From System Level Synthesis to Robust Closed-loop Data-Enabled Predictive Control

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Abstract— **Willems' fundamental lemma and system level synthesis both characterize a linear dynamic system by its input/output sequences. In this work, we show that the extension of the fundamental lemma in uncertain LTI systems is equivalent to the system level synthesis. Inspired by this observation, a robust closed-loop data-enabled predictive control scheme is proposed, where a novel formulation of causal feedback control law is further derived. Two numerical experiments, including the temperature control of a single-zone building, are carried out to validate the effectiveness of the proposed data-driven controller.**

I. INTRODUCTION

Although linear control theory is well-developed, the Willems' fundamental lemma [19] and the system level synthesis [1] still spark significant research interest. The Willems' fundamental lemma enables a data-driven characterization of a deterministic linear time invariant (LTI) system under reasonable assumptions on controllability and persistent excitation. This result has been successfully applied in controller design [5], [6], [14], [15], which shows better data efficiency than, for example, reinforcement learning. Further results in [21] relax the condition of the fundamental lemma, which improves the scalability of Willems' fundamental lemma in practical applications. Beyond the scope of LTI systems, [11], [4], [3], [16] attempt to extend the lemma to various nonlinear systems.

Different from Willems' fundamental lemma, system level synthesis (SLS) [1] is a model-based framework built for general uncertain linear systems, which are not necessarily time-invariant. The main benefit of the SLS scheme is that it provides an explicit link between a system's response and a linear feedback control law. As a model-based method, a direct integration of SLS into well-developed linear control theory allows the consideration of model structure and parametric uncertainty [10], [18].

The link between these two methods motivates this work. In [20, Theorem 2], a link between SLS and the fundamental lemma is derived for deterministic LTI systems. In this work, we further generalize this result to uncertain LTI systems in Lemma 2. Inspired by this observation, we develop a causal robust closed-loop data-enabled predictive control scheme, which results in the same scale of computational cost as a model-based robust MPC with a linear feedback law and full

state measurements. The major contributions of this work are summarized as follows:

- Show the equivalence between the Willems' fundamental lemma and the SLS for uncertain LTI systems.
- A robust closed-loop data-enabled predictive control scheme is proposed with a causal feedback structure.

Notation

colspan(A) denotes the column space (*e.g.* range) of the matrix A. I is the identity matrix and $\mathbf{O}$ is a matrix of all zeros. $x := \{x_i\}_{i=1}^T$ denotes a set of size T indexed by i. x_i denotes the measurement of x at time i, and $x_{0:L} := [x_0^\top, x_1^\top \ldots x_L^\top]^\top$ denotes a concatenated sequence from x_0 to x_L Meanwhile, the subscript $_\mathbf{d}$ indicates that a measured datapoint lies within the dataset. For the sake of consistency, L is reserved for the length of the system responses and n_c denotes the number of columns of a Hankel matrix.

II. PRELIMINARIES

A. Willems' Fundamental Lemma

Definition 1: A Hankel matrix of depth L associated with a signal sequence $\{s_i\}_{i=0}^T$, $s_i \in \mathbb{R}^{n_s}$ is

$$\mathfrak{H}_L(s) := \begin{bmatrix} s_0 & s_1 & \ldots & s_{T-L} \\ s_1 & s_2 & \ldots & s_{T-L+1} \\ \vdots & \vdots & & \vdots \\ s_{L-1} & s_L & \ldots & s_T \end{bmatrix}.$$

A deterministic LTI system, dubbed $\mathfrak{B}(A, B, C, D)$, is defined as

$$x_{i+1} = Ax_i + Bu_i , \ y_i = Cx_i + Du_i , \tag{1}$$

whose order is denoted by $\mathfrak{D}(\mathfrak{B}(A, B, C, D)) := n_x$, with $u_i \in \mathbb{R}^{n_u}$ and $y_i \in \mathbb{R}^{n_y}$. An L-step trajectory generated by this system is

$$[u_{1:L}^\top, y_{1:L}^\top] := [u_1^\top, \ldots, u_L^\top, y_1^\top, \ldots, y_L^\top]^\top .$$

The set of all possible L-step trajectories generated by $\mathfrak{B}(A, B, C, D)$ is denoted by $\mathfrak{B}_L(A, B, C, D)$.

Given a sequence of input-output measurements $\{u_{\mathbf{d},i}, y_{\mathbf{d},i}\}_{i=0}^T$, we call the input sequence persistently exciting of order L if $\mathfrak{H}_L(u_\mathbf{d})$ is full row rank. By building the following n_c-column stacked Hankel matrix

$$\mathfrak{H}_L(u_\mathbf{d}, y_\mathbf{d}) := \begin{bmatrix} \mathfrak{H}_L(u_\mathbf{d}) \\ \mathfrak{H}_L(y_\mathbf{d}) \end{bmatrix} , \tag{2}$$

we state the **Fundamental Lemma** as

Lemma 1: [19, Theorem 1] Consider a controllable linear time-invariant system and assume $\{u_{\mathbf{d}}\}_{i=0}^{T}$ is persistently exciting of order $L + \mathfrak{O}(\mathfrak{B}(A,B,C,D))$. Then $\text{colspan}(\mathfrak{H}_L(u_{\mathbf{d}}, y_{\mathbf{d}})) = \mathfrak{B}_L(A,B,C,D)$.

The Willems' fundamental lemma is applied to design data-driven predictive controller under the name Data-enabled predictive control (DeePC) [5], [14]. With a noise-free dataset measured offline, $u_{\mathbf{d}}$ and $y_{\mathbf{d}}$, a predictive control problem of horizon n_h is formulated as

$$
\min_{y_{pred}, u_{pred}, g} \quad J(y_{pred}, u_{pred})
$$
$$
\text{s.t.} \quad
\begin{bmatrix}
\mathfrak{H}_{L,init}(u_{\mathbf{d}}) \\
\mathfrak{H}_{L,init}(y_{\mathbf{d}}) \\
\mathfrak{H}_{L,pred}(u_{\mathbf{d}}) \\
\mathfrak{H}_{L,pred}(y_{\mathbf{d}})
\end{bmatrix} g =
\begin{bmatrix}
u_{init} \\
y_{init} \\
u_{pred} \\
y_{pred}
\end{bmatrix}, \qquad (3)
$$
$$
F_u u_{pred} \le f_u, \ F_y y_{pred} \le f_y
$$

where $J(\cdot, \cdot)$ is a convex objective function. F_u, f_u, F_y and f_y model polytopic input and output constraints. u_{init}, y_{init} are a fixed-length sequence of measured inputs and outputs preceding the current point in time. The matrix $\mathfrak{H}_L(u_{\mathbf{d}})$ is split into two sub-Hankel matrices as

$$
\mathfrak{H}_L(u_{\mathbf{d}}) =
\begin{bmatrix}
\mathfrak{H}_{L,init}(u_{\mathbf{d}}) \\
\mathfrak{H}_{L,pred}(u_{\mathbf{d}})
\end{bmatrix}.
$$

The matrix $\mathfrak{H}_{L,init}(u_{\mathbf{d}})$ is of depth t_{init} and $\mathfrak{H}_{L,pred}(u_{\mathbf{d}})$ is of depth n_h such that $t_{init} + n_h = L$. The matrices $\mathfrak{H}_{L,init}(y_{\mathbf{d}})$, $\mathfrak{H}_{L,pred}(y_{\mathbf{d}})$ are defined accordingly. The choice of t_{init} is made to ensure a unique estimation of the initial state; please refer to [15] for more details.

B. System Level Synthesis

For the sake of simplicity, system level synthesis (SLS) is introduced for a fully observed uncertain LTI system and a more general setup can be found in [1].

$$
x_{i+1} = Ax_i + Bu_i + w_i, \qquad (4)
$$

where w_i is the process noise. To proceed, we further define

$$
\mathcal{A} := \text{blkdiag}(\overbrace{A, A, \ldots, A}^{L-1}, \mathbf{O})
$$
$$
\mathcal{B} := \text{blkdiag}(\underbrace{B, B, \ldots, B}_{L-1}, \mathbf{O}), \ Z\mathcal{A} =
\begin{bmatrix}
\mathbf{O} & \mathbf{O} & \mathbf{O} & \ldots & \mathbf{O} & \mathbf{O} \\
A & \mathbf{O} & \mathbf{O} & \ldots & \mathbf{O} & \mathbf{O} \\
\mathbf{O} & A & \mathbf{O} & \ddots & \mathbf{O} & \mathbf{O} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\mathbf{O} & \mathbf{O} & \mathbf{O} & \ldots & A & \mathbf{O}
\end{bmatrix}
$$

where Z is the shift operator. The response of system (4) is

$$
x_{0:L} = Z\mathcal{A}x_{0:L} + Z\mathcal{B}u_{0:L} + w_{0:L-1}.
$$

A linear time-varying feedback control law is defined as

$$
u_{0:L} =
\begin{bmatrix}
K_{0,0} & & & \\
K_{1,0} & K_{1,1} & & \\
\vdots & \ddots & \ddots & \\
K_{L,0} & \ldots & K_{L,L-1} & K_{L,L}
\end{bmatrix}
x_{0:L} := \mathbf{K}x_{0:L}, \quad (5)
$$

where $K_{i,j}$ is the feedback law injecting into u_i with respect to the measurement x_j. System level synthesis (SLS)

characterizes all trajectories driven by some linear feedback control law and it is stated as

Theorem 1: ([1, Theorem 2.1]) Over a horizon $t = 0, \ldots, L-1$, the system dynamics (4) with block-lower-triangular state feedback law $\mathbf{K}$ defining the control action as $u_{0:L} = Kx_{0:L}$, the following are true

1) The affine subspace defined by

$$
\begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix}
\begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix} = I, \qquad (6)
$$

parametrizes all possible system responses as

$$
\begin{bmatrix} x_{0:L} \\ u_{0:L} \end{bmatrix} =
\begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix}
\begin{bmatrix} x_0 \\ w_{0:L-1} \end{bmatrix} \qquad (7)
$$

2) For any block-lower-triangular matrices Φ_x, Φ_u satisfying (6), the controller $K = \Phi_u \Phi_x^{-1}$ achieves the desired response.

III. EQUIVALENCE BETWEEN FUNDAMENTAL LEMMA AND SLS IN UNCERTAIN LTI SYSTEMS

For the sake of clarity, the equivalence between the fundamental lemma and the SLS is established by the system (4), its general equivalence for the following uncertain LTI system will be discussed in Remark 1,

$$
\begin{aligned}
x_{i+1} &= Ax_i + Bu_i + Ew_i \\
y_i &= Cx_i + Du_i + Fw_i,
\end{aligned} \qquad (8)
$$

where the process noise w is of dimension n_w. Because the disturbance can be considered as an uncontrolled input [9], [12], the fundamental lemma can be generalized to the system (8). Given an augmented input/output sequence $\{u_{\mathbf{d},i}, w_{\mathbf{d},i}, y_{\mathbf{d},i}\}_{i=0}^{T}$, by defining $\mathfrak{H}_L(y_{\mathbf{d}}, w_{\mathbf{d},i}, u_{\mathbf{d}})$ and $\mathfrak{B}_L(A,B,C,D,E,F)$ in a form similar to the fundamental lemma (Lemma 1), the extension of the fundamental lemma is concluded as follows.

Corollary 1: Consider a controllable linear system and assume $u_{\mathbf{d}}$ and $w_{\mathbf{d}}$ are both persistently exciting of order $L + \mathfrak{O}(\mathfrak{B}(A,B,C,D,E,F))$. Then $\text{colspan}(\mathfrak{H}_L(y_{\mathbf{d}}, w_{\mathbf{d}}, u_{\mathbf{d}})) = \mathfrak{B}_L(A,B,C,D,E,F)$.

Consider the fully observed system (4), Corollary 1 implies that for each trajectory $x_{0:L}$ driven by inputs u and disturbance w, there exists a $g \in \mathbb{R}^{n_c}$ such that

$$
\begin{bmatrix}
\mathfrak{H}_L(u_{\mathbf{d}}) \\
\mathfrak{H}_L(w_{\mathbf{d}}) \\
\mathfrak{H}_{L+1,1}(x_{\mathbf{d}}) \\
\mathfrak{H}_{L+1,2:L+1}(x_{\mathbf{d}})
\end{bmatrix} g =
\begin{bmatrix}
u_{0:L-1} \\
w_{0:L-1} \\
x_{0:L}
\end{bmatrix}, \qquad (9)
$$

where $\mathfrak{H}_{L+1}(x_{\mathbf{d}})$ is split into two parts as

$$
\mathfrak{H}_{L+1}(x_{\mathbf{d}}) :=
\begin{bmatrix}
\mathfrak{H}_{L+1,1}(x_{\mathbf{d}}) \\
\mathfrak{H}_{L+1,2:L}(x_{\mathbf{d}})
\end{bmatrix}.
$$

wher we adopt the MATLAB-style notation, $\mathfrak{H}_{L,i:j}(x_{\mathbf{d}})$ denotes the i-th to j-th row block of $\mathfrak{H}_L(x_{\mathbf{d}})$. In particular, $\mathfrak{H}_{L+1,1}(x_{\mathbf{d}}) := [x_{\mathbf{d},0}, x_{\mathbf{d},1} \ldots, x_{\mathbf{d},n_c-1}]$ are the first row block of $\mathfrak{H}_{L+1}(x_{\mathbf{d}})$ corresponding to the initial components. It is noteworthy to point out that the Hankel matrix of $x_{\mathbf{d}}$ is of depth $L+1$, because system (4) has a one-step delay from u to x. We conclude with the following Lemma

Lemma 2: The subspace defined in (6) and (7) is the same as the subspace defined by (9).

Proof. Equation (7) implies that

$$\begin{bmatrix} x_{0:L} \\ u_{0:L-1} \end{bmatrix} \in \operatorname{colspan}\left(\begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix}\right) .$$

By Corollary 1, there exists a linear map G such that

$$\begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix} = \begin{bmatrix} \mathfrak{H}_{L+1}(x_{\mathbf{d}}) \\ \mathfrak{H}_L(u_{\mathbf{d}}) \end{bmatrix} G ,$$

therefore equation (6) is rewritten as

$$\begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} \mathfrak{H}_{L+1}(x_{\mathbf{d}}) \\ \mathfrak{H}_L(u_{\mathbf{d}}) \end{bmatrix} G = I$$

$$\overset{(a)}{\Longrightarrow} \begin{bmatrix} \mathfrak{H}_{L+1,1}(x_{\mathbf{d}}) \\ \mathfrak{H}_L(w_{\mathbf{d}}) \end{bmatrix} G = I , \qquad (10)$$

where (a) comes from the substitution of dynamics (4). By denoting $\hat{g} = G[x_0^\top, w]^\top$, we have

$$\begin{bmatrix} x_0 \\ w_{0:L-1} \end{bmatrix} = I \begin{bmatrix} x_0 \\ w_{0:L-1} \end{bmatrix}$$

$$\overset{(b)}{=} \begin{bmatrix} \mathfrak{H}_{L+1,1}(x_{\mathbf{d}}) \\ \mathfrak{H}_L(w_{\mathbf{d}}) \end{bmatrix} G \begin{bmatrix} x_0 \\ w_{0:L-1} \end{bmatrix} = \begin{bmatrix} \mathfrak{H}_{L+1,1}(x_{\mathbf{d}}) \\ \mathfrak{H}_L(w_{\mathbf{d}}) \end{bmatrix} \hat{g} ,$$

where (b) results from (10). Meanwhile, we have

$$\begin{bmatrix} x_{0:L} \\ u_{0:L-1} \end{bmatrix} = \begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix} \begin{bmatrix} x_0 \\ w \end{bmatrix}$$

$$= \begin{bmatrix} \mathfrak{H}_{L+1}(x_{\mathbf{d}}) \\ \mathfrak{H}_L(u_{\mathbf{d}}) \end{bmatrix} G \begin{bmatrix} x_0 \\ w_{0:L-1} \end{bmatrix} = \begin{bmatrix} \mathfrak{H}_{L+1}(x_{\mathbf{d}}) \\ \mathfrak{H}_L(u_{\mathbf{d}}) \end{bmatrix} \hat{g} ,$$

which shows that every element in the subspace (6) and (7) corresponds to an element in the subspace (9) by linear transformation G defined in (10). Finally, since the feedback control law used in the SLS can be arbitrary linear feedback, the subspace (6) and (7) is therefore of dimension $n_x + L(n_u + n_w)$, which is equal to the dimension of subspace (9). Hence, these two subspaces are equivalent. ∎

Remark 1: Lemma 2 holds intuitively as they both represent the same system, this lemma further shows the explicit linear map between the fundamental lemma and the SLS. To generalize the equivalence to system (8), we only need to further assume the observability of the system (8). The observability implies that the sequence of $\{u_{0:L}, w_{0:L}, y_{0:L}\}$ uniquely determines the sequence of $x_{0:L}$, a further application of Lemma 2 shows the general equivalence.

IV. ROBUST CLOSED-LOOP DEEPC

As the SLS has successful applications in robust controller design, the equivalence established by Lemma 2 motivates us to develop a data-driven controller based on the fundamental lemma. According to the Corollary 1, we first modify prediction component in DeePC (3) to

$$\forall w_{pred} \in \mathcal{W} := \{w | F_w w \leq f_w\}$$

$$\begin{bmatrix} \mathfrak{H}_{L,init}(u_{\mathbf{d}}) \\ \mathfrak{H}_{L,init}(w_{\mathbf{d}}) \\ \mathfrak{H}_{L,init}(y_{\mathbf{d}}) \\ \mathfrak{H}_{L,pred}(u_{\mathbf{d}}) \\ \mathfrak{H}_{L,pred}(w_{\mathbf{d}}) \\ \mathfrak{H}_{L,pred}(y_{\mathbf{d}}) \end{bmatrix} g = \begin{bmatrix} u_{init} \\ w_{init} \\ y_{init} \\ u_{pred} \\ w_{pred} \\ y_{pred} \end{bmatrix} , \qquad (11)$$

whose elements are all defined in an approach similar to the standard DeePC (3) and the disturbance is assumed to be bounded in a polytope. For the sake of compactness, we denote the prediction part in (11) as

$$\begin{bmatrix} \mathcal{H}_{init} \\ \mathfrak{H}_{L,pred}(u_{\mathbf{d}}) \\ \mathfrak{H}_{L,pred}(w_{\mathbf{d}}) \\ \mathfrak{H}_{L,pred}(y_{\mathbf{d}}) \end{bmatrix} g = \begin{bmatrix} h_{init} \\ u_{pred} \\ w_{pred} \\ y_{pred} \end{bmatrix}$$

In this section, we will first formulate a general, but not necessarily causal, data-driven robust controller. Its causal realization, which we coin **Robust DeePC**, is further introduced in Theorem 2.

A. General Robust DeePC

Unlike a feedback controller defined by a state space model, in a DeePC scheme, inputs u_{pred} and y_{pred} are coupled indirectly through g. Hence, we propose to define a feedback control law with respect to g and then show that this is equivalent to state feedback. Similar to most feedback laws used in robust MPC, g is decomposed into a nominal part $\overline{g}$ and a linear feedback part $K_d w_{pred}$ as

$$g = \overline{g} + K_d w_{pred} . \qquad (12)$$

Based on this control law, a robust data-driven control is stated as

Lemma 3: If $\overline{g}$ and K_d satisfy following constraints, then the control law (12) guarantees n_h-step robust feasibility.

$$\begin{bmatrix} \mathcal{H}_{init} \\ \mathfrak{H}_{L,pred}(w_{\mathbf{d}}) \end{bmatrix} \overline{g} = \begin{bmatrix} h_{init} \\ \mathbf{0} \end{bmatrix} ,$$

$$\mathcal{H}_{init} K_d = \mathbf{O} , \ \mathfrak{H}_{L,pred}(w_{\mathbf{d}}) K_d = I ,$$

$$\forall w_{pred} \in \mathcal{W} , \qquad (13)$$

$$\begin{bmatrix} \mathfrak{H}_{L,pred}(u_{\mathbf{d}}) \\ \mathfrak{H}_{L,pred}(y_{\mathbf{d}}) \end{bmatrix} (\overline{g} + K_d w_{pred}) = \begin{bmatrix} \overline{u}_{pred} + u_{fb} \\ \overline{y}_{pred} + y_{fb} \end{bmatrix}$$

$$\begin{bmatrix} F_u & \mathbf{O} \\ \mathbf{O} & F_y \end{bmatrix} \begin{bmatrix} \overline{u}_{pred} + u_{fb} \\ \overline{y}_{pred} + y_{fb} \end{bmatrix} \leq \begin{bmatrix} f_u \\ f_y \end{bmatrix} .$$

Proof. In the control law (12), the nominal $\overline{g}$ generates a disturbance-free n_h step prediction. Hence, based on the prediction equation (11), we enforce

$$\forall w_{pred} \in \mathcal{W}$$

$$\begin{bmatrix} \mathcal{H}_{init} \\ \mathfrak{H}_{L,pred}(u_{\mathbf{d}}) \\ \mathfrak{H}_{L,pred}(w_{\mathbf{d}}) \\ \mathfrak{H}_{L,pred}(y_{\mathbf{d}}) \end{bmatrix} \Big(\underbrace{\overline{g}}_{(a)} + \underbrace{K_d w_{pred}}_{(b)}\Big) = \begin{bmatrix} h_{init} \\ \overline{u}_{pred} \\ \mathbf{0} \\ \overline{y}_{pred} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ u_{fb} \\ w_{pred} \\ y_{fb} \end{bmatrix} , \qquad (14)$$

where the matrix products of (a) and (b) correspond to the components on the right-hand side accordingly. As the future disturbance w_{pred} is unknown and arbitrary within the polytope $\mathcal{W}$, the matrix product of term (b) in (14) implies

$$\mathcal{H}_{init} K_d = \mathbf{O} , \ \mathfrak{H}_{L,pred}(w_{\mathbf{d}}) K_d = I. \qquad (15)$$

Due to the perturbation of the unknown future disturbance, the actual input and the actual output under the control law (12) are $\overline{u}_{pred} + u_{fb}$ and $\overline{y}_{pred} + y_{fb}$ respectively, which summarizes the proof with the robust constraints (13). ∎

Lemma 3 results in a robust data-driven control problem,

$$\min_{\substack{\overline{g},K_d,\\u_{pred},y_{pred}}} \max_{w_{pred}\in\mathcal{W}} J(\overline{y}_{pred},\overline{u}_{pred})$$

$$\text{s.t}\quad \begin{bmatrix} \mathcal{H}_{init} \\ \mathfrak{H}_{L,pred}(w_{\mathbf{d}}) \end{bmatrix}\overline{g} = \begin{bmatrix} h_{init} \\ \mathbf{0} \end{bmatrix},$$

$$\mathcal{H}_{init}K_d = \mathbf{O}\ ,\ \mathfrak{H}_{L,pred}(w_{\mathbf{d}})K_d = I\ ,$$

$$\forall\, w_{pred}\in\mathcal{W}\ ,$$

$$\begin{bmatrix} \mathfrak{H}_{L,pred}(u_{\mathbf{d}}) \\ \mathfrak{H}_{L,pred}(y_{\mathbf{d}}) \end{bmatrix}(\overline{g}+K_dw_{pred}) = \begin{bmatrix} \overline{u}_{pred}+u_{fb} \\ \overline{y}_{pred}+y_{fb} \end{bmatrix}$$

$$\begin{bmatrix} F_u & \mathbf{O} \\ \mathbf{O} & F_y \end{bmatrix}\begin{bmatrix} \overline{u}_{pred}+u_{fb} \\ \overline{y}_{pred}+y_{fb} \end{bmatrix} \leq \begin{bmatrix} f_u \\ f_y \end{bmatrix}.$$

(16)

This problem can be solved by a convex reformulation based on the dualization technique [2]. To clarify this procedure, we define

$$\begin{bmatrix} \tilde{f}_u \\ \tilde{f}_y \end{bmatrix} := \begin{bmatrix} f_u \\ f_y \end{bmatrix} - \begin{bmatrix} F_u & \mathbf{O} \\ \mathbf{O} & F_y \end{bmatrix}\begin{bmatrix} \mathfrak{H}_{L,pred}(u_{\mathbf{d}}) \\ \mathfrak{H}_{L,pred}(y_{\mathbf{d}}) \end{bmatrix}\overline{g};.$$

The resulting convex optimization is

$$\min_{\substack{\overline{g},K_d,\Lambda\\u_{pred},y_{pred}}} J(\overline{y}_{pred},\overline{u}_{pred})$$

$$\text{s.t.}\quad \begin{bmatrix} \mathcal{H}_{init} \\ \mathfrak{H}_{L,pred}(w_{\mathbf{d}}) \\ \mathfrak{H}_{L,pred}(u_{\mathbf{d}}) \\ \mathfrak{H}_{L,pred}(w_{\mathbf{d}}) \end{bmatrix}\overline{g} = \begin{bmatrix} h_{init} \\ \mathbf{0} \\ \overline{u}_{pred} \\ \overline{y}_{pred} \end{bmatrix},$$

$$\mathcal{H}_{init}K_d = \mathbf{O}\ ,\ \mathfrak{H}_{L,pred}(w_{\mathbf{d}})K_d = I\ ,$$

$$\Lambda^\top f_w \leq \begin{bmatrix} \tilde{f}_u \\ \tilde{f}_y \end{bmatrix},$$

$$F_w^\top \Lambda = K_d^\top \begin{bmatrix} F_u & \mathbf{O} \\ \mathbf{O} & F_y \end{bmatrix}^\top,\Lambda \geq \mathbf{O}\ ,\qquad (17)$$

where constraint (17) is imposed element-wise. The dual variable matrix is $\Lambda \in \mathbb{R}^{n_w \times n_{neq}}$ with n_{neq} the total number of inequality constraints imposed on inputs and outputs, each column of Λ corresponds to the dual variable of one inequality constraint of the outputs or the inputs.

B. Causal Robust DeePC

Before discussing the details of this section, we first recall some notations used in the previous sections, i-th row block of $\mathfrak{H}_{L,pred,i}(x_{\mathbf{d}})$ is $[x_{\mathbf{d},i+t_{init}}, x_{\mathbf{d},i+1+t_{init}}\dots, x_{\mathbf{d},i+n_c-1+t_{init}}]$. And we define $K_{d,:,j}$ as the j-th block column of feedback law K_d, which corresponds to the feedback generated by $w_{pred,i}$ and that is the $(n_w \times (i-1)) + 1$-th to $n_w \times i$-th entry of w_{pred}

Here starts the main result of this section. If the feedback matrix K_d is arbitrary, then the feedback control law is not necessarily causal. In particular, the feedback computed from $w_{pred,i}$ should not be able to change $u_{pred,1:i}$ and $y_{pred,1:i-1}$, because those events happen before $w_{pred,i}$.

Consider now a causal linear feedback control law on the disturbance

$$u_{pred} = \overline{u}_{pred} + K_w w_{pred}\ ,$$

where the feedback law K_w has a causal structure as

$$K_w := \begin{bmatrix} \mathbf{O} & & & \\ K_{w,1,0} & \mathbf{O} & & \\ \vdots & \ddots & \ddots & \\ K_{w,n_h,0} & \dots & K_{w,n_h,n_h-1} & \mathbf{O} \end{bmatrix}.$$

The sub-matrices $K_{w,i,j}$ are of size $\mathbb{R}^{n_u \times n_w}$. We define a standard robust MPC controller based on this causal control law as

$$\min_{\substack{u_{pred},K_w\\y_{pred}}} \max_{w_{pred}} J(y_{pred},u_{pred})$$

$$\text{s.t.}\quad x_{pred,0} = x_{t_{init}}$$

$$\forall\, i = 0,1\dots,n_h$$

$$x_{pred,i+1} = Ax_{pred,i} + Bu_{pred,i} + Ew_{pred,i}\qquad (18)$$

$$y_{pred,i} = Cx_{pred,i} + Du_{pred,i} + Fw_{pred,i}$$

$$\forall\, w_{pred}\in\mathcal{W}$$

$$u_{pred} = \overline{u}_{pred} + K_w w_{pred}$$

$$F_u u_{pred} \leq f_u\ F_y y_{pred} \leq f_y\ ,$$

where $u_{pred} := [u_{pred,0}^\top,\dots,u_{pred,n_h}^\top]$, y_{pred} and w_{pred} are defined accordingly. The goal of this section is to design a data-driven robust controller, whose resulting control law is identical to the model-based controller (18).

To construct the causal data-driven control law, we define

$$H = \begin{bmatrix} \mathfrak{H}_{L,init}(u_{\mathbf{d}}) \\ \mathfrak{H}_{L,init}(w_{\mathbf{d}}) \\ \mathfrak{H}_{L,init}(y_{\mathbf{d}}) \\ \mathfrak{H}_{L,pred,1}(u_{\mathbf{d}}) \\ \mathfrak{H}_{L,pred,1}(y_{\mathbf{d}}) \\ \vdots \\ \mathfrak{H}_{L,pred,n_h}(u_{\mathbf{d}}) \\ \mathfrak{H}_{L,pred,n_h}(y_{\mathbf{d}}) \end{bmatrix},\qquad (19)$$

with $H \in \mathbb{R}^{n_r \times n_c}$ and n_r the number of rows. The QR decomposition [17] of its transpose is

$$H^\top = \begin{bmatrix} Q_a & Q_b \end{bmatrix}\begin{bmatrix} R \\ \mathbf{O} \end{bmatrix},$$

Theorem 2: If the system (8) is observable, then the model-based robust control law in (18) is identical to the following data-driven control law,

$$\min_{\substack{\overline{g},K_p,\\u_{pred},y_{pred}}} \max_{w_{pred}\in\mathcal{W}} J(y_{pred},u_{pred})$$

$$\text{s.t.}\quad \begin{bmatrix} \mathcal{H}_{init} \\ \mathfrak{H}_{L,pred}(w_{\mathbf{d}}) \end{bmatrix}\overline{g} = \begin{bmatrix} h_{init} \\ \mathbf{0} \end{bmatrix}$$

$$K_d = \begin{bmatrix} Q_{a,:,n_{init}+1:n_r} & Q_b \end{bmatrix}K_p\ ,$$

$$\mathcal{H}_{init}K_d = \mathbf{O}\ ,\ \mathfrak{H}_{L,pred}(w_{\mathbf{d}})K_d = I$$

$$\forall\, w_{pred}\in\mathcal{W}$$

$$\begin{bmatrix} \mathfrak{H}_{L,pred}(u_{\mathbf{d}}) \\ \mathfrak{H}_{L,pred}(y_{\mathbf{d}}) \end{bmatrix}(\overline{g}+K_dw_{fb}) = \begin{bmatrix} \overline{u}_{pred}+u_{pred} \\ \overline{y}_{pred}+y_{fb} \end{bmatrix}$$

$$\begin{bmatrix} F_u & \mathbf{O} \\ \mathbf{O} & F_y \end{bmatrix}\begin{bmatrix} \overline{u}_{pred}+u_{fb} \\ \overline{y}_{pred}+y_{fb} \end{bmatrix} \leq \begin{bmatrix} f_u \\ f_y \end{bmatrix}$$

(20)

where $n_{init} := t_{init} \times (n_u + n_y) + n_u$ and K_p has a lower block triangular structure as

$$K_p = \begin{bmatrix} K_{p,1,1} & \mathbf{O} & \mathbf{O} & \ldots & \mathbf{O} \\ K_{p,2,1} & K_{p,2,2} & \mathbf{O} & \ldots & \mathbf{O} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ K_{p,n_h,1} & K_{p,n_h,2} & K_{p,n_h,3} & \ldots & K_{p,n_h,n_h} \end{bmatrix} .$$

$K_{p,i,j}$ are dense matrix blocks, whose sizes are $(n_u + n_y) \times n_w$ for $\forall\, i < n_h$ and are $[n_y + (n_c - n_r)] \times n_w$ when $i = n_h$.

Proof. First, it is observed that the data-driven formulation is based on the robust controller (16), where $\overline{y}_{pred}$ and $\overline{u}_{pred}$ are both well-defined. By Lemma 2, the control law will be equivalent once the causality is enforced.

We recall a useful property of QR decomposition [17]: the range of the first n rows of H is spanned by the first n columns of Q_a. Because $[Q_a, Q_b]$ is an unitary matrix, the remaining $n_r - n$ columns in Q_a and the matrix Q_b forms the null space of the first n rows in matrix H. Considering the constraint $\mathcal{H}_{init} K_d = \mathbf{O}$ in problem (16), each column of matrix K_d must lie within the null space of $\mathcal{H}_{init}$. Meanwhile, the feedback ingredients from w_{pred} cannot change the value of the first input $u_{pred,0}$. In conclusion, we enforce

$$\mathrm{colspan}(K_d) \subset \mathrm{colspan}([Q_{a,:,n_{init}+1:n_r}, Q_b]) ,$$

with $n_{init} := t_{init} \times (n_u + n_y) + n_u$. The j-th block column in matrix K_d, $K_{d,:,j}$, defines the feedback ingredient with respect to $w_{pred,j}$. By causality, the feedback from $w_{pred,j}$ should not be able to change the inputs and outputs that occur before $w_{pred,j}$. In particular, $K_{d,:,j}$ should further lie in the null space of the matrices $\mathfrak{H}_{L,pred,1:i}(u_\mathbf{d})$, $\mathfrak{H}_{L,pred,1:i-1}(y_\mathbf{d})$, we therefore enforce

$$K_{d,:,j} \subset \mathrm{colspan}([Q_{a,:,n_{init}+(j-1)(n_u+n_y)+1:n_r}, Q_b]) .$$

All the constraints on the null spaces can be reformulated as

$$K_d = \begin{bmatrix} Q_{a,:,n_{init}+1:n_r} & Q_b \end{bmatrix} K_p ,$$

which concludes the proof. ∎

We call the controller proposed in Theorem 2 a **robust data-enabled predictive controller (robust DeePC)**. The robust DeePC problem can also be reformulated into a convex optimization problem with the dualization technique discussed in the Section IV-A.

C. Discussion

In comparison with a model-based robust controller (18), the proposed controller distinguishes itself by a data-driven convex formulation. Meanwhile, the proposed scheme has the same scale of computational cost. In particular, the size of the optimization problem (20) only differs in the formulation of the feedback, where the number of decision variables in the feedback control law is $O(n_h \times (n_u + n_y))$ due to the causal reformulation. Hence, the computational cost of the robust DeePC control scheme is similar to the robust MPC.

In terms of the online data-driven control, the computational cost of the Hankel matrix update is low. In particular, the computational cost of the QR decompostion update by adding or removing a column scales linearly with respect to the size of the Hankel matrix [7, Section 6.5].

V. Numerical Example

In this section, numrical experiments are carried out to validate the proposed robust DeePC. First, a second order system is used to show the equivalence between the proposed robust DeePC and a the MPC (18) with full state measurement. After that, we test the proposed scheme in a building control problem to adapt power consumption with respect to the occupation patterns. The code is implemented with YALMIP [13] interfacing the GUROBI solver [8]

A. Second Order System

The proposed scheme is compared against a robust MPC controller (18) , and is tested on a second order system:

$$x_{i+1} = \begin{bmatrix} 0.9535 & 0.0761 \\ -0.8454 & 0.5478 \end{bmatrix} x_i + \begin{bmatrix} 0.0465 \\ 0.8454 \end{bmatrix} u_i + \begin{bmatrix} 0.0465 \\ 0.8454 \end{bmatrix} w_i$$

$$y_i = \begin{bmatrix} 1 & 0 \end{bmatrix} x_i ,$$

the process noise w is bounded within $[-0.1, 0.1]$, the inputs and outputs are constrained by $u \in [-5, 5], y \in [-0.5, 0.5]$. A quadratic stage cost is used

$$J(y_{pred}, u_{pred}) = \sum_{i=1}^{n_h} (y_{pred,i+1} - r)^\top Q (y_{pred,i+1} - r) + u_{pred,i}^\top R u_{pred,i} ,$$

where $Q = 10$, $R = 0.1$ and r is the reference. The Hankel matrices in the robust DeePC are built with a sequence of length 100. The tracking performance of the proposed controller is shown in Figure 1, where the robust MPC has full state measurement. As we claimed in Section IV, the response of the proposed controller is the same as the robust MPC (18) with full state measurement. The controller can safely protect the system away from the constraint by considering the perturbation caused by the future disturbance.
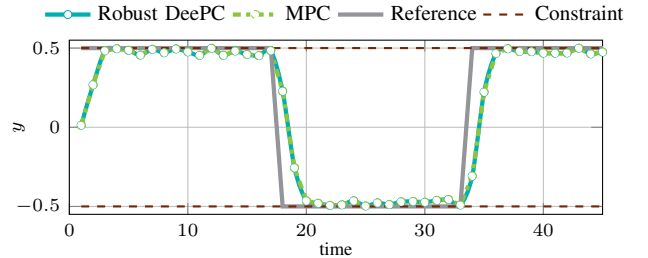

Fig. 1: Comparison of the Robust DeePC and the MPC controller. Note that the two responses are the same.

B. Building Control

We consider a single zone building model, which is disturbed by internal heat gain, solar radiation and external

temperature. The model used to generate the data is

$$x_{i+1} = \begin{bmatrix} 0.8511 & 0.0541 & 0.0707 \\ 0.1293 & 0.8635 & 0.0055 \\ 0.0989 & 0.0032 & 0.7541 \end{bmatrix} x_i + \begin{bmatrix} 0.0035 \\ 0.0003 \\ 0.0002 \end{bmatrix} u_i$$

$$+ 10^{-3} \begin{bmatrix} 22.2170 & 1.7912 & 42.2123 \\ 1.5376 & 0.6944 & 2.29214 \\ 103.1813 & 0.1032 & 196.0444 \end{bmatrix} w_i \ ,$$

$$y_i = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x_i \ ,$$

where x models the indoor temperature, wall temperature and the corridor temperature respectively. In a building control problem, the controller is designed to maintain occupant comfort while minimizing energy consumption. During the heating season, the indoor temperature is kept above $23°C$ to maintain occupant comfort during the day. When the room is not used at night, the room temperature is only required stay above $17°C$. Beyond the control requirements, the disturbances also show a time dependent pattern. Without loss of generality, we assume that during the day, the solar radiation and the internal heat gain are bounded within $[4, 6]$ with an external temperature fluctuating between $[6°C, 8°C]$. During the night, the solar radiation is 0 with much lower internal heat gain ranging between $[0, 2]$. Meanwhile, the external temperature is also lower at around $[2°C, 4°C]$. As the controller is designed to minimize power consumption, the loss function is

$$J(y_{pred}, u_{pred}) = \|u_{pred,i}\|_1 \ .$$

By building all the relevant Hankel matrices with a 100-step measurement sequence, the performance of the proposed controller is shown in Figure 2, where the operation starts from 6 A.M, and it is already overheated at that point before the controller effectively lowers the indoor temperature. It is also observed that the controller pre-heats the room to slightly above $23°C$ before the 6 A.M before the second morning. This whole cycle shows the effectiveness of the controller.
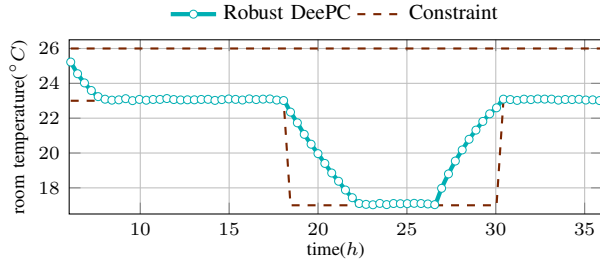

Fig. 2: Temperature control with rboust DeePC

VI. CONCLUSION

In this work, we show the equivalence between the SLS and the Willems' fundamental lemma for uncertain LTI systems. A convex data-driven controller is further proposed, which has a control law identical to the robust MPC. A toy example is used to show the equivalence between the robust DeePC and the robust MPC. The robust DeePC is further validated through a building control problem, which maintains occupants' comfort with minimal power consumption.

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