,

$$\begin{aligned}
I_{+,1} &\lesssim |\tilde{\tau}|^k |(\phi|_k A)(\tilde{\tau})| \int_{1/3\varepsilon}^{\operatorname{Im}(\tilde{\tau})-1/4} \frac{|\lambda(\tilde{\tau})\|\lambda(it)|}{|\lambda(it) - \lambda(\tilde{\tau})|} (\psi|_{2-k}A)(it) t^k t^{-|\nu|/2} dt \\
&\lesssim |\tilde{\tau}|^k e^{-\pi \operatorname{Im}(\tilde{\tau})\nu_1(\phi)} e^{-\pi \operatorname{Im}(\tilde{\tau})} \int_{1/3\varepsilon}^{\operatorname{Im}(\tilde{\tau})-1/4} \frac{|e^{-\pi i(it)}\|\lambda(it)|}{|1 - e^{\pi i(\tilde{\tau} - it)}|} e^{-\pi t \nu_1(\psi)} t^{|\nu|/2-k} dt \\
&\lesssim |\tilde{\tau}|^k e^{-\pi \operatorname{Im}(\tilde{\tau})(1+\nu_1(\phi))} \int_{1/3\varepsilon}^{\operatorname{Im}(\tilde{\tau})-1/4} \frac{1}{1 - e^{-\pi/4}} e^{-\pi t \nu_1(\psi)} t^{|\nu|/2-k} dt \\
&\lesssim |\tilde{\tau}|^k e^{-\pi \operatorname{Im}(\tilde{\tau})(1+\nu_1(\phi)-\nu_1(\psi))} \int_{1/3\varepsilon}^{\operatorname{Im}(\tilde{\tau})-1/4} t^{|\nu|/2-k} dt,
\end{aligned}$$

where all implied constants depend at most on ε, Y, k, j and where we used that $\nu_1(\psi) = \lfloor k/2 \rfloor - k/2 \le 0$. Since $\nu_1(\phi) = k/2 - \lfloor k/2 \rfloor$, the exponential is bounded by 1 and since in addition $|\tilde{\tau}| \asymp \operatorname{Im}(\tau)^{-1} \asymp \operatorname{Im}(\tilde{\tau})$, it is clear that $I_{+,1}$ may be bounded by some (negative) power of $\operatorname{Im}(\tau)$, depending on k and ν.

Now we turn to $I_{+,3}$, where we proceed similarly, with the roles of $\tilde{\tau}$ and $z = it$ interchanged in the lower bound of $|\lambda(it) - \lambda(\tilde{\tau})|$ (since this time, $t \ge \operatorname{Im}(\tilde{\tau}) + 1/4$). We get

$$\begin{aligned}
I_{+3} &\lesssim |\tilde{\tau}|^k e^{-\pi \operatorname{Im}(\tilde{\tau})\nu_1(\phi)} \int_{\operatorname{Im}(\tilde{\tau})+1/4}^{\infty} e^{-\pi t(\nu_1(\psi)+1)} t^{|\nu|/2-k} dt \\
&\lesssim |\tilde{\tau}|^k e^{-\pi \operatorname{Im}(\tilde{\tau})\nu_1(\phi)} e^{-\pi(\operatorname{Im}(\tilde{\tau})+1/4)(\nu_1(\psi)+1)} \times (\text{a polynomial in } \operatorname{Im}(\tilde{\tau}))
\end{aligned}$$

via integration by parts (if necessary, i.e. if $|\nu|/2 - k > 0$). Again, the exponential in $\operatorname{Im}(\tilde{\tau})$ will be bounded by 1, since $\nu_1(\psi) + \nu_1(\phi) + 1 \ge 0$ so that we get a bound for $I_{+,3}$ which is polynonomial in $\operatorname{Im}(\tau)^{-1}$ and all implied constants depend at most on k, ε, Y and j.

It remains to estimate $I_{+,2}$, where we use $\inf_{z \in p_\tau} |\lambda(z) - \lambda(\tilde{\tau})| > 0$ (by Lemma 2.9 and the choice of p_τ) and

$$I_{+,2} \lesssim |\tilde{\tau}|^k \sup_{z \in p_\tau} \left(e^{-\pi \operatorname{Im}(z)(1+\nu_1(\psi))} e^{-\pi(1+\nu_1(\phi))(\operatorname{Im}(\tilde{\tau}))} |z|^{\nu-k} \operatorname{Im}(z)^{-\nu/2} \right)$$

Bounding $\mathrm{Im}(z)$ from below by $\mathrm{Im}(\tilde{\tau}) - 1/4$ here and again using that $1 + \nu_1(\phi) + \nu_1(\psi) \geq 0$ as well as $|z| \asymp |\tilde{\tau}|$ or and $\mathrm{Im}(z) \asymp \mathrm{Im}(\tilde{\tau})$ we may also bound $I_{+,2}$ by a polynomial in $\mathrm{Im}(\tau)^{-1}$.

Finally, for the analysis of I_-, we run exactly the same analysis with $A = ST^{-1}$ mapping $i(1/3\varepsilon, \infty)$ to $-1 + i(0, 3\varepsilon)$ with almost no changes (replace $1 - 1/z$ by $1 + 1/z$ in some integrals).

2.3.6.5 Propagation of moderate growth

Let (as usual) $d \geq 1$ be an integer, $k = d/2$, and let $\ell, \hat{\ell} \geq 0$ be integers such that $\ell + \hat{\ell} = 1 + \lfloor k/2 \rfloor$. Abbreviate (as before) $F(\tau, x) = F_{k,\ell,\hat{\ell}}(\tau, x)$ and let

$$\tilde{F}(\tau, x) = g_d(\tau, x)|S - F(\tau, x)|_k S = (\tau/i)^{-k} \left(e^{\pi i(-1/\tau)|x|^2} - F(-1/\tau, x) \right)$$

We know, by Proposition 2.9 that F is of moderate growth on $\overline{\mathcal{M}}$ and we now wish to show that this property "propagates" to all of $\mathbb{H}$ via (repeated application) of the functional equations.

For a relatively clean (but not necessarily enlightening) proof, we will follow the approach in [CKM+21] adapted to our specific situation. In particular, we will need [CKM+21, Lemma 3.2], whose relevant parts we copy here for convenience. Exclusively for the following two Lemmas, we use the notations

$$A = T^2 \in \Gamma(2), \qquad B = ST^2S = V^{-2} \in \Gamma(2).$$

Lemma 2.10. *Let $M \in \Gamma(2)$ be an element of the form $M = A^{e_1} B^{f_1} T^{e_2} V^{f_2} \cdots$ with $e_i, f_i \in \mathbb{Z} - \{0\}$, except possibly $e_1 = 0$. The following holds:*

(i) The word length $|e_1| + |f_1| + |e_2| + |f_2| + \cdots \leq \|M\|_{Fr}$.

(ii) The initial subwords

$$A^{\mathrm{sgn}(e_1)}, A^{2\,\mathrm{sgn}(e_2)}, \ldots, A^{e_2}, A^{e_2} B^{\mathrm{sgn}(f_1)}, A^{e_2} B^{2\,\mathrm{sgn}(f_2)}, \ldots, A^{e_2} B^{f_2}, A^{e_2} B^{f_2} B^{\mathrm{sgn}(e_1)}, \ldots$$

have (strictly) increasing Frobenius norms.

Lemma 2.11. *Let $k \in \mathbb{R}$ and let let $\mathcal{S}$ be a complex vector space equipped with a semi-norm $\|\cdot\| : \mathcal{S} \to \mathbb{R}_{\geq 0}$. Let $F, G : \mathbb{H} \to \mathcal{S}$ be two functions. Assume that:*

(i) F is of moderate growth on $\overline{\mathcal{M}}$ with respect to $\|\cdot\|$: There exist constants $C_1, N_1 \geq 0$ such that for all $g \in \mathrm{SL}_2(\mathbb{R})$ one has

$$gi \in \overline{\mathcal{M}} \quad \Rightarrow \quad \|F(gi)\| \leq C_1 \|g\|_{Fr}^{N_1}.$$

(ii) G is of moderate growth on $\mathbb{H}$ with respect to $\|\cdot\|$.

(iii) The functional equations $F|_k(B-1) = G|_k(B-1)$ and $F|_k(A-I) = 0$ hold.

Then F is of moderate growth on $\mathbb{H}$.

Proof. For the proof, we omit the subscript k in the vertical bars denoting the slash action. Fix an element $g \in \mathrm{SL}_2(\mathbb{R})$. We wish to estimate $\|F(gi)\|$. Pick $\gamma \in \Gamma(2)$ such that $\gamma gi \in \overline{\mathcal{M}}$. Trivially, we have

$$F = (F - F|\gamma) + F|\gamma \tag{2.68}$$

and hence

$$\|F(gi)\| \leq \|(F - F|\gamma)(gi)\| + \|(F|\gamma)(gi)\|.$$

We will estimate the second term using assumptions (i) on F and the first term by repeatedly using (ii) and (iii). One of the main difficulties is to translate dependencies on γ to a dependence on $\|g\|_{\mathrm{Fr}}$ given that $\gamma gi \in \overline{\mathcal{M}}$. This is where Lemma 2.10 will be used. We first introduce some notations that will be useful in the proof. We define cocycles

$$\Phi, \Psi : \Gamma(2) \to \{ \text{ functions } \mathbb{H} \to \mathcal{S}\} \quad \text{by} \quad \Phi(\gamma) := F - F|\gamma, \quad \Psi(\gamma) := G - G|\gamma.$$

By assumption and definition, we have $\Phi(A) = 0$ and $\Phi(B) = \Psi(B)$. A repeated use of the cocycle property shows moreover that $\Phi(B^f) = \sum_{i=0}^{|f|-1} \Phi(B^{\mathrm{sgn}(f)})|B^{\mathrm{sgn}(f)i}$ for all $f \in \mathbb{Z} - \{0\}$. For $\epsilon \in \{\pm 1\}$ define

$$\mathcal{D}_1^\epsilon := \{z \in \mathbb{H} \,:\, |z| \geq 1, \,\epsilon \operatorname{Re}(z) \in [0, 1/2]\} \quad \text{so that} \quad \mathcal{D}_* := \mathcal{D}_1^{+1} \cup T\mathcal{D}_1^{-1}.$$

is also a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$. The elements $(h_1, \ldots, h_6) = (I, T^{-1}, S, TS, ST^{-1}, STS)$ are such that $\overline{\mathcal{M}} = \cup_{i=1}^{6} h_i \overline{\mathcal{D}_*}$ and $\max_{1\leq i\leq 6} \|h_i\|_{\mathrm{Fr}} \leq 3$. It is verified in [CKM+21, Prop 4.2] that for all $b \in \mathrm{SL}_2(\mathbb{R})$ and all $\delta \in \mathrm{SL}_2(\mathbb{Z})$ one has

$$bi \in \mathcal{D}_* \quad \Rightarrow \quad \|b\|_{\mathrm{Fr}} \leq 3\|\delta b\|_{\mathrm{Fr}}. \tag{2.69}$$

With these preparations, we now start by estimating the term $\|F|\gamma(gi)\|$ appearing in (2.68). By Lemma 2.4 and assumption (i), we have

$$\|F|\gamma(gi)\| \leq C_1 \|\gamma g\|_{\mathrm{Fr}}^{N_1+|k|} \|g\|_{\mathrm{Fr}}^{|k|}.$$

Let $i \in \{1, \ldots, 6\}$ be such that $h_i^{-1}\gamma gi \in \overline{\mathcal{D}_*}$. We use (2.69) with $\delta = \gamma^{-1}h_i$ and $b = h_i^{-1}\gamma g$ to estimate

$$\|\gamma g\|_{\mathrm{Fr}} = \|h_i h_i^{-1}\gamma g\|_{\mathrm{Fr}} \leq 3\|h_i^{-1}\gamma g\|_{\mathrm{Fr}} \leq 9\|g\|_{\mathrm{Fr}} \tag{2.70}$$

and deduce

$$\|F|\gamma(gi)\| \lesssim_{N_1,k} \|g\|_{\mathrm{Fr}}^{N_1+2|k|}. \tag{2.71}$$

Before proceeding, let us note that (2.70) allows us also to estimate $\|\gamma\|_{\mathrm{Fr}}$ in terms of $\|g\|_{\mathrm{Fr}}$, because

$$\|\gamma\|_{\mathrm{Fr}} = \|\gamma g g^{-1}\|_{\mathrm{Fr}} \leq \|\gamma g\|_{\mathrm{Fr}} \|g^{-1}\|_{\mathrm{Fr}} \leq 9\|g\|_{\mathrm{Fr}}^2. \tag{2.72}$$

Now we turn to the estimate of $\|\Phi(\gamma)\| = \|F - F|\gamma\|$ in (2.68). Let us assume that γ is of the form

$$\gamma = A^{e_1}B^{f_1}A^{e_2}B^{f_2}\cdots A^{e_m}B^{f_m}.$$

The three other cases either reduce to this case or are exactly similar. Define the *tail* of the word γ, starting after B^{f_i} by $W_i := A^{e_{i+1}}B^{f_{i+1}}\cdots A^{e_m}B^{f_m}$. Applying the cocycle property and the assumptions, we get

$$\Phi(\gamma) = \sum_{i=1}^{m} \sum_{t_i=0}^{|f_i|-1} \Psi(B^{\mathrm{sgn}(f_i)})|A^{\mathrm{sgn}(f_i)t_i}W_i.$$

Recall that $\Psi(B^\epsilon) = G - G|B^\epsilon$, for $\sigma \in \{\pm 1\}$, which is of moderate growth on $\mathbb{H}$ with respect to $\|\cdot\|$ (by Lemma 2.4). We may therefore choose $C \geq 0$ and $N \geq 0$ such that $\|\Psi(B^\epsilon)(b\cdot i)\| \leq C\|b\|_{\mathrm{Fr}}^N$ for all $b \in \mathrm{SL}_2(\mathbb{R})$ and all $\sigma \in \{\epsilon 1\}$. Thus, appealing once more to Lemma 2.4, we obtain

$$\|\Phi(\gamma)(gi)\| \lesssim \|g\|_{\mathrm{Fr}}^{N+2|k|} \sum_{i=1}^{m} \sum_{t_i=0}^{|f_i|-1} \|A^{\mathrm{sgn}(f_i)t_i}W_i\|_{\mathrm{Fr}}^{|k|}.$$

The Frobenius norm of the tail-word $A^{\mathrm{sgn}(f_i)t_i}W_i$ of γ equals the Frobenius norm of its inverse, which is an *initial* subword of γ^{-1}, which, by Lemma 2.10, has Frobenius norm bounded by $\|\gamma^{-1}\|_{\mathrm{Fr}} = \|\gamma\|_{\mathrm{Fr}}$. It follows that

$$\|\Phi(\gamma)(gi)\| \lesssim \|g\|_{\mathrm{Fr}}^{N+2|k|}\|\gamma\|_{\mathrm{Fr}}^{|k|}\sum_{i=1}^{m}\sum_{t_i=0}^{|f_i|-1} 1 \lesssim \|g\|_{\mathrm{Fr}}^{N+2|k|}\|\gamma\|_{\mathrm{Fr}}^{|k|+1} \lesssim \|g\|_{\mathrm{Fr}}^{N+4|k|+2},$$

where we used Lemma 2.10 again in the second inequality and (2.72) in the last step. Combining this last estimate with (2.71) we may can now go back to (2.68) to get the final bound $\|F(gi)\| \lesssim \|g\|_{\mathrm{Fr}}^{N+4|k|+2}$, which has the desired shape. □

Combining Proposition 2.9 and Lemma 2.11 with the continuity of the functions F and $\tilde{F}$ (and possibly with Lemma 2.4) we obtain the following proposition.

Proposition 2.10. *Let $k \geq 0$ be a real number and let ℓ, $\hat{\ell} \geq 0$ be integers such that $\ell + \hat{\ell} = 1 + \lfloor k/2 \rfloor$. The functions $\tau \mapsto F_{k,\ell,\hat{\ell}}(\tau, x)$ and $\tau \mapsto \tilde{F}_{k,\ell,\hat{\ell}}(\tau, x)$ are of moderate growth on all of $\mathbb{H}$ in the sense that any continuous semi-norm of $\mathcal{S}(\mathbb{R}^d)$ of them is.*

Remark 2.9. We could also prove the propagation of moderate growth of $(F, \tilde{F})$ in the following way. We replace this pair of functions by (F^+, F^-), where, as in §2.2.2.1,

$$F^+ := F - \tilde{F}, \qquad F^- := F + \tilde{F}.$$

Recall the twisted slash action $|_k^\epsilon$ from §2.2.2.1 and that these functions are both 2-periodic holomorphic and satisfy the functional equations

$$F^{-\epsilon}|_k^\epsilon(1 - S) = g_d|_k^\epsilon(1 - S), \qquad \epsilon \in \{\pm 1\}. \tag{2.73}$$

Let $\mathcal{D} = \mathcal{D}_2 = \{z \in \mathbb{H} : |z| > 1, \mathrm{Re}(z) \in (-1, 1)\}$ denote the standard fundamental domain for the theta group Γ_θ. It is clear that F^+, F^- are both of moderate growth on $\mathcal{D} \subseteq \mathcal{M}$. It is also clear that if we can show that F^+, F^- are of moderate growth on $\mathbb{H}$, then so will be functions F and $\tilde{F}$. To prove this, one can very closely follow the approach in [RV19], giving more geometric intuition. This is also rewritten in more detail in our preprint with Ramos [RS21, §2]. The argument appears moreover in [RdS20] and [BRS20].

Remark 2.10 (Vector-valued version)**.** In the previous remark, instead of using the reduction into "plus and minus", one could also consider the following vector-valued version of the argument, which works more directly with F and $\tilde{F}$ in the following way. Define a morphism of groups $\rho : \Gamma_\theta \to \mathrm{GL}_2(\mathbb{C})$ by $\rho(T^2) = 1$ and $\rho(S) := \left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$ and then a slash action on functions $\vec{F} : \mathbb{H} \to \mathbb{C}^2$ defined as $(\vec{F}\|\gamma)(z) := j_\Theta(\gamma, z)^{-2k}(\rho(\gamma)\vec{F}(\gamma z))$ for $\gamma \in \Gamma_\theta$ and $z \in \mathbb{H}$. Consider then the function

$$\vec{\Phi} : \mathbb{H} \to \mathbb{C}^2, \qquad \vec{\Phi}(\tau) := \begin{pmatrix} F(\tau) \\ \tilde{F}(\tau) \end{pmatrix}$$

satisfying the functional equations

$$\vec{\Phi}\|(1 - T^2) = \vec{0}, \qquad \vec{\Phi}\|(1 - S) = \begin{pmatrix} g \\ \hat{g} \end{pmatrix}.$$

Thus Φ is a vector valued modular integral on Γ_θ and of moderate growth on $\mathcal{D}$. Thus, by an adaption of the argument sketched in the previous remarks (for instance, replacing absolute values by a suitable norm on $\mathbb{C}^2$), propagation of moderate growth can be proved in a similar way.

2.3.7 Proof of Theorem 1

We are now ready to give the proof of Theorem 1. Fix an integer $d \geq 1$, set $k = d/2$ and let $n_0, \hat{n}_0$ be integers such that $n_0 + \hat{n}_0 = 1 + \lfloor k/2 \rfloor$. Set $\ell = n_0$, $\hat{\ell} = \hat{n}_0$. For all $n \in \mathbb{Z}$ and $x \in \mathbb{R}^d$, we define

$$a_{k,n}(x) := \frac{1}{2}\int_{iy-1}^{iy+1} F_{k,\ell,\hat{\ell}}(\tau,x)e^{-\pi i n\tau}d\tau, \qquad \tilde{a}_{k,n}(x) := \frac{1}{2}\int_{iy-1}^{iy+1} \tilde{F}_{k,\ell,\hat{\ell}}(\tau,x)e^{-\pi i n\tau}d\tau. \tag{2.74}$$

where $F_{k,\ell,\hat{\ell}}(\tau,x)$ is defined via Proposition 2.7 and

$$\tilde{F}_{k,\ell,\hat{\ell}}(\tau,x) = (\tau/i)^{-k}\left(e^{\pi i(-1/\tau)|x|^2} - F_{k,\ell,\hat{\ell}}(-1/\tau,x)\right),$$

which, by Proposition 2.8 is also the Fourier transform of $x \mapsto F_{k,\hat{\ell},\ell}(\tau,x)$ (for any fixed τ). To emphasize dependence on ℓ and $\hat{\ell}$, we will sometimes write

$$a_{k,n} = a_{k,n}^{(\ell,\hat{\ell})}, \qquad \tilde{a}_{k,n} = \tilde{a}_{k,n}^{(\ell,\hat{\ell})}$$

in this proof. By Proposition 2.8 and by Remark 2.5, the functions $a_{k,n}$ and $\tilde{a}_{k,n}$ are radial Schwartz functions on $\mathbb{R}^d$. By Proposition 2.10 (and by Lemma 2.3) these functions are zero if $n \leq 0$ and they have polynomial semi-norm growth. The interpolation formula (2.47) in part (i) thus holds, by the general discussion in §2.2.2. It only remains to prove assertion (ii) of Theorem 1 and that the $a_{k,n}, \tilde{a}_{k,n}$ are real-valued, because we already discussed the uniqueness aspect of Theorem 1 and the isomorphisms of $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ with spaces of sequences based on (ii) was proved in §2.3.2. To prepare for the proof of part (ii) of Theorem 1 , we take $y = 2$ in (2.74) and let (for the rest of this proof) $\gamma(t) = e^{\pi i(1-t)}$, $0 \leq t \leq 1$ denote the semi-circle joining -1 to 1 in clock-wise direction. Clearly, the path γ is (homotopic to a path which is) admissible for all points $\tau \in 2i + [-1,1]$ and we have

$$\begin{aligned} a_{k,n}^{(\ell,\hat{\ell})}(x) &= \frac{1}{2}\int_{2i+[-1,1]} \frac{1}{2}\int_\gamma \mathcal{K}_{k,\ell,\hat{\ell}}(\tau,z)e^{\pi i z|x|^2}dz e^{-\pi i n\tau}d\tau \\ &= \frac{1}{2}\int_\gamma \frac{1}{2}\int_{2i+[-1,1]} \mathcal{K}_{k,\ell,\hat{\ell}}(\tau,z)e^{-\pi i n\tau}d\tau e^{\pi i z|x|^2}dz = \frac{1}{2}\int_\gamma \vartheta_{k,n}^{(\ell,\hat{\ell})}(z)e^{\pi i z|x|^2}dz, \end{aligned} \tag{2.75}$$

where

$$\vartheta_{k,n}^{(\ell,\hat{\ell})}(z) = \Theta(z)^{4-2k}\lambda(Sz)^{1-\hat{\ell}}P_{k,n}^{(\ell,\hat{\ell})}(1/\lambda(z))$$

is the weakly holomorphic modular form of weight $2-k$ for $\Gamma(2)$ and $P_{k,n}^{(\ell,\hat{\ell})} \in X^\ell\mathbb{C}[X]$ has degree n and is zero for $n < \ell$. Changing the order of integration in the above computation is justified since $(\tau,z) \mapsto \mathcal{K}_{k,\ell,\hat{\ell}}(\tau,z)$ is continuous on (the compact domain) $(2i + [-1,1]) \times \gamma$. Recall also that the functions $\vartheta_{k,n}^{(\ell,\hat{\ell})}$ all vanish at the cusp 1. This has several consequences:

- The point-wise formula (2.75) shows again that each $a_{k,n}$ is a radial Schwartz function (since $\vartheta_{k,n}^{(\ell,\hat{\ell})}(z)g_d(z) \to 0$ in the Schwartz topology, as z tends to -1 or 1 within the fundamental domain, allowing for an application of integration theory in the Schwartz space, see Remark 2.5). To prove that $a_{k,n}(r) \in \mathbb{R}$, we may conjugate both sides of (2.75) and use that for $z \in \mathbb{H}$, with $|z| = 1$ (i.e. for z on γ), we have

$$\overline{\vartheta_{k,n}^{(\ell,\hat{\ell})}(z)} = \vartheta_{k,n}^{(\ell,\hat{\ell})}(-\overline{z}) = \vartheta_{k,n}^{(\ell,\hat{\ell})}(-1/z)$$

and then make a change of variables to see that $\overline{a_{k,n}(r)} = a_{k,n}(r)$ for all $r \in \mathbb{R}$. Here, the first equal sign holds in fact for all $z \in \mathbb{H}$, since $\vartheta_{k,n}^{(\ell,\hat{\ell})}$ is a 2-periodic holomorphic function with real Fourier coefficients. This is clear from the proof of Proposition 2.6 and the fact that Θ and λ have real (indeed integral) Fourier coefficients.

- Via a change of variables, we see that

$$\mathcal{F}(a_{k,n}^{(\ell,\hat{\ell})}) = \frac{1}{2}\int_\gamma \vartheta_{k,n}^{(\ell,\hat{\ell})}(z)\mathcal{F}(g_d(z))dz = \frac{1}{2}\int_\gamma (\vartheta_{k,n}^{(\ell,\hat{\ell})}|_{2-k}S)(z)g_d(z)dz. \tag{2.76}$$

- By changing the path of integration (as explained in Remark 2.7) from γ to the concatenation of line-segments joining -1, $-1+i$,$1+i$ and 1, we obtain for all $m \in \mathbb{N}_0$ that

$$a_{k,n}^{(\ell,\hat{\ell})}(\sqrt{m}) = \frac{1}{2}\int_{i+[-1,1]} \vartheta_{k,n}^{(\ell,\hat{\ell})}(z)e^{\pi izm}dz = \widehat{\vartheta_{k,n}^{(\ell,\hat{\ell})}}(-m). \tag{2.77}$$

Finally, recall from Proposition 2.8 that $\mathcal{F}_{\mathbb{R}^d} \circ F_{k,\hat{\ell},\ell} = \tilde{F}_{k,\ell,\hat{\ell}}$ which implies, by (2.76),

$$\tilde{a}_{k,n}^{(\ell,\hat{\ell})} = \mathcal{F}(a_{k,n}^{(\hat{\ell},\ell)}) = \frac{1}{2}\int_\gamma (\vartheta_{k,n}^{(\hat{\ell},\ell)}|_{2-k}S)(z)g_d(z)dz. \tag{2.78}$$

Here and in the following, we draw the reader's attention to the order of the indices ℓ and $\hat{\ell}$ in sub- or superscripts. Applying the Fourier transform to (2.78), we obtain

$$\mathcal{F}(\tilde{a}_{k,n}^{(\ell,\hat{\ell})}) = \frac{1}{2}\int_\gamma \vartheta_{k,n}^{(\hat{\ell},\ell)}(z)g_d(z)dz. \tag{2.79}$$

These formulas for $a_{k,n}, \tilde{a}_{k,n}$ as integral transforms allow us now to deduce the following properties. First, by (2.75) and (2.78) (and Fourier inversion) we deduce that

$$a_{k,n}^{(\ell,\hat{\ell})} = 0 \quad \text{if} \quad n < \ell \quad \text{ and } \quad \tilde{a}_{k,n}^{(\ell,\hat{\ell})} = 0 \quad \text{if} \quad n < \hat{\ell}.$$

By (2.77) and by Proposition 2.6, we have

$$\tilde{a}_{k,n}^{(\ell,\hat{\ell})}(\sqrt{m}) = \delta_{n,m} \quad \text{ for all } n, m \geq \ell$$

Similarly, by evaluating (2.79) at $\sqrt{m}$ and invoking Proposition 2.6, we get

$$\mathcal{F}(\tilde{a}_{k,n}^{(\ell,\hat{\ell})})(\sqrt{m}) = \delta_{n,m} \quad \text{ for all } n, m \geq \hat{\ell}.$$

By evaluating (2.76) at $\sqrt{m}$ we obtain

$$\mathcal{F}(a_{k,n}^{(\ell,\hat{\ell})})(\sqrt{m}) = 0 \quad \text{ for all } n, m \geq \hat{\ell}$$

since

$$(\vartheta_{k,n}^{(\ell,\hat{\ell})}|_{2-k}S)(z) = \Theta(z)^{4-2k}\lambda(z)^{1-\hat{\ell}}P_{k,n}^{(\ell,\hat{\ell})}(1/\lambda(Sz))$$

has valuation $\geq (1-\hat{\ell})$ at infinity. Similarly, by evaluating (2.78) at $\sqrt{m}$ we get

$$\tilde{a}_{k,n}^{(\ell,\hat{\ell})}(\sqrt{m}) = 0 \quad \text{ for all } \quad n, m \geq \ell.$$

This completes the proof of Theorem 1.

2.4 Modular integrals via Poincaré-type series

In this section, we present another method which can be used to solve the modular integral problem introduced in §2.2.2 via a construction closely related to the construction of Poincaré series. As indicated in the introduction, the underlying idea was shown to the author by M. Viazovska in fall 2019.

We start by recalling from §2.2.2 that, in order to prove a point-wise interpolation formula (2.39), it suffices to solve the following problem: Given a half-integer $k = d/2 \geq 0$, a real number $h \geq 2$ and a holomorphic function $\varphi : \mathbb{H} \to \mathbb{C}$ of moderate growth, find a holomorphic function $F = F_\varphi : \mathbb{H} \to \mathbb{C}$ of moderate growth such that:

(A) $F|_k(T^h - 1) = 0$,

(B) $F|_k(V^{-h} - 1) = \varphi|_k(V^{-h} - 1)$.

Let us also recall that it really suffices to solve this problem for each of the functions $\varphi(z) = g_d(z, x) = e^{\pi i z|x|^2}$ where $x \in \mathbb{R}^d$. Here and in the rest of this section, the slash action in weight k is defined as in §2.2.2.

We will write down a solution F of the above system as

$$F = -\sum_{\gamma \in \mathcal{V}_h} \varphi|_k \gamma \tag{2.80}$$

for a certain special subset $\mathcal{V}_h \subseteq \Gamma(h)$, which we define and study below in §2.4.1. For $h = 2$, this subset has the property that it is a complete set of pair-wise inequivalent representatives for the quotient space $\Gamma(2)_\infty \backslash \Gamma(2)$, except that the identity coset is not represented. In this sense, the construction is similar to that of a Poincaré series and we will see more closer connections below (note that if $|x|^2 = 2m/h$ for some integer $m \geq 0$, then $z \mapsto g(z, x)$ is h-periodic, and it would make sense to sum over cosets, whereas, for other values of h, this does not make sense).

2.4.1 Special subsets of $\Gamma_\theta(h)$ and $\Gamma(h)$

Here, we give the definition of the set $\mathcal{V}_h$, which will be used to define F as in (2.80) and establish some useful auxiliary properties of it.

Let $h \in \mathbb{C}$ with $|h| \geq 2$. Recall from Lemma 2.1 in §2.1.1 that the group $\Gamma(h) \leq \mathrm{PSL}_2(\mathbb{C})$ is *freely* generated by T^h, V^h. We define the subset $\mathcal{V}_h \subseteq \Gamma(h)$ to be the set of all $\gamma \in \Gamma(h)$ of the form

$$\gamma = V^{e_1 h} T^{f_1 h} V^{e_2 h} T^{f_2 h} \cdots V^{e_n h} T^{f_n h}, \tag{2.81}$$

where $n \geq 1$ and $e_1, \ldots, e_n, f_1, \ldots f_{n-1} \in \mathbb{Z} \setminus \{0\}$, $f_n \in \mathbb{Z}$. In other words, $\mathcal{V}_h$ is the set of all nonempty finite reduced words in V^h, V^{-h}, T^h, T^{-h} which *start in* V^h. Note that for all $\gamma, \gamma' \in \mathcal{V}_h$ and we have

$$\gamma \neq \gamma' \quad \Rightarrow \quad (c_\gamma, d_\gamma) \neq (c_{\gamma'}, d_{\gamma'})$$

where the last equality has to be interpreted in $(\mathbb{R}^2 \setminus \{(0,0)\})/\{\pm 1\}$. Indeed, this is so because if γ, γ' have the same bottom row (up to sign), then $\gamma' = T^{hm}\gamma$ for some $m \in \mathbb{Z}$, which is necessarily zero by definition of $\mathcal{V}_h$.

We also define two subsets $\mathcal{R}_h, \tilde{\mathcal{R}}_h \subseteq \{1\} \cup \mathcal{V}_h$ by

$$\mathcal{R}_h := \{\gamma \in \mathcal{V}_h \,:\, \gamma \text{ as in (2.81) with } f_n = 0\}, \tag{2.82}$$

$$\tilde{\mathcal{R}}_h := \{\gamma \in \mathcal{V}_h \,:\, \gamma \text{ as in (2.81) with } f_n \neq 0\} \cup \{1\}. \tag{2.83}$$

The set $\mathcal{V}_h$ is stable under right multiplication by powers of T^h and $\mathcal{R}_h$ is a complete set of pairwise inequivalent representatives for $\mathcal{V}_h/\langle T^h\rangle$. Similarly, $\{1\}\cup\mathcal{V}_h$ is stable under right multiplication by powers of V^h and $\tilde{\mathcal{R}}_h$ is a complete set of pairwise inequivalent representatives for $(\{1\}\cup\mathcal{V}_h)/\langle V^h\rangle$.

The reason why these sets are so useful is because the properties

$$\mathcal{V}_h V^h = (\mathcal{V}_h \smallsetminus \{V^h\}) \cup \{1\}, \qquad \mathcal{V}_h T^h = \mathcal{V}_h$$

immediately imply that any series of the form (2.80) indeed has the desired properties (A) and (B), provided of course that it converges absolutely. To prove that this series does converge absolutely (for sufficiently large k) and to prove various other properties, we need the following lemma.

Lemma 2.12. *Consider an element $\gamma \in \mathcal{V}_h$ as in (2.81) and write $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$, so that the entries a, b, c, d depend on n, e_i, f_i and h. Then the following holds:*

(i) If $f_n = 0$ (i.e. if $\gamma \in \mathcal{R}_h$), then $|c| \geq |d|$.

(ii) If $f_n \neq 0$ (i.e. if $\gamma \in \tilde{\mathcal{R}}_h$), then $|d| \geq |c|$.

(iii) $c \neq 0 \neq d$.

(iv) $|a| \leq |c|$ and $|b| \leq |d|$.

(v) Viewing h as a formal variable and the entries of γ as elements of $\mathbb{Z}[h]$, the degrees of the polynomials c and d are at least $2n-2$.

(vi) Viewed as functions of $h \in [2,\infty)$, the entries $|c|$ and $|d|$ are monotonically increasing on $[2,\infty)$.

Proof. We prove parts (i), (ii), and (iii) simultaneously, using induction on n, by multiplying on the right with a non-trivial power of V^h or T^h . The base case is $n = 1$, $f_1 = 0$, so $\gamma = V^{e_1 h}$ and the inequality in (i) holds trivially and certainly $c_\gamma d_\gamma \neq 0$. For the inductive step, assume $n \geq 2$. If $f_n \neq 0$, set $\gamma' = \gamma T^{-f_n h}$ and if $f_n = 0$, set $\gamma' = \gamma V^{-e_n h}$. Thus, we have

$$\text{either} \quad \gamma = \gamma' T^{f_n h} = \begin{pmatrix} * & * \\ c_{\gamma'} & d_{\gamma'} + f_n h c_{\gamma'} \end{pmatrix} \quad \text{or} \quad \gamma = \gamma' V^{e_n h} = \begin{pmatrix} * & * \\ c_{\gamma'} + e_n h d_{\gamma'} & d_{\gamma'} \end{pmatrix}.$$

If $f_n \neq 0$, then $|c_{\gamma'}| \geq |d_{\gamma'}| > 0$ by inductive hypothesis and hence

$$|d| = |d_{\gamma'} + h f_n c_{\gamma'}| \geq |f_n||h||c_{\gamma'}| - |d_{\gamma'}| \geq 2|c_{\gamma'}| - |c_{\gamma'}| = |c_{\gamma'}| = |c| > 0,$$

as desired. If $f_n = 0$, then $|d_{\gamma'}| \geq |c_{\gamma'}| > 0$ by inductive hypothesis and we deduce $|c| \geq |d| > 0$ in a similar way.

We prove part (iv) also by induction on n, but "in the reverse order", that is, by multiplying elements γ *from the left* by elements $V^{eh}T^{fh}$, starting with $(e, f) = (e_n, f_n)$ and $\gamma = 1$, then $(e, f) = (e_{n-1}, f_{n-1})$ and so on. To explain this more precisely, we first compute generally[15]

$$V^{eh}T^{fh}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} 1 & fh \\ eh & 1 + efh^2 \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a' + fhc' & b' + fhd' \\ a'eh + c' + efh^2c' & eb'h + d' + d'efh^2 \end{pmatrix}.$$

The base case of the above inductive scheme is the case $(e, f) \in (\mathbb{Z} \smallsetminus \{0\}) \times \mathbb{Z}$ and $a' = d' = 1$, $b' = c' = 0$, in which case we need to check that $1 \leq |eh|$ (clear) and that$|fh| \leq |1 + efh|$ (also

[15] the primed notation is unrelated to the one in the previous item

clear). For the inductive step, we assume that $|a'| \le |c'|$ and $|b'| \le |d'|$ and aim to show that for all $e, f \in \mathbb{Z} \smallsetminus \{0\}$, we have

$$|a' + fhc'| \le |a'eh + c' + efh^2c'|, \qquad |b' + fhd'| \le |eb'h + d' + d'efh^2|.$$

If $c' = 0$, the first equality holds trivially and if $d' = 0$, the second equality holds trivially. Dividing by c' and d' respectively, we reduce to proving that for all $x \in \mathbb{C}$ with $|x| \le 1$ and all nonzero $e, f \in \mathbb{Z}$, we have

$$|x + fh| \le |xeh + 1 + efh^2| = |eh(x + fh) + 1|$$

and this holds, since

$$|1 + eh(x + fh)| \ge |h||x + fh| - 1 \ge 2|x + fh| - 1 \ge |x + fh|,$$

because $|x + fh| \ge 2|f| - |x| \ge 2 - |x| \ge 1$.

Part (v) can also be proved by induction on n, as parts (i) and (ii). In fact, one has $\deg(c) = \deg(d) + 1$, if $f_n = 0$ and $\deg(d) = \deg(c) + 1$ if $f_n \neq 0$.

Part (vi) is easily verified for $n = 1$. For $n \ge 2$, note that parts (v) and (iii) together imply that the functions c and d are non-constant polynomial functions of h (with coefficients in $\mathbb{Z}$, depending upon e_i, f_i), all of whose complex zeros lie in the disc $|h| < 2$. It follows from the Gauss–Lucas theorem[16] that the zeros of their first derivatives also lie in that disc. In particular, the derivatives of the polynomials c and d have no *real* zeros in $\mathbb{R} \smallsetminus (-2, 2)$ and this implies the claim in (vi). □

The following fact was postponed from Remark 2.1 in §2.1.1.

Corollary 2.2. *Let $h \in \mathbb{C}$ with $|h| \ge 2$. Then $S \notin \Gamma(h)$ and the only relation in $\Gamma_\theta(h)$ is $S^2 = 1$.*

Proof. Note that for all $m \in \mathbb{Z}$, the lower right entry of $T^{mh}S$ is zero, so that $S \notin \Gamma(h)$ follows from part (iii) of Lemma 2.12 and the definition of the set $\mathcal{V}_h$. Proving the second part means proving that there cannot exist nonzero $m_i \in \mathbb{Z}$ and $\delta_1, \delta_2 \in \{0, 1\}$, so that

$$S^{\delta_1} T^{m_1 h} S T^{m_2 h} S T^{m_3 h} \cdots S T^{m_\ell h} S^{\delta_2} = 1.$$

If such a relation were to hold, we could conjugate it by S, so we can focus on $\delta_1 = 1$. In that case, a relation of the above form would produce an element of the set $\mathcal{V}_h$ which is equal to 1 or S, which is again excluded by part (iii) of Lemma 2.12. □

2.4.2 Convergence and definition of the generating functions

As explained above, given "any" function $\varphi : \mathbb{H} \to \mathbb{C}$, the formal series

$$F = -\sum_{\gamma \in \mathcal{V}_h} \varphi|_k \gamma \quad \text{and} \quad \tilde{F} := \varphi|_k S - F|_k S = \sum_{\gamma \in (\{1\} \cup \mathcal{V}_h) S} \varphi|_k \gamma$$

are both (formally) h-periodic functions on $\mathbb{H}$, which are related by $F + \tilde{F}|_k S = \varphi$. We prove next that, in the case $\varphi(z) = e^{\pi i z r^2}$ and $k > 2$, these series converge absolutely with some uniformity. We need one more preliminary observation. Consider any element $\gamma \in \mathcal{V}_h \cup \mathcal{V}_h S \cup \{S\}$, which appears in the series defining F or $\tilde{F}$. Then, by part (iii) of lemma 2.12, we have $|c_\gamma| \neq 0$ and hence, by part (vi) of the same Lemma $|c_\gamma| \ge 1$ (in fact, we have $|c_\gamma| \ge \lfloor h \rfloor$ for $\gamma \in \mathcal{V}_h$),

$$|\gamma z| = \left| \frac{a_\gamma}{c_\gamma} - \frac{1}{c_\gamma (c_\gamma z + d_\gamma)} \right| \le \frac{|a_\gamma|}{|c_\gamma|} + \frac{1}{|c_\gamma|^2 |z + d_\gamma / c_\gamma|} \le 1 + \operatorname{Im}(z)^{-1}, \tag{2.84}$$

where we also used part (iv) of Lemma 2.12.

[16] This theorem says that for every non-constant polynomial $P \in \mathbb{C}[x]$, the zeros of the derivative P' belong to the convex hull of the zeros of P.

Lemma 2.13. *Fix real numbers $k > 2, y_0 > 0, X \geq 1, h \geq 2$ and a compact subset $\Omega \subseteq \mathbb{C}$. There exists a constant C, depending upon these parameters (but not on h) such that for all $\gamma \in \mathcal{V}_h \cup (\mathcal{V}_h \cup \{1\})S$, all $z \in \mathbb{H}$ with $|\operatorname{Re}(z)| \leq X$, $\operatorname{Im}(z) \geq y_0$ and all $r \in \Omega$, we have*

$$\left| j_k(\gamma, z)^{-k} e^{\pi i (\gamma z) r^2} \right| \leq C(c_\gamma^2 + d_\gamma^2)^{-k/2}.$$

If we replace Ω by an arbitrary subset $\Omega \subseteq \mathbb{R}$, then C can be chosen independently of Ω.

Proof. Let $\gamma \in \mathcal{V}_h \cup (\mathcal{V}_h \cup \{1\})S$, $z \in \mathbb{H}$ with $\operatorname{Im}(z) \geq y_0 > 0$ and $r \in \Omega$ be arbitrary. Then, by (2.10) and by (2.84)

$$|j_k(\gamma, z)^{-k} e^{\pi i (\gamma z) r^2}| \leq |c_\gamma z + d_\gamma|^{-k} e^{\pi |\gamma z| |r|^2} \leq |c_\gamma z + d_\gamma|^{-k} e^{\pi (1 + y_0^{-1}) \sup_{r \in \Omega} |r|^2}.$$

Note that, in the case $\Omega \subseteq \mathbb{R}$, then we can simply use $|e^{\pi i \tau r^2}| \leq 1$ for all $\tau \in \mathbb{H}$ and $r \in \mathbb{R}$.

Consider now the $\mathbb{R}$-linear map $A_z : \mathbb{C} \to \mathbb{C}$ defined by $A_z(ci + d) = cz + d$ for $c, d \in \mathbb{R}$. For any $(c, d) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, we have

$$|ci + d| = |A_z^{-1}(cz + d)| \leq \|A_z^{-1}\|_{\text{op}} |cz + d|$$

and hence

$$|ci + d|^k \leq \|A_z^{-1}\|_{\text{op}}^k |cz + d|^k \leq C^k (1 + X(1 + y^{-1}))^k |cz + d|^k,$$

where we derived the last inequality from inversion of the matrix $M_z = \left(\begin{smallmatrix} 1 & \operatorname{Re}(z) \\ 0 & \operatorname{Im}(z) \end{smallmatrix}\right)$, representing A_z in the ordered $\mathbb{R}$-basis $(1, i)$ of $\mathbb{C}$ (plus the assumptions on z and the equivalence of norms). Inverting the last inequality and writing $|c_\gamma i + d_\gamma|^k = (c_\gamma^2 + d_\gamma^2)^{k/2}$ we finish the proof of the lemma. ☐

Having established Lemma 2.13, we may now define the following two functions, for real $k > 2$, $h \geq 2$, $z \in \mathbb{H}$ and $r \in \mathbb{C}$:

$$F_{k,h}(z, r) := -\sum_{\gamma \in \mathcal{V}_h} e^{\pi i z r^2}|_k \gamma, \qquad \tilde{F}_{k,h}(z, r) := \sum_{\gamma \in (\mathcal{V}_h \cup \{1\})S} e^{\pi i z r^2}|_k \gamma. \tag{2.85}$$

These series converge absolutely and uniformly on compact subsets of $\mathbb{H} \times \mathbb{C}$ by Lemma 2.13 and by part (vi) of Lemma 2.12. Indeed the two lemmas reduce the convergence question to the convergence of $\sum_{(0,0) \neq (c,d) \in \mathbb{Z}^2} (c^2 + d^2)^{-k/2}$ which is guaranteed for $k > 2$. As explained at the beginning of this subsection 2.4.2, the functions $F_{k,h}(z, r)$ and $\tilde{F}_{k,h}(z, r)$ are both h-periodic in z and are related by the functional equation

$$F_{k,h}(z, r) + (z/i)^{-k} \tilde{F}_{k,h}(-1/z, r) = e^{\pi i z r^2}. \tag{2.86}$$

Following the general strategy explained in §2.2.2 we now define, for every $r \in \mathbb{C}$, $n \in \mathbb{Z}$,

$$b_{k,h,n}(r) := \frac{1}{h} \int_{iy - h/2}^{iy + h/2} F_{k,h}(z, r) e^{-2\pi i n z/h} dz, \tag{2.87}$$

$$\tilde{b}_{k,h,n}(r) := \frac{1}{h} \int_{iy - h/2}^{iy + h/2} \tilde{F}_{k,h}(z, r) e^{-2\pi i n z/h} dz. \tag{2.88}$$

Recall that these integrals do not depend upon $y > 0$. It is clear (form general principles) that these define holomorphic even functions of $r \in \mathbb{C}$.

Remark 2.11 (Dependence on k and h)**.** In the above analysis, the parameters k, h were often considered as fixed. Let us remark that the dependence of $F_{k,h}(z,r)$ and $\tilde{F}_{k,h}(z,r)$ on (k,h) is (at the very least) continuous. To see why, let $\phi_h : \Gamma_\theta(2) \to \Gamma_\theta(h)$ denote the group isomorphism defined by

$$\phi_h(T^2) = T^h, \qquad \phi_h(S) = S.$$

We can then write

$$F_{k,h}(z,r) = -\sum_{\gamma \in \mathcal{V}_2} j_k(\phi_h(\gamma), z)^{-1} e^{\pi i (\gamma z) r^2}.$$

The proof of Lemma 2.13 shows that for all compact subsets $K \subseteq (2,\infty) \times [2,\infty) \times \mathbb{H} \times \mathbb{C}$ there are $\kappa > 2$ and $C = C_K > 0$ depending only on K so that for all $\gamma \in \mathcal{V}_2 \subseteq \Gamma(2)$, we have

$$\sup_{(k,h,z,r)\in K} |j_k(\phi_h(\gamma), z)^{-1} e^{\pi i(\gamma z) r^2}| \leq C_K (c_\gamma^2 + d_\gamma^2)^{-\kappa/2},$$

which proves uniform convergence of the series defining $F_{k,h}(z,r)$ in all parameters. Of course, the same holds for $\tilde{F}_{k,h}(z,r)$.

2.4.3 Main result

Let us now state the main result of §2.4.

Theorem 2. *Let $d \geq 5$ be an integer, $h \geq 2$ a real number and set $k = d/2$. Then the even entire functions $b_{k,h,n}, \tilde{b}_{k,h,n} : \mathbb{C} \to \mathbb{C}$, defined in* (2.87) *have the following properties.*

(i) For all $f \in \mathcal{S}(\mathbb{R}^d)$ and all $x \in \mathbb{R}^d$ we have

$$f(x) = \sum_{n=1}^{\infty} b_{k,h,n}(|x|) f(\sqrt{2n/h}) + \sum_{n=1}^{\infty} \tilde{b}_{k,h,n}(|x|) \hat{f}(\sqrt{2n/h}) \tag{2.89}$$

where the series converge absolutely and uniformly on $\mathbb{R}^d$.

(ii) For $n \leq 0$ we have $b_{k,h,n} = 0 = \tilde{b}_{k,h,n}$ and for $n \geq 1$ and $r \in \mathbb{R}$ we have

$$|b_{k,h,n}(r)| + |\tilde{b}_{k,h,n}(r)| \leq C_1 \left(C_2/(kh)\right)^{k/2} n^k, \tag{2.90}$$

where C_1, C_2 are absolute constants (independent of k, h, n, r). For $r > 0$ and $n \geq 1$ we have

$$|b_{k,h,n}(r)| + |\tilde{b}_{k,h,n}(r)| \leq C_3 n^{k/2+9/8} r^{-k+9/4}, \tag{2.91}$$

where $C_3 > 0$ is an absolute constant, independent of k, h, n, r.

(iii) The assignments $x \mapsto b_{k,h,n}(|x|), \tilde{b}_{k,h,n}(|x|)$ define smooth radial functions $\mathbb{R}^d \to \mathbb{C}$. For each fixed multi-index $\alpha \in \mathbb{N}_0^d$ and real number $R \geq 1$, there exist constants C_4, C_5, C_6, depending only on R and α, such that for all $|x| \leq R$, we have

$$|\partial_x^\alpha b_{k,h,n}(|x|)| + |\partial_x^\alpha \tilde{b}_{k,h,n}(|x|)| \leq C_4 (C_5/k)^{k/2} n^{k+|\alpha|} \tag{2.92}$$

and for $0 < |x| \leq R$ we have

$$|\partial_x^\alpha b_{k,h,n}(|x|)| + |\partial_x^\alpha \tilde{b}_{k,h,n}(|x|)| \leq C_6 n^{k/2+9/8+|\alpha|} |x|^{-k+9/4} \tag{2.93}$$

(iv) If $h > 2$, then, for every integer $n_0 \geq 1$, there exists an interpolation formula like (2.89)*, but with both series starting at $n = n_0$ and different functions $b_{k,n,h}$, $\tilde{b}_{k,h,n}$. See §2.4.4.5.*

Remark 2.12. In the case $h = 2$, Theorem 2 is covered by our published work [Sto21, Thm 2] which gives slightly stronger bounds on the functions in terms of the exponents of n and $r = |x|$. To avoid having to work with yet an additional parameter, we will not include this improvement here, but we remark that the only modification necessary to obtain it is a version of Lemma 2.14, in which we allow the parameter κ in its statement to be arbitrarily close to 2.

Remark 2.13. We will see in the next chapter that the decay of the uniform estimate (2.90) in terms of k as well as the exponents $k/2$ of n and $-k$ of r in the estimates (2.91) are absolutely crucial for our applications to non-radial functions, for large *and* small r.

Remark 2.14. There is an admissible constant C_3 in (2.91) that is uniformly bounded and decaying as a function of h and k, see Remark 2.15 below.

2.4.4 Proof of Theorem 2

This section is devoted to the proof of Theorem 2. We will obtain the bounds stated in Theorem 2 by first bounding the functions $F_{k,h}(z,r)$, $\tilde{F}_{k,h}(z,r)$ (and their derivatives with respect to x when $r = |x|$) in terms of suitable powers of $\mathrm{Im}(z)$ and depending on whether $r > 0$ or not. We will use these bounds together with the triangle inequality in (2.87) by choosing $y = c/n$ for suitable c (which may depend on k and r). To do so, we bound $F_{k,h}(z,r)$, $\tilde{F}_{k,h}(z,r)$ in terms of two auxiliary functions $U_{k,h}, \tilde{U}_{k,h}$ introduced below.

2.4.4.1 Bounds for two auxiliary functions

We start by defining the subset $\tilde{\mathcal{V}}_h \subseteq \Gamma_\theta(h)$, by $\tilde{\mathcal{V}}_h := (\mathcal{V}_h \cup \{1\})S$ and then, for any real[17] $\kappa > 2$, the following auxiliary functions $U_{\kappa,h}, \tilde{U}_{\kappa,h} : \mathbb{H} \to (0, +\infty)$, by

$$U_{\kappa,h}(z) := \sum_{\gamma \in \mathcal{V}_h} |j_\kappa(\gamma, z)|^{-1} = \sum_{\gamma \in \mathcal{V}_h} |c_\gamma z + d_\gamma|^{-\kappa}, \tag{2.94}$$

$$\tilde{U}_{\kappa,h}(z) := \sum_{\gamma \in \tilde{\mathcal{V}}_h} |j_\kappa(\gamma, z)|^{-1} = \sum_{\tilde{\gamma} \in \tilde{\mathcal{V}}_h} |c_{\tilde{\gamma}} z + d_{\tilde{\gamma}}|^{-\kappa} = \sum_{\gamma \in \{1\} \cup \mathcal{V}_h} |d_\gamma z - c_\gamma|^{-\kappa}. \tag{2.95}$$

By the proof of Lemma 2.13, these series converge absolutely and define continuous, h-periodic functions on $\mathbb{H}$.

Lemma 2.14. *There is a constant $C_0 > 0$ so that for all real $z = x + iy \in \mathbb{H}$, all $h \geq 2$ and all real $\kappa \geq 9/4 = 2 + 1/4$, we have*

$$\max\Big(|U_{\kappa,h}(z)|, |\tilde{U}_{\kappa,h}(z)|\Big) \leq C_0 2^{\kappa/2}(y^{-\kappa/2} + y^{-\kappa}).$$

Proof. By h-periodicity, it suffices to consider $z = x + iy \in \mathbb{H}$ with $|x| \leq h/2$. We start by proving the upper bound for $U_{\kappa,h}$. Recall that the set $\mathcal{R}_h$, defined in (2.82) and that it is a complete set of representatives for the quotient $\mathcal{V}_h/\langle T^h \rangle$. Therefore,

$$U_{k,h}(z) = \sum_{\gamma \in \mathcal{R}_h} \sum_{f \in \mathbb{Z}} |c_\gamma z + d_\gamma + fhc_\gamma|^{-\kappa} = \sum_{\gamma \in \mathcal{R}_h} |c_\gamma|^{-\kappa} \sum_{f \in \mathbb{Z}} |z + d_\gamma/c_\gamma + fh|^{-\kappa}.$$

[17] We will mostly use these functions for $\kappa = k$, but not always, hence the change in notation.

We next bound the terms

$$|z + d_\gamma/c_\gamma + fh|^2 = y^2 + (x + d_\gamma/c_\gamma + fh)^2$$

from below. If $|f| \le 1$ we bound them trivially from below by y^2 and if $|f| \ge 2$ we use part (i) of Lemma 2.12 as follows:

$$\begin{aligned} y^2 + (x + d_\gamma/c_\gamma + fh)^2 &\ge 2y|x + d_\gamma/c_\gamma + fh| \ge 2y(|f|h - (h/2) - 1) \\ &\ge 2yh(|f| - (1/2 + 1/h)) \ge 2yh(|f| - 1). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} U_{k,h}(z) &= \sum_{\gamma \in \mathcal{R}_h} (c_\gamma^2)^{-\kappa/2} \left(\sum_{|f| \le 1} y^{-\kappa} + \sum_{|f| \ge 2} (2yh(|f|-1))^{-\kappa/2} \right) \\ &\le \sum_{\gamma \in \mathcal{R}_h} ((c_\gamma^2 + d_\gamma^2)/2)^{-\kappa/2} \left(3y^{-\kappa} + (2h)^{-\kappa/2} 2\zeta(\kappa/2) y^{-\kappa/2} \right), \end{aligned}$$

where we used part (i) of Lemma 2.12 once more. By part (vi) of the same Lemma, the series $\sum_{\gamma \in \mathcal{R}_h} (c_\gamma^2 + d_\gamma^2)^{-\kappa/2}$ is dominated by $\sum_{(c,d) \in \mathbb{Z}^2_{\mathrm{prim}}} (c^2 + d^2)^{-\kappa/2}$, where $\mathbb{Z}^2_{\mathrm{prim}}$ denotes the set of row vectors $(c, d) \in \mathbb{Z}^2$ with $\gcd(c, d) = 1$. As we are assuming $\kappa \ge 2 + 1/4$ and $h \ge 2$, the last estimate for $U_{\kappa,h}(z)$ immediately implies the desired bound $U_{\kappa,h}(z) \le C_0 2^\kappa (y^{-\kappa/2} + y^{-\kappa})$.

The analysis for $\tilde{U}_{k,h}(x + iy)$ is very similar. Recalling the definition and properties of the set $\tilde{\mathcal{R}}_h$ defined in (2.83), we write

$$\tilde{U}_{\kappa,h}(z) = \sum_{\gamma \in \tilde{\mathcal{R}}_h} \sum_{e \in \mathbb{Z}} |d_\gamma z - (c_\gamma + ehd_\gamma)|^{-\kappa} = \sum_{\gamma \in \tilde{\mathcal{R}}_h} |d_\gamma|^{-\kappa} \sum_{e \in \mathbb{Z}} |z - (c_\gamma/d_\gamma + eh)|^{-\kappa}.$$

For all $e \in \mathbb{Z}$ we either bound $|z + (c_\gamma/d_\gamma + eh)|^2$ from below by y^2 (if $|e| \le 1$) or (if $|e| \ge 2$), by $2yh(|e| - 1)$ (using part (ii) of Lemma 2.12). Proceeding similarly as above for $U_{\kappa,h}$ we obtain the desired upper bound for $\tilde{U}_{\kappa,h}$. □

2.4.4.2 Proof of part (ii) of Theorem 2

Fix $k \ge 5/2$ and $h \ge 2$. By applying the triangle inequality to (2.87), using the trivial bound $|e^{\pi i z r^2}| \le 1$ and by Lemma 2.14, we see that for all $r \in \mathbb{R}$, $n \in \mathbb{Z}$, $y > 0$, we have

$$|b_{k,h,n}(r)| \le \frac{1}{h} \int_{-h/2}^{h/2} |F_{k,h}(x + iy, r)| e^{\pi n y} dx \le C_0 2^{k/2} (y^{-k/2} + y^{-k}) e^{2\pi n y/h}.$$

If $n \le 0$, we can let $y \to \infty$ in this estimate to deduce the vanishing of $b_{k,h,n}(r) = 0$ for all $r \in \mathbb{R}$ and hence for all $r \in \mathbb{C}$ by analyticity. For $n \ge 1$, we specialize the above bound to $y = \frac{kh}{2\pi n}$ and thus obtain

$$|b_{k,h,n}(r)| \le C_0 2^{k/2} \left(\left(\frac{2\pi n}{kh} \right)^{k/2} + \left(\frac{2\pi n}{kh} \right)^k \right) e^k \le C_1 \left(\frac{C_2}{kh} \right)^{k/2} n^k$$

for certain constants $C_1, C_2 > 0$, which establishes (2.90) for $b_{k,n,n}$. The argument for $\tilde{b}_{k,h,n}$ is completely analogous.

To establish (2.91), fix $r > 0$. Instead of the trivial bound $|F_{k,h}(z,r)| \le U_{k,h}(z)$, we now use, for some parameter $\beta > 0$ to be determined, the bound

$$\begin{aligned}
|F_{k,h}(z,r)| &\le \sum_{\gamma\in\mathcal{V}_h} |c_\gamma z + d_\gamma|^{-k} e^{-\pi \operatorname{Im}(\gamma z) r^2} \operatorname{Im}(\gamma z)^{\beta} \operatorname{Im}(\gamma z)^{-\beta} \\
&\le \sum_{\gamma\in\mathcal{V}_h} |c_\gamma z + d_\gamma|^{-k} \left(\frac{\beta}{\pi e r^2}\right)^{\beta} \operatorname{Im}(\gamma z)^{-\beta} \le \left(\frac{\beta}{\pi e r^2}\right)^{\beta} \operatorname{Im}(z)^{-\beta} U_{k-2\beta,h}(z).
\end{aligned}$$

Here, we used that $\sup_{t>0}(t^\beta e^{-\alpha t}) = \left(\frac{\beta}{e\alpha}\right)^{\beta}$. We take $\beta = k/2 - 9/8$, so that we can apply Lemma 2.14 with $\kappa = k - 2\beta = 9/4$, giving

$$|F_{k,h}(z,r)| \le \left(\frac{\beta}{\pi e}\right)^{\beta} r^{-2\beta} y^{-\beta} U_{2c,h}(z) \le C \left(\frac{\beta}{\pi e}\right)^{\beta} r^{-2\beta} y^{-\beta} (y^{-9/4} + y^{-9/8})),$$

where $C = C_0 2^{9/16}$, with C_0 as in the cited Lemma. From this and from (2.87) we deduce for all $n \ge 1$ all $y > 0$, we have

$$|b_{k,h,n}(r)| \le C \left(\frac{\beta}{\pi e}\right)^{\beta} r^{-2\beta} y^{-\beta} (y^{-9/4} + y^{-9/8}) e^{2\pi n y/h}.$$

By specializing this inequality to $y = \frac{\beta h}{2\pi n}$, we obtain, recalling that $\beta = k/2 - 9/8$,

$$\begin{aligned}
|b_{k,h,n}(r)| &\le C \left(\frac{\beta}{\pi e}\right)^{\beta} r^{-2\beta} \left(\frac{2\pi n}{\beta h}\right)^{\beta} \left(\left(\frac{2\pi n}{\beta h}\right)^{9/4} + \left(\frac{2\pi n}{\beta h}\right)^{9/8}\right) e^{\beta} \\
&= C r^{-2\beta} \left(\frac{2n}{h}\right)^{\beta} n^{9/4} \left(\left(\frac{2\pi}{\beta h}\right)^{9/4} + \frac{1}{n^{9/8}} \left(\frac{2\pi}{\beta h}\right)^{9/8}\right) \le C_3 n^{k/2+9/8} r^{-k+9/4}
\end{aligned}$$

for some absolute constants $C_3 > 0$. The argument for $\tilde{b}_{k,h,n}(r)$ is completely analogous.

Remark 2.15. We see from the above proof that for the bound (2.91), we can allow C_3 to be of the form

$$C_3 = C_3'(2/h)^{k/2-9/8}(h(k/2-9/8))^{-9/8}$$

which is not only bounded as a function of $k \ge 5/2$ and $h \ge 2$, but decaying in either of these parameters.

2.4.4.3 Proof of part (i) of Theorem 2

This part follows from the uniform bounds of part (ii), the functional equation (2.86) satisfied by $F_{k,h}$, $\tilde{F}_{k,h}$, the definition of $b_{k,h,n}(r)$, $\tilde{b}_{k,h,n}(r)$ as their Fourier coefficients and the general results of §2.2.2, in particular, the density of the Gaussians, which was Proposition 2.3.

2.4.4.4 Proof of part (iii) of Theorem 2

Let $k = d/2 \ge 5/2$ and $h \ge 2$. Fix some $R \ge 1$ and a multi-index $\alpha \in \mathbb{N}_0^d$. Consider $x \in \mathbb{R}^d$ such that $0 \le r = |x| \le R$. For $0 \le m \le |\alpha|$, let $P_{\alpha,m} \in \mathbb{Z}[2\pi i][x_1, \dots, x_d]$ be the polynomial satisfying

$$\partial_x^\alpha e^{\pi i \tau |x|^2} = \sum_{m=0}^{|\alpha|} P_{\alpha,m}(x)\tau^m, \quad x \in \mathbb{R}^d, \tau \in \mathbb{H}.$$

Using $\gamma z = \frac{a_\gamma}{c_\gamma} - \frac{1}{c_\gamma(c_\gamma z + d_\gamma)}$ and the binomial theorem, we write, using the variable $r = |x|$,

$$\partial_x^\alpha F_{k,h}(z,|x|) = -\sum_{m=0}^{|\alpha|} P_{\alpha,m}(x) \sum_{\gamma\in\mathcal{V}_h} |c_\gamma z + d_\gamma|^{-k} (\gamma z)^m e^{\pi i (\gamma z)|x|^2} \tag{2.96}$$

$$= -\sum_{m=0}^{|\alpha|} P_{\alpha,m}(x) \sum_{j=0}^{m} \binom{m}{j} \sum_{\gamma\in\mathcal{V}_h} (a_\gamma/c_\gamma)^{m-j} (-c_\gamma(c_\gamma z + d_\gamma))^{-j} |c_\gamma z + d_\gamma|^{-k} e^{-\pi \operatorname{Im}(\gamma z) r^2} \tag{2.97}$$

By Lemma 2.12, we know that $|a_\gamma/c_\gamma| \le 1$ and that $|c_\gamma| \ge 1$ for all $\gamma \in \mathcal{V}_h$. For the proof of (2.92), we bound the Gaussian trivially by 1 and thus obtain

$$\begin{aligned} |\partial_x^\alpha F_{k,h}(z,|x|)| &\le \sum_{m=0}^{|\alpha|} P_{\alpha,m}(x) \sum_{j=0}^{m} \binom{m}{j} U_{k+j,h}(z) \\ &\le C_0 2^{k/2+|\alpha|} \sum_{m=0}^{|\alpha|} P_{\alpha,m}(x) \sum_{j=0}^{m} \binom{m}{j} (y^{-k/2-j/2} + y^{-k-j}) \\ &= C_0 2^{k/2+|\alpha|} \sum_{m=0}^{|\alpha|} P_{\alpha,m}(x) \left(y^{-k/2}(1+y^{-1/2})^m + y^{-k}(1+y^{-1})^m \right) \end{aligned}$$

as in the proof of part (ii), we deduce that

$$|\partial_x^\alpha b_{k,h,n}(|x|)| \le C_0 2^{k/2+|\alpha|} \sum_{m=0}^{|\alpha|} |P_{\alpha,m}(x)| \left(y^{-k/2}(1+y^{-1/2})^m + y^{-k}(1+y^{-1})^m \right) e^{2\pi n y/h}$$

and by specializing this last inequality to $y = \frac{kh}{2\pi n}$ and bounding the polynomials $P_{\alpha,m}(x)$ by compactness and continuity for $|x| \le R$, we obtain (2.92).

Finally, for the proof of the bounds in (2.93), in which we assume that $0 < r = |x| < R$, we insert a term $1 = \operatorname{Im}(\gamma z)^{\beta_j} \operatorname{Im}(\gamma z)^{-\beta_j}$ into (2.97) for a suitable parameter β_j, to obtain, similarly to the proof of (2.91),

$$|\partial_x^\alpha F_{k,h}(z,|x|)| \le \sum_{m=0}^{|\alpha|} |P_{\alpha,m}(x)| \sum_{j=0}^{m} \binom{m}{j} \operatorname{Im}(z)^{-\beta_j} \left(\frac{\beta_j}{e\pi r^2}\right)^{\beta_j} U_{k+j-2\beta_j}(z).$$

Proceeding similarly as before, we obtain the bounds in (2.93). (See also §5.5 in [Sto21] for more details, at least in the case $h = 2$).

2.4.4.5 Proof of part (iv) of Theorem 2

We fix $h > 2$ and $k > 2$ and will not always display these parameters in our notation. Recall that part (iv) of Theorem 2 asserted that one can modify the interpolation formula (2.89) and let the summation start at $n = n_0$ for any given integer $n_0 \ge 1$.

To prove that this is possible, it suffices to find a new pair of generating functions $(F, \tilde{F})$ whose Fourier expansions start at $n = n_0$. In the paper with Radchenko [RS], we used decomposition into Fourier eigenspaces and the accompanying generating functions F^+, F^- as in §2.2.2.1, to reduce the problem to an old result of Hecke. Here, we will present this argument slightly differently. Recall from (2.2.2.1) the definition of the characters $\chi_\epsilon : \Gamma_\theta(h) \to \{\pm 1\}$ and the twisted slash action $\varphi|_k^\epsilon \gamma = \chi_\epsilon(\gamma) j_k(\gamma)^{-1}(\varphi \circ \gamma)$.

We define $M_k(\Gamma_\theta(h),\epsilon)$ as the space of all holomorphic functions $\varphi:\mathbb{H}\to\mathbb{C}$ of moderate growth satisfying $\varphi|_k^\epsilon\gamma=\varphi$ for all $\gamma\in\Gamma_\theta(h)$. We define $M_k(\Gamma(h))\supseteq M_k(\Gamma_\theta(h),\chi_+)$ to be the bigger space of all holomorphic $\varphi:\mathbb{H}\to\mathbb{C}$ of moderate growth such that $\varphi|_k\gamma=\varphi$ for all $\gamma\in\Gamma(h)$.

Proposition 2.11. *For every $n\geq 1$ and $\epsilon\in\{\pm1\}$, there exist $f_n^\epsilon\in M_k(\Gamma_\theta(h),\chi_\epsilon)$ such that $\widehat{f_n^\epsilon}(m)=0$ for all integers $m<n$ and such that $\widehat{f_n^\epsilon}(n)=1$.*

We defer the proof of Proposition 2.11 to the end of this section. Let us explain how it implies part (v) of Theorem 2. Fix an integer $n_0\geq 1$. By Proposition 2.11, we can construct $g_n^\epsilon\in M_k(\Gamma_\theta(h),\chi_\epsilon)$ such that we have

$$\widehat{g_n^\epsilon}(\nu)=\delta_{n,\nu},\qquad \text{for all}\quad n,\nu\in\{1,\ldots,n_0\}.$$

For $n\in\{1,\ldots,n_0\}$ we define

$$\varphi_n:=\frac{g_n^++g_n^-}{2}\in M_k(\Gamma(h))$$

whose Fourier expansions at the cusps 0 and ∞ are given as

$$\widehat{\varphi_n}(\nu)=\delta_{n,\nu},\quad \widehat{\varphi_n|_kS}(\nu)=0\qquad \text{for all}\quad n,\nu\in\{1,\ldots,n_0\}.$$

We may now define two new solutions $\Phi_{k,h}(z,r)$, $\tilde\Phi_{k,h}(z,r)$ to the functional equations by

$$\Phi_{k,h}(z,r):=F_{k,h}(z,r)-\sum_{n=1}^{n_0}b_{k,h,n}(r)\varphi_n(z)+\sum_{n=1}^{n_0}\tilde b_{k,h,n}(r)(\varphi_n|_kS)(z),$$

$$\tilde\Phi_{k,h}(z,r):=(g(z,r)-\Phi_{k,h}(z,r)|_kS=\tilde F_{k,h}(z,r)+\sum_{n=1}^{n_0}b_{k,h,n}(r)(\varphi_n|_kS)(z)-\sum_{n=1}^{n_0}\tilde b_{k,h,n}(r)\varphi_n(z).$$

We denote their Fourier coefficients by

$$c_{k,h,n}(r):=\frac{1}{h}\int_{iy-h/2}^{iy+h/2}\Phi_{k,h}(z,r)e^{2\pi inz/h},\quad \tilde c_{k,h,n}(r):=\frac{1}{h}\int_{iy-h/2}^{iy+h/2}\tilde\Phi_{k,h}(z,r)e^{2\pi inz/h}.$$

By construction, these are entire functions of $r\in\mathbb{C}$ which vanish identically if $n<n_0$ and there are constants $A,C>0$ depending on k,h so that for all $n\geq n_0$,

$$\sup_{r\in\mathbb{R}}|c_{k,h,n}(r)|+\sup_{r\in\mathbb{R}}|\tilde c_{k,h,n}(r)|\leq Cn^A.$$

Finally for all $f\in\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^{2k})$ and all $x\in\mathbb{R}^{2k}$, we have

$$f(x)=\sum_{n=n_0}^{\infty}c_{k,h,n}(|x|)f(\sqrt{2n/h})+\sum_{n=n_0}^{\infty}\tilde c_{k,h,n}(|x|)\hat f(\sqrt{2n/h}).\tag{2.98}$$

by the above functional equations satisfied by $\Phi_{k,h}$ and $\tilde\Phi_{k,h}$ and by the general results of §2.2.2.

Proof of Proposition 2.11. This proof is essentially due to Hecke [Hec36, §3] who proved the existence of such f_n^ϵ for all integers $n\geq k/2$. We will add a further observation (below near (2.100)) to his proof and show that the construction extends to all $n\geq 1$. Hecke's treatment in loc. cit. is somewhat brief and we refer to [BK08, Chapter 4] for more details and explanation, also for parts of the proof given below.[18]

[18] In terms of notation, note that we denote by h, the quantity λ in the cited references.

Let $B_1 := \{z \in \mathbb{C} : -h/2 < \mathrm{Re}(z) < 0, |z| > 1\}$, so that $B_1 \cap \mathbb{H}$ is the left half of the fundamental domain $\mathcal{D}_h$ (see figure 2a). Consider the following pieces of the boundary of B_1:

$$L_1 = -h/2 + i\mathbb{R}, \quad L_2 = i[1,\infty), \quad L_3 = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0, |z| = 1\}, \quad L_4 = i(-\infty, -1].$$

By the Riemann mapping theorem, there exists a biholomorphic map $\varphi : B_1 \to \mathbb{H}$. It may be chosen uniquely so that it extends continuously to the boundary of B_1 (minus the point $-i$), maps the latter to $\mathbb{R}$ and satisfies

$$\varphi(i) = 0, \quad \varphi(-i) = -\infty, \quad \varphi(i\infty) = 1, \quad \varphi(-i\infty) = a_0, \tag{2.99}$$

for some $a_0 > 1$, where the values at $\pm i\infty$ are understood in the limit $\mathrm{Im}(\tau) \to \pm\infty$. We then have

$$\varphi(L_1) = (1, a_0), \quad \varphi(L_2) = [0,1), \quad \varphi(L_3) = (-\infty, 0) \quad \text{and} \quad \varphi(L_4) = (a_0, \infty).$$

By the Schwarz reflection principle applied to L_1, L_2, and L_3, one may extend φ to an analytic function on $\mathbb{C}$ minus the set of points equivalent to $-i$ under the reflections just mentioned. Then $\varphi|_{\mathbb{H}}$ is bounded, $\Gamma(h)$-invariant and never takes the value 1.

We claim that there is $\delta > 0$ so that for all $\tau \in B_1$ with $|\mathrm{Im}(\tau)| \leq 2$ we have $|\varphi(\tau) - 1| \geq \delta$. To prove this, it suffices to show that for all $\tau \in B_1$ we have

$$\overline{\varphi(\overline{\tau})} = \frac{a_0}{\varphi(\tau)}, \tag{2.100}$$

because if we specialize the above to $\tau \in \mathbb{R} \cap B_1$, we get $|\varphi(\tau)|^2 = a_0 > 1$ and can then use continuity of φ to prove the claim. To prove (2.100), we note that both sides define biholomorphic mappings $B_1 \to \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$ and that they extend in the same way to the boundary points $\tau = \pm i, \pm i\infty$. Now Hecke proves the existence of a holomorphic function $\Phi : \mathbb{H} \to \mathbb{C}$ satisfying

$$\Phi(\tau + h) = \Phi(\tau), \quad \Phi(-1/\tau) = -\Phi(\tau), \quad \Phi(\tau)^2 = \varphi(\tau)$$

for all $\tau \in \mathbb{H}$ and then considers

$$\psi(\tau) := \frac{\varphi'(\tau)}{\Phi(\tau)(\varphi(\tau) - 1)},$$

which is holomorphic and nowhere vanishing on $\mathbb{H} \cup \{i\infty\}$ and transforms like a modular form in $M_2(h, +1)$ (we again refer to [BK08, Ch. 4] for justification and details). Using a suitable logarithm of ψ, Hecke defines

$$f_n^\epsilon(\tau) := \Phi(\tau)^{\frac{1-\epsilon}{2}} \psi(\tau)^{k/2} (\varphi(\tau) - 1)^n$$

and proves that $f_n^\epsilon \in M_k(h, \epsilon)$ for $n \geq k/2$. Note that since $\varphi(\tau) - 1$ vanishes to order 1 at $i\infty$ while Φ and ψ are non-vanishing at $i\infty$, each f_n^ϵ indeed vanishes to order exactly n at $i\infty$. It remains to be shown that f_n^ϵ belongs to $M_k(h, \epsilon)$ *for all* $n \geq 1$. For this, it suffices to show that the $\Gamma(h)$-invariant, continuous function $|f_n^\epsilon(\tau)| \,\mathrm{Im}(\tau)^{k/2}$ is bounded on the fundamental domain $\mathcal{D}_h$.

For $\tau \in \mathcal{D}_h$ with $\mathrm{Im}(\tau) \geq 2$ we have $\psi(\tau) - 1 = O(e^{-(2\pi/h)\,\mathrm{Im}(\tau)})$ while $\psi(\tau)^{k/2}$ and $\Phi(\tau)$ are both $O(1)$. For $\tau \in \mathcal{D}_h$ with $\mathrm{Im}(\tau) \leq 2$, we write

$$|f_n^\epsilon(\tau)| \,\mathrm{Im}(\tau)^{k/2} = |\varphi(\tau)|^{\frac{1-\epsilon}{4}} |\varphi(\tau) - 1|^{n-k/2} |f^*(\tau)|^{k/2}, \quad \text{where} \quad f^*(\tau) := \mathrm{Im}(\tau) |\psi'(\tau)/\Phi(\tau)|. \tag{2.101}$$

The function $f^* : \mathbb{H} \to \mathbb{R}_{\geq 0}$ is easily seen to be bounded (see [BK08, Ch. 4, p. 31]) and since we showed that $|\varphi(\tau) - 1|$ is bounded away from zero for $\tau \in \mathcal{D}_h$ with $\mathrm{Im}(\tau) \leq 2$ and that φ is bounded on $\mathbb{H}$, we are done. □

Remark 2.16 (on Hecke's proof [Hec36])**.** For comparison, if we restrict the discussion to $n \geq k/2$, then the function $|h(\tau) - 1|^{n-k/2}$ is obviously bounded on $\mathbb{H}$, since h is bounded on $\mathbb{H}$ and it is then enough to argue with the bounded function f^* only, as in (2.101) (on the whole fundamental domain $\mathcal{D}_h$). Presumably, Hecke was not concerned about the restriction $n \geq k/2$, since his primary goal was to show that the spaces $M_k(\Gamma_\theta(h), \chi_\epsilon)$ are infinite dimensional for all $h > 2$ and $k \in \mathbb{R}$. He used this result to show that certain spaces of Dirichlet series are infinite dimensional, while for other values of h, they are finite dimensional.

We finished the proof of part (iv) of Theorem 2 and thus the proof of the whole theorem.

2.4.5 Additional properties of the basis functions in the case $h = 2$

In this section, we will express each of the functions $b_{k,2,n}(r)$ as a sum of Bessel functions times Kloosterman-type sums, depending on the continuous parameter r. This highlights the connection to classical Poincaré series and we will use the connection to prove that (most of) these functions are *not* of rapid decay on $\mathbb{R}$. We assume throughout this section that $k \geq 3$ is an *integer* and that $h = 2$, so we work with modular forms for the group $\Gamma(2)$ of weight k and a certain character. The discussion which follows should apply analogously to the functions $\tilde{b}_{k,2,n}$.

Our arguments will be based on the following lemma, which establishes a further property of the set $\mathcal{V}_2 \subseteq \Gamma(2)$.

Lemma 2.15. *Let $\mathcal{P}$ denote the set of all pairs (c, d) of coprime integers c, d such that c is even, nonzero and d is (necessarily) odd. The bottom row assignment $\gamma \mapsto (c_\gamma, d_\gamma)$ yields a bijection $\mathcal{V}_2 \to \mathcal{P}/\mathbb{Z}^\times$. Here, quotienting by $\mathbb{Z}^\times = \{\pm 1\}$ signifies that we only consider the element of $\mathcal{P}$ up to sign, as we are working in $\mathrm{PSL}_2(\mathbb{Z})$.*

Proof. Since for $\gamma \in \Gamma(2)$ the entry c_γ is always even and d_γ always odd and since, by part (iii) of Lemma 2.12, c_γ is never zero, the map is well-defined. It is injective, since if $\gamma_1, \gamma_2 \in \mathcal{V}_2$ have the same bottom row, then $\gamma_1\gamma_2^{-1} = T^{2m}$ for some $m \in \mathbb{Z}$, hence $\gamma_1 = T^{2m}\gamma_2$, which forces $m = 0$, by the definition of $\mathcal{V}_2$ and the fact that $\Gamma(2)$ is freely generated by V^2 and T^2. Now we prove surjectvity. So let $(c, d) \in \mathcal{P}$ be arbitrary. Since $\gcd(2c, d) = 1$, we find $a, b' \in \mathbb{Z}$ so that $ad - b'(2c) = 1$. Then a is odd, $b := 2b'$ is even and

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(2)$$

has bottom row (c, d). It suffices to find $q \in \mathbb{Z}$ so that $T^{2q}M \in \mathcal{V}_2$, since $T^{2q}M$ and M have the same bottom row, both equal to (c, d). To prove the existence of such q, it suffices to find integers $m_1, \ldots, m_r$, nonzero integers $n_1, \ldots, n_r$ and some $p \in \mathbb{Z}$ such

$$MT^{2m_1}V^{2n_1} \cdots T^{2m_r}V^{2n_r} = T^{2p}.$$

Indeed, we then have $T^{-2p}M \in \mathcal{V}_2$ (so $q = -p$ has the desired property). To see that such integers exist, we write down the effect of right multiplication by general T^{2m}, V^{2n}:

$$\begin{bmatrix} * & * \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2m \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} * & * \\ c & d + 2cm \end{bmatrix}, \qquad \begin{bmatrix} * & * \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2n & 1 \end{bmatrix} = \begin{bmatrix} * & * \\ c + 2dn & d \end{bmatrix}. \tag{2.102}$$

We see from this that, via successive reduction of d mod $2c$ and c mod $2d$, we can make the lower left entry c of M equal to zero, by successively multiplying on the right by suitable elements $T^{2m_j}V^{2n_j}$. $\square$

Recall that the set $\mathcal{V}_2$ is also stable under right-multiplication by powers of T^2. By the first identity in (2.102) we see that $\gamma \mapsto (c_\gamma, d_\gamma)$ induces a well-defined map $\mathcal{V}_2/\langle T^2\rangle \to (\mathcal{P}/\mathbb{Z}^\times)/\mathbb{Z}$, where we let $\mathbb{Z}$ act on the right on $\mathcal{P}/\mathbb{Z}^\times$ by $(c,d)|m = (c, d+2mc)$ for $(c,d) \in \mathcal{P}$, $m \in \mathbb{Z}$. Since this induced map is clearly a bijection, we obtain the following consequence.

Corollary 2.3. *Retain the above set-up. For all even positive integers $c > 0$ and all odd integers $d \in \{1, \dots, 2c\}$, with $\gcd(c,d) = 1$, there is a unique element $\gamma(c,d) \in \mathcal{V}_2$ whose bottom row is (c,d) (up to sign).*

For later purposes it will be convenient to have notation for the upper left-entry of the element $\gamma(c,d)$ in the Corollary. Let us denote it as $a(c,d)$, so that we have

$$\gamma(c,d) = \begin{bmatrix} a(c,d) & * \\ c & d \end{bmatrix} \in \mathcal{V}_2, \tag{2.103}$$

for any c, d with the above properties.

These group-theoretic facts will allow us to imitate the computation for the formula of Fourier coefficients of classical Poincaré series in terms of Bessel functions and Kloosterman sums. The Kloosterman sum will be replaced by a certain analogue of it.

We start by recalling that for all $n \in \mathbb{Z}$, all $r \in \mathbb{C}$ and all $y > 0$, we have

$$b_{k,2,n}(r) = \frac{1}{2}\int_{iy-1}^{iy+1} F_{k,2}(\tau, r) e^{-\pi i n \tau} d\tau.$$

In the remainder of §2.4.5, we abbreviate $b_{k,2,n}(r)$ to $b_{k,n}(r)$ and $F_{k,2}(\tau,r)$ to $F_k(\tau,r)$ and we consider only $r \in \mathbb{R}_{\geq 0}$. In the above expression for $b_{k,n}(r)$ we write out the definition of $F_{k,2}(\tau,r)$ and split the sum over $\mathcal{V}_2$ into orbits modulo the right action of $\langle T^2\rangle$, giving

$$\begin{aligned} b_{k,n}(r) &= -\frac{1}{2}\sum_{\gamma\in\mathcal{V}_2/\langle T^2\rangle}\sum_{m\in\mathbb{Z}}\int_{iy-1}^{iy+1} g_{2k}(\tau,r)|_k(\gamma T^{2m}) e^{-\pi i n\tau} d\tau \\ &= -\frac{1}{2}\sum_{\gamma\in\mathcal{V}_2/\langle T^2\rangle}\int_{iy+\mathbb{R}} j_k(\gamma,\tau)^{-1} e^{\pi i (\gamma\tau) r^2} e^{-\pi i n \tau} d\tau. \end{aligned}$$

Here, the first equal sign is justified by absolute and uniform (on $iy + [-1,1]$) convergence of the series $F_k(\tau,r)$ and the second equal sign is justified by absolute convergence of the integral, which is easily seen from $|j_k(\gamma,\tau)| = |c_\gamma\tau + d_\gamma|^k$ and the fact that $c_\gamma \neq 0$ for all $\gamma \in \mathcal{V}_2$. Next, we use the the matrices $\gamma(c,d) \in \mathcal{V}_2$ as in (2.103) as set of representatives for $\mathcal{V}_2/\langle T^2\rangle$, as well as the identity $\gamma\tau = \frac{a_\gamma}{c_\gamma} - \frac{1}{c_\gamma(c_\gamma\tau+d_\gamma)}$, to rewrite the above as[19]

$$\begin{aligned} b_{k,n}(r) &= -\frac{1}{2}\sum_{\substack{c=1\\ c\equiv 0(2)}}^{\infty}\sum_{\substack{d=1\\ \gcd(2c,d)=1}}^{2c} e^{\pi i\frac{a(c,d)}{c}r^2}\int_{iy+\mathbb{R}}\frac{e^{\pi i\frac{-1}{c^2(\tau+d/c)r^2}}e^{-\pi i n\tau}}{j_k(\gamma(c,d),\tau)}d\tau \\ &= -\frac{1}{2}\sum_{\substack{c=1\\ c\equiv 0(2)}}^{\infty}\sum_{\substack{d=1\\ \gcd(2c,d)=1}}^{2c} e^{\pi i\frac{a(c,d)}{c}r^2}e^{\pi i\frac{dn}{c}}\int_{iy+\mathbb{R}}\frac{e^{\pi i\frac{-1}{c^2\tau}r^2}e^{-\pi i n\tau}}{j_k(\gamma(c,d),\tau-d/c)}d\tau. \end{aligned}$$

Next, we recall from (2.14) that for all $z \in \mathbb{H}$, and all c, d as in the above sum, we have

$$j_k(\gamma(c,d),z) = \Theta_3(\gamma(c,d)z)^{2k}/\Theta_3(z)^{2k} = g_c(d)^{2k}((z+d/c)/i)^k,$$

[19]For even c, the conditions $\gcd(2c,d) = 1$ and $\gcd(c,d) = 1$ are equivalent, but we prefer to write the first one.

where we recall that $g_c(d) = \frac{1}{2}G(d,2c) = \frac{1}{2}\sum_{m=0}^{2c-1} e^{\pi i d m^2/c}$. Applying that formula to $z = \tau + d/c$ yields

$$b_{k,n}(r) = -\frac{1}{2}\sum_{\substack{c=1\\ c\equiv 0(2)}}^{\infty}\ \sum_{\substack{d=1\\ \gcd(2c,d)=1}}^{2c} g_c(d)^{-2k} e^{\pi i \frac{a(c,d)}{c} r^2} e^{\pi i \frac{dn}{c}} \int_{iy+\mathbb{R}} \frac{e^{\pi i \frac{-1}{c^2\tau} r^2} e^{-\pi i n\tau}}{(\tau/i)^k} d\tau. \tag{2.104}$$

In a next step, we will express each of the above integrals in terms of the J-Bessel function

$$J_\alpha(x) = (x/2)^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)}(x/2)^{2m} \qquad x>0, \alpha>0.$$

We define, for y, k and n as above and for any real number $\beta \geq 0$, the integral

$$I(y,k,\beta,n) := \int_{iy+\mathbb{R}} \frac{e^{\pi i(-1/\tau)\beta} e^{-\pi i n\tau}}{(\tau/i)^k} d\tau$$

so that our integral of interest in (2.104) is obtained by setting $\beta = r^2/c^2$. First, let us remark that it is independent of $y>0$, as an application of Cauchy's theorem shows (apply it to a rectangle $[-R,R]+i[y_1,y_2]$, $y_1<y_2$ and let $R\to\infty$). For $n\leq 0$, we infer from this independence that $I(y,k,\beta,n) = \lim_{y\to\infty} I(y,k,\beta,n) = 0$, which reproves that $b_{k,2,n}(r) = 0$ for $n\leq 0$. Consider henceforth only $n>0$. To evaluate the integral in the case $\beta=0$, we write

$$I(y,k,0,n) = \int_{\mathbb{R}} \frac{e^{-\pi i n(t+iy)}}{(y-it)^k} dt = e^{\pi n y}\int_{\mathbb{R}} \frac{e^{\pi i n t}}{(y+it)^k} dt = e^{\pi n y}\frac{(2\pi)e^{-n\pi y}(\pi n)^{k-1}}{\Gamma(k)} = \frac{(2\pi)(\pi n)^{k-1}}{\Gamma(k)},$$

where we used the following formula from [ZMGR15, 8.315], applied with $\eta = y$ and $\nu = n\pi$:

$$\int_{\mathbb{R}} \frac{e^{i\nu t}}{(\eta+it)^k} dt = \frac{(2\pi)e^{-\nu\eta}\nu^{k-1}}{\Gamma(k)}.$$

(The argument of $\eta + it$ taken in $(-\pi/2,\pi/2)$, consistent with our standard convention made in item (4) of §1.1.) For $\beta>0$ we write out the exponential $e^{\pi i(-1/\tau)\beta}$ as a power series and obtain thus

$$\begin{aligned}
I(y,k,\beta,n) &= \sum_{m=0}^{\infty} \frac{(\pi i\beta)^m(-1)^m}{m!}\int_{iy+\mathbb{R}} \frac{e^{-\pi i n\tau}}{\tau^m(\tau/i)^k} d\tau = \sum_{m=0}^{\infty}\frac{(\pi\beta)^m(-1)^m}{m!} I(y,m+k,0,n)\\
&= \sum_{m=0}^{\infty}\frac{(\pi\beta)^m(-1)^m}{m!}\frac{(2\pi)(\pi n)^{k+m-1}}{\Gamma(m+k)}\\
&= (2\pi)(\pi n)^{k-1}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+k-1+1)}(\pi\beta)(\pi n)^m\\
&= (2\pi)(\pi n)^{k-1}(\pi\sqrt{n\beta})^{1-k}J_{k-1}(2\pi\sqrt{n\beta}) = (2\pi)n^{k/2-1/2}\sqrt{\beta}^{1-k}J_{k-1}(2\pi\sqrt{n\beta}).
\end{aligned}$$

We specialize this to $\beta = r^2/c^2$, and obtain from (2.104) that, for $r>0$,

$$\begin{aligned}
b_{k,n}(r) &= -\frac{1}{2}\sum_{\substack{c=1\\ c\equiv 0(2)}}^{\infty}\ \sum_{\substack{d=1\\ \gcd(2c,d)=1}}^{2c} g_c(d)^{-2k} e^{\pi i\frac{a(c,d)}{c}r^2} e^{\pi i\frac{dn}{c}}\Big((2\pi)n^{k/2-1/2}r^{1-k}c^{k-1}J_{k-1}(2\pi\sqrt{nr}/c)\Big)\\
&= -\pi n^{k/2-1/2}r^{1-k}\sum_{\substack{c=1\\ c\equiv 0(2)}}^{\infty}\frac{1}{c}S_k(r,n,c)J_{k-1}(2\pi\sqrt{nr}/c), \qquad (2.105)
\end{aligned}$$

where we define

$$S_k(r,n,c) := \sum_{\substack{d=1\\ \gcd(2c,d)=1}}^{2c} (g_c(d)/\sqrt{c})^{-2k} e^{\pi i \frac{a(c,d)}{c} r^2} e^{\pi i \frac{dn}{c}}. \tag{2.106}$$

We also obtain

$$b_{k,n}(0) = -\frac{1}{2} \sum_{\substack{c=1\\ c\equiv 0(2)}}^{\infty} \sum_{\substack{d=1\\ \gcd(2c,d)=1}}^{2c} g_c(d)^{-2k} e^{\pi i \frac{dn}{c}} \frac{(2\pi)(\pi n)^{k-1}}{\Gamma(k)} = -\pi \frac{(\pi n)^{k-1}}{\Gamma(k)} \sum_{\substack{c=1\\ c\equiv 0(2)}}^{\infty} \frac{1}{c^k} S_k(0,n,c)$$

Further above, we have assumed that k is an integer, but we have in fact not used this assumption in the above computations.

The above formula and (2.105) are interesting because they (almost) "interpolate" the ones for classical Poincaré-series in the parameter r. We can already see this property in the definition of $F_k(\tau, r)$, which "interpolates" classical Poincaré series P_m^k of weight k from integral parameters $m = \sqrt{m}^2$ to real non-negative (indeed, complex) parameters $m = r^2$, $r \in \mathbb{C}$, since the set $\mathcal{V}_2$ represents the coset space $\Gamma(2)_\infty \backslash \Gamma(2)$, up to the identity coset. We will now make this connection more precise and use it to prove not all of the functions $b_{k,n}(r)$ are of rapid decay as a function of $r \geq 0$. Indeed, we will show that infinitely many of them are not of rapid decay along the subset $r = \sqrt{m}$, $m \in \mathbb{Z}_{\geq 0}$ (provided the weight k is sufficiently large). This will be a consequence of the fact that the Fourier coefficients of a nonzero modular form cannot grow too slowly.

There is a somewhat technical but convenient step that we will take, which consists in switching from the group $\Gamma(2)$ to the group $\Gamma_0(4)$, which is conjugate to it. We do so for ease of reference to the literature. Note that if $f(z)$ is modular of weight k for the group $\Gamma(2)$, then $f(2z)$ will be modular of weight k for the group $\Gamma_0(4)$, where modularity for that group is defined with respect to the theta function $\theta(z) := \Theta_3(2z)$ (which agrees with the usual notion if k is an even integer).

Let us now specialize (2.105) to radii $r = \sqrt{m}$ with $m \in \mathbb{N}$. We replace c by $c/2$ in (2.105) and correspondingly sum over $c \in 4\mathbb{N}$, giving

$$b_{k,n}(\sqrt{m}) = -\pi \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c=1\\ c\equiv 0(4)}}^{\infty} \frac{2}{c} S_k(\sqrt{m}, n, c/2) J_{k-1}(4\pi\sqrt{nm}/c). \tag{2.107}$$

We have

$$S_k(\sqrt{m}, n, c/2) = \sum_{\substack{d=1\\ \gcd(c,d)=1}}^{c} (g_{c/2}(d)/\sqrt{c/2})^{-2k} e^{2\pi i \left(\frac{a(c/2,d)m}{c} + \frac{dn}{c}\right)}$$

Let $\chi : (\mathbb{Z}/4\mathbb{Z})^\times \to \{-1, 1\}$ denote the non-trivial character. For $4|c$ and d coprime to c, we have $G_c(d)^2 = i(2c)\chi(d)$ and since $g_{c/2}(d) = \frac{1}{2} G_c(d)$ (by our definition (2.14)) this implies that

$$(\sqrt{c/2}/g_{c/2}(d))^{2k} = i^{-k} \chi(d)^k$$

Note that for $4|c$, we can also view χ as a character on $(\mathbb{Z}/c\mathbb{Z})^\times$. Note also that the definition of $a(c/2, d)$ further implies that $a(c/2, d)d \equiv 1 \pmod{c}$. It follows that we can write

$$S_k(\sqrt{m}, n, c/2) = i^{-k} \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^\times} \chi(d)^k e(dm/c + \bar{d}n/c) =: i^{-k} S_{\chi^k}(m, n, c), \tag{2.108}$$

where $\bar{d}$ denotes the inverse of d mod c and where $e(w) = e^{2\pi i w}$. Let us introduce the notations

$$\sigma_k(m,n) := \sum_{\substack{c=1 \\ c\equiv 0(4)}}^{\infty} \frac{1}{c} S_{\chi^k}(m,n,c) J_{k-1}(4\pi\sqrt{nm}/c)$$

and

$$P_m^k(z) := \sum_{\gamma\in\Gamma_0(4)_\infty\backslash\Gamma_0(4)} \chi(\gamma)^k (c_\gamma z + d_\gamma)^{-k} e(m(\gamma z)), \tag{2.109}$$

for the mth Poincaré series of weight k for the group $\Gamma_0(4)$ (and character χ^k). It is known [IK04, Lemma 14.2][20] that its nth Fourier coefficient is given by

$$\widehat{P_m^k}(n) = 2\pi i^{-k} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} (\delta(m,n) + \sigma_k(m,n)) . \tag{2.110}$$

It follows from this formula and from (2.107) and (2.108) (not forgetting the factors 2 and i^{-k}) that, for all integers $m, n \geq 1$ such that $m \neq n$, we have

$$b_{k,n}(\sqrt{m}) = -\widehat{P_m^k}(n). \tag{2.111}$$

We comment that the minus sign is consistent with the minus sign in the definition of $F_k(\tau, \sqrt{m})$. Again, this relation is not surprising (in hindsight), because the series $F_k(\tau, \sqrt{m})$ equals $-P_m^k(\tau/2)$, up to the identity coset in the summation range and it is the identity coset and orthogonality of the exponentials $e(mx)$ on $\mathbb{R}/\mathbb{Z}$ that create the delta function $\delta(n,m)$ in the formula (2.110).

Now that we have established (2.111) we can move on to proving the following result. It shows that the upper bound $|b_{k,n}(r)| \lesssim n^{k/2+5/4} r^{-k+5/2} \lesssim_n r^{-k+5/2}$ from (2.91) can't be significantly improved and that (infinitely many of) the functions $b_{k,n}(r)$ are (in particular) not of rapid decay on $\mathbb{R}$.

Proposition 2.12. *Let $k \geq 3$ and $n \geq 1$ be integers. Assume that the nth Poincaré series P_n^k as defined in* (2.109) *does not vanish identically on $\mathbb{H}$. Then, for each $\varepsilon > 0$, the function $r \mapsto r^{k-1+\varepsilon} b_{k,n}(r)$ is unbounded on $(0, +\infty)$, in fact, unbounded on the subset of $r = \sqrt{m}$, $m \in \mathbb{N}$.*

Proof. Fix $k \geq 3$ and $n \geq 1$ and abbreviate $\varphi := P_n^k \in S_k(\Gamma_0(4), \chi^k) \smallsetminus \{0\}$. Let $A > 0$ and assume that $|b_{k,n}(r)| = O(r^{-A})$ as $r \to \infty$. We will show that $A \leq k-1$. By (2.111) and (2.110) we have, for all $m \in \mathbb{N} \smallsetminus \{n\}$,

$$|b_{k,n}(\sqrt{m})| = |\widehat{P_m^k}(n)| = \left(\frac{n}{m}\right)^{k-1} |\widehat{P_n^k}(m)| = \left(\frac{n}{m}\right)^{k-1} |\widehat{\varphi}(m)|.$$

It follows from our assumption that $\widehat{\varphi}(m) = O(m^{-A/2+k-1})$. On the other hand, the Rankin–Selberg method (see [Ran39, Theorem 1 and Remark B on page 364]) implies that for every $f \in S_k(\Gamma_0(4), \chi^k)$, the Fourier coefficients $\widehat{f}(m)$ satisfy

$$\sum_{m=1}^{M} |\widehat{\varphi}(m)|^2 = c_k(\varphi) M^k + O(M^{k-2/5}), \qquad \text{as } M \to \infty,$$

[20] in the cited reference, there may be a typo reversing the roles of n and m, as noted in a list of errata to the book.

where $c_k(\varphi)$ is proportional to the Petersson norm of φ which is > 0 since $\varphi \neq 0$. The bound $\widehat{\varphi}(m) = O(m^{-A/2+k-1})$ gives on the other hand that

$$\sum_{m=1}^{M} |\widehat{\varphi}(m)|^2 \lesssim M^{-A+2k-1} = M^k M^{k-1-A}, \qquad \text{as } M \to \infty,$$

which implies $A \leq k-1$, as desired. □

Remark 2.17 (More elementary arguments)**.** If we wish to avoid the input of the Rankin–Selberg method in the proof of the above proposition and are satisfied with a numerically weaker result, we may also argue as follows. Assume that $k \geq 3$ is an integer and that a nonzero form $\varphi \in S_k(\Gamma_0(4), \chi)$ has Fourier coefficients satisfying $\widehat{\varphi}(m) = O(m^{\alpha})$ for some $\alpha > 0$. Then, by a simple geometric series estimate, we obtain $\varphi(x+iy) \lesssim y^{-\alpha-1}$ as $y \to 0$. Thus, the $\Gamma_0(4)$-invariant function $\Phi(z) := |\varphi(z)| \operatorname{Im}(z)^{k/2}$ satisfies $\Phi(z) \lesssim \operatorname{Im}(z)^{k/2-\alpha-1}$ as $\operatorname{Im}(z) \to 0$. If we had $k/2 - \alpha - 1 > 0$, then, by considering the orbit of points (in a small disc) in a fundamental domain of $\Gamma_0(4)$ which accumulate (necessarily) at the real line, we would obtain that Φ and hence φ vanish at those points and hence identically. It follows that $\alpha \geq k/2 - 1$. In the notation of the proof of Proposition 2.12, we can apply this with $\alpha = -A/2 + k - 1$ and deduce $A \leq k$.

Remark 2.18 (On the assumption of Proposition 2.12)**.** If $S_k(\Gamma_0(4), \chi^k) \neq 0$, then there are infinitely many n such that $P_n^k \neq 0$. Indeed, if there is a nonzero cusp form $\varphi \in S_k(\Gamma_0(4), \chi^k)$ then $\widehat{\varphi}(n) \neq 0$ for infinitely many $n \geq 1$ (see the previous remark) and hence the Petersson product $\langle \varphi, P_n^k \rangle$, which is proportional to $\widehat{\varphi}(n)$ is not zero and hence $P_n^k \neq 0$. Thus, for large enough k, the assumption of Proposition 2.12 always holds for infinitely many n.

2.5 Some concluding remarks

We end this chapter with some remarks on Theorem 1 and Theorem 2.

2.5.1 Some coincidental identities between the functions $b_{k,2,n}$ and the $a_{k,n}$

If $d = 2k \geq 5$, then we can apply the interpolation formula from Theorem 2 to the basis functions $a_{k,n}$, $\tilde{a}_{k,n}$ from Theorem 1, for any choice of $n_0 = n_0(d)$, $\hat{n}_0 = \hat{n}_0(d)$. For instance, for any $m \geq n_0$, we thus obtain, because of part (ii) of Theorem 1,

$$\begin{aligned} a_{k,m}(r) &= \sum_{n=1}^{\infty} b_{k,2,n}(r) a_{k,m}(\sqrt{n}) + \sum_{n=1}^{\infty} \tilde{b}_{k,2,n}(r) \widehat{a_{k,m}}(\sqrt{n}) \\ &= \sum_{1 \leq n < n_0} b_{k,2,n}(r) a_{k,m}(\sqrt{n}) + b_{k,2,m}(r) + \sum_{1 \leq n < \hat{n}_0} \tilde{b}_{k,2,n}(r) \widehat{a_{k,m}}(\sqrt{n}), \end{aligned} \tag{2.112}$$

which expresses $a_{k,m}(r)$ as a finite linear combination of the $b_{k,2,n}$ and $\tilde{b}_{k,2,n}$. We may obtain similar expressions for the $\tilde{a}_{k,m}$ and for the Fourier transforms of these functions. For instance, for any integer $m \geq \hat{n}_0$, we have

$$\tilde{a}_{k,m}(r) = \sum_{1 \leq n < n_0} b_{k,2,n}(r) \tilde{a}_{k,m}(\sqrt{n}) + \sum_{1 \leq n < \hat{n}_0} \tilde{b}_{k,2,n}(r) \widehat{\tilde{a}_{k,m}}(\sqrt{n}) + \tilde{b}_{k,2,m}(r). \tag{2.113}$$

For some values of k, n_0 and $\hat{n}_0$, the finite sums over $1 \leq n < n_0$ and $1 \leq n \leq \hat{n}_0$ in (2.112) , (2.113) are empty. For instance, if $5 \leq d \leq 7$, then $1 + \lfloor d/4 \rfloor = 2$ and the choice $n_0 = \hat{n}_0 = 1$ leads to the identities

$$a_{k,m}(r) = b_{k,m}(r), \qquad \tilde{a}_{k,m}(r) = \tilde{b}_{k,m}(r) \qquad \text{for all } m \geq 1 \text{ and } 5/2 \leq k \leq 7/2.$$

Since the $a_{k,m}$ and $\tilde{a}_{k,m}$ are rapidly decaying, it follows from Proposition 2.12, that all Poincaré series of that weight for the group $\Gamma_0(4)$ vanish. To be precise, under our simplifying assumptions, we may deduce this for $k = 3$. This in turn implies that $S_3(\Gamma_0(4), \chi) = 0$, which is no contradiction. As the weight increases, we only have the more general linear relations described above, which perhaps makes them all the more interesting in view of Proposition 2.12.

We may also understand the relationships between $a_{k,n}$ and the $b_{k,2,n}$ functions at the level of their generating series. Indeed, for fixed k, $n_0, \hat{n}_0$ and r, the difference

$$F_{k,2}(\tau, r) - F_{k,n_0,\hat{n}_0}(\tau, r)$$

is a modular form of weight k for the group $\Gamma(2)$ which must vanish identically if the parameters k, n_0 and $\hat{n}_0$ are adjusted as in the previous example.

2.5.2 The parameter $h \geq 2$

One may wonder whether there is also a version of Theorem 1 with $\sqrt{n}$ replaced by $\sqrt{2n/h}$, for $h > 2$, or, in other words, whether there is a version of Theorem 2 in the case $d < 5$. We believe that this might be possible, in the sense that one might be able to construct a modular kernel function for $\Gamma(h)$ also when $h > 2$ and define an interpolation basis similar to the one in Theorem 1 via contour integrals. Such a (hypothetical) proof should then apply to all dimensions.

Let us also remark on the motivation for introducing the additional parameter $h \geq 2$ into our analysis. This parameter arose by specializing our more general analysis of Chapter 4, where we study necessary conditions on the existence of Fourier interpolation sets of the form $\sqrt{\Lambda} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : (x_1^2, \ldots, x_n^2) \in \Lambda\}$ for general lattices $\Lambda \subseteq \mathbb{R}^n$ to the case $n = 1$. This specialization amounts to considering (the possibility of existence of) Fourier interpolation formulas using nodes of the form $\sqrt{2n/h}$ for any real $h > 0$. As we will prove in Proposition 4.1 in Chapter 4 a necessary condition for the existence of such formula is that the group $\Gamma(h)$ is freely generated by T^h and V^h and another desirable (and likely necessary) condition is that it is moreover discrete. These conditions led us to consider $h \geq 2$. Let us also remark that N. Sardari proved several interesting results in the "non-free but discrete" case $h \in (0, 1)$, that is $h = 2\cos(\pi/q)$ with $q \in \mathbb{Z}_{\geq 3}$, to answer a question raised in [CKM+21].

2.5.3 Other period functions

The construction of the generating functions $F_{k,\ell,\hat{\ell}}(\tau, r)$ and $F_{k,h}(\tau, r)$ for the proof of Theorems 1 and 2 did not use any specific properties of the Gaussian $\varphi(z) = e^{\pi i z r^2}$. Both methods of proofs can be used to construct modular integrals with more general prescribed period functions φ. This flexibility of the method is used in the recent work [BRS20], where the authors consider $\varphi(z) = \varphi_s(z) := (z/i)^s$ for a complex parameter s. The Mellin transforms of the resulting modular integrals can be used to prove "Mellin-interpolation formulas" recovering any sufficiently well-behaved function $\Psi(s)$ defined and analytic for $\mathrm{Re}(s) \in (-\varepsilon, 1 + \varepsilon)$, for some $\varepsilon > 0$, from its values at the non-trivial zeros ρ of (the analytic continuation of) an L-function $L(s)$ (e.g. the Riemann zeta function) and the values of its inverse Mellin transform $\psi(y) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=c} \Psi(s) y^{-s} ds$, $y > 0$, at the positive integers $y = n$, $n \in \mathbb{N}$. The weight k of the modular integral is determined by the center of symmetry $k/2$ of the functional equation of the completed L-function $L^*(s) = \gamma(s)L(s) = L^*(k-s)$. Here, $\gamma(s)$ is the Γ-factor of the L-function, expressible in terms of the Gamma function.

We believe that, when $k > 2$, the construction of a Poincaré-type series with respect to the period function $\varphi(z) = (z/i)^s$ can be used to prove a similar result for L-functions $L(s) = L(s, f)$ attached to cusp forms f on $\Gamma_0(N)$ of weight $k \in \mathbb{Z}_{\geq 3}$. Given that such a result is already contained in [BRS20] and given that the series construction has several technical drawbacks, we have not pursued this in great detail.

It would be interesting to know whether there are other period functions φ, for which the resulting modular integral solves an interesting problem.

2.5.4 The parameters $n_0, \hat{n}_0$.

We formulate this last remark as an open question. Let $d \in \mathbb{N}$ and $E, \hat{E} \subseteq \mathbb{N}_0$ be any two finite subsets such that $|E| + |\hat{E}| = 1 + \lfloor d/4 \rfloor$. Is the pair

$$\left(\bigcup_{n \in \mathbb{N}_0 \setminus E} \sqrt{n} S^{d-1}, \bigcup_{n \in \mathbb{N}_0 \setminus \hat{E}} \sqrt{n} S^{d-1} \right)$$

a Fourier uniqueness pair for $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$? To prove such a result, it would be sufficient to construct certain modular forms in $M_{d/2}(\Gamma(2))$ with prescribed Fourier coefficients on the sets E and $\hat{E}$ with respect to the cusps 0 and ∞; compare with the arguments after Proposition 2.11.

3 Fourier interpolation from spheres

The main goal of this chapter is to establish Fourier interpolation formulas that recover any $f \in \mathcal{S}(\mathbb{R}^d)$ from the restrictions of f and $\hat{f}$ to the spheres $\sqrt{n}S^{d-1}$, $n \in \mathbb{N}_0$. We will do so by combining Theorems 1 and 2 from Chapter 2 with some general harmonic analysis results that we establish in §3.1. The main results are stated as Corollary 3.2 and Theorem 3.

In §3.3 we will use basic principles from functional analysis to prove perturbations of these results. More precisely, Theorem 4 shows that the Fourier uniqueness sets given by the spheres $\sqrt{n}S^{d-1}$ can be replaced by surfaces close to those spheres, or discrete sets of points contained in them. This exposes our joint work with Ramos [RS21].

In §3.4 we will indicate an application of our techniques by sketching a proof that the so-called magic functions for the sphere packing problems in 8 and 24 dimensions are necessarily radial functions and unique among all Schwartz functions which are admissible and optimal for the Cohn–Elkies linear programming method.

3.1 Some harmonic analysis results

Let $d \geq 2$ be an integer and let $0 \leq r_1 < r_2 < \dots$ be a sequence of radii, tending to infinity. The main purpose of §3.1 is to show how one can deduce Fourier uniqueness and -interpolation results for the union of the spheres $r_n S^{d-1}$ from Fourier uniqueness and -interpolation results for *radial* functions in dimensions $d, d+2, d+4, \dots$, with nodes r_n. In this sense, we reduce the problem for general functions to the same problem for radial functions in a sequence of higher dimensions. To this end, we first gather some general auxiliary results in §3.1.1 and then study and introduce certain spherical averages in §3.1.2.

3.1.1 Polynomials, harmonic polynomials and Gaussians

In this section, we recall some basic definitions and facts about harmonic polynomials that we will use later. Most of the material in this section can be found in [SW71] or [ABR01, Ch. 5]. We also prove that the the space of of functions given by "a Gaussians times a harmonic polynomial" is dense in the Schwartz space (Corollary 3.1).

Let $d \geq 2$. Let $\mathcal{P}(\mathbb{R}^d)$ denote the $\mathbb{C}$-algebra of all polynomial functions $P : \mathbb{R}^d \to \mathbb{C}$. For each $m \in \mathbb{N}_0$, let $\mathcal{P}_m(\mathbb{R}^d) \subseteq \mathcal{P}(\mathbb{R}^d)$ denote the subspace of those polynomial functions that are homogeneous of degree m and by $\mathcal{H}_m(\mathbb{R}^d) \subseteq \mathcal{P}_m(\mathbb{R}^d)$ the subspace of $u \in \mathcal{P}_m(\mathbb{R}^d)$ satisfying $\Delta u = 0$. Elements of $\mathcal{H}_m(\mathbb{R}^d)$ are called *harmonic polynomials of degree* m and elements of $\mathcal{H}(\mathbb{R}^d) := \oplus_{m \geq 0} \mathcal{H}_m(\mathbb{R}^d)$ are called *harmonic polynomials.* To simplify notation, we sometimes abbreviate

$$\mathcal{P}_m = \mathcal{P}_m(\mathbb{R}^d), \quad \mathcal{H}_m = \mathcal{H}_m(\mathbb{R}^d), \quad \mathcal{P} = \mathcal{P}(\mathbb{R}^d), \quad \mathcal{H} = \mathcal{H}(\mathbb{R}^d)$$

when d is understood from context.

When restricted to the unit sphere $S^{d-1} \subseteq \mathbb{R}^d$, harmonic polynomials of different degrees are mutually orthogonal with respect to the L^2-inner product induced by the surface measure and the set of all restrictions of these polynomials is dense in $C(S^{d-1})$ for the sup-norm. Thus $L^2(S^{d-1})$ decomposes as an orthogonal direct sum $\oplus_{m\geq 0}\mathcal{H}_m(S^{d-1})$ with dense L^2-closure. Here, we write $\mathcal{H}_m(S^{d-1}) := \{u|_{S^{d-1}} : u \in \mathcal{H}_m(\mathbb{R}^d)\}$.

We next give a formula for the dimension of $\mathcal{H}_m$. By definition, $\mathcal{H}_m$ is the kernel of the Laplacian, viewed as a linear map $\mathcal{P}_m \to \mathcal{P}_{m-2}$ (here, we assume $m \geq 2$). We will show that it is onto, which will allow us to deduce

$$\dim \mathcal{H}_m = \dim \mathcal{P}_m - \dim \mathcal{P}_{m-2} = \binom{d+m-1}{d-1} - \binom{d+m-3}{d-1} \sim \frac{2}{(d-2)!} m^{d-2}, \tag{3.1}$$

where the last asymptotic relation holds for $m \to \infty$ and d fixed. To prove that $\Delta(\mathcal{P}_m) = \mathcal{P}_{m-2}$, we will write Δ as the adjoint of an injection $\mathcal{P}_{m-2} \to \mathcal{P}_m$ with respect to a suitable inner product. To define the inner product, we first define a linear map $u \mapsto \partial(u)$, $\mathcal{P} \to \mathrm{End}(\mathcal{P})$ by $\partial(x^\alpha)v = \partial^\alpha v$ for $v \in \mathcal{P}$ and $\alpha \in \mathbb{N}_0^d$. Then define a sesquilinear form $(\cdot,\cdot) : \mathcal{P} \times \mathcal{P} \to \mathbb{C}$ by $(u, v) := (\partial(u)\overline{v})(x)|_{x=0}$. Denote its restriction to $\mathcal{P}_m \times \mathcal{P}_m$ by $(\cdot,\cdot)_m$. It is positive definite. Define $\iota(u)(x) = u(x)|x|^2$ for any $u \in \mathcal{P}$. Clearly, ι is an injective linear map $\mathcal{P}_{m-2} \to \mathcal{P}_m$. It is then straight-forward to check the promised adjointness

$$(\Delta u, v)_{m-2} = (u, \iota(v))_m \quad \text{for all } u \in \mathcal{P}_m, v \in \mathcal{P}_{m-2}.$$

As a consequence we have

$$\mathcal{P}_m = \ker(\Delta) \oplus \ker(\Delta)^\perp = \mathcal{H}_m \oplus \mathrm{Im}(\iota).$$

In other words, we may write any $u \in \mathcal{P}_m$ as $u(x) = u_0(x) + v_1(x)|x|^2$, where $u_0 \in \mathcal{H}_m$ and $v_1 \in \mathcal{P}_{m-2}$. Iterating this we find that any $u \in \mathcal{P}_m$ can be written as

$$u(x) = \sum_{0\leq j\leq m/2} |x|^{2j} u_j(x) \tag{3.2}$$

with $u_j \in \mathcal{H}_{m-2j}$.

We will often use of the zonal spherical harmonic (or reproducing kernel) $Z_m^d(\zeta,\omega)$, $\zeta,\omega \in S^{d-1}$, characterized by the property

$$\int_{S^{d-1}} u(\zeta)\overline{Z_m^d(\zeta,\omega)}d\zeta = u(\omega) \quad \text{for all } u \in \mathcal{H}_m(\mathbb{R}^d). \tag{3.3}$$

From this defining property one may deduce that $Z_m^d(h\zeta, h\omega) = Z_m^d(\zeta,\omega)$ for all $h \in \mathrm{O}(d)$ and all $\zeta, \omega \in S^{d-1}$ and that

$$Z_m^d(\omega,\omega) = \|Z_m^d(\cdot,\omega)\|_{L^2}^2 = \dim \mathcal{H}_m(\mathbb{R}^d), \tag{3.4}$$

for all $\omega \in S^{d-1}$. If $\mathcal{B}_m \subseteq \mathcal{H}_m(S^{d-1})$ is an orthonormal basis, then $Z_m^d(\zeta,\omega) = \sum_{u\in\mathcal{B}_m} u(\zeta)\overline{u(\omega)}$. We will frequently use the following L^∞-L^2 bound

$$\sup_{\zeta\in S^{d-1}} |u(\zeta)| \leq \dim(\mathcal{H}_m(\mathbb{R}^d))^{1/2}\|u\|_{L^2(S^{d-1})} \lesssim_d m^{d/2-1}\|u\|_{L^2(S^{d-1})}, \tag{3.5}$$

which is valid for all $u \in \mathcal{H}_m$ and follows by applying the Cauchy–Schwarz inequality to (3.3) and (3.4).

Next, we need to recall some formulas for the Fourier transforms of harmonic polynomials, times Gaussians.

Proposition 3.1. *Let $d \geq 2$, $m \geq 0$ be integers and let $u \in \mathcal{H}_m(\mathbb{R}^d)$. Then*

$$\mathcal{F}_{\mathbb{R}^d}(ug(z)) = i^{-m}(z/i)^{-d/2-m}ug(-1/z)$$

for all $z \in \mathbb{H}$, that is

$$\int_{\mathbb{R}^d} u(x)e^{\pi i z|x|^2}e^{-2\pi i\langle x,\xi\rangle}dx = i^{-m}(z/i)^{-d/2-m}u(\xi)e^{\pi i(-1/z)|\xi|^2}. \tag{3.6}$$

Proof. This is well-known, but we recall a proof for completeness and because the result is quite important for our purposes.

First, since both sides of (3.6) define holomorphic functions of $z \in \mathbb{H}$, it suffices to prove the identity for $z = iy$, $y > 0$. For those z, the proof reduces in fact to the case $z = i$, by homogeneity of u and properties of the Fourier transform. Thus, it suffices to prove the identity (3.6) when $z = i$, in which case it reads

$$\int_{\mathbb{R}^d} u(x)e^{-\pi|x|^2}e^{-2\pi i\langle x,\xi\rangle}dx = i^{-m}u(\xi)e^{-\pi|\xi|^2}. \tag{3.7}$$

To do that, we note that (for each fixed $\xi \in \mathbb{R}^d$), the difference of the RHS and the LHS of 3.7 defines an $\mathrm{O}(d)$-invariant linear form $\mathcal{H}_m(\mathbb{R}^d) \to \mathbb{C}$. Since $\mathcal{H}_m(\mathbb{R}^d)$ is an irreducible $\mathrm{O}(d)$-representation, it thus suffices to verify 3.7 for *some* nonzero $u_0 \in \mathcal{H}_m(\mathbb{R}^d)$. Define $u_{0,m}(x) = (x_1 + ix_2)^m$ for $m \in \mathbb{N}_0$. We prove by induction on m that 3.7 holds for $u = u_{0,m}$. The case $m = 0$, is well-known (Fourier transform of the Gaussian). For the inductive step, we recall the general identity

$$\left(\frac{1}{(-2\pi i)}\frac{\partial}{\partial\xi_1} + i\frac{1}{(-2\pi i)}\frac{\partial}{\partial\xi_2}\right)\mathcal{F}_{\mathbb{R}^d}(f(x))(\xi) = \mathcal{F}_{\mathbb{R}^d}((x_1 + ix_2)(f(x))(\xi)$$

for $f \in \mathcal{S}(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d$. Applying it to $f(x) = u_{0,m}(x)e^{-\pi|x|^2}$ and using the inductive hypothesis, we obtain

$$\begin{aligned}
\mathcal{F}_{\mathbb{R}^d}(u_{0,m+1}(x)e^{-\pi|x|^2})(\xi) &= \left(\frac{1}{(-2\pi i)}\frac{\partial}{\partial\xi_1} + i\frac{1}{(-2\pi i)}\frac{\partial}{\partial\xi_2}\right)\left(i^{-m}u_{0,m}(\xi)e^{-\pi|\xi|^2}\right) \\
&= \frac{i^{-m}}{(-2\pi i)}e^{-\pi|\xi|^2}\Big[\left(m(\xi_1+i\xi_2)^{m-1} + (\xi_1+i\xi_2)^m(-2\pi)\xi_1\right) \\
&\qquad + i\left(im(\xi_1+i\xi_2)^{m-1} + (\xi_1+i\xi_2)^m(-2\pi)\xi_2\right)\Big] \\
&= i^{-m-1}e^{-\pi|\xi|^2}(\xi_1+i\xi_2)^{m+1} = i^{-(m+1)}u_{0,m+1}(\xi)e^{-\pi|\xi|^2}
\end{aligned}$$

as desired. □

A key analytic input to Chapter 2 was the fact that the span of all Gaussians $g_d(z)$, $z \in \mathbb{H}$, is dense in the space of radial Schwartz functions on $\mathbb{R}^d$. For what we will do in the case of general Schwartz functions we will need a similar result in which we replace the Gaussians $g_d(z)$ by Gaussian times harmonic polynomials, that is, by functions of the form $g_d(z)(x)u(x)$, with $u \in \mathcal{H}(\mathbb{R}^d)$. The density of the span of those functions is the content of Corollary 3.1 below which is based on the following Proposition.

Proposition 3.2. *For each $c > 0$ the subspace $\mathcal{W}_c$ of $\mathcal{S}(\mathbb{R}^d)$ consisting of all Schwartz function of the form $p(x)e^{-c|x|^2}$ with $p \in \mathcal{P}(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$.*

Proposition 3.2 is presumably well known. For example, exercises 6 and 7 in chapter 2.1 on Hermite functions in [HT12] propose a proof for $d = 1$. For completeness, we include the proof we found here. It is inspired by a trick learned from [RV19], which was already used in the proof of Proposition 2.4. The "trick" consists in writing $f(x) = f_1(x)e^{-c|x|^2}$ and using that, if f is compactly supported and smooth, then so is $f_1(x) = f(x)e^{c|x|^2}$ so that we can express f_1 in terms of $\widehat{f_1}$ via Fourier inversion. In the resulting formula for $f(x)$, we "approximate" the exponential $e^{2\pi i\langle x,\xi\rangle}$ by its (1-dimensional) Taylor-series to get polynomial expressions in $\langle x, \xi\rangle$ (hence in x after integrating over ξ).

Before giving the proof of Proposition 3.2, let use record the following consequence of it, of Proposition 2.3 and the decomposition (3.2) of polynomials into even powers of the Euclidean norm and harmonic polynomials. This corollary is crucial in the proof of the later main results.

Corollary 3.1. *Let $\mathcal{V}$ denote the subspace of $\mathcal{S}(\mathbb{R}^d)$ spanned by functions of the form $f(x) = u(x)e^{\pi iz|x|^2}$ for some $z \in \mathbb{H}$ and $u \in \mathcal{H}(\mathbb{R}^d)$. Then $\mathcal{V}$ is dense in $\mathcal{S}(\mathbb{R}^d)$.*

Proof of Corollary 3.1. In this proof, the word "approximate" is used in reference to the Schwartz topology, meaning in the sense of approximating with respect to any given finite collection of semi-norms.

By Proposition 3.2, it suffices to approximate any given $f \in \mathcal{W}_\pi$ by elements of the space $\mathcal{V}$. By (3.2) any element of $\mathcal{W}_\pi$ is a finite linear combination of functions of the form $u(x)|x|^{2j}e^{-\pi|x|^2}$ for some $j \in \mathbb{N}_0$ and some $u \in \mathcal{H}(\mathbb{R}^d)$. By Proposition 2.3 we may approximate each function $|x|^{2j}e^{-\pi|x|^2}$ arbitrarily well by a linear combination of Gaussian $e^{\pi iz|x|^2}$. Since multiplication by a polynomial is a continuous linear operation on $\mathcal{S}(\mathbb{R}^d)$, this procedure will approximate the original function arbitrarily well too. ☐

Proof of Proposition 3.2. It is sufficient to prove that $\mathcal{W}_c$ is dense in $\mathcal{S}(\mathbb{R}^d)$ *for some* $c > 0$, since for each $\lambda > 0$, the linear map $S_\lambda : f(x) \mapsto f(\lambda x)$ is a continuous automorphism of $\mathcal{S}(\mathbb{R}^d)$ and we have $S_\lambda(\mathcal{W}_c) = \mathcal{W}_{c\lambda^2}$ and hence $S_\lambda(\overline{\mathcal{W}_c}) \subseteq \overline{S_\lambda(\mathcal{W}_c)} \subseteq \overline{\mathcal{W}_{c\lambda^2}}$. We will prove that $\mathcal{W}_2$ is dense in $\mathcal{S}(\mathbb{R}^d)$.

Fix $f_0 \in \mathcal{S}(\mathbb{R}^d)$ and fix a symmetric open zero-neighborhood $U_0 \subseteq \mathcal{S}(\mathbb{R}^d)$. It suffices to find $f \in \mathcal{W}_2$ such that

$$f \in f_0 + U_0 + U_0 + U_0.$$

Choose $f_1 \in C_c^\infty(\mathbb{R}^d)$ such that $f_1 \in f_0 + U_0$. Define $f_2 \in C_c^\infty(\mathbb{R}^d)$ by $f_2(x) = f_1(x)e^{2|x|^2}$. For $\varphi \in \mathcal{S}(\mathbb{R}^d)$ to be chosen in a moment, we write

$$f_1(x) = f_2(x)e^{-2|x|^2} = (f_2(x) - \varphi(x))e^{-2|x|^2} + \varphi(x)e^{-2|x|^2}.$$

The assignment $\varphi(x) \mapsto (f_2(x) - \varphi(x))e^{-2|x|^2}$ defines a continuous map $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$. The preimage of U_0 under this map is therefore an open subset of $\mathcal{S}(\mathbb{R}^d)$ and it contains f_2. We choose and fix $\varphi \in \mathcal{S}(\mathbb{R}^d)$ inside that open subset such that $\widehat{\varphi} \in C_c^\infty(\mathbb{R}^d)$, which is possible since the space of Schwartz functions with compactly supported Fourier transform is dense in the Schwartz space. Thus, we have

$$f_0(x) = (f_0(x) - f_1(x)) + (f_2(x) - \varphi(x))e^{-2|x|^2} + \varphi(x)e^{-2|x|^2} \in U_0 + U_0 + \varphi(x)e^{-2|x|^2}$$

and it is now sufficient to find $f \in \mathcal{W}_2$ such that $f(x) \in \varphi(x)e^{-2|x|^2} + U_0$ (abusing notation here). To this end, we define, for any $t \in \mathbb{C}$ and $N \in \mathbb{N}_0$, the function $E_N(t) := \sum_{n=0}^{N} \frac{t^n}{n!}$ and we claim that

$$e^{-2|x|^2}\varphi(x) = e^{-2|x|^2}\int_{\mathbb{R}^d} \widehat{\varphi}(\xi)e^{2\pi i\langle x,\xi\rangle}dx = \lim_{N\to\infty} e^{-2|x|^2}\int_{\mathbb{R}^d} \widehat{\varphi}(\xi)E_N(2\pi i\langle x,\xi\rangle)d\xi,$$

in the Schwartz topology. The proof of this claim will finish the proof of the proposition, since each of the integrals in the limit clearly belongs to $\mathcal{W}_2$, so that, for any sufficiently large N, the function $f(x) = e^{-2|x|^2} \int_{\mathbb{R}^d} \widehat{\varphi}(\xi) E_N(2\pi i\langle x, \xi\rangle) d\xi$ has the desired property.

So let $\alpha, \beta \in \mathbb{N}_0^d$ be given multi-indices and consider, for $N \in \mathbb{N}_0$ such that (without loss of generality) $N \geq |\beta| + 1$, the difference

$$\begin{aligned}
&x^\alpha \partial_x^\beta \left(e^{-2|x|^2} \int_{\mathbb{R}^d} \widehat{\varphi}(\xi) E_N(2\pi i\langle x, \xi\rangle) d\xi - e^{-2|x|^2} \varphi(x) \right) \\
&= x^\alpha \partial_x^\beta \left(e^{-2|x|^2} \int_{\mathbb{R}^d} \widehat{\varphi}(\xi) \left(E_N(2\pi i\langle x, \xi\rangle) - e^{2\pi i\langle x, \xi\rangle} \right) d\xi \right) \\
&= \sum_{\gamma_1+\gamma_2=\beta} \frac{\beta!}{\gamma_1!\gamma_2!} \left(x^\alpha \partial_x^{\gamma_2} e^{-2|x|^2} \right) \int_{\mathbb{R}^d} \widehat{\varphi}(\xi) \partial_x^{\gamma_1} \left(E_N(2\pi i\langle x, \xi\rangle) - e^{2\pi i\langle x, \xi\rangle} \right) d\xi \\
&= \sum_{\gamma_1+\gamma_2=\beta} \frac{\beta!}{\gamma_1!\gamma_2!} \left(x^\alpha P_{\gamma_1}(x) e^{-2|x|^2} \right) \int_{\mathbb{R}^d} \widehat{\varphi}(\xi) (2\pi i)^{|\gamma_2|} \xi^{\gamma_2} \left(E_{N-|\gamma_2|}(2\pi i\langle x, \xi\rangle) - e^{2\pi i\langle x, \xi\rangle} \right) d\xi,
\end{aligned}$$

where the P_{γ_1} are some polynomials and where we (repeatedly) used that

$$\partial_{x_j} E_N(2\pi i\langle x, \xi\rangle)) = 2\pi i \xi_j E_{N-1}(2\pi i\langle x, \xi\rangle).$$

By writing $e^{-2|x|^2} = e^{-|x|^2} e^{-|x|^2}$ we reduce to showing that for any fixed $\gamma \in \mathbb{N}_0^d$, we have

$$\sup_{x\in\mathbb{R}^d} \left| e^{-|x|^2} \int_{\mathbb{R}^d} \widehat{\varphi}(\xi) \xi^\gamma \left(E_M(2\pi i\langle x, \xi\rangle) - e^{2\pi i\langle x, \xi\rangle} \right) d\xi \right| \longrightarrow 0 \quad \text{as} \quad M \to \infty. \tag{3.8}$$

Choose $R = R_\varphi \geq 1$ such that $\widehat{\varphi}(\xi) = 0$ for $|\xi| \geq R$. We will divide the ξ-integral into two ranges depending on x: one where the truncated exponential series E_M approximates $e^{2\pi i\langle x, \xi\rangle}$ well and the complement. To do so, we use the following general estimate

$$|t| \leq 1 + M/2 \quad \Rightarrow \quad |E_M(t) - e^t| \leq 2\frac{|t|^{M+1}}{(M+1)!}, \tag{3.9}$$

which is valid for all $t \in \mathbb{C}$ and follows from the following simple computation

$$\begin{aligned}
|E_M(t) - e^t| &\leq \sum_{m=M+1}^\infty \frac{|t|^m}{m!} \leq \frac{|t|^{M+1}}{(M+1)!} \sum_{m=0}^\infty \frac{(1+M/2)^m}{(M+1+1)(M+1+2)\cdots(M+1+m)} \\
&\leq \frac{|t|^{M+1}}{(M+1)!} \sum_{m=0}^\infty (1/2)^m.
\end{aligned}$$

We denote the function of M and x inside the absolute values in (3.8) by $A(M,x)$ and write $A(M,x) = A_1(M,x) + A_2(M,x)$, where

$$A_1(M,x) := e^{-|x|^2} \int_{\substack{|\xi|\leq R \\ 2\pi|x||\xi|\leq(1+M/4)}} \widehat{\varphi}(\xi) \xi^\gamma \left(E_M(2\pi i\langle x, \xi\rangle) - e^{2\pi i\langle x, \xi\rangle} \right) d\xi. \tag{3.10}$$

Then (3.9) and Stirling's asymptotic formula imply that, uniformly in $x \in \mathbb{R}^d$, we have

$$|A_1(M,x)| \lesssim_{\gamma,\varphi,R} \frac{(1+M/4)^{M+1}}{(M+1)!} \sim \left(\frac{M+4}{4}\right)^{M+1} \left(\frac{e}{M+1}\right)^{M+1} \frac{1}{\sqrt{2\pi(M+1)}} = o(M^{-1/2}),$$

as $M \to \infty$ (Here, we used that $e/4 < 1$, which is the reason why we used $1 + M/4$ instead of $1 + M/2$ to apply (3.9)). It remains to estimate

$$A_2(M,x) = e^{-|x|^2} \int_{\substack{|\xi| \le R \\ 2\pi|x||\xi| > (1+M/4)}} \widehat{\varphi}(\xi)\xi^\gamma \left(E_M(2\pi i\langle x,\xi\rangle) - e^{2\pi i \langle x,\xi\rangle} \right) d\xi.$$

Introduce

$$X = X(M,R) = \frac{1+M/4}{2\pi R}.$$

Then the domain of integration is such that

$$|x| \le X \quad \Longrightarrow \quad A_2(M,x) = 0.$$

Therefore,

$$\begin{aligned} |A_2(M,x)| &\le \mathbf{1}_{(|x|>X)} e^{-|x|^2} \int_{|\xi|\le R} |\widehat{\varphi}(\xi)\xi^\gamma| (e^{2\pi|x||\xi|} + 1) d\xi \\ &\lesssim_{\varphi,\gamma,R} \mathbf{1}_{(|x|>X)} e^{-|x|^2} e^{2\pi R|x|} = \mathbf{1}_{(|x|>X)} e^{(\pi R)^2} e^{-(|x|-\pi R)^2}. \end{aligned}$$

Given that for $|x| \ge X$, we have

$$|x| - \pi R \ge X(M,R) - \pi R \longrightarrow +\infty \quad \text{as} \quad M \to \infty,$$

we obtain $\sup_{x\in\mathbb{R}^d} |A_2(M,x)| \to 0$, as desired. □

We conclude this section with the following lemma giving bounds for the L^2-norm of derivatives of harmonic polynomials. It will be used in the proof of Lemma 3.3 in Chapter 3.

Lemma 3.1. *Let $d \ge 2$, $m \ge 0$ and $\gamma \in \mathbb{N}_0^d$ and assume $(m,\gamma) \ne (0,0)$. Set $c = |\gamma|$. Then, for all $u \in \mathcal{H}_m(\mathbb{R}^d)$, we have*

$$\|\partial^\gamma u\|_{L^2(S)} \le \sqrt{d^c}\, m^c \|u\|_{L^2(S)}. \qquad (S = S^{d-1})$$

Proof. We may assume that $m \ge 1$ and that $c \le m$, as otherwise $\partial^\gamma u = 0$. By [ABR01, Thm 5.14] there exists a constant $\nu_d > 0$ so that for all $u, v \in \mathcal{H}_m(\mathbb{R}^d)$ of the form $u(x) = \sum_{|\alpha|=m} b_\alpha x^\alpha$, $v(x) = \sum_{|\alpha|=m} c_\alpha x^\alpha$, we have

$$\langle u, v\rangle_{L^2(S)} = \int_S u(\zeta)\overline{v(\zeta)} d\zeta = \nu_d \prod_{i=0}^{m-1} (d+2i)^{-1} \sum_{|\alpha|=m} \alpha! b_\alpha \overline{c_\alpha}.$$

Applying this with $u = v$ and computing $\partial^\gamma u(x) = \sum_{|\alpha|=m,\alpha\ge\gamma} c_\alpha \frac{\alpha!}{(\alpha-\gamma)!} x^{\alpha-\gamma}$, we obtain

$$\|\partial^\gamma u\|^2_{L^2(S)} \le \left(\prod_{i=m-c}^{m-1} (d+2i) \right) \left(\max_{\substack{|\alpha|=m \\ \gamma\le\alpha}} \frac{\alpha!}{(\alpha-\gamma)!} \right) \|u\|^2_{L^2(S)} \le (md)^c m^c \|u\|^2_{L^2(S)}. \qquad \square$$

3.1.2 Study of certain spherical averages

Let us start by giving some motivation for and overview of the definitions and propositions in this section, as it will be somewhat technical. A similar overview in the case $\mathbb{R}^2$ using Fourier series was already given in the introduction.

Consider a smooth function $f \in C^\infty(\mathbb{R}^d)$ and a point $x \in \mathbb{R}^d \smallsetminus \{0\}$. Let $\omega_x = x/|x| \in S^{d-1}$. Then the value $f(x) = f(|x|\omega_x)$ can be viewed as the value at $\omega = \omega_x$ of the function $\omega \mapsto f(|x|\omega)$ on S^{d-1}. Since the latter function is smooth, it admits a point-wise absolutely uniformly convergent expansion into spherical harmonics:

$$f(|x|\omega) = \sum_{m=0}^{\infty} \sum_{u \in \mathcal{B}_m} u(\omega) \int_{S^{d-1}} f(|x|\zeta)\overline{u(\zeta)} d\zeta,$$

where $\mathcal{B}_m \subseteq \mathcal{H}_m(S^{d-1})$ denotes a chosen orthonormal basis. Specializing to $\omega = \omega_x$ gives

$$f(x) = f(|x|\omega_x) = \sum_{m=0}^{\infty} \sum_{u \in \mathcal{B}_m} u(x) L_u f(|x|), \quad \text{where} \quad L_u f(t) = |t|^{-m} \int_{S^{d-1}} f(|t|\zeta)\overline{u(\zeta)} d\zeta, \quad (3.11)$$

for $t \in \mathbb{R}^\times$. For any $p \in \mathbb{N}$, we can also view $L_u f$ as a *radial* function on $\mathbb{R}^p \smallsetminus \{0\}$ by composing with Euclidean norm $\mathbb{R}^p \to [0, \infty)$. Call this composition $L_u^p f : \mathbb{R}^p \smallsetminus \{0\} \to \mathbb{C}$. Thus, $L_u^1 f = L_u f$ by definition. We will show that each $L_u^p f$ extends to a smooth function on $\mathbb{R}^p$. Moreover, we will show that, if $f \in \mathcal{S}(\mathbb{R}^d)$ and $p = d + 2m$, with m the degree of u, then the Fourier transform of $L_u^{d+2m} f$ *on* $\mathbb{R}^{d+2m}$ (exists and) equals $i^m L_u^{d+2m} \hat{f}$, where $\hat{f} = \mathcal{F}_{\mathbb{R}^d}(f)$. Thus, *if* we have Fourier interpolation formulas for *radial* functions on all spaces $\mathbb{R}^{d+2m}$, that use some fixed set of radii $0 \le r_1 < r_2 < r_3 \dots$ as nodes, then, by viewing each number $L_u f(|x|)$ in (3.11) as the value at the radius $|x|$ of the radial function $L_u^{d+2m} f$ on $\mathbb{R}^{d+2m}$, we can express $L_u f(|x|)$ only in terms of the values of $L_u^{d+2m} f$ and $L_u^{d+2m} \hat{f}$ at these radii r_n. Consequently, we can express $f(x)$ only in terms of the restrictions of f and $\hat{f}$ to the spheres $r_n S^{d-1}$.

To implement this strategy we start over with the definition of $L_u f$ and $L_u^p f$ and carefully prove some of the assertions mentioned above, one after the other.

Definition 3.1. For $m \in \mathbb{N}_0$, $d \ge 2$, $f \in C^\infty(\mathbb{R}^d)$ and $u \in \mathcal{H}_m(\mathbb{R}^d)$ we define $L_u f : \mathbb{R}^\times \to \mathbb{C}$ by

$$L_u f(t) := |t|^{-m} \int_{S^{d-1}} f(|t|\zeta)\overline{u(\zeta)} d\zeta \qquad t \in \mathbb{R}^\times.$$

Note that if $m = 0$ then $u \equiv c$ is constant and $L_u f(t) = c \int_{S^{d-1}} f(|t|\zeta) d\zeta$ equals c times the average of f over the sphere $|t| S^{d-1}$.

Proposition 3.3. *For each $f \in C^\infty(\mathbb{R}^d)$ and $u \in \mathcal{H}_m(\mathbb{R}^d)$, the function $L_u f : \mathbb{R}^\times \to \mathbb{C}$ extends uniquely to a smooth even function $L_u f : \mathbb{R} \to \mathbb{C}$ by*

$$L_u f(0) = \sum_{|\alpha|=m} \frac{\partial^\alpha f(0)}{\alpha!} \int_{S^{d-1}} \overline{u(\zeta)} \zeta^\alpha d\zeta.$$

The assignments $f \mapsto L_u f$ (parameterized by u) define continuous linear maps $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}_{even}(\mathbb{R})$.

Before we give the proof of Proposition 3.3 we remark that we can make sense of it and of Definition 3.1 in the case $d = 1$ in the following way. Equip $S^0 = \{-1, 1\}$ with the probability measure that assigns mass $1/2$ both to 1 and -1 so that the integral of any function $F : S^0 \to \mathbb{C}$

over S^0 is simply $(F(-1)+F(1))/2$. Note also that $\mathcal{H}_0(\mathbb{R}) = \mathbb{C}1$, $\mathcal{H}_1(\mathbb{R}) = \mathbb{C}u_1$ where $u_1(x) = x$ and $\mathcal{H}_m(\mathbb{R}) = 0$ for all $m \geq 2$. Thus for any $f \in C^\infty(\mathbb{R})$,

$$L_{u_1}f(t) = \frac{1}{2|t|}\left((-1)f(|t|) + (1)f(-|t|)\right) = \frac{f(|t|) - f(-|t|)}{2|t|}, \quad t \in \mathbb{R}^\times$$
$$L_{u_1}f(0) = f'(0)\left(\frac{(-1)(-1)+(1)(1)}{2}\right) = f'(0)$$

It is an exercise in the use of Taylor's Theorem to show the above function $L_{u_1}f$ is indeed smooth on $\mathbb{R}$. The following proof gives a solution of a more complicated version of this exercise.

Proof of Proposition 3.3. Let $d \geq 2$, $f \in C^\infty(\mathbb{R}^d)$ and $u \in \mathcal{H}_m(\mathbb{R}^d)$, $m \in \mathbb{N}_0$. We abbreviate $S = S^{d-1}$ and $Lf = L_u f$. To start, recall that by Taylor's theorem we have, for every $x \in \mathbb{R}^d$ and every $K \in \mathbb{N}_0$,

$$f(x) = \sum_{k=0}^{K} \sum_{|\alpha|=k} \frac{(\partial^\alpha f)(0)}{\alpha!} x^\alpha + \sum_{|\alpha|=K+1} \frac{K+1}{\alpha!} \int_0^1 (1-s)^K (\partial^\alpha f)(sx)ds\, x^\alpha.$$

We specialize this to $x = |t|\zeta$, where $(t,\zeta) \in \mathbb{R}^\times \times S$ and take $K \geq m+1$. Then we integrate over $\zeta \in S$ against $\overline{u}(\zeta/|t|)$ and use the decomposition (3.2), applied to monomials $P(x) = x^\alpha$, combined with orthogonality relations for spherical harmonics, to obtain

$$Lf(t) = \sum_{\substack{k=m \\ k \equiv m(2)}}^{K} |t|^{k-m} \sum_{|\alpha|=k} \frac{(\partial^\alpha f)(0)}{\alpha!} \int_S \zeta^\alpha \overline{u(\zeta)} d\zeta + |t|^{K+1-m} R_K(t), \tag{3.12}$$

with remainder term

$$R_K(t) = \sum_{|\alpha|=K+1} \frac{K+1}{\alpha!} \int_S \int_0^1 (1-s)^K (\partial^\alpha f)(|t|\zeta s) ds\, \overline{u(\zeta)} \zeta^\alpha d\zeta.$$

The first sum in (3.12) is a polynomial in t^2, hence in $C^\infty_{\mathrm{rad}}(\mathbb{R})$. It therefore suffices to show that there is $\ell(K) \in \mathbb{N}_0$ tending to infinity with K such that $t \mapsto |t|^{K+1-m} R_K(t)$ belongs to $C^{\ell(K)}(\mathbb{R})$. The reason why this should be true is of course that $t \mapsto |t|^{K+1-m} R_K(t)$ gains more and more regularity near zero, if K gets large and vanishes at zero to high order (consider for example $K = 2N-1+m$ for some large N). We note also that once we have proved this, we will also have proved that the polynomial in t^2 in (3.12) is the Taylor-polynomial of the even function $L_u f$.

To turn this into a rigorous argument, we first check that on $\mathbb{R}^\times$, we have

$$\frac{d^j}{dt^j}|t|^c = (t/|t|)^j \frac{c!}{(c-j)!}|t|^{c-j} \qquad (0 \leq j \leq c), \tag{3.13}$$

$$\frac{d^j}{dt^j}(\partial^\alpha f)(s\zeta|t|) = s^j (t/|t|)^j \sum_{|\beta|=j} (\partial^{\alpha+\beta} f)(|t|s\zeta)\zeta^\beta. \tag{3.14}$$

We now take K of the form $K = m+2N$ for $N \in \mathbb{N}$. Then we deduce from the Leibniz rule and the above formulas (3.13), (3.14) that, for $0 \leq j \leq N$, the derivative

$$\frac{d^j}{dt^j}|t|^{K-m+1}R_K(t) = \tag{3.15}$$

$$(t/|t|)^j \sum_{j_1+j_2=j} a_{j_1,j_2}|t|^{2N+1-j_1} \sum_{\substack{|\alpha|=K+1\\|\beta|=j_2}} \frac{K+1}{\alpha!} \int_S \int_0^1 s^{j_2}(1-s)^K(\partial^{\alpha+\beta}f)(s|t|\zeta)ds\,\zeta^{\alpha+\beta}\overline{u(\zeta)}d\zeta, \tag{3.16}$$

where we used $(t/|t|)^j = (t/|t|)^{j_1}(t/|t|)^{j_2}$ and where

$$a_{j_1,j_2} = \frac{j!}{j_1!j_2!}\frac{(2N+1)!}{(2N+1-j_1)!}.$$

All of these computations hold for $t \in \mathbb{R}^\times$. We deduce that $\frac{d^j}{dt^j}|t|^{K-m+1}R_K(t) \to 0$, as $t \to 0$ on $\mathbb{R}^\times$ and that the relevant difference quotients at $t = 0$ also tend to zero.

To prove the second assertion of the proposition, assume now that $f \in \mathcal{S}(\mathbb{R}^d)$ and fix integers $j, n \geq 0$ such that n is even. Define

$$A = \sup_{t\in[0,1]} |(1+t^n)(Lf)^{(j)}(t)|, \qquad B = \sup_{t\in[1,\infty)} |(1+t^n)(Lf)^{(j)}(t)|.$$

It suffices to show that A and B can be bounded in terms of finitely many continuous semi-norms of f. Here, we also used that $(Lf)^{(j)}$ is either even or odd, to be able to restrict to non-negative arguments y, for convenience.

To estimate the term A, we again take $K = 2N + m$ with $j \leq N$. We then read off from (3.12) that the jth derivative of the polynomial $Lf(t) - |t|^{2N+1}R_K(t)$ has degree at most $2N - j$, and that its coefficients are multiples of $\partial^\alpha f(0)$, with $|\alpha| \leq K$, so that the supremum over $y \in [0,1]$ of that derivative may be bounded in terms of finitely many continuous semi-norms of f. For the remainder term we note that inside the integrals appearing in (3.16), the vectors $t|y|\zeta \in \mathbb{R}^d$ have Euclidean norm at most 1 for all triples $(t, s, \zeta) \in [0,1]^2 \times S$ under consideration, so that we can bound these integrals in terms of suprema of partial derivatives of f, over the closed unit ball in $\mathbb{R}^d$.

To estimate the term B, we compute directly from the definition, using the Leibniz rule as well as (3.14) (with $\alpha = 0, t = 1$), that, for $m \geq 1$, $t \geq 1$,

$$(Lf)^{(j)}(t) = \sum_{j_1+j_2=j} b_{j,j_1,j_2}y^{-m-j_1} \sum_{|\beta|=j_2} \int_S (\partial^\beta f)(t\zeta)\zeta^\beta\overline{u(\zeta)}d\zeta, \tag{3.17}$$

where

$$b_{j,j_1,j_2} = \frac{j!}{j_1!j_2!}\frac{(-1)^{j_1}(m+j_1-1)!}{(m-1)!}$$

If $m = 0$, the formula for $(Lf)^{(j)}$ is simpler (namely only the inner sum in (3.17) with j_2 replaced by j and $\overline{u(\zeta)}$ replaced by 1). We may now multiply (3.17) with $1 + t^n$, and the bound

$$|(1+t^n)(\partial^\beta f)(t\zeta)| \leq \sup_{|x|\geq 1}(1+|x|^n)|\partial^\beta f(x)|.$$

Note here that $t^n = |t\zeta|^n$ for $\zeta \in S$. Thus, B can be bounded in terms of f as required. $\square$

Definition 3.2. For all $p, d \in \mathbb{N}$, $f \in C^\infty(\mathbb{R}^d)$ and $u \in \mathcal{H}_m(\mathbb{R}^d)$, we define $L_u^p f : \mathbb{R}^p \to \mathbb{C}$ by $L_u^p(x) := L_u(|x|)$ for $x \in \mathbb{R}^p$.

Proposition 3.4. *For all $f \in \mathcal{S}(\mathbb{R}^d)$ all $u \in \mathcal{H}_m(\mathbb{R}^d)$ and all $p \in \mathbb{N}$ we have $L_u^p f \in \mathcal{S}_{rad}(\mathbb{R}^p)$ and $f \mapsto L_u^p f$ is a continuous linear map $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}_{rad}(\mathbb{R}^d)$. Moreover, we have*

$$\mathcal{F}_{\mathbb{R}^{d+2m}}(L_u^{d+2m} f) = i^m L_u^{d+2m}(\mathcal{F}_{\mathbb{R}^d}(f)) \tag{3.18}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

Proof. The first two assertions follow from Proposition 3.3 and the general fact that the assignment $h(t) \mapsto f(|x|)$ gives a well-defined continuous linear map $\mathcal{S}_{\text{even}}(\mathbb{R}) \to \mathcal{S}_{\text{rad}}(\mathbb{R}^p)$. To prove (3.18), note that both sides depend continuously upon $f \in \mathcal{S}(\mathbb{R}^d)$, by continuity of $f \mapsto L_u^p f$ and by continuity the Fourier transform. By Corollary 3.1, it therefore suffices to prove (3.18) in the cases where $f(x) = v(x)e^{\pi i z|x|^2}$ for any given $v \in \mathcal{H}_\mu(\mathbb{R}^d)$, $\mu \in \mathbb{N}_0$ and $z \in \mathbb{H}$. We henceforth assume that f is of this form and we also fix some $\eta \in \mathbb{R}^{d+2m} \setminus \{0\}$. We will evaluate both sides of (3.18) separately at the point η and see that the values agree.

We start with the left hand side of (3.18). We note first that, by definition,

$$L_u^{d+2m} f(y) = \frac{1}{|y|^m} \int_S f(|y|\zeta)\overline{u(\zeta)} d\zeta = \frac{1}{|y|^m} \int_S e^{\pi i ||y|\zeta|^2} v(|y|\zeta)\overline{u(\zeta)} d\zeta = |y|^{\mu - m} e^{\pi i z|y|^2} \langle v, u \rangle_{L^2(S)}$$

for all $y \in \mathbb{R}^{d+2m} \setminus \{0\}$. It follows that

$$\mathcal{F}_{\mathbb{R}^{d+2m}}(L_u^{d+2m} f)(\eta) = \begin{cases} (z/i)^{-(d+2m)/2} e^{\pi i(-1/z)|\eta|^2} \langle v, u \rangle_{L^2(S)} & \text{if } \mu = m, \\ 0 & \text{otherwise.} \end{cases} \tag{3.19}$$

Now we turn to the right hand side of (3.18). By Proposition 3.1, we have

$$\mathcal{F}_{\mathbb{R}^d}(f)(\xi) = i^{-m}(z/i)^{-d/2-m} v(\xi) e^{\pi i(-1/z)|\xi|^2}$$

for all $\xi \in \mathbb{R}^d$. Then, by definition of L_u^{d+2m},

$$\begin{aligned} i^m L_u^{d+2m}(\mathcal{F}_{\mathbb{R}^d}(f))(\eta) &= i^m i^{-m} (z/i)^{-d/2-m} \frac{1}{|\eta|^m} \int_S v(|\eta|\zeta) e^{\pi i(-1/z)||\eta|\zeta|^2} \overline{u(\zeta)} d\zeta \\ &= (z/i)^{-d/2-m} |\eta|^{\mu-m} e^{\pi i(-1/z)|\eta|^2} \langle u, v \rangle_{L^2(S)}, \end{aligned}$$

which agrees with (3.19), as desired. ☐

For later purposes we conclude this subsection with the following lemma that gives a formula for the *radial* derivatives of the functions $L_u^p f$, for $f \in C^\infty(\mathbb{R}^d)$. By definition, this is just the the one-variable derivative of $L_u^1 f(t)$ with respect to t.

Lemma 3.2. *Let $m \geq\in \mathbb{N}_0$, $d \geq 2$, $f \in C^\infty(\mathbb{R}^d)$, $u \in \mathcal{H}_m(\mathbb{R}^d)$. Then, for all $t_0 > 0$,*

$$\frac{d}{dt}\Big|_{t=t_0} L_u f(t) = \frac{-m}{t_0} L_u f(t_0) + \frac{1}{t_0^m} \int_{S^{d-1}} \langle \nabla f(t_0\zeta), \zeta \rangle \overline{u(\zeta)} d\zeta. \tag{3.20}$$

Moreover, $(L_u f)'(0) = 0$.

Proof. The formula (3.20) follows from the product rule, the chain rule and differentiation under the integral. The property $(L_u f)'(0) = 0$ follows from the fact that $L_u f$ is even on $\mathbb{R}$. ☐

3.1.3 Relations between restrictions of Schwartz functions to spheres

This is an interlude on the the image restriction map $R = R_{A,B}$, using the notation of the introduction, in the setting where $A = B = \cup_{n\in\mathbb{N}_0}\sqrt{n}S^{d-1}$. The result we prove about the image will not be used elsewhere, but contrasts the precise result given by Proposition 2.5 in the radial case.

Fix an integer $d \geq 2$. We saw in §2.3.2 how the Fourier coefficients of any $\vartheta \in M_{d/2}(\Gamma(2))$ yield a summation formula for the values $f(\sqrt{n})$ and $\hat{f}(\sqrt{n})$, $n \in \mathbb{N}_0$ for any radial $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$. By means of Proposition 3.4 we can now upgrade this result to non-radial functions $f \in \mathcal{S}(\mathbb{R}^d)$ by applying the radial summation formulas to $L_u^{d+2m}f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^{d+2m})$, for any given $u \in \mathcal{H}_m(\mathbb{R}^d)$, with $m \in \mathbb{N}_0$. Given any $\varphi \in M_{d/2+m}(\Gamma(2))$, we then have, by §2.3.2, by definition of $L_u^{d+2m}f$ and by Proposition 3.4,

$$
\begin{aligned}
0 &= \sum_{n=0}^{\infty} \widehat{\vartheta}(n) L_u^{d+2m} f(\sqrt{n}) - \sum_{n=0}^{\infty} \widehat{\vartheta|S}(n) i^m L_u^{d+2m} \hat{f}(\sqrt{n}) \\
&= \widehat{\varphi}(0) \sum_{|\alpha|=m} \frac{\partial^\alpha f(0)}{\alpha!} \int_{S^{d-1}} \overline{u(\zeta)}\zeta^\alpha d\zeta - i^m \widehat{\varphi|S}(0) \sum_{|\alpha|=m} \frac{\partial^\alpha \widehat{f}(0)}{\alpha!} \int_{S^{d-1}} \overline{u(\zeta)}\zeta^\alpha d\zeta \\
&\quad + \sum_{n=1}^{\infty} \widehat{\vartheta}(n) n^{-m/2} \int_{S^{d-1}} f(\sqrt{n}\zeta)\overline{u(\zeta)} d\zeta - i^m \sum_{n=1}^{\infty} \widehat{\vartheta|S}(n) n^{-m/2} \int_{S^{d-1}} \hat{f}(\sqrt{n}\zeta)\overline{u(\zeta)} d\zeta.
\end{aligned}
$$

We now formulate the result of this computation a bit differently and also simplify by "assuming away" the terms involving partial derivatives of f at the origin. Let $M_k^{0,\infty}(\Gamma(2)) \subseteq M_k(\Gamma(2))$ denote the subspace of $\vartheta \in M_k(\Gamma(2))$ such that $\widehat{\vartheta}(0) = 0 = \widehat{\vartheta|S}(0)$. Let $V = V_d$ denote the space of pairs of sequences $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}}$ of smooth functions $f_n, g_n : S^{d-1} \to \mathbb{C}$ whose sup norms decay rapidly with n. Let $\Phi = \Phi_d : \mathcal{S}(\mathbb{R}^d) \to V$ denote the obvious map: $\Phi(f) = \Big((f(\sqrt{n}\zeta))_{n\in\mathbb{N}}, (\hat{f}(\sqrt{n}\zeta))_{n\in\mathbb{N}}\Big)$. For each $m \in \mathbb{N}_0$ we define a linear map $\Lambda_m : M_{d/2+m}^{0,\infty}(\Gamma(2)) \otimes \mathcal{H}_m(\mathbb{R}^d) \to V^*$ by

$$
\Lambda_m(\vartheta \otimes u)((f_n),(g_n)) = \sum_{n=1}^{\infty} \widehat{\vartheta}(n) n^{-m/2} \int_S f_n(\zeta)\overline{u(\zeta)} d\zeta - i^m \sum_{n=1}^{\infty} \widehat{\vartheta|S}(n) n^{-m/2} \int_S g_n(\zeta)\overline{u(\zeta)} d\zeta
$$

where we abbreviated $S = S^{d-1}$ for ease of type setting. We have shown that the image of each Λ_m annihilates the image of Φ_d. We claim that the natural extension of the Λ_m to the direct sum $M_d = \bigoplus_{m=0}^{\infty} (M_{d/2+m}^{0,\infty}(\Gamma(2)) \otimes \mathcal{H}_m(\mathbb{R}^d))$ is injective. Indeed if an element $F \in M_d$ is such that $\Lambda(F) = 0$, then, for any fixed $n_0 \in \mathbb{N}$, $m_0 \in N_0$ and $u_0 \in \mathcal{H}_{m_0}(\mathbb{R}^d)$, we can write the vanishing $0 = \Lambda(F)((f_n),(g_n))$ for the following two inputs

$$
\begin{cases} f_n(\zeta) = \delta_{n,n_0} u_0(\zeta) \\ g_n(\zeta) = 0 \end{cases}, \qquad \begin{cases} f_n(\zeta) = 0 \\ g_n(\zeta) = \delta_{n,n_0} u_0(\zeta) \end{cases}.
$$

We then readily obtain that $F = 0$. We may now deduce:

Proposition 3.5. *In the above notation, the image $W := \Phi_d(\mathcal{S}(\mathbb{R}^d)) \subseteq V$ is annihilated by the image $A = \Lambda(M_d) \subseteq V^*$ and thus $\dim_{\mathbb{C}}(V/W) = \infty$.*

Proof. Indeed, we have $\dim_{\mathbb{C}}(M_d) = \infty$ and hence $\dim_{\mathbb{C}}(A) = \infty$ since Λ is injective. Since A naturally injects into $(V/W)^*$ we have that the latter space is infinite dimensional, in particular V/W itself is infinite dimensional. ☐

3.2 Fourier interpolation from spheres

We can finally give a precise statement of the ideas outlined at the beginning of §3.1, §3.1.2 and the overview of this chapter given in the introduction (in the case $d = 2$).

Proposition 3.6. *Fix an integer $d \geq 2$ and two arbitrary sequences $\{r_n\}_{n\in\mathbb{N}_0}, \{\rho_n\}_{n\in\mathbb{N}_0} \subseteq [0, \infty)$. Suppose that for all $p \in \{d + 2m : m \in \mathbb{N}_0\}$, all integers $n \in \mathbb{N}_0$ and real numbers $t \geq 0$, there exist complex numbers $c_{p,n}(t), \tilde{c}_{p,n}(t)$ such that for all $g \in \mathcal{S}_{rad}(\mathbb{R}^p)$ and all $v \in \mathbb{R}^p$ we have*

$$g(v) = \sum_{n=0}^{\infty} g(r_n) c_{p,n}(|v|) + \sum_{n=0}^{\infty} (\mathcal{F}_{\mathbb{R}^p} g)(r_n) \tilde{c}_{p,n}(|v|)$$

and both of these series converge (not necessarily absolutely). Then, for every $f \in \mathcal{S}(\mathbb{R}^d)$ and every $x \in \mathbb{R}^d$, we have

$$\begin{aligned} f(x) = \sum_{m=0}^{\infty} \Big(\sum_{n=0}^{\infty} c_{d+2m,n}(|x|) \frac{1}{r_n^m} \int_S f(r_n\zeta) Z_m^d(x, \zeta) d\zeta \\ + \sum_{n=0}^{\infty} i^m \tilde{c}_{d+2m,n}(|x|) \frac{1}{\rho_n^m} \int_S \widehat{f}(\rho_n\zeta) Z_m^d(x, \zeta) d\zeta \Big) \end{aligned}$$

where, if $r_n = 0$ or $\rho_m = 0$, the integrals are defined by Proposition 3.3 and where we recall that $Z_m^d(x, y) = \sum_{u\in\mathcal{B}_m} u(x)\overline{u(y)}$ denotes the reproducing kernel on the space of spherical harmonics of degree m (where $\mathcal{B}_m$ can be any orthonormal basis). The series on the right converges in the indicated order of summation and such that $\sum_{m=0}^{\infty} |(\cdots)| < \infty$.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d \smallsetminus \{0\}$ be arbitrary (we consider the case $x = 0$ at the end). Recall the general expansion of $f(x)$ in terms of $L_u f$ given in (3.11). Here, we rewrite it in the form

$$f(x) = \sum_{m=0}^{\infty} \sum_{u\in\mathcal{B}_m} u(x) L_u^{d+2m} f(\iota_m(x)) \tag{3.21}$$

where $\iota_m(x) = (x, 0) \in \mathbb{R}^{d+2m}$ is the vector in $\mathbb{R}^{d+2m}$ whose first d coordinates are given by that of x and whose last $2m$ coordinates are all set to zero and where $L_u^{d+2m} f \in \mathcal{S}_{\text{rad}}(\mathbb{R}^{d+2m})$ is defined as in Definition 3.2 (and is indeed a radial Schwartz function by Proposition 3.4). Here, we could have replaced $L_u^{d+2m} f(\iota_m(x))$ by $L_u^p f(\iota_p(x))$ for any $p = p(m) \in \mathbb{N}$ and any $\iota_p(x) \in \mathbb{R}^p$ with the same norm as x, without changing the value of this expression. With the choice $p(m) = d + 2m$ we can write, using our assumption,

$$L_u^{d+2m} f(\iota_m(x)) = \sum_{n=0}^{\infty} L_u^{d+2m} f(r_n) c_{d+2m,n}(|x|) + \sum_{n=0}^{\infty} (\mathcal{F}_{\mathbb{R}^{d+2m}} L_u^{d+2m} f)(r_n) \tilde{c}_{d+2m,n}(|x|)$$

and then, by Proposition 3.4, rewrite the Fourier transform of $L_u^{d+2m} f$ as $i L_u^{d+2m} \hat{f}$. This finishes the proof for $x \neq 0$. For $x = 0$, the sum over $m \in \mathbb{N}_0$ in the claimed formula reduces to the single term $m = 0$ (since $Z_m^d(0, \zeta) = 0$ for all $m \geq 1$) and we thus need to show that

$$f(0) = \sum_{n=0}^{\infty} c_{d,n}(0) \int_S f(r_n\zeta) d\zeta + \sum_{n=0}^{\infty} \tilde{c}_{d,n}(0) \int_S \widehat{f}(\rho_n\zeta) d\zeta.$$

This last identity holds since it is the formula that expresses the radial function $L_1^d f$ at the point zero, using the assumed radial interpolation formula (here, the "1" in L_1^d means the constant polynomial 1). $\square$

Besides deducing Fourier interpolation formulas for general Schwartz functions in $\mathbb{R}^d$ from Fourier interpolation formulas for radial Schwartz functions in $\mathbb{R}^{d+2m}$, $m \in \mathbb{N}_0$, we can prove an analogous result that only addresses the Fourier *uniqueness* aspect.

Proposition 3.7. *Fix $d \geq 2$ and fix two sequences $(r_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ of positive real numbers. Suppose that for all $m \in \mathbb{N}_0$ the pair*

$$\left(\bigcup_{n\in\mathbb{N}} r_n S^{d+2m-1}, \bigcup_{n\in\mathbb{N}} \rho_n S^{d+2m-1}\right) \tag{3.22}$$

is a Fourier uniqueness set for $\mathcal{S}_{rad}(\mathbb{R}^{d+2m})$. Then it is also a Fourier uniqueness pair for $\mathcal{S}(\mathbb{R}^{d+2m})$ for all $m \in \mathbb{N}_0$.

Proof. It suffices to prove that (3.22) is a Fourier uniqueness pair for $\mathcal{S}(\mathbb{R}^{d+2m})$ in the case $m = 0$. So let $f \in \mathcal{S}(\mathbb{R}^d)$ be such that $f(r_n S^{d-1}) = \{0\} = \hat{f}(\rho_n S^{d-1})$ for all $n \in \mathbb{N}$. Fix $x \in \mathbb{R}^d \setminus \{0\}$. We aim to show that $f(x) = 0$. To do so, it suffices, by the general expansion (3.21), to show that each radial function $L_u^{d+2m} f \in \mathcal{S}(\mathbb{R}^{d+2m})$ vanishes identically. To show that, it suffices by assumption to show that $L_u^{d+2m} f(r_n) = 0$ and $0 = \mathcal{F}_{\mathbb{R}^{d+2m}}(L_u^{d+2m} f)(\rho_n) = i^{-m} L_u^{d+2m}(\hat{f})(\rho_n)$ for all $n \in \mathbb{N}, m \in \mathbb{N}_0$ and $u \in \mathcal{H}_m(\mathbb{R}^d)$ (here, we used Proposition 3.4). But this holds by assumption and the assumption of $L_u^{d+2m} f$ (as the integrand vanishes identically). $\square$

Remark 3.1. We could also allow ρ_n or r_n to be equal to zero in the statement of Proposition 3.7 at the cost of using also some partial derivatives of f and $\hat{f}$ at the origin, working with the the definition of $L_u^p f(0)$.

In a similar spirit, we could allow the assumed radial interpolation or -uniqueness result on $\mathbb{R}^{d+2m}$ to involve first order radial derivatives, by means of Lemma 3.2. We will use this in §3.4 below, where we will discuss a possible application to the uniqueness of the so-called magic functions for the sphere packing problem.

Corollary 3.2. *Let $d \geq 2$ and let $n_0, \hat{n}_0$ be integers such that $n_0 + \hat{n}_0 = 1 + \lfloor d/4 \rfloor$. Then*

$$\left(\bigcup_{n\geq n_0} \sqrt{n} S^{d-1}, \bigcup_{n\geq \hat{n}_0} \sqrt{n} S^{d-1}\right) \tag{3.23}$$

is a Fourier uniqueness pair for $\mathcal{S}(\mathbb{R}^d)$. If $d \geq 5$, then, for all integers $N \geq 1$ and all real numbers $h > 2$, the pair

$$\left(\bigcup_{n\geq N} \sqrt{2n/h} S^{d-1}, \bigcup_{n\geq N} \sqrt{2n/h} S^{d-1}\right) \tag{3.24}$$

is a Fourier uniqueness pair for $\mathcal{S}(\mathbb{R}^d)$.

Proof. The first assertion follows from Proposition 3.7 and Theorem 1. The second from part (iv) of Theorem 2 together with Proposition 3.7. $\square$

Remark 3.2. Given that we can remove an arbitrary number of spheres from (3.24), this Fourier uniqueness pair is not tight in any sense. While (3.23) is a tight uniqueness pair for *radial* Schwartz functions, it is not tight in the larger space $\mathcal{S}(\mathbb{R}^d)$. Indeed, as we will explain after Proposition 3.8, we may replace finitely many of the first few spheres with finite subsets contained in them. Furthermore, as will see in Theorem 4, we can, in fact, replace all spheres by discrete subsets contained in them.

3.2.1 Main theorem

In fact, we can upgrade Corollary 3.2 to the following Fourier interpolation result, which can be seen as the main result of this thesis.

Theorem 3. *Let $d \geq 2$ be an integer. There exist smooth functions $A_n^d, \tilde{A}_n^d : \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \to \mathbb{C}$ and radial Schwartz functions $c_d, \tilde{c}_d : \mathbb{R} \to \mathbb{C}$ such that for all $f \in \mathcal{S}(\mathbb{R}^d)$ and for all $x \in \mathbb{R}^d$ we have*

$$f(x) = c_d(|x|)f(0) + \sum_{n=1}^{\infty} \int_{S^{d-1}} A_n^d(x,\zeta) f(\sqrt{n}\zeta) d\zeta + \tilde{c}_d(|x|)\hat{f}(0) + \sum_{n=1}^{\infty} \int_{S^{d-1}} \tilde{A}_n^d(x,\zeta)\hat{f}(\sqrt{n}\zeta)d\zeta \tag{3.25}$$

and both series converge absolutely. The partial sums on the right hand side converge rapidly in $C^\infty(\mathbb{R}^d \setminus \{0\})$ to the left hand side. If $d \geq 4$, then we can take $c_d = 0 = \tilde{c}_d$.

The essence of the proof of this theorem is to combine Theorems 1 and 2 with Proposition 3.6 and to interchange the sums and integrals in the statement of that proposition. Let us now explain this sketch more precisely, not yet paying too close attention to convergence issues. Those will be dealt with later.

Fix $d \geq 2$. We define two sequences of functions $c_{d,n}(r), \tilde{c}_{d,n}(r)$, indexed by $n \in \mathbb{N}_0$, as follows:

- If $d \geq 5$, we define
$$c_{d,n}(r) = b_{d/2,2,n}(r), \quad \tilde{c}_{d,n} = \tilde{b}_{d/2,2,n}(r)$$
where $b_{k,h,n}, \tilde{b}_{k,h,n}$ are functions having the properties stated in Theorem 2. These are defined for all $r \in \mathbb{C}$, but in this section, we only need them when $r \in \mathbb{R}$.

- If $2 \leq d \leq 4$, we define
$$c_{d,n}(r) = a_{d/2,n}(r), \qquad \tilde{c}_{d,n} = \tilde{a}_{d/2,n}(r),$$
to be the functions in Theorem 1, with parameters
$$n_0 = n_0(d) = 1 + \lfloor d/8 \rfloor, \qquad \hat{n}_0 = \hat{n}_0(d) = \lfloor (d+4)/8 \rfloor = \hat{n}_0(d). \tag{3.26}$$
(i.e. $n_0(2) = n_0(3) = n_0(4) = 1$, $\hat{n}_0(2) = 0 = \hat{n}_0(3)$, $\hat{n}_0(4) = 1$.)

Let us then apply Proposition 3.6 with $\rho_n = \sqrt{n} = r_n$. Fix $x \in \mathbb{R}^d$ and write $u_{m,x}(y) = Z_m^d(x,y)$. Abbreviate also $S = S^{d-1}$ (to mildly increase the chances of making formulas fit on one line). Then, for any $f \in \mathcal{S}(\mathbb{R}^d)$, we have

$$f(x) = \sum_{m=0}^{\infty} \Big(\sum_{n=0}^{\infty} c_{d+2m,n}(|x|) L_{u_{m,x}}^{d+2m} f(\sqrt{n}) + \sum_{n=0}^{\infty} i^m \tilde{c}_{d+2m,n}(|x|) L_{u_{m,x}}^{d+2m} \hat{f}(\sqrt{n}) \Big) \tag{3.27}$$

$$= \sum_{m=0}^{\infty} \Big(c_{d+2m,0}(|x|) L_{u_{m,x}}^{d+2m} f(0) + i^m \tilde{c}_{d+2m,0}(|x|) L_{u_{m,x}}^{d+2m} \hat{f}(0) \Big) \tag{3.28}$$

$$+ \sum_{m=0}^{\infty} \Big(\sum_{n=1}^{\infty} c_{d+2m,n}(|x|) L_{u_{m,x}}^{d+2m} f(\sqrt{n}) + \sum_{n=1}^{\infty} i^m \tilde{c}_{d+2m,n}(|x|) L_{u_{m,x}}^{d+2m} \hat{f}(\sqrt{n}) \Big). \tag{3.29}$$

In the sum (3.28) we note that, if $m \geq 1$, then $d + 2m \geq 2 + 2 = 4$ and so $n_0(d + 2m) \geq 1$ and $\hat{n}_0(d + 2m) \geq 1$, so that this sum collapses to the term with $m = 0$. Furthermore, if $d = 4$, also the terms with $m=0$ vanish (since $c_{4,0}=0=\tilde{c}_{4,0}$). Interchanging sums and integrals formally in the other sum (3.29), we thus obtain

$$f(x) = c_{d,0}(|x|)f(0) + \tilde{c}_{d,0}(|x|)\hat{f}(0) + \sum_{n=1}^{\infty}\int_S \Big(\sum_{m=0}^{\infty} c_{d+2m,n}(|x|)n^{-m/2}Z_m^d(x,\zeta)\Big)f(\sqrt{n}\zeta)d\zeta$$
$$+\sum_{n=1}^{\infty}\int_S \Big(\sum_{m=0}^{\infty} i^m\tilde{c}_{d+2m,n}(|x|)n^{-m/2}Z_m^d(x,\zeta)\Big)\hat{f}(\sqrt{n}\zeta)d\zeta.$$

This (formal) computation makes it quite clear that the functions $A_n^d, \tilde{A}_n^d$ entering Theorem 3 should be given by the following (formal) series

$$A_n^d(x,y) = \sum_{m=0}^{\infty} c_{d+2m,n}(|x|)n^{-m/2}Z_m^d(x,y), \tag{3.30}$$
$$\tilde{A}_n^d(x,y) = \sum_{m=0}^{\infty} \tilde{c}_{d+2m,n}(|x|)n^{-m/2}Z_m^d(x,y) \tag{3.31}$$

and the proof of Theorem 1 consists in verifying that all involved series and integrals converge absolutely, so that the computations just presented can be justified.

Before we do this, let us pause to record an observation which holds regardless of such subtleties.

Proposition 3.8. *For $N\in\mathbb{N}_0$ and $d\geq 2$, let $V_d(N)$ denote the space of all $f\in\mathcal{S}(\mathbb{R}^d)$ such that $f(\sqrt{n}\zeta)=\hat{f}(\sqrt{n}\zeta)=0$ all integers $n\geq N$ and all $\zeta\in S^{d-1}$. Then $V_d(N)$ is finite dimensional. In fact, there is a finite dimensional space $W\subseteq\mathcal{S}_{rad}(\mathbb{R}^d)$ (depending only on d and N) such that any $f\in V_d(N)$ is a finite linear combination of functions of the form $u(x)b(x)$ for some $b\in W$ and $u\in\mathcal{H}_m(\mathbb{R}^d)$ with $m\leq 4N$.*

Proof. Exceptionally for this proof, we redefine the functions $c_{d,n}, \tilde{c}_{c,n}$ to be the functions $a_{d/2,n}, \tilde{a}_{d/2,n}$ given by Theorem 1 **for all** $d\geq 2$, with $n_0=n_0(d)$ and $\hat{n}_0=\hat{n}_0(d)$ given as in (3.26).

Suppose $f\in V_d(N)$ and apply formula (3.27) to f (with these new functions $c_{d,n}$, $\tilde{c}_{d,n}$). Since $f(0)=\hat{f}(0)=0$, the sum in (3.28) vanishes. For the terms in (3.29), we note that

$$\begin{aligned} m\geq 4n+1 \quad &\Longrightarrow \quad n\leq \frac{m-1}{4} = \frac{d+2m-2-d}{8} \leq \frac{d+2m+4}{8}-1 < \hat{n}_0(d+2m) \\ &\Longrightarrow \quad c_{d,n}=0=\tilde{c}_{d,n}. \end{aligned}$$

It follows that the sums in (3.29) reduce to finite sums and that f is a linear combination of tensors $b\otimes u$, where b is one the functions $c_{d+2m,n}$ or $c_{d+2m,n}$ with $n\leq N$ and $m\leq 4N$ and $u\in\mathcal{H}_m$ with $m\leq 4N$. ☐

Proposition 3.8 implies that the first uniqueness pair in Corollary 3.2 is *not* tight, as we will now explain. Fix an integer $N\geq 1+\lfloor d/4\rfloor$. Fix a finite subset $D=D_N\subseteq S^{d-1}$ with the property that for all $m\leq 4N+4$ and all $u\in\mathcal{H}_m(\mathbb{R}^d)$ we have $u|_D=0\Leftrightarrow u=0$. The existence of D_N follows from general linear algebra principles[21]. Consider any $f\in\mathcal{S}(\mathbb{R}^d)$ such that f and $\hat{f}$ vanish on all spheres $\sqrt{n}S^{d-1}$ for integers $n>N$ and such that f and $\hat{f}$ vanish on all of the finite sets $\sqrt{n}D_n$ where $1+\lfloor d/4\rfloor\leq n\leq N$. We claim that $f=0$. By Proposition 3.8, we have $f(x)=\sum_{j\in J}b_j(x)u_j(x)$ for

[21]For any finite-dimensional vector space X and any collection of subspaces $Y_i\subseteq X$, $i\in I$ such that $\operatorname{codim}(Y_i)=1$ for all $i\in I$ and such that $\cap_{i\in I}Y_i=\{0\}$, there is a finite subset $J\subseteq I$ such that $\cap_{j\in J}Y_j=\{0\}$. Apply this to $X=\oplus_{m\leq 4N}\mathcal{H}_m$, to $I=S^{d-1}$, and to the subpsaces $Y_\omega\subseteq X$, $\omega\in I$, consisting of all $u\in X$ such that $u(\omega)=0$.

some finite set J, some harmonic polynomials u_j of degree at most $4N$ and some radial Schwartz functions b_j. We also know that the Fourier transform $\widehat{f} = \sum_{j\in J} \tilde{b}_j(x)u_j(x)$ takes the same form (where $\tilde{b}_j$ is the Fourier transform of b_j on $\mathbb{R}^{d+2\deg(u_j)}$ up to a power of i, by Bochner's formula [Boc51]). Now let n be an integer such that $1 + \lfloor d/4 \rfloor \leq n \leq N$. By assumption, we have that for each $\omega \in D_N$,

$$0 = f(\sqrt{n}\omega) = \sum_{j\in J} b_j(\sqrt{n})n^{\deg(u_j)/2}u_j(\omega).$$

It follows that this holds in fact for all $\omega \in S^{d-1}$ and similarly for $\hat{f}$. By Corollary 3.2, we must have $f = 0$, as claimed.

3.2.2 Proof of Theorem 3

We now turn to the proof of Theorem 3. In order not to complicate the presentation too much, we will first consider the case $d \geq 5$ where we explain all steps in detail. We will explain the modifications that are needed for the cases $d \in \{2, 3, 4\}$ in §3.2.3. Note that, since $d \geq 5$, all the functions $c_{d,n}, \tilde{c}_{d,n}$ come from Theorem 2.

We first show that the series $A^d_n(x, y)$ and $\tilde{A}^d_n(x, y)$ defined in (3.30) and (3.31) converge absolutely and define smooth functions of $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$. Once this this is done, it will be easy to show that for any (merely) continuous function $f : \mathbb{R}^d \to \mathbb{C}$, the integrals $\int_{S^{d-1}} f(\sqrt{n}\zeta)A_n(x, \zeta)d\zeta$ (or those involving $\hat{f}$) define smooth functions of $x \in \mathbb{R}^d$ and estimate its derivatives in terms of the sup norm of f (or $\hat{f}$) over $\sqrt{n}S^{d-1}$. We then prove the interpolation formula (3.25) point-wise by rewriting those integrals as the RHS of (3.29). In other words, we will do the above formal calculations "backwards" (and make them rigorous). Since we also keep track of all partial derivatives, various technicalities arise, which might make the proof a bit hard to read. In case the reader shares this feeling, we recommend to first look at the sketch we give in §3.2.3 for $d \in \{2, 3, 4\}$ below, where we explain the argument without derivatives.

To quantify absolute and uniform convergence of partial derivatives in x and y of the series $A^d_n(x, y)$ and $\tilde{A}^d_n(x, y)$, we introduce the following notation. For all sextuples of parameters

$$T = (n, \alpha, \beta, \delta, R, s) \in \mathcal{T} := \mathbb{N} \times \mathbb{N}^d_0 \times \mathbb{N}^d_0 \times [0, \infty) \times [0, \infty) \times (0, 1], \tag{3.32}$$

satisfying $\delta \leq R$ and for each $m \in \mathbb{N}_0$, we define[22]

$$S_m(T) = \sup_{\substack{\delta\leq|x|\leq R\\ s\leq|y|\leq s^{-1}}} \left|\partial^\alpha_x \partial^\beta_y c_{d+2m,n}(|x|)Z^d_m(x, y)n^{-m/2}\right|.$$

We define $\tilde{S}_m(T)$ analogously by replacing $c_{d+2m,n}$ by $\tilde{c}_{d+2m,n}$. Let us also define, for each $T \in \mathcal{T}$,

$$\mathcal{A}(T) = \sum_{m=0}^{\infty} S_m(T), \qquad \tilde{\mathcal{A}}(T) = \sum_{m=0}^{\infty} \tilde{S}_m(T).$$

The following Lemma will give bounds for these quantities.

Lemma 3.3. *Assume $d \geq 5$. Fix multi-indices $\alpha, \beta \in \mathbb{N}^d_0$.*

[22]The definition of $S_m(T)$ comes with an obvious abuse of notation in that the RHS does not seem to depend upon the LHS. We ask the reader to make the association $n = n_T$, $\alpha = \alpha_T$, etc. themselves. The same comments apply to $\mathcal{A}(T)$ and other notations involving tuples $T \in \mathcal{T}$.

(i) For every $s \in (0,1]$, $R > 0$ and $n \in \mathbb{N}$, the tuple $T = (n, \alpha, \beta, 0, R, s)$ satisfies $\mathcal{A}(T) < \infty$ and $\tilde{\mathcal{A}}(T) < \infty$. Note here that $\delta = 0$.

(ii) For all $0 < \delta < R < \infty$, there exists a constant $C > 0$, depending on δ, α, R and d, such that for every $n \in \mathbb{N}$, the tuple $T = (n, \alpha, 0, \delta, R, 1)$ satisfies

$$\max\left((\mathcal{A}(T), \tilde{\mathcal{A}}(T)\right) \leq C n^{\frac{5d}{4} + \frac{1}{8} + |\alpha|}.$$

(Note here that $s = 1$, $\beta = 0$, so we restrict attention to the sup over $y \in S = S^{d-1}$ without partial derivatives in that variable).

For the the proof and the rest of this section, let us choose orthonormal bases $\mathcal{B}_m$ of each of the spaces $\mathcal{H}_m$ and recall that $Z_m^d(x,y) = \sum_{u \in \mathcal{B}_m} u(x)\overline{u(y)}$.

Proof of Lemma 3.3. To be able to refer to them later, let us first record the following computations, which follow directly from the generalized Leibniz rule

$$\partial_x^\alpha \partial_y^\beta c_{d+2m,n}(|x|) Z_m^d(x,y) = \sum_{\gamma_1+\gamma_2=\alpha} \frac{\alpha!}{\gamma_1!\gamma_2!} \partial_x^{\gamma_1} c_{d+2m,n}(|x|) \partial_x^{\gamma_2} \partial_y^\beta Z_m^d(x,y) \tag{3.33}$$

$$= \sum_{u \in \mathcal{B}_m} \partial_y^\beta \overline{u(y)} \sum_{\gamma_1+\gamma_2=\alpha} \frac{\alpha!}{\gamma_1!\gamma_2!} \partial_x^{\gamma_1} c_{d+2m,n}(|x|) \partial_x^{\gamma_2} u(x). \tag{3.34}$$

Whenever an estimate below involves the γ_2th or βth derivative of a harmonic polynomial of degree m, we may assume that $|\gamma_2| \leq m$ or $|\beta| \leq m$, as otherwise the derivative vanishes. For the reader's and author's convenience, we rewrite the bounds (2.92) and (2.93) from Theorem 2 in a convenient form here. These bounds say that for each $R \geq 1$ and each multi-index $\gamma \in \mathbb{N}_0^d$, there are constants $C_4, C_6 > 0$ depending only on d, γ and R such that for all $x \in \mathbb{R}^d$ with $|x| \leq R$ we have

$$|\partial_x^\gamma c_{d+2m,n}(|x|)| + |\partial_x^\gamma \tilde{c}_{d+2m,n}(|x|)| \leq C_4 \left(\frac{2C_5}{d+2m}\right)^{d/4+m/2} n^{d/2+m+|\gamma|} \tag{3.35}$$

and for $0 < |x| \leq R$ we have

$$|\partial_x^\gamma c_{d+2m,n}(|x|)| + |\partial_x^\gamma \tilde{c}_{d+2m,n}(|x|)| \leq C_6 n^{d/4+m/2+9/8+|\gamma|} |x|^{-d/2-m+9/4} \tag{3.36}$$

Here, $C_5 \geq 1$ is an absolute constant (in our paper [Sto21], we computed that $C_5 = 47$ is admissible).

Now let us turn to the proof of part (i), which will basically follow from the rapid decay with respect to the parameter m in (3.35). So let $\alpha, \beta \in \mathbb{N}_0^d$, $R > 0$, $s \in (0,1]$, $n \in \mathbb{N}$ be as in (i) and let $T = (n, \alpha, \beta, 0, R, s)$ be the tuple under consideration. We wish to obtain a bound for $S_m(T)$ which is summable over $m \in \mathbb{N}_0$. Looking at (3.34) we first bound, for any $\gamma_1, \gamma_2 \in \mathbb{N}_0$ such that $\gamma_1 + \gamma_2 = \alpha$, and for $|x| \leq R$, $s \leq |y| \leq s^{-1}$,

$$\begin{aligned}\left|\partial_y^\beta \overline{u(y)} \partial_x^{\gamma_1} c_{d+2m,n}(|x|) \partial_x^{\gamma_2} u(x)\right| &= |y|^{|\beta|-m} \sup_{\zeta_1 \in S} |\partial^\beta u(\zeta_1)| |\partial_x^{\gamma_1} c_{d+2m,n}(|x|)| |x|^{|\gamma_2|-m} \sup_{\zeta_2 \in S} |\partial^{\gamma_2} u(\zeta_2)| \\ &\lesssim_{d,\gamma_2,\beta,n,s,R} \left(s^{-m}(1+m)^{|\beta|+d-2}\right) m^{-m/2} (1+m)^{|\gamma_2|+d-2}\end{aligned}$$

where we used Lemma 3.1 combined with the standard L^∞-L^2 norm bound for harmonic polynomials (3.5). Summing that bound over γ_1, γ_2 and then over $u \in \mathcal{B}_m$ we obtain (using that $|\mathcal{B}_m| = \dim(\mathcal{H}_m) \lesssim m^{d-2}$)) a bound for $S_m(T)$ of the form

$$S_m(T) \lesssim X^m m^Y m^{-m/2},$$

where $X, Y > 0$ and the implied constant depends on T (in particular on n). This proves part (i) (the modifications for $\tilde{c}_{d+2m,n}$ are clear).

For the proof of part (ii) we proceed similarly but we will track the dependence on n more carefully.

Let $0 < \delta < R < \infty$ and set $T = (n, \alpha, 0, \delta, R, 1)$. We may and will assume that $\delta < 1 \leq R$. Let $M \geq 1$ be an integral parameter, to be chosen later. We define start and end[23] sums

$$\mathcal{A}_{\mathrm{start}}(T) = \sum_{m=0}^{M} S_m(T), \qquad \mathcal{A}_{\mathrm{end}}(T) = \sum_{m=M+1}^{\infty} S_m(T).$$

We begin with the analysis of $\mathcal{A}_{\mathrm{end}}$, which is similar to the proof of part (i) and we will not yet use that $|x| \geq \delta$. As in the proof of part (i), we use Lemma 3.1 to bound the derivatives with respect to x of $Z_m^d(x, y)$ appearing in (3.33) by

$$\begin{aligned} |\partial_x^{\gamma_2} Z_m^d(x,\zeta)| &\ll_{d,|\gamma_2|} |x|^{m-|\gamma_2|}(m-|\gamma_2|)^{\frac{d-2}{2}} m^{|\gamma_2|} \|Z_m^d(\cdot,\zeta)\|_{L^2(S)} \\ &\ll_{d,|\gamma_2|} |x|^{m-|\gamma_2|} m^{d-2+|\gamma_2|}, \end{aligned} \tag{3.37}$$

where we used that $\|Z_m^d(\cdot,\zeta)\|_{L^2(S)}^2 = \dim \mathcal{H}_m(\mathbb{R}^d)$ and where the implied constants depend neither on x, nor on ζ. We have $|x|^{m-|\gamma_2|} \leq R^m$ in (3.37) and combined with (3.35) we see that

$$\begin{aligned} \mathcal{A}_{\mathrm{end}}(T) &\lesssim_{d,R,\alpha} \sum_{m=M+1}^{\infty} n^{-m/2} \left(\frac{C_5}{d+2m}\right)^{d/4+m/2} n^{\frac{d+2m}{2}} R^m m^{d-2} \sum_{\gamma_1+\gamma_2=\alpha} \frac{\alpha!}{\gamma_1!\gamma_2!} n^{|\gamma_1|} m^{|\gamma_2|} \\ &\lesssim_{d,R,\alpha} n^{d/2+|\alpha|} \sum_{m=M+1}^{\infty} \left(\frac{C_5 R^2 n}{d+2m}\right)^{m/2} m^{d-2}(1+m)^{|\alpha|}, \end{aligned}$$

where we absorbed the term $(C_5/(d+2m))^{d/4} \lesssim_d 1$ into the implied constant and used that the inner sum over γ_1, γ_2 is equal to

$$(n+m)^{|\alpha|} = (n(1+m/n))^{|\alpha|} \leq n^{|\alpha|}(1+m)^{|\alpha|}.$$

We now take $M = \lfloor C_5 R^2 n \rfloor + 2$. Then $\frac{C_5 R^2 n}{d+2m} \leq \frac{1}{2}$ for all $m \geq M+1$ and hence

$$\mathcal{A}_{\mathrm{end}}(T) \lesssim_{d,R,\alpha} n^{d/2+|\alpha|} \sum_{m=1}^{\infty} 2^{-m/2} m^{d-2}(1+m)^{|\alpha|} \ll_{d,\alpha,R} n^{d/2+|\alpha|}.$$

It remains to bound the finite sum $\mathcal{A}_{\mathrm{start}}(T)$. At this point, the restriction $|x| \geq \delta > 0$ becomes important. By (3.36) we have, for $\delta \leq |x| \leq R$,

$$|\partial^{\gamma_1} c_{d+2m,n}(|x|)| \lesssim_{\gamma_1,R} n^{9/8+d/4+m/2+|\gamma_1|} |x|^{-d/2-m+9/4}. \tag{3.38}$$

Crucially, the term $n^{m/2}$ in (3.38) cancels with the term $n^{-m/2}$ in the definition of $S_m(T)$ and the term $|x|^{-m}$ in (3.38) cancels with $|x|^m$ in (3.37). This implies

$$\begin{aligned} \mathcal{A}_{\mathrm{start}}(T) &\lesssim_{d,R,\alpha} \sum_{m=0}^{M} \sup_{\delta \leq |x| \leq R} \sum_{\gamma_1+\gamma_2=\alpha} \frac{\alpha!}{\gamma_1!\gamma_2!} n^{9/8+d/4+|\gamma_1|} |x|^{-d/2+9/4} |x|^{-|\gamma_2|} m^{d-2+|\gamma_2|} \\ &\leq \left(\sup_{\delta \leq |x| \leq R} |x|^{-d/2+9/4}\right) n^{d/4+9/8} \sum_{m=0}^{M} (n+m/\delta)^{|\alpha|} m^{d-2}. \end{aligned} \tag{3.39}$$

[23] the words "head" and "tail" could also be used

For $m \le M$ we can bound

$$(n+m/\delta)^{|\alpha|} = n^{|\alpha|}\delta^{-|\alpha|}(\delta + \tfrac{m}{n})^{|\alpha|} \le n^{|\alpha|}\delta^{-|\alpha|}(1+\tfrac{C_5 R^2 n+2}{n})^{|\alpha|} \lesssim_{R,\alpha} n^{|\alpha|}.$$

Inserting this into (3.39), we get

$$\mathcal{A}_{\mathrm{start}}(T) \lesssim_{d,R,\delta,\alpha} n^{d/4+9/8+|\alpha|}(M+1)M^{d-2} \lesssim_{R,d} n^{d/4+9/8+|\alpha|+(d-1)} = n^{5d/4+1/8+|\alpha|}.$$

Thus $\mathcal{A}_{\mathrm{start}}(T)$ dominates $\mathcal{A}_{\mathrm{end}}(T)$ and this proves part (ii). $\square$

Let us now turn to the proof of Theorem 3 (still assuming $d \ge 5$). First of all, by part (i) of Lemma 3.3, the series defining $A_n^d(x,y)$, $\tilde{A}_n^d(x,y)$ converge absolutely and define smooth functions of $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$. Next, we note that if $x = 0$, then $A_n^d(0,y) = c_{d,n}(0)$ and $\tilde{A}_n^d(0,y) = \tilde{c}_{d,n}(0)$ for all $n \in \mathbb{N}$ and the formula (3.25) reads

$$f(0) = \sum_{n=1}^{\infty} c_{d,n}(0) \int_{S^{d-1}} f(\sqrt{n}\zeta)d\zeta + \sum_{n=1}^{\infty} \tilde{c}_{d,n}(0) \int_{S^{d-1}} \hat{f}(\sqrt{n}\zeta)d\zeta,$$

which holds for all $f \in \mathcal{S}(\mathbb{R}^d)$ by Theorem 2 applied to the radial Schwartz function $L_1^d f(x) = \int_{S^{d-1}} f(|x|\zeta)d\zeta$; we already encountered this "collapse" in the proof of Proposition 3.6.

Next, we define for $f \in \mathcal{S}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$, the integrals

$$I_n(f)(x) := \int_{S^{d-1}} A_n^d(x,\zeta) f(\sqrt{n}\zeta)d\zeta, \qquad \tilde{I}_n(\hat{f})(x) := \int_{S^{d-1}} A_n^d(x,\zeta)\hat{f}(\sqrt{n}\zeta)d\zeta. \tag{3.40}$$

By part (i) of Lemma 3.3, each of these defines a smooth functions of $x \in \mathbb{R}^d$. We next verify that (3.25) holds point-wise for any fixed $x \in \mathbb{R}^d \setminus \{0\}$. Define $U_n(f) := \sup_{\zeta \in S^{d-1}} |f(\sqrt{n}\zeta)|$. Trivially, we have

$$|A_n^d(x,\zeta) f(\sqrt{n}\zeta)| \le U_n(f) \sum_{m=0}^{\infty} |c_{d+2m,n}(|x|) n^{-m/2} Z_m^d(x,\zeta)| \le U_n(f)\mathcal{A}(n,0,0,|x|,|x|,1)$$

for all $n \in \mathbb{N}$ and $\zeta \in S^{d-1}$ and therefore

$$|I_n(f)(x)| \le U_n(f)\mathcal{A}(n,0,0,|x|,|x|,1) \lesssim_d U_n(f) n^{5d/4+1/8}. \tag{3.41}$$

It follows that for all $n \in \mathbb{N}$, we have

$$I_n(f)(x) = \sum_{m=0}^{\infty} c_{d+2m,n}(|x|) \frac{1}{\sqrt{n}^m} \int_{S^{d-1}} f(\sqrt{n}\zeta) Z_m^d(x,\zeta)d\zeta$$

and this series converges absolutely. Since $U_n(f) \lesssim_{f,B} n^{-B}$ for any given $B > 0$, we also see that the double series

$$\sum_{n=1}^{\infty} I_n(f)(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{d+2m,n}(|x|) \frac{1}{\sqrt{n}^m} \int_{S^{d-1}} f(\sqrt{n}\zeta) Z_m^d(x,\zeta)d\zeta$$

converges absolutely, so that its value is unchanged if we first sum over m and then over n. It is clear that all the same bounds and convergence statements hold for $\tilde{I}_n(\hat{f})(x)$ (replacing $U_n(f)$ by $U_n(\hat{f})$, which is also rapidly decaying). It follows that

$$\sum_{n=1}^{\infty} I_n(f)(x) + \sum_{n=1}^{\infty} \tilde{I}_n(\hat{f})(x)$$

is equal to the RHS of (3.29), hence equal to the LHS of that equation, which is $f(x)$ (note that (3.28) vanishes since $d \geq 5$). As for the uniform convergence of partial derivatives, note that we we can bound, similarly to (3.41),

$$\sup_{\delta \leq |x| \leq R} \partial_x^\alpha I_n(f)(x) \lesssim_{d,\alpha} U_n(f)\mathcal{A}(n,\alpha,0,\delta,R,1) \lesssim U_n(f) n^{5d/4+1/8}$$

by part (ii) of Lemma 3.3, for all fixed $\alpha \in \mathbb{N}_0^d$ fixed $f \in \mathcal{S}(\mathbb{R}^d)$ and fixed $0 < \delta \leq R < \infty$. This finishes the proof of Theorem 3 (in the case $d \geq 5$).

Remark 3.3 (Reformulation of the proof)**.** We could also have proved Theorem 1 in the following way, which is a little bit more direct and does not rely on Proposition 3.6. For $x \in \mathbb{R}^d \smallsetminus \{0\}$, the above estimates imply that the RHS of (3.25) converges absolutely and defines a tempered distribution on $\mathcal{S}(\mathbb{R}^d)$. Thus, it suffices to prove equality with the tempered distribution $f \mapsto f(x)$ on a generating set of a dense subspace of $f \in \mathcal{S}(\mathbb{R}^d)$. To that end, it suffices by Corollary 3.1 to consider functions of the form $f(x) = u(x)e^{\pi i z|x|^2}$ for $z \in \mathbb{H}$ and $u \in \mathcal{B}_m$. In that case, by orthogonality of spherical harmonics and by Proposition 3.1, the desired equality reduces to the validity of the radial interpolation formula on $\mathbb{R}^{d+2m}$ for the Gaussian – in other words, to the functional equation $F_{k,2}(z,|x|) + (z/i)^k \tilde{F}_{k,2}(-1/z,|x|) = e^{\pi i z|x|^2}$, where $k = d/2 + m$. The proof of the formula for $x = 0$ remains the same.

3.2.3 Further explanation and remarks in the case $2 \leq d \leq 4$ and $\sqrt{2n/h}$

It is clear from the above proof that for $d \geq 5$, an analogue of Theorem 3 holds if we replace $\sqrt{n}\zeta$ by $\sqrt{2n/h}\zeta$ everywhere in (3.25) if we modify the kernels A_n^d and $\tilde{A}_n^d$ by using the corresponding functions $b_{k,h,n}$ and $\tilde{b}_{k,h,n}$ from Theorem 2. Indeed all the above argument used about these functions are the radial interpolation formulas they satisfy and their bounds in term of k, h, n and r, which only improve as $h \geq 2$ becomes larger. The result is obviously weaker for $h > 2$, so we see no reason to give further details. E.g., in the case $h = 4$ the information needed to reconstruct $f(x)$ "doubles".

To prove Theorem 3 in the case $2 \leq d \leq 4$ (and $h = 2$), the above proof only requires little adjustment. The definition of the kernels A_n^d, $\tilde{A}_n^d$, $n \geq 1$, remain as in (3.30), (3.31) but note that they now include at least one (and at most two) of the functions from Theorem 1, as $m = 0$- or $m = 1$-term. An analogue of Lemma 3.3 still holds (possibly weaker from the numerical point of view). In the estimates of the series $\mathcal{A}(T)$, $\tilde{A}(T)$ (with $\alpha = \beta = 0$) we bound the terms $c_{d+2m,n}(|x|)|x|^m$ appearing for $m = 0, 1$ in $\mathcal{A}_{\text{start}}(T)$ "exceptionally" uniformly on $\mathbb{R}^d$ by using that all Schwartz semi-norms of the functions coming from Theorem 1 have polynomial growth in n (in particular, their sup norms).

Since we have an essentially unlimited amount of space to write this thesis, let us verify the key convergence of the double series (3.27) in the case $2 \leq d \leq 4$, but only point-wise and without partial derivatives. We hope that by doing so, we streamline the basic mechanism of the proof and convince even the most skeptical reader (that is, the author) a bit more of the truth of (3.25).

So let us once again fix a Schwartz function $f \in \mathcal{S}(\mathbb{R}^d)$ and a point $x \in \mathbb{R}^d \smallsetminus \{0\}$. One key convergence property to establish is finiteness of the series

$$X(\hat{f},x) := \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} |\tilde{c}_{d+2m,n}(|x|)| \frac{1}{\sqrt{n}^m} \int_S |\hat{f}(\sqrt{n}\zeta) Z_m^d(x,\zeta)|\, d\zeta. \tag{3.42}$$

and for a similarly defined $X(f,x)$ involving $c_{d+2m,n}$ and f. Once this is known, we can interchange the sum over n and m and in a second (easier) step, interchange the m-sum with the integral. We focus on $\hat{f}$ and $\tilde{c}_{d+2m,n}$, since the analysis of f and $c_{d+2m,n}$ is completely analogous.

We use the notations[24]

$$U_n(\hat{f}) := \sup_{\zeta\in S^{d-1}} |\hat{f}(\sqrt{n}\zeta)|, \qquad H_d := \sup_{m\in\mathbb{N}_0} \dim\left(\mathcal{H}_m(\mathbb{R}^d)\right) m^{2-d}$$

and bound the integral (3.42) by

$$\left|\int_S |\hat{f}(\sqrt{n}\zeta) Z_m^d(x,\zeta)d\zeta\right| \le |x|^m U_n(\hat{f}) \sup_{\zeta\in S^{d-1}} |Z_m^d(x/|x|,\zeta)| \le H_d |x|^m U_n(\hat{f}) m^{d-2}.$$

We remark that via "integration by parts", this bound could be improved in the sense that m^{d-2} could be replaced by m^{-A} for any fixed $A \ge 0$, at the cost of introducing sup norms of partial derivatives of f over $\sqrt{n}S^{d-1}$. For what we wish to do here, this does not seem to help.

Next, we recall the following bounds from Theorems 1 and 2:

$$|\tilde{c}_{d+2m,n}(|x|)| \le \begin{cases} C_1 n^\beta |x|^{-m} & \text{for } 0 \le m \le 1, n \ge 0,\\ C_2 n^{d/4+m/2+9/8}|x|^{-d/2-m+9/4} & \text{for } m \ge 2, n \ge 1,\\ C_3 \left(\frac{B}{(d/2+m)}\right)^{d/4+m/2} n^{d/2+m} & \text{for } m \ge 2, n \ge 1,\end{cases}$$

where the constants C_1, C_2, C_3, B and β are positive and do not depend on any of the parameters $m, |x|, d$ or n (and are *not* labeled as in Theorem 2). To verify that $X(\hat{f},x)$ is finite, we show that, if we first sum over n and then over m, the sum is finite. We do so with the help of auxiliary integers $M_n = M_n(x,d) \ge 2$ and subdivide the series in the following way:

$$\begin{aligned} X(\hat{f},x) &\le H_d \sum_{n=1}^{\infty}\left(\sum_{m=0}^{1} + \sum_{m=2}^{M_n} + \sum_{m=M_n+1}^{\infty}\right) |\tilde{c}_{d+2m,n}(|x|)||x|^m n^{-m/2} m^{d-2} U_n(\hat{f})\\ &\le X_1(\hat{f},x) + X_2(\hat{f},x) + X_3(\hat{f},x)\end{aligned}$$

where, by the above bounds for $|\tilde{c}_{d+2m,n}(|x|)|$, we have

$$\begin{aligned} X_1(\hat{f},x) &\le H_d C_1 \sum_{n=1}^{\infty} n^\beta U_n(\hat{f})\\ X_2(\hat{f},x) &\le H_d C_2 \sum_{n=1}^{\infty}\sum_{m=2}^{M_n} n^{d/4+9/8}|x|^{-d/2+9/4} m^{d-2} U_n(\hat{f})\\ X_3(\hat{f},x) &\le H_d C_3 \sum_{n=1}^{\infty}\sum_{m=M_n+1}^{\infty} \left(\frac{2B}{d+2m}\right)^{d/4+m/2} |x|^m n^{-m/2} n^{d/2+m} m^{d-2} U_n(\hat{f})\end{aligned}$$

Here, we point out the "miraculous" cancellations in $X_2(\hat{f},x)$ of the terms $|x|^m n^{-m/2}$ with the factors $|x|^{-m} n^{m/2}$ appearing in the second bound for $\tilde{c}_{d+2m,n}(|x|)$ listed above. To choose M_n suitably, we write the summands of $X_3(\hat{f},x)$ as

$$n^{d/2} U_n(\hat{f}) \left(\frac{2B}{d+2m}\right)^{d/4} \left(\frac{2B|x|^2 n}{d+2m}\right)^{m/2} m^{d-2} \le H_d' n^{d/2} U_n(\hat{f}) \left(\frac{2B|x|^2 n}{d+2M_n+2}\right)^{m/2} m^{d-2}$$

[24] here and in the following, an expression of the form 0^0 is to be interpreted as 1. For instance when $d = 2$, we have $\dim(\mathcal{H}_m(\mathbb{R}^2)) \le 2$ for all $m \in \mathbb{N}_0$.

where $H_d' := \sup_{d\geq 2}\left(\frac{2B}{d+6}\right)^{d/4}$. We choose $M_n := \lceil 2B|x|^2 n\rceil + 1 \geq 2$ so that the expression in brackets, which is being raised to the power $m/2$ is at most $1/2$. With this choice, we have

$$X_3(\hat{f},x) \leq H_d H_d' C_3 \sum_{n=1}^{\infty} n^{d/2} U_n(\hat{f}) \sum_{m=2}^{\infty} 2^{-m/2} m^{d-2} < \infty$$

and $X_2(\hat{f},x)$ can be estimated

$$\text{by} \quad X_2(\hat{f},x) \leq H_d C_2 \sum_{n=1}^{\infty} |x|^{9/4-d/2} n^{d/4} U_n(\hat{f}) \sum_{m=2}^{M_n} m^{d-2}$$
$$\text{and} \quad \sum_{m=2}^{M_n} m^{d-2} \leq M_n^{d-1} \leq (2B|x|^2 n + 2)^{d-2}$$

which shows that $X_2(\hat{f},x) < \infty$ too. Thus, all $X_j(\hat{f},x)$, $j = 1,2,3$ are finite and this proves convergence of $X(\hat{f},x)$ itself.

Remark 3.4 (Remarks on uniform convergence)**.** While the proof for $2 \leq d \leq 4$ is overall a bit more involved than the one for $d \geq 5$ in terms of "book keeping" and in terms of required input (Theorem 1 *and* 2), there is one technical aspect which is better but also a bit puzzling to the author. Namely, if $2 \leq d \leq 4$, then the the supremum $\sup_{\delta\leq|x|\leq R}|x|^{-d/2+9/4}$ which appeared in (3.39), in the proof of Lemma 3.3 equals $R^{9/4-d/2}$ and is thus independent of δ. An analysis of the proof then shows that the RHS of the interpolation formula (3.25) in Theorem 1 converges in fact uniformly on *all* compact subsets of $\mathbb{R}^d$ to f (not only on compact sets avoiding zero). We believe that this is an artifact of our proof and that uniform convergence holds on all compact subsets for all $d \geq 2$.

Remark 3.5 (Extension of Theorem 3 to functions outside the Schwartz class)**.** We remark that the interpolation formula (3.25) also holds for functions outside the Schwartz class. More precisely, we proved in [Sto21, Cor. 6.1] the following result.

Proposition 3.9. *Let $d \geq 5$ and let B be a real number such that $B > 5d + 9/2$. Let $f : \mathbb{R}^d \to \mathbb{C}$ be a continuous integrable function such that $|f(x)| + |\hat{f}(x)| \lesssim |x|^{-B}$ for $|x| \geq 1$. The interpolation formula* (3.25) *holds for the function f with point-wise absolute convergence and uniform convergence on compact subsets avoiding the origin.*

Here, the number "5" in $5d+9/2$ comes comes from the number $5d/4+1/8$ in part (ii) of Lemma 3.3 and could be improved if that exponent could be improved. Let us briefly sketch the argument for the proof of Proposition 3.9. We refer to §6 in our paper [Sto21] for a more complete treatment. The proof of Lemma 3.6 below involves a similar approximation argument with convolutions (but serves a different purpose).

Let $\phi_\varepsilon(x) = \varepsilon^{-d} e^{-\pi|x/\varepsilon|^2}$, $x \in \mathbb{R}^d$, $\varepsilon \in (0,1)$ denote the Gaussian approximate identity. For $f : \mathbb{R}^d \to \mathbb{C}$ as in the Proposition, consider the functions $J_\varepsilon f \in \mathcal{S}(\mathbb{R}^d)$ defined as

$$J_\varepsilon f(x) := (\widehat{\phi_\varepsilon} \cdot (\phi_\varepsilon * f))(x) = e^{-\pi|\varepsilon x|^2} \int_{\mathbb{R}^d} \phi_\varepsilon(y) f(x-y) dy, \quad x \in \mathbb{R}^d$$

We have $\widehat{J_\varepsilon f} = \phi_\varepsilon * (\widehat{\phi_\varepsilon} \cdot \hat{f}) =: \tilde{J}_\varepsilon \hat{f}$. With $I_n(f)$ and $\tilde{I}_n(\hat{f})$ defined as in (3.40), we write, using Theorem 3 applied to $J_\varepsilon f$,

$$\begin{aligned} f = (f - J_\varepsilon f) + J_\varepsilon f &= (f - J_\varepsilon f) + \sum_{n=1}^{\infty} I_n(J_\varepsilon f) + \sum_{n=1}^{\infty} \tilde{I}_n(\widehat{J_\varepsilon f}) \\ &= (f - J_\varepsilon f) + \sum_{n=1}^{\infty} I_n(J_\varepsilon f - f) + \sum_{n=1}^{\infty} \tilde{I}_n(\tilde{J}_\varepsilon(\hat{f}) - \hat{f}) + \sum_{n=1}^{\infty} I_n(f) + \sum_{n=1}^{\infty} \tilde{I}_n(\hat{f}). \end{aligned}$$

One can show that all terms depending on ε tend to zero as $\varepsilon \to 0$ (see §6 in [Sto21]) and thus that f equals $\sum_{n=1}^{\infty} (I_n(f) + \tilde{I}_n(\hat{f}))$, as desired.

3.3 Perturbed Fourier uniqueness and interpolation results

The ideas presented in this section were conceived and implemented jointly with João Ramos in the article [RS21]. Our goal here is to present them in a way that is coherent with the other contents of the thesis. We do not aim at giving a precise quantitative result. For a version of the latter and all technical details that its proof entails, we refer to our article with Ramos [RS21]. We mention that what we present in this section is based on Theorem 1 from Chapter 2, but that in [RS21], we used another version of that theorem, based on the work [BRS20]. We would also like to mention that Ramos and Sousa [RdSar] [RdS20] previously obtained various results on the Fourier uniqueness problem and on perturbations of interpolation formulas using methods form functional analysis.

Theorem 4 below says that we may perturb the Fourier uniqueness sets given by Corollary 3.2 in two ways: by replacing the spheres $\sqrt{n}S^{d-1}$ by *surfaces* close to those spheres and by *discrete subsets* contained in them. We can think of these perturbations respectively as being in "radial" and "spherical" directions. After gathering some technical results about suitable function spaces in §3.3.2, we will give the proof of Theorem 4 in §3.3.3. Then, in §3.3.4, we will state and prove Theorem 5 which is a perturbed version of Theorem 1 pertaining to radial Schwartz functions.

Theorem 4. *Fix an integer $d \geq 2$. There exist positive real numbers $\sigma_n, \hat{\sigma}_n > 0$, indexed by $n \in \mathbb{N}_0$ such that the following holds.*

For all vectors $\varepsilon_0, \hat{\varepsilon}_0 \in \mathbb{R}^d$ satisfying $|\varepsilon_0| \leq \sigma_0$, $|\hat{\varepsilon}_0| \leq \hat{\sigma}_0$ and for all sequences of continuous[25] *functions $\varepsilon_n, \hat{\varepsilon}_n : S^{d-1} \to \mathbb{R}$ whose sup-norms are at most σ_n and $\hat{\sigma}_n$ respectively, the pair*

$$\left(\{\varepsilon_0\} \cup \bigcup_{n \in \mathbb{N}} \{(\sqrt{n} + \varepsilon_n(\zeta))\zeta \,:\, \zeta \in S^{d-1}\}, \{\hat{\varepsilon}_0\} \cup \bigcup_{n \in \mathbb{N}} \{(\sqrt{n} + \hat{\varepsilon}_n(\zeta))\zeta \,:\, \zeta \in S^{d-1}\} \right) \tag{3.43}$$

is a Fourier uniqueness pair for $\mathcal{S}(\mathbb{R}^d)$. Moreover, for any sequences of finite partitions

$$\bigsqcup_{i \in I(n)} \Omega_{n,i} = S^{d-1} = \bigsqcup_{j \in J(n)} \hat{\Omega}_{n,j}$$

of the unit sphere S^{d-1} into closed[26] *subsets $\Omega_{n,i}, \hat{\Omega}_{n,j} \subseteq S^{d-1}$ whose diameters*[27] *are bounded by σ_n and $\hat{\sigma}_n$ respectively and for all choices of points $\zeta_{n,i} \in \Omega_{n,i}$, $\hat{\zeta}_{n,j} \in \hat{\Omega}_{n,j}$, the pair*

$$\left(\{\varepsilon_0\} \cup \{\sqrt{n}\zeta_{n,i} \,:\, n \in \mathbb{N}, i \in I(n)\}, \{\hat{\varepsilon}_0\} \cup \{\sqrt{n}\hat{\zeta}_{n,j} \,:\, n \in \mathbb{N}, j \in J(n)\} \right) \tag{3.44}$$

[25] we could allow them to be bounded and measurable

[26] we could allow them to be measurable

[27] by diameter of a subset $\Omega \subseteq \mathbb{R}^d$ we mean $\mathrm{diam}(\Omega) := \sup\{|x - y| \,:\, x, y \in \Omega\}$.

is a Fourier uniqueness pair for $\mathcal{S}(\mathbb{R}^d)$.

Remark 3.6. As will be clear from the proof, for all non-negative integers $n_0, \hat{n}_0 \geq 0$ satisfying $n_0 + \hat{n}_0 = 1 + \lfloor d/4 \rfloor$, both (3.43) and (3.44) remain uniqueness sets if we remove the points near (or on) the first n_0 spheres of the "function side" and the first $\hat{n}_0$ spheres on the "Fourier side".

Remark 3.7. In [RS21] we proved that there are constants $\delta = \delta_d > 0$ and $c > 0$ (not depending on d) such that Theorem 4 holds if

$$\max\left(|\sigma_n|, |\hat{\sigma}_n|\right) \leq \delta(n+1)^{-10n-2.5d-c}.$$

This numerical result relied on estimates for the basis functions for radial Fourier interpolation with $\sqrt{n}$-nodes derived from [BRS20], which are also explicit in the weight k. We believe that the statements of Theorem 4 remain true under numerically weaker assumptions on $|\sigma_n|, |\hat{\sigma}_n|$, but we cannot offer a proof.

Remark 3.8. In fact, it is possible to obtain more perturbations of the spheres $\sqrt{n}S^{d-1}$ than the ones stated in Theorem 4. A more general formulation, which unifies the two types of perturbations described above, is given in Remark 3.10. (This was observed at a late stage of writing this text, so we kept the formulation of the theorem as it is).

3.3.1 The basic idea

Before delving into more technical matters, let us abstractly sketch the main idea of the proof technique. This is going to be an informal, rough sketch, so we will not be precise. The sketch is going to be rigorously implemented in §3.3.4. Section 3.3.3 contains a similar version of that idea with point-evaluations replaced by integration over spheres.

Let V be a space of functions on $\mathbb{R}^d$. Suppose given some "nodes" $y_n \in \mathbb{R}^d$, some $a_n \in V$ and an interpolation formula $f = \sum_{n=1}^{\infty} f(y_n)a_n$, valid for all $f \in V$. In our case, such a formula would also involve information about $\hat{f}$ but in order to sketch the basic idea, we may ignore this. We can interpret such a formula as a way of expressing the identity map on V. Given small perturbations $\varepsilon_n \in \mathbb{R}^d$, consider the following operator T on V:

$$Tf = \sum_{n=1}^{\infty} f(y_n + \varepsilon_n)a_n$$

(assuming it can be defined via this expression, in a suitable way). If V is a Banach space, all of whose elements are Lipschitz continuous functions on $\mathbb{R}^d$, with Lipschitz constant controlled by the norm $\|\cdot\|_V$ of V, then

$$\|Tf - f\|_V \leq \sum_{n=1}^{\infty} |f(y_n + \varepsilon_n) - f(y_n)| \|a_n\|_V \lesssim \|f\|_V \sum_{n=0}^{\infty} |\varepsilon_n| \|a_n\|_V,$$

which shows that T is invertible provided that ε_n is sufficiently small. (As we will see, in practice, we also need to assume that the ε_n are sufficiently small to even *define* T). It follows that

$$f = T^{-1}Tf = \sum_{n=1}^{\infty} (T^{-1}a_n) f(y_n + \varepsilon_n),$$

provided of course, that this can be justified. This gives us a perturbed interpolation formula.

3.3.2 Preliminaries on function spaces

We collect here some preliminary results of technical nature which will be used in the proofs of Theorems 4 and 5. We remark that we essentially only need the basic Lemma 3.4 of these preliminaries for the proof of Theorem 4, but all of them for Theorem 5.

For $x \in \mathbb{R}^d$, let $\langle x \rangle := (1+|x|^2)^{1/2}$. This quantity is comparable in size to the Euclidean norm $|x|$ (for large $|x|$), its main advantage is that it is always ≥ 1 so that any real power of it is defined so that we have the monotonicty $\langle x \rangle^s \leq \langle x \rangle^{s'}$ whenever $0 \leq s \leq s'$. For any $s \in \mathbb{R}$, we abbreviate by $a_s(x) := \langle x \rangle^s$ and define

$$V^s(\mathbb{R}^d) := \left\{ f \in L^1(\mathbb{R}^d) \,:\, a_s f, a_s \hat{f} \in L^1(\mathbb{R}^d) \right\}.$$

For $f \in V^s(\mathbb{R}^d)$, we define

$$\|f\|_{V^s(\mathbb{R}^d)} := \|f\|_{V^s} := \|f a_s\|_{L^1(\mathbb{R}^d)} + \|\hat{f} a_s\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \langle x \rangle^s |f(x)| dx + \int_{\mathbb{R}^d} \langle \xi \rangle^s |\hat{f}(\xi)| d\xi.$$

This is clearly a norm on $V^s(\mathbb{R}^d)$. We denote by $V^s_{\mathrm{rad}}(\mathbb{R}^d) \subseteq V^s(\mathbb{R}^d)$ the subspace of radial functions. It is closed (since, by Fourier inversion, point-evaluations on $V^s(\mathbb{R}^d)$ are continuous).

We now establish some basic properties of these spaces: We prove that they are complete, that they contain $\mathcal{S}(\mathbb{R}^d)$ as a dense subspace and that $\cap_{s \geq 1} V^s(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d)$. The last property is a corollary of a more precise statement, Proposition 3.5 below. These results are very likely known and not surprising for an experienced analyst, but we include their proofs as an exercise for the author.

Let us also remark that it is likely possible to work instead with a similarly defined *Hilbert* space, defined based on the standard Sobolev space $H^s(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) \,:\, a_s \hat{f} \in L^2(\mathbb{R}^d)\}$ by $\mathcal{H}^s(\mathbb{R}^d) := \{f \in L^2(\mathbb{R}^d) \,:\, f, \hat{f} \in H^s(\mathbb{R}^d)\}$, which makes a comparison to yet unpublished work of Kulikov–Nazarov–Sodin.

Remark 3.9. In our paper [RS21] with Ramos, we actually used the function $m_s(x) := 1 + |x|^s$ instead of $a_s(x)$ and defined $V^s(\mathbb{R}^d)$ to be the space of all $f \in L^1(\mathbb{R}^d)$ such that $m_s f$, $m_s \hat{f} \in L^1(\mathbb{R}^d)$. Since $m_s(x) \asymp 1 \asymp \langle x \rangle^s$ for $|x| \leq 1$ and $m_s(x) \langle x \rangle^{-s} \asymp 1$ for $|x| \geq 1$, the definitions are equivalent.

Lemma 3.4. *For each $s \geq 0$, the space $V^s(\mathbb{R}^d)$, equipped with the above norm, is a Banach space. In particular, $V^s_{rad}(\mathbb{R}^d)$ is also a Banach space.*

Proof. Let $\{f_n\}_{n \in \mathbb{N}} \subseteq V^s(\mathbb{R}^d)$ be a Cauchy sequence. Then $\{f_n a_s\}_{n \in \mathbb{N}}$ and $\{\hat{f}_n a_s\}_{n \in \mathbb{N}}$ are L^1-Cauchy sequences and therefore admit L^1-limits $\varphi, \tilde{\varphi} \in L^1(\mathbb{R}^d)$ repsectively. Define $f := \varphi a_s^{-1}$ and $\tilde{f} := \tilde{\varphi} a_s^{-1}$. We claim that $f \in V^s(\mathbb{R}^d)$ and $\|f - f_n\|_{V^s} \to 0$ as $n \to \infty$. To prove this, it suffices to show that $\hat{f}(\xi) = \tilde{f}(\xi)$ for almost every $\xi \in \mathbb{R}^d$. Since $\hat{f}_n \to \tilde{f}$ (in particular) in $L^1(\mathbb{R}^d)$, we find a measurable set $N \subseteq \mathbb{R}^d$ of measure zero and $1 \leq n_1 < n_2 < \ldots$ so that for all $\xi \in \mathbb{R}^d \smallsetminus N$ we have $|\widehat{f_{n_j}}(\xi) - \tilde{f}_{n_j}(\xi)| \to 0$ as $j \to \infty$. For $\xi \in \mathbb{R}^d \smallsetminus N$ we then write

$$|\hat{f}(\xi) - \tilde{f}(\xi)| \leq |\hat{f}(\xi) - \widehat{f_{n_j}}(\xi)| + |\widehat{f_{n_j}}(\xi) - \tilde{f}(\xi)| \leq \|f - f_{n_j}\|_{L^1(\mathbb{R}^d)} + |\widehat{f_{n_j}}(\xi) - \tilde{f}(\xi)|$$

and note that both terms tend to zero as $j \to \infty$. □

Lemma 3.5. *Fix $d \geq 1$. For $s \geq 1$ and $f \in V^s(\mathbb{R}^d)$ we have, for $|x| \geq 1$,*

$$|f(x)| + |\widehat{f}(x)| \lesssim_{d,s} \|f\|_{V^s}^{\frac{d}{d+1}} |x|^{-\frac{s}{d+1}}, \tag{3.45}$$

where the implied constant depends only on d and s. Moreover, if $s \geq 2$, then, for $|x| \geq 1$,

$$|\nabla f(x)| + |\nabla \widehat{f}(x)| \lesssim_{d,f} |x|^{-\frac{s}{2(d+1)}}. \tag{3.46}$$

Proof. Fix a nonzero function $f \in V^s(\mathbb{R}^d)$. We first note that Fourier inversion implies the following bounds, valid for all $x \in \mathbb{R}^d$:

$$|f(x)| \leq \|f\|_{V^s}, \qquad |\nabla f(x)| = \Big(\sum_{j=1}^{d} |\partial_j f(x)|^2 \Big)^{1/2} \leq 2\pi\sqrt{d}\|f\|_{V^s}.$$

Fix $x \in \mathbb{R}^d$ with $|x| \geq 1$ and define

$$r_x := \frac{|f(x)|}{4\pi\sqrt{d}\|f\|_{V^s}} \leq \frac{1}{4\pi\sqrt{d}} \leq \frac{|f(x)|}{2}.$$

Let $B_x := B_{r_x}(x)$ denote the closed ball centered at x with radius r_x. For any $y \in B_x$, we have

$$|f(y) - f(x)| \leq 2\pi\sqrt{d}\|f\|_{V^s}|x-y| \leq 2\pi\sqrt{d}\|f\|_{V^s} r_x = \frac{1}{2}$$

and hence $|f(y)| \geq \frac{|f(x)|}{2}$. Moreover, for any $y \in B_x$, we have

$$|y| \geq |x| - r_x \geq |x| - 1/2 \geq |x|/2 \qquad \text{since } \ |x| \geq 1.$$

These inequalities imply that, with c_d denoting the volume of the unit ball in $\mathbb{R}^d$,

$$\begin{aligned} \|a_s f\|_{L^1(\mathbb{R}^d)} &\geq \int_{B_x} (1+|y|^2)^{s/2}|f(y)|dy \geq 2^{-s/2-1} \int_{B_x} (2+|x|^2)^{s/2}|f(x)|dy \\ &= \frac{c_d}{2^{s/2+1}} r_x^d |f(x)||x|^s = \frac{c_d}{2^{s/2+1}} \frac{|f(x)|^d}{(4\pi)^d d^{d/2}\|f\|_{V^s}^d}|f(x)||x|^s. \end{aligned}$$

This proves (3.45) for the function f. By symmetry, one obtains the same estimate for $\widehat{f}$. The bound (3.46) then follows from (3.45) via an argument using Taylor's theorem with remainder (see [Sto21, Lemma 6.1] for a sketch). ☐

Lemma 3.6. *For each $s \geq 0$, the space $\mathcal{S}(\mathbb{R}^d)$ is dense in $V^s(\mathbb{R}^d)$ and $\mathcal{S}_{rad}(\mathbb{R}^d)$ is dense in $V^s_{rad}(\mathbb{R}^d)$.*

Proof. We fix a smooth, non-negative, radial bump function $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \phi = 1$ and such that $\phi(x) = 0$ if $|x| \geq 1$. For $\varepsilon \in (0, 1/2]$ and $x, \xi \in \mathbb{R}^d$, we define

$$\phi_\varepsilon(x) = \varepsilon^{-d}\phi(x/\varepsilon) \qquad \text{and} \qquad \psi_\varepsilon(\xi) := \widehat{\phi_\varepsilon}(\xi) = \widehat{\phi}(\varepsilon\xi).$$

Let $f \in V^s(\mathbb{R}^d)$ (or $V^s_{\mathrm{rad}}(\mathbb{R}^d)$) be arbitrary. We claim that:

(i) $\psi_\varepsilon \cdot (\phi_\varepsilon * f) \in \mathcal{S}(\mathbb{R}^d)$ (or $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$) for all $\varepsilon > 0$,

(ii) $\lim_{\varepsilon\to 0} \|\psi_\varepsilon \cdot (\phi_\varepsilon * f) - f\|_{V^s(\mathbb{R}^d)} = 0$.

Part (i) holds more generally for bounded continuous functions f. To prove (ii), we will show that

$$\lim_{\varepsilon\to 0} \| \left(\psi_\varepsilon \cdot (\phi_\varepsilon * f) - f\right) a_s\|_{L^1} = 0, \tag{3.47}$$

$$\lim_{\varepsilon\to 0} \|(\phi_\varepsilon * (\psi_\varepsilon \cdot \hat{f}) - \hat{f})a_s\|_{L^1} = 0. \tag{3.48}$$

We start with the proof of (3.47). We take an auxiliary test function $\alpha \in C_c^\infty(\mathbb{R}^d)$ and write

$$\begin{aligned}\psi_\varepsilon(\phi_\varepsilon * f) - f &= \psi_\varepsilon(\phi_\varepsilon * f - f) + \psi_\varepsilon f - f \\ &= \psi_\varepsilon(\phi_\varepsilon * (f-\alpha) + \phi_\varepsilon * \alpha - \alpha + \alpha - f) + (\psi_\varepsilon - 1)f \\ &= \psi_\varepsilon(\phi_\varepsilon * (f-\alpha)) + \psi_\varepsilon(\phi_\varepsilon * \alpha - \alpha) + \psi_\varepsilon(\alpha - f) + (\psi_\varepsilon - 1)f \\ &= \psi_\varepsilon(\phi_\varepsilon * (f-\alpha)) + \psi_\varepsilon(\phi_\varepsilon * \alpha - \alpha) + (\psi_\varepsilon - 1)\alpha + (\alpha - f)\end{aligned}$$

We multiply each of these four terms with the function a_s and then take the L^1-norm. We suppose given a $\delta > 0$ and will show that all there is $\varepsilon_0(\delta)$ such that all of these four norms are at most $\delta/4$ when $\varepsilon < \varepsilon_0(\delta)$. We first choose α depending only on δ, s, d such that

$$\|(\alpha - f)a_s\|_{L^1} \leq \frac{\delta}{4M}$$

where $M \geq 1$ is another parameter at our disposal. This is possible since $C_c^\infty(\mathbb{R}^d)$ is dense in L^1 and since multiplication by a_s preserves $C_c^\infty(\mathbb{R}^d)$. We may then estimate

$$\begin{aligned}\|\psi_\varepsilon(\phi_\varepsilon * (f-\alpha))a_s\|_{L^1} &= \int_{\mathbb{R}^d} \widehat{\phi}(\varepsilon x) \left| \int_{\mathbb{R}^d} \phi_\varepsilon(y)(f(x-y) - \alpha(x-y))\, dy \right| a_s(x)\, dx \\ &\leq \|\phi\|_{L^1} \int_{|y|\leq\varepsilon} \phi_\varepsilon(y) \int_{\mathbb{R}^d} (f(x-y) - \alpha(x-y)) a_s(x-y) \frac{a_s(x)}{a_s(x-y)}\, dx\, dy \\ &\leq C_\varepsilon \|\phi\|_{L^1} \|(f-\alpha)a_s\|_{L^1} \int_{|y|\leq\varepsilon} \phi_\varepsilon(y)\, dy \\ &\leq C_\varepsilon \|\phi\|_{L^1}^2 \|(f-\alpha)a_s\|_{L^1} \leq \frac{C_\varepsilon \delta}{4M}\end{aligned}$$

where

$$C_\varepsilon = \sup_{x\in\mathbb{R}^d, |y|\leq\varepsilon} \frac{a_s(x)}{a_s(x-y)} < \infty,$$

which is uniformly bounded as a function of $\varepsilon \in (0, 1/2]$. We also used that $\|\phi\|_{L^1} = 1$ (but we will sometimes write these terms to see what we used). Thus, by choosing M suitably, we get that $\|\psi_\varepsilon(\phi_\varepsilon * (f-\alpha))a_s\|_{L^1} \leq \frac{\delta}{4}$, for all $\varepsilon \in (0, 1/2]$.

Next, we choose $R_\alpha \geq 1$ such that for all $x, y \in \mathbb{R}^d$ we have

$$(|x| \geq R_\alpha \quad \text{and} \quad |y| \leq 1/2) \implies \alpha(x-y) = 0 = \alpha(x).$$

Then

$$\begin{aligned}\|\psi_\varepsilon(\phi_\varepsilon * \alpha - \alpha)a_s\|_{L^1} &= \int_{|x|\leq R_\alpha} a_s(x)|\psi_\varepsilon(x)| \left| \int_{|y|\leq\varepsilon} \phi_\varepsilon(y)(\alpha(x-y) - \alpha(x))\, dy \right| dx \\ &\leq (1+R_\alpha^2)^{s/2} |B_{R_\alpha}(0)| \|\phi\|_{L^1} \sup_{\mathbb{R}^d} |\nabla\alpha| \int_{|y|\leq\varepsilon} |y|\phi_\varepsilon(y)\, dy \lesssim_{\alpha,s,d} \varepsilon\end{aligned}$$

where the last step follows from a change of variables $y \leftrightarrow y/\varepsilon$. Thus, for sufficiently small ε (depending on α, s, d and hence on δ, s, d), this will also be at most $\delta/4$, as desired.

The remaining term is

$$\begin{aligned}\|(\psi_\varepsilon - 1)\alpha a_s\| &= \int_{|x|\leq R_\alpha} |(\widehat{\phi}(\varepsilon x) - \widehat{\phi}(0))||\alpha(x)| a_s(x)\, dx \lesssim_{\alpha,s,d} \int_{|x|\leq R_\alpha} |(\widehat{\phi}(\varepsilon x) - \widehat{\phi}(0))|\, dx \\ &\lesssim_\phi \int_{R_\alpha} |\varepsilon x|\, dx \lesssim_\alpha \varepsilon.\end{aligned}$$

which is also at most $\delta/4$ for sufficiently small ε, as desired.

We turn to the proof of (3.48), for which we write, using an auxiliary text function $\beta \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}
\phi_\varepsilon * (\psi_\varepsilon \hat{f}) - \hat{f} &= \phi_\varepsilon * (\psi_\varepsilon \hat{f}) - \beta + \beta - \hat{f} \\
&= \phi_\varepsilon * (\psi_\varepsilon (\hat{f} - \beta) + \psi_\varepsilon \beta) - \beta + \beta - \hat{f} \\
&= \phi_\varepsilon * (\psi_\varepsilon (\hat{f} - \beta)) + (\phi_\varepsilon * (\psi_\varepsilon \beta) - \beta) + (\beta - \hat{f}) \\
&= \phi_\varepsilon * (\psi_\varepsilon (\hat{f} - \beta)) + (\phi_\varepsilon * (\psi_\varepsilon \beta) - (\psi_\varepsilon \beta)) + (\psi_\varepsilon - 1)\beta + (\beta - \hat{f})
\end{aligned}$$

We give ourselves a $\delta > 0$ and will show that the $L^1(a_s(x)dx)$-norm of the four terms above are all at most $\delta/4$ provided ε is sufficiently small in terms of δ. We first choose β, depending only on δ, s, d and M such that

$$\|(\beta - \hat{f})a_s\|_{L^1} \leq \frac{\delta}{4M},$$

where $M \geq 1$ is another parameter at our disposal. Then, similarly to the proof of (3.47),

$$\begin{aligned}
\|(\phi_\varepsilon * (\psi_\varepsilon (\hat{f} - \beta)))a_s\|_{L^1} &= \int_{\mathbb{R}^d} a_s(x) \left| \int_{|y|\leq\varepsilon} \phi_\varepsilon(y)\psi_\varepsilon(x-y)(\hat{f}(x-y) - \beta(x-y))\, dy \right| dx \\
&\lesssim_{d,s} \|(\hat{f} - \beta)a_s\|_{L^1}
\end{aligned}$$

independently of ε, which we can thus make at most $\delta/4$, provided M is chosen large enough in terms of s, d. (Here, as in the proof of the first part, we use that $|\psi(x-y)| \leq 1$ and we artificially insert a term $a_s(x-y)$).

Next, we choose $R_\beta \geq 1$ such that for all $x, y \in \mathbb{R}^d$ we have

$$(|x| \geq R_\beta \quad \text{and} \quad |y| \leq 1/2) \quad \Longrightarrow \quad \beta(x-y) = 0 = \beta(x).$$

Then

$$\|\left(\phi_\varepsilon * (\psi_\varepsilon \beta) - (\psi_\varepsilon \beta)\right) a_s\|_{L^1} \leq \int_{|x|\leq R_\beta} a_s(x) \int_{|y|\leq\varepsilon} \phi_\varepsilon(y) |\psi_\varepsilon(x-y)\beta(x-y) - \psi_\varepsilon(x)\beta(x)|\, dydx$$

For any x, y in the above integral, we have

$$\begin{aligned}
|\psi_\varepsilon(x-y)\beta(x-y) - \psi_\varepsilon(x)\beta(x)| &\leq |\psi_\varepsilon(x-y) - \psi_\varepsilon(x)||\beta(x-y)| + |\psi_\varepsilon(x)||\beta(x-y) - \beta(x)| \\
&\lesssim_{\phi,\beta} \varepsilon|y| + |y|
\end{aligned}$$

and from this we obtain, similarly as in the proof of (3.47), the estimate

$$\|\left(\phi_\varepsilon * (\psi_\varepsilon \beta) - (\psi_\varepsilon \beta)\right) a_s\|_{L^1} \lesssim_{s,d,\beta} \varepsilon,$$

as desired. Finally, we have, exactly as at the end of the proof for (3.47), $\|a_s(\psi_\varepsilon - 1)\beta\|_{L^1} \lesssim_{s,d,\beta} \varepsilon$, also as desired. □

3.3.3 Proof of Theorem 4

In this section we prove Theorem 4. Fix $d \geq 2$. Whenever convenient, we will abbreviate $S = S^{d-1}$. As in the statement of the Theorem, we consider the following objects:

- two vectors $\varepsilon_0, \hat{\varepsilon}_0 \in \mathbb{R}^d$,

- for each $n \in \mathbb{N}$, two continuous functions $\varepsilon_n, \hat{\varepsilon}_n : S \to \mathbb{R}$,
- two vectors $\zeta_0, \hat{\zeta}_0 \in \mathbb{R}^d$ (these are just new names for the vectors ε_0, $\hat{\varepsilon}_0$ in (3.44), to separate the arguments more clearly),
- for each $n \in \mathbb{N}$, two finite nonempty sets $I(n), J(n)$ of indices,
- for each $n \in \mathbb{N}$ and $i \in I(n)$ a closed subset $\Omega_{n,i} \subseteq S^{d-1}$ and a point $\zeta_{n,i} \in \Omega_{n,i}$,
- for each $n \in \mathbb{N}$ and $j \in J(n)$ a closed subset $\hat{\Omega}_{n,j} \subseteq S^{d-1}$ and a point $\hat{\zeta}_{n,j} \in \hat{\Omega}_{n,j}$.

We assume that for all $n \in \mathbb{N}$ we have

$$\bigsqcup_{i \in I(n)} \Omega_{n,i} = S^{d-1} = \bigsqcup_{j \in J(n)} \hat{\Omega}_{n,j}.$$

We define, for each $n \in \mathbb{N}$, two functions $\delta_n, \hat{\delta}_n : S^{d-1} \to S^{d-1}$ by

$$\delta_n(\zeta) := \sum_{i \in I(n)} \mathbf{1}_{\Omega_{n,i}}(\zeta)\zeta_{n,i}, \qquad \hat{\delta}_n(\zeta) := \sum_{j \in J(n)} \mathbf{1}_{\hat{\Omega}_{n,j}}(\zeta)\hat{\zeta}_{n,j}. \tag{3.49}$$

Thus, $\delta_n(\zeta) = \zeta_{n,i}$ if $\zeta \in \Omega_{n,i}$ and $\hat{\delta}_n(\zeta) = \hat{\zeta}_{n,j}$ if $\zeta \in \hat{\Omega}_{n,j}$. We denote

$$\sigma_0 := |\varepsilon_0|, \qquad \sigma_n := \sup_{\zeta \in S^{d-1}} |\varepsilon_n(\zeta)|, \qquad \hat{\sigma}_0 := |\hat{\varepsilon}_0|, \qquad \hat{\sigma}_n := \sup_{\zeta \in S^{d-1}} |\hat{\varepsilon}_n(\zeta)|$$

and

$$\omega_0 := |\zeta_0|, \qquad \omega_n := \sup_{i \in I(n)} \operatorname{diam}(\Omega_{n,i}), \qquad \hat{\omega}_0 := |\hat{\zeta}_0|, \qquad \hat{\omega}_n := \sup_{j \in J(n)} \operatorname{diam}(\hat{\Omega}_{n,j}).$$

Next, we define, as earlier in this chapter, the non-negative integers

$$n_0(d) = 1 + \lfloor d/8 \rfloor, \qquad \hat{n}_0(d) = \lfloor (d+4)/8 \rfloor.$$

and then the kernels $K_n^d, \tilde{K}_n^d : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ exactly as $A_n^d, \tilde{A}_n^d$ in (3.30) and (3.31) but with the functions $a_{d,n}, \tilde{a}_{d,n}$ replaced by the functions $a_{d/2,n}, \tilde{a}_{d/2,n} \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ provided by Theorem 1 with the above choices of $n_0 = n_0(d)$, $\hat{n}_0 = \hat{n}_0(d)$. Thus, the sums only go from $m = 0$ to $4n$ (compare with the proof of Proposition 3.8).

We pause to explain why Remark 3.6 is true. As will become clear, the following proof also works with any other choices of $n_0(d)$ and $\hat{n}_0(d)$ with sum $1 + \lfloor d/4 \rfloor$, provided that $n_0(d)$ and $\hat{n}_0(d)$ both grow linearly with d.

Note that for any continuous (or even only bounded and measurable) function $h : S^{d-1} \to \mathbb{C}$, the integral $\int_S K_n^d(x,\zeta)h(\zeta)d\zeta$ defines a Schwartz function of x (in fact, just a linear combination of radial functions times harmonic polynomials) and in particular an element of $V^s(\mathbb{R}^d)$ for all $s \geq 1$.

We will next prove that if the quantities $\sigma_n, \hat{\sigma}_n, \omega_n, \hat{\omega}_n$ are sufficiently small, then the following two formulas define bounded linear operators $T, Q : V^1(\mathbb{R}^d) \to V^1(\mathbb{R}^d)$ such that $\|T - \mathrm{id}\| < 1$ and $\|Q - \mathrm{id}\| < 1$. How these operators are useful for the proof of Theorem 4 will become clear in a moment. The formula defining[28] T is

$$f(x) - Tf(x) = a_{d/2,0}(|x|)\,[f(0) - f(\varepsilon_0)] + \sum_{n=1}^{\infty} \int_S K_n^d(x,\zeta)\left[f(\sqrt{n}\zeta) - f(\sqrt{n}\zeta + \varepsilon_n(\zeta)\zeta)\right] d\zeta$$

$$+ \tilde{a}_{d/2,0}(|x|)\left[\hat{f}(0) - \hat{f}(\hat{\varepsilon}_0)\right] + \sum_{n=1}^{\infty} \int_S \tilde{K}_n^d(x,\zeta)\left[\hat{f}(\sqrt{n}\zeta) - \hat{f}(\sqrt{n}\zeta + \hat{\varepsilon}_n(\zeta)\zeta)\right] d\zeta.$$

The formula defining Q is

$$f(x) - Qf(x) = a_{d/2,0}(|x|)\,[f(0) - f(\zeta_0)] + \sum_{n=1}^{\infty} \int_S K_n^d(x,\zeta)\left[f(\sqrt{n}\zeta) - f(\sqrt{n}\delta_n(\zeta))\right] d\zeta$$

$$+ \tilde{a}_{d/2,0}(|x|)\left[\hat{f}(0) - \hat{f}(\hat{\delta}_0)\right] + \sum_{n=1}^{\infty} \int_S \tilde{K}_n^d(x,\zeta)\left[\hat{f}(\sqrt{n}\zeta) - \hat{f}(\sqrt{n}\hat{\delta}_n(\zeta))\right] d\zeta.$$

To prove absolute convergence (in the $V^1(\mathbb{R}^d)$-norm) of these series and to prove boundedness of the operators, we will use the following general estimate

$$f(u) - f(v) = \int_{\mathbb{R}^d} \hat{f}(\xi)\left(e^{2\pi i\langle \xi, u\rangle} - e^{2\pi i\langle \xi, v\rangle}\right) d\xi \lesssim |u-v| \int_{\mathbb{R}} |\hat{f}(\xi)||\xi| d\xi \lesssim |u-v| \|f\|_{V^1}, \quad (3.50)$$

valid for all $u, v \in \mathbb{R}^d$. We have the analogous bound $|\hat{f}(u) - \hat{f}(v)| \lesssim |u-v|\|f\|_{V^1}$. Using them we can estimate all summands in the definition of Tf by $\|f\|_{V^1(\mathbb{R}^d)}$ and the quantities $\sigma_n, \hat{\sigma}_n$ and similarly all summands in the series defining Q in terms of ω_n, $\hat{\omega}_n$. For example, we have

$$\left\| \int_S \tilde{K}_n^d(\cdot,\zeta)\left[\hat{f}(\sqrt{n}\zeta) - \hat{f}(\sqrt{n}\hat{\delta}_n(\zeta))\right] d\zeta \right\|_{V^1} = \int_{\mathbb{R}^d} \left| \int_S \tilde{K}_n^d(x,\zeta)\left[\hat{f}(\sqrt{n}\zeta) - \hat{f}(\sqrt{n}\hat{\delta}_n(\zeta))\right] d\zeta \right| \langle x\rangle dx$$

$$= \int_{\mathbb{R}^d} \left| \sum_{j\in J(n)} \int_{\hat{\Omega}_{n,j}} \tilde{K}_n^d(x,\zeta)\left[\hat{f}(\sqrt{n}\zeta) - \hat{f}(\sqrt{n}\hat{\zeta}_{n,j})\right] d\zeta \right| \langle x\rangle dx$$

$$\lesssim \int_{\mathbb{R}^d} \sum_{j\in J(n)} \int_{\hat{\Omega}_{n,j}} |\tilde{K}_n^d(x,\zeta)|\left[\|f\|_{V^1}\sqrt{n}\hat{\omega}_n\right] d\zeta \langle x\rangle dx = \|f\|_{V^1}\sqrt{n}\hat{\omega}_n \int_S \|\tilde{K}_n^d(\cdot,\zeta)\|_{V^1} d\zeta.$$

Thus, if we define

$$k_n := \int_S \|K_n^d(\cdot,\zeta)\|_{V^1} d\zeta, \qquad \tilde{k}_n := \int_S \|\tilde{K}_n^d(\cdot,\zeta)\|_{V^1} d\zeta, \quad (3.51)$$

then we obtain from such estimates and similar ones the bounds

$$\|f - Tf\|_{V^1} \lesssim_d \|f\|_{V^1}\left(\|a_{d/2,0}\|_{V^1}\sigma_0 + \sum_{n=1}^{\infty}\sigma_n k_n + \|\tilde{a}_{d/2,0}\|_{V^1}\hat{\sigma}_0 + \sum_{n=1}^{\infty}\hat{\sigma}_n\tilde{k}_n\right), \quad (3.52)$$

$$\|f - Qf\|_{V^1} \lesssim_d \|f\|_{V^1}\left(\|a_{d/2,0}\|_{V^1}\omega_0 + \sum_{n=1}^{\infty}\omega_n\sqrt{n}k_n + \|\tilde{a}_{d/2,0}\|_{V^1}\hat{\omega}_0 + \sum_{n=1}^{\infty}\hat{\omega}_n\sqrt{n}\tilde{k}_n\right). \quad (3.53)$$

It is now clear that if we choose $\sigma_n, \hat{\sigma}_n$ small enough in terms of $k_n, \tilde{k}_n$, $\|a_{d/2,0}\|_{V^1}, \|\tilde{a}_{d/2,0}\|_{V^1}$ and the implied constant in (3.52) (which depends only on d), then the series defining $Tf(x)$ converge absolutely, define elements of $V^1(\mathbb{R}^d)$ and we moreover have, in operator norm, $\|\operatorname{id}_{V^1} - T\|_{V^1} < 1$.

[28] the reason why we don't write directly the expression for Tf is a typographical one and also has to do with clarity of exposition (to be appreciated later).

Similarly, if we choose $\omega_n, \hat{\omega}_n$ small enough in terms of $k_n, \tilde{k}_n$, $\|a_{d/2,0}\|_{V^1}, \|\tilde{a}_{d/2,0}\|_{V^1}$ and the implied constant in (3.53) (which depends only on d), then the series defining $Qf(x)$ converge absolutely, define elements of $V^1(\mathbb{R}^d)$ and moreover $\|\operatorname{id}_{V^1} - Q\|_{V^1} < 1$, in operator norm.

We assume for the remainder of this proof that we made these choices. In particular, the operators T and Q are continuous and invertible on $V^1(\mathbb{R}^d)$. Consider now a Schwartz function $f_1 \in \mathcal{S}(\mathbb{R}^d)$ such that f_1 and $\widehat{f_1}$ vanish on the sets displayed in (3.43) (i.e. on the "graphs" of the functions $\varepsilon_n, \hat{\varepsilon}_n$ (over) $\sqrt{n}S^{d-1}$). Likewise, consider a Schwartz function $f_2 \in \mathcal{S}(\mathbb{R}^d)$ such that f_2 and $\widehat{f_2}$ vanish on the discrete sets displayed in (3.44) (i.e. f_2 vanishes on all points $\zeta_{n,i}$ and $\widehat{f_2}$ vanishes at all points $\hat{\zeta}_{n,j}$). We want to show that $f_1 = 0$ and that $f_2 = 0$. For this it suffices to show that $f_1 - Tf_1 = f_1$ and that $f_2 - Qf_2 = f_2$ (because these equalities are equivalent to $Tf_1 = 0$ and $Qf_2 = 0$ respectively and T and Q are injective). Looking at the defining expression for $f_1(x) - Tf_1(x)$, we see that

$$\begin{aligned} &f_1(x) - Tf_1(x) \\ &= a_{d/2}(|x|)f_1(0) + \sum_{n=1}^{\infty} \int_S K_n^d(x,\zeta) f_1(\sqrt{n}\zeta) d\zeta + \tilde{a}_{d/2}(|x|)\widehat{f}_1(0) + \sum_{n=1}^{\infty} \int_S \tilde{K}_n^d(x,\zeta)\widehat{f}_1(\sqrt{n}\zeta) d\zeta. \end{aligned}$$

By assumption on f_1 and $\widehat{f_1}$, and by (3.50), we have

$$\sup_{\zeta \in S^{d-1}} |f_1(\sqrt{n}\zeta)| \lesssim_d \|f_1\|_{V^1} \varepsilon_n, \qquad \sup_{\zeta \in S^{d-1}} |\widehat{f}_1(\sqrt{n}\zeta)| \lesssim_d \|f_1\|_{V^1} \hat{\varepsilon}_n.$$

By imposing ε_n and $\hat{\varepsilon}_n$ to be even smaller if necessary, we can interchange the (implicit) sum over $m \in \{0, 1, \ldots, 4n\}$ and the sum over $n \in \mathbb{N}$ in the above expression for $f_1(x) - Tf_1(x)$ and rewrite this expression in the form (3.27) to conclude that it indeed equals $f_2(x)$. Similarly, we can show (by imposing ω_n and $\hat{\omega}_n$ to be even smaller if necessary) that $f_2(x) - Qf_2(x) = f_2(x)$ via (3.27) and thus finish the proof of Theorem 4.

Remark 3.10. It is possible to unify and generalize the results and the proof of Theorem 4 in the following way. Consider *bounded measurable* functions $\beta_n, \hat{\beta}_n : S^{d-1} \to \mathbb{R}^d$ and small positive real numbers $\sigma_n, \hat{\sigma}_n$ such that for all $n \in \mathbb{N}$ we have

$$\sup_{\zeta \in S^{d-1}} |\zeta - \beta_n(\zeta)| \le \sigma_n, \quad \sup_{\zeta \in S^{d-1}} |\zeta - \hat{\beta}_n(\zeta)| \le \hat{\sigma}_n$$

and consequently for all $f \in V^1(\mathbb{R}^d)$,

$$\sup_{\zeta \in S^{d-1}} |f(\sqrt{n}\zeta) - f(\sqrt{n}\beta_n(\zeta))| \lesssim \|f\|_{V^1} \sqrt{n}\sigma_n$$

and similarly with $\hat{\beta}_n$ and $\hat{f}$. With k_n defined as in (3.51) we thus have

$$\left\| \int_S \tilde{K}_n^d(\cdot, \zeta) \left[f(\sqrt{n}\zeta) - f(\sqrt{n}\beta_n(\zeta)) \right] d\zeta \right\|_{V^1} \lesssim \|f\|_{V^1} k_n \sqrt{n}\sigma_n$$

and similarly for $\hat{f}, \hat{\beta}_n, \tilde{K}_n^d, \tilde{k}_n$. By defining an operator $R : V^1(\mathbb{R}^d) \to V^1(\mathbb{R}^d)$ in the same way as Q but with the functions $\delta_n, \hat{\delta}_n$ replaced by β_n, $\hat{\beta}_n$, we can prove that the union of all perturbed spheres $\{\sqrt{n}\beta_n(\zeta)\}_{\zeta \in S}$ and $\{\sqrt{n}\hat{\beta}_n(\zeta)\}_{\zeta \in S}$ (together with two points close to the origin) forms a Fourier uniqueness pair for $\mathcal{S}(\mathbb{R}^d)$, provided σ_n and $\hat{\sigma}_n$ are sufficiently small. This subsumes the two previous results by taking $\beta_n(\zeta) = \delta_n(\zeta)$, $\hat{\beta}_n(\zeta) = \delta_n(\zeta)$ with δ_n and $\hat{\delta}_n$ defined in terms of subsets $\Omega_{n,i}, \hat{\Omega}_{n,j} \subseteq S^{d-1}$ as above, or by taking

$$\beta_n(\zeta) = \left(1 + \frac{\varepsilon_n(\zeta)}{\sqrt{n}}\right)\zeta, \quad \hat{\beta}_n(\zeta) = \left(1 + \frac{\hat{\varepsilon}_n(\zeta)}{\sqrt{n}}\right)\zeta,$$

for continuous functions $\varepsilon_n, \hat{\varepsilon}_n : S^{d-1} \to \mathbb{R}$ as in the statement of Theorem 4.

3.3.4 Perturbation in the radial case

The idea of inverting an operator close to the identity, expressed in terms of an interpolation formula, can also be used to perturb the interpolation formulas for radial Schwartz functions in Theorem 1. Let us explain how this works, as it will give us the chance to use more of the properties of the spaces $V^s(\mathbb{R}^d)$. More precisely, we will be working with the subspace $V^s_{\mathrm{rad}}(\mathbb{R}^d) \subseteq V^s(\mathbb{R}^d)$ of radial functions. It is closed and hence complete, so it is a Banach space itself by Lemma 3.4 and $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ is dense in $V^s_{\mathrm{rad}}(\mathbb{R}^d)$ and the intersection of all of these Banach spaces.

Let $d \geq 1$ be an integer and let $n_0, \hat{n}_0 \geq 0$ be integers such that $n_0 + \hat{n}_0 = 1 + \lfloor d/4 \rfloor$. Let $a_{d/2,n}\tilde{a}_{d/2,n} \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ be the attached radial Schwartz functions coming from Theorem 1. Fix some parameter $s \geq 1$. For all $n \in \mathbb{N}_0$, let $\varepsilon_n, \hat{\varepsilon}_n$ be two real numbers. We will show that if these numbers are sufficiently small, then we can make sense of the following operator on $V^s_{\mathrm{rad}}(\mathbb{R}^d)$ and invert it.[29]

$$Tf := \sum_{n=0}^{\infty} f(\sqrt{n} + \varepsilon_n) a_{d/2,n} + \sum_{n=0}^{\infty} \hat{f}(\sqrt{n} + \hat{\varepsilon}_n) \tilde{a}_{d/2,n} \tag{3.54}$$

This doesn't directly make sense as an operator on $V^s_{\mathrm{rad}}(\mathbb{R}^d)$, but we can make sense of it abstractly in the following way. First, it is clear from Theorem 1 that T defines a continuous linear endomorphism of $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$. We recall that the latter space is dense in $V^s_{\mathrm{rad}}(\mathbb{R}^d)$ and we can extend T to that space by continuously extending $f \mapsto f - Tf$ to it. To show that this is possible, we estimate, for $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$, using (3.50), and the interpolation formula (2.47),

$$\|f - Tf\|_{V^s} \lesssim_d \|f\|_{V^s} \sum_{n=0}^{\infty} \left(|\varepsilon_n| \|a_{d/2,n}\|_{V^s} + |\hat{\varepsilon}_n| \|\tilde{a}_{d/2,n}\|_{V^s} \right) \tag{3.55}$$

On the other hand we know by Theorem 1 that

$$\|a_{d/2,n}\|_{V^s} + \|\tilde{a}_{d/2,n}\|_{V^s} \lesssim (1+n)^{\alpha_s}$$

for some $\alpha_s = \alpha_s(d) \geq 0$. Thus, by assuming (for instance)

$$|\varepsilon_n| + |\hat{\varepsilon}_n| \leq \delta(1+n)^{-\alpha_s - 1.1} \tag{3.56}$$

for some $\delta > 0$ small enough in terms of the implied constant in (3.55), we may deduce that $f \mapsto f - Tf$ is continuous $\mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d) \to V^s_{\mathrm{rad}}(\mathbb{R}^d)$, hence extends continuously to a continuous endomorphism of $V^s_{\mathrm{rad}}(\mathbb{R}^d)$, and so does T itself with $\|T - \mathrm{id}_{V^s}\| < 1$. Only for this last property do we need to take δ small. In particular, T is invertible and we can define, for every $n \in \mathbb{N}_0$, the functions

$$c_{d/2,n} := T^{-1}(a_{d/2,n}) \in V^s_{\mathrm{rad}}(\mathbb{R}^d), \qquad \tilde{c}_{d/2,n} := T^{-1}(\tilde{a}_{d/2,n}) \in V^s_{\mathrm{rad}}(\mathbb{R}^d).$$

[29] As opposed to [RS21] or [RdS20], we write the perturbations of $\sqrt{n}$ as $\sqrt{n} + \varepsilon_n$ instead of $\sqrt{n + \varepsilon_n}$, which allows for a slightly quicker treatment

Since $f = T^{-1}(Tf)$ for all $f \in V^s_{\mathrm{rad}}(\mathbb{R}^d)$, we would like to say that (by pulling T^{-1} inside the sum (3.54)) we have

$$f = \sum_{n=0}^{\infty} f(\sqrt{n} + \varepsilon_n) c_{d/2,n} + \sum_{n=0}^{\infty} \hat{f}(\sqrt{n} + \hat{\varepsilon}_n) \tilde{c}_{d/2,n} \tag{3.57}$$

for all $f \in V^s_{\mathrm{rad}}(\mathbb{R}^d)$ with convergence in that space. However, this can't quite be justified and is also unlikely to be true abstract reasons, as it would express $\mathrm{id}_{V^s_{\mathrm{rad}}}$ as a limit of finite rank operators in the strong operator topology[30]. Instead, we claim that there is $s' = s'(s) > s$ so that (3.57) holds for all $f \in V^{s'}_{\mathrm{rad}}$ with convergence in the V^s-topology. Indeed, for $f \in V^{s'}_{\mathrm{rad}}$, we have, by Lemma 3.5 that

$$|f(\sqrt{n} + \varepsilon_n)| + |\hat{f}(\sqrt{n} + \hat{\varepsilon}_n)| \lesssim (1+n)^{-\frac{s'}{2(d+1)}}$$

and hence the series in the definition of Tf (3.54) converges in the V^s topology, provided $f \in V^{s'}_{\mathrm{rad}}$ and

$$\frac{-s'}{2(d+1)} + \alpha_s < -1 \quad \Leftrightarrow \quad s' > 2(d+1)(\alpha_s + 1). \tag{3.58}$$

In this case, the proof of (3.57) by writing $f = T^{-1}(Tf)$ and interchanging the series in (3.54) with the operator T^{-1} is justified. We have proved the following theorem.

Theorem 5. *Let $d, n_0, \hat{n}_0$ and $s \geq 1$ be as above. Then, provided ε_n and $\hat{\varepsilon}_n$ are sufficiently small (as quantified in* (3.56)*) and provided s' is sufficiently large (as quantified in* (3.58)*), the perturbed interpolation formula* (3.57) *holds for all $f \in V^{s'}_{rad}(\mathbb{R}^d)$ with convergence in $V^s_{rad}(\mathbb{R}^d)$.*

Remark 3.11. Let us return to the expression defining T in (3.54). We defined T abstractly as a continuous linear operator on $V^s_{\mathrm{rad}}(\mathbb{R}^d)$, but let us remark that Lemma 3.5 clearly implies that the series defining T defines "directly" a bounded linear operator $V^{s'}_{\mathrm{rad}}(\mathbb{R}^d) \to V^s_{\mathrm{rad}}(\mathbb{R}^d)$ for some $s' = s'(d,s) < s$. Our remark is that this latter fact is compatible with the abstract result that any continuous linear map defined on a limit of Banach spaces, with dense image in each of its limitands, necessarily factors through one of these limitands, provided the image is a normed vector space.[31] We illustrate this in the following commutative diagram

$$\begin{tikzcd}
\varprojlim V^s_{\mathrm{rad}}(\mathbb{R}^d) = \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d) \arrow[r] \arrow[d, hook] & \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d) \arrow[dd, hook] \\
V^{s'}_{\mathrm{rad}}(\mathbb{R}^d) \arrow[dr, dashed, "T"] & \\
& V^s_{\mathrm{rad}}(\mathbb{R}^d)
\end{tikzcd}$$

The actual operator T we use is an endomorphism of $V^s(\mathbb{R}^d)$ (which restricts to the operator T labeling the dashed arrow above)

[30] albeit not necessarily in the operator norm topology, which is we say "unlikely" instead of "impossible". If the RHS of (3.57) converged in the operator norm topology, then it would mean that $\mathrm{id}_{V^s(\mathbb{R}^d)}$ is compact, which is impossible

[31] If the target of the map is $\mathbb{C}$, then we do not need to assume density. A proof of these claims is given in [Gar18, ch. 13, claim 13.14.4]

3.4 Application towards uniqueness of magic functions

By "magic" functions in the title of this section we mean certain radial Schwartz functions on $\mathbb{R}^8$ and $\mathbb{R}^{24}$ that were constructed by Viazovska [Via17] and Cohn, Kumar, Miller, Radchenko and Viazovska [CKM$^+$17] as integral transforms of weakly holomorphic (quasi-) modular forms to resolve the sphere packing problem in 8 and 24 dimensions respectively. These functions are solutions to a convex Fourier optimization problem that we briefly recall below. The subsequent work [CKM$^+$21] proved a much stronger result about the optimal sphere packing configurations in those dimensions, which are given by the E_8 and the Leech lattice respectively. Namely, it was proved that they are universally optimal energy minimizing configurations for a large class of potential functions. We don't need to know the details of that result − what is important for us is that the main tools in the proof were Fourier interpolation formulas, applying to all $f \in \mathcal{S}(\mathbb{R}^p)$, $p = 8, 24$, of the from

$$f(x) = \sum_{n=n_0(p)}^{\infty} \Big(f(\sqrt{2n}) a_{p,n}(|x|) + f'(\sqrt{2n}) b_{p,n}(|x|) + \widehat{f}(\sqrt{2n}) \tilde{a}_{p,n}(|x|) + \hat{f}'(\sqrt{2n}) \tilde{b}_{p,n}(|x|) \Big) \tag{3.59}$$

in which $n_0(p) \in \mathbb{N}$, $n_0(8) = 1$, $n_0(24) = 2$ and $a_{p,n}, b_{p,n}, \tilde{a}_{p,n}, \tilde{b}_{p,n}$ are even Schwartz function on $\mathbb{R}$ (and $x \in \mathbb{R}^p$ is arbitrary). Moreover, these are free interpolation formulas in the sense that if we replace $f(\sqrt{2n}), f'(\sqrt{2n}), \hat{f}(\sqrt{2n}), \hat{f}'(\sqrt{2n})$ on the RHS by *any* quadruple of rapidly decaying sequences of complex numbers then the series will define a radial Schwartz function f on $\mathbb{R}^p$ such that f and $\hat{f}$ and its radial derivatives evaluate at $\sqrt{2n}$ to those given four sequences.

As announced above, we now briefly recall the Cohn–Elkeis linear programming method for sphere packing bounds and its relation to the above formulas. As opposed to any other work using this method, we will not assume that our functions are radial.[32] Instead, we will show that any *optimal* function is *necessarily* radial, provided formulas as the one in (3.59) exist in all even dimensions $d \geq 8$ or 24 respectively.

For any dimension $d \geq 1$, we write $\mathcal{A}_{\mathrm{LP}}(d)$ for the set of all continuous, even and integrable functions $f : \mathbb{R}^d \to \mathbb{R}$ having the following properties:

(1) $\widehat{f}(\xi) \geq 0$ for all $\xi \in \mathbb{R}^d$.

(2) $f(0) = \hat{f}(0) > 0$.

(3) The quantity $r_-(f) := \inf \{ r > 0 : \forall x \in \mathbb{R}^d : |x| \geq r \Rightarrow f(x) \leq 0 \}$ is finite. In other words, f is eventually non-positive.

Under an additional decay assumption[33], the main result in [CE03][Thm 3.2] implies that the sphere packing constant Δ_d in $\mathbb{R}^d$ (see the cited reference for a definition) satisfies

$$\Delta_d \leq |B^d_{1/2}(0)| \inf_{f \in \mathcal{A}_{LP}(d)} r_-(f)^d, \tag{3.60}$$

where $B^d_r(x)$ denotes the open ball of radius r, centered at $x \in \mathbb{R}^d$ and $|E|$ denotes the Lebesgue measure of a measurable subset $E \subseteq \mathbb{R}^d$.

Now let $d = 8$ or $d = 24$ and let $\Lambda_d \subseteq \mathbb{R}^d$ be the E_8-, or the Leech lattice respectively. We assume that these lattices are scaled so that the shortest vectors have lengths $\sqrt{2n_0(d)}$, where

[32] for the purpose of proving the existence of am optimal function for this program, the assumption of being radial can be made without loss of generality in the sense that if an extremizer exists, then also a radial one, by averaging over the orthogonal group.

[33] it enables an application of Poisson summation, but can be removed, see [CE03][Sec. 9] or [CZ14][Thm 3.3].

$n_0(8) = 1$, $n_0(24) = 2$. We suppose given a function $\varphi = \varphi_d \in \mathcal{S}(\mathbb{R}^d) \cap \mathcal{A}_{\mathrm{LP}}(d)$ (maybe not radial) such that $r_-(\varphi) \leq \sqrt{2n_0(d)}$. By (3.60) the existence of φ would prove that Λ_d gives the optimal sphere packing in $\mathbb{R}^d$. (By the cited works, we know such φ exist, but we pretend that we don't know this in the following analysis, to emphasize the uniqueness statement that we will prove.)

The Poisson summation formula and the fact that Λ_d is self-dual and (hence) unimodular imply

$$\varphi(0) + \sum_{0 \neq \lambda \in \Lambda_d} \varphi(\lambda) = \widehat{\varphi}(0) + \sum_{0 \neq \lambda^* \in \Lambda_d} \widehat{\varphi}(\lambda^*). \tag{3.61}$$

Here, we can subtract $\varphi(0) = \widehat{\varphi}(0)$ from both sides and are then left with the equality of a sum of non-positive real numbers on the left with a sum of non-negative real numbers on the right, which forces all of these numbers to be zero. Given any orthogonal transformation $\rho \in \mathrm{O}(d)$, we can repeat the same argument with Λ_d replaced by the self-dual lattice $\rho(\Lambda_d)$. By varying ρ over $\mathrm{O}(d)$, and using the assumptions on φ, we deduce that any such φ must have the following properties:

(i) $\varphi(\sqrt{2n}\zeta) = \widehat{\varphi}(\sqrt{2n}\zeta) = 0$ for all $\zeta \in S^{d-1}$ and all integers $n \geq n_0(d)$,

(ii) $\nabla\widehat{\varphi}(\sqrt{2n}\zeta) = 0$ for all $\zeta \in S^{d-1}$ and all integers $n \geq n_0(d)$ since $\widehat{\varphi}$ is C^1 and non-negative,

(iii) $\nabla\varphi(\sqrt{2n}\zeta) = 0$ for all $\zeta \in S^{d-1}$ and all integers $n \geq n_0(d) + 1$ since f is C^1 and non-negative on the open set $\{x \in \mathbb{R}^d \, : \, |x|^2 > n_0(d)\}$.

In words, φ and $\widehat{\varphi}$ and their gradients vanish on all spheres $\sqrt{2n}S^{d-1}$ for integers $n \geq n_0(d)$, except $\nabla\varphi$ might not vanish on $\sqrt{2n_0(d)}S^{d-1}$. We now make the following assumption.

Assumption 1. *Assume that for every even dimension $p \geq d$, there exist four sequences of functions $a_{p,n}, b_{p,n}, \tilde{a}_{p,n}, \tilde{b}_{p,n} : \mathbb{R} \to \mathbb{R}$, indexed by $n \in \mathbb{N}_0$, so that for all $x \in \mathbb{R}^p$ and all $f \in \mathcal{S}_{rad}(\mathbb{R}^p)$ the formula* (3.59) *holds with point-wise convergence*[34] *and with a monotonically increasing function $n_0 : 2\mathbb{N} \to \mathbb{N}$, $p \mapsto n_0(p)$, satisfying:*

- $n_0(8) = 1$ *and* $n_0(p) > 1$ *for all* $p > 8$,
- $n_0(24) = 2$ *and* $n_0(p) > 2$ *for all* $p > 24$.

We continue working under this assumption and with a function $\varphi = \varphi_d \in \mathcal{S}(\mathbb{R}^d)$ as above. For each $m \in \mathbb{N}_0$, let $\mathcal{B}_m \subseteq \mathcal{H}_m(\mathbb{R}^d)$ be an orthonormal basis. Fix any $x \in \mathbb{R}^d \setminus \{0\}$. Recall from (3.21) that

$$\varphi(x) = \sum_{m=0}^{\infty} \sum_{u \in \mathcal{B}_m} u(x)(L_u^{d+2m}\varphi)(\iota_m(x)) \tag{3.62}$$

where $\iota_m(x) \in \mathbb{R}^{d+2m}$ any vector of norm $|x|$ and $L_u^{d+2m}\varphi \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^{d+2m})$ is defined as in Definition 3.2. By Assumption 1, we have, for fixed $m \in \mathbb{N}_0$ and $u \in \mathcal{B}_m$ in the above expansion (3.62)

$$\begin{aligned}(L_u^{d+2m}\varphi)(\iota_m(x)) = \sum_{n=n_0(d+2m)}^{\infty} \Big(&\alpha_{m,n}(\varphi)a_{d+2m,n}(|x|) \; + \; \beta_{m,n}(\varphi)b_{d+2m,n}(|x|) \\ &+ \tilde{\alpha}_{m,n}(\varphi)\tilde{a}_{d+2m,n}(|x|) \; + \; \tilde{\beta}_{m,n}(\varphi)\tilde{b}_{d+2m,n}(|x|)\Big)\end{aligned}$$

[34] We do not assume absolute convergence in (3.59) nor *any* regularity or decay of the functions $a_{p,n}, b_{p,n}, \tilde{a}_{p,n}, \tilde{b}_{p,n}$ whatsoever. Also, in order to have well-defined expressions in all of the following formulas, we set all of these functions equal to zero if $n < n_0(p)$.

where, by Proposition 3.4 and by Lemma 3.2, we have, for every integer $n \geq n_0(d+2m)$,

$$\begin{aligned}
\alpha_{m,n}(\varphi) &= (L_u^{d+2m}\varphi)(\sqrt{2n}) = \frac{1}{\sqrt{2n}^m}\int_S \varphi(\sqrt{2n}\zeta)\overline{u(\zeta)}d\zeta \\
\tilde{\alpha}_{m,n}(\varphi) &= \mathcal{F}_{\mathbb{R}^{d+2m}}(L_u^{d+2m}\varphi)(\sqrt{2n}) = i^m L_u^{d+2m}(\widehat{\varphi})(\sqrt{2n}) = \frac{i^m}{\sqrt{2n}^m}\int_S \widehat{\varphi}(\sqrt{2n}\zeta)\overline{u(\zeta)}d\zeta \\
\beta_{m,n}(\varphi) &= (L_u^{d+2m}\varphi)'(\sqrt{2n}) = \frac{-m}{\sqrt{2n}}(L_u^{d+2m}\varphi)(\sqrt{2n}) + \frac{1}{\sqrt{2n}^m}\int_S \langle\nabla\varphi(\sqrt{2n}\zeta),\zeta\rangle\overline{u(\zeta)}d\zeta \\
\tilde{\beta}_{m,n}(\varphi) &= \frac{-i^m m}{\sqrt{2n}}(L_u^{d+2m}\widehat{\varphi})(\sqrt{2n}) + \frac{i^m}{\sqrt{2n}^m}\int_S \langle\nabla\widehat{\varphi}(\sqrt{2n}\zeta),\zeta\rangle\overline{u(\zeta)}d\zeta
\end{aligned}$$

It follows from these formulas and the properties (i), (ii) and (iii) of φ listed above, that

$$\alpha_{m,n}(\varphi) = \tilde{\alpha}_{m,n}(\varphi) = \tilde{\beta}_{m,n}(\varphi) = 0,$$

for every $m \in \mathbb{N}_0$ and $n \geq n_0(d+2m)$ and that

$$\beta_{m,n}(\varphi) = \frac{1}{\sqrt{2n}^m}\int_S \langle\nabla\varphi(\sqrt{2n}\zeta),\zeta\rangle\overline{u(\zeta)}d\zeta = 0 \quad \text{if } n > n_0(d).$$

Consequently, we have

$$\varphi(x) = \varphi_d(x) = \sum_{m=0}^{\infty}\sum_{u\in\mathcal{B}_m} u(x) \sum_{n=n_0(d+2m)}^{n_0(d)} b_{d+2m,n}(|x|)\frac{1}{\sqrt{2n}^m}\int_S \langle\nabla\varphi(\sqrt{2n}\zeta),\zeta\rangle\overline{u(\zeta)}d\zeta, \tag{3.63}$$

where the inner sum over n is either empty (and hence zero) or reduced to $n = n_0(d)$, as we are assuming that $n_0(d) \leq n_0(d+2m)$ for all $m \in \mathbb{N}_0$. In fact, by our more precise assumption on the function $p \mapsto n_0(p)$, the following holds:

- if $d = 8$, then $b_{p,1} = 0$ for all even $p \geq 10$ by Assumption 1 and hence

$$\varphi_8(x) = b_{8,1}(|x|)\int_S \langle\nabla\varphi(\sqrt{2}\zeta),\zeta\rangle d\zeta, \tag{3.64}$$

- if $d = 24$, then $b_{p,1} = b_{p,2} = 0$ for all even $p \geq 26$ by Assumption 1 and hence

$$\varphi_{24}(x) = b_{24,2}(|x|)\int_S \langle\nabla\varphi(2\zeta),\zeta\rangle d\zeta. \tag{3.65}$$

We have assumed that $x \in \mathbb{R}^d \smallsetminus \{0\}$, but (3.64) (3.65) must then clearly also hold for $x = 0$ by continuity.

Let us summarize our analysis in the following theorem.

Theorem 6 (conditional on Assumption 1)**.** *Let the set up and assumptions be as above. In particular, $d \in \{8, 24\}$ and we suppose $\varphi_d \in \mathcal{S}(\mathbb{R}^d) \cap \mathcal{A}_{LP}(d)$ is an optimal Cohn–Elkies function. Then φ_8 must be given by* (3.64) *and φ_{24} by* (3.65)*. In particular, φ_d is radial and a multiple of to the function $x \mapsto b_{d,n_0(d)}(|x|)$.*

We end with a few remarks. First, we should remark that the existence and uniqueness of optimal Cohn-Elkies functions *among radial* Schwartz functions in known: The works [Via17], [CKM+17] prove existence and [CKM+21] proves existence and uniqueness among radial Schwartz functions. Theorem 6 thus represents a strengthening of that result.

We remark that we believe that Assumption 1 holds true. The conditions on the function n_0 we impose are quite specific, but even if they can't be met and we only know that $p \mapsto n_0(p)$ is increasing (and tending to infinity), then the above argument shows that the space of optimal φ_d, as above, is a finite dimensional subspace of the space spanned by all functions of the form $x \mapsto b_{d+2m,n}(|x|)u(x)$ with $m \in \mathbb{N}_0$ and $u \in \mathcal{H}_m(\mathbb{R}^d)$.

Finally, we observe that the above arguments show that, if Assumption 1 holds, then the "basis function" $b_{d,n_0(d)}$ is necessarily given by a nonzero multiple of the magic function.

4 Fourier non-uniqueness sets and totally real number fields

This chapter is based on joint work with Danylo Radchenko [RS][35]. We will prove two theorems which give two types of families $\{A_K\}_K$ of discrete Fourier *non*-uniqueness sets $A_K \subseteq \mathbb{R}^n$ parameterized by totally real number fields $K/\mathbb{Q}$ of degree n. These theorems contrast our main Fourier uniqueness results from Chapter 3. In one theorem (Theorem 8), the sets A_K are given as "component-wise square roots" of lattices given by fractional ideals in K and we have $A_K \subseteq \cup_{m\in\mathbb{N}_0}\sqrt{m}S^{n-1}$. More precisely, for any lattice $\Lambda \subseteq \mathbb{R}^n$, we set

$$\sqrt{\Lambda} := \{(x_1,\dots,x_n) \in \mathbb{R}^n \,:\, (x_1^2,\dots,x_n^2) \in \Lambda\} \tag{4.1}$$

and we consider lattices $\Lambda = \Lambda_K$ which is the image of the co-different $\mathcal{O}_K^\vee$ of K under the real embeddings of K in $\mathbb{R}^n$. In the other Theorem (Theorem 7) the sets A_K are discrete subsets of ellipsoids defined in another way in terms of the co-different.

Before discussing those results in detail, we will start in §4.1 with a general discussion depending on two arbitrary lattices $\Lambda_1, \Lambda_2 \subseteq \mathbb{R}^n$ and ask when there can exist an interpolation formula with nodes given by the sets $\sqrt{\Lambda_1}, \sqrt{\Lambda_2}$, on the function- and the Fourier side respectively. We will do so by attaching to those lattices a certain subgroup $\Gamma(2\Lambda_1^\vee, 2\Lambda_2^\vee) \leq \mathrm{PSL}_2(\mathbb{R})^n$ and formulate necessary conditions for the existence of the interpolation formulas in terms of that group. In particular, we show that if $n \geq 2$ and if that group is discrete (which we also conjecture to be a necessary condition), then there is no such interpolation formula.

4.1 Interpolation formulas with square roots of lattices

4.1.1 Set up

In §4.1 we will be working in the following generality. Let $n, d, d_1, \dots, d_n \geq 1$ be integers such that $d = d_1 + \dots + d_n$. We often view $\mathbb{R}^d$ as a product $\mathbb{R}^d = \prod_{j=1}^n \mathbb{R}^{d_j}$ and denote its elements correspondingly as n-tuples $x = (x_1, \dots, x_n)$ with $x_j \in \mathbb{R}^{d_j}$. We view the product of orthogonal groups $\mathrm{O}(d_1) \times \dots \times \mathrm{O}(d_n)$ embedded block-diagonally in $\mathrm{O}(d)$ and acting diagonally on $\mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_n}$. For $z = (z_1, \dots, z_n) \in \mathbb{H}^n$, we define the Gaussian

$$g(z) \in \mathcal{S}(\mathbb{R}^d)^{\mathrm{O}(d_1)\times\dots\times\mathrm{O}(d_n)} \quad \text{by} \quad g(z)(x) = g(z,x) = e^{\pi z_1 |x_1|^2} \cdots e^{\pi i z_n |x_n|^2}, x \in \mathbb{R}^d. \tag{4.2}$$

The following elementary lemma will be useful on different occasions in this chapter. Note that it implies the linear independence of the above functions $g(z)$ for different values of $z \in \mathbb{H}^n$.

[35]Other chapters of this thesis also contain some parts of the cited paper, for instance the addition of the parameter $h \geq 2$ in the set up of §2.2.2 and in Theorem 2.

Lemma 4.1. *Let $m \geq 1$ and let $c_1, \ldots, c_m \in \mathbb{C}^n$ be pair-wise distinct n-tuples of complex numbers. Then the functions $g_i : \mathbb{R}^n \to \mathbb{C}$, $1 \leq i \leq m$, defined by $g_i(r) = e^{\sum_{j=1}^m c_{ij} r_j^2}$ are linearly independent.*

Proof. We induct on m, the case $m = 1$ being clear. Assume that $m \geq 2$ and that $t_1, \ldots, t_m \in \mathbb{C}$ are such that $\sum_{i=1}^m t_i g_i(r) = 0$ for all $r \in \mathbb{R}^n$. We divide by $g_m(r)$ and obtain

$$0 = \sum_{i=1}^{m} t_i g_i(r) g_m(r)^{-1} = \sum_{i=1}^{m-1} t_i e^{\sum_{j=1}^n (c_{ij} - c_{mj}) r_j^2} + t_m$$

for all $r \in \mathbb{R}^n$. We differentiate with respect to any variable r_k and obtain

$$0 = \sum_{i=1}^{m-1} 2 t_i r_k (c_{ik} - c_{mk}) e^{\sum_{j=1}^n (c_{ij} - c_{mj}) r_j^2}.$$

By continuity, this remains true, if we divide through by r_k. By induction, (since the $m-1$ elements $c_i' = c_i - c_m \in \mathbb{C}^n$ are pairwise distinct), we deduce $t_i(c_{ik} - c_{mk}) = 0$ for all $1 \leq i \leq m-1$ and all $1 \leq k \leq n$, hence $t_i = 0$ for all $1 \leq i \leq m-1$ and then also $t_m = 0$, as desired. ☐

4.1.2 Generating series and functional equations

The following discussion is a generalization of the one in §2.2.2. Consider two lattices $\Lambda_1, \Lambda_2 \subseteq \mathbb{R}^n$. For $i = 1, 2$, we define

$$\Lambda_{i,+} := \Lambda_i \cap [0, \infty)^n.$$

We also define $\mathrm{sq} : [0, \infty)^n \to [0, \infty)^n$ by $\mathrm{sq}(x_1, \ldots, x_n) := (\sqrt{x_1}, \ldots, \sqrt{x_n})$.

We want to know whether there exist functions $a_\lambda, \tilde{a}_\mu : \mathbb{R}^n \to \mathbb{C}$ such that for all $f \in \mathcal{S}(\mathbb{R}^d)^{\mathrm{O}(d_1) \times \cdots \times \mathrm{O}(d_n)}$ and all $x = (x_1, \ldots, x_n) \in \mathbb{R}^d$, we have

$$f(x_1, \ldots, x_n) = \sum_{\lambda \in \Lambda_{1,+}} a_\lambda(|x_1|, \ldots, |x_n|) f(\mathrm{sq}\,(\lambda)) + \sum_{\mu \in \Lambda_{2,+}} \tilde{a}_\mu(|x_1|, \ldots, |x_n|) \widehat{f}(\mathrm{sq}\,(\mu)). \tag{4.3}$$

Let us first *assume* that such functions $a_\lambda, \tilde{a}_\mu$ exist and that, for each fixed $r = (r_1, \ldots, r_n) \in [0, +\infty)^n$, they grow at most polynomially in their index parameters $\lambda \in \Lambda_1$ and $\mu \in \Lambda_2$ respectively (formally speaking, we assume that for each r, there are $C = C_r$ and $N = N_r$ so that $|a_\lambda(r)| \leq C(1 + |\lambda|)^N$ and similarly for $\tilde{a}_\mu$). We consider the generating functions

$$F(z, r) = \sum_{\lambda \in \Lambda_{+,1}} a_\lambda(r) e^{\pi i \sum_{i=1}^n z_i \lambda_i}, \quad \tilde{F}(z, r) = \sum_{\mu \in \Lambda_{2,+}} \tilde{a}_\mu(r) e^{\pi i \sum_{i=1}^n z_i \lambda_i}, \quad z \in \mathbb{H}^n. \tag{4.4}$$

By construction, each of these functions is holomorphic in $z \in \mathbb{H}^n$ and periodic with respect to the lattices $2\Lambda_1^\vee$ or $2\Lambda_2^\vee$ respectively. Moreover, applying the formula (4.3) to the Gaussian $f = g(z)$, as defined in (4.2) shows that

$$g(z, r) = F(z, r) + (z_1/i)^{-d_1/2} \cdots (z_n/i)^{-d_n/2} \tilde{F}(-1/z, r). \tag{4.5}$$

where $-1/z$ is a shorthand for $(-1/z_1, \ldots, -1/z_n)$. Conversely, no longer assuming the existence of $a_\lambda, \tilde{a}_\mu$, but the existence of holomorphic $2\Lambda_1^\vee$-periodic functions $z \mapsto F(z, r)$ and holomorphic $2\Lambda_2^\vee$-periodic functions $z \mapsto \tilde{F}(z, r)$ satisfying suitable growth conditions[36], which are related via the functional equations (4.5) and with Fourier expansions indexed over $\Lambda_{i,+}$ only (instead of the whole Λ_i), we can deduce an interpolation formula (4.3) by appealing to Proposition 2.4 (which asserted the density of the span of the functions $g(z)$ in $\mathcal{S}(\mathbb{R}^d)^{\mathrm{O}(d_1) \times \cdots \times \mathrm{O}(d_n)}$).

[36] we don't have to be as precise about this point as in §2.2.2, because we will have no chance of making use of this direction of the (implicitly formulated) equivalence.

4.1.3 Conditions (D) and (F)

We continue in the set up of the previous section and assume the existence of a formula like (4.3) and hence the functions F and $\tilde{F}$, defined as above. Similarly to what we did in the case $n = 1$ in §2.2.2, we will now describe the transformation laws of F and $\tilde{F}$ more group theoretically, by describing them with a subgroup of $\mathrm{PSL}_2(\mathbb{R})^n$ depending on Λ_1, Λ_2 (but not on the dimensions d_j). Eventually, our discussion here will lead us to prove the non-existence of certain interpolation formulas of the form (4.3) for certain classes of lattices (e.g. $\Lambda_1 = \Lambda_2 = \mathbb{Z}^n$).

We will often freely switch between $\mathrm{PSL}_2(\mathbb{R})^n$ and the isomorphic group $G := \mathrm{PSL}_2(\mathbb{R}^n)$, where we view $\mathbb{R}^n = \mathbb{R}\times\cdots\times\mathbb{R}$ as a commutative ring with component-wise addition and multiplication. For $x \in \mathbb{R}^n$ we define

$$T^x := \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in G, \qquad V^x := \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \in G, \tag{4.6}$$

where $0 = (0,\dots,0)$, $1 = (1,\dots,1)$. We also define the element $S := \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \in G$, so that $ST^xS = V^{-x}$. For any lattice $L \subseteq \mathbb{R}^n$, we define the following subgroups of G:

$$\Gamma_{\mathrm{upp}}(L) := \{T^x \,:\, x \in L\} \cong L, \qquad \Gamma_{\mathrm{low}}(L) := \{V^y \,:\, y \in L\} \cong L$$

and then, for any two lattices $L_1, L_2 \subseteq \mathbb{R}^n$, the subgroup

$$\Gamma(L_1, L_2) := \langle \Gamma_{\mathrm{upp}}(L_1) \cup \Gamma_{\mathrm{low}}(L_2)\rangle \leq G.$$

To relate this group to the assumed interpolation formula, we now set $L_i = 2\Lambda_i^\vee$ for $i = 1, 2$. To proceed, let us suppose that we are given a cocycle $J : G \to \mathrm{Hol}(\mathbb{H}^n, \mathbb{C}^\times)$ satisfying

$$J(S)(z) = \prod_{j=1}^{n} (z_j/i)^{d_j/2} =: \mu(z), \qquad \text{and} \quad J(T^x) = 1 \quad \text{for all } x \in \mathbb{R}^n.$$

We may then define a slash action of G (and its group algebra $\mathbb{C}[G]$) on functions f on $\mathbb{H}^n$ by $f|\gamma := J(\gamma)^{-1}\cdot(f\circ\gamma)$, $\gamma \in G$.

Remark 4.1. In practice, it suffices that J can be defined only on the subgroup of G, generated by $\Gamma(L_1, L_2)$ and S, but its existence is non-trivial and may not always be guaranteed. On the other hand, when $8|d_j$ for all j, such a cocycle J can be defined on the full group G, namely, we can define

$$J_{G;d_1,\dots,d_n}(g) = \prod_{j=1}^{n} (g_j')^{-d_j/4}, \quad \text{where} \quad g = (g_1,\dots,g_n) \in \mathrm{PSL}_2(\mathbb{R})^n$$

and g_j' is the derivative of the Möbius transformation g_j. So more explicitly,

$$J_{G;d_1,\dots,d_n}(g)(z) = J_{G;d_1,\dots,d_n}(g_1,\dots,g_n)(z_1,\dots,z_n) = \prod_{j=1}^{n} (c_{g_j} z_j + c_{g_j})^{-d_j/2}.$$

The results we prove about general lattices in this section do not depend upon the existence of J.

We return to the functions F, $\tilde{F}$ introduced in (4.4). In what follows we will suppress the parameters $r \in [0,\infty)^n$ and $z \in \mathbb{H}^n$ from the notation. Using the slash action just introduced, F and $\tilde{F}$ (as functions on $\mathbb{H}^n$) must satisfy (besides certain growth conditions)

(1) $F|(T^x - 1) = 0$ for all $x \in L_1$. This says that F is L_1-periodic.

(2) $\tilde{F}|(T^y - 1) = 0$ for all $y \in L_2$. This says that $\tilde{F}$ is L_2-periodic.

(3) $F + \tilde{F}|S = g$. The functional equation (4.5) holds, where g denotes the Gaussian, as defined in (4.2).

The same arguments as in the case $n = 1$ in §2.2.2 show that it suffices to find only F satisfying

(a) $F|(T^x - 1) = 0$ for all $x \in L_1$.

(b) $F|(V^y - 1) = g|(V^y - 1)$ for all $y \in L_2$

Indeed, we can then define $\tilde{F}$ as $\tilde{F} = g|S - F|S$ and this function will be L_2-periodic.

We see from the above cohomological formalism that every relation between elements in the group $\Gamma = \Gamma(L_1, L_2)$ imposes a condition on the 1-cocycle

$$\Phi : \gamma \mapsto F|(\gamma - 1), \quad \gamma \in \Gamma.$$

There are trivial relations that come from the free abelian subgroups $\Gamma_{\mathrm{upp}}(L_1)$ and $\Gamma_{\mathrm{low}}(L_2)$ that are always respected. There is, however, no reason why a "mixed" relation between elements of these two groups should be preserved by Φ, as any such relation translates to non-trivial conditions for the Gaussian g. Thus, one would like that

$$\boxed{\Gamma(L_1, L_2) \textit{ is the free inner product of } \Gamma_{\mathrm{upp}}(L_1) \textit{ and } \Gamma_{\mathrm{low}}(L_2).} \tag{F}$$

A natural further desideratum is:

$$\boxed{\Gamma(L_1, L_2) \textit{ is discrete in } G \cong \mathrm{PSL}_2(\mathbb{R})^n.} \tag{D}$$

In fact, the existence of F and $\tilde{F}$ with the above transformation properties *implies* (F) by the following proposition.

Proposition 4.1. *Assume that there exist functions F and $\tilde{F}$ as in* (4.4) *satisfying* (4.5). *Then condition* (F) *holds.*

We give the proof of Proposition 4.1 below, but we start by giving a proof in the special case, where the existence of a cocycle J is guaranteed (e.g. when $8|d_j$ for all j). We hope that this may help in understanding the general proof.

Proof of Proposition 4.1, assuming existence of J. By way of contradiction, assume that (F) fails and consider a non-trivial relation

$$V^{y_1} T^{x_1} V^{y_2} T^{x_2} \cdots V^{y_m} T^{x_m} = 1$$

with $m \geq 1$ minimal and with $x_1, \ldots, x_m \in L_1$, $y_1, \ldots, y_m \in L_2$, all nonzero (by conjugation with some T^x or V^y if necessary, we can bring any minimal non-trivial relation into the above form). Consider the cocycle $\Phi(\gamma) = F|(\gamma - 1)$ as above and apply the cocycle property $\Phi(\gamma_1\gamma_2) = \Phi(\gamma_1)|\gamma_2 + \Phi(\gamma_2)$ repeatedly, to obtain

$$0 = \Phi(1) = \sum_{i=1}^{m} \Phi(V^{y_i})|P_i = \sum_{i=1}^{m} (g|V^{y_i}P_i - g|P_i), \quad P_i := T^{x_i} V^{y_{i+1}} \cdots V^{y_m} T^{x_m}. \tag{4.7}$$

Since $x_m \neq 0$, all $2m$ group elements $V^{y_i}P_i, P_i$ are pairwise distinct by minimality of m. Thus, we have an identity $0 = \sum_{j=1}^{2m} \delta_j J(\gamma_j)^{-1} g(\gamma_j)$ with $\delta_j \in \{\pm 1\}$ and with pairwise distinct $\gamma_j \in \Gamma(L_1, L_2)$. We obtain the desired contradiction by specializing this identity to some point $z \in \mathbb{H}^n$, which is not fixed by any $\gamma_i \gamma_j^{-1}$ for $i \neq j$ and invoking Lemma 4.1. $\square$

Proof of Proposition 4.1 in general. Consider the abstract free product $\widetilde{\Gamma} = \widetilde{\Gamma}(L_1, L_2) = \Gamma_{\text{upp}}(L_1) * \Gamma_{\text{low}}(L_2)$ and define $\widetilde{J} \colon \widetilde{\Gamma} \to \text{Hol}(\mathbb{H}^n, \mathbb{C}^\times)$ by

$$\widetilde{J}(T^x)(z) = 1, \qquad \widetilde{J}(V^y)(z) = \mu(T^{-y}Sz)\mu(z),$$

for $x \in L_1$, $y \in L_2$, where, as above, $\mu(z) = \prod_{j=1}^n (z_j/i)^{d_j/2}$. Since $\mu(z)\mu(Sz) = 1$, the cocycle $\tilde{J}$ is well-defined on $\Gamma_{\text{low}}(L_2)$ and is trivially well-defined on $\Gamma_{\text{upp}}(L_1)$, hence on all of $\tilde{\Gamma}$. Let $\pi : \widetilde{\Gamma} \to \Gamma(L_1, L_2)$ denote the natural morphism. We may define a right action of $\widetilde{\Gamma}$ on functions $f \colon \mathbb{H}^n \to \mathbb{C}$ by $f|\gamma := \widetilde{J}(\gamma)^{-1}(f \circ \pi(\gamma))$. We define a $\widetilde{\Gamma}$-cocycle $\widetilde{\Phi}$ by $\widetilde{\Phi}(\gamma) := F|(\gamma - 1)$.

The morphism π is surjective and we need to show that it is also injective. So let $R = V^{y_1}T^{x_1}V^{y_2}T^{x_2} \cdots V^{y_m}T^{x_m} \in \ker \pi$. Assume, by way of contradiction, that $R \neq 1$ and that $m \geq 1$ is minimal. Since $J_{G;8d_1,\ldots,8d_n} \circ \pi$ agrees with $\widetilde{J}^8$ on the generators of $\widetilde{\Gamma}$, we see that $J(R)$ is constant and equal to some 8th root of unity. Thus, instead of (4.7), we obtain

$$(\widetilde{J}(R)^{-1} - 1)F = \widetilde{\Phi}(R) = \sum_{i=1}^m \widetilde{\Phi}(V^{y_i})|P_i = \sum_{i=1}^m (g|V^{y_i}P_i - g|P_i)\,,$$

where P_i is as in (4.7). Since F is L_1-periodic, by acting on both sides of the above equation by $T^x - 1$ for a suitable $x \in L_1$ (so that the resulting linear combination of Gaussians on the right-hand side involves $4m$ distinct elements) and again invoking Lemma 4.1 for suitable $z \in \mathbb{H}^n$ (not fixed by any element in a finite set of non-trivial group elements) we arrive at the desired contradiction. □

Thus, condition (F) is necessary for the existence of F and $\tilde{F}$. On the other hand, we don't know whether condition (D) is necessary as well. We comment on this condition separately in §4.1.4 below. We proceed by stating and proving the next (somewhat disappointing) result, which tells us that whenever (D) holds, then no interpolation formula of the form (4.3) can exist. Interestingly enough, totally real number fields will appear in the proof, even though our set up did not involve them up until now.

Proposition 4.2. *Let $n \geq 2$ and let $L_1, L_2 \subseteq \mathbb{R}^n$ be arbitrary lattices. Suppose the group $\Gamma(L_1, L_2) \leq G$ is discrete. Then condition* (F) *does not hold.*

Proof. Consider the following property (irreducibility) of a lattice $L \subseteq \mathbb{R}^n$

$$L \smallsetminus \{0\} \subseteq (\mathbb{R}^\times)^n. \tag{I}$$

Note that if L is the image of a fractional ideal in a totally real number field under the natural embedding, then (I) holds. The proof distinguishes two cases, according to whether both L_1, L_2 satisfy (I) or one of them does not.

Case 1: Both L_1, L_2 have property (I). In this case, by a result of A. Selberg (sketched in [Sel69]), generalized by Benoist-Oh [BH10, Cor. 1.2], there exists a totally real number field K of degree n such that $\Gamma(L_1, L_2)$ is commensurable to a conjugate of the group $\text{PSL}_2(\mathcal{O}_K)$ embedded in G. Since Hilbert modular groups of totally real number fields are known to be irreducible lattices[37] in $\text{PSL}_2(\mathbb{R})^n$, it follows that $\Gamma = \Gamma(L_1, L_2)$ is an irreducible lattice in $G \cong \text{PSL}_2(\mathbb{R})^n$. Since $\text{PSL}_2(\mathbb{R})^n$ has trivial center (and $n \geq 2$) Margulis' normal subgroup theorem [Mar91, Thm

[37]We use the definition that a lattice Γ in a connected, real semi-simple Lie group G with finite center is *irreducible* if for all non-discrete closed normal subgroups N of G the subgroup ΓN is dense in G. The set of such irreducible lattices in G is closed under the equivalence relations given by conjugation and commensurability. See [Mor15, §4.3].

4.9] implies that the commutator subgroup $[\Gamma,\Gamma]\lhd\Gamma$ is of finite index in Γ, so that the abelianization of Γ is finite. On the other hand, under condition (F) we have[38]

$$\Gamma^{\mathrm{ab}} \cong (\Lambda_1 * \Lambda_2)^{\mathrm{ab}} \cong \Lambda_1 \oplus \Lambda_2,$$

which is infinite.

Case 2: One of the lattices L_1, L_2 does not have property (I). Let us first suppose that L_1 does not have property (I). Fix a nonzero element $x_0 \in L_1$ whose (say) first coordinate is zero. We will construct a sequence of lattice vectors $y_k \in L_2 \smallsetminus \{0\}$ such that the commutators

$$[T^{x_0}, V^{y_k}] = T^{x_0} V^{y_k} T^{-x_0} V^{-y_k} \in \Gamma(L_1, L_2)$$

tend to $1 \in G$, as $k \to \infty$. As we are assuming that $\Gamma(L_1, L_2)$ is discrete, the sequence must be stationary and so (F) would not hold. To produce the sequence y_k, we apply Minkowski's lattice point theorem to the convex, compact, centrally symmetric bodies

$$C_k := \left\{(t_1, \ldots, t_n) \in \mathbb{R}^n \;:\; |t_1| \le 1 + k^{n-1} 2^n \operatorname{covol}(L_2),\ \max_{2\le j\le n} |t_j| \le 1/k \right\},$$

whose volumes are $> 2^n \operatorname{covol}(L_2)$. We may thus choose $0 \ne y_k \in L_2 \cap C_k$ and with this choice, we have $[T^{x_0}, V^{y_k}] \to 1$ as $k \to \infty$.

Finally, if L_2 does not have property (I), we can modify the argument just given in an obvious way, by taking a fixed nonzero element $y_0 \in L_2$ with some vanishing coordinate and a sequence of nonzero lattice vectors $x_k \in L_1$ all of whose coordinates tend to zero, except in the coordinate where y_0 is zero. ☐

4.1.4 Remarks on the necessity of condition (D)

Let us explore what happens when condition (D) fails. We focus on the case $L_1 = L_2 =: L$ and abbreviate $\Gamma := \Gamma(L, L) \le G \cong \mathrm{PSL}_2(\mathbb{R})^n$. Suppose that Γ is not discrete. Then the closure $H := \overline{\Gamma} \le G$ is a non-discrete, closed subgroup of G and in particular a positive-dimensional Lie subgroup of G (by Cartan's closed subgroup theorem). The Lie algebra $\mathfrak{h}$ of H is thus a nonzero subalgebra of the Lie algebra $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{R})^{\oplus n}$ of G. It is moreover stable under conjugation by all elements of H an in particular stable under conjugation by all elements T^x, V^y, $x, y \in L$.

Let us for simplicity suppose that $n = 1$. In that case, we actually know a bit more about the structure of the group(s) Γ (form §2.1.1). The arguments that follow should be soft enough to generalize to $n \ge 2$.

We work with the basis

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of $\mathfrak{sl}_2(\mathbb{R})$. With respect to this basis, the automorphisms $\alpha_x = \mathrm{Ad}(T^x)$ and $\beta_x = \mathrm{Ad}(V^y)$ of $\mathfrak{sl}_2(\mathbb{R})$ are respectively represented by

$$\begin{pmatrix} 1 & -2x & -x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ -y^2 & -2y & 1 \end{pmatrix}.$$

[38] it is easy to check that the abelianization of the free product of two abelian groups is isomorphic to the direct sum of these groups.

Suppose that a nonzero subspace $V \subseteq \mathfrak{sl}_2(\mathbb{R})$ is stable under some α_x and some β_y, $x, y \in \mathbb{R}^\times$. We will show that $V = \mathfrak{sl}_2(\mathbb{R})$. We have

$$\begin{aligned} \ker(\alpha_x - \mathrm{id}) = \mathbb{R}e \ &\subseteq \ \ker(\alpha_x - \mathrm{id})^2 = \mathbb{R}e + \mathbb{R}h \ \subseteq \ \ker(\alpha_x - \mathrm{id})^3 = \mathfrak{sl}_2(\mathbb{R}), \\ \ker(\beta_y - \mathrm{id}) = \mathbb{R}f \ &\subseteq \ \ker(\beta_y - \mathrm{id})^2 = \mathbb{R}f + \mathbb{R}h \ \subseteq \ \ker(\beta_y - \mathrm{id})^3 = \mathfrak{sl}_2(\mathbb{R}). \end{aligned}$$

Now fix $0 \neq v \in V$. If $(\alpha_x - \mathrm{id})^2 v \neq 0$, then the three vectors $v, (\alpha_x - \mathrm{id})v, (\alpha_x - \mathrm{id})^2 v$ are linearly independent, so $V = \mathfrak{sl}_2(\mathbb{R})$. Similarly, if $(\beta_y - \mathrm{id})^2 v \neq 0$, then $V = \mathfrak{sl}_2(\mathbb{R})$. Assume therefore that

$$(\alpha_x - \mathrm{id})^2 v = 0 = (\beta_y - \mathrm{id})^2 v.$$

Then, by the above computations of the kernels, we have

$$v \in (\mathbb{R}e + \mathbb{R}h) \cap (\mathbb{R}f + \mathbb{R}h) = \mathbb{R}h,$$

hence $h \in V$ and therefore also

$$\alpha_x h = h - 2xe \in V, \qquad \beta_y h = h - 2yf \in V$$

and finally $e, f \in V$, so $V = \mathfrak{sl}_2(\mathbb{R})$. Applying this to $V = \mathfrak{h}$, we deduce that $H = \overline{\Gamma} = G$. In other words, Γ is dense in G.

To proceed, recall that an element $g \in G$ is elliptic if $|\operatorname{Tr}(g)| < 2$. The set of elliptic elements is thus open in G, so that Γ contains many elliptic elements. We know what these look like: they are conjugate to elements

$$k(\theta) := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \in \mathrm{PSO}(2), \quad \theta \in \mathbb{R}.$$

It is thus likely that Γ contains an elliptic element $\gamma_0 \in \Gamma$ of infinite order (corresponding to irrational θ). Let $z_0 \in \mathbb{H}$ denote its fixed point. To simplify notation, suppose that $z_0 = i$, so that $\gamma_0 = k(\theta_0)$, for some irrational $\theta_0 \in \mathbb{R}$. Suppose moreover that we can work with the usual slash-action in weight $k \in 2\mathbb{Z}$ of $\mathrm{PSL}_2(\mathbb{R})$ on functions in $\mathbb{H}$, that is to say, the one introduced at the beginning of §2.1.1.3 via the standard automorphic factor μ, defined in (2.5), so $\mu(g, z) = c_g z + d_g$ for $g \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathbb{H}$ (not to be confused with the function μ, previously used in this section). Under these assumptions on Γ, the generating function $F : \mathbb{H} \to \mathbb{C}$ is uniquely determined by the condition $\Phi(\gamma_0) = F|_k \gamma_0 - F$, in the sense that there can be at most one F meeting such a condition. Note also that the left hand side can be expressed in terms of linear combinations of Gaussians, since $\gamma_0 \in \Gamma$. Since there should to be many such elements $\gamma_0 \in \Gamma$, it seems unlikely that F can meet all of these defining conditions simultaneously.

To explain why F is uniquely determined by such a single condition, suppose that $F|_k\gamma_0 - F = 0$. We will show that $F = 0$. Note that, by continuity and the assumption that the closure of $\langle \gamma_0 \rangle$ is $\mathrm{PSO}(2)$, we obtain $F|_k(k(\theta)) - F = 0$ for all $\theta \in \mathbb{R}$. Next, consider the Cayley-map $\delta(z) = \frac{z-i}{z+i}$ giving a biholomorphic map $\delta : \mathbb{H} \to \mathbb{D}$ mapping i to 0. Note that δ "conjugates" the stabilizers of i in $\mathrm{Aut}(\mathbb{H})$ and 0 in $\mathrm{Aut}(\mathbb{D})$. Define the holomorphic function $G : \mathbb{D} \to \mathbb{C}$ by $G(w) := F(\delta^{-1}(w))$ for $w \in \mathbb{D}$. A computation then shows that the condition $F|_k k(\theta) = F$ for all $\theta \in \mathbb{R}$ is expressed in terms of G as

$$G(e^{i\theta} w) = \mu(k(\theta), \delta^{-1}(w))^k G(w) \quad \text{for all } \theta \in \mathbb{R}, w \in \mathbb{D}. \tag{4.8}$$

Setting $w = 0$, we obtain $G(0) = e^{-k\theta} G(0)$ for all $\theta \in \mathbb{R}$, so $G(0) = 0$. We can inductively show that $G^{(m)}(0) = 0$ for all $m \geq 0$. Indeed, suppose this holds for some $m \geq 0$, then, by differentiating both sides of (4.8) $m + 1$ times, we obtain

$$G^{(m+1)}(e^{i\theta} w) e^{i(m+1)\theta} = \sum_{j=0}^{m+1} \binom{m+1}{j} G^{(j)}(w) \frac{d^{m+1-j}}{dw^{m+1-j}} \mu(k(\theta), \delta^{-1}(w))^k$$

Evaluating at $w = 0$ yields

$$G^{(m+1)}(0)e^{i(m+1)\theta} = G^{(m+1)}(0)e^{-k\theta},$$

for all θ, so that $G^{(m+1)}(0) = 0$ follows. This shows that $G = 0$, hence $F = 0$ as claimed.

While the above arguments included many simplifying assumptions and do not prove that condition (D) is necessary for the existence of an interpolation formula of the shape (4.3), we nevertheless believe that this condition is necessary.

4.1.5 Lattices having property (F)

The reader may wonder whether there exist any lattices $L_1, L_2 \subseteq \mathbb{R}^n$ for which the group $\Gamma(L_1, L_2)$ is *not* discrete, or equivalently, any for which condition (F) *does* hold (by Proposition 4.2). Such lattices do exist, as the following lemma shows.

Lemma 4.2. *Let $L_1, L_2 \subseteq \mathbb{R}^n$ be lattices with bases $v_1, \dots, v_n$ for L_1 and $w_1, \dots, w_n$ for L_2. Write these bases vectors as $v_i = (v_{i,1}, \dots, v_{i,n})$, $w_i = (w_{i,1}, \dots, w_{i,n})$. Assume that there is $j_0 \in \{1, \dots, n\}$ so that the real numbers*

$$v_{1,j_0}, \dots, v_{n,j_0}, w_{1,j_0}, \dots, w_{n,j_0}$$

are algebraically independent over $\mathbb{Z}$ (equivalently, over $\mathbb{Q}$). Then $\Gamma(L_1, L_2)$ is the free product of $\Gamma_{upp}(L_1)$ and $\Gamma_{low}(L_2)$. In particular, if $n \geq 2$, the group $\Gamma(L_1, L_2) \subseteq \mathrm{PSL}_2(\mathbb{R})^n$ is not discrete.

Proof. Assume, for a contradiction, that for some $N \geq 1$ and $x_1, \dots, x_N \in L_1$, $y_1, \dots, y_N \in L_2$, we have a non-trivial relation in $\Gamma(L_1, L_2)$, of the form

$$T^{x_1}V^{y_1} \cdots T^{x_N}V^{y_N} = 1 \tag{4.9}$$

with $x_\nu \neq 0 \neq y_\nu$ for all $\nu \in \{1, \dots, N\}$, except possibly $y_N = 0$. We use the viewpoint $G \cong \mathrm{PSL}_2(\mathbb{R})^n$ and project this relation onto the j_0th factor to obtain a relation in $\mathrm{PSL}_2(\mathbb{R})$ of the form

$$\begin{bmatrix} 1 & \phi_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \psi_1 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & \phi_N \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \psi_N & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.10}$$

in which the real numbers

$$\phi_\nu = \phi_\nu(v_{1,j_0}, \dots, v_{n,j_0}), \quad \psi_\nu = \psi_\nu(w_{1,j_0}, \dots, w_{n,j_0})$$

are some $\mathbb{Z}$-linear combinations of the indicated arguments. They are all nonzero (except possibly ψ_N), because the v_{i,j_0} and w_{i,j_0} are in particular $\mathbb{Z}$-linearly independent. Because they are in fact *algebraically* independent, the identity in (4.10) must be a polynomial one, in the sense that it holds if we view $\phi_\nu \in \mathbb{Z}[X_1, \dots, X_n]$, $\psi_\nu \in \mathbb{Z}[Y_1, \dots, Y_n]$ and correspondingly, (4.10) holds in $\mathrm{PSL}_2(\mathbb{Z}[X_1, \dots, X_n, Y_1, \dots, Y_n])$. But, by specializing the variables X_i, Y_i to some *even* integers so that ϕ_1 and ψ_1 are nonzero, we obtain a contradiction to the fact that $\Gamma(2)$ is free on two generators (see Lemma 2.1). □

4.2 Fourier non-uniqueness sets from totally real number fields

We now turn from non-existence of interpolation formulas to (stronger) Fourier non-uniqueness results for sets of the form $\sqrt{\Lambda}$ with lattices Λ coming from totally real number fields. Before stating the main results, we fix some notation and recall basic facts about algebraic number fields (restricted to the totally real case).

Let $n \geq 2$ be an integer and let $K/\mathbb{Q}$ be a totally real number field of degree $n = [K : \mathbb{Q}]$. Let $\sigma_1, \ldots, \sigma_n : K \to \mathbb{R}$ denote the real embeddings. Recall that an element $x \in K$ is said to be totally positive (resp. totally non-negative) if $\sigma_j(x) > 0$ for all j (resp. $\sigma_j(x) \geq 0$ for all j). Let $\sigma : K \to \mathbb{R}^n$ be defined by $\sigma(x) = (\sigma_1(x), \ldots, \sigma_n(x))$, which is sometimes called the Minkowski embedding. Recall that the trace of an element $\alpha \in K$ is given by $\mathrm{Tr}(\alpha) = \mathrm{Tr}_{K/\mathbb{Q}}(\alpha) = \sum_{j=1}^{n} \sigma_j(\alpha)$ (and also equals the trace of the $\mathbb{Q}$-linear map $x \mapsto \alpha x$). We denote the ring of integers of K by $\mathcal{O}_K$. For any $\mathcal{O}_K$-submodule $\mathfrak{a} \subseteq K$, we write

$$\mathfrak{a}^{\vee} = \{x \in K \,:\, \mathrm{Tr}(xa) \in \mathbb{Z} \text{ for all } a \in \mathfrak{a}\}$$

for its dual with respect to the trace pairing. If $\mathfrak{a}$ is a fractional ideal, then so is $\mathfrak{a}^{\vee}$. Recall that the inverse different or co-different of K is defined as the fractional ideal $\mathcal{O}_K^{\vee}$ and that the different of K is defined as $\mathfrak{d} := \mathfrak{d}_K = (\mathcal{O}_K^{\vee})^{-1}$. One has then the general formula

$$\mathfrak{a}^{\vee} = \mathfrak{d}^{-1}\mathfrak{a}^{-1} = \mathcal{O}_K^{\vee}\mathfrak{a}^{-1},$$

valid for any fractional ideal $\mathfrak{a} \subseteq K$. As is well-known, if $\mathfrak{a}$ is a fractional ideal in K, then $\sigma(\mathfrak{a}) \subseteq \mathbb{R}^n$ is a lattice and $\sigma(\mathfrak{a}^{\vee}) = \sigma(\mathfrak{a})^{\vee}$, where on the right we mean the dual lattice in the usual sense (see item (14) in Section 1.1). Moreover, the covolume of $\sigma(\mathfrak{a})$ is given by

$$\mathrm{covol}\,(\sigma(\mathfrak{a})) = \mathrm{N}(\mathfrak{a})\sqrt{|\operatorname{disc}(K)|}, \tag{4.11}$$

where $\mathrm{N}(\mathfrak{a}) \in \mathbb{Q}_{>0}$ is the ideal norm of $\mathfrak{a}$ (the unique extension of the absolute norm on integral ideals to all fractional ideals of K) and $|\operatorname{disc}(K)| = \mathrm{covol}(\sigma(\mathcal{O}_K))^2$ is the discriminant of K. For any fractional ideal $\mathfrak{a} \subseteq K$ we use the shorthand

$$\sqrt{\mathfrak{a}} = \sqrt{\sigma(\mathfrak{a})} := \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, (x_1^2, \ldots, x_n^2) = \sigma(\alpha) \;\text{ for some } \alpha \in \mathfrak{a}\right\} \subseteq \mathbb{R}^n \tag{4.12}$$

(which is not to be confused with the radical of an ideal)

A theorem of Hecke [Hec54, §63, Satz 176] asserts that the different $\mathfrak{d}$ defines a square in the ideal class group of K. This implies that we can find $c \in K^{\times}$ and a fractional ideal $\mathfrak{a} \subseteq K$ so that

$$\mathfrak{d}^{-1} = c\mathfrak{a}^2. \tag{4.13}$$

The pair $(c, \mathfrak{a})$ is not uniquely defined by this equation. For any unit $\varepsilon \in \mathcal{O}_K^{\times}$, we may replace it by $(\varepsilon c, \mathfrak{a})$. Let us also mention that there is a large class of number fields, where we can compute admissible $c, \mathfrak{a}$ easily. Namely, if $f \in \mathbb{Z}[X]$ is irreducible and monic, has n distinct real roots $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$ and square-free discriminant, then, as is well-known, the totally real number field $K = \mathbb{Q}[X]/(f)$ has ring of integers $\mathcal{O}_K = \oplus_{0 \leq i \leq n-1}\mathbb{Z}\alpha^i$ and co-different $\mathcal{O}_K^{\vee} = \frac{1}{f'(\alpha)}\mathcal{O}_K$, so that (4.13) holds with $c = \frac{1}{f'(\alpha)}$ and $\mathfrak{a} = \mathcal{O}_K$.

In general, given $c, \mathfrak{a}$ satisfying (4.13), we define the following subset of $\mathbb{R}^n$:

$$E(c, \mathfrak{a}) := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, (x_1^2/|\sigma_1(c)|, \ldots, x_n^2/|\sigma_n(c)|) = \sigma(\alpha) \,\text{for some } \alpha \in \mathfrak{a}^2\} \tag{4.14}$$

Note that this is a discrete subset of a union of ellipsoids defined by the equations $\sum_{i=1}^{n} x_i^2/|\sigma_i(c)| = m$, where m runs over a set of non-negative integers, the set of traces of totally non-negative integers in $\mathfrak{a}^2$.

4.2.1 Statement of non-uniqueness results

We can now state the two main Fourier non-uniqueness results of Chapter 4.

Theorem 7. *Let $n \geq 2$ and let $K/\mathbb{Q}$ be a totally real number field of degree n. Let $V \subseteq \mathcal{S}(\mathbb{R}^n)$ denote the subspace linearly spanned by all Gaussians $e^{\pi i z_1 x_1^2} \cdots e^{\pi i z_n x_n^2}$, with $z_j \in \mathbb{H}$ and $x_j \in \mathbb{R}$. Let $c, \mathfrak{a}$ be such that (4.13) holds. Then, for each sign $\epsilon \in \{\pm 1\}$, the space*

$$\left\{ f \in V \,:\, \hat{f} = \epsilon f \text{ and } f|_{E(\mathfrak{a},c)} = 0 \right\} \tag{4.15}$$

is infinite dimensional.

The content of the next theorem is similar to the one of Theorem 7, but with $E(c, \mathfrak{a})$ replaced by $\sqrt{\mathcal{O}_K^\vee}$. In fact, it holds in the slightly more general setting introduced at the beginning of §4.1, that is to say, in the space of $\mathrm{O}(d_1) \times \cdots \times \mathrm{O}(d_n)$-invariant (Schwartz) functions on $\mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_n}$. We say that such a function f *vanishes* on $\sqrt{\mathcal{O}_K^\vee}$, if we have $f(x) = 0$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^d$ for which there is $\alpha \in \mathcal{O}_K^\vee$ such that $(|x_1|^2, \ldots, |x_n|^2) = \sigma(\alpha)$.

Theorem 8. *In the above notation, let V denote the subspace of Schwartz functions on $\mathbb{R}^d$ linearly generated by all Gaussians $e^{\pi i z_1 |x_1|^2} \cdots e^{\pi i z_n |x_n|^2}$, with $z_j \in \mathbb{H}$ and $x_j \in \mathbb{R}^{d_j}$. Let $K/\mathbb{Q}$ be a totally real number field of degree $n \geq 2$. Then, for every $\epsilon \in \{\pm 1\}$, the space*

$$\left\{ f \in V \,:\, \hat{f} = \epsilon f \text{ and } f \text{ vanishes on } \sqrt{\mathcal{O}_K^\vee} \right\} \tag{4.16}$$

is infinite dimensional.

Let us add a few remarks and comments on these theorems. The first remark is that the functions we produce in the spaces (4.15) and (4.16) are quite explicit. More precisely, if one can write down two non-trivial units of $\mathcal{O}_K$ in the congruence classes $1 + 4\mathcal{O}_K$ and $1 + 3\mathcal{O}_K$, then one can also write down 16 elements of the Hilbert modular group $\mathrm{PSL}_2(\mathcal{O}_K)$ and a linear combination of 16 Gaussians $e^{\pi i z_1 x_1^2} \cdots e^{\pi i z_n x_n^2}$ in these spaces, where the parameters $z = (z_1, \ldots, z_n) \in \mathbb{H}^n$ are the translates of some generic (non-elliptic) point $\tau \in \mathbb{H}^n$ by these elements. Eight of these 16 elements are written down explicitly in the proof of Proposition 4.3. The other eight are obtained easily from those, in a way corresponding to the fact that the functions f we produce are of the form $f = h + \epsilon \hat{h}$. Example 1 below illustrates this over $\mathbb{Q}(\sqrt{5})$ where we express these matrices in terms of the golden ratio $\frac{1+\sqrt{5}}{2}$.

Remark 4.2 (Linear algebra)**.** Let X be an arbitrary nonempty set and let $X' \subseteq X$ be a finite subset. Let F be an infinite dimensional subspace of the space of all functions $f : X \to \mathbb{C}$. Then the subspace $F' \subseteq F$ consisting of all $f \in F$ such that $f|_{X'} = 0$, is also infinite dimensional. Indeed, there is an obvious exact sequence

$$0 \to F' \to F \to \mathbb{C}^{X'} \to 0$$

In particular, we can add any finite subset $X' \subseteq X = \mathbb{R}^n$ to the sets $E(c, \mathfrak{a})$ or $\sqrt{\mathcal{O}_K^\vee}$ in Theorems 7 or Theorem 8, without changing the conclusion.

Remark 4.3 (On the number of points in $\sqrt{m} S^{n-1} \cap \sqrt{\mathfrak{d}^{-1}}$)**.** The cardinality of $\sqrt{\mathfrak{d}^{-1}} \cap \sqrt{m} S^{n-1}$ is 2^n times the number of totally non-negative elements in $\mathfrak{d}^{-1}$ of trace $m \geq 0$. By choosing a $\mathbb{Z}$-basis for $\mathcal{O}_K$ containing 1 and considering the element $\alpha_1 \in K$ such that the $\mathbb{Q}$-linear functional $y \mapsto \mathrm{Tr}(\alpha_1 y)$ takes the value 1 on $y = 1$ and zero on all other elements of the basis, we see that $\mathrm{Tr}(\alpha_1) = 1$ and $\alpha_1 \in \mathfrak{d}^{-1}$. It follows that for all $m \in \mathbb{Z}$, we have

$$\{\alpha \in \mathfrak{d}^{-1} \,:\, \mathrm{Tr}(\alpha) = m\} = m\alpha_1 + (\mathfrak{d}^{-1})_0, \qquad (\mathfrak{d}^{-1})_0 := \{\alpha \in \mathfrak{d}^{-1} \,:\, \mathrm{Tr}(\alpha) = 0\}.$$

Thus, for $m \geq 0$, the subset of $\mathbb{R}^n$ whose cardinality we are interested in, can be written as

$$\big(m\sigma(\alpha_1) + \sigma((\mathfrak{d}^{-1})_0)\big) \cap [0,\infty)^n$$

whose cardinality equals that of

$$\sigma((\mathfrak{d}^{-1})_0) \cap m([0,\infty)^n - \sigma(\alpha_1))$$

which is the set of lattice points of $\sigma((\mathfrak{d}^{-1})_0) \subseteq \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 0\}$ in a homogeneously expanding $(n-1)$-dimensional region, allowing for an application of a standard estimate of the number of such points, as $m \to \infty$. The necessary volume computations are done (for any fractional ideal, in fact) in the work of Ash and Friedberg, see [AF07, §5, Prop. 5.1 and §6]. From the cited parts of their work, we deduce that

$$\left|\sqrt{\mathfrak{d}^{-1}} \cap \sqrt{m}S^{n-1}\right| = 2^n \frac{\sqrt{|\operatorname{disc}(K)|}}{(n-1)!} m^{n-1} + O(m^{n-2}), \quad m \to \infty, \tag{4.17}$$

where the implied constant may depend on K and n. We point out the following features of this asymptotic formula:

- The surface area of $\sqrt{m}S^{n-1}$ grows like $m^{\frac{n-1}{2}}$, so the points are more densely spaced than a constant number of points per unit surface area on S^{n-1}.
- We may increase the density of points by a constant factor, by taking the discriminant of K arbitrarily large, while keeping the degree n fixed.
- For small m, there may be no points in $\sqrt{\mathfrak{d}^{-1}} \cap \sqrt{m}S^{n-1}$, but note that we can add *any* finite set of points on these small spheres by Remark 4.2.

Remark 4.4 (Relations between Theorem 7 and Theorem 8)**.** Note that, if the number c in (4.13) can be chosen in such a way that all its conjugates $\sigma_j(c)$ are positive, then $E(c,\mathfrak{a}) = \sqrt{\mathcal{O}_K^\vee}$ so that both theorems assert the same (in the case that all d_j are equal). As already mentioned, we are free to replace c by εc for any unit $\varepsilon \in \mathcal{O}_K^\vee$ and so the implication Theorem 7 $\Rightarrow$ Theorem 8 holds, provided the number field K has units $\varepsilon \in \mathcal{O}_K^\times$ of all possible sign patterns $(\sigma_j(\varepsilon))_{1\leq j\leq n} \in \{\pm 1\}^n$. Such conditions on unit groups are studied more generally in the literature, via the notion of signature rank (being defined as the dimension of the $\mathbb{F}_2$-vector space of signs of units). In the real quadratic case, there are units of all possible sign-patterns, if and only if the fundamental unit has norm -1.

Let us also remark that if the absolute values $|\sigma_j(c)|$ are all equal to some constant $\theta > 0$, then $E(c,\mathfrak{a})$ is contained in the union of spheres $\sqrt{n\theta}$, for $n \in \mathbb{N}_0$. We will see an example of this in the real quadratic case, in Remark 4.5.

In fact, it can happen that $c = 1$, so that the different $\mathfrak{d}_K$ is exactly equal to the square of another fractional ideal. An easy sufficient condition is when $K/\mathbb{Q}$ is Galois and has *odd* degree n. To see this, consider the factorization $\mathfrak{d}_K = \prod_{i=1}^g \mathfrak{p}_i^{e_i}$ of the different into prime ideals $\mathfrak{p}_i$ of $\mathcal{O}_K$. Fix any $i \in \{1,\ldots,g\}$ and let $e = e_i$, $\mathfrak{p} = \mathfrak{p}_i$. A formula attributed to Hilbert then says that

$$e = \sum_{t=0}^{\infty} (|V_t(\mathfrak{p})| - 1),$$

where $V_0(\mathfrak{p}) \rhd V_1(\mathfrak{p}) \rhd V_2(\mathfrak{p}) \rhd \ldots$ is a decreasing filtration of the inertia group of $\operatorname{Gal}(K/\mathbb{Q})$ defined by

$$V_t(\mathfrak{p}) = \left\{\sigma \in \operatorname{Gal}(K/\mathbb{Q}) : \sigma(x) - x \in \mathfrak{p}^t \quad \text{for all } x \in \mathcal{O}_K\right\}.$$

Now, since $n = |\operatorname{Gal}(K/\mathbb{Q})|$ is assumed to be odd, all the numbers $|V_t(\mathfrak{p})| - 1$ must be even, so that e is even. We refer to [Mol11, Ex. 5.45, p. 253] for more details.

Remark 4.5 (Specialization to real quadratic fields)**.** To illustrate the theorems in the case $n = 2$, consider a real quadratic field $K = \mathbb{Q}(\sqrt{D})$ as a subfield of $\mathbb{R}$ of discriminant D, $\sqrt{D} > 0$ and for $x \in K$ write $\sigma_1(x) = x$ and $\sigma_2(x) =: x'$ so that $\sqrt{D}' = -\sqrt{D}$. Define $\omega := (D + \sqrt{D})/2$ and $c := 1/\sqrt{D}$. Then $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\omega$ and $\mathcal{O}_K^\vee = c\mathcal{O}_K = c\mathcal{O}_K^2$ (square of a fractional ideal). Thus, every element of $\mathcal{O}_K^\vee$ may be written as $\alpha = c(\ell + m\omega)$ for $\ell, m \in \mathbb{Z}$ and has $\mathrm{Tr}(\alpha) = \ell\,\mathrm{Tr}(c) + m\,\mathrm{Tr}(\omega c) = m$. The element α is totally non-negative if and only if $m \geq 0$ and $-m\omega \leq \ell \leq -m\omega'$. This shows that

$$\left|\sqrt{\mathcal{O}_K^\vee} \cap \sqrt{m}S^1\right| = 2\,|\mathbb{Z} \cap [-m\omega, -m\omega']| \sim 2m\sqrt{D}, \quad m \to \infty,$$

which exemplifies (4.17) and Theorem 8 in the simplest case.

Let us now illustrate Theorem 7 with $\mathfrak{a} = \mathcal{O}_K$ and the above value of c, which is *not* totally positive and satisfies $|\sigma_1(c)| = |\sigma_2(c)| = \frac{1}{\sqrt{D}}$. We assume that $4|D$ and set $d := D/4 \equiv 2, 3 \pmod 4$, so that $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\sqrt{d}$. Then $E(c, \mathfrak{a})$ is the set of $x = (x_1, x_2) \in \mathbb{R}^2$ such that

$$(x_1^2, x_2^2) = \frac{1}{2\sqrt{d}}\left(a + b\sqrt{d}, a - b\sqrt{d}\right)$$

for some $a, b \in \mathbb{Z}$ satisfying $|b\sqrt{d}| \leq a$ and $x_1^2 + x_2^2 = \frac{a}{\sqrt{d}}$. In other words, $E(c, \mathfrak{a})$ is a discrete subset of a union of circles of radii $\sqrt{a/\sqrt{d}}$, for all integers $a \geq 0$ with about $a/\sqrt{d}$ many points on each. If $D \equiv 1 \pmod 4$, then $E(c, \mathfrak{a})$ is a discrete subset of the union of all circles of radii $\sqrt{t/\sqrt{D}}$ for all integers $t \geq 0$ with about $2t/\sqrt{D}$ points on each.

4.3 Proof of Theorem 7

The goal of this section is to prove Theorem 7. In §4.3.1 we introduce some notation and define a "theta-subgroup" Γ_ϑ of the Hilbert modular group $\mathrm{PSL}_2(\mathcal{O}_K)$. In §4.3.2 we define a slash action of the group algebra $\mathbb{C}[\Gamma_\vartheta]$ on complex-valued functions on a product of upper and lower half planes, via theta functions. The examples of non-trivial functions satisfying the vanishing conditions of Theorem 7 will be given as Gaussians slashed with suitable elements in $\mathbb{C}[\Gamma_\vartheta]$. Lemmas 4.3 and 4.4 will show that "suitable" means to belong to the intersection of two right ideals in $\mathbb{C}[\Gamma_\vartheta]$. In §4.3.3, we will show that this intersection is infinite dimensional and conclude the proof of Theorem 7 in §4.3.4.

4.3.1 Hilbert modular groups and subgroups

We consider a totally real number field K of degree $n = [K : \mathbb{Q}] \geq 2$. As in (4.13), we choose and fix $c \in K^\times$ and a fractional ideal $\mathfrak{a} \subseteq K$ so that $\mathfrak{d}^{-1} = c\mathfrak{a}^2$, where $\mathfrak{d}$ is the different of K. Depending upon these quantities we define signs $\delta_j := \mathrm{sgn}(\sigma_j(c))$, a vector of signs $\delta = (\delta_j)_{1 \leq j \leq n} \in \{\pm 1\}^n$ and the product space

$$\mathbb{H}_\delta^n := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \,:\, \mathrm{Im}(\delta_j z_j) > 0 \text{ for all } j\}.$$

For all of §4.3, we will work with Gaussians $g_\delta(z) \in \mathcal{S}(\mathbb{R}^n)$ defined by[39]

$$g_\delta(z)(x) := g_\delta(z, x) := e^{\pi i \sum_{j=1}^n \delta_j z_j x_j^2}, \quad z \in \mathbb{H}_\delta^n,\, x \in \mathbb{R}^n. \tag{4.18}$$

[39]The subscript δ has not the same meaning as the subscript d of g_d, used in other parts of this thesis

We consider the Hilbert modular group $\Gamma := \mathrm{PSL}_2(\mathcal{O}_K)$ and denote

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T^{\beta} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \quad \beta \in \mathcal{O}_K, \qquad M(\varepsilon) = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \quad \varepsilon \in \mathcal{O}_K^{\times},$$

viewing these as elements of Γ. Next we embed Γ into $\mathrm{PSL}_2(\mathbb{R})^n$ via the real embeddings σ_j. The latter group and hence Γ itself, acts on $\mathbb{H}_\delta^n$ via fractional linear transformations. This action is faithful and we sometimes identify a group element with the associated automorphism of $\mathbb{H}_\delta^n$, in particular when writing compositions of maps. Define

$$\Gamma_\vartheta := \left\langle \{S\} \cup \{T^{2\beta}\}_{\beta \in \mathcal{O}_K} \cup \{M(\varepsilon)\}_{\varepsilon \in \mathcal{O}_K^\times} \right\rangle \leq \Gamma.$$

Remark 4.6. Let $\tilde{\Gamma}_\vartheta$ denote the image in Γ of the group of matrices in $\mathrm{SL}_2(\mathcal{O}_K)$ which reduce to $\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathcal{O}_K/2\mathcal{O}_K)$. By definition, $\Gamma_\vartheta \leq \tilde{\Gamma}_\vartheta$ and equality is known to hold (at least) in the case $K = \mathbb{Q}(\sqrt{5})$ by works of Maass [Maa41, §1]. Even though it would be convenient, we do not need to know equality in general and mainly mention it to provide context, but also because we will refer back to the group $\tilde{\Gamma}$ in the proof of Proposition 4.3 below.

Remark 4.7. Note that if $\Lambda_1 = \Lambda_2 = \sigma(\mathcal{O}_K^\vee)$, then the group $\Gamma(2\Lambda_1^\vee, 2\Lambda_2^\vee)$ studied in §4.1 identifies with the sugroup $\Gamma_K(2)$ of Γ_ϑ generated by the elements $T^{2\alpha}$ and $ST^{2\alpha}S$, $\alpha \in \mathcal{O}_K$. The group $\Gamma_K(2)$ is certainly discrete in $\mathrm{PSL}_2(\mathbb{R})^n$, but it never satisfies condition (F). This follows from Proposition 4.2, but we can also write down the following explicit example (which was found by hand). Take any $0 \neq \beta \in \mathcal{O}_K$ such that $1 + 5\beta \in \mathcal{O}_K^\times$, then

$$T^{-2\beta(1+5\beta)^{-1}} V^2 T^2 V^{2\beta} T^{-2(1+5\beta)^{-1}} V^{-2(1+5\beta)} = 1. \tag{4.19}$$

The point of departure of our work with D. Radchenko was to find out whether groups such as $\Gamma_K(2)$ can satisfy condition (F), which we first investigated via Magma computations based on a group presentation of $\mathrm{PSL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{5})})$ computed by H. Yoshida in [Yos11]. These computations then revealed a list of many complicated relations such as (4.19), which indicated that interpolation is likely not possile in this setting and led us to prove the opposite results.

4.3.2 Automorphic factors and slash action

Our task here is to define a suitable automorphic factor and a corresponding slash action of Γ_ϑ on spaces of functions on $\mathbb{H}_\delta^n$ so that the action of S matches with the Fourier transform acting on Gaussians and so that $T^{2\beta}$ simply acts as translation by $2\sigma(\beta)$, $\beta \in \mathcal{O}_K$. We will use theta functions attached to fractional ideals in K. Essentially the same functions were already studied by Hecke [Hec54, §56].

We define the function $\vartheta : \mathbb{H}_\delta^n \to \mathbb{C}$ by the absolutely and normally convergent series

$$\vartheta(z) := \vartheta(z_1, \ldots, z_n) := \sum_{\alpha \in \mathfrak{a}} e^{\pi i \sum_{j=1}^n z_j \sigma_j(c\alpha^2)},$$

where we recall that $\mathfrak{d}^{-1} = c\mathfrak{a}^2$. We next determine the transformation behavior of ϑ under the generators of Γ_ϑ. These are certainly not new, but we include their proofs to keep the presentation self-contained. First, since $\mathfrak{a}$ is an $\mathcal{O}_K$-submodule of K, we have, for every $\varepsilon \in \mathcal{O}_K^\times$, and every $z \in \mathbb{H}_\delta^n$

$$\vartheta(M(\varepsilon)z) = \vartheta(\sigma_1(\varepsilon)^2 z_1, \ldots, \sigma_n(\varepsilon)^2 z_n) = \vartheta(z).$$

Next, $\vartheta(T^{2\beta}z) = \vartheta(z)$ for all $z \in \mathbb{H}_\delta^n$ and all $\beta \in \mathcal{O}_K$, since for all $\alpha \in \mathfrak{a}$,

$$\sum_{j=1}^n (z_j + 2\sigma_j(\beta))\sigma_j(c\alpha^2) = \sum_{j=1}^n z_j\sigma_j(c\alpha^2) + 2\operatorname{Tr}_{K/\mathbb{Q}}(\beta c\alpha^2)$$

and the above trace is an integer. To study the effect of ϑ under S note that, by definition, $\vartheta(z)$ is the sum over the lattice $\sigma(\mathfrak{a})$ of the Schwartz function $f_z = g_\delta(|\sigma_1(c)|z_1, \ldots, |\sigma_n(c)|z_n)$ whose Fourier transform is

$$\begin{aligned}\widehat{f}_z(\xi) &= \prod_{j=1}^n (\delta_j|\sigma_j(c)|z_j/i)^{-1/2} e^{\pi i\delta_j(-1/(|\sigma_j(c)|z_j))\xi_j^2} \\ &= |\operatorname{N}_{K/\mathbb{Q}}(c)|^{-1/2} \prod_{j=1}^n (\delta_j z_j/i)^{-1/2} e^{\pi i(-1/z_j)(1/\sigma_j(c))\xi_j^2}.\end{aligned}$$

By applying Poisson summation to the function f_z and the lattice $\sigma(\mathfrak{a}) \subseteq \mathbb{R}^n$, we get

$$\begin{aligned}\vartheta(z) &= \frac{1}{\operatorname{covol}(\sigma(\mathfrak{a}))} \sum_{\lambda^* \in \sigma(\mathfrak{a})^\vee} \widehat{f}_z(\lambda^*) \\ &= \frac{1}{|\operatorname{N}_{K/\mathbb{Q}}(c)|^{1/2}\operatorname{covol}(\sigma(\mathfrak{a}))} \prod_{j=1}^n (\delta_j z_j/i)^{-1/2} \sum_{\beta \in c\mathfrak{a}} e^{\pi i\sum_{j=1}^n (-1/z_j)(1/\sigma_j(c))\sigma_j(\beta)^2},\end{aligned}$$

where we used that $\mathfrak{a}^\vee = c\mathfrak{a}$, which follows from multiplying the relation $c\mathfrak{a}^2 = \mathfrak{d}^{-1}$, by $\mathfrak{a}^{-1}$ and using the general formula $\mathfrak{b}^\vee = \mathfrak{d}^{-1}\mathfrak{b}^{-1}$. Writing $\beta = c\alpha$ and summing over $\alpha \in \mathfrak{a}$, the above computation proves

$$\vartheta(z) = (\delta_1 z_1/i)^{-1/2} \cdots (\delta_n z_n/i)^{-1/2}\vartheta(Sz)$$

provided that $|\operatorname{N}_{K/\mathbb{Q}}(c)|\operatorname{covol}(\sigma(\mathfrak{a}))^2 = 1$ holds. This in turn follows again from the relation $c\mathfrak{a}^2 = \mathfrak{d}^{-1}$, the general volume formula (4.11) and properties of the ideal norm. To see this, square the relation

$$\operatorname{covol}(\sigma(\mathfrak{a})) = \operatorname{N}(\mathfrak{a})\operatorname{covol}(\sigma(\mathcal{O}_K)).$$

to get

$$\operatorname{covol}(\sigma(\mathfrak{a}))^2 = \operatorname{N}(\mathfrak{a})^2(\operatorname{covol}\sigma(\mathcal{O}_K))^2 = \operatorname{N}(\mathfrak{a}^2)(\operatorname{covol}\sigma(\mathcal{O}_K))^2$$

and multiply by $|\operatorname{N}_{K/\mathbb{Q}}(c)|$, giving

$$|\operatorname{N}_{K/\mathbb{Q}}(c)|\operatorname{covol}(\sigma(\mathfrak{a}))^2 = \operatorname{N}(c\mathfrak{a}^2)(\operatorname{covol}\sigma(\mathcal{O}_K))^2 = \operatorname{N}(\mathfrak{d}^{-1})(\operatorname{covol}\sigma(\mathcal{O}_K))^2.$$

The latter ratio is known to equal 1, since the ideal norm of the different is the absolute value of discriminant, which also equals the squared covolume of $\sigma(\mathcal{O}_K)$.

We now define $\Omega_\delta^n := \{z \in \mathbb{H}_\delta^n : \vartheta(z) \neq 0\}$, a nonempty open subset of $\mathbb{H}_\delta^n$ containing the product of the imaginary axes. Note that Ω_δ^n is stable under the action of Γ_θ. We consider the 1-cocycle $j_\vartheta : \Gamma_\vartheta \to \operatorname{Hol}(\Omega_\delta^n, \mathbb{C}^\times)$, defined by

$$j_\vartheta(\gamma)(z) := j_\vartheta(\gamma, z) := \frac{\vartheta(\gamma z)}{\vartheta(z)}. \tag{4.20}$$

Here, $\operatorname{Hol}(\Omega_\delta^n, \mathbb{C}^\times)$ denotes the abelian group of all nowhere vanishing, holomorphic functions on Ω_δ^n. Our computations from above and the definitions imply that, for all $\beta \in \mathcal{O}_K$, all $\varepsilon \in \mathcal{O}_K^\times$, all $z \in \Omega_\delta^n$ and all $\gamma_1, \gamma_2 \in \Gamma_\vartheta$,

$$j_\vartheta(T^{2\beta}) = 1,\ j_\vartheta(M(\varepsilon)) = 1,\ j_\vartheta(S, z) = \prod_{j=1}^n (\delta_j z_j/i)^{1/2},\ j_\vartheta(\gamma_1\gamma_2) = (j_\vartheta(\gamma_1) \circ \gamma_2) \cdot j_\vartheta(\gamma_1). \tag{4.21}$$

It not strictly necessary for our purposes but, for convenience, we will lift j_ϑ to a cocycle $j_\vartheta : \Gamma_\vartheta \to \mathrm{Hol}\,(\mathbb{H}^n_\delta, \mathbb{C}^\times)$. To explain how, note that, by our definition of Γ_ϑ via generators, and by (4.21), each function $j_\vartheta(\gamma)$ can we written as a finite product of functions $j_\vartheta(S) \circ \gamma'$ over some $\gamma' \in \Gamma_\vartheta$ and all of these are everywhere defined, holomorphic and nowhere vanishing on $\mathbb{H}^n_\delta$. Thus, we can (re-)define j_ϑ on *generators* by requiring that (4.21) holds. Any relation in Γ_ϑ will be respected in $\mathrm{Hol}\,(\mathbb{H}^n_\delta, \mathbb{C}^\times)$ since the functions expressing the relation must agree on the non-empty open subset $\Omega^n_\delta \subseteq \mathbb{H}^n_\delta$. We return to this technical point in Remark 4.8.

Finally, for any function f on $\mathbb{H}^n_\delta$ with values in a complex vector space and any $\gamma \in \Gamma_\vartheta$, we define a new function $f|\gamma$ on $\mathbb{H}^n_\delta$ by

$$f|\gamma := j_\vartheta(\gamma)^{-1} \cdot (f \circ \gamma), \quad \text{that is} \quad (f|\gamma)(z) = j_\vartheta(\gamma, z)^{-1} f(\gamma \cdot z). \tag{4.22}$$

We extend this group action to the group algebra $\mathcal{R} := \mathbb{C}[\Gamma_\vartheta]$ in the usual way.

The next two lemmas hint at the usefulness of the action we just introduced, for the proof of Theorem 7. Indeed, these Lemmas will essentially reduce the proof of Theorem 7 to a purely algebraic statement about a right ideal in the algebra $\mathcal{R}$, which will be addressed in the next section.

Lemma 4.3. *For every $A \in \mathcal{R}$ and $z \in \mathbb{H}^n_\delta$ we have $\mathcal{F}_{\mathbb{R}^n}((g_\delta|A)(z)) = (g_\delta|SA)(z)$.*

Proof. By linearity, we may assume that $A \in \Gamma_\vartheta$. Given that $\widehat{g_\delta(z)} = j_\vartheta(S,z)^{-1} g_\delta(Sz)$ and the properties (4.21),

$$\begin{aligned} \mathcal{F}((g_\delta|A)(z)) &= j_\vartheta(A,z)^{-1}\mathcal{F}(g_\delta(Az)) = j_\vartheta(A,z)^{-1} j_\vartheta(S,Az)^{-1} g_\delta(S(Az)) \\ &= j_\vartheta(SA,z)^{-1} g_\delta(SAz) = (g_\delta|SA)(z), \end{aligned}$$

as claimed. □

We denote by $\mathcal{I} = \sum_{\beta \in \mathcal{O}_K} (1 - T^{2\beta})\mathcal{R}$ the right ideal generated by all elements $(1 - T^{2\beta})$, $\beta \in \mathcal{O}_K$.

Lemma 4.4. *For all $A \in \mathcal{I}$ and all $z \in \mathbb{H}^n_\delta$, the function $(g_\delta|A)(z) : \mathbb{R}^n \to \mathbb{C}$ vanishes at all points $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ for which there is $\alpha \in \mathfrak{a}^2$ such that $x_j^2 = |\sigma_j(c)|\sigma_j(\alpha)$ for all j, that is to say, at all points of the set $E(c, \mathfrak{a})$, defined in* (4.14).

Proof. By linearity, may assume that $A = (T^{2\beta} - 1)\gamma$ for some $\gamma \in \Gamma_\vartheta$ and some $\beta \in \mathcal{O}_K$. By definition and by (4.21), we have

$$(g_\delta|(T^{2\beta} - 1)\gamma)(z) = (g_\delta|T^{2\beta}\gamma)(z) - (g_\delta|\gamma)(z) = j_\vartheta(\gamma, z)^{-1}(g_\delta(\gamma z + 2\sigma(\beta)) - g_\delta(\gamma z)).$$

Set $\tau := \gamma z$. Then, for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$,

$$g_\delta(\tau + 2\sigma(\beta))(x) - g_\delta(\tau)(x) = \left(e^{2\pi i \sum_{j=1}^n \delta_j \sigma_j(\beta) x_j^2} - 1\right) g_\delta(\tau)(x).$$

If there is $\alpha \in \mathfrak{a}^2$ so that $x_j^2 = |\sigma_j(c)|\sigma_j(\alpha)$ for all j, then, since $\delta_j = \sigma_j(c)/|\sigma_j(c)|$, we have

$$\sum_{j=1}^n \delta_j \sigma_j(\beta) x_j^2 = \sum_{j=1}^n \sigma_j(c)\sigma_j(\beta)\sigma_j(\alpha) = \mathrm{Tr}_{K/\mathbb{Q}}(c\beta\alpha) \in \mathbb{Z},$$

because $c\alpha \in \mathcal{O}_K^\vee$ and $\beta \in \mathcal{O}_K$. This proves what we want. □

4.3.3 Ideals in the group algebra $\mathcal{R} = \mathbb{C}[\Gamma_\vartheta]$

Lemma 4.3 and Lemma 4.4 together show that, for any element $A \in \mathcal{R}$ which belongs to the ideal $\mathcal{I}$ and which can also be written as $A = (1 + \epsilon S)A_1$ for some $A_1 \in \mathcal{R}$ and $\epsilon \in \{\pm 1\}$ is such that, for any $z \in \mathbb{H}^n_\delta$, the Schwartz function $f = (g_\delta | A)(z)$ vanishes at all points of the set $E(c, \mathfrak{a})$ and has Fourier transform $\widehat{f} = \epsilon f$. The next proposition will show that there are plenty of such elements A. It lies at the heart of our proof of Theorem 7 (and Theorem 8).

Proposition 4.3. *We have $(1 - S)\mathcal{R} \cap \mathcal{I} \neq 0$ and $(1 + S)\mathcal{R} \cap \mathcal{I} \neq 0$. Moreover, these intersections are infinite dimensional vector spaces over $\mathbb{C}$.*

Proof. We first note that if $\mathcal{J} \subseteq \mathcal{R}$ is any *nonzero* right ideal, then, since the group Γ_ϑ is infinite, we can produce an arbitrarily high number of right translates of a single nonzero element in $\mathcal{J}$ that have disjoint supports (say), showing that $\dim_{\mathbb{C}}(\mathcal{J}) = \infty$. So we only need to show that $(1 \pm S)\mathcal{R} \cap \mathcal{I} \neq 0$.

To do that, we note that if two elements $\gamma_1, \gamma_2 \in \Gamma_\vartheta$ have the same bottom row (possibly up to sign), then $\gamma_1 - \gamma_2 = (1 - \gamma_2\gamma_1^{-1})\gamma_1 \in \mathcal{I}$. It thus suffices to construct $A_+, A_- \in \mathcal{R}$ such that $(1 - S)A_-$ and $(1 + S)A_+$ can be written as non-trivial finite sums of differences of group elements with equal bottom row. We also know that left multiplication by S interchanges the rows of a matrix and switches the sign on the top. Guided by these two observations, we make the Ansatz

$$A_- = \sum_{r \in \mathbb{Z}/2n\mathbb{Z}} \gamma_r, \quad \gamma_r = \begin{pmatrix} c_{r-1} & d_{r-1} \\ c_r & d_r \end{pmatrix}, \qquad A_+ = \sum_{r \in \mathbb{Z}/2n\mathbb{Z}} (-1)^r \gamma'_r, \quad \gamma'_r = \begin{pmatrix} c'_{r-1} & d'_{r-1} \\ c'_r & d'_r \end{pmatrix},$$

where $n \geq 1$ and $c_r, d_r, c'_r, d'_r \in \mathcal{O}_K$ are to be found so that all elements γ_r, γ'_r belong to Γ_ϑ and such that $0 \neq (1 \pm S)A_\pm$ because these elements always belong to $\mathcal{I}$. Some experimentation shows that there are no non-trivial examples for $n = 1, 2, 3$ and further experimentation yields an example for $n = 4$ as follows. Choose $a, b, x, y \in \mathcal{O}_K$ such that

$$(1 + 4a)(1 + 4x) = 1 = (1 - 3b)(1 - 3y), \qquad axby \neq 0. \tag{4.23}$$

This is possible by Dirichlet's unit Theorem, which implies that for all non-zero integral ideals $\mathfrak{a} \subseteq \mathcal{O}_K$, the kernel of the natural map $\mathcal{O}_K^\times \to (\mathcal{O}_K/\mathfrak{a})^\times$ is infinite (use this for $\mathfrak{a} = 4\mathcal{O}_K$ or $3\mathcal{O}_K$). Consider then the elements $\gamma_r = \gamma'_r$ defined by

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 2a \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} -1 & 2a \\ 2 & -(1+4a) \end{pmatrix} \qquad \gamma_3 = \begin{pmatrix} 2 & -(1+4a) \\ \frac{1-4b}{1+4a} & 2b \end{pmatrix}$$

$$\gamma_4 = \begin{pmatrix} \frac{1-4b}{1+4a} & 2b \\ 2y & \frac{1-4y}{1+4x} \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} 2y & \frac{1-4y}{1+4x} \\ -(1+4x) & 2 \end{pmatrix}$$

$$\gamma_6 = \begin{pmatrix} -(1+4x) & 2 \\ 2x & -1 \end{pmatrix} \qquad \gamma_7 = \begin{pmatrix} 2x & -1 \\ 1 & 0 \end{pmatrix}.$$

We claim that: (i) each γ_r belongs to Γ_ϑ and (ii) that $(1 \pm S)A_\pm \neq 0$. To prove (i), we first verify, by computing determinants and using (4.23), that each γ_r belongs to the congruence group $\tilde{\Gamma}_\vartheta \supseteq \Gamma_\vartheta$ defined in Remark 4.6. On the other hand, for $r \neq 4$, either one of the diagonal or off-diagonal entries of γ_r is a unit, so that, by multiplying γ_r from the right or the left by $S^{\delta_1} T^{2\alpha} S^{\delta_2}$ with suitable $\alpha \in \mathcal{O}_K$, $\delta_1, \delta_2 \in \{0, 1\}$, we obtain a matrix in $\tilde{\Gamma}_\vartheta$ one of whose diagonal or off-diagonal entries is zero and hence belongs to Γ_ϑ. For γ_4, note that $\gamma_4 T^{2(1+4a)}$ has lower right entry equal to $1+4a$, which is a unit.

To verify (ii) note that, since none of a, b, x, y is zero, we have $\{\gamma_r\}_{r\in\mathbb{Z}/8\mathbb{Z}} \cap \{S, 1\} = \{1\}$, so that the coefficient of $1 \in \Gamma_\vartheta$ in the finite sum $(1 \pm S)A_\pm$ is $1 \in \mathbb{C}$. □

Having proved Proposition 4.3 it remains to show that we can produce any number of linearly independent functions $(g_\delta|A)(z)$ by varying $A \in \mathcal{I} \cap (1 \pm S)\mathcal{R}$ and $z \in \mathbb{H}^n_\delta$ suitably. This is essentially a consequence of Lemma 4.1, as we will explain in a moment. First, let us call a point $z \in \mathbb{H}^n_\delta$ a *generic* point (for the field K) if for all $\gamma, \gamma' \in \Gamma = \mathrm{PSL}_2(\mathcal{O}_K)$ we have

$$\gamma \neq \gamma' \quad \Rightarrow \quad \gamma z \neq \gamma' z$$

We contend that the set of generic points of K is dense and open in $\mathbb{H}^n_\delta$. Indeed, it is equal to

$$\bigcap_{\gamma\in\Gamma} \{z \in \mathbb{H}^n_\delta \,:\, \gamma z \neq z\} = \mathbb{H}^n_\delta \setminus \bigcup_{\gamma\in\Gamma} \{z \in \mathbb{H}^n_\delta \,:\, \gamma z = z\} \tag{4.24}$$

and each set in the union on the right is either empty or a singleton set. That the above set is dense and open in $\mathbb{H}^n_\delta$ follows from Baire's theorem, but for the proof of Theorem 7 below it suffices that there exists at least *one* generic point while for the proof of Theorem 8 we will only use that it is infinite.

Lemma 4.5. *Let $z \in \mathbb{H}^n_\delta$ be a generic point for K. Then the linear map $\Phi_z : \mathbb{C}[\Gamma_\vartheta] \to \mathcal{S}(\mathbb{R}^n)$, defined by $\Phi_z(A) = (g_\delta|A)(z)$ is injective.*

Proof. Let $A = \sum_{i=1}^m a_j\gamma_j \in \ker(\Phi_z)$, where $a_j \in \mathbb{C}$ and the $\gamma_j \in \Gamma_\vartheta$ are pairwise distinct. This means that

$$0 = \Phi_z(A) = \sum_{j=1}^n a_j j_\vartheta(\gamma_j, z)^{-1} g_\delta(\gamma_j z).$$

Since z is generic for K, Lemma 4.1 applied with $c_\mu = \pi i(\gamma_\mu z_\mu) \in \mathbb{C}^n$ implies $a_j j_\vartheta(\gamma_j, z)^{-1} = 0$ for all j hence $a_j = 0$ for all j, as desired. □

4.3.4 Conclusion

We are now ready to give the proof of Theorem 7. Recall that $V \subseteq \mathcal{S}(\mathbb{R}^n)$ denotes the linear span of all Gaussians $g_\delta(z)$, $z \in \mathbb{H}^n_\delta$.

Proof of Theorem 7. Fix a generic point $z \in \mathbb{H}^n_\delta$ for K and fix a sign $\epsilon \in \{\pm 1\}$. Let $V(c, \mathfrak{a}) \subseteq V$ denote the space of $f \in V$ such that $\hat{f} = \epsilon f$ and $f|_{E(c,\mathfrak{a})} = 0$. We must show that it is infinite dimensional.

By Lemma 4.5, the linear map $\Phi_z : \mathbb{C}[\Gamma_\vartheta] \to V$ is injective. It thus follows from Proposition 4.3 that the image of $(1 + \epsilon S)\mathcal{R} \cap \mathcal{I})$ under Φ_z is infinite dimensional. On the other hand, that image is contained in $V(c, \mathfrak{a})$ by by Lemma 4.4 and Lemma 4.3, finishing the proof. □

Remark 4.8. We return to the discussion after (4.21) where we lifted the cocycle j_ϑ to $\mathrm{Hol}(\mathbb{H}^n_\delta, \mathbb{C}^\times)$ and said that this was not strictly necessary. Indeed, we could have replaced $\mathbb{H}^n_\delta$ by Ω^n_δ everywhere in the above arguments and used that this set also contains infinitely many generic points (as it is open in $\mathbb{H}^n_\delta$). As a side remark, we note that $\Omega^n_\delta \neq \mathbb{H}^n_\delta$, since ϑ^8 is a Hilbert modular form of positive (parallel) weight 4, so necessarily has a zero in $\mathbb{H}^n_\delta$, as a consequence of the the so-called Köcher principle.

Example 1. For the real quadratic field $K = \mathbb{Q}(\sqrt{5})$ with fundamental unit $\varepsilon_0 = \frac{1+\sqrt{5}}{2}$ given by the golden ratio, we find the following solutions to (4.23):

$$\varepsilon_0^6 = 1 + 4(1 + 2\varepsilon_0) = 1 + 4a, \qquad \varepsilon_0^{-6} = 1 + 4(3 - 2\varepsilon_0) = 1 + 4x,$$
$$\varepsilon_0^8 = 1 - 3(-4 - 7\varepsilon_0) = 1 - 3b, \qquad \varepsilon_0^{-8} = 1 - 3(-11 + 7\varepsilon_0) = 1 - 3y.$$

This gives the matrices

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 2 + 4\varepsilon_0 \end{pmatrix}$$
$$\gamma_2 = \begin{pmatrix} -1 & 2 + 4\varepsilon_0 \\ 2 & -5 - 8\varepsilon_0 \end{pmatrix} \qquad \gamma_3 = \begin{pmatrix} 2 & -5 - 8\varepsilon_0 \\ -3 + 4\varepsilon_0 & -8 - 14\varepsilon_0 \end{pmatrix}$$
$$\gamma_4 = \begin{pmatrix} -3 + 4\varepsilon_0 & -8 - 14\varepsilon_0 \\ -22 + 14\varepsilon_0 & 1 - 4\varepsilon_0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} -22 + 14\varepsilon_0 & 1 - 4\varepsilon_0 \\ -13 + 8\varepsilon_0 & 2 \end{pmatrix}$$
$$\gamma_6 = \begin{pmatrix} -13 + 8\varepsilon_0 & 2 \\ 6 - 4\varepsilon_0 & -1 \end{pmatrix} \qquad \gamma_7 = \begin{pmatrix} 6 - 4\varepsilon_0 & -1 \\ 1 & 0 \end{pmatrix}$$

Let us end this section by formulating an open question. Consider the group algebra $\mathbb{C}[\Gamma]$ and the right ideal $\mathcal{I}_1$ generated by all elements $(1 - T^{\alpha})(1 - T^{\beta})$ with $\alpha, \beta \in \mathcal{O}_K$. Do we have $(1 \pm S)\mathbb{C}[\Gamma] \cap \mathcal{I}_1 \neq 0$ for some choice of sign? It is very likely that an affirmative answer to this question would produce linear combinations of Gaussians $f \in V$ such that $f, \hat{f}$ *and all of their first order partial derivatives* vanish at $\sqrt{2\mathcal{O}_K^{\vee}}$ which is a subset of the union of all spheres $\sqrt{2m}S^{n-1}$ over integers $m \geq 0$. Such a (hypothetical) non-uniqueness result would contrast the (hypothetical, but very plausible) uniqueness result sketched in §3.4.

4.4 Proof of Theorem 8

In this section we give the proof of Theorem 8. We will use some of the notation and results of §4.3, in particular, the eight elements γ_r, $r \in \mathbb{Z}/8\mathbb{Z}$ given in the proof of Proposition 4.3, Lemma 4.1 and the notion of a *generic* point for K, as defined near (4.24) (but with $\mathbb{H}^n_\delta$ replaced by $\mathbb{H}^n$). The entries of the matrices γ_r depend on a non-trivial solution $a, b, x, y \in \mathcal{O}_K$ to the equation (4.23). We fix one such solution that satisfies in addition that

$$\textit{all four units} \quad (1 + 4a), (1 + 4x), (1 - 3b), (1 - 3y) \in \mathcal{O}_K^{\times} \quad \textit{are totally positive}. \tag{4.25}$$

This is possible, since the subgroup of totally positive units in $\mathcal{O}_K^{\times}$ is infinite (indeed, already the subgroup of squared units is infinite, by Dirichlet's unit Theorem).

We work for all of §4.4, on $\mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_n}$ and with the corresponding Gaussians $g(z) : \mathbb{R}^d \to \mathbb{C}$, defined as $g(z)(x) = g(z, x) = e^{\pi i z_1 |x_1|^2} \cdots e^{\pi i z_n |x_n|^2}$ with $x_j \in \mathbb{R}^{d_j}$ and $z_j \in \mathbb{H}$. We also fix a sign $\epsilon \in \{\pm 1\}$ and consider a generic point $z = (z_1, \dots, z_n) \in \mathbb{H}^n$. We use the short hand notation $\mu(z) := \prod_{j=1}^n (z_j/i)^{d_j/2} \in \mathbb{C}^{\times}$.

For a set of coefficients $\{\lambda_r(z)\}_{r \in \mathbb{Z}/8\mathbb{Z}} \subseteq \mathbb{C}$ which we will determine later, consider the linear combination of Gaussians

$$h_z = \sum_{r \in \mathbb{Z}/8\mathbb{Z}} \lambda_r(z) g(\gamma_r z),$$

where the matrices $\gamma_r \in \Gamma$, $r \in \mathbb{Z}/8\mathbb{Z}$, are as in the proof of Proposition 4.3. Consider the function

$$\begin{aligned} f_z := h_z + \epsilon\widehat{h_z} &= \sum_{r\in\mathbb{Z}/8\mathbb{Z}} \lambda_r(z)g(\gamma_r z) + \sum_{r\in\mathbb{Z}/8\mathbb{Z}} \epsilon\lambda_r(z)\mu(\gamma_r z)^{-1}g(S\gamma_r z) \\ &= \sum_{r\in\mathbb{Z}/8\mathbb{Z}} \big(\lambda_{r-1}(z)g(\gamma_{r-1}z) + \epsilon\lambda_r(z)\mu(\gamma_r z)^{-1}g(S\gamma_r z)\big). \end{aligned}$$

where we used that $\widehat{g(\tau)} = \mu(\tau)^{-1}g(-1/\tau)$ for all $\tau \in \mathbb{H}$. and shifted indices. By construction, we have $\widehat{f_z} = \epsilon f_z$. We claim that the coefficients $\lambda_r(z)$ can be chosen in such a way that

$$\lambda_r(z) \neq 0 \quad \text{and} \quad \epsilon\lambda_r(z)\mu(\gamma_r z)^{-1} = -\lambda_{r-1}(z) \qquad \text{for all } r \in \mathbb{Z}/8\mathbb{Z}. \tag{4.26}$$

We postpone the proof of this claim to a later stage. Assuming its truth for the moment, we get

$$f_z = \sum_{r\in\mathbb{Z}/8\mathbb{Z}} \lambda_{r-1}(z)\left(g(\gamma_{r-1}z) - g(S\gamma_r z)\right). \tag{4.27}$$

Each difference $g(\gamma_{r-1}z) - g(S\gamma_r z)$ vanishes (in the sense defined before Theorem 8) on $\sqrt{\mathcal{O}_K^\vee}$, since, by construction, $S\gamma_r = T^{2\beta_r}\gamma_{r-1}$ for some $\beta_r \in \mathcal{O}_K$ and so

$$S\gamma_r z = T^{2\beta_r}\gamma_{r-1}z = \gamma_{r-1}z + 2\sigma(\beta_r),$$

implying that, if there is $\alpha \in \mathcal{O}_K^\vee$ so that $|x_j|^2 = \sigma_j(\alpha)$ for all j, then

$$f_z(x) = \sum_{r\in\mathbb{Z}/8\mathbb{Z}} \lambda_{r-1}(z)e^{\pi i\sum_{j=1}^n \sigma_j(\gamma_{r-1})z_j|x_j|^2}\left(1 - e^{2\pi i\sum_{j=1}^n \sigma_j(\beta_r)\sigma_j(\alpha)}\right) = 0,$$

because $\sum_{j=1}^n \sigma_j(\beta_r)\sigma_j(\alpha) = \mathrm{Tr}(\alpha\beta_r) \in \mathbb{Z}$.

So far, z was an arbitrary generic point. We now verify that $f_z \neq 0$ and that we can produce an arbitrary number of linearly independent functions of this form. Since z is generic for K, we have

$$\{r \in \mathbb{Z}/8\mathbb{Z} : \gamma_r z = z \text{ or } S\gamma_r z = z\} = \{r \in \mathbb{Z}/8\mathbb{Z} : \gamma_r = 1 \text{ or } S\gamma_r = 1\} = \{0\}$$

and this shows $f_z \neq 0$ via Lemma 4.1 and $\lambda_r(z) \neq 0$ for all r. Assume we have constructed linearly independent $f_{\tau_1}, \dots, f_{\tau_m}$ of this form with generic $\tau_j \in \mathbb{H}^n$ (here, the subscripts do *not* denote coordinates). Since the set of generic points for K is infinte (indeed uncountable), we can choose a generic point $\tau_{m+1} \in \mathbb{H}^n \smallsetminus \{\tau_1, \dots, \tau_m\}$ and the functions $f_{\tau_1}, \dots, f_{\tau_{m+1}}$ are then linearly independent as well. Indeed, if $0 = \sum_{i=1}^{m+1} t_i f_{\tau_i} = \sum_{w\in\mathbb{H}^n} a_w g(w)$, for $t_i \in \mathbb{C}$ and (unique) $a_w \in \mathbb{C}$ we find that $0 = a_{\tau_i} = t_i$ for all i, as desired.

Proof of the claim made in (4.26)

To finish the proof of Theorem 8, it remains to prove the claim made in (4.26). A short calculation shows that this claim is equivalent to

$$1 = \prod_{r\in\mathbb{Z}/8\mathbb{Z}} \mu(\gamma_r z) = \prod_{r\in\mathbb{Z}/8\mathbb{Z}} \prod_{j=1}^n ((\sigma_j(\gamma_r)z_j)/i)^{d_j/2}. \tag{4.28}$$

Indeed, if (4.28) holds, we can choose an arbitrary constant $\lambda_0 = \lambda_0(z) \in \mathbb{C}^\times$ and put

$$\lambda_{k+8\mathbb{Z}}(z) = (-\epsilon)^k\lambda_0 \prod_{1\le i\le k} \mu(\gamma_{i+8\mathbb{Z}}z) \quad \text{for } 1 \le k \le 7.$$

Let us denote the product on the right of (4.28) by $\rho(z)$. From the specific shape of the γ_r, it is clear that $\rho(z)^8 = 1$. Since $\mathbb{H}^n$ is connected, we deduce that the continuous function $z \mapsto \rho(z)$ is constant, with constant value given by an eighth root of unity ρ. To determine ρ, we will take the points z_j to $i\infty$. For this we need the following lemma.

Lemma 4.6. *For any $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, we have*

$$\lim_{y\to\infty} \frac{((g\cdot(iy))/i)^{1/2}}{|g\cdot(iy)|^{1/2}} = \exp(-\tfrac{\pi i}{4}\operatorname{sgn}(ac)) = e(-\tfrac{1}{8}\operatorname{sgn}(ac)), \tag{4.29}$$

where we write $e(w) = \exp(2\pi i w)$ and where the power functions are defined as in Item (4) of Section 1.1.

We defer the proof of Lemma 4.6 to the end of this section. Writing

$$\rho = \frac{\rho}{|\rho|} = \lim_{y\to\infty} \frac{\rho((iy,\dots,iy))}{|\rho((iy,\dots,iy))|}$$

and applying formula (4.29) (and using the fact that the d_j are integers[40]) , we see that

$$\rho = \prod_{r\in\mathbb{Z}/8\mathbb{Z}} \prod_{j=1}^{n} e\left(-\tfrac{d_j}{8}\operatorname{sgn}(\sigma_j(c_r c_{r-1}))\right), \tag{4.30}$$

where we recall that c_r denotes the lower left entry of γ_r and c_{r-1} the upper left entry of γ_r. Let us write down the eight products $c_r c_{r-1}$ appearing in (4.30). For $\alpha_1, \alpha_2 \in K$, we write $\alpha_1 \equiv \alpha_2$ to express that there is a totally positive $\beta \in K^\times$ so that $\alpha_2 = \alpha_1\beta$. Then, by assumption (4.25),

$$\begin{aligned}
c_0c_7 &= 0 & c_1c_0 &= 0\\
c_2c_1 &= -2 \equiv -1 & c_3c_2 &= 2\frac{1-4b}{1+4a} \equiv 1-4b\\
c_4c_3 &= \frac{1-4b}{1+4a}(2y) \equiv (1-4b)y & c_5c_4 &= -(2y)(1+4x) \equiv -y\\
c_6c_5 &= -2x(1+4x) \equiv -x & c_7c_6 &= 2x \equiv x.
\end{aligned}$$

We introduce the shorthands

$$\eta_j := \operatorname{sgn}(1-4\sigma_j(b)), \qquad \xi_j = \operatorname{sgn}(\sigma_j(y)).$$

Interchanging the order of multiplication in (4.30), using the above list of identities and noting that c_0c_7, c_1c_0 don't contribute, while the contributions of c_6c_5 and c_7c_6 cancel, we arrive at the formula

$$\rho = e(-\tfrac{1}{8}\Sigma), \quad \text{where} \quad \Sigma = \sum_{j=1}^{n} d_j(-1+\eta_j+\xi_j\eta_j-\xi_j) = \sum_{j=1}^{n} d_j(\eta_j-1)(\xi_j+1).$$

We claim that for each j we have $(\eta_j-1)(\xi_j+1) = 0$, or equivalently

$$1-4\sigma_j(b) > 0 \qquad \text{or} \qquad \sigma_j(y) < 0. \tag{4.31}$$

[40]We arrived at a minor conflict of notation: There are dimensions $d_j \in \mathbb{N}$, $j \in \{1,\dots,n\}$ and elements $d_r \in \mathcal{O}_K$, $r \in \mathbb{Z}/8\mathbb{Z}$, the entries of the right columns of the elements γ_r. The $d_r \in \mathcal{O}_K$ won't play a role in the remaining argument.

By (4.23), we have $(1-3b)(1-3y)=1$ and hence

$$(1-3\sigma_j(b))(1-3\sigma_j(y))=1.$$

By assumption (4.25), both factors in this product are positive. Assume now that $\sigma_j(y)>0$. Then the factor $(1-3\sigma_j(y))$ belongs to the interval $(0,1)$, implying that the factor $(1-3\sigma_j(b))$ belongs to the interval $(1,\infty)$ and so $-\sigma_j(b)>0$. But $-\sigma_j(b)>0$ implies $1-4\sigma_j(b)>1>0$. We assumed that $\sigma_j(y)>0$ and deduced $1-4\sigma_j(b)>0$, which proves (4.31). This finishes the proof of $\rho=1$, hence the proof of the claim made in (4.26) and thus the proof of Theorem 8. It only remains to prove Lemma 4.6.

Proof of Lemma 4.6. We need to show that for all $g=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{R})$, we have

$$\lim_{y\to\infty} \arg_{(-\pi/4,\pi/4)}\left[((g\cdot(iy))/i)^{1/2}\right] = -\tfrac{\pi}{4}\operatorname{sgn}(ac). \tag{4.32}$$

Both sides of (4.32) are unchanged if we replace g by $-g$, so we may assume $c\geq 0$ for the verification. For $y>0$, we abbreviate

$$w(y) := (g\cdot(iy))/i = \frac{aiy+b}{-cy+id} \in \mathcal{H} := \{w\in\mathbb{C} : \operatorname{Re}(w)>0\}.$$

In this proof, any asymptotic notation refers to taking $y\to\infty$. If $c=0$, then $ad=1$ and we have

$$w(y) = \frac{a(iy)+b}{id} = \frac{a^2(iy)+ab}{i} = a^2y-iab, \quad \text{hence} \quad \frac{\operatorname{Im}(w(y))}{\operatorname{Re}(w(y))} = \frac{-b}{ay} \longrightarrow 0,$$

which shows that the argument of $w(y)$ and hence that of $w(y)^{1/2}$, goes to zero, as claimed. If $c>0$ and $a=0$, then $-bc=1$ and we have

$$w(y) = \frac{b}{-cy+di} = \frac{b^2}{y+dbi} = \frac{b^2y}{y^2+(db)^2} - \frac{b^2(db)i}{y^2+(db)^2}, \quad \text{hence} \quad \frac{\operatorname{Im}(w(y))}{\operatorname{Re}(w(y))} = \frac{-db}{y} \longrightarrow 0,$$

as claimed. Assume now that $c>0$ and that $a\neq 0$. Then

$$w(y) = \frac{1}{i}\left(\frac{a}{c} - \frac{1}{c(c(iy)+d)}\right) = (-i)(a/c)+o(1).$$

We deduce that

- if $a>0$, then $\arg(w(y))\to -\pi/2$, hence $\arg(w(y)^{1/2})\to -\pi/4$, as claimed.
- if $a<0$, then $\arg(w(y))\to \pi/2$, hence $\arg(w(y)^{1/2})\to \pi/4$, as claimed.

This finishes the proof of (4.32) and thus the proof of Lemma 4.6. □

We conclude with the following speculation. We have just proved that $\sqrt{\mathfrak{d}_K}$ is a Fourier non-uniqueness set in $\mathbb{R}^n$. In particular, there can't be a Fourier interpolation formula using theses nodes. Although it seems quite unlikely, we have not logically ruled out the possibility of the existence of a Fourier interpolation "basis" with respect to these nodes. That is to say (roughly speaking), a collection of (Schwartz-) functions $a_\lambda : \mathbb{R}^n \to \mathbb{C}$ (even in each variable), indexed by $\lambda\in\mathfrak{d}_K$, such that $a_\lambda(\sqrt{\mu})=\delta_{\lambda,\mu}$ for all $\lambda,\mu\in\mathfrak{d}_{K,+}$ and such that the Fourier transforms $\widehat{a_\lambda}$ vanish at all points of $\sqrt{\mathfrak{d}_K}$. Indeed, perhaps these can be constructed as certain integrals of modular objects on $\mathbb{H}^n$ against the Gaussians $e^{\pi i z_j x_1^2}\cdots e^{\pi i z_n x_n^2}$, similarly to what we did in §2.3.

Bibliography

[AB77] W.O Amrein and A.M Berthier. On support properties of L^p-functions and their Fourier transforms. *Journal of Functional Analysis*, 24(3):258–267, 1977. (cited on page 8)

[ABR01] Sheldon Axler, Paul Bourdon, and Wade Ramey. *Harmonic function theory.* Springer, 2001. (cited on pages 81 and 86)

[AF07] Avner Ash and Solomon Friedberg. Hecke L-functions and the distribution of totally positive integers. *Canadian Journal of Mathematics*, 59(4):67–695, 2007. (cited on page 128)

[BH10] Yves Benoist and Oh Hee. Discreteness criterion for subgroups of products of SL(2). *Transformation Groups*, 15:503–515, 2010. (cited on page 122)

[Bie75] Edward Bierstone. Local properties of smooth maps equivariant with respect to finite group actions. *J. Differential Geom.*, 4(10):523–540, 1975. (cited on page 33)

[BK08] B. Berndt and M. Knopp. *Hecke's theory of modular forms and Dirichlet series.* World Scientific, 2008. (cited on pages 20, 21, 71, and 72)

[Boc51] S. Bochner. Theta relations with spherical harmonics. *Proceedings of the National Academy of Sciences*, 37(12):804–808, 1951. (cited on page 96)

[BRS20] Andry Bondarenko, Danylo Radchenko, and Kristian Seip. Fourier interpolation with zeros of zeta and other L-functions. *arXiv e-prints*, 2020. `https://arxiv.org/abs/2005.02996`. (cited on pages 8, 10, 41, 59, 79, 80, 103, and 104)

[CE03] Henry Cohn and Noam Elkies. New upper bounds for sphere packing. *Annals of Mathematics*, 157(2):689–714, 2003. (cited on page 114)

[CKM+17] Henry Cohn, Abhinav Kumar, Stephen D. Miller, Danylo Radchenko, and Maryna Viazvoska. The sphere packing problem in dimension 24. *Annals of Mathematics*, pages 1017–1033, 2017. (cited on pages 9, 13, 37, 114, and 117)

[CKM+21] Henry Cohn, Abhinav Kumar, Stephen D. Miller, Danylo Radchenko, and Maryna Viazvoska. Universal optimality of the E_8- and Leech lattice and interpolation formulas. *Annals of Mathematics*, 2021. `https://arxiv.org/abs/1902.05438`. (cited on pages 9, 13, 14, 30, 35, 55, 57, 58, 79, 114, and 117)

[CZ93] Young Ju Choie and Don Zagier. Rational period functions for PSL(2,Z). *Contemporary Mathematics*, pages 89–108, 1993. (cited on page 29)

[CZ14] Henry Cohn and Yufei Zhao. Sphere packing bounds via spherical codes. *Duke Mathematical Journal*, 163:1965–2002, 2014. (cited on page 114)

[DJ08] William Duke and Paul Jenkins. On the zeros and coefficients of certain weakly holomorphic modular forms. *Pure and applied mathematics quarterly*, 4(4):1327–1340, January 2008. (cited on page 45)

[Eic65] Martin Eichler. Grenzkreisgruppen und kettenbruchartige Algorithmen. *Acta Arithmetica*, 11, 1965. (cited on page 29)

[Gar18] Paul Garrett. *Modern analysis of automorphic forms by example.* Cambridge University Press, 2018. (cited on pages 36 and 113)

[Gra08] Loukas Grafakos. *Classical Fourier analysis.* Springer, New York, NY, 2008. (cited on page 37)

[GT11] Loukas Grafakos and Gerald Teschl. On Fourier transforms of radial functions and distributions. *Journal of Fourier Analysis and Applications*, 19, 12 2011. (cited on page 32)

[Hec36] Erich Hecke. Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung. *Mathematische Annalen*, 112:664–699, 1936. (cited on pages 20, 21, 71, and 73)

[Hec54] Erich Hecke. *Vorlesungen über die Theorie der algebraischen Zahlen.* Leipzig: Akad. Verlag, 1923, 1954. (cited on pages 126 and 130)

[HMR11] Haakan Hedenmalm and Alfonso Montes-Rodríguez. Heisenberg uniqueness pairs and the Klein–Gordon equation. *Annals of Mathematics*, 173(3):1507–1527, 2011. (not cited)

[HT12] Roger E. Howe and Eng C. Tan. *Non-abelian harmonic analysis: Applications of SL(2,R).* Universitext. Springer New York, 2012. (cited on page 84)

[IK04] Henryk Iwaniec and Emmanuel Kowalski. *Analytic number theory.* AMS Cooloquium Publications, vol 53, 2004. (cited on page 77)

[Iwa97] Henryk Iwaniec. *Topics in classical automorphic forms.* American Mathematical Society, 1997. (cited on page 20)

[Jam07] Philippe Jaming. Nazarov's uncertainty principles in higher dimension. *Journal of Approximation Theory*, May 2007. (cited on page 8)

[JM12] Nathan K. Johnson-McDaniel. A dimensionally continued Poisson summation formula. *Journal of Fourier Analysis and Applications*, 18:367–285, 2012. (cited on page 35)

[Kno74] Marvin I. Knopp. Some new results on the Eichler cohomology of automorphic forms. *Bulletin of the American mathematical society*, 80(4):607–631, July 1974. (cited on pages 29, 30, and 40)

[Kno78] Marvin I. Knopp. Rational period functions for the modular group. *Duke mathematical journal*, 45(1), March 1978. (cited on page 29)

[Kno81] Marvin I. Knopp. Rational period functions of the modular group II. *Glasgow Mathematical Journal*, 22(2):185–197, 1981. (cited on page 29)

[Kno89] Marvin I. Knopp. Recent developments in the theory of rational period functions. In David V. Chudnovsky, Gregory V. Chudnovsky, Harvey Cohn, and Melvyn B. Nathanson, editors, *Number Theory*, pages 111–122, Berlin, Heidelberg, 1989. Springer Berlin Heidelberg. (cited on page 29)

[Kul21] Aleksei Kulikov. Fourier interpolation and time-frequency localization. *Journal of Fourier Analysis and Applications*, 2021. (cited on page 14)

[Lan85] Serge Lang. $\mathrm{SL}(2,\mathbb{R})$. Springer, New York, 1985. (cited on page 34)

[Maa41] Hans Maass. Modulformen und quadratische Formen über dem quadratischen Zahlkörper $R(\sqrt{5})$. *Mathematicshe Annalen*, 118:65–84, 1941. (cited on page 130)

[Mar91] Gregory Margulis. *Discrete subgroups of semisimple Lie groups.* Springer, 1991. (cited on page 123)

[Mol11] Richard A. Mollin. *Algebraic Number Theory.* CRC Press, 2011. (cited on page 128)

[Mor15] Dave Morris. *Introduction to arithmetic groups.* Deductive Press, 2015. (cited on page 122)

[Mum94] David Mumford. *Tata lectures on Theta, I.* Birkhauesuser, 1994. (cited on pages 20 and 24)

[Naz93] Fedor Nazarov. Local estimates for exponential polynomials and their applications to inequalities of the uncertainty principle type. *Algebra i Analiz*, 5:3–66, 1993. (cited on page 8)

[Ran39] R. A. Rankin. Contributions to the theory of Ramanujan's function τ(n) and similar arithmetical functions: II. the order of the Fourier coefficients of integral modular forms. *Mathematical Proceedings of the Cambridge Philosophical Society*, 35(3):357–372, 1939. (cited on page 77)

[Ran77] Robert A. Rankin. *Modular Forms and functions.* Cambridge University Press, 1977. (cited on page 20)

[RdS20] João P.G. Ramos and Matheus de Sousa. Perturbed interpolation formuale and applications. *arXiv e-prints*, 2020. `https://arxiv.org/abs/2005.10337`. (cited on pages 59, 103, and 112)

[RdSar] João P.G. Ramos and Matheus de Sousa. Fourier uniqueness pairs of powers of integers. *Journal of the European mathematical societiy*, to appear. (cited on page 103)

[RS] Danylo Radchenko and Martin Stoller. Fourier non-uniquness sets from totally real number fields. *Comentarii Mathematici Helvetici.* to appear (arXiv link `https://arxiv.org/abs/2108.11828`). (cited on pages 13, 15, 70, 118, and 143)

[RS21] João P.G. Ramos and Martin Stoller. Perturbed Fourier uniqueness and -interpolation results in higher dimensions. *Journal of functional analysis*, 282(12), June 2021. (cited on pages 13, 15, 41, 52, 59, 81, 103, 104, 105, 112, and 143)

[RV19] Danylo Radchenko and Maryna Viazovska. Fourier interpolation on the real line. *Publications mathématiques de l'IHÉS*, 29:51–81, 2019. (cited on pages 8, 9, 10, 11, 20, 35, 37, 40, 41, 45, 59, and 84)

[Sar21] Naser Sardiri. Higher Fourier interpolation on the plane. *arXiv e-prints*, April 2021. `https://arxiv.org/abs/2102.08753`. (cited on page 14)

[Sch75] Gerald W. Schwarz. Smooth functions invariant under the action of a compact Lie group. *Topology*, 14(1):63–68, 1975. (cited on pages 32 and 33)

[Sel69] Alte Selberg. Recent developments in the theory of discontinuous groups of motions of symmetric spaces. In *Proceedings of the 15th Scandinavian Congress Oslo 1968*, 1969. (cited on page 122)

[Ser73] Jean-Pierre Serre. *A course in arithmetic*. Springer, 1973. (cited on page 20)

[Sto21] Martin Stoller. Fourier interpolation from spheres. *Transactions of the American Mathematical Soeciety*, 374(11):8045–8079, November 2021. (cited on pages 11, 15, 67, 70, 97, 102, 103, 106, and 143)

[SW71] Elias M. Stein and Guido Weiss. *Introduction to Fourier analysis on Euclidean spaces*. Mathematical Series. Princeton University Press, 1971. (cited on page 81)

[Via17] Maryna Viazovska. The sphere packing problem in dimension 8. *Annals of Mathematics*, pages 991–1015, 2017. (cited on pages 9, 13, 114, and 116)

[Whi43] Hassler Whitney. Differentiable even functions. *Duke Math. Journal*, 1943. (cited on page 32)

[Yos11] H Yoshida. On some problems cconcerning discrete subgroups. *Commentarii Mathematici Universitatis Sancti Pauli*, 60, 2011. (cited on page 130)

[ZMGR15] Daniel Zwillinger, Victor Moll, I.S. Gradshteyn, and I.M. Ryzhik, editors. *Special functions*. Academic Press, eighth edition edition, 2015. (cited on pages 37 and 75)

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